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Generalized Euler–Poincaré Equations on Lie Groups and Homogeneous Spaces, Orbit Invariants and Applications

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Abstract. We develop the necessary tools, including a notion of logarithmic derivative for curves in homogeneous spaces, for deriving a general class of equations including Euler–Poincaré equations on Lie groups and homogeneous spaces. Orbit invariants play an important role in this context and we use these invariants to prove global existence and uniqueness results for a class of PDE. This class includes Euler–Poincaré equations that have not yet been considered in the literature as well as integrable equations like Camassa–Holm, Degasperis–Procesi, μCH and μDP equations, and the geodesic equations with respect to right-invariant Sobolev metrics on the group of diffeomorphisms of the circle.

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1. Introduction

The Euler–Poincaré (EP) equations on a Lie group G are the Euler–Lagrange equations for G-invariant Lagrangian $L : TG \to \mathbb{R}$ [18,19]. They are written in the right-invariant case for the reduced Lagrangian l on the Lie algebra $\mathfrak{g}$ of G as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta u} = -\operatorname{ad}_u^*\frac{\delta l}{\delta u}, \tag{1.1}$$

while the left-invariant case has no-sign. We consider more general equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta u} = -\theta_u^*\frac{\delta l}{\delta u}, \tag{1.2}$$

where θ^* is the infinitesimal action of $\mathfrak{g}$ on $\mathfrak{g}^*$ associated with a right G-action on $\mathfrak{g}^*$n which replaces the coadjoint action. We call them generalized EP equations. This generalization is motivated by [17], where the action of the diffeomorphism group of the circle on λ-densities is considered, and the Degasperis–Procesi (DP) equation and the μDP equation are obtained for $\lambda = 3$. With the action of the diffeomorphism group of a manifold on 1-form λ-densities, we obtain a generalized EPDiff equation. We generalize the formalism in [10] and prove an abstract Noether theorem for the generalized EP equations (1.2).

Geodesic equations with right G-invariant metrics on homogeneous spaces of G are studied in [13] and [16], the main examples being the Hunter–Saxton equation and the multidimensional Hunter–Saxton equation, which are geodesic equations for right-invariant $\dot{H}^1$ metrics on the homogeneous spaces $S^1\backslash\operatorname{Diff}(S^1)$ and $\operatorname{Diff}_{\mathrm{vol}}(M)\backslash\operatorname{Diff}(M)$, respectively. We introduce EP equations on the homogeneous space $H\backslash G$ as Euler–Lagrange equations with general right G-invariant Lagrangian functions. For this purpose we introduce a right logarithmic derivative suited to homogeneous spaces of right cosets: the logarithmic derivative of a curve $\bar{\gamma}$ in $H\backslash G$ is the orbit of the logarithmic derivative of γ, an arbitrary lift of $\bar{\gamma}$, for the left action of $C^\infty(I, H)$ on $C^\infty(I, \mathfrak{g})$ given by $h \cdot u = \operatorname{Ad}(h)u + h'h^{-1}$. The reduced Lagrangians l on $\mathfrak{g}$ have to be at the same time $\mathfrak{h}$-invariant under addition and H-invariant under the adjoint action. An example is the $\dot{H}^{-1}$ Lagrangian on loop Lie algebras, which leads to the Landau–Lifschitz equation.

The replacement of the coadjoint action ad^* by a Lie algebra action θ^* of $\mathfrak{g}$ on $\mathfrak{g}^*$ produces generalized EP equations on the homogeneous space $H\backslash G$ if the group $C^\infty(I, H)$ is a symmetry group of the equation (1.2). This is accomplished if θ^* is H-equivariant and if the restriction of θ^* to the Lie algebra $\mathfrak{h}$ of H is the same as the restriction of the coadjoint action ad^* to $\mathfrak{h}$. The typical examples are the Lie algebra of vector fields on the circle acting on λ-densities and the Lie algebra of vector fields on a manifold with a volume form acting on 1-form $(\lambda - 1)$-densities. These lead us to the μBurgers equation [17] and to a multidimensional μBurgers equation.

We consider a class of nonlinear partial differential equations

$$\partial_t \Phi u + u\partial_x \Phi u + \lambda(\partial_x u)\Phi u = 0 \tag{1.3}$$

where $\Phi = \sum_{j=0}^{r}(-1)^j \partial_x^{2j}$ or $\Phi = \mu - \partial_x^2$. These equations are generalized right Euler–Poincaré equations (1.2) on the diffeomorphism group of the circle for the reduced Lagrangian $l(u) = \frac{1}{2}\int_{S^1} u\Phi u \mathrm{d}x$. Integrable equations such as Camassa–Holm, Degasperis–Procesi, μCH and μDP are notable members of this class of PDE. One also obtains all geodesic equations with respect to right-invariant Sobolev metrics on the group of diffeomorphisms of the circle. Using orbit invariants we prove global (in time) existence and uniqueness of classical solutions of the periodic Cauchy problem for Equation (1.3).

2. Generalized Euler–Poincaré Equations and an Abstract Noether Theorem

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The Euler–Lagrange equation (1.1) for a right-invariant Lagrangian $L: TG \to \mathbb{R}$, with value $l: \mathfrak{g} \to \mathbb{R}$ at the identity, is called the *right Euler–Poincaré* (EP) *equation* [18,19]. Here, $u = \gamma'\gamma^{-1} = \delta^r\gamma$ is the right logarithmic derivative of a curve γ in G, and $\frac{\delta l}{\delta u}$ is a curve in $\mathfrak{g}^*$.

For quadratic Lagrangians we obtain geodesic equations on Lie groups with right-invariant Riemannian metric determined by a non-degenerate inner product on $\mathfrak{g}$. In this case, the reduced Lagrangian writes $l(\xi) = \frac{1}{2}\langle\xi,\xi\rangle_{\mathfrak{g}} = \frac{1}{2}(A\xi,\xi)$, where the symmetric *inertia operator* $A: \mathfrak{g} \to \mathfrak{g}^*$ is injective. The derivative $\frac{\delta l}{\delta \xi} = A\xi = \langle\xi,\cdot\rangle_{\mathfrak{g}}$ can be identified with ξ, and the EP equation (1.1) is no other than the Euler equation $\frac{\mathrm{d}}{\mathrm{d}t}u = -\operatorname{ad}(u)^\top u$ under this identification [15]. Here, $\operatorname{ad}(\xi)^\top$ denotes the adjoint of ad_ξ with respect to the inner product on $\mathfrak{g}$, i.e. $A(\operatorname{ad}(\xi)^\top\eta) = \operatorname{ad}^*_\xi(A\eta)$ [21].

We consider a *generalized EP equation* associated with a right G-action Θ^* on $\mathfrak{g}^*$, namely (1.2), where u is the right logarithmic derivative of a curve γ in G, θ^* is the infinitesimal action associated with the group action Θ^*.

PROPOSITION 2.1. *The quantity* $\Theta^*_\gamma\left(\frac{\delta l}{\delta u}\right)$ *is preserved pointwise along the solutions u of the generalized EP equation* (1.2), *for any curve γ in G satisfying* $u = \gamma'\gamma^{-1}$.

With the classical notation $m = \frac{\delta l}{\delta u}$ for the momentum, the generalized EP equation becomes $\frac{\mathrm{d}}{\mathrm{d}t}m = -\theta^*_u m$ and the conserved quantity is $\Theta^*_\gamma(m)$. The meaning of this conservation law is that the solution remains on the same orbit of the (right) group action Θ^* in the dual space $\mathfrak{g}^*$. An abstract Noether theorem that formalizes the connection between invariance of coadjoint orbits and conserved quantities along the flow of the EP equation is proved in [10].

THEOREM 2.2. *Given a G-manifold $\mathscr{C}$ and a G-equivariant map $\kappa: \mathscr{C} \to \mathfrak{g}^{**}$, the Kelvin quantity* $I(c,u) = \left(\kappa(c), \frac{\delta l}{\delta u}\right)$ *is conserved for u solution of the generalized EP equation* (1.2) *and c driven by the flow: $c = \gamma\cdot c_0$ for γ a curve in G with $u = \gamma'\gamma^{-1}$.*

A tensor density of weight $\lambda \geq 0$ (respectively $\lambda < 0$) on S^1 is a section of the bundle $\bigotimes^\lambda T^*S^1$ (respectively, $\bigotimes^{-\lambda} TS^1$). There is a right action of the diffeomorphism group $\operatorname{Diff}(S^1)$ on each density module $\mathscr{F}_\lambda = \{m\mathrm{d}x^\lambda : m \in C^\infty(S^1)\}$ [22] given by $\gamma\cdot(m\mathrm{d}x^\lambda) = (m\circ\gamma)(\gamma')^\lambda\mathrm{d}x^\lambda$, which naturally generalizes the coadjoint action Ad^* on the regular dual of the Lie algebra $\mathfrak{X}(S^1)$, identified with the space of quadratic differentials. The infinitesimal action $L^\lambda_u(m\mathrm{d}x^\lambda) = (um' + \lambda u'm)\,\mathrm{d}x^\lambda$ can be thought of as the Lie derivative of tensor densities. It represents the right action of $\mathfrak{X}(S^1)$ on $\mathscr{F}_\lambda$. The adjoint action $\operatorname{Ad}_\gamma(u) = (u\circ\gamma^{-1})((\gamma^{-1})')^{-1}$ is the left action on $\mathscr{F}_{-1}$ and the coadjoint action Ad^* is the right action on $\mathscr{F}_2$.

EP equations on the group of diffeomorphisms on the circle with restricted Lagrangian $l: \mathfrak{X}(S^1) \to \mathbb{R}$ take the familiar form $\partial_t m = -um' - 2u'm$, $m = \frac{\delta l}{\delta u}$. For

l the quadratic Lagrangian given by the L^2 inner product $\langle u_1, u_2\rangle = \int_{S^1} u_1 u_2 \mathrm{d}x$, the H^1 inner product $\langle u_1, u_2\rangle_{H^1} = \int_{S^1}(u_1u_2 + u_1'u_2')\mathrm{d}x$, resp. the inner product $\langle u_1, u_2\rangle_\mu = \int_{S^1}(\mu(u_1)\mu(u_2) + u_1'u_2')\mathrm{d}x$, where $\mu(u) = \int_{S^1} u\mathrm{d}x$ is the mean of the function u on S^1, we get $m = u, m = u - u''$, resp. $m = \mu(u) - u''$, and geodesic equations:

$$\partial_t u = -3uu', \quad \partial_t u - \partial_t u'' + 3uu' - uu''' - 2u'u'' = 0, \quad \partial_t u'' = 2\mu(u)u' - 2u'u'' - uu''',$$

namely *Burgers*' equation, *Camassa–Holm* equation (CH) [4] resp. the μCH equation (it is introduced as μHS in [14]). The orbit invariant in Proposition 2.1 assures that $\mathrm{Ad}^*_\gamma m = (m \circ \gamma)(\gamma')^2$ is conserved, for a geodesic γ. Other geodesic equations on diffeomorphism groups can be found in [24].

Replacing the coadjoint action $\mathrm{ad}^*_u m = um' + 2u'm$, which is the action on 2-densities on the circle, with the action on λ-densities, we write down generalized EP equations on $\mathrm{Diff}(S^1)$ as $\partial_t m = -um' - \lambda u'm$.

EXAMPLE 2.3. The *Degasperis–Procesi* (DP) equation [6]

$$\partial_t u - \partial_t u'' + 4uu' - uu''' - 3u'u'' = 0 \tag{2.1}$$

admits a Lax pair and a bihamiltonian structure. It has a geometric interpretation on the space of tensor densities on the circle [17]. Let Θ be the left action of $\mathrm{Diff}(S^1)$ on $\mathscr{F}_{-2}$ and Θ^* the right action of $\mathrm{Diff}(S^1)$ on $\mathscr{F}_3$ its dual. The corresponding generalized EP equation on $\mathrm{Diff}(S^1)$ for the right-invariant H^1 Lagrangian is the DP equation $\partial_t m = -um' - 3u'm$ where $m = u - u''$. Applying Proposition 2.1 we obtain the conserved quantity $\Theta^*_\gamma(m) = (m \circ \gamma)(\gamma')^3, m = u - u''$ for $\gamma'\gamma^{-1} = u$. This conserved quantity is observed both in [8] and [17].

EXAMPLE 2.4. The μDP equation [17]

$$\mu(\partial_t u) - \partial_t u'' + 3\mu(u)u' - 3u'u'' - uu''' = 0 \tag{2.2}$$

is an integrable equation with Lax pair formulation and bihamiltonian structure. It can be written in the form $\partial_t m = -um' - 3u'm$ with $m = \mu(u) - u''$, as a generalized EP equation with $\theta^*_u = u\partial_x + 3u'$ and reduced Lagrangian $l(u) = \int_{S^1}(\mu(u)^2 + (u')^2)\mathrm{d}x$. Applying Proposition 2.1 to the μDP equation, we obtain the conserved quantity $\Theta^*_\gamma(m) = (m \circ \gamma)(\gamma')^3$, $m = \mu(u) - u''$ for $\gamma'\gamma^{-1} = u$. This conservation law is used in [17] to prove a global existence theorem for the periodic Cauchy problem for the μDP equation.

EXAMPLE 2.5. A quadratic Lagrangian l on the Lie algebra of vector fields $\mathfrak{X}(M)$ on a compact Riemannian manifold (M, g) can be expressed through a positive-definite symmetric operator Φ on $\mathfrak{X}(M)$ by $l(u) = \frac{1}{2}\int_M g(u, \Phi(u))\nu$, where ν is the canonical volume form on M. The *EPDiff* equation is the geodesic equation on $\mathrm{Diff}(M)$ with right-invariant metric defined with the symmetric operator

Φ. The regular dual of $\mathfrak{X}(M)$ is the space $\Omega^1(M)\otimes\mathrm{Den}(M)$ of 1-form densities. The momentum density of the fluid is $m=\frac{\delta l}{\delta u}=\Phi(u)^\flat\otimes\nu$, where the operator $\flat$ is associated with the Riemannian metric g. Since the coadjoint action is the Lie derivative, the EPDiff equation is $\partial_t m+L_u m=0$, and written for $\mathbf{m}=\Phi(u)$ in $\mathfrak{X}(M)$ it takes the well-known form [9]

$$\partial_t\mathbf{m}+u\cdot\nabla\mathbf{m}+(\nabla u)^\top\cdot\mathbf{m}+(\operatorname{div}u)\mathbf{m}=0. \tag{2.3}$$

In the special case $\Phi(u)=u-\Delta u$, one obtains a *higher dimensional CH* equation: the geodesic equation on $\mathrm{Diff}(M)$ for the right-invariant H^1 metric.

A generalized EP equation on $\mathrm{Diff}(M)$ can be obtained by considering the right action $\Theta^*_\gamma\alpha=J(\gamma)^{\lambda-1}\gamma^*\alpha$ of $\mathrm{Diff}(M)$ on $\Omega^1(M)$, where $J(\gamma)$ denotes the Jacobian of the diffeomorphism γ with respect to the volume form ν, i.e. $\gamma^*\nu=J(\gamma)\nu$. The infinitesimal action is $\theta^*_u\alpha=L_u\alpha+(\lambda-1)(\operatorname{div}u)\alpha$. For $\lambda=1$, it is the canonical action on 1-forms. For $\lambda=2$ we recover the coadjoint action $\mathrm{Ad}^*_\gamma\alpha=J(\gamma)\gamma^*\alpha$, while identifying the space of 1-form densities, the regular dual of $\mathfrak{X}(M)$, with $\Omega^1(M)$. In analogy to the case of tensor densities on the circle, we say that the above action Θ^* is the action of $\mathrm{Diff}(M)$ on the space $\Omega^1(M)\otimes\mathscr{F}_{\lambda-1}(M)$ of 1-form $(\lambda-1)$-densities. The generalized right EP equation (1.2) is simply $\partial_t\alpha+L_u\alpha+(\lambda-1)(\operatorname{div}u)\alpha=0$, $\alpha=\Phi(u)^\flat$. We call this the *generalized EPDiff equation*. A more familiar form is

$$\partial_t\mathbf{m}+u\cdot\nabla\mathbf{m}+(\nabla u)^\top\cdot\mathbf{m}+(\lambda-1)(\operatorname{div}u)\mathbf{m}=0,\quad \mathbf{m}=\Phi(u). \tag{2.4}$$

In the special case $\lambda=3$ and $\Phi(u)=u-\Delta u$, it extends the DP equation to higher dimensions.

A known circulation result says that the quantity $\int_c\frac{1}{\rho}\frac{\delta l}{\delta u}$ is conserved, for a loop c in M and a density ρ on M, both driven by the EPDiff flow, i.e. $c=\gamma\circ c_0$ and $\rho=(\gamma^{-1})^*\rho_0$ for $\gamma'\gamma^{-1}=u$ and fixed c_0 and ρ_0 [10]. The result fits in the setting of the abstract Noether theorem when choosing $\mathscr{C}=\mathscr{L}(M)\times\mathrm{Den}(M)$ the product of the space of loops and the space of densities on M, acted on by $\mathrm{Diff}(M)$: the equivariant map $\kappa:\mathscr{L}(M)\times\mathrm{Den}(M)\to\mathfrak{X}(M)^{**}$, $(\kappa(c,\rho),m)=\int_c\frac{1}{\rho}m$, $\quad m\in\mathfrak{X}(M)^*$ provides the Kelvin quantity $I=\int_c\frac{1}{\rho}\frac{\delta l}{\delta u}$. This means that for $\rho=f\nu$, $f\in C^\infty(M)$, driven by the EPDiff flow, the quantity $I=\int_c\frac{1}{f}\mathbf{m}^\flat$ is constant along (2.3). There is a circulation result also for the generalized EPDiff equation, obtained for the equivariant map $\kappa:\mathscr{L}(M)\times\mathscr{F}_{\lambda-1}(M)\to\mathfrak{X}(M)^{**}$, $(\kappa(c,f),\alpha)=\int_c\frac{1}{f}\alpha$, with $\alpha\in\Omega^1(M)$ the regular dual of $\mathfrak{X}(M)$, where the group of diffeomorphisms acts from the left on the space $\mathscr{F}_{\lambda-1}(M)$ of $(\lambda-1)$-densities by $\gamma^{-1}\cdot f=J(\gamma)^{\lambda-1}(f\circ\gamma)$, $f\in C^\infty(M)$.

3. Euler–Poincaré Equations on Homogeneous Spaces

In this section we study the Euler–Lagrange equations for left (respectively, right) invariant Lagrangians on the tangent bundle of a homogeneous space of left (respectively, right) cosets. For writing corresponding EP equations, we need a kind of logarithmic derivative for curves in homogeneous spaces.

Smooth curves in G/H can always be lifted to smooth curves in G, since $\pi: G\to G/H$ is a principal bundle. Given a smooth curve $\bar\gamma: I=[0,1]\to G/H$, we compare the left logarithmic derivatives of two smooth lifts $\gamma,\gamma_1: I\to G$ of $\bar\gamma$, i.e. $\bar\gamma=\pi\circ\gamma=\pi\circ\gamma_1$. There exists a smooth curve $h: I\to H$ such that $\gamma_1=\gamma h$; hence $u_1=\delta^l\gamma_1=\delta^l(\gamma h)=h^{-1}\gamma^{-1}(\gamma' h+\gamma h')=\mathrm{Ad}(h^{-1})u+\delta^l h$ for $u=\delta^l\gamma: I\to\mathfrak{g}$. We notice that u_1 is obtained from u via a right action of the group element $h\in C^\infty(I,H)$:

$$u\cdot h=\mathrm{Ad}(h^{-1})u+\delta^l h. \tag{3.1}$$

This means one can define the *left logarithmic derivative* $\bar\delta^l$ of a curve $\bar\gamma$ in G/H as an orbit under the right action (3.1) of $C^\infty(I,H)$ on $C^\infty(I,\mathfrak{g})$, namely the orbit $\bar u=u\cdot C^\infty(I,H)$ of the left logarithmic derivative u of an arbitrary lift $\gamma: I\to G$ of $\bar\gamma$:

$$\bar\delta^l: C^\infty(I,G/H)\to C^\infty(I,\mathfrak{g})/C^\infty(I,H),\quad \bar\delta^l\bar\gamma=\delta^l\gamma\cdot C^\infty(I,H).$$

When the subgroup H is trivial, we recover the ordinary logarithmic derivative on G.

Remark 3.1. In the same way one defines a *right logarithmic derivative* for curves on the homogeneous space $H\backslash G$ of right cosets: $\bar\delta^r\bar\gamma=C^\infty(I,H)\cdot\delta^r\gamma$, where the group $C^\infty(I,H)$ acts on $C^\infty(I,\mathfrak{g})$ from the left by $h\cdot u=\mathrm{Ad}(h)u+\delta^r h$. The rigid rotations of the circle form a subgroup isomorphic to S^1 of the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle. The left action of the group $C^\infty(I,S^1)$ on $C^\infty(I,\mathfrak{X}(S^1))$ is

$$(a\cdot u)(t)(x)=u(t)(x-\tilde a(t))+\tilde a'(t),\quad t\in I, x\in\mathbb{R}, \tag{3.2}$$

where $\tilde a\in C^\infty(I,\mathbb{R})$ is any lift of the group element $a\in C^\infty(I,S^1)$ and vector fields on S^1 are identified with periodic functions on $\mathbb{R}$. This action is involved in the definition of the right logarithmic derivative $\bar\delta^r$ on the homogeneous space $S^1\backslash\mathrm{Diff}(S^1)$.

The tangent bundle TG of a Lie group G carries a natural group multiplication, the tangent map of the group multiplication on G. Given a subgroup H of G, its tangent bundle TH is a subgroup of TG and the submersion $T\pi: TG\to T(G/H)$ induces a diffeomorphism between TG/TH and $T(G/H)$. A left G-invariant Lagrangian $\bar L: T(G/H)\to\mathbb{R}$ determines a left G-invariant and right TH-invariant Lagrangian $L=\bar L\circ T\pi: TG\to\mathbb{R}$. The left G-invariance and right TH-invariance of L translates into H-invariance under adjoint action and $\mathfrak{h}$-invariance under vector addition of its restriction $l:\mathfrak{g}\to\mathbb{R}$. The left G-invariant

Lagrangian $\bar{L}:T(G/H)\to\mathbb{R}$ is uniquely determined by its restriction $\bar{l}$ to the tangent space $T_o(G/H)=\mathfrak{g}/\mathfrak{h}$ at $o=eH\in G/H$. If $p:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ denote the canonical projection, then $l=\bar{l}\circ p$. The adjoint action of H on $\mathfrak{g}$ induces an adjoint action of H on $\mathfrak{g}/\mathfrak{h}$, and the H-invariance of l translates into an H-invariance of $\bar{l}$. These results are summarized in the next proposition.

PROPOSITION 3.2. *The following are equivalent data: left G-invariant function $\bar{L}$ on $T(G/H)$, right TH-invariant and left G-invariant function L on TG, $\mathfrak{h}$-invariant and* $\mathrm{Ad}(H)$-*invariant function l on $\mathfrak{g}$, and H-invariant function $\bar{l}$ on $\mathfrak{g}/\mathfrak{h}$.*

THEOREM 3.3. *A solution of the Euler–Lagrange equation for a left G-invariant Lagrangian $\bar{L}:T(G/H)\to\mathbb{R}$ is a curve $\bar{\gamma}$ in G/H such that the left logarithmic derivative $u=\gamma^{-1}\gamma'$ of a lift γ of $\bar{\gamma}$ satisfies the left EP equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta u}=\mathrm{ad}_u^*\frac{\delta l}{\delta u}, \tag{3.3}$$

for l the ($\mathfrak{h}$-*invariant and* $\mathrm{Ad}(H)$-*invariant*) *restriction of $L=\bar{L}\circ T\pi$ to $\mathfrak{g}$.*

Proof. A variation with fixed endpoints of the curve $\bar{\gamma}$ in G/H can be lifted to a variation with endpoints in H of a lift γ in G of $\bar{\gamma}$, i.e. $\bar{\gamma}=\pi\circ\gamma$. Denoting by $u=\gamma^{-1}\gamma'$ the left logarithmic derivative of γ and by v the left logarithmic derivative of the variation of γ, we get $\delta u=\frac{\mathrm{d}v}{\mathrm{d}t}+[u,v]$, so

$$\begin{aligned}0&=\delta\int\bar{L}(\bar{\gamma}'(t))\mathrm{d}t=\delta\int L(\gamma'(t))\mathrm{d}t=\delta\int l(u)\mathrm{d}t=\int\Big(\frac{\delta l}{\delta u},\delta u\Big)\mathrm{d}t\\&=\int\Big(\frac{\delta l}{\delta u},\frac{\mathrm{d}v}{\mathrm{d}t}+\mathrm{ad}_u v\Big)\mathrm{d}t=\int\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\delta l}{\delta u},v\Big)\mathrm{d}t+\int\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta u}+\mathrm{ad}_u^*\frac{\delta l}{\delta u},v\Big)\mathrm{d}t.\end{aligned}$$

The variation of γ has endpoints in H and l is $\mathfrak{h}$-invariant, so the first term vanishes. Hence, u satisfies the EP equation (3.3). It is easy to verify that if the EP equation is satisfied by the logarithmic derivative of one lift of $\bar{\gamma}$, it is satisfied by the logarithmic derivative of any lift of $\bar{\gamma}$. □

We call (3.3) the left EP equation on the homogeneous manifold G/H. One has an orbit invariant for this equation, similar to the one in Proposition 2.1, as well as an abstract Noether theorem for homogeneous spaces.

PROPOSITION 3.4. *The quantity $\mathrm{Ad}^*_{\gamma^{-1}}\frac{\delta l}{\delta u}\in\mathfrak{g}^*$ is conserved along the left EP equation* (3.3) *on G/H with $\mathfrak{h}$-invariant and H-invariant Lagrangian function l on $\mathfrak{g}$, where γ is any lift of $\bar{\gamma}$ and $u=\delta^l\gamma$.*

THEOREM 3.5. *Considering a G-manifold $\mathscr{C}$ and a map $\kappa:\mathscr{C}\to\mathfrak{g}^{**}$ which is G-equivariant, the Kelvin quantity $I:\mathscr{C}\times\mathfrak{g}\to\mathbb{R}$, $I(c,u)=\Big(\kappa(c),\frac{\delta l}{\delta u}\Big)$ is conserved along solutions $\bar{\gamma}$ of the left Euler–Lagrange equation on G/H.*

Remark 3.6. A special Euler–Lagrange equation is the Euler equation on homogeneous spaces: geodesic equation for a left G-invariant Riemannian metric on G/H. It was studied in [13], using the Hamiltonian point of view.

Let $A:\mathfrak{g}\to\mathfrak{g}^*$ be a symmetric degenerate (inertia) operator with kernel $\mathfrak{h}$, such that A is H-equivariant. We consider the Lagrangian $l(\xi)=\frac{1}{2}(A\xi,\xi)$ on $\mathfrak{g}$. Taking into account the symmetry of A, the $\mathfrak{h}$-invariance of l is easily checked. The H-equivariance of A ensures the H-invariance of l, so l descends to an H-invariant Lagrangian $\bar{l}:\mathfrak{g}/\mathfrak{h}\to\mathbb{R}$, as needed for the EP equation (3.3) on homogeneous spaces. Now $m=\frac{\delta l}{\delta u}=Au$, so the left EP equation on G/H is the image under the inertia operator A of the left-invariant Euler equation: $\frac{\mathrm{d}}{\mathrm{d}t}u=\mathrm{ad}(u)^\top u$. This can be interpreted as the geodesic equation on G/H for the left-invariant Riemannian metric coming from the degenerate inner product $\langle\xi,\eta\rangle_{\mathfrak{g}}=(A\xi,\eta)$.

EXAMPLE 3.7. The *Hunter–Saxton* equation describing weakly nonlinear unidirectional waves [11]

$$\partial_t u''=-2u'u''-uu''' \tag{3.4}$$

is a geodesic equation on the homogeneous space $S^1\backslash\mathrm{Diff}(S^1)$ of right cosets with the right-invariant metric defined by the degenerate $\dot{H}^1$ inner product $\langle u_1,u_2\rangle=\int_{S^1}u_1'u_2'\mathrm{d}x$ on $\mathfrak{X}(S^1)$ [13].

In this case $l(u)=\frac{1}{2}\langle u,u\rangle=\frac{1}{2}\int_{S^1}(u')^2\mathrm{d}x$, so $m=\frac{\delta l}{\delta u}=-u''$ satisfies $\partial_t m=-um'-2u'm$, which gives Hunter–Saxton equation (3.4). It has to be read as an equation for the orbit $\bar{u}=C^\infty(I,S^1)\cdot u$ of $u\in C^\infty(I,\mathfrak{X}(S^1))$ under the left action (3.2), which plays the role of the right logarithmic derivative of a curve $\bar{\gamma}:I\to S^1\backslash\mathrm{Diff}(S^1)$. A conserved quantity for the Hunter–Saxton equation is $\mathrm{Ad}^*_\gamma m=-(u''\circ\gamma)(\gamma')^2$, where $\gamma:I\to\mathrm{Diff}(S^1)$ is any lift of the curve $\bar{\gamma}$.

EXAMPLE 3.8. This example concerns the *multidimensional Hunter–Saxton* equation from [16]. Let M be a compact manifold and let ν be a fixed volume form on M. We consider the homogeneous space $\mathrm{Diff}_{\mathrm{vol}}(M)\backslash\mathrm{Diff}(M)$ of right cosets. The Lagrangian $l(u)=\frac{1}{2}\int_M(\mathrm{div}\,u)^2\nu$ on $\mathfrak{X}(M)$ is both $\mathfrak{X}_{\mathrm{vol}}(M)$-invariant and $\mathrm{Diff}_{\mathrm{vol}}(M)$-invariant, so we have a corresponding EP equation

$$\partial_t m=-L_u m,\quad m=\frac{\delta l}{\delta u}=-d(\mathrm{div}\,u)\nu\in\mathfrak{X}(M)^*_{reg}=\Omega^1(M)\otimes\mathrm{Den}(M)$$

on the homogeneous space $\mathrm{Diff}_{\mathrm{vol}}(M)\backslash\mathrm{Diff}(M)$.

We replace the special form of the momentum m and we drop the constant density ν to obtain the equation $\partial_t d(\mathrm{div}\,u)=-dL_u(\mathrm{div}\,u)-(\mathrm{div}\,u)\mathrm{d}(\mathrm{div}\,u)$ in $\Omega^1(M)$. It coincides with the Hunter–Saxton equation when $M=S^1$: the subgroup of volume preserving diffeomorphisms of the circle is the subgroup of rigid rotations of the circle.

EXAMPLE 3.9. Let K be a Lie group with Lie algebra $\mathfrak{k}$ possessing a K-invariant inner product $\langle,\rangle_{\mathfrak{k}}$. The Lie algebra of the loop group $LK := C^\infty(S^1, K)$ is the loop algebra $L\mathfrak{k} = C^\infty(S^1, \mathfrak{k})$ with bracket $[\ ,\ ]_{\mathfrak{k}}$. The inner product permits the identification of the regular dual of $L\mathfrak{k}$ with $L\mathfrak{k}$. The subgroup of constant loops, identified with K, defines the homogeneous space of right cosets $K\backslash LK$. Each $\mathfrak{k}$-invariant and K-invariant Lagrangian l on $L\mathfrak{k}$ determines a right EP equation on $K\backslash LK$:

$$\partial_t m = [u, m]_{\mathfrak{k}}, \quad m = \frac{\delta l}{\delta u}, \tag{3.5}$$

If the Lagrangian is defined by the $\dot{H}^1$ inner product $l(u) = \frac{1}{2}\int_{S^1}\langle u', u'\rangle_{\mathfrak{k}} \mathrm{d}x$, then $m = -u''$ and the EP equation (3.5) becomes $\partial_t u'' = [u, u'']_{\mathfrak{k}}$. Another possibility is the Lagrangian $l(u) = \frac{1}{2}\int_{S^1}\langle u - \mu(u), u - \mu(u)\rangle_{\mathfrak{k}} \mathrm{d}x$. This time $m = u - \mu(u)$, so $\partial_t u - \mu(\partial_t u) = -[u, \mu(u)]_{\mathfrak{k}}$. More important is the $\dot{H}^{-1}$ Lagrangian

$$l(u) = \frac{1}{2}\int_{S^1}\langle \partial_x^{-1} u, \partial_x^{-1} u\rangle_{\mathfrak{k}} \mathrm{d}x = -\frac{1}{2}\int_{S^1}\langle \partial_x^{-2} u, u\rangle_{\mathfrak{k}} \mathrm{d}x \tag{3.6}$$

because it leads to the *Landau–Lifschitz* equation [1,12]. It is defined on the homogeneous space $\mathfrak{k}\backslash L\mathfrak{k}$, which can be identified with the space of all derivatives of loops in $\mathfrak{k}$. It is also K-invariant, so it fits well into our setting for EP equations on homogeneous spaces. We get $m = -\partial_x^{-2} u$, so $u = -m''$, and (3.5) becomes $\partial_t m = [m, m'']_{\mathfrak{k}}$. In the special case $K = SO(3)$ we get the Landau–Lifschitz equation $\partial_t L = L \times L''$, where one identifies the Lie algebras $(\mathfrak{so}(3), [\ ,\ ])$ and $(\mathbb{R}^3, \times)$. This equation is the vortex filament equation $\partial_t c = c' \times c''$, for $L = c'$ the tangent vector to the filament, a closed arc-parametrized time-dependent curve c in $\mathbb{R}^3$.

4. Generalized EP Equations on Homogeneous Spaces

The coadjoint action ad^* in the right EP equation on the homogeneous space $H\backslash G$ of right cosets can be replaced with another action θ^*. The result is a generalized right EP equation on $H\backslash G$, similarly to the generalized EP equation on Lie groups in Section 2. This time we have to impose some conditions on θ^*, so that the following holds: if the right logarithmic derivative $u = \delta^r \gamma$ of one lift of the curve $\bar{\gamma}$ satisfies the generalized right EP equation $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta u} = -\theta_u^* \frac{\delta l}{\delta u}$, then the right logarithmic derivatives of all the other lifts of $\bar{\gamma}$ satisfy the same equation. In other words the group $C^\infty(I, H)$ with the left action in Remark 3.1 has to be a symmetry group of the above-generalized right EP equation (obviously true for $\theta^* = \mathrm{ad}^*$).

PROPOSITION 4.1. *Let H be a subgroup of G with Lie algebra $\mathfrak{h}$, and let θ^* be a Lie algebra action of $\mathfrak{g}$ on $\mathfrak{g}^*$. If the map $\theta^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is H-equivariant and if the action θ^* restricted to $\mathfrak{h}$ equals the coadjoint action* ad^* *restricted to $\mathfrak{h}$, then*

$C^\infty(I,H)$ *with the left action in Remark* 3.1 *is a symmetry group of*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta l}{\delta u}=-\theta_u^*\frac{\delta l}{\delta u}, \tag{4.1}$$

Proof. We have to show that for any solution $u\in C^\infty(I,\mathfrak{g})$ of (4.1), and for any $h\in C^\infty(I,H)$, the curve $h\cdot u=\mathrm{Ad}(h)u+\delta^r h\in C^\infty(I,\mathfrak{g})$ is again a solution of (4.1). In the computation below, we will use the fact that h acts on $m=\frac{\delta l}{\delta u}$ by the coadjoint action: $h\cdot m=\mathrm{Ad}^*_{h^{-1}}m$. The H-equivariance of θ^* means that $\theta^*_{\mathrm{Ad}(h)u}(\mathrm{Ad}^*_{h^{-1}}m)=\mathrm{Ad}^*_{h^{-1}}\theta^*_u m$. Knowing also that $\theta^*_{\delta^r h}=\mathrm{ad}^*_{\delta^r h}$ for any curve $h\in C^\infty(I,H)$, we compute

$$\begin{aligned}\partial_t(h\cdot m)+\theta^*_{h\cdot u}(h\cdot m)&=\partial_t(\mathrm{Ad}^*_{h^{-1}}m)+\theta^*_{\mathrm{Ad}(h)u+\delta^r h}(\mathrm{Ad}^*_{h^{-1}}m)\\&=\mathrm{Ad}^*_{h^{-1}}(\partial_t m)-\mathrm{ad}^*_{\delta^r h}(\mathrm{Ad}^*_{h^{-1}}m)+\theta^*_{\delta^r h}(\mathrm{Ad}^*_{h^{-1}}m)+\theta^*_{\mathrm{Ad}(h)u}(\mathrm{Ad}^*_{h^{-1}}m)\\&=\mathrm{Ad}^*_{h^{-1}}(\partial_t m+\theta^*_u m).\end{aligned}$$

This shows the symmetry of the generalized right EP equation under $C^\infty(I,H)$. □

Remark 4.2. This proposition ensures that if the two conditions on θ^* (i.e. to be H-equivariant and to extend ad^* on $\mathfrak{h}$) are satisfied, then (4.1) is an equation on the homogeneous space $H\backslash G$. The orbit-invariant $\Theta^*_\gamma\frac{\delta l}{\delta u}$ is an orbit invariant for the generalized right EP equation on homogeneous spaces.

When $G=\mathrm{Diff}(S^1)$ and $H=S^1$, the action of $\mathfrak{X}(S^1)$ on λ-densities, i.e. $\theta^*_u m=um'+\lambda u'm$, satisfies the two conditions required in Proposition 4.1. This means we get a generalized EP equation on the homogeneous space $S^1\backslash\mathrm{Diff}(S^1)$ in the form

$$\partial_t m=-um'-\lambda u'm,\quad m=\frac{\delta l}{\delta u}. \tag{4.2}$$

The same thing can be done in higher dimensions too. Let $G=\mathrm{Diff}(M)$ be the diffeomorphism group of a connected compact manifold, and $H=\mathrm{Diff}_{\mathrm{vol}}(M)$ the subgroup of volume preserving diffeomorphisms, where ν is a fixed volume form on M. As in Example 2.5, we identify $\Omega^1(M)$ with the regular dual of $\mathfrak{X}(M)$ using ν, so the coadjoint action can be written as $\mathrm{ad}^*_u\alpha=L_u\alpha+(\mathrm{div}\,u)\alpha$, an action on 1-form densities. The action θ^* on 1-form $(\lambda-1)$-densities: $\theta^*_u\alpha=L_u\alpha+(\lambda-1)(\mathrm{div}\,u)\alpha$ satisfies the conditions required in Proposition 4.1and hence it provides a generalized EP equation on the homogeneous space $\mathrm{Diff}_{\mathrm{vol}}(M)\backslash\mathrm{Diff}(M)$:

$$\partial_t\alpha=-L_u\alpha-(\lambda-1)(\mathrm{div}\,u)\alpha,\quad \alpha=\frac{\delta l}{\delta u}. \tag{4.3}$$

The first condition on θ^* can be verified as follows:

$$\begin{aligned}\theta^*_{\mathrm{Ad}_{h^{-1}}u}\mathrm{Ad}^*_h\alpha&=\theta^*_{h^*u}(h^*\alpha)=L_{h^*u}(h^*\alpha)+(\lambda-1)(\mathrm{div}\,h^*u)h^*\alpha\\&=h^*(L_u\alpha+(\lambda-1)(\mathrm{div}\,u)\alpha)=h^*(\theta^*_u\alpha)=\mathrm{Ad}^*_h(\theta^*_u\alpha),\end{aligned}$$

using the fact that $J(h)=1$ and $\operatorname{div}(h^*u)=h^*\operatorname{div}u$ for all $h\in\operatorname{Diff}_{\mathrm{vol}}(M)$. The second condition follows from $\theta_w^*\alpha=L_w\alpha=\operatorname{ad}_w^*\alpha$ for all $w\in\mathfrak{X}_{\mathrm{vol}}(M)$.

EXAMPLE 4.3. If the Lagrangian is given by the $\dot{H}^1$ inner product: $l(u)=\frac{1}{2}\int_{S^1}(u')^2\mathrm{d}x$, the generalized EP equation (4.2) becomes $\partial_t m=-um'-\lambda u'm$, $m=-u''$. For $\lambda=2$ one obtains a geodesic equation: the Hunter–Saxton equation from Example 3.7. For $\lambda=3$ one obtains a generalized EP equation: the *μBurgers* equation

$$-\partial_t u''-3u'u''-uu'''=0, \tag{4.4}$$

which is shown to admit a Lax pair formulation and a bihamiltonian structure in [17]. This terminology is related to a reformulation of this equation as $(\partial_t u+uu')'=0$ and hence as $\partial_t u+uu'=-\mu(\partial_t u)$, where μ denotes the mean of a function on the circle.

Applying Remark 4.2, we get the conserved quantity $\Theta_\gamma^*(m)=(m\circ\gamma)(\gamma')^3$ where $m=-u''$ for any lift $\gamma:I\to\operatorname{Diff}(S^1)$ of the solution curve $\bar{\gamma}:I\to S^1\backslash\operatorname{Diff}(S^1)$ whose right logarithmic derivative is $\bar{u}=C^\infty(I,S^1)\cdot u$, i.e. $\gamma'\circ\gamma^{-1}=u$.

EXAMPLE 4.4. As observed for the multidimensional Hunter–Saxton equation from Example 3.8, the Lagrangian $l(u)=\frac{1}{2}\int_M(\operatorname{div}u)^2\nu$ on $\mathfrak{X}(M)$ is both $\mathfrak{X}_{\mathrm{vol}}(M)$-invariant and $\operatorname{Diff}_{\mathrm{vol}}(M)$-invariant. The generalized EP equation (4.3) on the homogeneous space $\operatorname{Diff}_{\mathrm{vol}}(M)\backslash\operatorname{Diff}(M)$ becomes

$$\partial_t d(\operatorname{div}u)=-dL_u(\operatorname{div}u)-(\lambda-1)(\operatorname{div}u)d(\operatorname{div}u). \tag{4.5}$$

With the notation $\omega=\operatorname{div}u$, this equation is $d(\partial_t\omega+L_u\omega+\frac{\lambda-1}{2}\omega^2)=0$.

For $\lambda=3$, this generalized EP equation is the *multidimensional μBurgers* equation. It can be rewritten as $d\operatorname{div}(\partial_t u+(\operatorname{div}u)u)=0$. We deduce that the time-dependent function $\operatorname{div}(\partial_t u+(\operatorname{div}u)u)$ is constant on M. Its integral over M vanishes, so it is the zero function, and we have a simpler expression for the multidimensional μBurgers equation: $\operatorname{div}(\partial_t u+(\operatorname{div}u)u)=0$, thus generalizing the μBurgers equation on the circle $(\partial_t u+u'u)'=0$. With Remark 4.2, we get a conserved quantity: $\Theta_\gamma^*\alpha=J(\gamma)^2\gamma^*\alpha$ for $\alpha=-d(\operatorname{div}u)$.

EXAMPLE 4.5. In the setting of Example 3.9, with a linear function $\operatorname{tr}:\mathfrak{k}\to\mathbb{R}$ vanishing on all brackets in $\mathfrak{k}$, we build a Lie algebra action of $L\mathfrak{k}$ on its regular dual, $\theta_u^*(m)=\operatorname{ad}_u^*(m)+(\operatorname{tr}u')m$, which satisfies both conditions from Proposition 4.1. T corresponding generalized EP equation on $K\backslash LK$ is $\partial_t m=[u,m]_{\mathfrak{k}}+(\operatorname{tr}u')m$, $m=\frac{\delta l}{\delta u}$, after the identification of $L\mathfrak{k}$ with its regular dual using the given K-invariant inner product. For the $\dot{H}^{-1}$ Lagrangian (3.6) we get $u=-m''$, and the generalized EP equation above becomes $\partial_t m=[m,m'']_{\mathfrak{k}}-(\operatorname{tr}m''')m$.

5. Applications: Orbit Invariants in Global Existence Results

In this section we consider equations from mathematical physics that are generalized EP equations and prove global existence and uniqueness of solutions to the associated periodic Cauchy problems using Proposition 2.1.

The four integrable equations: CH, μCH, DP and μDP from Section 2 are special cases of the equation

$$\partial_t m = -um' - \lambda u' m, \quad m = \Phi u, \tag{5.1}$$

where the operator Φ on the space of smooth functions on the circle is either a linear differential operator of the form $\sum_{j=0}^{r}(-1)^j \partial_x^{2j}$ or the linear operator $\mu - \partial_x^2$, where $\mu(u)$ is the mean of the function u on S^1. The equation (5.1) is a generalized right Euler–Poincaré equations 1.2 on the group of diffeomorphisms of the circle for the reduced Lagrangian $l(u) = \frac{1}{2}\int_{S^1} u\Phi u \mathrm{d}x$. In this case the $\mathrm{Diff}(S^1)$ action Θ^* is the action on λ-densities on the circle, with associated infinitesimal action $\theta_u^* f = uf' + \lambda u' f$. The coadjoint action is obtained for $\lambda = 2$ and in this special case (5.1) is the geodesic equation on $\mathrm{Diff}(S^1)$ with respect to the right-invariant metric defined by the H^r inner product.

We consider the periodic Cauchy problem for (5.1):

$$\partial_t u + uu' = -\Phi^{-1}\left([u, \Phi]u' + \lambda u' \Phi u\right), \quad u(0, x) = u_0(x), \quad x \in S^1, t \in \mathbb{R}^+ \tag{5.2}$$

where $\Phi : H^s \to H^{s-r}$, $\Phi = \sum_{j=0}^{r}(-1)^j \partial_x^{2j}$ and λ is an arbitrary real number. The main result of this section is the following global (in time) existence and uniqueness theorem.

THEOREM 5.1. *Let $s > 2r + \frac{1}{2}$. Assume that the initial data $u_0 \in H^s(S^1)$ satisfy $\Phi u_0 \geq 0$. Then the Cauchy problem* (5.2) *has a unique global solution $u \in C(\mathbb{R}^+, H^s(S^1)) \cap C^1(\mathbb{R}^+, H^{s-1}(S^1))$.*

We postpone the proof until the end of this section and proceed to establish local well-posedness (existence, uniqueness and continuous dependence on initial data of solutions for a short time) and persistence of solutions of the Cauchy problem (5.2).

THEOREM 5.2. *Let $s > 2r + \frac{1}{2}$. Then the periodic Cauchy problem* (5.2) *has a unique solution $u \in C([0, T), H^s(S^1)) \cap C^1([0, T), H^{s-1}(S^1))$ for some $T > 0$ and the solution depends continuously on initial data.*

Our proof of this theorem uses an approach developed in [7] for Euler and Navier–Stokes equations. Let $\gamma(t)$ denote the flow of $u(t)$, i.e. $u(t, x) = \dot{\gamma}(t, \gamma^{-1}(t, x))$. For convenience we use the notation $\Psi_\gamma \xi := \left(\Psi(\xi \circ \gamma^{-1})\right) \circ \gamma$ for a pseudodifferential operator Ψ. We write the Cauchy problem (5.2) as an initial value problem for an ODE in the form

$$\ddot{\gamma}=-\Phi_{\gamma}^{-1}\left([\dot{\gamma},\Phi_{\gamma}]\partial_{x\gamma}\dot{\gamma}+\lambda\dot{\gamma}'\Phi_{\gamma}\dot{\gamma}\right),\quad \dot{\gamma}'(0,x)=u_0(x),\quad \gamma(0,x)=x \tag{5.3}$$

and the local well-posedness of (5.2) follows from Picard iterations if $F(\gamma,\dot{\gamma})=-\Phi_{\gamma}^{-1}\left([\dot{\gamma},\Phi_{\gamma}]\partial_{x\gamma}\dot{\gamma}+\lambda\dot{\gamma}'\Phi_{\gamma}\dot{\gamma}\right)$ defines a continuously differentiable map from $\mathscr{D}^s\times H^s(S^1)$ into $H^s(S^1)$. Here, $\mathscr{D}^s$ denotes orientation preserving circle diffeomorphisms of Sobolev class H^s.

Proof of Theorem 5.2. We use the symbol $\lesssim$ to denote $\le C_{\gamma}$ where C_{γ} is a constant that depends on λ and the H^s norms of γ and γ^{-1}. Three important results about Sobolev spaces are used in the following estimates repeatedly: The algebra property of H^s for $s>1/2$, the Sobolev imbedding theorem $C^1\hookrightarrow H^s$ for $s>3/2$ and the composition lemma (see the Appendix in [2] and lemma 4.1 in [20]).

Our first estimate

$$\begin{aligned}\|F(\gamma,\dot{\gamma})\|_{H^s}&\lesssim\|[\dot{\gamma}\circ\gamma^{-1},\Phi]\partial_x(\dot{\gamma}\circ\gamma^{-1})\|_{H^{s-2r}}+\|\partial_x(\dot{\gamma}\circ\gamma^{-1}).\Phi(\dot{\gamma}\circ\gamma^{-1})\|_{H^{s-2r}}\\&\lesssim\|\dot{\gamma}\circ\gamma^{-1}\|_{H^s}^2+\|\dot{\gamma}\circ\gamma^{-1}\|_{H^{s-2r+1}}\|\dot{\gamma}\circ\gamma^{-1}\|_{H^s}\lesssim\|\dot{\gamma}\|_{H^s}^2\end{aligned}$$

follows from composition lemma and algebra property of Sobolev spaces and establishes that F is a bounded map from $\mathscr{D}^s\times H^s$ into H^s.

The directional derivatives of F are given by the formulas

$$\begin{aligned}\partial_{\dot{\gamma}}F_{(\gamma,\dot{\gamma})}(X)=&-\Phi_{\gamma}^{-1}\big([X,\Phi_{\gamma}]\partial_{x\gamma}\dot{\gamma}+[\dot{\gamma},\Phi_{\gamma}]\partial_{x\gamma}X\big)\\&-\lambda\Phi_{\gamma}^{-1}\big((\Phi_{\gamma}\dot{\gamma})\partial_{x\gamma}X+(\Phi_{\gamma}X)\partial_{x\gamma}\dot{\gamma}\big),\end{aligned} \tag{5.4}$$

$$\begin{aligned}\partial_{\gamma}F_{(\gamma,\dot{\gamma})}(X)=&[X,\Phi_{\gamma}^{-1}]\big((\Phi_{\gamma}\dot{\gamma})\partial_{x\gamma}^2\dot{\gamma}\big)+\Phi_{\gamma}^{-1}[\dot{\gamma},\Phi_{\gamma}]\big((\partial_{x\gamma}X)\partial_{x\gamma}\dot{\gamma}\big)\\&-[X,\Phi_{\gamma}^{-1}[\dot{\gamma},\Phi_{\gamma}]]\partial_{x\gamma}^2\dot{\gamma}-(\lambda+1)[X,\Phi_{\gamma}^{-1}]\partial_{x\gamma}\big((\Phi_{\gamma}\dot{\gamma})\partial_{x\gamma}\dot{\gamma}\big)\\&+\lambda\Phi_{\gamma}^{-1}\big((\partial_{x\gamma}\dot{\gamma})[\Phi_{\gamma},X]\partial_{x\gamma}\dot{\gamma}\big).\end{aligned} \tag{5.5}$$

The H^s norm of the first term on the right-hand side of (5.4) is bounded by the sum $\|[X,\Phi_{\gamma}]\partial_{x\gamma}\dot{\gamma}\|_{H^{s-2r}}+\|[\dot{\gamma},\Phi_{\gamma}]\partial_{x\gamma}X\|_{H^{s-2r}}$ which is bounded by $\|X\|_{H^s}\|\dot{\gamma}\|_{H^s}$. Similarly, the second term on the right-hand side of (5.4) is estimated by $\|\dot{\gamma}\|_{H^s}\|X\|_{H^{s-2r+1}}+\|X\|_{H^s}\|\dot{\gamma}\|_{H^{s-2r+1}}$ which is bounded by $\|\dot{\gamma}\|_{H^s}\|X\|_{H^s}$. Therefore, we have the estimate $\|\partial_{\dot{\gamma}}F_{(\gamma,\dot{\gamma})}(X)\|_{H^s}\lesssim\|\dot{\gamma}\|_{H^s}\|X\|_{H^s}$. To estimate the H^s norm of $\partial_{\gamma}F_{(\gamma,\dot{\gamma})}(X)$ we use the same tools along with commutator estimates to get $\|\partial_{\gamma}F_{(\gamma,\dot{\gamma})}(X)\|_{H^s}\lesssim\|\dot{\gamma}\|_{H^s}^2\|X\|_{H^s}$, so both $\partial_{\dot{\gamma}}F_{(\gamma,\dot{\gamma})}$ and $\partial_{\gamma}F_{(\gamma,\dot{\gamma})}$ are bounded linear operators on the space of H^s functions.

In order to complete the proof of the theorem, it is sufficient to establish that $\partial_{\dot{\gamma}}F_{(\gamma,\dot{\gamma})}$ and $\partial_{\gamma}F_{(\gamma,\dot{\gamma})}$ depend continuously on $(\gamma,\dot{\gamma})$ in some neighbourhood of $(\mathrm{id},0)$ in $\mathscr{D}^s\times H^s$. Note that $\|\partial_{\gamma}F_{(\gamma,\dot{\gamma})}(X)-\partial_{\gamma}F_{(\mathrm{id},\dot{\gamma})}(X)\|_{H^s}$ is a sum of differences corresponding to each term in (5.5). For instance, the difference corresponding to the first term on the right-hand side of (5.5) is $[X,\Phi_{\gamma}^{-1}]\big((\Phi_{\gamma}\dot{\gamma})\partial_{x\gamma}^2\dot{\gamma}\big)-[X,\Phi^{-1}]\big((\Phi\dot{\gamma})\partial_x^2\dot{\gamma}\big)$. We add and subtract appropriate terms to this difference to bound its H^s norm:

$$\begin{aligned}
&\|[X,\Phi_\gamma^{-1}]\big((\Phi_\gamma\dot\gamma)\partial_{x_\gamma}^2\dot\gamma\big)-[X,\Phi^{-1}]\big((\Phi\dot\gamma)\partial_x^2\dot\gamma\big)\|_{H^s}\\
&\quad\le\|[X,\Phi_\gamma^{-1}]\big((\Phi_\gamma\dot\gamma)\partial_{x_\gamma}^2\dot\gamma\big)-[X,\Phi_\gamma^{-1}]\big((\Phi_\gamma\dot\gamma)\partial_{x_\gamma}^2\dot\gamma\big)\circ\gamma^{-1}\|_{H^s} &(5.6)\\
&\qquad+\|[X\circ\gamma^{-1},\Phi^{-1}]\big(\Phi(\dot\gamma\circ\gamma^{-1}).\partial_x^2(\dot\gamma\circ\gamma^{-1})-(\Phi\dot\gamma)\partial_x^2\dot\gamma\big)\|_{H^s} &(5.7)\\
&\qquad+\|[X\circ\gamma^{-1}-X,\Phi^{-1}]\big((\Phi\dot\gamma)\partial_x^2\dot\gamma\big)\|_{H^s}. &(5.8)
\end{aligned}$$

Here, we use another property of composition of H^s functions with H^s class diffeomorphisms, Lemma 4.2 in [20] (see [2] for the proof in the case of s integer), to estimate all three terms (5.6)–(5.8). For (5.6) we have the bound

$$\begin{aligned}
&\|[X\circ\gamma^{-1},\Phi^{-1}]\big(\Phi(\dot\gamma\circ\gamma^{-1})\|_C^2\|\gamma-\mathrm{id}\|_{H^s}\\
&\quad\lesssim\|X\|_{H^s}\|\dot\gamma\|_{H^s}\|\dot\gamma\|_{H^{s-2r+2}}\|\gamma-\mathrm{id}\|_{H^s}. &(5.9)
\end{aligned}$$

The term in (5.7) is bounded by the expression $\|X\|_{H^s}\|\Phi(\dot\gamma\circ\gamma^{-1})\big(\partial_x^2(\dot\gamma\circ\gamma^{-1})-\partial_x^2\dot\gamma\big)\|_{H^{s-2r}}+\|X\|_{H^s}\|\big(\Phi(\dot\gamma\circ\gamma^{-1})-\Phi\dot\gamma\big)\partial_x^2\dot\gamma\|_{H^{s-2r}}\lesssim\|X\|_{H^s}\|\dot\gamma\|_{H^s}^2\|\gamma-\mathrm{id}\|_{H^s}$. The estimate on (5.8) is given by $\|[X\circ\gamma^{-1}-X,\Phi^{-1}]\big((\Phi\dot\gamma)\partial_x^2\dot\gamma\big)\|_{H^s}\lesssim\|X\circ\gamma^{-1}-X\|_{H^s}\|\dot\gamma\|_{H^s}\ \|\dot\gamma\|_{H^{s-2r+2}}\lesssim\|X\|_{H^s}\|\gamma-\mathrm{id}\|_{H^s}\|\dot\gamma\|_{H^s}^2$. For all the difference terms corresponding to (5.5) are bounded similarly by $\|X\|_{H^s}\|\gamma-\mathrm{id}\|_{H^s}\|\dot\gamma\|_{H^s}^2$, hence we have

$$\|\partial_\gamma F_{(\gamma,\dot\gamma)}(X)-\partial_\gamma F_{(\mathrm{id},\dot\gamma)}(X)\|_{H^s}\lesssim\|X\|_{H^s}\|\gamma-\mathrm{id}\|_{H^s}\|\dot\gamma\|_{H^s}^2. \qquad (5.10)$$

Furthermore, for $\|\partial_{\dot\gamma}F_{(\gamma,\dot\gamma)}(X)-\partial_\gamma F_{(\mathrm{id},\dot\gamma)}(X)\|_{H^s}$ a similar estimate follows using the same techniques.

Hence, $(\dot\gamma,F(\gamma,\dot\gamma))$ defines a continuously differentiable vector field in a neighbourhood of $(\mathrm{id},0)$ in the space $\mathscr{D}^s\times H^s$. Therefore the classical Picard iterations apply to the Cauchy problem (5.3). □

For $\lambda=2$, Equation (5.2) is an equation for geodesics for the right-invariant metric induced by the H^r inner product. The local well-posedness is proved in [5] in this case. Another case of interest is when $\Phi=\mu-\partial_x^2$ where $\mu(u)=\int_{S^1}u(x)dx$. Both local well-posedness and global existence results of Theorems 5.2 and 5.1 are shown in [17] for $\lambda>0$ in this case.

The following proposition is a persistence result for Sobolev class solutions of (5.2); i.e. it provides a condition under which the short-time solutions persist for all time. It is in the spirit of the persistence result of Beale et al. in [3] for Euler equations of hydrodynamics.

PROPOSITION 5.3. *Let $s>2r+\frac{1}{2}$ and let $u\in C([0,T),H^s(S^1))$ be a solution of* (5.2). *If there exists a $K>0$ such that $\|u(t)\|_{C^1}\le K<\infty$ for all t, then u can be extended to a solution of* (5.2) *that exists for all time.*

The idea of the proof is to use mollifiers [23] to obtain the differential inequality $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s} \lesssim \|u\|_{C^1}\|u\|_{H^s}^2$ and then Proposition 5.3 follows from Gronwalls inequality.
Now we have all the ingredients for the global existence and uniqueness result.

Proof of Theorem 5.1. The Sobolev embedding theorem implies $\|\partial_x u\|_\infty \lesssim \|\Phi u\|_{L^1}$. The orbit invariant mentioned in Proposition 2.1 guarantees that $\Phi u_0 \geq 0$ implies $\Phi u \geq 0$. Furthermore, the integral $\int_{S^1} \Phi u \mathrm{d}x$ is conserved; hence, we have $\|\Phi u\|_{L^1} = \int_{S^1} \Phi u \mathrm{d}x = \int_{S^1} \Phi u_0 \mathrm{d}x$. Therefore, by Proposition 5.3, the solution of theorem 5.2 persists for all time. □

As an anonymous referee pointed out, the results of this section hold for the more general case $\Phi = \sum_{j=0}^{r} a_j (-1)^j \partial_x^{2j}$ where $a_j \in \mathbb{R}$.

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