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Two Positive Solutions of a Quasilinear Elliptic Dirichlet Problem

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Abstract. For a class of second order quasilinear elliptic equations we establish the existence of two non-negative weak solutions of the Dirichlet problem on a bounded domain, Ω. Solutions of the boundary value problem are critical points of C^1-functional on $H_0^1(\Omega)$. One solution is a local minimum and the other is of mountain pass type.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let X denote the Sobolev space $H_0^1(\Omega)$ with the norm $\|u\| = \{\int_\Omega |\nabla u|^2 dx\}^{1/2}$. The usual norm on $L^p(\Omega)$ is denoted by $|\cdot|_p$.

In this paper we consider critical points of the functional $\Phi_{\lambda,h} : X \to \mathbb{R}$ defined by

$$\Phi_{\lambda,h}(u) = \int_\Omega \Gamma(\frac{1}{2}[u^2 + |\nabla u|^2])dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega uh\, dx \tag{1.1}$$

where $\lambda \in \mathbb{R}$ and the functions Γ and h satisfy the following conditions.

(g1) (a) $\Gamma \in C^1([0,\infty),\mathbb{R})$ with $\Gamma(0) = 0$, $\gamma = \Gamma'$ non-increasing on $[0,\infty)$ and $\gamma(\infty) = \lim_{t\to\infty}\gamma(t) > 0$.

(b) $h \in L^2(\Omega)$ and $h \geq 0$ a.e. on Ω.

The idea of studying the critical points of (1.1) under the hypotheses (g1) came from earlier work with H.-S. Zhou [11, 12] where a more complicated functional with some similar features was treated on $(0,\infty)$ instead of on a bounded subset of $\mathbb{R}^N$. Other types of quasilinear equations have been discussed using min-max methods in [2, 13].

Proposition 3.1 shows hypothesis (g1) ensures that $\Phi_{\lambda,h} \in C^1(X,\mathbb{R})$ for all $\lambda \in \mathbb{R}$ and critical points of $\Phi_{\lambda,h}$ correspond to weak solutions of the Dirichlet problem

$$-\nabla\cdot[\gamma(\tfrac{1}{2}[u^2+|\nabla u|^2])\nabla u]+\gamma(\tfrac{1}{2}[u^2+|\nabla u|^2])u=\lambda u+h \quad \text{in } \Omega$$
$$u=0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Clearly $u\equiv 0$ is a solution of this problem for all λ when $h\equiv 0$.

If $\gamma(0)=\gamma(\infty)$ and $h\equiv 0$, γ is constant and (1.2) becomes the linear eigenvalue problem

$$-\Delta u+u=\frac{\lambda}{\gamma(\infty)}u \text{ in } \Omega, \qquad u=0 \text{ on } \partial\Omega. \tag{1.3}$$

The non-zero critical points of (1.1) are precisely the eigenfunctions of the Laplacian on Ω with the Dirichlet boundary condition. In particular, setting

$$\lambda_1=\inf\{\int_\Omega |\nabla u|^2 : u\in X=H_0^1(\Omega) \text{ with } \int_\Omega u^2dx=1\},$$

it is well-known that $\lambda_1>0$ is the lowest eigenvalue and there exists an eigenfunction $\phi\in H_0^1(\Omega)\cap C^2(\Omega)$ with $\phi>0$ and $-\Delta\phi=\lambda_1\phi$ in Ω. Furthermore, λ_1 is a simple eigenvalue and all eigenfunctions corresponding to higher eigenvalues change sign on Ω. Hence, when $\gamma(\infty)=\gamma(0)$ and $h\equiv 0$, (1.2) has a positive solution if and only if $\lambda=\gamma(\infty)[1+\lambda_1]$. In fact, the set of all positive solutions is $\{t\phi : t>0\}$. When $\gamma(\infty)=\gamma(0)$ and $h\not\equiv 0$, (1.2) has a positive solution if and only if $\lambda<\gamma(\infty)[1+\lambda_1]$.

From now on we deal with the case $\gamma(\infty)<\gamma(0)$ where (1.2) has a quasilinear structure and (g1) is no longer sufficient to ensure its ellipticity. Indeed, setting $F(z,p)=\Gamma(\frac{1}{2}[z^2+|p|^2])$ for $z\in\mathbb{R}$ and $p\in\mathbb{R}^N$, it is well-known, see Chapter 10 of [7], that the ellipticity of (1.2) is equivalent to the convexity of $F(z,p)$ with respect to p for all (z,p). It is easily seen that this corresponds to the convexity of $g(t)=\Gamma(t^2)$ on $[0,\infty)$. As is shown in an appendix, the following hypothesis ensures the uniform ellipticity of (1.2).

(g2) For Γ satisfying (g1) and with $g(t)=\Gamma(t^2)$, there exists $\rho>0$ such that $g(t)\geq g(s)+g'(s)(t-s)+\rho(t-s)^2$ for all $t,s\geq 0$.

It is easily checked that (g2) is equivalent to the property $\frac{g'(t)-g'(s)}{t-s}\geq 2\rho$ for $t>s\geq 0$. Thus we see that (g1) requires $\Gamma(t)$ to be a concave function of t, whereas (g2) imposes a kind of uniform, strict convexity in t on $\Gamma(t^2)$. As is shown in Section 3, (g2) ensures that $\Phi_{\lambda,h}$ is weakly sequentially lower semi-continuous on X, although a weaker form of convexity would suffice for this purpose. The full strength of (g2) is invoked in Section 8.

For our treatment of the homogenous case $h\equiv 0$ the hypotheses (g1) and (g2) are sufficient. However for the inhomogeneous problem, we require one further condition on the function Γ. Setting $K(t)=\Gamma(t)-\Gamma'(t)t=\int_0^t[\gamma(s)-\gamma(t)]\,ds$ for $t\geq 0$, it follows from (g1) that

$$K\in C([0,\infty)), K(0)=0 \text{ and } K \text{ non-decreasing on } [0,\infty).$$

(g3) For Γ satisfying (g1), $\lim_{t\to\infty}\{\Gamma(t)-\Gamma'(t)t\}<\infty$.

If (g1) holds and $\Gamma \in C^2([0,\infty))$, then (g2) is satisfied provided that $\inf_{t\geq 0}\{\gamma(t)+2t\gamma'(t)\} > 0$ and (g3) is satisfied provided that $\int_0^\infty |\gamma'(t)|t\,dt < \infty$.

Example Consider $\gamma(t) = A + (1+t)^{-\alpha}$ for $t \geq 0$ where $A > 0$ and $\alpha > 0$. Then (g1) is satisfied by $\Gamma(t) = \int_0^t \gamma(s)\,ds$. For fixed α, (g2) holds for large enough values of A and (g3) holds if $\alpha > 1$. Here are some typical cases.

For $\alpha = 1/2$, $\Gamma(t) = At + 2[\sqrt{1+t} - 1]$ and (g2) holds for all $A > 0$ whereas (g3) always fails.

For $\alpha = 1$, $\Gamma(t) = At + \ln(1+t)$ and (g2) holds if and only if $A > 1/8$ but (g3) is never satisfied.

For $\alpha = 3$, $\Gamma(t) = At - \frac{1}{2}[(1+t)^{-2} - 1]$ and (g2) is satisfied for $A > 1/3$ whereas (g3) is satisfied for all $A > 0$. The value 1/3 is not optimal for obtaining (g2), the sharp condition being $A > 5^4/2^{11} \sim 0.3052$.

We can now state the main result of this paper.

Theorem 1.1. *Let* (g1) *and* (g2) *be satisfied with* $\gamma(\infty) < \gamma(0)$ *and consider* λ *such that* $\gamma(\infty) + \gamma(\infty)\lambda_1 < \lambda < \gamma(0) + \gamma(\infty)\lambda_1$.

(I) *For* $h \equiv 0$, $u_1 \equiv 0$ *is a solution of* (1.2) *with* $\Phi_{\lambda,0}(u_1) = 0$ *and there is another non-negative weak solution* $u_2 \in H_0^1(\Omega)$ *with* $\Phi_{\lambda,0}(u_2) > 0$.

(II) *If in addition* (g3) *is satisfied, there exists* $H_\lambda > 0$ *such that* $0 < |h|_2 < H_\lambda$ *ensures that* (1.2) *has at least two distinct non-negative weak solutions* u_1, $u_2 \in H_0^1(\Omega)$ *with* $\Phi_{\lambda,h}(u_2) > 0 > \Phi_{\lambda,h}(u_1)$.

In both parts, u_1 *is a strict local minimum of* $\Phi_{\lambda,h}$ *whereas* u_2 *is characterized by a min-max principle of mountain pass type. Furthermore,* u_1 *is not a global minimum since* $\inf_{u\in X}\Phi_{\lambda,h}(u) = -\infty$ *under our hypotheses.*

Proof. This follows from Theorem 4.2 and Corollary 8.2. In Theorem 4.2 the solution u_1 is obtained as a local minimum of $\Phi_{\lambda,h}$ in a neighbourhood of the origin. For this we assume that Γ satisfies (g1) and (g2) and that $\lambda < \gamma(0)+\gamma(\infty)\lambda_1$. The positivity of u_1 is deduced from its characterization as a local minimum. Then, with the additional assumption that $\lambda > \gamma(\infty)[1+\lambda_1]$, we show that $\Phi_{\lambda,h}$ has a mountain pass geometry. Referring to a result presented in Section 2, this implies the existence of a sequence of approximate critical points of $\Phi_{\lambda,h}$ having some additional properties. We use these properties to show that the sequence is bounded in X, provided that (g3) is satisfied in the case $h \not\equiv 0$. Using the hypothesis (g2), we establish in Corollary 8.2 the existence of a second solution u_2 as the limit of a subsequence of these approximate critical points. The positivity of u_2 is a consequence of the localization of the approximating sequence derived from the result in Section 2. □

Remark. We have already observed that, for $h \equiv 0$, $u \equiv 0$ is a solution of (1.2) for all λ. It follows from Lemma 4.1 that it is a strict local minimum of $\Phi_{\lambda,0}$ for all $\lambda < \gamma(0) + \gamma(\infty)\lambda_1$. For $\lambda < \gamma(\infty)[1+\lambda_1]$, $\mu \equiv \min\{0, \frac{\gamma(\infty)-\lambda}{\lambda_1}\} + \gamma(\infty) > 0$ and for all $u \in X$,

$$\Phi_{\lambda,0}(u) \geq \frac{1}{2}\int_\Omega [\gamma(\infty) - \lambda]u^2 + \gamma(\infty)|\nabla u|^2 dx \geq \frac{\mu}{2}\|u\|^2$$

showing that 0 is a strict global minimum of $\Phi_{\lambda,0}$ in this case. In fact, for $\lambda < \gamma(\infty)[1+\lambda_1]$, 0 is the only solution of (1.2) for $h \equiv 0$ because $\Phi'_{\lambda,0}(u) = 0$ implies

$$0 = \Phi'_{\lambda,0}(u)u \geq \int \gamma(\infty)[u^2 + |\nabla u|^2] - \lambda u^2 \, dx \geq \mu \|u\|^2$$

by Proposition 3.1.

2. Cerami sequences with localization

Here we recall a result from [10] concerning the existence of special sequences of approximate critical points of a functional having what is usually known as a mountain pass geometry. In this section, $(X, \|\cdot\|)$ denotes any real Banach space and its dual space is denoted by $(X^*, \|\cdot\|_*)$.

Proposition 2.1. *Let* $\Phi \in C^1(X, \mathbb{R})$ *and* $e \in X\backslash\{0\}$. *Set* $P(e) = \{p \in C([0,1], X) : p(0) = 0$ *and* $p(1) = e\}$ *and then* $c(e) = \inf_{p\in P(e)} \max_{t\in[0,1]} \Phi(p(t))$. *Suppose that*

$$(MPG) \qquad \max\{\Phi(0), \Phi(e)\} < c(e) \tag{2.1}$$

and let $\{p_n\} \subset P(e)$ *be a sequence of paths such that* $M_n = \max_{t\in[0,1]} \Phi(p_n(t)) \to c(e)$. *Then there exists a sequence* $\{u_n\} \subset X$ *such that*

$$\Phi(u_n) \to c(e), (1 + \|u_n\|)\|\Phi'(u_n)\|_* \to 0 \text{ and } \frac{d(u_n, p_n([0,1]))}{(1 + \|u_n\|)} \to 0 \text{ as } n \to \infty.$$

Hence, if S *is a subset of* X *such that* $p_n([0,1]) \subset S$ *for all* n, *then* $\frac{d(u_n,S)}{1+\|u_n\|} \to 0$.

This result was proved in [10] as Corollary 1.2. Subsequently, P.J. Rabier [8] showed how it can be deduced from the much earlier work of Ghoussoub [6].

Remarks. (1) A functional Φ satisfying the condition (MPG) is said to have a mountain pass geometry since the mountain pass theorem then shows that $c(e)$ is a critical level of Φ provided that the Palais-Smale condition is also satisfied at this level [1, 6].

(2) A sequence having the properties $\Phi(u_n) \to c(e)$ and $(1+\|u\|)\|\Phi'(u_n)\|_* \to 0$ is usually called a Cerami sequence for the level $c(e)$ because she seems to have been first to exploit weights of this kind in critical point theory, [3, 4]. See also [5].

(3) The property $d(u_n, S)/(1+\|u_n\|) \to 0$ gives some, albeit limited, information about the location of the sequence $\{u_n\}$ which may be helpful in proving that it is bounded. If one is ultimately able to prove the convergence of this special Cerami sequence, then one obtains a critical point in S. An example in [10] shows that, under our hypotheses, there may be no Cerami sequence for which $d(u_n, S)$ is bounded as $n \to \infty$.

(4) Suppose that $\|u_n\| \to \infty$. In the case where $p_n([0,1]) \subset S$ for all $n \in \mathbb{N}$ and $S = tS$ for all $t > 0$, the information that $\frac{d(u_n,S)}{1+\|u_n\|} \to 0$ can be exploited in the

following way. Let $K > 0$ be fixed and set $z_n = t_n u_n$ where $t_n = K/\|u_n\|$. Then, for $u_n \neq 0$,

$$d(u_n, S) = \inf_{v \in S} \|u_n - v\| = \frac{1}{t_n} \inf_{v \in S} \|z_n - t_n v\|$$
$$= \frac{1}{t_n} \inf_{z \in t_n S = S} \|z_n - z\| = \frac{\|u_n\|}{K} d(z_n, S).$$

and so

$$d(z_n, S) = \frac{K(1 + \|u_n\|)}{\|u_n\|} \frac{d(u_n, S)}{1 + \|u_n\|} \to 0.$$

If in addition, X is reflexive and S is a cone (i.e. a closed convex subset of X such that $tS = S$ for all $t > 0$), we can go further by passing to a subsequence such that $z_n \rightharpoonup z$ weakly in X. Since $d(z_n, S) \to 0$, there exist $s_n \in S$ and $r_n \in X$ such that $z_n = s_n + r_n$ and $\|r_n\| \to 0$. This implies that $s_n \rightharpoonup z$ and, since S is closed and convex, we conclude that $z \in S$.

3. Basic properties of $\Phi_{\lambda,h}$

The condition (g1) implies that Γ is concave and $\gamma(\infty)t \leq \Gamma(t) \leq \gamma(0)t$ for all $t \geq 0$. In particular,

$$\int_\Omega \Gamma(\frac{1}{2}[u^2 + |\nabla u|^2])dx \leq \frac{\gamma(0)}{2} \int_\Omega u^2 + |\nabla u|^2 dx$$

and, for any $\lambda \in \mathbb{R}$, we can define a functional $\Phi_{\lambda,h} : X = H_0^1(\Omega) \to \mathbb{R}$ by (1.1). Furthermore, since $\Gamma \in C^1([0, \infty))$ with $\Gamma(0) = 0$ we have that

$$\Gamma(t) = \Gamma'(0)t - G(t)t \text{ where } G \in C^1((0, \infty)) \text{ with } \lim_{t \to 0} G(t) = 0. \tag{3.1}$$

Setting $G(0) = 0$ we have $G \in C([0, \infty))$ and $0 \leq G(t) \leq \gamma(0) - \gamma(\infty)$.

Proposition 3.1. (I) *If Γ satisfies* (g1)*, then $\Phi_{\lambda,h} \in C^1(X, \mathbb{R})$ for all $\lambda \in \mathbb{R}$ with*

$$\Phi'_{\lambda,h}(u)v = \int_\Omega \gamma(\frac{1}{2}[u^2 + |\nabla u|^2])[uv + \nabla u \cdot \nabla v] - \lambda uv - hv\, dx \tag{3.2}$$

for all $u, v \in X$.

(II) *If in addition* (g2) *is satisfied, $\Phi_{\lambda,h} : X \to \mathbb{R}$ is weakly sequentially lower semi-continuous.*

Proof. (I) Defining $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by $F(x, s, p) = \Gamma(\frac{1}{2}[s^2 + |p|^2]) - \frac{\lambda}{2}s^2 - sh(x)$ one easily checks that the hypotheses of C.1 Theorem in [9] are satisfied.

(II) Let $w_n \rightharpoonup w$ weakly in X. Then $\int_\Omega (w_n - w)h\, dx \to 0$ and, by the compactness of the Sobolev embedding, $\int_\Omega (w_n - w)^2 dx \to 0$. Hence it is enough to prove that

$$\Phi_{0,0}(w) \leq \liminf_{n \to \infty} \Phi_{0,0}(w_n).$$

For $u, v \in X$, let $y = (u, \nabla u)$ and $z = (v, \nabla v)$ and so that $y \cdot z = uv + \nabla u \cdot \nabla v$. Then, using (g2),

$$\Phi_{0,0}(u) - \Phi_{0,0}(v) = \int_\Omega g(\frac{|y|}{\sqrt{2}}) - g(\frac{|z|}{\sqrt{2}})\, dx$$

$$\geq \int_\Omega g'(\frac{|z|}{\sqrt{2}})\{\frac{|y|-|z|}{\sqrt{2}}\}dx = \int_\Omega \gamma(\frac{|z|^2}{2})\sqrt{2}|z|\{\frac{|y|-|z|}{\sqrt{2}}\}dx.$$

From part(I) we also have that

$$\Phi'_{0,0}(v)(u-v) = \int_\Omega \gamma(\frac{|z|^2}{2})\{v(u-v) + \nabla v \cdot \nabla(u-v)\}\, dx$$

$$= \int_\Omega \gamma(\frac{|z|^2}{2}) z \cdot (y-z)\, dx.$$

Hence we obtain

$$\Phi_{0,0}(u) - \Phi_{0,0}(v) - \Phi'_{0,0}(v)(u-v) \geq \int_\Omega \gamma(\frac{|z|^2}{2})\{|z|(|y|-|z|) - z\cdot(y-z)\}$$

$$= \int_\Omega \gamma(\frac{|z|^2}{2})\{|z||y| - z \cdot y\}\, dx \geq 0.$$

Putting $u = w_n$ and $v = w$, this yields

$$\Phi_{0,0}(w) \leq \Phi_{0,0}(w_n) - \Phi'_{0,0}(w)(w_n - w),$$

where $\Phi'_{0,0}(w)(w_n - w) \to 0$ since $\Phi'_{0,0}(w) \in X^*$ and $w_n \rightharpoonup w$ weakly in X. □

Remark. For any $u \in X$, we have that $\Phi'_{\lambda,h}(u) = 0 \Leftrightarrow$

$$\int_\Omega \gamma(\frac{1}{2}[u^2 + |\nabla u|^2])[uv + \nabla u \cdot \nabla v] - \lambda uv - hv\, dx = 0 \text{ for all } v \in X$$

and this precisely what is meant by a weak solution of (1.2).

4. Existence of a local minimum

Lemma 4.1. *Let* (g1) *be satisfied and consider* $\lambda < \gamma(0) + \gamma(\infty)\lambda_1$. *Set* $\mu_\lambda = \min\{\gamma(\infty), \gamma(\infty) + \frac{\gamma(0)-\lambda}{\lambda_1}\}$.

(I) *Then* $\mu_\lambda > 0$ *and there exists* $R_\lambda > 0$ *such that* $\Phi_{\lambda,0}(u) \geq \frac{\mu_\lambda}{4}\|u\|^2$ *for all* $u \in X$ *with* $\|u\| \leq R_\lambda$.

(II) *For* $\|u\| = R_\lambda$ *and* $|h|_2 \leq \frac{\mu_\lambda}{8} R_\lambda \sqrt{\lambda_1} \equiv H_\lambda$, *we also have* $\Phi_{\lambda,h}(u) \geq \frac{\mu_\lambda}{8} R_\lambda^2 \equiv \varepsilon_\lambda$.

Proof. For all $u \in X$, the concavity of Γ implies that

$$\Gamma(\frac{1}{2}[u^2 + |\nabla u|^2]) \geq \frac{1}{2}\Gamma(u^2) + \frac{1}{2}\Gamma(|\nabla u|^2) \geq \frac{1}{2}[\gamma(0)u^2 - G(u^2)u^2] + \frac{1}{2}\gamma(\infty)|\nabla u|^2,$$

where G is defined by (3.1). Also, for $1 < p < 2^*/2$ we have that

$$0 \le \int_\Omega G(u^2)u^2 dx \le \{\int_\Omega G(u^2)^{p'} dx\}^{\frac{1}{p'}} \{\int_\Omega u^{2p} dx\}^{\frac{1}{p}}$$
$$\le \{\int_\Omega G(u^2)^{p'} dx\}^{\frac{1}{p'}} C_{2p}^2 \|u\|^2 \tag{4.1}$$

where 2^* and C_{2p} are the critical Sobolev exponent and constant for the embedding of X in $L^{2p}(\Omega)$. Furthermore,

$$[\gamma(0) - \lambda] \int_\Omega u^2 dx \ge 0 \text{ if } \lambda \le \gamma(0) \text{ and}$$
$$[\gamma(0) - \lambda] \int_\Omega u^2 dx \ge \frac{\gamma(0) - \lambda}{\lambda_1} \int_\Omega |\nabla u|^2 dx \text{ if } \lambda > \gamma(0).$$

Hence

$$\int_\Omega \gamma(\infty)|\nabla u|^2 + [\gamma(0) - \lambda]u^2 dx \ge \mu_\lambda \int_\Omega |\nabla u|^2 dx$$

and so

$$\Phi_{\lambda,0}(u) \ge \frac{\mu_\lambda}{2}\|u\|^2 - \frac{1}{2}\{\int_\Omega G(u^2)^{p'} dx\}^{\frac{1}{p'}} C_{2p}^2 \|u\|^2 \text{ for all } u \in X. \tag{4.2}$$

But $0 \le G(t) \le \gamma(0) - \gamma(\infty)$ for all $t \in \mathbb{R}$ and so $u \mapsto G(u^2)^{p'}$ is continuous from $L^2(\Omega)$ into $L^1(\Omega)$. Since $G(0) = 0$, it follows that there exists $\delta_\lambda > 0$ such that $\{\int_\Omega G(u^2)^{p'} dx\}^{\frac{1}{p'}} \le \frac{\mu_\lambda}{2C_{2p}^2}$ for $u \in L^2(\Omega)$ with $|u|_2 \le \delta_\lambda$. Therefore, setting $R_\lambda = \sqrt{\lambda_1}\delta_\lambda$,

$$\Phi_{\lambda,0}(u) \ge \frac{\mu_\lambda}{4}\|u\|^2 \text{ for } u \in X \text{ with } \|u\| \le R_\lambda.$$

(II) For all $u \in X$, $|u|_2^2 \le \|u\|^2/\lambda_1$ and $|\int_\Omega uh\, dx| \le |u|_2|h|_2$, so it follows from part (I) that

$$\Phi_{\lambda,h}(u) \ge \frac{\mu_\lambda}{4}\|u\|^2 - \frac{1}{\sqrt{\lambda_1}}\|u\||h|_2 = \|u\|\{\frac{\mu_\lambda}{4}\|u\| - \frac{1}{\sqrt{\lambda_1}}|h|_2\} \ge \frac{\mu_\lambda}{8}R_\lambda^2$$

for $\|u\| = R_\lambda$ and $|h|_2 \le \frac{\mu_\lambda}{8}R_\lambda\sqrt{\lambda_1}$. □

In the notation of the lemma, we now consider the open ball, $B(0, R_\lambda)$ in X with centre 0 and radius R_λ. Its closure and boundary are denoted by $\overline{B(0, R_\lambda)}$ and $\partial B(0, R_\lambda)$, respectively.

Theorem 4.2. *Let* (g1) *and* (g2) *be satisfied with* $\lambda < \gamma(0) + \gamma(\infty)\lambda_1$ *and* $|h|_2 \le H_\lambda$, *where* H_λ *and* R_λ *are given by Lemma* 4.1. *There exists* $u_1 = u_1(\lambda, h) \in B(0, R_\lambda)$ *such that* $\Phi_{\lambda,h}(u_1) \le \Phi_{\lambda,h}(v)$ *for all* $v \in \overline{B(0, R_\lambda)}$. *Furthermore,* $u_1 \ge 0$ *on* Ω *and* $\Phi'_{\lambda,h}(u_1) = 0$. *Also,* $\Phi_{\lambda,h}(u_1) < 0$ *if* $h \not\equiv 0$ *whereas* $u_1 \equiv 0$ *for* $h \equiv 0$.

Proof. With λ and h fixed, let $m = \inf\{\Phi_{\lambda,h}(u) : u \in \overline{B(0, R_\lambda)}\}$ and consider a sequence $\{v_n\} \subset \overline{B(0, R_\lambda)}$ such that $\Phi_{\lambda,h}(v_n) \to m$. By passing to a subsequence, we may suppose that $v_n \rightharpoonup v$ weakly in X for some element $v \in \overline{B(0, R_\lambda)}$ since X is reflexive and $\overline{B(0, R_\lambda)}$ is closed and convex. By Proposition 3.1(II), $\Phi_{\lambda,h}(v) = m$

and by Lemma 4.1, $m \le \Phi_{\lambda,h}(0) = 0 < \varepsilon_\lambda \le \inf\{\Phi_{\lambda,h}(u) : u \in \partial B(0, R_\lambda)\}$, showing that $v \in B(0, R_\lambda)$.

Set $u_1 = |v|$. Recalling that, for all $u \in X$, $|u| \in X$ and $|\nabla u| = |\nabla |u||$ a.e. on Ω, we have that $u_1 \in X$ with $\|u_1\| = \|v\|$ and $\Phi_{\lambda,0}(u_1) = \Phi_{\lambda,0}(v)$. Since $h \ge 0$ a.e. on Ω, we have that $\int_\Omega vh\,dx \le \int_\Omega u_1 h\,dx$ and so $\Phi_{\lambda,h}(u_1) \le \Phi_{\lambda,h}(v) = m$, too. Thus $u_1 \in B(0, R_\lambda)$ and $\Phi_{\lambda,h}(u_1) = m \le 0$. It follows from Proposition 3.1(I) that $\Phi'_{\lambda,h}(u_1) = 0$. For $h \equiv 0$, Lemma 4.1(I) shows that $u_1 = 0$ and hence $\Phi_{\lambda,h}(u_1) = 0$. If $h \not\equiv 0$ and $\Phi_{\lambda,h}(u_1) = 0$, we have that $m = 0 = \Phi_{\lambda,h}(0)$ and so $\Phi'_{\lambda,h}(0) = 0$. But this means that $u \equiv 0$ satisfies (1.2) which implies that $h \equiv 0$. Hence $\Phi_{\lambda,h}(u_1) < 0$ when $h \not\equiv 0$. □

5. The mountain pass geometry of $\Phi_{\lambda,h}$

Lemma 5.1. *Suppose that* Γ *satisfies* (g1). *For all* $\lambda \in \mathbb{R}$,

$$\frac{\Phi_{\lambda,h}(t\phi)}{t^2} \to \frac{\gamma(\infty)[1+\lambda_1] - \lambda}{2} \int_\Omega \phi(x)^2 dx \text{ as } t \to \infty$$

where ϕ *is the positive eigenfunction associated with* λ_1.

Proof. Since $\phi \in C^1(\Omega)$ and $\phi(x) > 0$ for all $x \in \Omega$, we have that

$$\frac{\Gamma(\frac{t^2}{2}[\phi(x)^2 + |\nabla\phi(x)|^2])}{t^2} \to \frac{1}{2}\gamma(\infty)[\phi(x)^2 + |\nabla\phi(x)|^2] \text{ as } t \to \infty$$

for all $x \in \Omega$. On the other hand,

$$0 \le \frac{\Gamma(\frac{t^2}{2}[\phi(x)^2 + |\nabla\phi(x)|^2])}{t^2} \le \frac{1}{2}\gamma(0)[\phi(x)^2 + |\nabla\phi(x)|^2]$$

for all $x \in \Omega$ and $t \ne 0$ where $\phi \in H_0^1(\Omega)$. As $t \to \infty$, the Dominated Convergence Theorem yields

$$\begin{aligned}\frac{\Phi_{\lambda,h}(t\phi)}{t^2} &= \int_\Omega \frac{\Gamma(\frac{t^2}{2}[\phi(x)^2 + |\nabla\phi(x)|^2])}{t^2} - \frac{1}{2}\lambda\phi(x)^2 - \frac{\phi(x)h(x)}{t}\,dx \\ &\to \frac{1}{2}\int_\Omega [\gamma(\infty) - \lambda]\phi(x)^2 + \gamma(\infty)|\nabla\phi(x)|^2 dx \\ &= \frac{1}{2}\int_\Omega [\gamma(\infty) - \lambda + \gamma(\infty)\lambda_1]\phi(x)^2 dx.\end{aligned}$$ □

Combining Lemmas 4.1 and 5.1 we see that $\Phi_{\lambda,h}$ has a mountain pass geometry for some values of λ. We use the notation introduced in Section 2.

Corollary 5.2. *Suppose that* (g1) *holds and consider* (λ, h) *such that*

$$\gamma(\infty) + \gamma(\infty)\lambda_1 < \lambda < \gamma(0) + \gamma(\infty)\lambda_1 \quad \textit{and} \quad |h|_2 \le H_\lambda.$$

Then $\Phi_{\lambda,h}$ *satisfies (MPG) with* $c(e) > 0$ *for* $e = T\phi$, *where* $T > 0$ *is large enough so that* $\Phi_{\lambda,h}(T\phi) < 0$.

Proof. By Lemma 4.1(II), there exists $R_\lambda > 0$ such that $\inf_{\|u\|=R_\lambda} \Phi_{\lambda,h}(u) \geq \varepsilon_\lambda > 0$. Since $\lambda > \gamma(\infty)[1+\lambda_1]$ it follows from Lemma 5.1 that we can choose $T > 0$ so large that $T\|\phi\| > R_\lambda$ and $\Phi_{\lambda,h}(T\phi) < 0$. Then $\max\{\Phi_{\lambda,h}(0), \Phi_{\lambda,h}(T\phi)\} \leq 0$. But for any $p \in P(T\phi)$ there exists $t \in (0,1)$ such that $\|p(t)\| = R_\lambda$ and so $\max_{t\in[0,1]} \Phi_{\lambda,h}(p(t)) \geq \varepsilon_\lambda$, showing that (MPG) is satisfied with $c(T\phi) \geq \varepsilon_\lambda$. □

6. Existence of a special Cerami sequence for $\Phi_{\lambda,h}$

Using Proposition 2.1 we obtain a Cerami sequence for $\Phi_{\lambda,h}$ having some additional properties which will enable us to show that it has a convergent subsequence.

Proposition 6.1. *Let* (g1) *be satisfied and consider* (λ, h) *such that* $\gamma(\infty)[1 + \lambda_1] < \lambda < \gamma(0) + \gamma(\infty)\lambda_1$ *and* $|h|_2 \leq H_\lambda$. *Let* $S = \{u \in X : u \geq 0$ *a.e. in* $\Omega\}$. *Then there exists a sequence* $\{u_n\} \subset X$ *such that*

$$\Phi_{\lambda,h}(u_n) \to c > 0,\ (1 + \|u_n\|)\Phi'_{\lambda,h}(u_n) \to 0 \text{ and } \frac{d(u_n, S)}{1 + \|u_n\|} \to 0. \tag{6.1}$$

Furthermore, one of the following cases occurs: either (i) $u_n \rightharpoonup u \in S$ *weakly in* X *or* (ii) $\|u_n\| \to \infty$. *In case* (ii), *for any* $K > 0$, *the sequence can be chosen so that* $w_n = K\frac{u_n}{\|u_n\|} \rightharpoonup w \in S$ *weakly in* X.

Proof. By Proposition 3.1(I) and Corollary 5.2, the hypotheses of Proposition 2.1 are satisfied by $\Phi_{\lambda,h}$ with $c = c(e) > 0$ and $e = T\phi$ for $T > 0$ sufficiently large. Let $\{p_n\} \subset P(e)$ be a sequence of paths such that $M_n \to c$. Since $e \in S$ and $u \mapsto |u|$ is a continuous mapping from X into itself (see Proposition 5.1 in [10], $|p_n| \in P(e)$ for all n and $|p_n|(t) = |p_n(t)| \in S$ for all $n \in \mathbb{N}$ and $t \in [0,1]$. But, for all $u \in X$, $|\nabla u| = |\nabla|u||$ a.e. on Ω and so $\Phi_{\lambda,0}(u) = \Phi_{\lambda,0}(|u|)$ for all $u \in X$. Since $h \geq 0$ a.e. on Ω, we also have that $\int_\Omega uh\,dx \leq \int_\Omega |u|h\,dx$ for all $u \in X$ and so $\Phi_{\lambda,h}(|u|) \leq \Phi_{\lambda,h}(u)$, too. Hence $\Phi_{\lambda,h}(|p_n|(t)) \leq \Phi_{\lambda,h}(p_n(t))$ for all $t \in [0,1]$ and $\max_{t\in[0,1]} \Phi_{\lambda,h}(|p_n|(t)) = M_n \to c$. It follows from Proposition 2.1 that there exists a sequence $\{u_n\} \in X$ satisfying (6.1) with $\Phi = \Phi_{\lambda,h}$.

If $\|u_n\| \not\to \infty$, by passing to a subsequence, we can suppose that $u_n \rightharpoonup u$ weakly in X and $d(u_n, S) \to 0$. Then there exists a sequence $\{v_n\} \subset S$ such that $\|u_n - v_n\| \to 0$ and consequently, $v_n \rightharpoonup u$ weakly in X. Since S is closed and convex, this implies that $u \in S$. Thus we obtain case (i) whenever case (ii) does not occur. In case (ii), the Remark 4 following Proposition 2.1 shows that the subsequence can be chosen so that $w_n = K\frac{u_n}{\|u_n\|} \rightharpoonup w \in S$. □

7. The special Cerami sequence is bounded in X

The next step is to show that case (ii) in Proposition 6.1 cannot occur and for this the following properies of $\Phi_{\lambda,h}$ will be useful.

Lemma 7.1. (a) *If* (g1) *is satisfied, then*

$$\Phi_{\lambda,0}(tu) \leq \Phi_{\lambda,0}(u) + \frac{t^2-1}{2}\Phi'_{\lambda,0}(u)u \text{ for all } \lambda, t \in \mathbb{R} \text{ and } u \in X.$$

(b) *If in addition* (g3) *holds, then*

$$\Phi_{\lambda,h}(tu) \leq t(2-t)\Phi_{\lambda,h}(u) + t(t-1)\Phi'_{\lambda,h}(u)u + (1-t)^2 K(\infty)|\Omega|$$

for all $\lambda, t \in \mathbb{R}$, $u \in X$ *and* $h \in L^2(\Omega)$, *where* $K(\infty) = \lim_{t\to\infty}[\Gamma(t) - \gamma(t)t]$.

Proof. (a) Fix $\lambda \in \mathbb{R}$ and $u \in X$. For $s \geq 0$, consider the function $q : [0, \infty) \to \mathbb{R}$ defined by

$$q(s) = \Phi_{\lambda,0}(\sqrt{s}\, u) = \int_\Omega \Gamma(\frac{s}{2}[u^2 + |\nabla u|^2]) - \frac{\lambda}{2} s u^2 dx.$$

By Proposition 3.1, $q \in C^1((0,\infty)) \cap C([0,\infty))$ with $q'(s) = \Phi'_{\lambda,0}(\sqrt{s}\, u)\frac{1}{2\sqrt{s}}u$ for $s > 0$. But, by (g1), q is also concave on $[0,\infty)$ and so $q(s) \leq q(1) + q'(1)(s-1)$ for all $s \geq 0$. Putting $s = t^2$ we get

$$\Phi_{\lambda,0}(tu) = \Phi_{\lambda,0}(|t|u) = q(s) \leq \Phi_{\lambda,0}(u) + \frac{t^2-1}{2}\Phi'_{\lambda,0}(u)u$$

as required.

(b) Since $\Phi_{\lambda,h}(u) = \Phi_{\lambda,0}(u) - \int_\Omega uh\, dx$ and $\Phi'_{\lambda,h}(u)u = \Phi'_{\lambda,0}(u)u - \int_\Omega uh\, dx$, it follows from part (a) that

$$\Phi_{\lambda,h}(tu) + t\int_\Omega uh\, dx \leq \Phi_{\lambda,h}(u) + \int_\Omega uh\, dx + \frac{t^2-1}{2}\{\Phi'_{\lambda,h}(u)u + \int_\Omega uh\, dx\}$$

and hence

$$\Phi_{\lambda,h}(tu) \leq \Phi_{\lambda,h}(u) + \frac{t^2-1}{2}\Phi'_{\lambda,h}(u)u + \frac{(1-t)^2}{2}\int_\Omega uh\, dx$$

for all $\lambda, t \in \mathbb{R}$ and $u \in X$. But

$$2\Phi_{\lambda,h}(u) - \Phi'_{\lambda,h}(u)u = \int_\Omega 2K(\frac{1}{2}[u^2 + |\nabla u|^2]) - uh\, dx,$$

where $K(t) = \Gamma(t) - \gamma(t)t$ is non-decreasing and so

$$\int_\Omega uh\, dx \leq 2K(\infty)|\Omega| - 2\Phi_{\lambda,h}(u) + \Phi'_{\lambda,h}(u)u,$$

proving part (b). □

Lemma 7.2. *Suppose that* (g1) *is satisfied. Let* $\{w_n\}$ *be a sequence in* X *such that* $w_n \rightharpoonup w$ *weakly in* X *and let* $\{t_n\} \subset \mathbb{R}\backslash\{0\}$ *be a sequence such that* $t_n \to 0$ *and* $t_n\Phi'_{\lambda,h}(\frac{w_n}{t_n}) \to 0$ *in* X^*. *Then*

$$\int_\Omega \gamma(\infty)[wv + \nabla w \cdot \nabla v] - \lambda wv\, dx = 0 \text{ for all } v \in X.$$

Proof. By (3.2) we have

$$t_n\Phi'_{\lambda,h}(\frac{w_n}{t_n})v = \int_\Omega \gamma(\frac{1}{2t_n^2}[w_n^2+|\nabla w_n|^2])[w_n v + \nabla w_n \cdot \nabla v] - \lambda w_n v - t_n v h\, dx$$

and so

$$\int_\Omega \gamma(\frac{1}{2t_n^2}[w_n^2+|\nabla w_n|^2])[w_n v + \nabla w_n \cdot \nabla v] - \lambda w_n v\, dx \to 0.$$

Since

$$\int_\Omega w_n v\, dx \to \int_\Omega wv\, dx \text{ and } \int_\Omega [w_n v + \nabla w_n \cdot \nabla v]dx \to \int_\Omega [wv + \nabla w \cdot \nabla v]dx,$$

it is enough to prove that

$$\int_\Omega \{\gamma(\frac{1}{2t_n^2}[w_n^2+|\nabla w_n|^2]) - \gamma(\infty)\}[w_n v + \nabla w_n \cdot \nabla v]dx \to 0.$$

Let $z_n = (w_n, \nabla w_n)$ and $y = (v, \nabla v)$. Then $z_n, y \in [L^2(\Omega)]^{N+1}$ and there exists a constant Z such that $\int_\Omega |z_n|^2 dx \le Z$ for all n since $\{w_n\}$ is bounded in X. Also

$$\begin{aligned}&\int_\Omega \{\gamma(\frac{1}{2t_n^2}[w_n^2+|\nabla w_n|^2]) - \gamma(\infty)\}[w_n v + \nabla w_n \cdot \nabla v]dx\\ &= \int_\Omega \{\gamma(\frac{1}{2t_n^2}|z_n|^2) - \gamma(\infty)\} z_n \cdot y\, dx\\ &= (\int_{A_n^m} + \int_{B_n^m})\{\gamma(\frac{1}{2t_n^2}|z_n|^2) - \gamma(\infty)\} z_n \cdot y\, dx\end{aligned}$$

where $A_n^m = \{x \in \Omega : |z_n|^2 \le 1/m\}$ and $B_n^m = \Omega \backslash A_n^m$ for $m, n \in \mathbb{N}$. For all $m, n \in \mathbb{N}$,

$$\begin{aligned}&\int_{A_n^m} |\gamma(\frac{1}{2t_n^2}|z_n|^2) - \gamma(\infty)||z_n||y|\, dx \le \frac{\gamma(0)-\gamma(\infty)}{\sqrt{m}} \int_{A_n^m} |y|\, dx\\ &\le \frac{\gamma(0)-\gamma(\infty)}{\sqrt{m}} |\Omega|^{1/2} \{\int_\Omega |y|^2 dx\}^{1/2}.\end{aligned}$$

But, for every $m \in \mathbb{N}$, there exists $S_m > 0$ such that $|\gamma(s) - \gamma(\infty)| < 1/m$ for all $s \ge S_m$. Since $t_n \to 0$, there exists $N(m) > 0$ such that $t_n^2 \le \frac{1}{2mS_m}$ for all $n \ge N(m)$. Hence, on B_n^m, $|\gamma(\frac{1}{2t_n^2}|z_n|^2) - \gamma(\infty)| < 1/m$ for $n \ge N(m)$ and so

$$\begin{aligned}&\int_{B_n^m} |\gamma(\frac{1}{2t_n^2}|z_n|^2) - \gamma(\infty)||z_n||y|\, dx \le \frac{1}{m}\{\int_\Omega |z_n|^2 dx\}^{1/2}\{\int_\Omega |y|^2 dx\}^{1/2}\\ &\le \frac{Z^{1/2}}{m}\{\int_\Omega |y|^2 dx\}^{1/2}\end{aligned}$$

for all $n \ge N(m)$. Thus

$$\int_\Omega |\gamma(\frac{1}{2t_n^2}|z_n|^2) - \gamma(\infty)||z_n||y|\, dx \le \{\frac{\gamma(0)-\gamma(\infty)}{\sqrt{m}}|\Omega|^{1/2} + \frac{Z^{1/2}}{m}\}\{\int_\Omega |y|^2 dx\}^{1/2}$$

for all $m \in \mathbb{N}$ and all $n \ge N(m)$. It follows that

$$\int_\Omega |\gamma(\frac{1}{2t_n^2}|z_n|^2) - \gamma(\infty)||z_n||y|\, dx \to 0 \text{ as } n \to \infty,$$

completing the proof. □

Proposition 7.3. *Under the hypotheses of Proposition* 6.1*, and with the additional assumption that* (g3) *holds when* $h \not\equiv 0$*, there exist a sequence* $\{u_n\} \subset X$ *and* $u \in S$ *such that* $u_n \rightharpoonup u$ *weakly in* X *and* (6.1) *is satisfied.*

Proof. We need only show that case (ii) of Proposotion 6.1 cannot occur. Arguing by contradiction, we suppose that there is a sequence $\{u_n\}$ satisfying (6.1) such that $\|u_n\| \to \infty$. Set

$$w_n = K\frac{u_n}{\|u_n\|} \text{ where } K > 0 \text{ is a constant to be chosen later .}$$

Then $t_n = K/\|u_n\| \to 0$ as $n \to \infty$ and, as noted in Proposition 6.1, by passing to a subsequence we can assume that $w_n \rightharpoonup w$ weakly in X where $w \in S$. Since $\Phi_{\lambda,h}(w_n) = \Phi_{\lambda,h}(t_n u_n)$ it follows from Lemma 7.1 that

$$\limsup_{n\to\infty} \Phi_{\lambda,0}(w_n) \le c \text{ and } \limsup_{n\to\infty} \Phi_{\lambda,h}(w_n) \le K(\infty)|\Omega| \text{ when } h \not\equiv 0.$$

On the other hand,

$$\begin{aligned}\Phi_{\lambda,h}(w_n) &= \int_\Omega \Gamma(\frac{1}{2}[w_n^2 + |\nabla w_n|^2]) - \frac{\lambda}{2}w_n^2 - w_n h\, dx \\ &\ge \frac{1}{2}\int_\Omega [\gamma(\infty) - \lambda]w_n^2 + \gamma(\infty)|\nabla w_n|^2 dx - \int_\Omega w_n h\, dx \\ &= \frac{\gamma(\infty) - \lambda}{2}\int_\Omega w_n^2 dx + \frac{\gamma(\infty)}{2}K^2 - \int_\Omega w_n h\, dx,\end{aligned}$$

since $\|w_n\| = K$ for all n. If $w = 0$, we have that $\int_\Omega w_n^2 dx \to 0$, by the compactness of the embedding of X in $L^2(\Omega)$, and consequently

$$\liminf_{n\to\infty} \Phi_{\lambda,h}(w_n) \ge \frac{\gamma(\infty)}{2}K^2.$$

But the constant $K > 0$ is arbitrary and can be chosen so large that $\frac{\gamma(\infty)}{2}K^2 > c$, when $h \equiv 0$, and $\frac{\gamma(\infty)}{2}K^2 > K(\infty)|\Omega|$, when $h \not\equiv 0$, contradicting our earlier conclusion about $\limsup_{n\to\infty} \Phi_{\lambda,h}(w_n)$.

Hence we have that $w \not\equiv 0$ and $w \ge 0$ since $w \in S$. Therefore $\int_\Omega w\phi\, dx > 0$ where ϕ is the positive eigenfunction associated with λ_1. By Lemma 7.2, we have that

$$\lambda \int_\Omega w\phi\, dx = \gamma(\infty)\int_\Omega [w\phi + \nabla w \cdot \nabla\phi]\, dx = \gamma(\infty)\int_\Omega [w\phi + \lambda_1 w\phi]\, dx$$

from which it follows that $\lambda = \gamma(\infty)[1 + \lambda_1]$, contradicting our hypothesis that $\lambda > \gamma(\infty)[1 + \lambda_1]$. Hence our assumption that $\|u_n\| \to \infty$ leads to a contradiction and the proof is complete. □

8. Convergence of the special Cerami sequence

Lemma 8.1. *If* (g1) *and* (g2) *are satisfied, then*

$$\frac{\eta}{2}\|u-v\|^2 \le \Phi_{\lambda,h}(u) - \Phi_{\lambda,h}(v) + \|\Phi'_{\lambda,h}(v)\|_*\|u-v\| + \frac{\lambda}{2}|u-v|_2^2$$

for all $u, v \in X$ *and all* $\lambda \in \mathbb{R}$, *where* $\eta = \min\{\rho, \gamma(\infty)\} > 0$.

Proof. For $u, v \in X$, set $w = (u, \nabla u)$ and $z = (v, \nabla v)$ so that $w \cdot z = uv + \nabla u \cdot \nabla v$. Then

$$\begin{aligned}
\Phi_{0,0}(u) - \Phi_{0,0}(v) &= \int_\Omega g(\frac{|w|}{\sqrt{2}}) - g(\frac{|z|}{\sqrt{2}})\, dx \\
&\ge \int_\Omega g'(\frac{|z|}{\sqrt{2}})\{\frac{|w|-|z|}{\sqrt{2}}\} + \frac{\rho}{2}(|w|-|z|)^2 dx \\
&= \int_\Omega \gamma(\frac{|z|^2}{2})\sqrt{2}|z|\{\frac{|w|-|z|}{\sqrt{2}}\} + \frac{\rho}{2}(|w|-|z|)^2 dx \\
&\ge \int_\Omega \gamma(\frac{|z|^2}{2})|z|(|w|-|z|) + \frac{\eta}{2}(|w-z|^2 + 2w\cdot z - 2|w||z|) dx
\end{aligned}$$

We also have that

$$\begin{aligned}
\Phi'_{0,0}(v)(u-v) &= \int_\Omega \gamma(\frac{|z|^2}{2})\{v(u-v) + \nabla v \cdot \nabla(u-v)\}\, dx \\
&= \int_\Omega \gamma(\frac{|z|^2}{2}) z\cdot(w-z)\, dx.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\Phi_{\lambda,h}(u) - \Phi_{\lambda,h}(v) - \Phi'_{\lambda,h}(v)(u-v) \\
&= \Phi_{0,0}(u) - \Phi_{0,0}(v) - \Phi'_{0,0}(v)(u-v) - \frac{\lambda}{2}\int_\Omega u^2 - v^2 - 2v(u-v) \\
&\ge \int_\Omega \gamma(\frac{|z|^2}{2})\{|z|(|w|-|z|) - z\cdot(w-z)\} + \frac{\eta}{2}(|w-z|^2 + 2w\cdot z - 2|w||z|) \\
&\quad - \frac{\lambda}{2}(u-v)^2\, dx \\
&= \int_\Omega \{\gamma(\frac{|z|^2}{2}) - \eta\}\{|z||w| - z\cdot w\} + \frac{\eta}{2}|w-z|^2 - \frac{\lambda}{2}(u-v)^2\, dx \\
&\ge \int_\Omega \{\gamma(\infty) - \eta\}\{|z||w| - z\cdot w\} + \frac{\eta}{2}|w-z|^2 - \frac{\lambda}{2}(u-v)^2\, dx \\
&\ge \int_\Omega \frac{\eta}{2}|w-z|^2 - \frac{\lambda}{2}(u-v)^2\, dx.
\end{aligned}$$

Hence

$$\frac{\eta}{2}\int_\Omega |w-z|^2 dx \le \Phi_{\lambda,h}(u) - \Phi_{\lambda,h}(v) - \Phi'_{\lambda,h}(v)(u-v) + \frac{\lambda}{2}\int_\Omega (u-v)^2\, dx.$$

But $|w-z|^2 = (u-v)^2 + |\nabla(u-v)|^2$ and so we have

$$\frac{\eta}{2}\|u-v\|^2 \le \Phi_{\lambda,h}(u) - \Phi_{\lambda,h}(v) - \Phi'_{\lambda,h}(v)(u-v) + \frac{\lambda-\eta}{2}\int_\Omega (u-v)^2\,dx$$

from which the desired conclusion follows easily. □

Corollary 8.2. *Under the hypotheses of Proposition* 7.3*, consider the special Cerami sequence* $\{u_n\}$ *for* $\Phi_{\lambda,h}$ *and its weak limit* $u \in S$ *given by that result. Suppose that* Γ *satisfies* (g2)*. Then* $\|u_n - u\| \to 0$ *and consequently,* $\Phi'_{\lambda,h}(u) = 0$ *with* $\Phi_{\lambda,h}(u) = c > 0$.

Proof. By Lemma 8.1 we have

$$\frac{\eta}{2}\|u_n-u_m\|^2 \le \Phi_{\lambda,h}(u_n) - \Phi_{\lambda,h}(u_m) + \|\Phi'_{\lambda,h}(u_m)\|_*\|u_n-u_m\| + \frac{\lambda}{2}|u_n-u_m|_2^2 \quad (8.1)$$

for all n, m. Since $u_n \rightharpoonup u$ weakly in X, the compactness of the embedding into $L^2(\Omega)$ implies that $\{u_n\}$ is a Cauchy sequence in $L^2(\Omega)$. From (6.1) we have that $\lim_{m\to\infty}\|\Phi'_{\lambda,h}(u_m)\|_* = 0$. But the weak convergence of $\{u_n\}$ in X also implies that $\{\|u_n\|\}$ is a bounded sequence and hence $\|\Phi'_{\lambda,h}(u_m)\|_*\|u_n-u_m\| \to 0$ as $m,n\to\infty$. Finally by (6.1), $\lim_{n\to\infty}\Phi_{\lambda,h}(u_n) = c$, and so from (8.1) we can now conclude that $\{u_n\}$ is a Cauchy sequence in X. Thus $u_n \to u$ strongly in X. Returning to (6.1) and using Proposition 3.1 we obtain $\Phi_{\lambda,h}(u) = c$ and $\Phi'_{\lambda,h}(u) = 0$. □

9. Appendix

If $\Gamma \in C^2([0,\infty))$, the partial differential equation in (1.2) can be written as

$$-\sum_{i,j=1}^{N} a_{ij}(u,\nabla u)\partial^2_{ij}u + \{\gamma(\frac{1}{2}[u^2+|\nabla u|^2]) - \gamma'(\frac{1}{2}[u^2+|\nabla u|^2])|\nabla u|^2 - \lambda\}u = h$$

where $a_{ij}(u,\nabla u) = \gamma(\frac{1}{2}[u^2+|\nabla u|^2])\delta_{ij} + \gamma'(\frac{1}{2}[u^2+|\nabla u|^2])\partial_i u\partial_j u$.

The conditions (g1) and (g2) imply uniform ellipticity in the following sense. For $z\in\mathbb{R}$ and $p,\xi\in\mathbb{R}^N$, setting $s = \frac{1}{2}[z^2+|p|^2]$, we have that

$$\begin{aligned}\sum_{i,j=1}^{\infty} a_{ij}(z,p)\xi_i\xi_j &= \gamma(s)|\xi|^2 + \gamma'(s)(p\cdot\xi)^2 \\ &\ge \gamma(s)|\xi|^2 + \gamma'(s)|p|^2|\xi|^2 \text{ since } \gamma'(s)\le 0 \text{ by (g1)} \\ &\ge \gamma(s)|\xi|^2 + \gamma'(s)2s|\xi|^2 = \frac{1}{2}g''(t)|\xi|^2 \text{ where } t=\sqrt{s} \\ &\ge \rho|\xi|^2 \text{ by (g2).}\end{aligned}$$

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