# Relative Projectivity and Relative Endotrivial Modules

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## Abstract / Résumé


Abstract.

This dissertation is concerned with modular representation theory of finite groups, and more precisely, with the study of classes of representations, which we shall term *relative endotrivial modules*. Given a prime number $p$, a finite group $G$ of order divisible by $p$, we shall say that a $kG$-module $M$ is endotrivial relatively to the $kG$-module $V$ if its endomorphism algebra $\mathrm{End}_k(M)$ is isomorphic, as a $kG$-module, to a direct sum of a trivial module and another module which is projective relatively to $V$, i.e. in short $\mathrm{End}_k(M) \cong k \oplus (V - projective)$.

More accurately, in the first part of the text projectivity relative to $kG$-modules is used to define groups of relative endotrivial modules, which are obtained by replacing the notion of projectivity with that of relative projectivity in the definition of ordinary endotrivial modules. However, in order to achieve this goal we first need to develop the theory of projectivity relative to modules, in particular with respect to standard group operations such as induction, restriction and inflation. Then, for finite groups having a cyclic Sylow $p$-subgroup, using the structure of the group $T(G)$ of endotrivial modules described in [MT07], we give a complete classification of the groups of relative endotrivial modules. We also study the case of groups that have a Sylow $p$-subgroup isomorphic to a Klein group $C_2 \times C_2$, as well as the case of $p$-nilpotent groups.

In a second part of the text, it is shown how our new groups of relative endotrivial modules provide a natural context to generalise the Dade group of a $p$-group $P$ to an arbitrary finite group. The classification of endo-permutation modules and the complete description of the structure of the Dade group $D(P)$ was completed in 2004 by S. Bouc with [Bou06]. This adventure had started about 25 years earlier with the first papers and results by E. Dade in [Dad78a] and [Dad78b] in 1978, and the final classification was in fact achieved through the non-effortless combined work of several (co)-authors between 1998 and 2004, including S. Bouc, J. Carlson, N. Mazza and J. Thévenaz. It is most interesting to note that crucial building pieces for this classification are indeed the endotrivial modules, which are particular cases of endo-permutation modules. Yet, for an arbitrary finite group $G$, no satisfying equivalent group structure to the Dade group on a class of $kG$-modules has been defined so far. With the goal to fill this gap, we turn the problem upside down, in some sense, and show how one can regard an endo-permutation module as an *endotrivial module*, of course not in the ordinary sense, but in the relative sense. This shall enable us to endow a set of isomorphism classes of endo-$p$-permutation modules with a group structure, similar to that of the Dade group. We shall call this new group, the *generalised Dade group of the group* $G$, explicitly compute its structure and show how it is closely related to that of the $G$-stable points of the Dade group of a Sylow $p$-subgroup of $G$.



**Keywords**: relative projectivity to modules, relative endotrivial modules, endotrivial modules, endo-permutation modules, endo-$p$-permutation modules, Dade group, modular representations of finite groups.

Résumé.

Cette thèse se place dans la théorie des représentations modulaires des groupes finis. On y étudie des familles de représentations que l'on appellera *modules endo-triviaux relatifs*. Etant donné un nombre premier $p$, un groupe fini $G$ d'ordre divisible par $p$ et un corps algébriquement clos $k$, un $kG$-module $M$ est appelé *endo-trivial relativement au $kG$-module $V$* si son algèbre des endomorphismes $\mathrm{End}_k(M)$ est isomorphe, comme $kG$-module, à la somme directe d'un module trivial et d'un module projectif relativement à $V$, i.e. $\mathrm{End}_k(M) \cong k \oplus (V - projectif)$.

Dans un premier temps, on y traite en particulier de projectivité relative à un $kG$-module, laquelle notion servira à définir des groupes de modules endo-triviaux relatifs, obtenus en remplaçant la notion de projectivité ordinaire par la notion de projectivité relative dans la définition classique d'un module endo-trivial. Pour atteindre ce but, on y développe la théorie de la projectivité relative à un module, dont en particulier certains aspects d'algèbre homologique relative et les comportements par rapport aux opérations de groupes standards comme la restriction, l'induction et l'inflation. Pour les groupes finis avec un $p$-sous-groupe de Sylow cyclique, on y donne une classification complète des groupes d'endo-triviaux relatifs. On y traite aussi plus précisément les cas de groupes $p$-nilpotents et des groupes possédant un 2-sous-groupe de Sylow isomorphe au groupe de Klein $C_2 \times C_2$.

Dans une deuxième partie de ce travail, on utilise l'approche des modules endo-triviaux relatifs afin de fournir un contexte naturel qui permet de généraliser la structure de groupe de Dade d'un $p$-groupe à un groupe fini arbitraire. La classification complète des modules d'endo-permutations, via la description de la structure du groupe de Dade $D(P)$ s'est achevée en 2004 avec S. Bouc. Cette aventure avait commencé un quart de siècle plus tôt avec les premiers articles sur le sujet [Dad78a] et [Dad78b] par E. Dade en 1978. La classification finale est en fait le résultat du travail combiné et de longue haleine de plusieurs (co)-auteurs entre 1998 et 2004, incluant S. Bouc, J. Carlson, N. Mazza et J. Thévenaz. On notera en particulier, avec grand intérêt, que les pièces de construction élémentaires de cette classification sont les modules endo-triviaux, qui sont des cas particuliers de modules d'endo-permutation. Cependant, jusqu'à présent, aucun équivalent du groupe de Dade n'a été défini pour un groupe fini arbitraire. Pour palier à ce manque, on propose dans ce texte, de regarder ce problème, comme depuis l'hémisphère sud, c'est-à-dire la tête à l'envers, et l'on montre comment voir un module d'endo-permutation comme un *module endo-trivial*, bien sûr non pas au sens ordinaire, mais au sens relatif. Cette approche nous permet de définir un *groupe de Dade généralisé* $D(G)$ pour n'importe quel groupe fini $G$, à partir d'une sous-classe de la classe des $kG$-modules d'endo-$p$-permutation. Finalement on donne une description explicite de la structure du groupe $D(G)$ et montre qu'elle est étroitement liée à celle des points $G$-stables du groupe de Dade d'un $p$-sous-groupe de Sylow de $G$.



**Mots-clés**: projectivité relative à un module, modules endo-triviaux relatifs, modules endo-triviaux, modules d'endo-permutation, modules d'endo-$p$-permutation, groupe de Dade, représentations modulaires des groupes finis.

*En mémoire de* Daniel Lassueur
*L'Ingénieur qui me contait*
*la théorie de l'intégration.*

*Son soutien inconditionnel fût présent*
*jusqu'aux avants-dernières minutes*
*de l'écriture de ce texte.*

*Few are the mathematicians who can still be sure that no triangulated category is floating in their ink-pot.*
Paul Balmer

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**Apologies.**
First of all I ought an apology to my advisor for choosing not to hyphen the word endotrivial.
Then I ought apologies to native English speakers for torturing their beautiful language with awkward French-influenced sentences, as well as for having involuntarily introduced Kiwi expressions every now and then.

## Table of Contents

# Notation

| | |
|---|---|
| Coker | cokernel |
| $C_G(H)$ | centraliser of the subgroup $H$ in $G$ |
| $C_n$ | cyclic group of order $n$ |
| $\Gamma(X(N_G(P)))$ | group of $kG$-Green correspondents of the modules in $X(N_G(P))$ |
| dim | dimension |
| $D(P)$ | Dade group of the $p$-group $P$ |
| $D(G)$ | generalised Dade group of the finite group $G$ |
| $\mathrm{End}_k$ | $k$-endomorphism ring |
| $[G, G]$ | commutator subgroup of the group $G$ |
| $\mathrm{Hom}_k(M, N)$ | $k$-homomorphisms from $M$ to $N$ |
| $\mathrm{Inf}_{G/N}^G$ | inflation from $G/N$ to $G$, inflation map |
| $\mathrm{Ind}_H^G$ | induction from $H$ to $G$ |
| ker | kernel |
| $kG$ | group algebra of the group $G$ over the field $k$ |
| $\mathsf{mod}(kG)$ | category of f.g. left $kG$-modules |
| $\mathsf{Mod}(kG)$ | category of all left $kG$-modules |
| $\mathbb{N}$ | the natural numbers, 0 included |
| $N_G(H)$ | normaliser of the subgroup $H$ in $G$ |
| $Proj(V)$ | subcategory of $V$-projective modules |
| $\mathrm{Res}_H^G$ | restriction from $G$ to $H$, restriction map |
| $\Sigma_n$ | symmetric group of degree $n$ |
| $\mathsf{stmod}(kG)$ | stable category of f.g. left $kG$-modules |
| $\mathsf{stmod}_V(kG)$ | relative stable category of f.g. left $kG$-modules |
| $\mathrm{Tr}_V$ | trace map of the module $V$ |
| $T(G)$ | group of endotrivial modules of $G$ |
| $T_V(G)$ | group of $V$-endotrivial modules of $G$ |
| $\mathcal{V}_G(M)$ | support variety of the $kG$-module $M$ |
| $X(G)$ | group of one-dimensional representations of the group $G$ |
| $\mathbb{Z}$ | the Integers |
| $\Omega^n(M)$ | $n$-th syzygy module of the module $M$ |
| $\Omega_V^n(M)$ | $n$-th $V$-relative syzygy module of the module $M$ |
| $\widetilde{\Omega}_V^n(M)$ | $\Omega_V^n(M) \oplus (V - proj)$ |
| $\Omega_{\mathcal{H}}^n(M)$ | $n$-th relative syzygy module of the module $M$ relative to the family of subgroups $\mathcal{H}$ |
| $\Omega_V$ | class in $T_V(G)$ of the relative syzygy module $\Omega_V(k)$ |
| $\Omega$ | class in $T(G)$ of the syzygy module $\Omega(k)$ |
| $M^*$ | $k$-dual of the $kG$-module $M$ |
| ${}^gV$ | conjugate of the module $V$ by the element $g$ |
| ${}^gH$ | conjugate of the subgroup $H$ by the element $g$ |
| $a \mid b$ | $a$ divides $b$ |
| $a \nmid b$ | $a$ does not divide $b$ |
| $M \mid N$ | $M$ is a direct summand of $N$ |
| $\oplus, \otimes$ | direct sum, tensor product |
| $\cong$ | isomorphism |
| $\leq_G$ | subgroup relation up to conjugacy in $G$ |
| $\downarrow_H^G$ | restriction from $G$ to $H$ |
| $\uparrow_H^G$ | induction from $H$ to $G$ |
| $\uparrow_{\otimes H}^{G}$ | tensor induction from $H$ to $G$ |
| $\hookrightarrow$, $\twoheadrightarrow$ | injective morphism, surjective morphism |
| $\forall$, $\exists$, $\exists!$ | universal symbols “for all”, “there exists”, “there exists a unique” |

# Introduction

The field of mathematics this thesis is concerned with is the theory of modular representations of finite groups, which studies the modules over the group algebra $kG$ of a finite group $G$ over a field $k$ of characteristic $p$ dividing the order of $G$. We shall even require that the field is algebraically closed.

One of the main goals of the algebraist is to understand his field of study by classifying the objects he works with. Thus in module theory, one wants to understand the indecomposable modules. Unfortunately, in modular representation theory of finite groups it was been proven that in general it is impossible to classify the indecomposable modules. Said in mathematical terms, but for exceptions, the group algebra of a finite group has wild representation type. In consequence, the philosophy becomes that described by E. Dade in his celebrated introduction to his paper [Dad78a]: [*...we are reduced to looking for subfamilies which are, at the same time, small enough to be classified and large enough to be useful.*]

Given a $p$-group $P$, in [Dad78a], Dade describes two *interesting* families of $kP$-modules by requiring their endomorphism algebra to fulfill a certain property, namely the family of *endo-permutation modules* and the family *endotrivial modules.* A $kP$-module is called *endo-permutation* if its endomorphism algebra is a permutation module, and it is called *endotrivial* if its endomorphism algebra decomposes as the direct sum of a trivial module and a projective module. Endotrivial modules are particular cases of endo-permutation modules. These two notions give rise to two group structures $D(P)$ and $T(P)$ called, the Dade group of the $p$-group $P$ and the group of endotrivial modules of $P$, respectively. In fact, $T(P) \leq D(P)$. Moreover, the problems of classifying the endo-permutation and endotrivial modules are equivalent to describing the structure of the groups $D(P)$ and $T(P)$, respectively. As mentioned in the abstract, this classification started with Dade in 1978 and was completed in 2004 through the joint efforts of several mathematicians.

A first encounter with these notions raises two questions:
(1) Why require a trivial summand $k$ in the endomorphism algebra of an endotrivial module?
(2) The group of endotrivial modules $T(G)$ can be defined for an arbitrary finite group $G$, so why isn't there a Dade group for an arbitrary finite group?

These rather informal questions are in fact the real starting points of the work presented in this text. The answers are both hidden in the following result by D. Benson and J. Carlson [BC86, Thm. 2.1]:

Theorem 0.0.1 (Benson-Carlson).
*Let $k$ be an algebraically closed field of characteristic $p$ (possibly $p = 0$), and let $\Lambda$ be a Hopf algebra with antipode over $k$ (e.g. $\Lambda = kG$). If $M$ and $N$ are finite-dimensional indecomposable $\Lambda$-modules, then*

$$k \mid M \otimes N \text{ if and only if } \begin{cases} (1)\ M \cong N^*\,; \\ (2)\ p \nmid \dim_k(N). \end{cases}$$

*Moreover, if $k$ is a direct summand of $N^* \otimes N$ then it has multiplicity one, i.e. $k \oplus k$ is not a summand.*

This result is yet another evidence of a splitting in the theory between the indecomposable $kG$-modules with $k$-dimension coprime to $p$, and those with $k$-dimension divisible by $p$. The philosophy of this thesis being in some sense to further develop Dade's idea to define interesting families of modules by requiring a fixed property on the endomorphism algebras, this result is crucial. Let's dissect it deeper. First recall that if $M$ is a $kG$-module, then $\mathrm{End}_k(M) \cong M^* \otimes M$. Thus the endomorphisms of an indecomposable $kG$-module $M$ with $k$-dimension coprime to $p$ have the form

$$\mathrm{End}_k(M) \cong k \oplus X$$

for some $kG$-module $X$. Therefore, asking that the trivial module is a direct summand in the endomorphisms is very natural, because it is there anywhere, as soon as the dimension of the module is coprime to $p$. The module $M$ is endotrivial if $X$ is a projective module, and when $G$ is a $p$-group, it is endo-permutation if $X$ is a permutation module. But, asking that $X$ is projective or permutation are extremely restricting requirements. The main idea of this piece of work is to relax these hypothesis and to allow the module $X$ to be projective relatively to some other f.g. $kG$-module $V$. We shall call such a module *endotrivial relatively to the module $V$*. We shall soon discover that, for this notion to be interesting, it is also necessary to require that the trivial summand $k$ cannot appear a second time as summand a of $X$, which is equivalent to require that all the indecomposable summands of $X$ have $k$-dimension divisible by $p$. In consequence with relative endotrivial modules we shall study families of modules, all with dimension equal to $\pm 1$ modulo $p$, and with projectivity relative to $V$, families of modules all with dimension divisible by $p$.

From the categorical point of view, one reason for interest in endotrivial modules comes from the fact that the tensor product with an endotrivial module always induces a self-equivalence of the stable category $\mathsf{stmod}(kG)$. Likewise the tensor product with an endotrivial module always induces a self-equivalence of the associated relative stable category.

The text itself is built according to the following pattern. In a first part (in Chapter 2) we develop the theory of projectivity relative to $kG$-modules. It originates in the early 1990's in an unpublished piece of work by T. Okuyama [Oku91] and was then further developed by J. Carlson and co-authors, essentially with a view to cohomological properties. Here we need to develop other aspects of this theory, such as the behaviour of relative projectivity with respect to restrictions, inflations, inductions, or vertices, sources and Green correspondence. We also need to develop the associated relative homological algebra, and in particular properties of the relative syzygy modules. Finally, [Oku91] being unpublished, this chapter is also an opportunity to bring together the known and new material on relative projectivity to modules.

In a second part we define our main objects of study, that is the groups $T_V(G)$ of relatively $V$-endotrivial modules of a group $G$. These groups generalise naturally the ordinary group $T(G)$ of endotrivial modules as they are simply obtained by replacing projectivity with relative projectivity, according to the process described above. As a consequence, the study of relative endotrivial modules is always two-fold: on the one hand it is necessary to develop the properties of these relative endotrivial modules themselves, and on the other hand, in order to reach this aim, it is crucial to master the behaviours of the associated relative projectivity. Hence the reason to be of the first part. In a first time, we generalise many properties of the ordinary case to the relative case, whereas in a second time we examine more accurately the three cases of $p$-nilpotent groups, of groups with Klein Sylow 2-subgroups, and of groups having cyclic Sylow $p$-subgroups. In the latter case, we provide a description of all the different subcategories of relatively projective modules, and secondly give a complete classification of the groups of relative endotrivial modules.

In a third part, we come to a main motivation for an interest in the families of relative endotrivial modules: they provide a natural context to build a version of the Dade group for an arbitrary finite group $G$, and in the meantime, shed new light on the class of endo-$p$-permutation modules. Oddly enough, for $p$-groups $P$, it took about a quarter of a century to classify endo-permutation modules via the description of the structure of the Dade group and yet there is no version of this group for an arbitrary finite group $G$.

In fact, one attempt to build such a group was made by J.-M. Urfer in [Urf06], where the author generalises Dade's compatibility equivalence relation to the class of endo-$p$-permutation $kG$-modules and obtains a group structure on the resulting set of equivalence classes, which is induced by the tensor product. Although this group has many similarities with the Dade group of a $p$-group, it is unsatisfying in the sense that the classes constituting its elements do not have a unique indecomposable representative, up to isomorphism.

Hence the idea that perhaps one should not use the whole class of endo-$p$-permutation modules, but restrict to a subclass that has more similarities with the class of capped endo-permutation modules.

This is where our relative endotrivial modules come in the picture. In the classical theory, endotrivial modules are always regarded as special cases of endo-permutation modules. Here the idea is to take this description upside down and realise that endo-permutation modules are endotrivial modules relatively to the module $V := \bigoplus_{Q \lneq P} k\!\uparrow_Q^P$, or in other words, relatively to the family of all proper subgroups of the group $P$. As the latter condition easily passes to arbitrary finite groups, it allows of a definition of a group structure $D(G)$, induced by the tensor product $\otimes_k$, on the set of isomorphisms classes of indecomposable endo-$p$-permutation modules which are also endotrivial relatively to the module $W := \bigoplus_{Q \lneq P} k\!\uparrow_Q^G$, where $P$ is a Sylow $p$-subgroup of $G$. The result is that this new group $D(G)$ generalises naturally the Dade group in many ways. We shall call it the *generalised Dade group* of the group $G$. Furthermore, using the general results we have developed for the groups of relative endotrivial modules, it becomes easy to express the structure of $D(G)$ in terms of the Dade group of the Sylow $p$-subgroup $P$ and in terms of the Green correspondents of the one-dimensional representations of the normaliser $N_G(P)$.

# CHAPTER 1

# Background Material

**Conventions and Notations**

Throughout this text we assume the reader is acquainted with elementary group, ring, module and algebra theory. Unless otherwise specified, $k$ shall denote an algebraically closed field of prime characteristic $p$, $G$ an arbitrary finite group with order divisible by $p$, $P$ a finite $p$-group. We write $M^* = \mathrm{Hom}_k(M, k)$ for the $k$-dual of a $kG$-module $M$, $\otimes$ instead of $\otimes_k$ for the tensor product balanced over $k$ when no confusion is to be made. Modules are considered to be finitely generated left modules. Moreover, we shall always consider modules up to isomorphism.

## 1.1. Modules for group algebras

We briefly review a few of the most fundamental properties of group algebras and their modules. The reader is refered to [CR90] and [Ben98a] for proofs and statements of these results in wider generality. In particular, most of the results hereafter hold if the characteristic of the field is zero or does not divide the order of the group, or if the field $k$ is replaced with a ring with unity.

Let $G$ be a finite group and $k$ a field. The *group algebra* is the $k$-algebra $kG$ whose elements are the formal sums $\sum_{g\in G} a_g \cdot g$ for $a_g \in k$. Addition is defined component-wise, while the multiplication derives from the group law:

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g\,;$$

$$(\sum_{g\in G} a_g g) \cdot (\sum_{g\in G} b_g g) = \sum_{g\in G} (\sum_{h\in G} a_{gh^{-1}} b_h) g\,.$$

In addition $kG$ is a $k$-vector space with a $k$-basis given by the elements of $G$. The data of a $kG$-module is equivalent to the data of a representation of the group $G$ over $k$. The group algebra $kG$ is a finite dimensional algebra and hence the Krull-Schmidt Theorem holds, and allows one to reduce the study of $kG$-modules to that of indecomposable $kG$-modules.

Krull-Schmidt Theorem.
*Let $A$ be a $k$-algebra, finitely generated as a $k$-module, where $R$ is complete commutative noetherian local ring (such as for example a field or a complete discrete valuation ring). Then every finitely generated left $A$-module $M$ is expressible as a finite direct sum of indecomposable submodules. Furthermore, if $M = \bigoplus_{i=1}^{r} M_i \cong \bigoplus_{j=1}^{s} N_j$ are two such expressions, then $r = s$ and there is a permutation $\sigma \in \Sigma_n$ such that $M_i = N_{\sigma(i)}$ for every integer $1 \leq i \leq r$.*

Maschke's Theorem.
*The group algebra $kG$ is semi-simple if and only if the characteristic of the field $k$ does not divide the order of the group $G$.*

As a consequence, the theory diverges into two parts, depending on whether the characteristic of the field $k$ divides the order of the group. In case $kG$ is semi-simple, any irreducible $kG$-module is projective. In this piece of work, as announces the title we are interested in the other framework, where $char(k) = p > 0$ divides $|G|$, and where not every module is projective. This part of the theory is called *modular representation theory of finite groups*. Also recall that the algebra $kG$ is symmetric and therefore the class of projective $kG$-modules coincides with the class of injective $kG$-modules.

**Module categories.** Let $\mathsf{mod}(kG)$ denote the category of finitely generated left $kG$-modules and $\mathsf{Mod}(kG)$ the category of all left $kG$-modules. Moreover, let $\mathsf{stmod}(kG)$ denote the stable category of finitely generated $kG$-modules modulo projectives. The objects in $\mathsf{stmod}(kG)$ are the same as those in $\mathsf{mod}(kG)$ and the morphisms from a module $M$ to a module $N$ are given by

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N),$$

where $\mathrm{PHom}_{kG}(M,N)$ denotes the set of all morphisms from $M$ to $N$ that factor through a projective module. A morphism $f : M \longrightarrow N$ is said to *factor through a projective module* if there exists $P \in \mathsf{mod}(kG)$ and two morphisms $\alpha \in \mathrm{Hom}_{kG}(M,P)$ and $\beta \in \mathrm{Hom}_{kG}(P,N)$ such that $f = \beta\alpha$.

A useful property of $\mathsf{mod}(kG)$ is the following.

Lemma 1.1.1.
*A short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ in $\mathsf{mod}(kG)$ splits if and only if $B \cong A \oplus C$.*

Of course, the necessary condition is true in any abelian category. The converse is a particularity of module categories over finite-dimensional algebras. See [Car96, Lem. 6.12] for a proof.

**Group operations, tensor products,** Hom**s and duality.**

**Restriction.** If $H \leq G$, then $kH$ is a subring of $kG$ and so any $kG$-module $M$ can be restricted to a $kH$-module, which we write as $M{\downarrow}_H^G$, or $\mathrm{Res}_H^G(M)$. In other words, restriction can be seen as the forgetful functor

$$\begin{array}{rccc} \mathrm{Res}_H^G : & \mathsf{mod}(kG) & \longrightarrow & \mathsf{mod}(kH) \\ & M & \longmapsto & M{\downarrow}_H^G\,. \end{array}$$

This functor is exact, for obvious trivial reasons.

**Induction.** If $L$ is a $kH$-module, then the induced module in $\mathsf{mod}(kG)$ is defined as the extension of scalars $L{\uparrow}_H^G := kG \otimes_{kH} L$, also denoted $\mathrm{Ind}_H^G(M)$. Since $kG$ is a free right $kH$-module, of rank $|G : H|$, there is an isomorphism of $k$-vector spaces $L{\uparrow}_H^G \cong \bigoplus_{x \in [G/H]} x \otimes L$, where $[G/H]$ denotes a complete set of representatives of the left cosets of $H$ in $G$, and where $x \otimes L$ denotes the conjugate module of $L$ by $x$ ($x \otimes L$ is also denoted ${}^xL$ and it is a $k[{}^xH]$-module). Moreover, induction can be seen as the functor

$$\begin{array}{rccc}\mathrm{Ind}_H^G : & \mathsf{mod}(kH) & \longrightarrow & \mathsf{mod}(kG) \\ & L & \longmapsto & L{\uparrow}_H^G\,,\end{array}$$

which is exact, again because $kG$ is a free right $kH$-module.

Note that if a short sequence of $kH$-modules $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is exact, then so is the induced sequence $0 \longrightarrow A{\uparrow}_H^G \xrightarrow{\alpha{\uparrow}_H^G} B{\uparrow}_H^G \xrightarrow{\beta{\uparrow}_H^G} C{\uparrow}_H^G \longrightarrow 0$. Moreover, one splits if and only if the other splits.

**Inflation.** If $N \trianglelefteq G$, any $k[G/N]$-module $V$ can be seen as a $kG$-module, denoted $\mathrm{Inf}_{G/N}^G(V)$ and called the *inflation of the module $V$ from $G/N$ to $G$*. This operation also yields an exact functor $\mathrm{Inf}_{G/N}^G : \mathsf{mod}(k[G/N]) \longrightarrow \mathsf{mod}(kG)$.

**Tensor induction.** Let $H \leq G$ and $L$ be a $kH$-module. Let $|G : H| =: n$ and consider the wreath product $\Sigma_n \wr H^n$, whose elements are of the form $(\sigma; h_1, \ldots, h_n)$ with $\sigma \in \Sigma_n$, $(h_1, \ldots, h_n) \in H^n$, and whose multiplication is given by: $(\sigma'; h'_1, \ldots, h'_n) \cdot (\sigma; h_1, \ldots, h_n) = (\sigma'\sigma; h'_{\sigma(1)}h_1, \ldots, h'_{\sigma(n)}h_n)$.

Given cosets representatives $g_1, \ldots, g_n$ of $H$ in $g$, for $g \in G$, we can write $g \cdot g_j = g_{\sigma(j)}h_j$ for uniquely defined elements $\sigma \in \Sigma_n$ and $(h_1, \ldots, h_n) \in H^n$. In other words, there is an injective group homomorphism $i : G \longrightarrow \Sigma_n \wr H^n : g \longrightarrow (\sigma; h_1, \ldots, h_n)$. Now, if $M \in \mathsf{mod}(kH)$, then the $n$-fold tensor product $M^{\otimes n}$ can be made into a $\Sigma_n \wr H^n$-module via

$$(\sigma; h_1, \ldots, h_n) \cdot (m_1 \otimes \cdots \otimes m_n) := h_{\sigma^{-1}(1)}m_{\sigma^{-1}(1)} \otimes \cdots \otimes h_{\sigma^{-1}(n)}m_{\sigma^{-1}(n)}\,.$$

The *tensor induced* module is defined to be $L{\uparrow}_{\otimes H}^{G} := i^*(M) = (M^{\otimes n}){\downarrow}_G^{\Sigma_n \wr H^n}$. The $kG$-module structure on $L{\uparrow}_{\otimes H}^{G}$ does not depend on the choice of the coset representatives. Tensor induction is well-behaved with respect to the tensor product $\otimes_k$, but not to the direct sum: if $L_1$ and $L_2$ are $kH$-modules, then $(L_1 \otimes L_2){\uparrow}_{\otimes H}^{G} \cong L_1{\uparrow}_{\otimes H}^{G} \otimes L_2{\uparrow}_{\otimes H}^{G}$ and $(L_1 \oplus L_2){\uparrow}_{\otimes H}^{G} \cong L_1{\uparrow}_{\otimes H}^{G} \oplus L_2{\uparrow}_{\otimes H}^{G} \oplus L'$, where $L'$ is the direct sum of modules induced from subgroups $K$ containing the intersection of the conjugates of $H$.

Here are summarized some of the most useful relations between tensor products, $\mathrm{Hom}_k(-,-)$ and group operations.

Proposition 1.1.2.
*Let $H, K$ be subgroups of $G$, $N$ a normal subgroup of $G$. Let $M, S \in \mathsf{mod}(kG)$, $L \in \mathsf{mod}(kH)$, $U \in \mathsf{mod}(kK)$ and $V \in \mathsf{mod}(k[G/N])$.*

(a) *There is a natural isomorphism of $kG$-modules $\theta_{M,S} : M^* \otimes_k S \longrightarrow Hom_k(M,S)$ defined by $\theta_{M,S}(f \otimes s)(m) := f(m) \cdot s$. In particular* $\mathrm{End}_k(M) \cong M^* \otimes M$.

(b) **Frobenius Reciprocity.** *There is a natural isomorphism of $kG$-modules:*

$$L{\uparrow}_H^G \otimes M \;\cong\; (L \otimes M{\downarrow}_H^G){\uparrow}_H^G$$

(c) **Nakayama Relations.** *There are natural isomorphisms of $kG$-modules*

$$\begin{array}{rcl}\mathrm{Hom}_{kG}(L{\uparrow}_H^G, M) & \cong & \mathrm{Hom}_{kH}(L, M{\downarrow}_H^G) \\ \mathrm{Hom}_{kG}(M, L{\uparrow}_H^G) & \cong & \mathrm{Hom}_{kH}(M{\downarrow}_H^G, M)\end{array}$$

*or, in other words, the functors* $\mathrm{Ind}_H^G$ *and* $\mathrm{Res}_H^G$ *are adjoint functors on both sides.*

(d) **Mackey Decomposition Formula.** *There is an isomorphism of $kK$-modules*

$$L{\uparrow}_H^G{\downarrow}_K^G \cong \bigoplus_{x \in [K\backslash G/H]} ({}^xL){\downarrow}^{{}^xH}_{{}^xH \cap K}{\uparrow}^K_{{}^xH \cap K}$$

*where $[K\backslash G/H]$ denotes a set of representatives of the $(K,H)$ double cosets in $G$.*

(e) *A consequence of the Mackey formula is the following:*

$$L{\uparrow}_H^G \otimes U{\uparrow}_K^G \cong \bigoplus_{x\in[K\backslash G/H]} ((^xL){\downarrow}^{^xH}_{^xH\cap K} \otimes U{\downarrow}^K_{^xH\cap K}){\uparrow}^G_{^xH\cap K}$$

(f) *Let $\varphi : H/H\cap N \longrightarrow HN/N$ be the canonical group isomorphism. Restriction and inflation commute in the following way:*

$$\mathrm{Res}^G_H \circ \mathrm{Inf}^G_{G/N}(V) \cong \mathrm{Inf}^H_{H/N\cap N} \circ \mathrm{Iso}(\varphi^{-1}) \circ \mathrm{Res}^{G/N}_{HN/N}(V)$$

(g) *If moreover $N \leq H$ and $W \in \mathsf{mod}(k[H/N])$, then induction and inflation commute in the following way:*

$$\mathrm{Ind}^G_H \circ \mathrm{Inf}^H_{H/N}(W) = \mathrm{Inf}^G_{G/N} \circ \mathrm{Ind}^{G/N}_{H/N}(W)$$

**Vertices, sources and Green correspondence.** The theory of vertices and sources establishes relationships between representation theory and the $p$-local structure of the group $G$. We shall make extensive use it.

Proposition-Definition 1.1.3.
*Let $M$ be a $kG$-module and $H$ a subgroup of $G$. Then $M$ is called projective relative to $H$ iff it is a direct summand of some module induced from $H$, or equivalently, iff it is a direct summand of $M{\downarrow}_H^G{\uparrow}_H^G$.*

Notice that a $kG$-module is projective if and only if it is projective relative to the trivial subgroup $\{1\}$. It is also a corollary that if $H$ is a subgroup of $G$ containing a Sylow $p$-subgroup of $G$, then every $kG$-module is projective relatively to $H$.

Definition 1.1.4.
Let $H \leq G$. A short exact sequence of $kG$-modules $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is called *$H$-split* if the restricted sequence $0 \longrightarrow A{\downarrow}_H^G \xrightarrow{\alpha} B{\downarrow}_H^G \xrightarrow{\beta} C{\downarrow}_H^G \longrightarrow 0$ splits.

Proposition-Definition 1.1.5.
*Let $U$ be an indecomposable $kG$-module.*

(a) *There is a $p$-subgroup $Q$ of $G$, unique up to conjugacy in $G$, such that $U$ is relatively $H$-projective for a subgroup $H$ of $G$ if and only if $H$ contains a $G$-conjugate of $Q$. Such a subgroup $Q$ is called a vertex of $U$.*

(b) *There is an indecomposable $kQ$-module $S$, unique up to conjugacy in $N_G(Q)$, such that $U \mid S{\uparrow}_Q^G$. Then, the module $S$ is called a source of $U$.*

One of the main tools of modular representation theory is Green's correspondence theorem that transfers the study of the indecomposable $kG$-modules from $G$ to proper subgroups of $G$. We refer the reader to [CR90, §20] for a proof, as well as for a statement of the result in full generality and use this section to set up notation for the next chapters.

An admissible triple $(G, H, D)$ for the Green correspondence consists of a finite group $G$, a $p$-subgroup $D$ and a subgroup $H$ containing $N_G(D)$. For each such triple, we define three families of subgroups:

$$\mathcal{X} := \{^xD \cap D \mid x \in G \setminus H\}\,, \quad \mathcal{Y} := \{^xD \cap H \mid x \in G \setminus H\} \quad \mathcal{A} := \{D^* \leq D \mid D^* \not\leq_G \mathcal{X}\}\,.$$

Theorem 1.1.6 (Green Correspondence).
*Let $(G, H, D)$ be an admissible triple as above. Then, there exists a bijection*

$$\Gamma : [M] \rightleftarrows [N] : Gr$$

*from the set of isomorphism classes of indecomposable $kG$-modules $M$ with vertex in $\mathcal{A}$ to the set of isomorphism classes of indecomposable $kH$-modules $N$ with vertex in $\mathcal{A}$. An indecomposable $kG$-module $M$ with vertex in $\mathcal{A}$ corresponds to an indecomposable $kH$-module $N$ with the same vertex if and only if the following equivalent conditions hold:*

*(i) $M\downarrow^G_H \cong N \oplus (\mathcal{Y} - proj)$; (ii) $N\uparrow^G_H \cong M \oplus (\mathcal{X} - proj)$.*

*Furthermore, corresponding modules have the same source as well as the same vertex.*

**Module varieties.** Sources for a complete introduction to module varieties are [Ben98b] and [CTVEZ03].

By the Evens-Venkov Theorem, the cohomology ring $H^*(G, k)$ is a finitely generated $k$-algebra, and thus it is noetherian. Define $V_G(k)$, or simply $V_G$, to be the maximal ideal spectrum of $H^*(G, k)$, topologized by the Zariski topology. This is a homogeneous affine variety.

For modules $M, N \in \mathsf{mod}(kG)$, $\mathrm{Ext}^*_{kG}(M, N)$ is a finitely generated module over $H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$. For any $kG$-module $M$, let $J_G(M)$ be the annihilator in $H^*(G, k)$ of the cohomology ring $\mathrm{Ext}^*_{kG}(M, M)$. We can take $J_G(M)$ to be the annihilator in $H^*(G, k)$ of the identity element $\mathrm{Id}_M$. By definition, the *support variety* of the module $M$ is the closed subset $V_G(M) := V_G(J_G(M))$ of $V_G(k)$ consisting of all maximal ideals that contain $J_G(M)$. The variety $V_G(M)$ is homogeneous. Moreover varieties have the following properties:

Properties 1.1.7.
*Let $M$ and $N$ be $kG$-modules.*

(a) $V_G(M) = 0$ *if and only if $M$ is projective.*
(b) $V_G(M) = V_G(M^*)$.
(c) $V_G(M \oplus N) = V_G(M) \cup V_G(N)$.
(d) $V_G(M \otimes N) = V_G(M) \cap V_G(N)$.

If $G$ is an elementary abelian $p$-group, then the notion of *support variety* coincides with the notion of *rank variety* which is defined as follows. Let $G =< x_1, \dots, x_n >$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n$, let $u_\alpha = 1 + \sum_{i=1}^n \alpha_i(x_i - 1)$. For a $kG$-module $M$, define the *rank variety* of $M$ to be

$$V^r_G(M) = \{\alpha \in k^n \mid M\downarrow_{\langle u_\alpha\rangle} \text{ is not a free } k\langle u_\alpha\rangle\text{-module}\} \cup \{0\}$$

where $M\downarrow_{\langle u_\alpha\rangle}$ is the restriction of $M$ to the subalgebra $k\langle u_\alpha\rangle$ of $kG$.

## 1.2. Endo-permutation modules, endotrivial modules and relatives

Our main goal is the study of classes of modules, that we shall call *relative endotrivial modules*, and that will turn out to be closely related to endotrivial and endo-permutation modules. We review here the definitions and basic properties of the two latter classes of modules. For a nice and shallow introduction to the subject we refer the reader to the survey paper *Endo-permutation modules, a guided tour*, by J. Thévenaz [Thé07], which gives an overview of the main steps of the recent classification of endo-permutation modules, and is also a great source for further material related to the subject.

Definition 1.2.1.
Let $G$ be a finite group and $P$ be a finite $p$-group.

(a) A $kG$-module is called a *permutation* module iff it possesses a $G$-invariant $k$-basis.
(b) A $kG$-module is a *p-permutation* module iff it is isomorphic to a direct summand of permutation $kG$-module.
(c) A $kP$-module $M$ is called an *endo-permutation module* iff its endomorphism algebra $\mathrm{End}_k(M)$ is a permutation $kP$-module. Furthermore, an endo-permutation module $M$ is *capped* if it possesses an indecomposable summand with vertex $P$.
(c) A $kG$-module $M$ is called an *endo-p-permutation module* iff $\mathrm{End}_k(M)$ is a $p$-permutation $kG$-module.
(d) A $kG$-module $M$ is termed *endotrivial* iff $\mathrm{End}_k(M) \cong M^* \otimes_k M \cong k \oplus (proj)$, where $(proj)$ denotes a projective module.

Remarks 1.2.2.
It follows from the definitions that:
— For a $p$-group the notions of permutation and $p$-permutation module coincide. So do the notions of endo-permutation and endo-$p$-permutation modules.
— Any permutation $kP$-module is an endo-permutation $kP$-module and any $p$-permutation $kG$-module is an endo-$p$-permutation $kG$-module.
— Any endotrivial $kG$-module is a an endo-$p$-permutation $kG$-module.
— If $X$ is a $G$-invariant $k$-basis of a permutation $kG$-module $M$, then the decomposition of $X$ into $G$-orbits yields a direct sum decomposition of $M$ into summands, each isomorphic to a permutation module $k[G/H]$ with $H$ the stabiliser of an element $x \in X$ in the considered $G$-orbit.
— For a $p$-group $P$, the modules $k[P/Q]$ are indecomposable for every subgroup $Q$ of $P$ (since their socle is clearly indecomposable). By contrast this is not true in general if $G$ is not a $p$-group and the indecomposable direct summands of $k[G/H]$ are, in general, not permutation modules any more, hence the notion of a $p$-permutation module. — Permutation modules are stable under the operations of restriction, induction and conjugation.
— For instance, $kP$ is a permutation module and thus an endo-permuation module but it is not endotrivial.
— For instance, the trivial $kG$-module $k$ is endo-$p$-permutation, $p$-permutation and endotrivial.

Proposition 1.2.3 (Endo-permutation modules and the Dade group, [Dad78a]).

(a) *The class of endo-permutation modules is closed under taking direct summands, duals, tensor products (over $k$), Heller translates, restriction to a subgroup and tensor induction to an overgroup.*
(b) *An endo-permutation $kP$-module $M$ is capped if and only if the trivial module is a direct summand of* $\mathrm{End}_k(M)$.
(c) *If $M$ is capped, then any two indecomposable summands of $M$ with vertex $P$ are isomorphic. This unique summand, up to isomorphism, is called the cap of $M$ and is written $M_0$.*
(d) *An equivalence relation $\sim$ on the class of endo-permutation module is defined by: $M \sim N$ if and only if $M_0 \cong N_0$.*
(e) *Let $D(P)$ denote the resulting set of equivalence classes. Then $D(P)$ is an abelian group for the following law:*
$$[M] + [N] \cong [M \otimes N]$$
*The zero element is the class $[k]$ of the trivial $kP$-module, while the opposite of a class $[M]$ is the class of the dual module $[M^*]$. This group is called the Dade group of the group $P$.*

Proposition 1.2.4 (Endotrivial modules).

(a) *The class of endotrivial modules is closed under taking duals, tensor products (over $k$), Heller translates, restriction to a subgroup.*
(b) *An endotrivial $kG$-module $M$ can be written in a unique way (up to isomorphism) as $M = M_0 \oplus F$, where $M_0$ is indecomposable and endotrivial and $F$ is a projective $kG$-module.*
(c) *An equivalence relation $\sim$ on the class of endotrivial modules is defined by: $M \sim N$ if and only if $M_0 \cong N_0$.*
(d) *Let $T(G)$ denote the resulting set of equivalence classes. Then $T(G)$ is an abelian group for the following law:*
$$[M] + [N] \cong [M \otimes N]$$
*The zero element is the class $[k]$ of the trivial $kG$-module, while the opposite of a class $[M]$ is the class of the dual module $[M^*]$. This group is called the group of endotrivial modules of the group $G$.*
(e) *If $G$ is a $p$-group, then an endotrivial $kG$-module is a capped endo-permutation module with cap $M_0$, and the group $T(G)$ embeds as a subgroup of the Dade group $D(G)$.*

The classification of endo-permutation modules, through the description of the structure of the Dade group, started with [Dad78a], [Dad78b] and independently [Alp77]. It was completed in 2004 by S. Bouc in [Bou06]. In between, crucial steps for this classification include the classification of the endotrivial modules of a $p$-group. All this was achieved through the work of [Pui90], [BT00], [CT00], [CT04], [CT05], [Bou04] and [BM04].
At this stage there is no classification for endotrivial modules in general. Some cases are treated in the following articles: groups with a cyclic Sylow $p$-subgroup in [MT07], groups with a normal Sylow $p$-subgroup in [Maz07], symmetric and alternating groups in [CMN09], finite groups of Lie type in [CMN06], $p$-solvable groups in [CMT11a], groups with quaternion or semi-dihedral Sylow 2-subgroups in [CMT11b].

The class of endo-$p$-permutation modules was mainly studied in [Urf06] and [Urf07]. Main properties are the following:

Proposition 1.2.5 (Endo-$p$-permutation modules).
*The class of endo-$p$-permutation modules is closed under taking direct summands, duals, tensor products (over $k$), restriction to a subgroup and tensor induction to an overgroup.*

More detailed properties of endotrivial, endo-permutation and endo-$p$-permutation modules shall be recalled in the following chapters in relevant situations.

# CHAPTER 2

# Projectivity Relative to a Module

The purpose of this *first* chapter is to provide a wide-ranging treatment of the subject of relative projectivity with respect to a $kG$-module. It originates in the early 1990's in T. Okuyama's unpublished piece of work [Oku91]. Afterwards, it was further developed and used by J. Carlson, C. Peng and W.Wheeler in [CP96] and [CPW98]. A more detailed reference is Carlson's Lectures in Mathematics [Car96].

Relative projectivity to a module, however, is just a special case of the relative homological algebra generated by a projective class of epimorphisms as described in [HS71, Chap. 10]. It also coincides with the relative projectivity relative to a pair of adjoint functors as described in [HS71, Chap. 9]. For the sake of completeness, sections 1,2,3,4,5, 11 and 12 essentially give an overview of the properties of relative projectivity expounded in the aforementioned references, whereas the other sections develop further material.

## 2.1. Definitions, notation, terminology

Definition 2.1.1 ([Oku91]).
Let $V$ be a $kG$-module.

(a) A finitely generated $kG$-module $M$ is termed *projective relative to the module* $V$ or *relatively* $V$*-projective*, or simply $V$*-projective* if there exists a $kG$-module $N$ such that $M$ is isomorphic to a direct summand of $V \otimes_k N$.

(b) A short exact sequence $E : 0 \longrightarrow A \stackrel{\alpha}{\longrightarrow} B \stackrel{\beta}{\longrightarrow} C \longrightarrow 0$ in mod $(kG)$ is termed $V$*-split* if the tensored sequence $V \otimes E : 0 \longrightarrow V \otimes A \xrightarrow{V\otimes\alpha} V \otimes B \xrightarrow{V\otimes\beta} V \otimes C \longrightarrow 0$ splits.

Notation and Terminology.
The subcategory of all $V$-projective modules of mod($kG$) shall be denoted by $Proj(V)$ and the class of $V$-projective indecomposable modules by $IProj(V)$. A module $U$ is said to be a generator for $Proj(V)$ iff $Proj(U) = Proj(V)$. Moreover, in computations we shall often denote by $(V-proj)$ a module in $Proj(V)$, which does not need to be specified, and simply $(proj)$ for a projective module. We shall also always consider the modules in $Proj(V)$ and $IProj(V)$ up to isomorphism.

**Categorical setting.** From the point of view of category theory, relative projectivity has the following two characteristics:

Firstly, projectivity relative to a $kG$-module $V$ gives rise to an analogue of the stable module category $\mathsf{stmod}(kG)$. Indeed, define the *$V$-stable category* to be the category $\mathsf{stmod}_V(kG)$ in which the objects are those in $\mathsf{mod}(kG)$ and the morphisms from a module $M$ to a module $N$ are given by

$$\underline{\mathrm{Hom}}^V_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}^V_{kG}(M,N),$$

where $\mathrm{PHom}^V_{kG}(M,N)$ denotes the subspace of $\mathrm{Hom}_{kG}(M,N)$ consisting of all morphisms from $M$ to $N$ that factor through a $V$-projective module.

[CPW98, Sect. 6] proves that $\mathsf{stmod}_V(kG)$ is a triangulated category. (See [CPW98] for a description of the distinguished triangles.) Moreover $(\mathsf{stmod}_V(kG), \otimes_k, k)$ can be seen as a $\otimes$-triangulated category , when equipped with the symmetric monoidal structure given by the tensor product over $k$,

$$\otimes_k : \mathsf{stmod}_V(kG) \times \mathsf{stmod}_V(kG) \longrightarrow \mathsf{stmod}_V(kG)\,,$$

which is exact in each variable, and with unit $k$.

Secondly, the subcategories $\mathit{Proj}(V)$ are functorially finite in the sense of [AS80] and therefore the homological algebra generated by $\mathit{Proj}(V)$ also coincides with their notion of homological algebra generated by a functorially finite subcategory.

## 2.2. The trace map and first properties

To any $kG$-module $V$ one can associate a $kG$-homomorphism called a *trace map* and defined by:

$$\begin{array}{rccc} \mathrm{Tr}_V\colon & V^*\otimes V & \longrightarrow & k \\ & f\otimes v & \longmapsto & f(v) \end{array}$$

Indeed $\mathrm{Tr}_V \circ \theta^{-1}_{V,V}$ is the ordinary trace of matrices. (Where $\theta^{-1}_{V,V}$ is the natural isomorphism of proposition 1.1.2.)

As Okuyama points out in [Oku91], the real point of origin of the notion of relative projectivity to a module is the following crucial lemma due to Auslander and Carlson in [AC86].

Lemma 2.2.1 ([AC86], [Car96]).
*Let $V$ be a $kG$-module.*

(a) *If $p\nmid \dim_k(V)$, then the trace short exact sequence*

$$0\longrightarrow \ker(\mathrm{Tr}_V) \longrightarrow V^*\otimes V \xrightarrow{\mathrm{Tr}_V} k \longrightarrow 0$$

*splits. In consequence, $V^*\otimes V\cong k\oplus \ker(\mathrm{Tr}_V)$.*

(b) *Furthermore, the trace s.e.s. is always $V$-split, that is the s.e.s.*

$$0\longrightarrow V\otimes\ker(\mathrm{Tr}_V) \to V\otimes V^*\otimes V \xrightarrow{V\otimes \mathrm{Tr}_V} V\otimes k\longrightarrow 0$$

*splits. In consequence, $V\mid V\otimes V^*\otimes V$.*

(c) *Furthermore, if $p\mid \dim_k(V)$, then $V\oplus V\mid V\otimes V^*\otimes V$.*

Proof.

(a) Let $n := \dim_k(V)$, let $\{v_i\}_{1\leq i\leq n}$ be a $k$-basis for $M$ and let $\{v_i^*\}_{1\leq i\leq n}$ be its dual basis. Then, $\mathbf{r} := \sum_{i=1}^n v_i^*\otimes v_i$ is the element in $V^*\otimes V$ corresponding to $\mathrm{id}_V\in \mathit{End}_k(V)\cong V^*\otimes V$ and there is a $kG$-homomorphism

$$\begin{array}{rccc} I\colon & k & \longrightarrow & V^* \otimes V \\ & 1_k & \longmapsto & \mathbf{r}\,. \end{array}$$

It is easy to check that the homomorphism $\frac{1}{\dim_k(V)} I$ is a $kG$-section for $\mathrm{Tr}_V$.

(b) The map

$$\begin{array}{ccc} V \otimes k & \longrightarrow & V \otimes V^* \otimes V \\ v \otimes 1 & \longmapsto & \sum_{i=1}^n v \otimes v_i^* \otimes v_i \end{array}$$

is a $kG$-section for the map $V \otimes \mathrm{Tr}_V$, whatever the value of $\dim_k(V)$ modulo $p$.

(c) Define

$$\begin{array}{rccc} \psi\colon & V \otimes V^* \otimes V & \longrightarrow & V \oplus V \\ & v \otimes f \otimes v' & \longmapsto & (f(v)v', f(v')v) \end{array}$$

which is a surjective $kG$-homomorphism. In addition, the assumption that $\dim_k(V)$ is 0 in $k$ implies that the $kG$-homomorphism

$$\begin{array}{rccc} \theta\colon & V \oplus V & \longrightarrow & V \otimes V^* \otimes V \\ & (v, v') & \longmapsto & \sum_{i=1}^n v \otimes v_i^* \otimes v_i + \sum_i v_i \otimes v_i^* \otimes v' \end{array}$$

is a section for $\psi$. The result follows.

$\square$

The following omnibus proposition sums up elementary properties of relative projectivity, that we shall use extensively in the sequel of this text.

PROPOSITION 2.2.2 (**Omnibus properties**).
*Let $A, B, M, U, V$ be $kG$-modules.*

(a) *Any direct summand of a $V$-projective module is $V$-projective.*
(b) *If $U \in Proj(V)$, then $Proj(U) \subseteq Proj(V)$.*
(c) *If $p \nmid \dim_k(V)$ then $Proj(V) = \mathsf{mod}(kG)$.*
(d) $Proj(V) = Proj(V^*)$.
(e) $Proj(U \oplus V) = Proj(U) \oplus Proj(V)$.
(f) $Proj(U) \cap Proj(V) = Proj(U \otimes V) \supseteq Proj(U) \otimes Proj(V)$ .
(g) $Proj(\bigoplus_{j=1}^n V) = Proj(V) = Proj(\bigotimes_{j=1}^m V)\ \forall\, m, n \in \mathbb{N}\backslash\{0\}$.
(h) *$C \cong A \oplus B$ is $V$-projective if and only if both $A$ and $B$ are $V$-projective.*
(i) *$Proj(V) = Proj(\Omega^n(V))$ for all $n \in \mathbb{Z}$.*
(j) $Proj(V) = Proj(V^* \otimes V)$.
(k) *$M \in Proj(V)$ if and only if $\mathrm{End}_k(M) \cong M^* \otimes M \in Proj(V)$.*
(l) *Let $H \trianglelefteq G$, let $g \in G$, and let $W \in \mathsf{mod}(kH)$. Then ${}^gProj(W) = Proj({}^gW)$. In particular, if $W$ is $G$-invariant, then $Proj({}^gW) = Proj(W)$ for all $g \in G$ and a $kH$-module $M$ is $W$-projective if and only if all its $G$-conjugates ${}^gM$ are $W$-projective.*
(m) *$Proj(kG) \subseteq Proj(V)$ for every $kG$-module $V$. Moreover, $Proj(kG)$ is equal to the whole collection of projective modules in $\mathrm{mod}(kG)$ and $Proj(kG) = Proj(P)$ for any projective $kG$-module $P$.*

Apart from (l), all these properties appear either in [Oku91], or in [Car96, Sect. 8], or in [CP96, Sect. 3.3]. However, they do not necessarily come with a proof.

Moreover we note that statement (f) was mistyped (and not proven) in [CP96, Lem. 3.3(iii)] as $Proj(U) \otimes Proj(V) = Proj(U \otimes V)$ instead of $Proj(U) \cap Proj(V) = Proj(U \otimes V)$. We note that in general, $Proj(U) \otimes Proj(V) \neq Proj(U \otimes V)$. For instance, take $G := C_9$ the cyclic group of order 9, $U := k{\uparrow}_{C_3}^{C_9}$ and $V := kG$. Then, $Proj(V) = Proj(U \otimes V)$, the set of projective modules, whereas it will be easy to compute from the results we obtain in Chapter 5 for cyclic $p$-groups that $Proj(U) \otimes Proj(V) = \{kG^{\oplus 3n} \,|\, n \in \mathbb{N}\}$.

Proof.

(a) Is straightforward from the definition.

(b) Let $M \in Proj(U)$, then $M \,|\, U \otimes N$ for some $kG$-module $N$. In addition $U \in Proj(V)$ means that $U \,|\, V \otimes L$ for some $kG$-module $L$. Thus, $M \,|\, V \otimes N \otimes L$, i.e. $M \in Proj(V)$.

(c) If $p \nmid \dim_k(V)$, the trace map $\mathrm{Tr}_V : V^* \otimes V \twoheadrightarrow k$ splits by 2.2.1. Therefore $k \,|\, V^* \otimes V$. So that for every $M \in \mathsf{mod}(kG)$, $M \,|\, V^* \otimes V \otimes M$. Hence $Proj(V) = \mathsf{mod}(kG)$.

(d) By part (b) of 2.2.1, $V \,|\, V \otimes V^* \otimes V$. Thus, $V \in Proj(V^*)$ and dually $V^* \in Proj(V)$. Hence by (b) $Proj(V) = Proj(V^*)$.

(e) Let $M \in Proj(U \oplus V)$. Then there is $N \in mod(kG)$ such that

$$M \,|\, (U \oplus V) \otimes N \cong (U \otimes N) \oplus (V \otimes N)\,.$$

By Krull-Schmidt we can write $M \cong M_U \oplus M_V$ with $M_U \,|\, U \otimes N$ and $M_V \,|\, V \otimes N$. Then $M_U \in Proj(U)$ and $M_V \in Proj(V)$, so that $M \in Proj(U) \oplus Proj(V)$. For the reverse inclusion, let $M \in Proj(U) \oplus Proj(V)$. Write $M \cong M_U \oplus M_V$ with $M_U \in Proj(U)$ and $M_V \in Proj(V)$. Thus $M_U \,|\, U \otimes N_U$ and $M_V \,|\, V \otimes N_V$ for some $N_U, N_V \in \mathsf{mod}(kG)$. Then,

$$M_U \oplus M_V \,|\, (U \oplus V) \otimes N_U \oplus (V \oplus U) \otimes N_V \cong (U \oplus V) \otimes (N_U \oplus N_V)$$

whence $M \in Proj(U \oplus V)$, and the result follows.

(f) $Proj(U \otimes V) \subseteq Proj(U) \cap Proj(V)$ by the very definition of $U \otimes V$-projectivity. If $M \in Proj(U) \cap Proj(V)$, then there are $kG$-modules $N$ and $L$ such that $M \,|\, U \otimes N$ and $M \,|\, V \otimes L$. By 2.2.1 $M \,|\, M \otimes M^* \otimes M$. In consequence:

$$M \,|\, M \otimes M^* \otimes M \,|\, U \otimes N \otimes M^* \otimes V \otimes L \cong U \otimes V \otimes N \otimes M^* \otimes L\,.$$

Hence $Proj(U) \cap Proj(V) = Proj(U \otimes V)$. In addition, if $M \in Proj(U) \otimes Proj(V)$, that is $M \cong M_U \otimes M_V$ with $M_U \in Proj(V)$ and $M_V \in Proj(V)$, then there are modules $N_U, N_V \in \mathsf{mod}(kG)$ such that $M_U \,|\, U \otimes N_U$ and $M_V \,|\, V \otimes N_V$. This yields

$$M \cong M_U \otimes M_V \,|\, U \otimes N_U \otimes V \otimes N_V \cong U \otimes V \otimes N_U \otimes N_V\,.$$

Hence $Proj(U \otimes V) \supseteq Proj(U) \otimes Proj(V)$.

(g) Follows from (e) and (f).

(h) The necessary condition is given by (a). For the converse, assume that $A, B \in Proj(V)$, then by (e), $C \cong A \oplus B \in Proj(V) \oplus Proj(V) = Proj(V \oplus V) = Proj(V)\,.$

(i) For all $n \in \mathbb{Z}$, $\Omega^n(k) \otimes V \cong \Omega^n(V) \oplus (proj)$, whence $Proj(\Omega^n(V)) \subseteq Proj(V)$ by (b). Now $\Omega^n(V) \otimes \Omega^{-n}(V) \cong \Omega^0(V) \oplus (proj)$, so that $Proj(\Omega^0(V)) \subseteq Proj(\Omega^n(V))$. Moreover $V \cong \Omega^0(V) \oplus (proj) \in Proj(\Omega^0(V))$ by (m) below. Hence $Proj(V) \subseteq Proj(\Omega^0(V))$ and the result follows.

(j) By (d) and (g), $Proj(V^* \otimes V) = Proj(V^*) \cap Proj(V) = Proj(V) \cap Proj(V) = Proj(V)$.

(k) By (j), $Proj(M) = Proj(M^* \otimes M)$. Thus by (c), if $M \in Proj(V)$, then $Proj(V) \supseteq Proj(M) = Proj(M^* \otimes M)$, so that in particular $M^* \otimes M \in Proj(V)$, and conversely.

(l) Let $M$ be a $kH$-module. Then $M \in Proj(W)$ if and only if $M \,|\, W \otimes N$ for some $N \in \mathsf{mod}(kG)$ if and only if ${}^gM \,|\, {}^g(W \otimes N) \cong {}^gM \otimes {}^gN$ if and only if ${}^gM \in Proj({}^gW)\,.$

(m) Recall that $V \otimes kG \cong kG^{\oplus dim_k V}$, hence $kG$ is $V$-projective and $Proj(kG) \subseteq Proj(V)$ by (b). In particular, if $P$ is a projective module, then $Proj(kG) \subseteq Proj(P)$. In addition, $P$ is a direct summand of $kG^n$ for some $n \in \mathbb{N}$, hence $P \in Proj(kG)$ and by (b) $Proj(P) \subseteq Proj(kG)$.

□

## 2.3. $V$-split short exact sequences

Proposition 2.3.1 ([Car96]).
*Let* $E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *be a* $V$*-split short exact sequence in* $\mathsf{mod}(kG)$ *and* $M$ *a* $V$*-projective module. Then, the tensored sequence*

$$E \otimes M : 0 \longrightarrow A \otimes M \xrightarrow{\alpha \otimes M} B \otimes M \xrightarrow{\beta \otimes M} C \otimes M \longrightarrow 0$$

*splits.*

Proof (Sketch): Since the $M \mid V \otimes N$ for some module $N \in \mathsf{mod}(kG)$, the exact sequence $E \otimes M$ is a direct summand of the exact sequence $E \otimes V \otimes N$, which splits, and therefore so does $E \otimes M$. □

Corollary 2.3.2.
*Let* $E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *be a short exact sequence in* $\mathsf{mod}(kG)$.

(a) *Let* $U, V \in \mathsf{mod}(kG)$ *such that* $Proj(U) = Proj(V)$*. Then,* $E$ *is* $U$*-split if and only if* $E$ *is* $V$*-split.*
*In particular,* $E$ *is* $V$*-split if and only if* $E$ *is* $V^*$*-split if and only if* $E$ *is* $\Omega^n(V)$*-split.*
(b) $E$ *is* $V$*-split if and only if* $E^*$ *is* $V$*-split.*

Proof.
(a) If $E$ is $U$-split then $E \otimes W$ splits for every $W \in Proj(U)$ by 2.3.1, in particular $E \otimes V$ splits, i.e. $E$ is $V$-split. Swap the roles of $U$ and $V$ to obtain the converse.
(b) It is clear that $E$ is $V$-split if and only if $E^*$ is $V^*$-split. Then (a) yields the result because $Proj(V) = Proj(V^*)$.

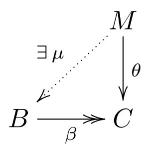
□

## 2.4. Equivalent definitions

Finally, we have all the tools in hand to establish the following equivalent definition for $V$-projectivity.

Proposition 2.4.1.
*Let* $M$ *and* $V$ *be* $kG$*-modules. Then the following statements are equivalent:*

(a) $M$ *is* $V$*-projective;*
(b) $M \mid V^* \otimes V \otimes M$*;*
(c) ***Universal property of*** $V$***-projective modules****: for every surjective* $V$*-split* $kG$*-homomorphism* $\beta : B \longrightarrow C$ *and every* $kG$*-homomorphism* $\theta : M \longrightarrow C$ *with* $M \in Proj(V)$*, there exists a* $kG$*-homomorphism* $\mu : M \longrightarrow B$ *such that the following diagram commutes:*




Proof.
(b) ⇒ (a): is obvious.

(c) $\Rightarrow$ (b): Since $Tr_V$ is $V$-split, so is $Tr_V \otimes M$. Therefore, taking $\theta = \mathrm{Id}_M$ in (c) yields the existence of a $kG$-section $\mu : M \longrightarrow V^* \otimes V \otimes M$ for $Tr_V \otimes M$. Hence $V^* \otimes V \otimes M \cong M \oplus \ker(\mathrm{Tr}_V \otimes M)$.

(a) $\Rightarrow$ (c): There is a commutative diagram

$$\begin{array}{ccc} Hom_{kG}(M,B) & \xrightarrow{\beta_*} & Hom_{kG}(M,C) \\ \cong\big\downarrow & \circlearrowleft & \big\downarrow\cong \\ Hom_{kG}(k, M^* \otimes B) & \xrightarrow[(1\otimes\beta)_*]{} & Hom_{kG}(k, M^* \otimes C) \end{array}$$

Now, since $M$ is $V$-projective, so is $M^*$ by the omnibus properties of $V$-projectivity. Thus the sequence

$$0 \longrightarrow M^* \otimes \ker(\beta) \longrightarrow M^* \otimes B \xrightarrow{1\otimes\beta} M^* \otimes C \longrightarrow 0$$

splits. Therefore, the $kG$-homomorphism $(1 \otimes \beta)_*$ is onto and, by commutativity of the diagram, so is $\beta_*$. Thus there exists a map $\mu \in Hom_{kG}(M,B)$ such that $\theta = \beta_*(\mu) = \beta\mu$. □

REMARK 2.4.2.
A $kG$-module $M$ shall be termed $V$*-injective* if and only if it is $V$-projective. In fact, a proper definition should be established by dualizing the universal property of $V$-projective modules. However, since $kG$ is symmetric, as in the case of ordinary projectivity, this would produce a class of modules coinciding with the class of $V$-projective modules.

## 2.5. Projectivity relative to subgroups

The notion of projectivity relative to a module encompasses the notion of projectivity relative to a subgroup, widely used in the theory of vertices and sources. We refer to [Alp86] and [CR90] for presentations of this theory.

LEMMA 2.5.1.
*Let $G$ be a finite group and $H$ be a subgroup. Let $M$ be a $kG$-module. Then*

(a) *$M$ is projective relative to the subgroup $H$ if and only if $M$ is projective relative to the $kG$-module $k\uparrow_H^G$;*
(b) *a short exact sequence $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathsf{mod}(kG)$ is $H$-split if and only if it is $k\uparrow_H^G$-split.*

PROOF.

(a) $M$ is projective relatively to $H$ if and only if

$$M \mid M\downarrow_H^G\uparrow_H^G \cong (k \otimes M\downarrow_H^G)\uparrow_H^G \cong k\uparrow_H^G \otimes M$$

therefore $M \in Proj(k\uparrow_H^G)$. Conversely, $M \in Proj(k\uparrow_H^G)$ if and only if there exists a module $N \in \mathsf{mod}(kG)$ such that $M \mid k\uparrow_H^G \otimes N \cong (k \otimes N\downarrow_H^G)\uparrow_H^G$ which is a module induced from $H$, thus $M$ is $H$-projective.

(b) Recall that a short exact sequence is $H$-split if and only if it splits upon restriction to $H$. There is a commutative diagram:

$$\begin{array}{cccccccc} E\otimes k\uparrow_H^G: & 0 \longrightarrow & A\otimes k\uparrow_H^G & \xrightarrow{\alpha\otimes 1} & B\otimes k\uparrow_H^G & \xrightarrow{\beta\otimes 1} & C\otimes k\uparrow_H^G & \longrightarrow 0 \\ & & \downarrow\cong & \circlearrowleft & \downarrow\cong & \circlearrowleft & \downarrow\cong & \\ E\downarrow_H^G\uparrow_H^G: & 0 \longrightarrow & A\downarrow_H^G\uparrow_H^G & \longrightarrow & B\downarrow_H^G\uparrow_H^G & \longrightarrow & C\downarrow_H^G\uparrow_H^G & \longrightarrow 0 \end{array}$$

Thus, the top sequence splits if and only if the bottom sequence splits. In addition $E\downarrow_H^G\uparrow_H^G$ splits if and only if $E\downarrow_H^G$ splits (see 1.1). Hence the result.

□

Furthermore, the notion of projectivity relative to a module also encompasses the notion of projectivity relative to a familly of subgroups as described in [Thé85] or [Knö78] . With the advantage that it becomes somewhat less cumbersome when we look at it as projectivity relative to a single module.

Lemma 2.5.2.

*Let $G$ be a finite group and $\mathcal{H}$ be a family of subgroups of $G$. Let $M$ be a $kG$-module. Then*

(a) *$M$ is projective relative to the family $\mathcal{H}$ if and only if $M$ is projective relative to the $kG$-module $V(\mathcal{H}) := \bigoplus_{H\in\mathcal{H}} k\uparrow_H^G$;*

(b) *a short exact sequence $E : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathsf{mod}(kG)$ is $\mathcal{H}$-split if and only if it is $V(\mathcal{H})$-split.*

Proof.

(a) If $M \in Proj(\mathcal{H})$ then $M \cong \bigoplus_{i\in I} M_i$, with $I$ a finite indexing set and for all $i \in I$ $M_i \in Proj(H_i) = Proj(k\uparrow_{H_i}^G)$ for some $H_i \in \mathcal{H}$. Therefore

$$M \in \bigoplus_{i\in I} Proj(k\uparrow_{H_i}^G) = Proj(\bigoplus_{i\in I} k\uparrow_{H_i}^G) \subseteq Proj(\bigoplus_{H\in\mathcal{H}} k\uparrow_H^G)$$

since $\bigoplus_{i\in I} k\uparrow_{H_i}^G \mid \bigoplus_{H\in\mathcal{H}} k\uparrow_H^G$.

Conversely, if $M \in Proj(\bigoplus_{H\in\mathcal{H}} k\uparrow_H^G) = \bigoplus_{H\in\mathcal{H}} Proj(k\uparrow_H^G)$, then $M$ decomposes as $M \cong \bigoplus_{H\in\mathcal{H}} M_H$ with $M_H \in Proj(k\uparrow_H^G)$, that is $M$ is projective relatively to $\mathcal{H}$.

(b) The sequence $E$ is $\mathcal{H}$-split if and only if it has a $kH$-linear section for all $H \in \mathcal{H}$, that is if and only if it is $k\uparrow_H^G$-split for all $H \in \mathcal{H}$ by the preceeding lemma. But

$$E\otimes V(\mathcal{H}) \cong E \otimes (\bigoplus_{H\in\mathcal{H}} k\uparrow_H^G) \cong \bigoplus_{H\in\mathcal{H}} E\otimes k\uparrow_H^G$$

as short exact sequences, thus if $E$ is $\mathcal{H}$-split, then it is also $V(\mathcal{H})$-split. Indeed, taking the direct sum of the sections on each summand $E\otimes k\uparrow_H^G$ gives a section for $E\otimes V(\mathcal{H})$. For the converse recall that if $E$ is assumed to be $V(\mathcal{H})$-split, then by Proposition 2.3.1, $E\otimes N$ splits for any $V(\mathcal{H})$-projective module $N$. In particular, $k\uparrow_H^G \in Proj(\bigoplus_{H\in\mathcal{H}} k\uparrow_H^G)$ for all $H \in \mathcal{H}$, hence $E\otimes k\uparrow_H^G$ splits for all $H\in\mathcal{H}$, or in other words, by the preceding lemma, $E$ has a $kH$-linear section for every $H \in \mathcal{H}$.

□

Notation.

In the sequel of this text, if $\mathcal{H}$ is a family of subgroups, then we always write $V(\mathcal{H}) := \bigoplus_{H\in\mathcal{H}} k\uparrow_H^G$ and for simplicity, we also sometimes write $Proj(\mathcal{H})$ instead of $Proj(V(\mathcal{H}))$.

Remark 2.5.3.
As two different families of subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ can generate the same subcategories $Proj(\mathcal{H}_1)$ and $Proj(\mathcal{H}_2)$ of modules, the following observation shall be useful in computations. Recall that equivalently a $kG$-module $M$ is projective relatively to a subgroup $H \leq G$ or to any conjugate subgroup ${}^gH$ for $g \in G$. In other words, $M \in \mathsf{mod}(kG)$ is projective relatively to $H$ if and only if it is projective relatively to the family of subgroups of $G$ defined by the conjugacy class of $H$ (or to any subfamily of the latter). Indeed, $Proj(k\!\uparrow_H^G) = Proj(k\!\uparrow_{{}^gH}^G)$ implies that

$$\begin{aligned} Proj(\bigoplus_{g\in G} k\!\uparrow_{{}^gH}^G) &= \bigoplus_{g\in G} Proj(k\!\uparrow_{{}^gH}^G) \\ &= \bigoplus_{g\in G} Proj(k\!\uparrow_H^G) = Proj(k\!\uparrow_H^G) \end{aligned}$$

by the omnibus properties for relative projectivity. In consequence, a family $\mathcal{H}$ of subgroups of a group $G$ may always be replaced by a larger family

$$\overline{\mathcal{H}} := \{\, {}^gH \mid g \in G, H \in \mathcal{H}\}$$

closed under conjugation, or, on the contrary, by a subfamily $\underline{\mathcal{H}}$ containing only one representative for the conjugacy classes of subgroups in $\mathcal{H}$, without altering the resulting relative projectivity:

$$Proj(\mathcal{H}) = Proj(\overline{\mathcal{H}}) = Proj(\underline{\mathcal{H}})$$

## 2.6. Operations on groups

We now establish some notation and basic facts concerning projectivity relative to modules with respect to standard operations on groups.

Lemma 2.6.1.
*Let $H$ be a subgroup of $G$ and $N$ be a normal subgroup of $G$ such that $p$ divides $|G/N|$.*

(a) ***Restriction**: Let $Z$ be a $V$-projective $kG$-module, then $Z\!\downarrow_H^G$ is a $V\!\downarrow_H^G$-projective $kH$-module. We shall use the following short notation:*

$$Proj(V)\!\downarrow_H^G \subseteq Proj(V\!\downarrow_H^G)$$

(b) ***Induction**: Let $Z$ be a $V$-projective $kH$-module, then $Z\!\uparrow_H^G$ is a $V\!\uparrow_H^G$-projective $kG$-module. We shall use the following short notation:*

$$Proj(V)\!\uparrow_H^G \subseteq Proj(V\!\uparrow_H^G)$$

(c) ***Tensor induction**: Let $Z$ be a $V$-projective $kH$-module, then $Z\!\uparrow_{\otimes H}^G$ is a $V\!\uparrow_{\otimes H}^G$-projective $kG$-module.*

(d) ***Inflation**: let $Z$ be a $V$-projective $k[G/N]$-module, then $\mathrm{Inf}_{G/N}^G(Z)$ is an $\mathrm{Inf}_{G/N}^G(V)$-projective $kG$-module. We shall use the following short notation:*

$$\mathrm{Inf}_{G/N}^G(Proj(V)) \subseteq Proj(\mathrm{Inf}_{G/N}^G(V))$$

(e) ***Isomorphism**: let $\varphi : G \longrightarrow \widetilde{G}$ be a group homomorphism and $Z$ be a $V$-projective $kG$-module. Then $V$ and $Z$ can be viewed as $k\widetilde{G}$-modules via $\varphi^{-1}$, denoted $\mathrm{Iso}(\varphi)(V)$ and $\mathrm{Iso}(\varphi)(Z)$. Then $\mathrm{Iso}(\varphi)(Z)$ is an $\mathrm{Iso}(\varphi)(V)$-projective $k\widetilde{G}$-module.*

PROOF.

(a) $Z$ is $V$-projective if and only if $Z \mid V \otimes L$ for some $kG$-module $L$, thus

$$Z \downarrow_H^G \mid (V \otimes L) \downarrow_H^G \cong V \downarrow_H^G \otimes L \downarrow_H^G,$$

i.e. $Z$ is $V \downarrow_H^G$-projective.

(b) $Z$ is $V$-projective if and only if $Z \mid V \otimes L$ for some $kH$-module $L$, hence

$$Z \uparrow_H^G \mid (V \otimes L) \uparrow_H^G \mid V \uparrow_H^G \otimes L \uparrow_H^G$$

since $V \uparrow_H^G \otimes L \uparrow_H^G \cong \bigoplus_{[HgH]} [{}^g V \downarrow^H_{{}^g H \cap H} \otimes L \downarrow^H_{{}^g H \cap H}] \uparrow^G_{{}^g H \cap H}$. Thus $Z \uparrow_H^G$ is $V \uparrow_H^G$-projective.

(c) $Z$ is $V$-projective if and only if $Z \mid V \otimes L$ for some $kH$-module $L$. Write $V \otimes L \cong Z \oplus Z'$, then

$$(V \otimes L)\uparrow^G_{\otimes H} \cong Z\uparrow^G_{\otimes H} \oplus Z'\uparrow^G_{\otimes H} \oplus S$$

where $S$ is a $kG$-module that does not need to be described for the purpose of this argument. However, it follows that

$$Z\uparrow^G_{\otimes H} \mid (V \otimes L)\uparrow^G_{\otimes H} \cong V\uparrow^G_{\otimes H} \otimes L\uparrow^G_{\otimes H}$$

i.e. $Z\uparrow^G_{\otimes H}$ is a $V\uparrow^G_{\otimes H}$-projective $kG$-module.

(d) $Z$ is $V$-projective if and only if $Z \mid V \otimes L$ for some $k[G/N]$-module $L$. Then

$$Inf^G_{G/N}(Z) \mid Inf^G_{G/N}(V \otimes L) \cong Inf^G_{G/N}(V) \otimes Inf^G_{G/N}(L),$$

i.e. $Inf^G_{G/N}(Z)$ is an $Inf^G_{G/N}(V)$-projective $kG$-module.

(e) is clear enough.

□

REMARK 2.6.2.

As we shall use restriction extensively, note that the reverse inclusion for (a) does not hold in general. For instance, if $G = C_3 \times C_3$, let $h$ be one of its generators, $H := < h >$ and $V := k\uparrow_H^{C_3 \times C_3}$, then $V\downarrow_H^G \cong k \oplus k \oplus k$. It follows that

$$Proj(V\downarrow_H^G) = Proj(k^{\oplus 3}) = Proj(k) = \mathsf{mod}(kH)\,,$$

whereas using Green's indecomposability theorem it is easy to compute that

$$Proj(V)\downarrow_H^G = \{M \in \mathsf{mod}(kH) \mid M \cong a_1 k \oplus a_2\Omega(k) \oplus a_3 kH, a_1, a_2, a_3 \in 3\mathbb{Z}\}\,,$$

where $\Omega(k)$ denotes the kernel of a projective cover of the trivial $kH$-module.

Next we focus on the behaviour of relatively projective modules with respect to restrictions and inductions.

LEMMA 2.6.3.

*Let $G$ be a finite group and $X \leq H \leq G$ be subgroups. Let $U, V$ be $kG$-modules and $W, Z$ be $kX$-modules. The following inclusions and equalities hold:*

(a) $Proj(V\downarrow_H^G)\downarrow_X^H = Proj(V\downarrow_X^G)$;

(b) $Proj(W\uparrow_X^H)\uparrow_H^G = Proj(W\uparrow_X^G)$;

(c) $Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V\downarrow_H^G\uparrow_H^G) \subseteq Proj(V)$. *If, moreover, $V$ is $H$-projective, then* $Proj(V\downarrow_H^G\uparrow_H^G) = Proj(V)$;

(d) *If* $Proj(V) \subseteq Proj(U)$*, then* $Proj(V\downarrow_H^G) \subseteq Proj(U\downarrow_H^G)$ *and if* $Proj(V) = Proj(U)$*, then* $Proj(V\downarrow_H^G) = Proj(U\downarrow_H^G)$*.*

(e) *If* $Proj(W) \subseteq Proj(Z)$*, then* $Proj(W\uparrow_X^G) \subseteq Proj(Z\uparrow_X^G)$ *and if* $Proj(W) = Proj(Z)$*, then* $Proj(W\uparrow_X^G) = Proj(Z\uparrow_X^G)$*.*

Proof.

(a)/(b) In both cases the inclusion $\subseteq$ was stated in Lemma 2.6.1. The reverse inclusion is a straightforward consequence of the transitivity of restrictions and inductions. E.g. $V\downarrow_X^G = (V\downarrow_H^G)\downarrow_X^H$ so that $V\downarrow_X^G \in Proj(V\downarrow_H^G)\downarrow_X^H$ and by the omnibus properties of relative projectivity $Proj(V\downarrow_X^G) \subseteq Proj(V\downarrow_H^G)\downarrow_X^H$. A similar argument can be carried through for induction.

(c) The inclusion $Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V\downarrow_H^G\uparrow_H^G)$ is a special case of Lemma 2.6.1, part (a). In addition, Frobenius reciprocity yields $V\downarrow_H^G\uparrow_H^G \cong V \otimes k\uparrow_H^G$, thus by 2.2.2.(f),

$$Proj(V\downarrow_H^G\uparrow_H^G) = Proj(V) \cap Proj(k\uparrow_H^G) \subseteq Proj(V)\,.$$

Moreover, if $V$ is $H$-projective, then $Proj(V) \subseteq Proj(k\uparrow_H^G)$ by 2.2.2.(b) again. Consequently,

$$Proj(V\downarrow_H^G\uparrow_H^G) = Proj(V) \cap Proj(k\uparrow_H^G) = Proj(V)\,.$$

(d)/(e) If $Proj(V) \subseteq Proj(U)$, then, in particular, $V \in Proj(U)$ so that $V\downarrow_H^G \in Proj(U\downarrow_H^G)$ by 2.6.1.(a), hence $Proj(V\downarrow_H^G) \subseteq Proj(U\downarrow_H^G)$ by 2.2.2. Swap the roles of $V$ and $U$ for the reverse inclusion. Property (e) is obtained likewise.

□

The following lemma partly restates (a) and (b) of the two preceding ones, respectively, but focuses on a particular module rather than on a whole subcategory of relatively projective modules.

Lemma 2.6.4.
*Let* $G$ *be a finite group and* $H$ *be a subgroup of* $G$*. Let* $M$ *be an* $H$*-projective module. Then, the following conditions are equivalent:*

(a) $M$ *is* $V$*-projective;*

(b) $M\downarrow_H^G$ *is* $V\downarrow_H^G$*-projective;*

(c) $M\downarrow_H^G\uparrow_H^G$ *is* $V$*-projective.*

Proof.

(a)⇒ (b): is given by Lemma 2.6.1 (a).

(b)⇒ (c): Again by 2.6.1, $M\downarrow_H^G \in Proj(V\downarrow_H^G)$ implies that $M\downarrow_H^G\uparrow_H^G \in Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V)$.

(c)⇒ (a): By $H$-projectivity and by Lemma 2.6.3, $M \mid M\downarrow_H^G\uparrow_H^G \in Proj(V)$, therefore $M \in Proj(V)$.

□

As a consequence, one sees that the set of $V$-projective modules is actually determined by restriction to a Sylow $p$-subgroup, in the sense that two different $kG$-modules generate the same set of relative projectivity if and only if their restriction to Sylow $p$-subgroup generate the same set of relative projectivity:

Corollary 2.6.5.
*Let* $G$ *be a finite group and* $P$ *a Sylow* $p$*-subgroup of* $G$*. Let* $V$ *and* $W$ *be two* $kG$*-modules. Then:*

(a) $Proj(V) \subseteq Proj(W)$ *if and only if* $Proj(V\downarrow_P^G) \subseteq Proj(W\downarrow_P^G)$*;*

(b) $Proj(V) = Proj(W)$ *if and only if* $Proj(V\downarrow_P^G) = Proj(W\downarrow_P^G)$*.*

PROOF. In both cases the necessary condition was established in 2.6.3. For the sufficient condition, assume that $Proj(V\downarrow_P^G) = Proj(W\downarrow_P^G)$. Applying 2.6.4 twice yields the following equivalences: $M \in Proj(V)$ if and only if $M\downarrow_P^G \in Proj(V\downarrow_P^G) = Proj(W\downarrow_P^G)$ if and only if $M \in Proj(W)$. In other words, $Proj(V) = Proj(W)$. This proves (b). The same argument can be carried through with an inclusion instead of an equality, it proves (a). $\square$

Notice that inflation has similar properties:

LEMMA 2.6.6.
*Let $N$ be a normal subgroup of the group $G$ such that $p \mid |G/N|$. Let $V, W$ be $k[G/N]$-modules. Then:*

(a) *$Proj(V) \subseteq Proj(W)$ if and only if $Proj(\mathrm{Inf}_{G/N}^G(V)) \subseteq Proj(\mathrm{Inf}_{G/N}^G(W))$;*
(b) *$Proj(V) = Proj(W)$ if and only if $Proj(\mathrm{Inf}_{G/N}^G(V)) = Proj(\mathrm{Inf}_{G/N}^G(W))$.*

PROOF. We have the equivalences:

$$\begin{aligned} Proj(V) \subseteq Proj(W) &\iff V \text{ is } W\text{-projective} \\ &\iff V \mid V \otimes W^* \otimes W \text{ by Lemma 2.2.1} \\ &\iff \mathrm{Inf}_{G/N}^G(V) \mid \mathrm{Inf}_{G/N}^G(V) \otimes \mathrm{Inf}_{G/N}^G(W)^* \otimes \mathrm{Inf}_{G/N}^G(W) \\ &\iff \mathrm{Inf}_{G/N}^G(V) \text{ is } \mathrm{Inf}_{G/N}^G(W)\text{-projective} \\ &\iff Proj(\mathrm{Inf}_{G/N}^G(V)) \subseteq Proj(\mathrm{Inf}_{G/N}^G(W)) \text{ by Proposition 2.2.2.} \end{aligned}$$

The reverse inclusion is obtained by exchanging the roles of $V$ and $W$. $\square$

LEMMA 2.6.7.
*Let $G$ be a group and $p$ a prime that divides $|G|$. Assume $G$ is a $p$-group or possesses a normal subgroup $H$ such that the factor group $G/H$ is a $p$-group. Then, the indecomposable modules projective relatively to the subgroup $H$ are described as follows:*

$$IProj(k\uparrow_H^G) = \{M\uparrow_H^G \mid M \text{ is an indecomposable } kH\text{-module}\}$$

PROOF. Let $N$ be a $kG$-module, then,

$$k\uparrow_H^G \otimes N \cong N\downarrow_H^G\uparrow_H^G \cong \bigoplus_{I \text{ finite}} N_i\uparrow_H^G$$

with $N\downarrow_H^G \cong \bigoplus_I N_i$ a decomposition of $N\downarrow_H^G$ into indecomposable modules. By assumption $G$ is such that Green's indecomposability criterion applies, therefore $N_i\uparrow_H^G$ is indecomposable for all $i \in I$. Now, by definition, an indecomposable $kG$-module is projective relatively to $H$ if it is a direct summand of $k\uparrow_H^G \otimes N$ for some $kG$-module $N$, therefore the above and the Krull-Schmidt theorem yield the inclusion

$$IProj(k\uparrow_H^G) \subseteq \{M\uparrow_H^G \mid M \text{ is an indecomposable } kH\text{-module}\}.$$

On the other hand, recall that a $kG$-module is projective relatively to $H$ if and only if it is a direct summand of a module induced from $H$. In particular, $M\uparrow_H^G \in Proj(k\uparrow_H^G)$ for all indecomposable $kH$-module $M$. Now by assumption Green's indecomposability criterion applies so that $M\uparrow_H^G$ is indecomposable for all indecomposable $kH$-module $M$. Hence the reverse inclusion. $\square$

REMARK 2.6.8.
In particular, in case $H$ is a cyclic $p$-group of order $p^n$, then the set $IProj(k\uparrow_H^G)$ of indecomposable

$kG$-modules projective relatively to the subgroup $H$ is finite of order $p^n$:

$IProj(k\uparrow_H^G) = \{M_i\uparrow_H^G \mid M_i$ is the unique indecomposable $kH$-module of dimension $i$, $\forall 1 \leq i \leq p^n\}$

## 2.7. Vertices, sources and Green correspondence

We establish here links between the notion of $V$-projectivity, vertices, sources and the Green correspondence.

Lemma 2.7.1.
*Let $M$ be an indecomposable $kG$-module and $(D, S)$ a vertex-source pair for $M$. Moreover, let $V \in \mathsf{mod}(kG)$ and $W \in \mathsf{mod}(kD)$.*

(a) *If $S \in Proj(W)$, then $M \in Proj(W\uparrow_D^G)$.*
(b) *If $M \in Proj(V)$, then $S \in Proj(V\downarrow_D^G)$.*

Proof.
(a) It follows from the definitions of vertices and sources of a module that if $S \in Proj(W)$, then $M \mid S\uparrow_D^G$. Moreover, $S\uparrow_D^G \in Proj(W\uparrow_D^G)$ by Lemma 2.6.1, hence $M \in Proj(W\uparrow_D^G)$ by 2.2.2.(a).
(b) Likewise, if $M \in Proj(V)$, then $S \mid M\downarrow_D^G \in Proj(V\downarrow_D^G)$. Hence the result.

□

Lemma 2.7.2.
*Let $(G, H, Q)$ be an admissible triple for the Green correspondence.*

(a) *Let $U$ be an indecomposable $kG$-module with vertex $Q$ and $Gr(U)$ be its $kH$-Green correspondent. Let $V \in \mathsf{mod}(kG)$.*
*Then $U \in Proj(V)$ if and only if $Gr(U) \in Proj(V\downarrow_H^G)$.*
(b) *Let $T$ be an indecomposable $kH$-module with vertex $Q$ and $\Gamma(T)$ be its $kG$-Green correspondent. Let $W \in \mathsf{mod}(kH)$. Then the following holds:*
*if $T \in Proj(W)$ then $\Gamma(T) \in Proj(W\uparrow_H^G)$ and if $\Gamma(T) \in Proj(W\uparrow_H^G)$, then $T \in Proj(W\uparrow_H^G\downarrow_H^G)$.*

Proof.
(a) If $U \in Proj(V)$, then by 2.6.1 $U\downarrow_H^G \in Proj(V\downarrow_H^G)$, therefore so does $Gr(U)$ as a direct summand of $U\downarrow_H^G$. Conversely, if $Gr(U) \in Proj(V\downarrow_H^G)$, then by 2.6.3,

$$Gr(U)\uparrow_H^G \in Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V)\,.$$

Hence $U \in Proj(V)$, as a direct summand of $Gr(U)\uparrow_H^G$.
(b) In like manner, if $T \in Proj(W)$, then by 2.6.1 $T\uparrow_H^G \in Proj(W\uparrow_H^G)$, therefore so does $\Gamma(T)$ since $\Gamma(T) \mid T\uparrow_H^G$. Now, if $\Gamma(T) \in Proj(W\uparrow_H^G)$, then

$$\Gamma(T)\downarrow_H^G \in Proj(W\uparrow_H^G)\downarrow_H^G \subseteq Proj(W\uparrow_H^G\downarrow_H^G)$$

and so does $T$ as a direct summand of $\Gamma(T)$.

□

## 2.8. Absolute $p$-divisibility

Many arguments shall use the next result by D. Benson and J. Carlson [BC86, Thm. 2.1], which we shall often refer to as the *Benson-Carlson theorem*:

THEOREM 2.8.1.
*Let $k$ be an algebraically closed field of characteristic $p$ (possibly $p = 0$). Let $M, N$ be finite-dimensional indecomposable $kG$-modules, then*

$$k \mid M \otimes N \text{ if and only if } \begin{cases} (1)\ M \cong N^*\,; \\ (2)\ p \nmid \dim_k(N). \end{cases}$$

*Moreover, if $k$ is a direct summand of $N^* \otimes N$ then it has multiplicity one, i.e. $k \oplus k$ is not a summand.*

REMARK 2.8.2.
In general, if $M$ and $N$ are finite-dimensional decomposable modules, write $M \cong \bigoplus_{i\in I} M_i$ and $N \cong \bigoplus_{j\in J} M_j$ as direct sums of indecomposable modules, then,

$$k \mid M \otimes N \text{ if and only if } \exists\, i \in I, j \in J \text{ such that } M_i \cong N_j^* \text{ and } p \nmid \dim_k(N_j).$$

In particular, if $p$ divides the $k$-dimension of all direct summands of $N$ then $k$ is not a summand of $N^* \otimes N = \mathrm{End}_k(N)$.

Moreover it is worth keeping in mind that the implication ($p \nmid \dim_k(N) \Rightarrow k \mid N^* \otimes N$) is always true, that is even if $N$ is decomposable, since in this case the trace map splits. Futhermore, the theorem enables us to characterize those $kG$-modules $V$ relatively to which the trivial module is projective, which shall be essential later on to define the promised-since-the-title groups of relative endotrivial modules.

PROPOSITION 2.8.3.
*Let $V \in \mathsf{mod}(kG)$. Then, the following are equivalent:*

(a) *The trivial $kG$-module $k$ is $V$-projective;*
(b) *$p = \mathrm{char}(k)$ does not divide the $k$-dimension of at least one of the indecomposable direct summands of $V$;*
(c) *the subcategory $Proj(V)$ is equal to the whole category of finite-dimensional $kG$-modules $\mathsf{mod}(kG)$.*

PROOF.

(a)⇒(b): By 2.4.1, $k \in Proj(V)$ if and only if $k \mid V^* \otimes V$. Thus, according to the remark above, $V$ has an indecomposable direct summand whose $k$-dimension is not divisible by $p$.

(b)⇒(c): Since $V$ is finitely generated, write $V = \bigoplus_{j\in J} V_j$ as a direct sum of indecomposable modules. Then by 2.2.2,

$$Proj(V) = \bigoplus_{j\in J} Proj(V_j)\,.$$

By assumption, there exists $j_0 \in J$ such that $p$ does not divide $\dim_k(V_{j_0})$ so that by Proposition 2.2.2, $Proj(V_{j_0}) = \mathsf{mod}(kG)$. Therefore $Proj(V) = \mathsf{mod}(kG)$ as well.

(c)⇒(a): is trivial.

□

In other words, the proposition shows that projectivity relative to a module $V$ is interesting essentially if the $k$-dimensions of all the indecomposable direct summands of $V$ are divisible by $p = \operatorname{char} k$, that is when $Proj(V)$ is not equal to the whole category of finite-dimensional $kG$-modules $\mathsf{mod}(kG)$. To use the terminology introduced in [BC86], in the sequel, such a $kG$-module $V$ shall be called *absolutely $p$-divisible*.

As another consequence of Theorem 2.8.1 we can rephrase [Ben98a, Prop. 5.8.1] to get the following characterisation for dimensions of $V$-projective $kG$-modules.

LEMMA 2.8.4.
*Let $V$ be an absolutely $p$-divisible $kG$-module and $U \in Proj(V)$. Then $p$ divides $\dim_k U$.*

PROOF. Assume that $p \nmid \dim_k(U)$, then by Proposition 2.2.2 the trace map $Tr_U : U^* \otimes U \longrightarrow k$ splits. Whence $k \,|\, U^* \otimes U$. But $U$ is $V$-projective, thus by definition there exists a $kG$-module $N$ such that $U \,|\, V \otimes N$. Therefore

$$k \,|\, U^* \otimes U \,|\, U^* \otimes N \otimes V$$

so that $k \in Proj(V)$ and by Proposition 2.8.3, $V$ is not absolutely $p$-divisible. □

LEMMA 2.8.5.
*Let $V, M, N \in \mathsf{mod}(kG)$ such that $M$ is not absolutely $p$-divisible.Then $N \in Proj(V)$ if and only if $M \otimes N \in Proj(V)$.*

PROOF. Since $M$ is not absolutely $p$-divisible, $Proj(M) = \mathsf{mod}(kG)$ by Proposition 2.8.3 above. Therefore, by the omnibus properties of $V$-projectivity:

$$Proj(M \otimes N) = Proj(M) \cap Proj(N) = \mathsf{mod}(kG) \cap Proj(N) = Proj(N)$$

Thus $Proj(M \otimes N) \subseteq Proj(V)$ if and only if $Proj(N) \subseteq Proj(V)$ and in consequence $N \in Proj(V)$ if and only if $M \otimes N \in Proj(V)$. □

## 2.9. Absolute $p$-divisibility and operations on groups

**Restriction.** The aim of this section is mainly to describe the behaviour of absolute $p$-divisibility with respect to restrictions. It shall turn out to be a key argument for the forthcoming study of relative endotrivial modules.

LEMMA 2.9.1.
*Let $V$ be a $kG$-module whose restriction $V\downarrow^G_H$ to some subgroup $H \leq G$ is absolutely $p$-divisible, then $V$ is absolutely $p$-divisible itself.*

PROOF. Let $V_i$ be an indecomposable summand of $V$, then $V_i\downarrow^G_H$ is absolutely $p$-divisible by assumption. Thus its $k$-dimension is divisible by $p$, since the $k$-dimensions of all its direct summands are, and consequently so is $dim_k(V_i) = dim_k(V_i\downarrow^G_H)$, as required. □

As shows the following counterexample, the converse statement to the lemma is not true in general.

COUNTEREXAMPLE 2.9.2.
Let $G := C_2 \times C_2$ be the Klein Group. Then the module $k\uparrow^{C_2\times C_2}_{C_2\times 1}$ is indecomposable and its

$k$-dimension is 2, therefore it is absolutely 2-divisible, but $k\uparrow_{C_2\times 1}^{C_2\times C_2}\downarrow_{C_2\times 1}^{C_2\times C_2}\cong k\oplus k$ which is certainly not absolutely 2-divisible.

Notwithstanding, it is always true that a restriction to a subgroup containing a Sylow $p$-subgroup preserves absolute $p$-divisibility. Besides, depending on the vertices of the module considered, it is even possible to restrict to $p$-subgroups and preserve absolute $p$-divisibility. Precisely, the result is stated as follows.

THEOREM 2.9.3.
*Let $P$ be a Sylow $p$-subgroup of $G$. Let $V$ be a $kG$-module with vertex $Q \leq P$.*

(a) *Then for every subgroup $H \geq P$, the module $V$ is absolutely $p$-divisible if and only if $V\downarrow_H^G$ is absolutely $p$-divisible.*
(b) *Furthermore, if $Q \lneq P$, then for every subgroup $R$ of $P$ such that $P \geq R \gneq Q$, the module $V$ is absolutely $p$-divisible if and only if $V\downarrow_R^G$ is absolutely $p$-divisible.*

PROOF. In both cases the sufficient condition is given by lemma 2.9.1 and we are left with the necessary condition to prove.

(a) Let $P \leq H \leq G$ be a subgroup and assume that $V\downarrow_H^G$ is not absolutely $p$-divisible. Then, by Proposition 2.8.3, $Proj(V\downarrow_H^G) = \mathsf{mod}(kH)$. Besides Lemma 2.6.3 yields:

$$\mathsf{mod}(kH)\uparrow_H^G = Proj(V\downarrow_H^G)\uparrow_H^G \subseteq Proj(V\downarrow_H^G\uparrow_H^G) = Proj(V)$$

We deduce that, in particualr, $k\uparrow_H^G \in Proj(V)$. Finally since $p \nmid \dim_k(k\uparrow_H^G) = |G : H|$, it follows from Lemma 2.8.4 that $V$ is not absolutely $p$-divisible.

(b) Let $R$ be a subgroup of $P$. By assumption, $V \in Proj(k\uparrow_Q^G)$, so that $V\downarrow_R^G \in Proj(k\uparrow_Q^G\downarrow_R^G)$ and the Mackey formula yields:

$$k\uparrow_Q^G\downarrow_R^G \cong \bigoplus_{g\in[R\backslash G/Q]} ({}^g k)\downarrow^{{}^gQ}_{{}^gQ\cap R}\uparrow^R_{{}^gQ\cap R} = \bigoplus_{g\in[R\backslash G/Q]} k\uparrow^R_{{}^gQ\cap R}$$

Therefore,

$$V\downarrow_R^G \in \bigoplus_{g\in[R\backslash G/Q]} Proj(k\uparrow^R_{{}^gQ\cap R})$$

and so do all its direct summands. Now, the assumption that $Q \lneq R$ implies that ${}^gQ \cap R \lneq R$ for every $g \in [R\backslash G/Q]$. Thus any direct summand of $V\downarrow_R^G$ has a vertex strictly smaller than $R$ and it is well-known that the $k$-dimension of such indecomposable modules is divisible by $p$ (see Lemma 3.5.1 in Chapter 3). Hence the result.

□

**Inflation.**
LEMMA 2.9.4.
*Let $G$ be a finite group with a normal subgroup $N$ such that $p \mid |G/N|$ and $V \in \mathsf{mod}(k[G/N])$ be an indecomposable module. Then $V$ is absolutely $p$-divisible if and only if $\mathrm{Inf}_{G/N}^G(V)$ is.*

PROOF. Since inflation does not alter dimensions and $\mathrm{Inf}_{G/N}^G(V)$ is indecomposable if and only if $V$ is, the result is straightforward. □

**Induction.** Absolute $p$-divisibility is not well-behaved with respect to induction. First, contrary to restriction, the fact the an induced module $M\uparrow_H^G$ is absolutely $p$-divisible does not imply that the initial module $M$ is itself absolutely $p$-divisible.

Counterexample 2.9.5.
Let $P$ be a $p$-group with $|P| \geq 2$ and $Q$ be a subgroup such that $1 \lneq Q \lneq P$. Then the permutation module $k\uparrow_Q^P$ is absolutely $p$-divisible. Indeed, its $k$-dimension is divisible by $p$ and moreover it is indecomposable since its socle clearly is. However, the trivial $kQ$-module is not absolutely $p$-divisible.

Nonetheless, we have the following criterion for an induced module to be absolutely $p$-divisible.

Lemma 2.9.6.
*Let $G$ be a finite group with a subgroup $H$ and $V \in \mathsf{mod}(kH)$ be an arbitrary module. If $G$ is a $p$-group, or else if $H$ is normal in $G$ and $G/H$ is a $p$-group, then $V\uparrow_H^G$ is absolutely $p$-divisible.*

Proof. Since induction and direct sums commute, we may as well assume that $V$ is indecomposable and the assumptions of the lemma imply that $V\uparrow_H^G$ is indecomposable as well by Green's indecomposability theorem. Furthermore, $p \mid \dim_k V\uparrow_H^G = |G:H| \dim_k V$ because $p \mid |G:H|$. Hence the result. □

## 2.10. Relative projectivity and module varieties.

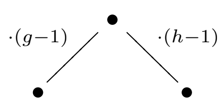
One can think of varieties as a way for grouping $kG$-modules into families of modules all with a given variety (or one contained in it). Now, given a fixed closed homogeneous subvariety $\mathcal{V}$ of $\mathcal{V}_G(k)$, if we pick a module $M$ affording $\mathcal{V}$, then the subcategory $Proj(M)$ is made up of modules all affording a variety contained in $\mathcal{V}$. In this respect relative projectivity to modules is a finer notion than that of variety.

Lemma 2.10.1.
(a) *If $Proj(M) \subseteq Proj(M')$, then $\mathcal{V}_G(M) \subseteq \mathcal{V}_G(M')$;*
(b) *If $Proj(M) = Proj(M')$, then $\mathcal{V}_G(M) = \mathcal{V}_G(M')$;*
(c) *The converse of statement (b) does not hold. In particular, $\mathcal{V}_G(M) = \mathcal{V}_G(k)$ does not imply that $Proj(M) = \mathsf{mod}(kG)$.*

Proof.
(a) In particular $M \in Proj(M')$, hence there exists two $kG$-modules $N$ and $U$ such that $M \oplus U \cong M' \otimes N$. Taking varieties yields:

$$\mathcal{V}_G(M) \subseteq \mathcal{V}_G(M) \cup \mathcal{V}_G(U) = \mathcal{V}_G(M') \cap \mathcal{V}_G(N) \subseteq \mathcal{V}_G(M').$$

(b) Applying (a) twice yields the double inclusion.
(c) (Author's favourite example.) For $G = C_3 \times C_3 =: <g> \times <h>$, $k$ of characteristic 3, the module $M$ defined by the following diagram




has a variety equal to $\mathcal{V}_G(k)$. Indeed, simply for dimensional reasons, it can never be free on restriction to a shifted subgroup. However, $Proj(M) \subsetneq \mathrm{mod}(kG)$. For $M$ is indecomposable and 3-dimensional, hence absolutely 3-divisible.

□

## 2.11. Relative homological algebra

We give here a review of the basic relative homological algebra linked to the projectivity relative to a $kG$-module $V$. All the results presented are generalisation of the non-relative case and their proofs are similar (see for example [HS71]). Most of the material here comes from [Oku91] or [Car96] where the results are stated but often not proven. For completeness we give sketches of the proofs.

Definition 2.11.1 ([Car96], Sect. 8).
Let $V \in \mathsf{mod}(kG)$ be a module.

(a) A *$V$-projective resolution* of a module $M \in \mathsf{mod}(kG)$ is a non-negative complex $\mathbf{P}_*$ of $V$-projective modules together with a surjective $kG$-homomorphism $P_0 \xrightarrow{\varepsilon} M$ such that the sequence
$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$
is exact and *totally $V$-split*, that is such that for all $i \geq 1$, the short exact sequences
$$0 \longrightarrow \ker(\partial_i) \longrightarrow P_i \xrightarrow{\partial_i} \mathrm{Im}(\partial_i) \longrightarrow 0\,,$$
$$0 \longrightarrow \ker(\varepsilon) \longrightarrow P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$
are $V$-split. The latter sequence is called a *$V$-projective presentation* of $M$.

(b) Similarly, there is a notion of *$V$-injective presentation* and a notion of *$V$-injective resolution* $M \xrightarrow{\imath} \mathbf{I}_*$.

Existence of $V$-projective resolutions.([Car96, Prop. 8.7])
Every module $M \in \mathsf{mod}(kG)$ has a $V$-projective resolution and a $V$-injective resolution.

Proof. Since the trace map $\mathrm{Tr}_V : V^* \otimes V \longrightarrow k$ is $V$-split, so is the tensored exact sequence
$$\ker(\mathrm{Tr}_V) \otimes M \hookrightarrow V^* \otimes V \otimes M \xrightarrow{\mathrm{Tr}_V \otimes M} M\,.$$
Iterate this construction to get, for every $n \geq 2$, $V$-split exact sequences:
$$\ker(\mathrm{Tr}_V)^{\otimes n} \otimes M \hookrightarrow V^* \otimes V \otimes (\ker(\mathrm{Tr}_V)^{\otimes(n-1)} \otimes M) \xrightarrow{\mathrm{Tr}_V \otimes \ker(\mathrm{Tr}_V)^{\otimes(n-1)} \otimes M} \ker(\mathrm{Tr}_V)^{\otimes(n-1)} \otimes M$$
Obtain a $V$-projective resolution of $M$ by splicing these sequences together:
$$\begin{array}{ccccccccc} \cdots & \longrightarrow & & V^* \otimes V \otimes \ker(\mathrm{Tr}_V) \otimes M & & \longrightarrow & & V^* \otimes V \otimes M & \twoheadrightarrow M \\ & \searrow & & \nearrow & \searrow & & \nearrow & & \\ & & \ker(\mathrm{Tr}_V)^{\otimes 2} \otimes M & & & \ker(\mathrm{Tr}_V) \otimes M & & & \end{array}$$
To obtain a $V$-injective resolution of $M$, start with a $V$-projective resolution of the dual $M^*$ and dualize! (I.e. apply the functor $\mathrm{Hom}_k(-,k)$.) $\square$

Lemma 2.11.2 ([Car96], Sect. 8).
*Let $M, N \in \mathsf{mod}(kG)$ and let $\boldsymbol{P}_* \xrightarrow{\varepsilon} M$ be a $V$-projective resolution of $M$. Then the tensored complex $\boldsymbol{P}_* \otimes N \xrightarrow{\varepsilon \otimes N} M \otimes N$ is a $V$-projective resolution of $M \otimes N$.*

Proof. Since $P_i$ is $V$-projective, so is $P_i \otimes N$ by 2.2.2, and since the complex $P_* \xrightarrow{\varepsilon} M$ is exact and totally $V$-split, so is the complex $P_* \otimes N \xrightarrow{\varepsilon \otimes N} M \otimes N$. $\square$

Lemma 2.11.3 ([Car96], Sect. 8).
*Let $M, N, U, V \in \mathsf{mod}(kG)$. Let $\boldsymbol{P}_* \overset{\varepsilon}{\twoheadrightarrow} M$ be a $U$-projective resolution of $M$ and let $\boldsymbol{Q}_* \overset{\eta}{\twoheadrightarrow} N$ be a $V$-projective resolution of $N$. Then $\boldsymbol{P}_* \otimes \boldsymbol{Q}_* \overset{\varepsilon\otimes\eta}{\twoheadrightarrow} M \otimes N$ is a $U \otimes V$-projective resolution of $M \otimes N$.*

Proof. By assumption $H_*(\mathbf{P}_*) = H_0(\mathbf{P}_*) \cong M$ and $H_*(\mathbf{Q}_*) = H_0(\mathbf{Q}_*) \cong N$, because $\mathbf{P}_*$ and $\mathbf{Q}_*$ are exact complexes in positive degrees. Therefore, by the Künneth formula, $H_*(\mathbf{P}_* \otimes \mathbf{Q}_*) = H_0(\mathbf{P}_* \otimes \mathbf{Q}_*) \cong H_0(\mathbf{P}_*) \otimes H_0(\mathbf{Q}_*) \cong M \otimes N$. Thus $\mathbf{P}_* \otimes \mathbf{Q}_* \overset{\varepsilon\otimes\eta}{\twoheadrightarrow} M \otimes N$ is a resolution of $M \otimes N$. Moreover, for every $n \geq 0$, $(\mathbf{P}_* \otimes \mathbf{Q}_*)_n = \bigoplus_{i+j=n} P_i \otimes Q_j \in Proj(U \otimes V)$ since $P_i \in Proj(U)$ and $Q_j \in Proj(V)$ for all $i, j \geq 0$. Finally to see that $\mathbf{P}_* \otimes \mathbf{Q}_*$ is totally $(U \otimes V)$-split, it suffices to notice that $(\mathbf{P}_* \otimes \mathbf{Q}_*) \otimes (U \otimes V) \cong (\mathbf{P}_* \otimes U) \otimes (\mathbf{Q}_* \otimes V)$ is totally split, because both $(\mathbf{P}_* \otimes U)$ and $(\mathbf{Q}_* \otimes V)$ are. □

Relative Schanuel's Lemma 2.11.4.
*Let $V, M \in \mathsf{mod}(kG)$.*

(a) *Let $P \overset{\varepsilon}{\twoheadrightarrow} M$ and $Q \overset{\theta}{\twoheadrightarrow} M$ be two $V$-projective presentations of $M$. Then there is an isomorphism of $kG$-modules $P \oplus \ker\theta \cong Q \oplus \ker\varepsilon$.*
(b) *Let $M \overset{\imath}{\longrightarrow} I$ and $M \overset{\jmath}{\longrightarrow} J$ be two $V$-injective presentations of $M$. Then there is an isomorphism of $kG$-modules $I \oplus \operatorname{Coker}\jmath \cong J \oplus \operatorname{Coker}\imath$.*

Proof. Form $B := \{(p;q) \in P \times Q \mid \varepsilon(p) = \theta(q)\}$ the pullback of the pair of maps $\varepsilon : P \twoheadrightarrow M$ and $\theta : Q \twoheadrightarrow M$, and let $\pi_P : B \longrightarrow P$ and $\pi_Q : B \longrightarrow Q$ be the canonical projections. Since $P, Q \in Proj(V)$ and $\varepsilon, \theta$ are $V$-split, by the universal property of $V$-projective modules there are $kG$-homomorphisms $\mu : Q \longrightarrow P$, $\nu : P \longrightarrow Q$ such that $\varepsilon\mu = \theta$ and $\theta\nu = \varepsilon$. Then the universal property of the pull-back applied twice yields $kG$-linear sections for $\pi_Q$ and $\pi_P$. Hence $P \oplus \ker\theta \cong B \cong Q \oplus \ker\varepsilon$. This proves (a), and (b) is proved in similar fashion using a pushout. □

Relative Lifting Theorem 2.11.5.
*Let $V \in \mathsf{mod}(kG)$. Let $\boldsymbol{P}_*$, $\boldsymbol{Q}_*$ be two non-negative complexes, such that $P_n \in Proj(V)$ for all $n \geq 0$ and $\boldsymbol{Q}_*$ is totally $V$-split and exact, except possibly in degree zero. Let $f : H_0(\boldsymbol{P}_*) \longrightarrow H_0(\boldsymbol{Q}_*)$ be a $kG$-linear map. Then there exists a chain map $\mu_* : \boldsymbol{P}_* \longrightarrow \boldsymbol{Q}_*$ inducing $f$ in degree zero. Furthermore $\mu_*$ is unique up to homotopy.*

Relative Comparison Theorem 2.11.6.
*Let $V \in \mathsf{mod}(kG)$. Let $\boldsymbol{P}_* \overset{\varepsilon}{\twoheadrightarrow} M$ and $\boldsymbol{Q}_* \overset{\eta}{\twoheadrightarrow} M$ be two $V$-projective resolutions of the module $M \in \mathsf{mod}(kG)$. Then $\boldsymbol{P}_*$ and $\boldsymbol{Q}_*$ are canonically homotopy equivalent, that is, there are chain maps $\mu_* : \boldsymbol{P}_* \longrightarrow \boldsymbol{Q}_*$, $\psi_* : \boldsymbol{Q}_* \longrightarrow \boldsymbol{P}_*$ lifting $\mathrm{id}_M$, unique up to homotopy, such that $\psi_*\mu_* \sim \mathrm{id}_{\boldsymbol{P}_*}$ and $\mu_*\psi_* \sim \mathrm{id}_{\boldsymbol{Q}_*}$. (Where $\sim$ means that the maps are homotopic chain maps.)*

These theorems can be dualized to obtain similar statements for $V$-injective modules and resolutions. The proofs are identical to those of the non-relative case.

Definition 2.11.7.
Let $V, M \in \mathsf{mod}(kG)$.

(a) A *$V$-projective cover* of $M$ is a *minimal* $V$-projective presentation $\varepsilon : P_M \twoheadrightarrow M$ satisfying the following property: if $\theta : Q \twoheadrightarrow M$ is another surjective $V$-split $kG$-homomorphism, then there exists an injective $kG$-homomorphism $\sigma : P_M \longrightarrow Q$ such

that $\varepsilon = \theta\sigma$, and likewise there is a surjective $kG$-homomorphism $\tau : Q \longrightarrow P_M$ such that $\varepsilon\tau = \theta$.
A *minimal $V$-projective resolution* of $M$ is a $V$-projective resolution $\mathbf{P}_* \overset{\varepsilon}{\twoheadrightarrow} M$ such that if $\mathbf{Q}_* \overset{\theta}{\twoheadrightarrow} M$ is another $V$-projective resolution of $M$, then there exists an injective chain map $\mu_* : (\mathbf{P}_* \overset{\varepsilon}{\twoheadrightarrow} M) \longrightarrow (\mathbf{Q}_* \overset{\theta}{\twoheadrightarrow} M)$ and likewise a surjective chain map $\mu'_* : (\mathbf{Q}_* \overset{\theta}{\twoheadrightarrow} M) \longrightarrow (\mathbf{P}_* \overset{\varepsilon}{\twoheadrightarrow} M)$, both lifting the identity on $M$.

(b) *$V$-injective hulls* and *minimal $V$-injective resolutions* are defined similarly.

Remark 2.11.8.
The relative Schanuel's Lemma shows that $V$-projective covers and minimal resolutions, and $V$-injective hulls and minimal resolutions, if they exist, are unique up to isomorphism.
Moreover, as in the non-relative case, one can also define a relative version of the notion of essential $kG$-homomorphism and use it to define $V$-projective covers.

Proposition 2.11.9 ([Car96], Sect. 8).
*Every module $M \in \mathsf{mod}(kG)$ has a minimal $V$-projective resolution and a minimal $V$-injective resolution.*

Proof. Choose $P_M \in \mathsf{mod}(kG)$ to be a $V$-projective module of smallest $k$-dimension such that there exists a surjective $kG$-homomorphism $\varepsilon : P_M \twoheadrightarrow M$ (in the worst case take $\mathrm{Tr}_V \otimes M : V^* \otimes V \otimes M \twoheadrightarrow M$). Then, as in the non-relative case, use Fitting's Lemma to prove that it is a $V$-projective cover of $M$. Likewise, build a $V$-projective cover of $\ker(\varepsilon)$. Iterate the process to get a minimal $V$-projective resolution. Notice that the dual of a minimal $V$-projective resolution of $M^*$ is a minimal $V$-injective resolution of $M$. □

A $V$-projective cover of the trivial module provides us with a canonical generator for $Proj(V)$:

Corollary 2.11.10 ([Car96], Sect. 8).
*Let $\varepsilon : V_k \longrightarrow k$ be a $V$-projective cover of the trivial module $k$. Then $Proj(V) = Proj(V_k)$.*

Proof. By definition $V_k \in Proj(V)$, so $Proj(V_k) \subseteq Proj(V)$ by 2.2.2. In addition the sequence $\ker(\varepsilon) \longrightarrow V_k \overset{\varepsilon}{\twoheadrightarrow} k$ is $V$-split so that $V \mid V_k \otimes V$, thus $Proj(V) \subseteq Proj(V_k)$. □

Remark 2.11.11.
A $V$-projective resolution $\mathbf{P}_* \overset{\varepsilon}{\twoheadrightarrow} M$ is minimal if and only if for all $n \geq 1$, $\mathrm{Im}(\partial_n)$ is *$V$-projective free*, i.e. it has no non-zero $V$-projective direct summands. This leads to the following definition of *relative syzygy modules.*

Definition 2.11.12 ([Car96], Sect. 8).
Let $V, M \in \mathsf{mod}(kG)$. Let $(\mathbf{P}_*, \partial_*) \overset{\varepsilon}{\twoheadrightarrow} M$ and $M \overset{\imath}{\hookrightarrow} (\mathbf{I}_*, \partial^*)$ be minimal $V$-projective and $V$-injective resolutions of $M$, respectively. Define for $n \geq 1$:

$$\Omega_V^n(M) := \ker \partial_{n-1} \text{ and } \Omega_V^{-n}(M) := \mathrm{Coker}(\partial^{n-1})$$

For $n = 0$, define $\Omega_V^0$ to be the $V$-projective free part of $M$. The modules $\Omega_V^n(M)$ are called the *relative syzygy modules* of $M$ and $\Omega_V^n$, the *relative Heller operators*. Moreover, if $V$ is projective (e.g. if $V = kG$), we drop the index $V$ and write $\Omega^n(M)$ instead of $\Omega_V^n(M)$, and if $n = 1$, then we drop the exponent and write $\Omega_V(M)$ instead of $\Omega_V^1(M)$.

Remark 2.11.13.
If $G =: P$ is a $p$-group and if $V = k{\uparrow}_Q^P$ for some subgroup $Q$ of $P$, then the notion of projectivity relative to the module $V$ coincides with the notion of projectivity relative to the subgroup $Q$, as we have already noticed, but it also coincides with the notion of projectivity relative to the $P$-set $X = P/Q$ described in [Bou00]. In consequence, for any $M \in \mathsf{mod}(kP)$, we have $\Omega_V(M) \cong \Omega_X(M)$ in the sense of [Bou00]. Also, if $Y$ is any non transitive finite $P$-set, it can be decomposed as a disjoint union of transitive $P$-sets $\bigsqcup_{i\in I} P/Q_i$ and projectivity relative to $Y$ coincides with projectivity relative to the family of subgroups $\{Q_i\}_{i\in I}$. In the sequel we shall juggle with these three visions in order to use the one that reveals to be the most adapted to the situation.

Because two minimal $V$-projective resolutions of the same module $M$ are isomorphic, the modules $\Omega_V^n(M)$ do not depend on the choice of the $V$-projective resolution (i.e. up to isomorphism). They do not depend either on the choice of the generator $V$ for $Proj(V)$:

Lemma 2.11.14.
*Let $V, W \in \mathsf{mod}(kG)$ such that $Proj(V) = Proj(W)$. Then $\Omega_V(M) \cong \Omega_W(M)$ for every module $M \in \mathsf{mod}(kG)$.*

Proof. Let $M \in \mathsf{mod}(kG)$ and $P_V \xrightarrow{\varepsilon} M$ be a $V$-projective cover of $M$. Then, on the one hand $P_V \in Proj(V) = Proj(W)$, and on the other hand, by 2.3.2, $\varepsilon$ is $V$-split if and only if it is $W$-split. As a consequence, $\Omega_V(M) = \ker(\varepsilon) = \Omega_W(M)$. □

## 2.12. Arithmetic of relative syzygies

Relative syzygy modules behave in much the same way as ordinary syzygy modules do.

Proposition 2.12.1 (**Omnibus properties**).
*Let $M, N, V \in \mathsf{mod}(kG)$ and let $m, n \in \mathbb{Z}$.*

(a) $\Omega_V^{-n}(M) \cong (\Omega_V^n(M^*))^*$.
(b) $\Omega_V^n(M)$ *is $V$-projective free.*
(c) *If $M \in Proj(V)$, then $\Omega_V^n(M) = 0$.*
(d) $\Omega_V^1(\Omega_V^{-1}(M)) \cong \Omega_V^0(M) \cong \Omega_V^{-1}(\Omega_V^1(M))$.
(e) *If the word* minimal *is dropped in Definition 2.11.12, then we get modules*

$$\widetilde{\Omega}_V^n(M) \cong \Omega_V^n(M) \oplus (V - proj)\,.$$

(f) $\Omega_V^n(M \oplus N) \cong \Omega_V^n(M) \oplus \Omega_V^n(N)$.
(g) $\Omega_V^n(\Omega_V^m(M)) \cong \Omega_V^{n+m}(M)$.
(h) *Suppose $M \notin Proj(V)$. If $M$ is indecomposable, then so is $\Omega_V^n(M)$.*
(i) $\Omega_V^m(M) \otimes N \cong \Omega_V^m(M \otimes N) \oplus (V - proj)$.
(j) $\Omega_V^m(M) \otimes \Omega_V^n(N) \cong \Omega_V^{m+n}(M \otimes N) \oplus (V - proj)$.
(k) *If $\Omega_V^n(M) = 0$, then $M \in Proj(V)$.*

Most of these properties can be found in [Car96, Sect. 8] or are more general versions of [Car96, Prop. 4.4], in which case the proofs are similar and obtained by replacing projectivity with relative projectivity.

Proof. (Sketches)

These properties hold for $n = 0$, almost trivially. Moreover, once (a) is proven, then it is enough to prove the other formulae for $n \geq 1$. Dualizing yields the result for $n \leq 1$.

(a) is straightforward from the fact that the dual of a minimal $V$-projective resolution of $M$ is a minimal $V$-injective resolution of $M^*$ and conversely.

(b) follows from the minimality of the $V$-projective resolution in Definition 2.11.12.

(c) A minimal $V$-projective resolution of $M$ is given by $(\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id}} M)$. Hence $\Omega^n_V(M) = 0$ for all $n \in \mathbb{Z}$.

(d) Let $\Omega^1_V(\Omega^{-1}_V(M)) \hookrightarrow P \twoheadrightarrow \Omega^{-1}_V(M)$ be a $V$-projective cover and let $M \hookrightarrow I \twoheadrightarrow \Omega^{-1}_V(M)$ be a $V$-injective hull. The latter is also a $V$-projective cover of $\Omega^{-1}_V(M)$, therefore the relative Schanuel's Lemma yields $I \oplus \Omega^1_V(\Omega^{-1}_V(M)) \cong P \oplus M$, where $I, P \in Proj(V)$ and $\Omega^1_V(\Omega^{-1}_V(M)$ is $V$-projective free by (b). Thus the Krull-Schmidt Theorem forces $\Omega^1_V(\Omega^{-1}_V(M) \cong \Omega^0_V(M)$. A similar arguments yields the second isomorphism.

(e) follows from the relative Schanuel's Lemma and part (b).

(f) follows from the fact that $V$-projective covers are additive.

(g) follows by truncating a minimal $V$-projective resolution of $M$ to a minimal $V$-projective resolution of $\Omega^m_V(M)$.

(h) If $\Omega^n_V(M)$ were decomposable, then so would be $M$ by (f).

(i) If $\mathbf{P}_* \overset{\varepsilon}{\twoheadrightarrow} M$ is a minimal $V$-projective resolution of $M$, then by Lemma 2.11.2, $\mathbf{P}_* \otimes N \overset{\varepsilon\otimes N}{\twoheadrightarrow} M \otimes N$ is a $V$-projective resolution of $M \otimes N$, but it is not necessarily minimal. Then by (e):

$$\Omega^m_V(M) \otimes N \cong \mathrm{Im}(\partial_m \otimes N) = \widetilde{\Omega}^m_V(M \otimes N) \cong \Omega^m_V(M \otimes N) \otimes (V - proj)$$

(j) Using (i) twice, compute:

$$\begin{aligned}\Omega^m_V(M) \otimes \Omega^n_V(N) &\cong \Omega^m_V(M \otimes \Omega^n_V(N)) \oplus (V - proj)\\ &\cong \Omega^m_V(\Omega^n_V(M \otimes N) \oplus (V - proj)) \oplus (V - proj)\\ &\cong \Omega^m_V(\Omega^n_V(M \otimes N)) \oplus \Omega^m_V(V - proj) \oplus (V - proj)\\ &\cong \Omega^{m+n}_V(M \otimes N) \oplus (V - proj)\end{aligned}$$

The last equality follows from (c) because $\Omega^m_V(V - proj) = 0$.

(k) If $\Omega^n_V(M) = 0$, then $\Omega^0_V(M) \cong \Omega^{-n}_V(\Omega^n_V(M)) = 0$, that is the $V$-projective free part of $M$ is zero.

□

Lemma 2.12.2.

(a) *Let $H$ be a subgroup of $G$ and $M, V$ be $kG$-modules, then:*

$$\Omega_V(M)\downarrow^G_H \cong \Omega_{V\downarrow^G_H}(M\downarrow^G_H) \oplus (V\downarrow^G_H -proj)$$

(b) *Let $N$ be a normal subgroup of $G$ and $M$ be a $k[G/N]$-module, then :*

$$\Omega_{k\uparrow^G_N}(\mathrm{Inf}^G_{G/N}(M)) \cong \mathrm{Inf}^G_{G/N}(\Omega(M))$$

Proof.

(a) The restriction of a minimal $V$-projective resolution is a $V\downarrow^G_H$-projective resolution of $M\downarrow^G_H$, it is not necessarily minimal though. Thus the formula follows from the relative Schanuel's lemma.

(b) This formula is a version for projectivity relative to modules of a formula given in [Bou00, Cor. 4.1.2] for relative syzygies of $P$-sets, with $P$ a $p$-group. The proof is identical.

□

**Relative syzygies, vertices, sources and Green correspondence.**

LEMMA 2.12.3 ([Oku91], Cor 9.9).
*Let $V, W \in \mathsf{mod}(kG)$.*

(a) *Let $M \in Proj(W)$. Then $\Omega_V^n(M) \in Proj(W)$ for all $n \in \mathbb{Z}$.*
(b) *Let $M$ be an indecomposable non-$V$-projective $kG$-module. Then, for all $n \in \mathbb{Z}$, $M$ and $\Omega_V^n(M)$ have the same vertices.*

PROOF. Let $n \in \mathbb{Z}$.

(a) If $M \in Proj(W)$, then $M \mid W \otimes N$ for some module $N \in \mathsf{mod}(kG)$. Moreover, by Lemma 2.12.1 part (i), $\Omega_V^n(M) \mid \Omega_V^n(k) \otimes M$. Thus
$$\Omega_V^n(M) \mid \Omega_V^n(k) \otimes M \mid \Omega_V^n(k) \otimes W \otimes N$$
and so $\Omega_V^n(M) \in Proj(W)$.
(b) If $D$ is a vertex for $M$, then $M$ is $k{\uparrow}_D^G$-projective and by part (a) so is $\Omega_V^n(M)$. Moreover $D$ is also a vertex for $\Omega_V^n(M)$, otherwise there would exist a subgroup $Q \lneq_G D$ such that $\Omega_V^n(M)$ is $k{\uparrow}_D^G$-projective and therefore so would be $M \cong \Omega_V^{-n}(\Omega_V^n(M))$ (by 2.12.1), contradicting the minimality of $D$.

□

From this result, one easily concludes that relative Heller operators commute with the Green correspondence.

COROLLARY 2.12.4.
*Let $V$ be a $kG$-module.*

(a) *Let $(G, H; Q)$ be an admissible triple for the Green correspondence. Let $U$ be a non $V$-projective indecomposable $kG$-module with vertex $Q$. If $T$ is the $kH$-Green correspondent of $U$, then $\Omega_{V\downarrow_H^G}(T)$ is the $kH$-Green correspondent of $\Omega_V(U)$.*
(b) *Let $M$ be an indecomposable non-$V$-projective $kG$-module and $(D, S)$ a vertex-source pair for $M$. Then $\Omega_{V\downarrow_D^G}(S)$ is a source for $\Omega_V(M)$.*

PROOF.

(a) First, the assumption that $U$ is non $V$-projective ensures that neither $\Omega_V(U)$, nor $\Omega_{V\downarrow_H^G}(T)$ is zero. Indeed, by 2.7.2 $U \notin Proj(V)$ if and only if $Gr(U) \notin Proj(V\downarrow_H^G)$. Then, by assumption, both the modules $U$ and $T$ have vertex $Q$, thus, by the lemma, so do the modules $\Omega_V(U)$ and $\Omega_{V\downarrow_H^G}(T)$. Therefore, it suffices to prove that $\Omega_{V\downarrow_H^G}(T)$ is a direct summand of $\Omega_V(U)\downarrow_H^G$. Indeed, as seen before $\Omega_{V\downarrow_H^G}(U\downarrow_H^G) \mid \Omega_V(U)\downarrow_H^G$. In addition, by the Green correspondence, $T \mid U\downarrow_H^G$, so that, by the properties of relative syzygies, $\Omega_{V\downarrow_H^G}(T) \mid \Omega_{V\downarrow_H^G}(U\downarrow_H^G)$.
(b) Let $\Omega_{V\downarrow_D^G}(S) \hookrightarrow P_{V\downarrow_D^G}(S) \twoheadrightarrow S$ be a minimal $V\downarrow_D^G$-projective presentation of $S$. Then $\Omega_{V\downarrow_D^G}(S){\uparrow}_D^G \hookrightarrow P_{V\downarrow_D^G}(S){\uparrow}_D^G \twoheadrightarrow S{\uparrow}_D^G$ is a $V$-projective presentation of $S{\uparrow}_D^G$, but it is not necessarily minimal though. Nonetheless, the relative version of Shanuel's lemma yields:
$$\Omega_{V\downarrow_D^G}(S){\uparrow}_D^G \cong \Omega_V(S{\uparrow}_D^G) \oplus (V - proj)\,.$$
By assumption, $S$ is a source of $M$, thus $M$ is a direct summand of $S{\uparrow}_D^G$ and so $\Omega_V(M)$ is a direct summand of $\Omega_V(S{\uparrow}_D^G)$, which is, as seen above, in turn a direct summand of $\Omega_{V\downarrow_D^G}(S){\uparrow}_D^G$. Furthermore, according to the previous lemma, $M$ and $\Omega_V(M)$ have a common vertex. It follows that $\Omega_{V\downarrow_D^G}(S)$ is a source for $\Omega_V(M)$.

□

**The composition formula.** Finally we focus on compositions of relative Heller operators, taken relatively to different modules.

Proposition 2.12.5 ([Oku91], Thm. 9.10).
*Let $V, W \in \mathsf{mod}(kG)$. Then for any $kG$-module $M$ the following formula holds:*

$$\Omega_V \circ \Omega_W(M) \cong \Omega_{V\oplus W} \circ \Omega_{V\otimes W}(M)$$

*In particular, relative syzygy operators "commute" with each other, that is:*

$$\Omega_V \circ \Omega_W(M) \cong \Omega_W \circ \Omega_V(M)$$

This formula is the version for relative projectivity to modules of the same formula for projectivity relative to $P$-sets entitled "Thévenaz' Lemma" in [Bou00]. Since the document [Oku91] is unpublished we give a proof to this result which shall turn out to be very handy.

Proof. Start with $V$- and $W$-projective covers of $k$ and $M$:

$$\begin{aligned} E_{V,k} &: 0 \longrightarrow \Omega_V(k) \longrightarrow P_V \xrightarrow{\varepsilon} k \longrightarrow 0 \\ E_{W,M} &: 0 \longrightarrow \Omega_W(M) \longrightarrow P_W \xrightarrow{\tau} M \to 0 \ . \end{aligned}$$

Form the augmented tensored complex $E_{V,k} \boxtimes E_{W,M} = [(E_{V,k})_* \otimes (E_{W,M})_* \xrightarrow{\varepsilon\otimes\tau} M]$ described in remark 2.12.7 below:

$$0 \to \Omega_V(k) \otimes \Omega_W(M) \xrightarrow{h} (\Omega_V(k) \otimes P_W) \oplus (P_V \otimes \Omega_W(M)) \xrightarrow{g} P_V \otimes P_W \xrightarrow{\varepsilon\otimes\tau} M \to 0$$

Since the sequences $E_{V,k} \otimes V$ and $E_{W,M} \otimes W$ both split by definition of relative projective covers, and $(E_{V,k} \boxtimes E_{W,M}) \otimes (V \otimes W) \cong (E_{V,k} \otimes V) \boxtimes (E_{W,M} \otimes W)$, the sequence

$$0 \longrightarrow \ker(\varepsilon \otimes \tau) \longrightarrow P_V \otimes P_W \xrightarrow{\varepsilon\otimes\tau} M \longrightarrow 0$$

is $(V \otimes W)$-split. As $P_V \otimes P_W \in \mathit{Proj}(V \otimes W)$, this sequence is a $V \otimes W$-projective presentation of $M$, but it is not necessarily minimal. Thus, the relative Schanuel's Lemma yields

$$\ker(\varepsilon \otimes \tau) \cong \Omega_{V\otimes W}(M) \oplus ((V \otimes W) - \mathit{proj})\,.$$

On the other hand, the tail of the sequence $E_{V,k} \boxtimes E_{W,M}$:

$$0 \to \Omega_V(k) \otimes \Omega_W(M) \xrightarrow{h} (\Omega_V(k) \otimes P_W) \oplus (P_V \otimes \Omega_W(M)) \to \mathrm{Coker}(h) \longrightarrow 0$$

is a $V \oplus W$-projective presentation of $\mathrm{Coker}(h)$. Indeed, the middle term is clearly $V \oplus W$-projective by 2.2.2. It is also $V \oplus W$-split: first it is $V$-split since $(E_{V,k} \boxtimes E_{W,M}) \otimes V \cong (E_{V,k} \otimes V) \boxtimes E_{W,M}$ and $E_{V,k} \otimes V$ splits, and second it is $W$-split by a similar argument.
Then, using the fact that $\ker(\varepsilon \otimes \tau) \cong \mathrm{Coker}(h)$, Lemma 2.12.3 and the omnibus properties of relative syzygies and relative projectivity, compute:

$$\begin{aligned} \Omega_V \circ \Omega_W(M) \oplus (V - \mathit{proj}) &\cong \Omega_V(k) \otimes \Omega_W(M) \\ &\cong \Omega_{V\oplus W}(\mathrm{Coker}(h)) \oplus ((V \oplus W) - \mathit{proj}) \\ &\cong \Omega_{V\oplus W}(\Omega_{V\otimes W}(M) \oplus (V \otimes W) - \mathit{proj})) \oplus ((V \oplus W) - \mathit{proj}) \\ &\cong \Omega_{V\oplus W} \circ \Omega_{V\otimes W}(M) \oplus ((V \oplus W) - \mathit{proj}) \end{aligned}$$

If $M \in \mathit{Proj}(V)$ or $M \in \mathit{Proj}(W)$, then $\Omega_V \circ \Omega_W(M) \cong 0 \cong \Omega_{V\oplus W} \circ \Omega_{V\otimes W}(M)$. So assume that $M$ is indecomposable, $M \notin \mathit{Proj}(V)$ and $M \notin \mathit{Proj}(W)$. Thus both $\Omega_V \circ \Omega_W(M)$ and $\Omega_{V\oplus W} \circ \Omega_{V\otimes W}(M)$ are indecomposable, and they are neither $V$-projective nor $V \oplus W$-projective. The result follows from the Krull-Schmidt Theorem. □

In terms of families of subgroups, this formula has the following user-friendly form:

Corollary 2.12.6 ([Oku91], Cor. 9.11).
*Let $\mathcal{F}$ and $\mathcal{H}$ be families of subgroups of the group $G$. Then, for any $M \in \mathsf{mod}(kG)$, the following formula holds:*

$$\Omega_{\mathcal{F}} \circ \Omega_{\mathcal{H}}(M) \cong \Omega_{\mathcal{F}\cup\mathcal{H}} \circ \Omega_{{}^G\mathcal{F}\cap\mathcal{H}}(M)$$

*where* ${}^G\mathcal{F} \cap \mathcal{H} = \{\, {}^gF \cap H \mid F \in \mathcal{F}, H \in \mathcal{H}\}$.

Proof. Let $V(\mathcal{F}) = \bigoplus_{F\in\mathcal{F}} k\!\uparrow_F^G$ and $V(\mathcal{H}) = \bigoplus_{H\in\mathcal{H}} k\!\uparrow_H^G$ be the modules associated with the families $\mathcal{F}$ and $\mathcal{H}$. The proposition yields

$$\Omega_{V(\mathcal{F})} \circ \Omega_{V(\mathcal{H})}(M) \cong \Omega_{V(\mathcal{F})\oplus V(\mathcal{H})} \circ \Omega_{V(\mathcal{F})\otimes V(\mathcal{H})}(M)\,.$$

Besides,

$$V(\mathcal{F}) \oplus V(\mathcal{H}) = (\bigoplus_{F\in\mathcal{F}} k\!\uparrow_F^G) \oplus (\bigoplus_{H\in\mathcal{H}} k\!\uparrow_H^G) = \bigoplus_{H\in\mathcal{F}\cup\mathcal{H}} k\!\uparrow_H^G = V(\mathcal{F}\cup\mathcal{H})\,,$$

and using Frobenius reciprocity and the Mackey formula, we compute

$$\begin{aligned} V(\mathcal{F}) \otimes V(\mathcal{H}) &\cong (\bigoplus_{F\in\mathcal{F}} k\!\uparrow_F^G) \otimes (\bigoplus_{H\in\mathcal{H}} k\!\uparrow_H^G) \cong \bigoplus_{F\in\mathcal{F}} \bigoplus_{H\in\mathcal{H}} (k\!\uparrow_F^G \otimes k\!\uparrow_H^G) \\ &\cong \bigoplus_{F\in\mathcal{F}} \bigoplus_{H\in\mathcal{H}} (k\!\uparrow_F^G\!\downarrow_H^G \otimes k)\!\uparrow_H^G \\ &\cong \bigoplus_{F\in\mathcal{F}} \bigoplus_{H\in\mathcal{H}} (\bigoplus_{x\in[H\backslash G/F]} {}^x k\!\downarrow^{{}^xF}_{{}^xF\cap H}\!\uparrow^H_{{}^xF\cap H})\!\uparrow_H^G \\ &\cong \bigoplus_{F\in\mathcal{F}} \bigoplus_{H\in\mathcal{H}} \bigoplus_{x\in[H\backslash G/F]} k\!\uparrow^G_{{}^xF\cap H} \end{aligned}$$

Finally it follows from Proposition 2.2.2 that $Proj(V(\mathcal{F}) \otimes V(\mathcal{H})) = Proj(V({}^G\mathcal{F}\cap\mathcal{H}))$, as required. ∎

Remark 2.12.7.
If $E: 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ and $F: 0 \longrightarrow S \xrightarrow{\sigma} T \xrightarrow{\tau} U \longrightarrow 0$ are short exact sequences, then one can define chain complexes $\mathbf{E}_*: 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ and $\mathbf{F}_*: 0 \longrightarrow S \xrightarrow{\sigma} T \longrightarrow 0$ with $B$ and $T$ in degree zero, and with homology $H_*(\mathbf{E}_*) = H_0(\mathbf{E}_*) = B/A \cong C$ and $H_*(\mathbf{F}_*) = H_0(\mathbf{F}_*) \cong U$. Then $\mathbf{E}_* \otimes \mathbf{F}_*$ is exact with homology $H_*(\mathbf{E}_* \otimes \mathbf{F}_*) = H_0(\mathbf{E}_* \otimes \mathbf{F}_*) \cong C \otimes U$. In other words, the augmented complex $[\mathbf{E}_* \otimes \mathbf{F}_* \xrightarrow{\beta\otimes\tau} C \otimes U] =: E \boxtimes F$ is a four-term exact sequence:

$$0 \longrightarrow A \otimes S \xrightarrow{h} (A \otimes T) \oplus (B \otimes S) \xrightarrow{g} B \otimes T \xrightarrow{\beta\otimes\tau} C \otimes U \longrightarrow 0$$

where $h = \begin{pmatrix} A \otimes \sigma \\ \alpha \otimes S \end{pmatrix}$ and $g = (-\alpha \otimes T, B \otimes \tau)$. Moreover, if $\beta$ and $\tau$ are split $kG$-homomorphisms, then so is $\beta \otimes \tau$, and if $\alpha$ or $\sigma$ is a split $kG$-homomorphisms, then so is $h$.

# CHAPTER 3

# The Groups of Relative Endotrivial Modules

Recall that a module $M \in \mathsf{mod}(kG)$ is called *endotrivial* if its endomorphism algebra, considered as a $kG$-module, has the form $\mathrm{End}_k(M) \cong k \oplus (proj)$. In this chapter we generalise this family of modules to weaker versions by replacing ordinary projectivity with projectivity relative to a $kG$-module. This operation enables us to build groups of representations which naturally generalise the group of endotrivial modules $T(G)$.

In the sequel, unless otherwise stated, $V$ shall denote a fixed absolutely $p$-divisible $kG$-module so that the subcategory $Proj(V)$ is not the whole category $\mathsf{mod}(kG)$ of $kG$-modules, which is equivalent to requiring that the trivial module $k$ is not projective relatively to $V$.

## 3.1. Relative endotrivial modules

Definition 3.1.1.
Let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module. A $kG$-module $M$ is termed *endotrivial relative to the kG-module V* or *relatively V-endotrivial* or simply *V-endotrivial* if its $k$-endomorphism ring is the direct sum of a trivial module and a $V$-projective module. That is, $M$ is endotrivial relative to $V$ if and only if

$$\mathrm{End}_k(M) \cong M^* \otimes M \cong k \oplus (V - proj)\,.$$

Notice that, if such an isomorphism exists, it is provided by the trace map $\mathrm{Tr}_M : M^* \otimes M \twoheadrightarrow k$, and thus it is natural.

Also notice that an endotrivial module, in the usual sense, is, in this terminology, a $kG$-endotrivial module. In the sequel, we shall often refer to endotrivial modules as *ordinary* endotrivial modules in order to differentiate the usual notion from the relative notion. In addition, since for any $kG$-module $V$, the subcategory $Proj(V)$ contains the projective $kG$-modules, it follows that endotrivial modules are always $V$-endotrivial modules, however the choice of $V$.

Categorical setting 3.1.2.
Definition 3.1.1 is equivalent to requiring that $\mathrm{End}_k(M)$ is isomorphic to a trivial module in the

relative stable category $\mathsf{stmod}_V(kG)$. Furthermore, it follows, straightforwardly, from the Benson-Carlson Theorem (2.8.4) that an indecomposable module $M$ is $V$-endotrivial if and only if it is invertible in $\mathsf{stmod}_V(kG)$ endowed with its usual tensor (triangulated) structure[1], with inverse $M^*$.

To begin with, here is a rudimentary but extremely useful dimensional characterisation for relatively endotrivial modules.

Lemma 3.1.3.
*Let $V$ be an absolutely $p$-divisible $kG$-module and $M$ be a $V$-endotrivial module. Then:*

(a) $\dim_k(M)^2 \equiv 1 \mod p$.
(b) *In case $V = k\uparrow_Q^G$, that is if we consider projectivity relative to the $p$-subgroup $Q$ of $G$, then* $\dim_k(M)^2 \equiv 1 \mod |P:Q|$ *where $P$ is a Sylow $p$-subgroup of $G$ containing $Q$.*

Proof.
(a) By 2.8.4 the $k$-dimension of any $V$-projective module is divisible by $p$, hence
$$\dim_k(M)^2 = \dim_k(\mathrm{End}_k(M)) = \dim_k(k \oplus (V - proj)) \equiv 1 \mod p\,.$$
(b) As a consequence of Green's indecomposability theorem, the $k$-dimension of a module is divisible by the index of one of its vertices in the corresponding Sylow $p$-subgroup. (See [CR90].)

□

In the sequel, we shall often use statement (a) of this lemma without further mention.

Remark 3.1.4.
Notice that in particular, if $G$ is a $p$-group of order $p^n$ then statement (b) gives the standard dimensional characterization for an ordinary endotrivial $kG$-module $M$: $\dim_k(M)^2 \equiv 1 \mod p^n$.

## 3.2. Direct sum decomposition structure

Before giving the first examples, it is helpful to describe the direct summands of relative endotrivial modules.

Lemma 3.2.1.
*If $M \in \mathsf{mod}(kG)$ is a $V$-endotrivial module, then so is $M \oplus W$ for any $W \in Proj(V)$.*

Proof. Compute
$$\begin{aligned}\mathrm{End}_k(M \oplus W) &\cong \mathrm{End}_k(M) \oplus (M^* \otimes W) \oplus (W^* \otimes M) \oplus (W^* \otimes W)\\ &\cong k \oplus (V - proj) \oplus (V - proj) \oplus (V - proj)\\ &\cong k \oplus (V - proj)\,.\end{aligned}$$

□

The next easy result is the first step towards the construction of an abelian group structure on the class of relative endotrivial modules.

[1] The algebraic geometers ([Bal10]) say that the $\otimes$-invertible elements of $\mathsf{stmod}(kG)$ have been *dubbed* **endotrivial** by the representation theorists. We follow up and dub $V$-**endotrivial** the $\otimes$-invertible elements of $\mathsf{stmod}_V(kG)$.

LEMMA 3.2.2.
*Let $M$ be a $V$-endotrivial $kG$-module and assume there is a direct sum decomposition $M \cong M_0 \oplus M_1$, then one of $M_0$ or $M_1$ is $V$-endotrivial and the other is $V$-projective. In consequence, $M$ is $V$-endotrivial if and only if its $V$-projective-free part is indecomposable and $V$-endotrivial.*

PROOF. By assumption, we have

$$k \oplus (V - proj) \cong \mathrm{End}_k(M) \cong \mathrm{End}_k(M_0) \oplus \mathrm{Hom}_k(M_0, M_1) \oplus \mathrm{Hom}_k(M_1, M_0) \oplus \mathrm{End}_k(M_1)\,.$$

As a result, the Krull-Schmidt theorem forces the trivial module $k$ to be a direct summand of either $\mathrm{End}_k(M_0)$, or $\mathrm{End}_k(M_1)$. Indeed, if it were not the case, $k$ would be a direct summand of $\mathrm{Hom}_k(M_0, M_1)$ or $\mathrm{Hom}_k(M_1, M_0)$. But the two latter modules being dual to each other, $k \oplus k$ would be a direct summand of $\mathrm{End}_k(M)$, which is not possible because, by the assumption that $V$ is absolutely $p$-divisible, $k \notin Proj(V)$ (Proposition 2.8.3). Thus we may assume that $\mathrm{End}_k(M_0) \cong k \oplus (V - proj)$ and $\mathrm{End}_k(M_1) \in Proj(V)$. But, by 2.2.2, $M_1 \in Proj(V)$ if and only if $M_1 \otimes M_1^* \in Proj(V)$. In conclusion, $M_0$ is $V$-endotrivial and $M_1 \in Proj(V)$, as required. □

### 3.3. Examples and constructions

Ordinary endotrivial modules are endotrivial relatively to any $kG$-module $V$. In particular, so is any one-dimensional $kG$-module $\chi$. Indeed, $\chi^* \otimes_k \chi \cong k$ because it is one-dimensional and thus trace map $\mathrm{Tr}_\chi$ splits.

The other class of examples of $V$-endotrivial modules that springs to mind is given by the kernels (and cokernels) of $V$-projective resolutions of the trivial module and in particular, the relative syzygies $\Omega_V^n(k)$, $n \in \mathbb{Z}$. More generally, we have the following construction to manufacture new $V$-endotrivial modules from old ones :

LEMMA 3.3.1.

(a) *Let $P \in Proj(V)$ and $0 \longrightarrow L \longrightarrow P \longrightarrow N \longrightarrow 0$ be a $V$-split short exact sequence. Then $N$ is $V$-endotrivial if and only if $L$ is.*
(b) *Let $M$ be a $V$-endotrivial $kG$-module. Then the $kG$-modules $\Omega_W^n(M)$ are $V$-endotrivial modules for every $kG$-module $W \in Proj(V)$ and for every $n \in \mathbb{Z}$.*

PROOF.

(a) follows from (b). Indeed, the sequence $0 \longrightarrow L \longrightarrow P \longrightarrow N \longrightarrow 0$ can be seen as a $V$-projective presentation of $N$ as well as a $V$-injective presentation of $L$. In consequence, $L = \widetilde{\Omega}_V(N) \cong \Omega_V(N) \oplus (V - proj)$ and $N = \widetilde{\Omega}_V^{-1}(L) \cong \Omega_V^{-1}(L) \oplus (V - proj)$, by 2.12.1.
(b) Using the arithmetic of the relative syzygies developed in section 2.12 compute:

$$\begin{aligned}\mathrm{End}_k(\Omega_W^n(M)) &\cong \Omega_W^n(M)^* \otimes \Omega_W^n(M) \cong \Omega_W^0(M^* \otimes M) \oplus (W - proj)\\ &\cong \Omega_W^0(k \oplus (V - proj)) \oplus (W - proj)\\ &\cong \Omega_W^0(k) \oplus \Omega_W^0(V - proj) \oplus (W - proj)\\ &\cong k \oplus (V - proj) \oplus (W - proj) \cong k \oplus (V - proj)\end{aligned}$$

The last isomorphism comes from the assumption that $W \in Proj(V)$. For, the last-but-one isomorphism, $\Omega_W^0(k) \cong k$ because $k \notin Proj(W) \subseteq Proj(V) \neq \mathsf{mod}(kG)$ and because $\Omega_W^0(k)$ is $W$-projective-free, and $\Omega_W^0(V - proj) = (V - proj)$ by Lemma 2.12.3.

□

Furthermore, the class of $V$-endotrivial $kG$-modules is closed under taking tensor products, duals and thus under application of $Hom_k(-,-)$.

LEMMA 3.3.2.
*If $M, N$ are $V$-endotrivial $kG$-modules, then so are the modules $M \otimes N$, $M^*$ and $\mathrm{Hom}_k(M, N)$.*

PROOF. Compute:

$$\begin{aligned}\mathrm{End}_k(M \otimes N) &\cong \mathrm{End}_k(M) \otimes \mathrm{End}_k(N) \cong (k \oplus (V - proj)) \otimes (k \oplus (V - proj)) \\ &\cong k \oplus (V - proj)\end{aligned}$$

Moreover $M^*$ is $V$-endotrivial because $\mathrm{End}_k(M^*) \cong \mathrm{End}_k(M)$. Finally $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$ which is $V$-endotrivial by the preceding arguments. □

LEMMA 3.3.3.
*Let $V \in \mathsf{mod}(kG)$ be absolutely $p$-divisible. Let $M$ be a $kG$-module such that $M \cong N_1 \otimes N_2$ for some $N_1, N_2 \in \mathsf{mod}(kG)$ . Then $M$ is $V$-endotrivial if and only if both $N_1$ and $N_2$ are $V$-endotrivial.*

PROOF. If $M$ is $V$-endotrivial, then $k \oplus (V - proj) \cong M^* \otimes M \cong N_1^* \otimes N_1 \otimes N_2^* \otimes N_2$ . Since $V$ is absolutely $p$-divisible, then in the first place $k \notin Proj(V)$ and, in the second place, by Lemma 3.1.3, $\dim_k(M)$ is coprime to $p$. Hence so are $\dim_k(N_1)$ and $\dim_k(N_2)$. Thus it follows from Lemma 2.2.1 and the Krull-Schmidt theorem that $N_1^* \otimes N_1 \cong k \oplus (V - proj)$ and $N_2^* \otimes N_2 \cong k \oplus (V - proj)$. As required. □

Next, we investigate the behaviour of relative endotrivial modules with respect to standard group operations.

LEMMA 3.3.4.

(a) *If $H$ is a subgroup of $G$ and $M$ a $V$-endotrivial $kG$-module, then $M\downarrow_H^G$ is a $V\downarrow_H^G$-endotrivial module.*
(b) *If $N$ is a normal subgroup of $G$ and $M$ a $V$-endotrivial $k[G/N]$-module, then $\mathrm{Inf}_{G/N}^G(M)$ is an $\mathrm{Inf}_{G/N}^G(V)$-endotrivial module.*
(c) *Let $\varphi : G_1 \longrightarrow G_2$ be a group isomorphism and $M$ a $kG_1$-module. Then $M$ can be seen as a $kG_2$-module, denoted by $\mathrm{Iso}_{G_1}^{G_2}(M)$, the action of $G_2$ on $M$ being given via $\varphi^{-1}$. Furthermore, if $V$ is a $kG_1$-module and $M$ is a $V$-endotrivial $kG_1$-module then $\mathrm{Iso}_{G_1}^{G_2}(M)$ becomes an $\mathrm{Iso}_{G_1}^{G_2}(V)$-endotrivial $kG_2$-module.*

PROOF. Use Lemma 2.6.1 to compute:

$$\mathrm{End}_k(M\downarrow_H^G) \cong \mathrm{End}_k(M)\downarrow_H^G \cong (k \oplus (V - proj))\downarrow_H^G \cong k\downarrow_H^G \oplus (V - proj)\downarrow_H^G \cong k \oplus (V\downarrow_H^G - proj)$$

This proves (a). Likewise:

$$\begin{aligned}\mathrm{End}_k(\mathrm{Inf}_{G/N}^G(M)) &\cong \mathrm{Inf}_{G/N}^G(\mathrm{End}_k(M)) \cong \mathrm{Inf}_{G/N}^G(k \oplus (V - proj)) \\ &\cong \mathrm{Inf}_{G/N}^G(k) \oplus \mathrm{Inf}_{G/N}^G(V - proj) \cong k \oplus (\mathrm{Inf}_{G/N}^G(V) - proj)\end{aligned}$$

and

$$\begin{aligned}\mathrm{End}_k(\mathrm{Iso}_{G_1}^{G_2}(M)) &\cong \mathrm{Iso}_{G_1}^{G_2}(\mathrm{End}_k(M)) \cong \mathrm{Iso}_{G_1}^{G_2}(k \oplus (V - proj)) \\ &\cong \mathrm{Iso}_{G_1}^{G_2}(k) \oplus \mathrm{Iso}_{G_1}^{G_2}(V - proj) \cong k \oplus (\mathrm{Iso}_{G_1}^{G_2}(V) - proj)\,.\end{aligned}$$

□

**Tensor induction.** The tensor induction of a relative endotrivial module does not necessarily produce a relative endotrivial module either. A counterexample will be provided in section 7.6.

**Induction.** Relative endotrivial modules are, in general, not stable under induction. This is easily seen by considering the group $G := C_3 \times C_3$ and its index 3 subgroup $H := C_3 \times 1$. Then, the trivial $kH$-module $k$ is endotrivial, but the induced module $k\uparrow_H^G$ can't be endotrivial relatively to any $kG$-module since it is indecomposable and thus by the Benson-Carlson Theorem 2.8.1, the module $(k\uparrow_H^G)^* \otimes k\uparrow_H^G$ does not have the trivial module as a direct summand. (This example extends to any indecomposable relative endotrivial $kH$-module $M$ and any $G \geq H$ satisfying the hypothesis of Green's indecomposability criterion, since then $dim_k(M\uparrow_H^G) = |G : H| \dim_k(M)$ is divisible by $p$.)

Notwithstanding, if the problem is taken the other way around, here is a condition for an induced module to be relatively endotrivial:

Lemma 3.3.5.
*Let $V \in \mathsf{mod}(kG)$ be absolutely $p$-divisible. If $M$ is a $V$-endotrivial $kG$-module such that $M \cong L\uparrow_H^G$ for some proper subgroup $H$ of $G$ and some $L \in \mathsf{mod}(kH)$, then $H$ contains a Sylow $p$-subgroup of $G$ and $L$ is $V\downarrow_H^G$-endotrivial.*

Proof. First, because $M$ is $V$-endotrivial, $\dim_k(M)$ is coprime to $p$ and therefore so are $\dim_k L$ and $|G : H|$. Thus $H$ must contain a Sylow $p$-subgroup of $G$.
Then, Theorem 2.9.3 implies that $V\downarrow_H^G$ is absolutely $p$-divisible as well. Furthermore, the Mackey formula implies that $L \mid M\downarrow_H^G$, which, for dimensional reasons, is not $V\downarrow_H^G$-projective (Lemma 2.8.4). It follows that $L$ is $V\downarrow_H^G$-endotrivial. Indeed, by the previous lemma, $M\downarrow_H^G$ is $V\downarrow_H^G$-endotrivial, thus by 3.2.2 we can write $M\downarrow_H^G = M_0 \oplus (V\downarrow_H^G -proj)$ with $M_0$ its unique indecomposable and $V\downarrow_H^G$-endotrivial. Now, looking at dimensions, $\dim_k(M_0)$ is coprime to $p$ and any direct summand of the $V\downarrow_H^G$-projective part has dimension divisible by $p$, so that $L$, as a direct summand of $M\downarrow_H^G$, must be of the form $M_0 \oplus (V\downarrow_H^G -proj)$ as well. □

## 3.4. Self-equivalences of the relative stable category

From the categorical point of view, one reason for interest in endotrivial modules comes from the fact that the tensor product with an endotrivial module always induces a self-equivalence of the stable category $\mathsf{stmod}(kG)$. We establish in this section that, in like manner, the tensor product with a $V$-endotrivial module always induces a self-equivalence of the relative stable category $\mathsf{stmod}_V(kG)$ associated with the module $V$. To start with, we dissect the tensor product with a $V$-endotrivial module in $\mathsf{mod}(kG)$.

Lemma 3.4.1.
*Let $V$ be an absolutely $p$-divisible $kG$-module and $M$ be an indecomposable $kG$-module with dimension coprime to $p$.*
*Then, $M$ is a $V$-endotrivial $kG$-module if and only if for any indecomposable $kG$-module $N$, the tensor product $M \otimes N$ has at most one non-$V$-projective indecomposable direct summand.*
*More accurately, for the necessary condition we have:*

(a) *if $N \in Proj(V)$, then $M \otimes N \in Proj(V)$ and so do all its direct summands.*
(b) *if $N \notin Proj(V)$, then $M \otimes N$ has exactly one non-$V$-projective direct summand.*

PROOF. Note that because $M$ has dimension coprime to $p$, $M \notin Proj(V)$. For the necessary condition, statement (a) is given by Lemma 2.2.2, parts (a),(b) and (f). For statement (b), Lemma 2.8.5 ensures that if $N \notin Proj(V)$, then $M \otimes N \notin Proj(V)$ either, therefore $M \otimes N$ has at least one non-$V$-projective direct summand. In order to prove that it is unique, write

$$M \otimes N \cong \bigoplus_{i \in I} A_i \oplus (V - proj)$$

where $\bigoplus_{i \in I} A_i$ is a decomposition into indecomposable summands of the $V$-projective-free part of $M \otimes N$. Then, on the one hand, we get

$$M^* \otimes (M \otimes N) \cong M^* \otimes (\bigoplus_{i \in I} A_i \oplus (V - proj)) \cong \bigoplus_{i \in I} \underbrace{(M^* \otimes A_i)}_{\notin Proj(V)} \oplus (V - proj)$$

where, because $M$ is $V$-endotrivial, it is not absolutely $p$-divisible so that we can invoke Lemma 2.8.5 to obtain that none of the modules $M^* \otimes A_i$ belongs to $Proj(V)$. But, on the other hand, because $M$ is $V$-endotrivial,

$$M^* \otimes M \otimes N \cong (k \oplus (V - proj)) \otimes N \cong N \oplus (V - proj)$$

in which decomposition $N$ is the unique non-$V$-projective indecomposable summand. Therefore, using Krull-Schmidt to compare the two decompositions of $M^* \otimes M \otimes N$ yields $|I| = 1$, as required.

Conversely, taking $N = M^*$ means that $M^* \otimes M \cong L \oplus (V - proj)$ for some $L \in \mathsf{mod}(kG)$. Moreover, since $\dim_k M$ is coprime to $p$, Lemma 2.2.1 states that the trivial module $k$ is a summand in $M^* \otimes M$. Since $V$ is absolutely $p$-divisible, $k \notin Proj(V)$ and therefore the Krull-Schmidt Theorem forces the existence of a decomposition $M^* \otimes M \cong k \oplus (V - proj)$. In other words, $M$ is $V$-endotrivial. □

Passing to the relatively $V$-stable category $\mathsf{stmod}_V(kG)$ provides us with the desired nicer statement:

PROPOSITION 3.4.2.
*Let $V$ be an absolutely $p$-divisible $kG$-module and $M$ be a $V$-endotrivial module. The tensor product with $M$ induces a self-equivalence of the relative stable category $\mathsf{stmod}_V(kG)$ with inverse induced by the dual module $M^*$:*

$$\mathsf{stmod}_V(kG) \underset{M^* \otimes -}{\overset{M \otimes -}{\rightleftarrows}} \mathsf{stmod}_V(kG)$$

PROOF. First, by the lemma, the tensor product of a $V$-projective module with $M$ or $M^*$ is $V$-projective again. Then, let $N$ be an indecomposable non-$V$-projective $kG$-module. Because $M$ is $V$-endotrivial, we have seen in the lemma that $M^* \otimes M \otimes N \cong N \oplus (V - proj)$ in $\mathsf{mod}(kG)$, hence $M^* \otimes M \otimes N \cong N$ in $\mathsf{stmod}_V(kG)$. The same is true, if we swap the roles of $M$ and $M^*$. Since the isomorphism $M^* \otimes M \cong k \oplus (V - proj)$ is natural (obtained via the trace map $\mathrm{Tr}_M$), it follows that $M \otimes -$ and $M^* \otimes -$ are self-equivalence of $\mathsf{stmod}_V(kG)$. □

### 3.5. Vertices and sources of relative endotrivial modules

To begin with, here is a well-known characterization of the dimension of modules with vertices strictly smaller than Sylow p-subgroups. Having found no reference for it, for completeness we give a quick proof using the language of relative projectivity.

Lemma 3.5.1.
*Let $M$ be an indecomposable $kG$-module and assume that a vertex $Q$ of $M$ is strictly smaller than a Sylow $p$-subgroup $P$ of $G$. Then $p$ divides the $k$-dimension of $M$.*

Proof. By assumption $M \in Proj(k\uparrow_Q^G)$. Thus $Proj(M) \subseteq Proj(k\uparrow_Q^G)$ by 2.2.2 and $Proj(k\uparrow_Q^G) \neq \mathsf{mod}(kG)$ because $Q \lneq P$ (this is well-known from the theory of vertices and sources). In consequence $Proj(M) \subsetneq \mathsf{mod}(kG)$. Hence by Proposition 2.8.3, $M$ is absolutely $p$-divisible so that $p \mid dim_k(M)$ because it is indecomposable. □

As a consequence, this lemma allows us to characterize the vertices of relatively endotrivial modules and compute their sources.

Lemma 3.5.2.
*Let $V$ be an absolutely $p$-divisible $kG$-module and $M$ be an indecomposable $V$-endotrivial $kG$-module. Then:*

(a) *The vertices of $M$ are the Sylow $p$-subgroups of $G$.*
(b) *If $(P, S)$ is a vertex-source pair for $M$, then $S$ is a $V\downarrow_P^G$-endotrivial module $S$ has multiplicity one as a direct summand of $M\downarrow_P^G$.*
(c) *Assume moreover that $M\downarrow_P^G \cong k \oplus (V\downarrow_P^G -proj)$, then the trivial $kP$-module is a source for $M$.*

Proof.

(a) The contrapositive statement of the previous lemma asserts that the vertices of a module with dimension coprime to $p$ are the Sylow $p$-subgroups. Hence the result.
(b) By assumption $S \mid M\downarrow_P^G$, so that $S^* \mid M^*\downarrow_P^G$ and
$$S \otimes S^* \mid M\downarrow_P^G \otimes S^* \mid M\downarrow_P^G \otimes M^*\downarrow_P^G \cong (M \otimes M^*)\downarrow_P^G \cong k \oplus (V\downarrow_P^G -proj)\,.$$
Thus it remains to show that $k \mid S \otimes S^*$. Assume ab absurdo that it is not the case, then $S \otimes S^*$ has to be $V\downarrow_P^G$-projective by the above and therefore, so is $S$ by 2.2.2. In consequence,
$$M \mid S\uparrow_P^G \in (Proj(V\downarrow_P^G))\uparrow_P^G \subseteq Proj(V)$$
by Lemma 2.6.3, which contradicts the fact that for an absolutely $p$-divisible module $V$, an indecomposable $V$-endotrivial module is $V$-projective-free.
(c) Since $P$ is a Sylow $p$-subgroup, $M$ is $P$-projective so that
$$M \mid M\downarrow_P^G\uparrow_P^G \cong (k \oplus (V\downarrow_P^G -proj))\uparrow_P^G \cong k\uparrow_P^G \oplus (V\downarrow_P^G -proj))\uparrow_P^G = k\uparrow_P^G \oplus (V - proj),$$
by Lemma 2.6.3 (c). Moreover, $M$ is $V$-projective-free by assumption, thus the Krull-Schmidt theorem yields that $M \mid k\uparrow_P^G$. In consequence, $P$ being a vertex of $M$, $k$ is a source of $M$.

□

## 3.6. Group structure

We can now copy the group structure on the ordinary endotrivial modules. Let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module and set an equivalence relation $\sim_V$ on the class of $V$-endotrivial $kG$-modules as follows: for $M$ and $N$ two $V$-endotrivial modules let
$$M \sim_V N \text{ if and only if } M_0 \cong N_0\,,$$

where $M_0$ and $N_0$ are the unique $V$-endotrivial indecomposable summands of $M$ and $N$, respectively, given by 3.2.2. This amounts to requiring that $M$ and $N$ are isomorphic in $\mathsf{stmod}_V(kG)$. Then let $T_V(G)$ denote the resulting set of equivalence classes. In particular, any equivalence class in $T_V(G)$ consists of an indecomposable $V$-endotrivial module $M_0$ and all the modules of the form $M_0 \oplus (V - proj)$.

Proposition 3.6.1.
*The ordinary tensor product $\otimes_k$ induces an abelian group structure on the set $T_V(G)$ defined as follows:*

$$[M] + [N] := [M \otimes_k N]$$

*The zero element is $[k]$ and the opposite of a class $[M]$ is the class $[M^*]$. Moreover $T_V(G)$ is called the group of $V$-endotrivial modules.*

Notice that we use an additive notation, which is consistent with the choice made in [BT00] and related articles treating endo-permutation and endotrivial modules.

Proof. The composition law $+$ is clearly commutative. It is also clearly well-defined by Lemma 3.3.2. Furthermore, if $M \sim_V M'$ and $N \sim_V N'$ then $M \cong M_0 \oplus (V - proj)$, $M' \cong M_0 \oplus (V - proj)$, $N \cong N_0 \oplus (V - proj)$, $N' \cong N_0 \oplus (V - proj)$, with $M_0, N_0$ indecomposable $V$-endotrivial modules. The omnibus properties of relative projectivity imply that

$$M \otimes N \cong (M_0 \otimes N_0) \oplus (V - proj) \text{ and } M' \otimes N' \cong (M_0 \otimes N_0) \oplus (V - proj)\,,$$

that is $M \otimes N \sim_V M' \otimes N'$. □

Corollary 3.6.2.
*If $M$ is a self-dual, $V$-endotrivial $kG$-module, then $[M]$ has order one or two in $T_V(G)$. Moreover $[M]$ has order one if and only if $M \cong k$.*

Proof. By assumption $M \otimes M \cong M^* \otimes M \cong k \oplus (V - proj)$ so that $2[M] = [k]$. □

Example 3.6.3.
To give a first example, this simple observation can be applied at once to the concrete case of a cyclic $p$-group $C_{p^n}$, $n \geq 1$. Indeed, all the indecomposable $kC_{p^n}$-modules are self-dual. Therefore, whatever the choice of the absolutely $p$-divisible module $V$, it can be deduced that the group $T_V(C_{p^n})$ is an elementary abelian 2-group. We shall give a complete description of all the different groups of relative endotrivial modules for cyclic $p$-groups in Chapter 5.

Categorical setting 3.6.4.
The group $T_V(G)$ of $V$-endotrivial modules may as well be considered as the group $T(G)$ of $V$-stable isomorphism classes of $V$-endotrivial modules. Furthermore, following the ideas of [Bal10], from the point of view of a $\otimes$-triangular geometer, $T_V(G)$ is *nothing but* the Picard group $\mathrm{Pic}(\mathsf{stmod}_V(kG))$ of the $\otimes$-triangulated category $(\mathsf{stmod}_V(kG), \otimes_k, k)$.

We further note that here the terminology Picard group does not designate the group of self-equivalences of $\mathsf{stmod}_V(kG)$ as it would for some authors in representation theory. Here, if $(\mathsf{K}, \otimes, \mathbf{1})$ is a $\otimes$-triangulated category, then $\mathrm{Pic}(\mathsf{K})$ designates the abelian group of isomorphism classes $[x]$ of $\otimes$-invertible objects, with addition $[x] + [y] = [x \otimes y]$ and zero $0 = [\mathbf{1}]$. Hence the equivalence above.

## 3.7. Some subgroups of $T_V(G)$

LEMMA 3.7.1.
*Let $U, V \in \mathsf{mod}(kG)$ be absolutely $p$-divisible modules such that $Proj(V) \subseteq Proj(U)$. Then:*

(a) *Every $V$-endotrivial $kG$-module is $U$-endotrivial.*
(b) *If $M$ and $N$ are $V$-endotrivial modules such that $M \sim_V N$, then $M \sim_U N$ as well. In consequence, $T_V(G)$ can be identified with a subgroup of $T_U(G)$ via the injective group homomorphism*

$$\begin{array}{rccc} \imath: & T_V(G) & \longrightarrow & T_U(G) \\ & [M]_V & \longmapsto & [M]_U \end{array}$$

*By abuse of notation, we shall simply write $T_V(G) \leq T_U(G)$.*

PROOF.
(a) Let $M$ be a $V$-endotrivial module, then $\mathrm{End}_k(M) \cong k \oplus (V - proj) = k \oplus (U - proj)$, i.e. $M$ is $U$-endotrivial.
(b) Write $M \cong M_0 \oplus (V - proj)$ and $N \cong N_0 \oplus (V - proj)$ with $M_0$, $N_0$ indecomposable and $V$-endotrivial. Then $M \sim_V N$ implies that $M_0 \cong N_0$. By (a), $M_0$ and $N_0$ are $U$-endotrivial, so that $M \sim_U N$. In consequence $\imath$ is a well-defined group homomorphism. The injectivity follows from the uniqueness of the indecomposable summand $M_0$.

□

The study of relative endotrivial modules for the Klein Group $C_2 \times C_2$ will show that it is possible to have a strict inclusion $Proj(V) \subsetneq Proj(U)$ but an isomorphism $T_V(G) \cong T_U(G)$. Nevertheless, a strict inclusion $Proj(V) \subsetneq Proj(U)$ implies that the class of $V$-endotrivial modules is strictly contained in the class of $U$-endotrivial modules. Indeed, if $M \in Proj(U) \setminus Proj(V)$, then, on the one hand, $L := k \oplus M$ is $U$-endotrivial, since

$$\mathrm{End}_k(L) \cong k \oplus M \oplus M^* \oplus (M \otimes M^*) = k \oplus (U - proj),$$

but on the other hand it is not $V$-endotrivial, otherwise $M$ would be $V$-projective. Besides, this argument shows that there are more modules belonging to the class $[k]$ in $T_U(G)$ than in $T_V(G)$.

CONSEQUENCE 3.7.2.
The group $T(G)$ of ordinary endotrivial modules is a subgroup of $T_V(G)$ for every absolutely $p$-divisible $V \in \mathsf{mod}(kG)$. For $Proj \subseteq Proj(V)$, thus Lemma 3.7.1 yields $T(G) \leq T_V(G)$.

ONE-DIMENSIONAL REPRESENTATIONS.
If $G$ is a finite group, we shall follow the notation of [MT07] and denote by $X(G)$ the abelian group of all isomorphism classes of one-dimensional $kG$-modules endowed with the group law induced by $\otimes_k$, which can also be identified with the group $Hom(G, k^\times)$ of $k$-linear characters of $G$. It is a $p'$-group, isomorphic to the $p'$-part of the abelianization $G/[G,G]$ of $G$.

As mentioned before in Section 3.3, a one-dimensional module $\chi$ is $V$-endotrivial for every absolutely $p$-divisible $kG$-module $V$, because $\chi^* \otimes \chi \cong k$. Therefore there is an embedding

$$\begin{array}{ccc} X(G) & \longrightarrow & T_V(G) \\ \chi & \longmapsto & [\chi] \end{array} .$$

mapping a one-dimensional module to its class in $T_V(G)$. Formalism would require to denote by $X_V(G)$ the image of $X(G)$ in $T_V(G)$, where the law is written additively, nonetheless, in order to keep light notation, we shall simply use $X(G)$ instead of $X_V(G)$ and, in addition, consider it as a subgroup of $T_V(G)$. Thus there is always a chain of subgroups:

$$X(G) \leq T(G) \leq T_V(G)$$

### 3.8. Computing with relative syzygies.

NOTATION.
Let $V \in \mathsf{mod}(kG)$. For simplicity of notation, in the sequel of this text, we shall denote by $\Omega_V$ the class of the relative syzygy module $\Omega_V(k)$ in any group of relative endotrivial modules $T_W(G)$ such that $V \in Proj(W)$.

The following formulae are basic rules for computations with relative syzygies in groups of relative endotrivial modules:

LEMMA 3.8.1.
*Let $n \geq 2$ be an integer and $V_1, \dots, V_n \in \mathsf{mod}(kG)$ be pairwise non isomorphic absolutely $p$-divisble modules.*

(a) *In $T_{V_1}(G)$, for any indecomposable $V_1$-endotrivial $kG$-module $N$:*
$$[\Omega_{V_1}(N)] = [\widetilde{\Omega}_{V_1}(N)]$$
(b) *In $T_{V_1\oplus V_2}(G)$:*
$$\Omega_{V_1} + \Omega_{V_2} = [\Omega_{V_1} \circ \Omega_{V_2}(k)]$$
(c) *In $T_{V_1\oplus V_2}(G)$:*
$$\Omega_{V_1\oplus V_2} = \Omega_{V_1} + \Omega_{V_2} - \Omega_{V_1\otimes V_2}$$
(d) *More generally, in $T_{V_1\oplus \dots \oplus V_n}(G)$:*
$$\Omega_{V_1\oplus\dots\oplus V_n} = \sum_{s=1}^{n} (-1)^{s+1} ( \sum_{1\leq i_1 < \dots < i_s \leq n} \Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}} )$$

PROOF.

(a) $\widetilde{\Omega}_{V_1}(N)$ is a notation for any module of the form $\Omega_{V_1}(N) \oplus (V_1 - proj)$, thus, in any case, $\Omega_{V_1}(N)$ is the unique $V_1$-endotrivial summand in $\widetilde{\Omega}_{V_1}(N)$. Hence the equality $[\Omega_{V_1}(N)] = [\widetilde{\Omega}_{V_1}(N)] \in T_{V_1}(G)$ follows.

(b) By definition of the addition $[\Omega_{V_1}(k)]+[\Omega_{V_2}(k)] = [\Omega_{V_1}(k)\otimes\Omega_{V_2}(k)]$. Moreover by Lemma 2.12.1, $\Omega_{V_1}(k)\otimes\Omega_{V_2}(k) \cong \Omega_{V_1}(\Omega_{V_2}(k))\oplus(V_1-proj)$, hence the equality $[\Omega_{V_1}(k)\otimes\Omega_{V_2}(k)] = [\Omega_{V_1} \circ \Omega_{V_2}(k)]$ in $T_{V_1\oplus V_2}(G)$ since $Proj(V_1) \subseteq Proj(V_1 \oplus V_2)$.

(c) Let us compute:
$$\begin{aligned}\Omega_{V_1} + \Omega_{V_2} &= [\Omega_{V_1} \circ \Omega_{V_2}(k)] \text{ by part (b)}\\ &= [\Omega_{V_1\oplus V_2} \circ \Omega_{V_1\otimes V_2}(k)] \text{ by Proposition 2.12.5}\\ &= \Omega_{V_1\oplus V_2} + \Omega_{V_1\otimes V_2} \text{ by part (b) again.}\end{aligned}$$
Whence the formula $\Omega_{V_1\oplus V_2} = \Omega_{V_1} + \Omega_{V_2} - \Omega_{V_1\otimes V_2}$ .

(d) By part (c), the formula holds for $n = 2$. Thus we can proceed by induction on the integer $n$. Let $n \geq 3$ and assume as induction hypothesis that the formula holds for every $(n-1)$-tuple of absolutely $p$-divisible modules $V_1, \dots V_{n-1} \in \mathsf{mod}(kG)$. Then, applying part (b) to the modules $V_1 \oplus \dots \oplus V_{n-1}$ and $V_n$ yields:
$$\begin{aligned}\Omega_{V_1\oplus\dots\oplus V_n} &= \Omega_{V_1\oplus\dots\oplus V_{n-1}} + \Omega_{V_n} - \Omega_{(V_1\oplus\dots\oplus V_{n-1})\otimes V_n}\\ &= \sum_{s=1}^{n-1} (-1)^{s+1} ( \sum_{1\leq i_1<\dots<i_s\leq n-1} \Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}} ) + \Omega_{V_n} - \Omega_{(V_1\oplus\dots\oplus V_{n-1})\otimes V_n}\end{aligned}$$

by the induction hypothesis. Moreover, applying the induction hypothesis a second time to the class $\Omega_{(V_1\oplus\ldots\oplus V_{n-1})\otimes V_n}$ yields:

$$\begin{aligned}\Omega_{(V_1\oplus\ldots\oplus V_{n-1})\otimes V_n} &= \Omega_{(V_1\otimes V_n)\oplus\ldots\oplus(V_{n-1}\otimes V_n)}\\ &= \sum_{s=1}^{n-1}(-1)^{s+1}\Big(\sum_{1\le i_1<\ldots<i_s\le n-1}\Omega_{(V_{i_1}\otimes V_n)\otimes\cdots\otimes(V_{i_s}\otimes V_n)}\Big)\\ &= \sum_{s=1}^{n-1}(-1)^{s+1}\Big(\sum_{1\le i_1<\ldots<i_s\le n-1}\Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}\otimes V_n}\Big)\end{aligned}$$

since $Proj((V_n)^{\otimes(n-1)}) = Proj(V_n)$ by Proposition 2.2.2. As a result

$$\begin{aligned}\Omega_{V_1\oplus\ldots\oplus V_n} &= \sum_{s=1}^{n-1}(-1)^{s+1}\Big(\sum_{1\le i_1<\ldots<i_s\le n-1}\Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}}\Big) + \Omega_{V_n} - \Omega_{(V_1\oplus\ldots\oplus V_{n-1})\otimes V_n}\\ &= \sum_{s=1}^{n-1}(-1)^{s+1}\Big(\sum_{1\le i_1<\ldots<i_s\le n-1}\Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}}\Big) + \Omega_{V_n}\\ &\quad - \sum_{s=1}^{n-1}(-1)^{s+1}\Big(\sum_{1\le i_1<\ldots<i_s\le n-1}\Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}\otimes V_n}\Big)\\ &= \sum_{s=1}^{n}(-1)^{s+1}\Big(\sum_{1\le i_1<\ldots<i_s\le n}\Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}}\Big).\end{aligned}$$

□

Remark 3.8.2.
Depending on the situation it can be interesting to use the following more manageable form for formula (d) of the lemma:

$$\Omega_{V_1\oplus\ldots\oplus V_n} = \sum_{i=1}^{n}\Omega_{V_i} - \sum_{j=2}^{n}\Omega_{\oplus_{r=1}^{j-1}V_r\otimes V_j}$$

In particular, if $\mathcal{H} := \{H_1,\ldots,H_n\}$ is a family of subgroups of the group $G$ such that the $kG$-module $V(\mathcal{H})$ is absolutely $p$-divisible, then by Corollary 2.12.6 the latter formula reads

$$\Omega_{\mathcal{H}} = \sum_{i=1}^{n}\Omega_{\{H_i\}} - \sum_{j=2}^{n}\Omega_{\,{}^G\{H_1,\ldots,H_{j-1}\}\cap\{H_j\}} \text{ in } T_{V(\mathcal{H})}(G)\,.$$

## 3.9. Standard homomorphisms.

In order to make further links between different groups of relative endotrivial modules, we define group homomorphisms and actions induced by group operations.

**Restriction.** Let $H$ be a subgroup of $G$ and let $V$ an absolutely $p$-divisible $kG$-module such that $V\!\downarrow^G_H$ is absolutely $p$-divisible too. Then both the groups $T_V(G)$ and $T_{V\downarrow^G_H}(H)$ are well-defined. If $M$ is a $V$-endotrivial $kG$-module, then $M\!\downarrow^G_H$ is a $V\!\downarrow^G_H$-endotrivial $kH$-module. Therefore, in this case restriction to a subgroup induces a well-defined group homomorphism:

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & T_V(G) & \longrightarrow & T_{V\downarrow^G_H}(H) \\ & [M] & \longmapsto & [M\downarrow^G_H] \end{array}$$

Indeed, $\mathrm{Res}^G_H$ is a group homomorphism since restriction and $\otimes_k$ commute. Furthermore, by Lemma 2.6.3, the map $\mathrm{Res}^G_H$ is independent of the choice of the generator $V$ for $Proj(V)$.

**Inflation.** Let $N$ be a normal subgroup of a group $G$ such that $p \mid |G/N|$. If $V$ is an absolutely $p$-divisible $k[G/N]$-module, then both the groups $T_V(G/N)$ and $T_{\mathrm{Inf}^G_{G/N}(V)}(G)$ are well-defined. In addition, if $M$ is a $V$-endotrivial $k[G/N]$-module, then $\mathrm{Inf}^G_{G/N}(M)$ is $\mathrm{Inf}^G_{G/N}(V)$-endotrivial. Thus inflation induces an injective group homomorphism:

$$\begin{array}{rccc} \mathrm{Inf}^G_{G/N}\colon & T_V(G/N) & \hookrightarrow & T_{\mathrm{Inf}^G_{G/N}(V)}(G) \\ & [M] & \longmapsto & [\mathrm{Inf}^G_{G/N}(M)] \end{array}$$

Indeed, inflation and $\otimes_k$ commute. Furthermore, by Lemma 2.6.6, the map $\mathrm{Inf}^G_{G/N}$ is independent of the choice of the generator $V$ for $Proj(V)$.

**Isomorphism.** Let $\varphi : G_1 \longrightarrow G_2$ be a group isomorphism. If $M$ is a $kG_1$-module, then it can be seen as a $kG_2$-module, denoted by $\mathrm{Iso}(\varphi)(M)$, the action of $G_2$ being given via $\varphi^{-1}$. Furthermore, if $V$ is an absolutely $p$-divisible $kG_1$-module, then $\mathrm{Iso}(\varphi)(V)$ is an absolutely $p$-divisible $kG_2$-module, and if $M$ is a $V$-endotrivial $kG_1$-module, then $M$ becomes an $\mathrm{Iso}(\varphi)(V)$-endotrivial $kG_2$-module. This operation induces a group isomorphism:

$$\begin{array}{rccc} \mathrm{Iso}(\varphi)\colon & T_V(G_1) & \longrightarrow & T_{\mathrm{Iso}(\varphi)(V)}(G_2) \\ & [M] & \longmapsto & [\mathrm{Iso}(\varphi)(M)] \end{array}$$

A concrete example of such an isomorphism between groups of relative endotrivial modules is provided below by conjugation.

Remark 3.9.1.
It should be noted that the three cases of restriction, inflation, and isomorphism can be unified in the single case of *restriction along* a group homomorphism $G_1 \longrightarrow G_2$. Nonetheless, we do not do it in these terms because of restriction for which we need to require that the module $V\downarrow^G_H$ is absolutely $p$-divisible. This shows that an arbitrary group homomorphism, and in particular an inclusion of subgroups, would not necessarily induce a well-defined group homomorphism between the corresponding groups of relative endotrivial modules.

**Conjugation.** Let $H \trianglelefteq G$ be a normal subgroup and $V$ be an absolutely $p$-divisible $G$-invariant $kH$-module. Then, for all $g \in G$, ${}^g Proj(V) = Proj(V)$ and ${}^gH = H$. Therefore, conjugation induces a well-defined action of $G$ (or rather $G/H$), on the group $T_V(H)$ given by:

$$\begin{array}{ccc} G \times T_V(H) & \longrightarrow & T_V(H) \\ (g, [M]) & \longmapsto & [{}^gM] \end{array}$$

In case the subgroup $H$ and the module $V$ are not assumed to be normal nor $G$-invariant, then the above assignment does not yield a group action. Nevertheless, for any element $g \in G$, the conjugation isomorphism $\gamma_g : H \longrightarrow {}^gH$ induces a group isomorphism

$$\begin{array}{rccc} \gamma_g\colon & T_V(H) & \longrightarrow & T_{{}^gV}({}^gH) \\ & [M] & \longmapsto & [{}^gM] \end{array}$$

In particular, if $H \trianglelefteq G$, then $T_V(H) \cong T_{{}^gV}(H)$.

# CHAPTER 4

# Restriction Maps

The purpose of this chapter is to relate groups of relative endotrivial modules for a group $G$ to those for a Sylow $p$-subgroup $P$ of $G$ or a subgroup $H$ containing $P$. In particular, links between endotrivial modules for $G$ and the normaliser $N_G(P)$ of the Sylow subgroup can be obtained by Green correspondence. Most of the results presented in this chapter are generalisations of results concerning ordinary endotrivial modules which can be found in [MT07], [CMN06] and [Maz07].

## 4.1. Restriction to a Sylow $p$-subgroup

To begin with, we describe restrictions to a Sylow $p$-subgroup. The following easy properties generalise [CMN06, Prop. 2.6].

Lemma 4.1.1.
*Let $P$ be a Sylow $p$-subgroup of $G$ and $H$ a subgroup of $G$ containing $P$. Let $V$ be an absolutely $p$-divisible $kG$-module.*

(a) *Let $M$ be a $V$-endotrivial $kG$-module. Then $M$ is a direct summand of a $V\downarrow^G_H$-endotrivial module induced from $H$ to $G$, namely the module $M\downarrow^G_H\uparrow^G_H$.*
(b) *Let $M$ be a $kG$-module such that $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial, then $M$ is $V$-endotrivial.*

Proof.
(a) Since $H \geq P$, by $H$-projectivity $M \mid M\downarrow^G_H\uparrow^G_H$ where $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial by 3.3.4.
(b) As $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial and $V\downarrow^G_H$ is absolutely $p$-divisible,

$$(\dim_k M)^2 = (\dim_k M\downarrow^G_H)^2 \equiv 1 \mod p\,.$$

In consequence both the trace map and its restriction to $H$ split, so that

$$M^* \otimes M \cong k \oplus \ker(Tr)$$

and

$$(M\downarrow^G_H)^* \otimes M\downarrow^G_H \cong k \oplus \ker(Tr)\downarrow^G_H$$

where $\ker(Tr)\downarrow^G_H$ has to be $V\downarrow^G_H$-projective by the assumption that $M\downarrow^G_H$ is $V\downarrow^G_H$-endotrivial. Besides, by $H$-projectivity, $\ker(Tr) \mid \ker(Tr)\downarrow^G_H\uparrow^G_H \in Proj(V)$ by 2.6.4.

Therefore $\ker(Tr)$ is a $V$-projective module as well and $M^* \otimes M \cong k \oplus (V - proj)$ as required.

□

We now treat the special case of a normal Sylow $p$-subgroup. The next proposition and its corollary partly generalise [CMN06, Prop. 2.6, (d)] and [Maz07, Cor. 2.7].

PROPOSITION 4.1.2.
*Let $P$ be a normal Sylow $p$-subgroup of $G$. Let $V$ be an absolutely $p$-divisible $kG$-module. Then, an indecomposable $kG$-module $M$ is $V$-endotrivial if and only if its restriction to $P$ is an indecomposable $V{\downarrow}^G_P$-endotrivial module.*

PROOF. Suppose $M$ is an indecomposable $V$-endotrivial module. Let $M{\downarrow}^G_P \cong N_1 \oplus \cdots \oplus N_s$, $s \in \mathbb{N}$, be a decomposition into indecomposable summands. Since $P$ is a vertex of $M$ (see 3.5.1), one may assume, without loss of generality, that $N_1$ is a source for $M$, so that $M \mid N_1{\uparrow}^G_P$ as well. Thus, given that $P$ is normal in $G$, the Mackey formula yields

$$M{\downarrow}^G_P \mid N_1{\uparrow}^G_P{\downarrow}^G_P \cong \bigoplus_{g \in [G/P]} {}^gN_1 \,.$$

Now, on the one hand $M{\downarrow}^G_P$ is $V{\downarrow}^G_P$-endotrivial, which is more accurately the direct sum of an indecomposable $V{\downarrow}^G_P$-endotrivial module, whose $k$-dimension is coprime to $p$, and a $V{\downarrow}^G_P$-projective module, all of whose indecomposable summands have $k$-dimension divisible by $p$. On the other hand the $G$-conjugates ${}^gN_1$ of $N_1$ are all indecomposable with $k$-dimension equal to that of $N_1$. Therefore, this forces $M{\downarrow}^G_P$ to be indecomposable ($V{\downarrow}^G_P$-endotrivial). Conversely, let $M$ be such that $M{\downarrow}^G_P$ is an indecomposable $V{\downarrow}^G_P$-endotrivial module. Firstly the fact that $M{\downarrow}^G_P$ is indecomposable forces $M$ to be indecomposable as well, and secondly it follows from part (b) of Lemma 4.1.1 that $M$ is $V$-endotrivial. □

As a consequence, when the Sylow $p$-subgroup $P$ is normal in the group $G$, then the $V$-endotrivial modules are detected upon restriction to $P$. Since the restriction of a $V$-endotrivial module is $G$-invariant, at the level of groups of relatively endotrivial modules, there is an inclusion

$$\mathrm{Im}(\mathrm{Res}^G_P) \leq T_{V{\downarrow}^G_P}(P)^{N_G(P)/P} \,.$$

A natural question is to ask when this inclusion is indeed an equality, that is when the restriction map is actually surjective onto the $N_G(P)/P$-fixed points of $T_{V{\downarrow}^G_P}(P)$. We shall see further in Chapter 5 that, for instance, it is always the case for groups with cyclic Sylow $p$-subgroups.

COROLLARY 4.1.3.
*Let $P$ be a normal Sylow $p$-subgroup of $G$. Let $V$ be an absolutely $p$-divisible $kG$-module, $M$ be an indecomposable $V$-endotrivial module and $(X_*, \partial_*)$ be a $V$-projective resolution of $M$. Then:*

(a) *$(X_*, \partial_*)$ is minimal if and only if $(X_*{\downarrow}^G_P, \partial_*{\downarrow}^G_P)$ is a minimal $V{\downarrow}^G_P$-projective resolution of $M{\downarrow}^G_P$;*
(b) *in particular, $\Omega^n_V(M){\downarrow}^G_P \cong \Omega^n_{V{\downarrow}^G_P}(M{\downarrow}^G_P)$ for all integers $n$.*

PROOF. Given that $(X_*, \partial_*)$ is a minimal $V$-projective resolution, for each integer $n \geq 0$ there is a $V$-split short exact sequence

$$0 \longrightarrow \Omega^{n+1}_V(M) \longrightarrow X_n \xrightarrow{\partial_n} \Omega^n_V(M) \longrightarrow 0 \,.$$

Restricting it from $G$ to $P$ yields a $V{\downarrow}_P^G$-projective presentation of $\Omega^n_{V{\downarrow}_P^G}(M){\downarrow}_P^G$ :

$$0 \longrightarrow \Omega_V^{n+1}(M){\downarrow}_P^G \longrightarrow X_n{\downarrow}_P^G \xrightarrow{\partial_n{\downarrow}_P^G} \Omega_V^n(M){\downarrow}_P^G \longrightarrow 0$$

although, it is not necessarily minimal. However, by 2.12.1,

$$\Omega_V^{n+1}(M){\downarrow}_P^G \cong \Omega_V^{n+1}(M{\downarrow}_P^G) \oplus (V{\downarrow}_P^G\ -proj) \text{ and } \Omega_V^n(M){\downarrow}_P^G \cong \Omega_V^n(M{\downarrow}_P^G) \oplus (V{\downarrow}_P^G\ -proj)\,.$$

Besides, by the proposition, both these modules are indecomposable so that the $V{\downarrow}_P^G$-projective factors are zero. Therefore the above short exact sequence is indeed

$$0 \longrightarrow \Omega_V^{n+1}(M{\downarrow}_P^G) \longrightarrow X_n{\downarrow}_P^G \xrightarrow{\partial_n{\downarrow}_P^G} \Omega_V^n(M{\downarrow}_P^G) \longrightarrow 0\ .$$

Hence the minimality of $(X_*{\downarrow}_P^G, \partial_*{\downarrow}_P^G)$. The converse is trivial. □

## 4.2. Restriction to the normaliser of a Sylow $p$-subgroup

The goal is now to figure out the behaviour of restriction maps from a group $G$ to the normaliser of a Sylow $p$-subgroup $P$ or a subgroup $H$ containing $N_G(P)$. The picture to keep in mind is the following:

$$\begin{array}{c} G \\ | \\ H \\ | \\ N_G(P) \\ | \\ P \end{array}$$

It follows from Theorem 2.9.3 and Section 3.9 that for every absolutely $p$-divisible $kG$-module $V$ there is a well-defined restriction map

$$\mathrm{Res}_H^G : T_V(G) \longrightarrow T_{V{\downarrow}_H^G}(H)\,.$$

The following statement generalises [CMN06, Prop. 2.6.(a)].

Lemma 4.2.1.
*Let $P$ be a Sylow $p$-subgroup of $G$ and let $H$ be a subgroup of $G$ containing $N_G(P)$. Let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module. Then both the restriction maps $\mathrm{Res}_H^G : T_V(G) \longrightarrow T_{V{\downarrow}_H^G}(H)$ and $\mathrm{Res}_{N_G(P)}^H : T_{V{\downarrow}_H^G}(H) \longrightarrow T_{V{\downarrow}_{N_G(P)}^H}(N_G(P))$ are injective.*

Proof. Since $N_H(P) = N_G(P)$, it suffices to show that $\mathrm{Res}_H^G$ is injective. Let $M$ be an indecomposable $V$-endotrivial $kG$-module. By Lemma 3.5.2, $P$ is a vertex of $M$. Then, on the one hand, the Green correspondence for the triple $(G, H; P)$ yields:

$$M{\downarrow}_H^G \cong Gr(M) \oplus X$$

where $Gr(M)$ is an indecomposable $kH$-module with vertex $P$ and $X \in Proj(\mathcal{Y})$ with $\mathcal{Y} = \{{}^xP \cap H \,|\, x \in G \setminus H\}$. But ${}^xP \cap H \lneq {}^xP$ for all $x \in G \setminus H$, otherwise ${}^xP$ would be a Sylow $p$-subgroup of $H$ which is not possible, since then there would be $h \in H$ such that ${}^{hx}P = P$, that is $hx \in N_G(P) \subseteq H$ and $x \in H$. Therefore all the direct summands of $X$ have a vertex strictly smaller than $P$. On the other hand, $M{\downarrow}_H^G$ is a $V{\downarrow}_H^G$-endotrivial module, that is:

$$M{\downarrow}_H^G \cong M_0 \oplus (V{\downarrow}_H^G\ -proj)$$

with $M_0$ an indecomposable $V{\downarrow}_H^G$-endotrivial module, thus with vertex $P$ by 3.5.2. In consequence, the Krull-Schmidt theorem implies that $M_0 \cong Gr(M)$, the $kH$-Green correspondent of $M$, whose uniqueness yields the injectivity of $\mathrm{Res}_H^G$. □

REMARK 4.2.2.
As a scholium, note that in particular an indecomposable $V$-endotrivial module restricts to $N_G(P)$ as

$$M{\downarrow}_{N_G(P)}^G \cong M_0 \oplus (V{\downarrow}_{N_G(P)}^G - proj)$$

where the $V{\downarrow}_{N_G(P)}^G$-projective part is actually projective relatively to the family of subgroups $\mathcal{Y} = \{{}^xP \cap N_G(P) \,|\, x \in G \backslash N_G(P)\}$.

### 4.3. Cases in which restriction maps are isomorphisms

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Knowing that the restriction map $\mathrm{Res}_H^G : T_V(G) \longrightarrow T_{V\downarrow_H^G}(H)$ is injective for every subgroup $H$ containing the normaliser $N_G(P)$ of a Sylow $p$-subgroup $P$, the next question that arises is to understand when this map is an isomorphism. The last section on groups with cyclic Sylow $p$-subgroup shall provide us with examples in which the answer depends on the module $V$ to which relative projectivity is considered. Notwithstanding, one can show that in case the subgroup $H$ is strongly $p$-embedded in $G$, then $\mathrm{Res}_H^G$ is always an isomorphism, however the choice of the module $V$. This result generalises the similar result for ordinary endotrivial modules that can be found, for instance, in [MT07, Lem. 2.7]. Furthermore, the proof of this result provides us with the following more general sufficient condition on the module $V$ for the restriction map to be an isomorphism.

LEMMA 4.3.1.
*Let $P$ be a Sylow $p$-subgroup of $G$ and $H \leq G$ a subgroup containing the normaliser $N_G(P)$ of $P$. Let $V$ be an absolutely $p$-divisible $kG$-module. If $Proj(V{\downarrow}_H^G) \supseteq Proj(\mathcal{Y})$, where $\mathcal{Y}$ is the family of subgroups $\{{}^gP \cap H \,|\, g \in G \setminus H\}$ involved in the Green correspondence, then the restriction map $\mathrm{Res}_H^G : T_V(G) \longrightarrow T_{V\downarrow_H^G}(H)$ is an isomorphism. Furthermore, the inverse map is induced by induction, so that*

$$T_V(G) = \{[M{\uparrow}_H^G] \,|\, [M] \in T_{V\downarrow_H^G}(H)\} \cong T_{V\downarrow_H^G}(H)$$

*More accurately, on indecomposable $V{\downarrow}_H^G$-endotrivial modules, the inverse map is induced by the Green correspondence, that is, if $\Gamma(M)$ denotes the Green correspondent of an indecomposable $kH$-module $M$, then*

$$T_V(G) = \{[\Gamma(M)] \,|\, M \text{ is an indecomposable } V{\downarrow}_H^G \text{-endotrivial } kH\text{-module}\}\,.$$

PROOF. By 4.2.1, the map $\mathrm{Res}_H^G$ is one-to-one, therefore it only remains to show that it is onto as well. Let $L$ be an indecomposable $V{\downarrow}_H^G$-endotrivial module. The Mackey formula yields the isomorphism.

$$L{\uparrow}_H^G{\downarrow}_H^G \cong L \oplus \bigoplus_{\substack{g\in[H\backslash G/H] \\ g\in G-H}} ({}^gL){\downarrow}_{{}^gH\cap H}^{{}^gH}{\uparrow}_{{}^gH\cap H}^H =: L \oplus L'\,,$$

where, by the proof of the Green correspondence, $L' \in Proj(\mathcal{Y})$, so that

$$L{\uparrow}_H^G{\downarrow}_H^G \cong L \oplus (\mathcal{Y} - proj) = L \oplus (V{\downarrow}_H^G - proj)$$

by assumption. In other words, $L{\uparrow}_H^G{\downarrow}_H^G$ is $V{\downarrow}_H^G$-endotrivial and consequently $L{\uparrow}_H^G$ is $V$-endotrivial by 4.1.1. Therefore, $\mathrm{Res}_H^G([L{\uparrow}_H^G]) = [L{\uparrow}_H^G{\downarrow}_H^G] = [L]$. Hence the surjectivity of $\mathrm{Res}_H^G$. Moreover, the proof of the injectivity of $\mathrm{Res}_H^G$ shows that the unique indecomposable $V$-endotrivial summand of $L{\uparrow}_H^G$ has to be isomorphic to the $kG$-Green correspondent of $L$.
It follows from the proof of the injectivity (Lemma 4.2.1) that the inverse map is induced by Green correspondence on the indecomposable modules. To see that, alternatively, it is induced by induction, let $[M] \in T_{V\downarrow_H^G}(H)$ and write $M \cong M_0 \oplus (V{\downarrow}_H^G - proj)$ with $M_0$ an indecomposable $V{\downarrow}_H^G$-endotrivial module. Then,

$$M{\uparrow}_H^G \cong M_0{\uparrow}_H^G \oplus (V{\downarrow}_H^G - proj){\uparrow}_H^G \cong \Gamma(M_0) \oplus (\mathcal{X} - proj) \oplus (V - proj)$$

where $\mathcal{X}$ is the familly of subgroups involved in the Green correspondence, as described in section 2, and $Proj(V{\downarrow}_H^G){\uparrow}_H^G \subseteq Proj(V)$ by Lemma 2.6.3. As just mentioned above, $\Gamma(M_0)$ is $V$-endotrivial, therefore it remains to check that $Proj(\mathcal{X}) \subseteq Proj(V)$. But this is a consequence of the hypothesis that $Proj(V{\downarrow}_H^G) \supseteq Proj(\mathcal{Y})$. Indeed, at the level of $kH$-modules, $Proj(\mathcal{Y}) \supseteq Proj(\mathcal{X})$ by definition of the families $\mathcal{X}$ and $\mathcal{Y}$, thus $Proj(V{\downarrow}_H^G) \supseteq Proj(\mathcal{X})$. Inducing to $G$ yields in $\mathsf{mod}(kG)$ the required inclusions

$$Proj(V) \supseteq Proj(V{\downarrow}_H^G){\uparrow}_H^G \supseteq Proj(\mathcal{X}){\uparrow}_H^G = Proj(\mathcal{X})\,.$$

□

Corollary 4.3.2.
*If the subgroup $H$ is strongly $p$-embedded in $G$, then $\mathrm{Res}_H^G : T_V(G) \longrightarrow T_{V\downarrow_H^G}(H)$ is an isomorphism.*

Proof. If $H$ is strongly $p$-embedded in $G$, then for any $g \in G \setminus H$ the subgroup ${}^gH \cap H$ has order coprime to $p$, thus $\mathcal{Y} = \{\{1\}\}$. Therefore $\mathrm{Res}_H^G$ is an isomorphism, regardless of the module $V$, since then $Proj(V{\downarrow}_H^G) \supseteq Proj(\mathcal{Y}) = Proj$ for any $kG$-module $V$.

□

For instance, if the Sylow $p$-subgroup $P$ is a trivial intersection subgroup (TI), then $N_G(P)$ is strongly $p$-embedded in $G$. Moreover, any strongly $p$-embedded subgroup contains the normaliser of some Sylow $p$-subgroup of $G$.

Remark 4.3.3.

(a) The first explicit example that springs to mind for a module satisfying the hypotheses of the lemma is the absolutely $p$-divisible module

$$V := \bigoplus_{Q \in \mathcal{F}_G} k{\uparrow}_Q^G$$

where $\mathcal{F}_G := \{Q \lneq P\}$ is the family of all proper $p$-subgroups of the Sylow $p$-subgroup $P$. Indeed, for any $p$-subgroup $Q \lneq P \leq G$ it results from the Mackey formula that $k{\uparrow}_Q^H$ is a direct summand of $k{\uparrow}_Q^G{\downarrow}_H^G$, thus

$$Proj(V{\downarrow}_H^G) \supseteq Proj(\mathcal{F}_H) = Proj(\overline{\mathcal{F}_H}) \supseteq Proj(\mathcal{Y})\,,$$

where the middle equality was established in Remark 2.5.3.

(b) Finally, it is also worth emphasizing that in general the $kG$-Green correspondent $\Gamma(L)$ of an indecomposable $V{\downarrow}_H^G$-endotrivial module $L$ might or might not be a $V$-endotrivial module. Again, the section on groups with cyclic Sylow $p$-subgroups shall provide us with a handful of examples illustrating this phenomenon. Nonetheless, whenever the lemma applies $\Gamma(L)$ is always $V$-endotrivial.

### 4.4. On the kernels of the restriction maps

The next result gives conditions on the module $V$ under which the kernel of the restriction map $\mathrm{Res}^G_Q : T_V(G) \longrightarrow T_{V\downarrow^G_Q}(P)$ is exactly $X(G)$. This generalises [MT07, Lem. 2.6]. The proof is the same, it is only analysed more deeply in order to state the results in terms of $V$-projectivity, which is less restricting than ordinary projectivity. This criterion shall be especially useful in the forthcoming chapters dealing with groups having a cyclic Sylow $p$-subgroup and the Dade group of a finite group.

LEMMA 4.4.1.
*Let $G$ be a finite group and $P$ be a Sylow $p$-subgroup of $G$. Let $V$ be an absolutely $p$-divisible $kG$-module.*

(a) *Let $Q$ be any $p$-subgroup of $G$ such that the restriction map $\mathrm{Res}^G_Q : T_V(G) \longrightarrow T_{V\downarrow^G_Q}(Q)$ is well-defined, that is such that $V\downarrow^G_Q$ is absolutely $p$-divisible. Then $X(G) \leq \ker(\mathrm{Res}^G_Q)$.*

(b) *If all the direct summands of $V\downarrow^G_P$ have a vertex strictly included in ${}^xP \cap P$ for every $x \in G \setminus N_G(P)$, then $X(G) = \ker(\mathrm{Res}^G_P)$. In particular, if $P$ is normal in $G$, then $X(G) = \ker(\mathrm{Res}^G_P)$.*

PROOF.

(a) This is clear since the only one-dimensional $kQ$-module is the trivial module.

(b) It remains to show the reverse inclusion. So let $M$ be an indecomposable $V$-endotrivial $kG$-module such that $[M] \in \ker(\mathrm{Res}^G_P)$, i.e. $M\downarrow^G_P \cong k \oplus (V\downarrow^G_P -proj)$. Thus, by $P$-projectivity, we have:

$$M \mid M\downarrow^G_P\uparrow^G_P \cong k\uparrow^G_P \oplus (V\downarrow^G_P -proj)\uparrow^G_P = k\uparrow^G_P \oplus (V - proj)$$

where by 2.6.1 and 2.6.3 $Proj(V\downarrow^G_P)\uparrow^G_P \subseteq Proj(V\downarrow^G_P\uparrow^G_P) = Proj(V)$. Now, since by assumption $M$ is indecomposable and $V$-endotrivial, that is $V$-projective-free, $M$ must be a direct summand of $k\uparrow^G_P$, therefore restricting to $P$ and applying the Mackey formula yields:

$$M\downarrow^G_P \mid k\uparrow^G_P\downarrow^G_P \cong k^{\oplus|N_G(P):P|} \oplus \bigoplus_{\substack{x\in[P\backslash G/P] \\ x\notin N_G(P)}} k\uparrow^P_{{}^xP\cap P}$$

Each summand $k\uparrow^P_{{}^xP\cap P}$ has a vertex equal to ${}^xP \cap P$. Write $V\downarrow^G_P \cong \bigoplus_{i=1}^m V_i$, $m \in \mathbb{N}$, as a sum of indecomposable modules, and for all $1 \leq i \leq n$ let $Q_i$ be a vertex of $V_i$, then

$$Proj(V\downarrow^G_P) = \bigoplus_{i=1}^m Proj(V_i) \subseteq \bigoplus_{i=1}^m Proj(k\uparrow^G_{Q_i}) .$$

Assume then that $k\uparrow^P_{{}^xP\cap P} \in Proj(V\downarrow^G_P)$, thus $k\uparrow^P_{{}^xP\cap P} \in Proj(k\uparrow^G_{Q_i})$ for some $1 \leq i \leq m$. However, $Q_i \lneq_G {}^xP \cap P$ by assumption, contradicting the fact that ${}^xP \cap P$ is a vertex. Therefore, none of the summands $k\uparrow^P_{{}^xP\cap P}$ belongs to $Proj(V\downarrow^G_P)$, which forces $M\downarrow^G_P$ to be a direct summand of $k^{\oplus|N_G(P):P|}$. Using once more that $M\downarrow^G_P$ is $V\downarrow^G_P$-endotrivial allows us to deduce that $M\downarrow^G_P \cong k$, for $V\downarrow^G_P$ being absolutely $p$-divisible, $k \notin Proj(V\downarrow^G_P)$. Hence $[M] \in X(G)$.

□

In case $V = kG$, that is if we consider ordinary endotrivial modules, then condition (b) is equivalent to requiring that ${}^xP \cap P$ is non trivial for all $x \in G$, as is stated in [MT07, Lem. 2.6].

# CHAPTER 5

# Groups With Cyclic Sylow $p$-Subgroups

## Part A: Cyclic $p$-groups

Let $G := C_{p^n}$ be a cyclic $p$-group of order $p^n$, $n \geq 1$ and $k$ an algebraically closed field of characteristic $p$. Then $kC_{p^n} \cong k[X]/(X-1)^{p^n}$ as $k$-algebras and $M_i := k[X]/(X-1)^i$ is the unique indecomposable $kC_{p^n}$-module of dimension $i$, up to isomorphism, for each $1 \leq i \leq p^n$. Moreover, for $1 \leq i \leq p^n$ this provides a complete list of indecomposable $kC_{p^n}$-modules, up to isomorphism. In particular, $M_1 = k$, the trivial module, and $M_{p^n} = kC_{p^n}$ is the indecomposable projective module. (See [Thé95, Exercises 5.4, 17.2 and 28.3] for details.) Besides, for all $1 \leq i \leq p^n$, a simple comparison of dimensions yields $\Omega(M_i) \cong M_{p^n-i}$. Also note that the indecomposable absolutely $p$-divisible modules are the $M_i$'s with $p$ dividing their dimension $i$. Finally, according to notation used in [MT07], for all integers $0 \leq r \leq n$, we shall denote by $Z_r$ the unique cyclic subgroup of $P$ of order $p^r$, with $Z_0 = 1$, $Z_1 =: Z$ and $Z_n = P$. Thus there are isomorphisms $M_{p^r} \cong k\uparrow_{Z_{n-r}}^G$ and we keep in mind that projectivity relative to $M_{p^r}$ is the same thing as projectivity relative to the subgroup $Z_{n-r}$. In part A of this chapter we give a classification of the endotrivial modules relatively to any absolutely $p$-divisible $kG$-module $V$. In part $B$, we shall use this classification to treat the case of groups that are not necessarily cyclic $p$-groups any more, but have a cyclic Sylow $p$-subgroup.

### 5.1. Relative projectivity to modules

The goal of this chapter is first to describe all the absolutely $p$-divisible modules $V$ for which the subcategories $Proj(V)$ are strictly different and secondly to describe explicitly all the modules they contain (up to isomorphism).

Lemma 5.1.1.
*For every integer $1 \leq r \leq n$, $IProj(M_{p^r}) = \{M_{\alpha p^r} \mid \alpha \in \mathbb{N}, 1 \leq \alpha \leq p^{n-r}\}$.*

Proof. Let $1 \leq r \leq n$ and $1 \leq \alpha \leq p^{n-r}$ be integers. Consider the subgroup $Z_{n-r} \leq C_{p^n}$ of index $p^r$. By 2.2.2 (c), $\mathsf{mod}(kZ_{n-r}) = Proj(k)$. In particular, the $kZ_{n-r}$-module $M_\alpha \in Proj(k)$

thus, by Lemma 2.6.1, we get

$$M_\alpha \uparrow_{Z_{n-r}}^{C_{p^n}} \in Proj(k \uparrow_{Z_{n-r}}^{C_{p^n}}) .$$

In addition, by Green's indecomposability theorem, both $M_\alpha \uparrow_{Z_{n-r}}^{C_{p^n}}$ and $k \uparrow_{Z_{n-r}}^{C_{p^n}}$ are indecomposable. Because for every $1 \leq i \leq p^n$, there is a unique indecomposable $kC_{p^n}$-module with $k$-dimension $i$, it is necessary that $M_\alpha \uparrow_{Z_{n-r}}^{C_{p^n}} \cong M_{\alpha p^r}$ and $k \uparrow_{Z_{n-r}}^{C_{p^n}} \cong M_{p^r}$, so that $M_{\alpha p^r} \in Proj(M_{p^r})$. This yields the inclusion

$$\{M_{\alpha p^r} \,|\, 1 \leq \alpha \leq p^{n-r}\} \subseteq IProj(M_{p^r}) .$$

On the other hand, projectivity relative to the module $M_{p^r}$ is exactly the same thing as projectivity relative to the subgroup $Z_{n-r}$ of $C_{p^n}$. Therefore, if $M$ is projective relatively to $Z_{n-r}$, then by 3.1.3 the index $p^r = |C_{p^n} : Z_{n-r}|$ divides $\dim_k(M)$, which proves the second inclusion. □

Corollary 5.1.2.
*For every integer $1 \leq r \leq n$, the collection of $kC_{p^n}$-modules projective relatively to the $kC_{p^n}$-module $M_{p^r}$ is given as follows:*

$$Proj(M_{p^r}) = \{ \bigoplus_{I \, finite} M_{\alpha_i p^r} \,|\, \alpha_i \in \mathbb{N} \text{ and } 1 \leq \alpha_i \leq p^{n-r} \, \forall i \in I\}$$

Lemma 5.1.3.
*Let $M_i$ be an indecomposable $kC_{p^n}$-module such that $p^r$, with $1 \leq r \leq n-1$, is the largest power of $p$ dividing $\dim_k(M_i) = i$. Write $i := \alpha_i p^r$ with $1 \leq \alpha_i \leq p-1$ an integer. Then $Proj(M_i) = Proj(M_{p^r})$.*

Proof. By Lemma 5.1.1, $M_i = M_{\alpha_i p^r} \in IProj(M_{p^r})$. In consequence,

$$Proj(M_i) \subseteq Proj(M_{p^r}) .$$

In order to show the reverse inclusion, consider again the subgroup $Z_{n-r}$. Since $p \nmid \alpha_i$, by 2.2.2 (c), $Proj(M_{\alpha_i}) = \mathsf{mod}(kZ_{n-r})$. In particular, the trivial $kZ_{n-r}$-module $k \in Proj(M_{\alpha_i})$, hence

$$M_{p^r} = k \uparrow_{Z_{n-r}}^{C_{p^n}} \in Proj(M_{\alpha_i} \uparrow_{Z_{n-r}}^{C_{p^n}}) = Proj(M_{\alpha_i p^r})$$

by Green's indecomposability theorem again. Thus $Proj(M_i) \supseteq Proj(M_{p^r})$. □

We shall now show that in $\mathsf{mod}(kC_{p^n})$ projectivity relative to modules is indeed reduced to projectivity relative to subgroups. In other words:

Proposition 5.1.4.
*Let $V$ be an absolutely $p$-divisible $kC_{p^n}$-module. Then, there exists a subgroup $Z_{n-r}$ of $C_{p^n}$, with $r \geq 1$, such that*

$$Proj(V) = Proj(M_{p^r}) = Proj(k \uparrow_{Z_{n-r}}^{C_{p^n}}) .$$

Proof. If $V$ is indecomposable then $V \cong M_i$ for some $i$ divisible by $p$, and the result has been shown in the preceding lemma. If $V$ is decomposable, write $V := \bigoplus_{j=1}^s M_j$, $s \in \mathbb{N}$. Factor every $1 \leq j \leq s$, as $j := \alpha_j p^{r_j}$ with $1 \leq \alpha_j \leq p-1$ and $1 \leq r_j \leq n-1$. Let $m := \min_j \{r_j\}$. Then, using Proposition 2.2.2, compute:

$$Proj(V) = Proj(\bigoplus_{j=1}^s M_j) = \bigoplus_{j=1}^s Proj(M_j) = \bigoplus_{j=1}^s Proj(M_{p^{r_j}}) = \bigoplus_{j=1}^s Proj(M_{p^m}) = Proj(M_{p^m})$$

where clearly $Proj(M_{p^{r_j}}) \subseteq Proj(M_{p^m})$ either by a classical argument on projectivity relative to subgroups or by Lemma 5.1.1. $\square$

In particular, note that for $G = C_p$ a cyclic group of prime order, there is no relative projectivity to modules other than ordinary projectivity. More generally, we note that there is a unique chain of strict inclusions of subcategories of relatively projective $kC_{p^n}$-modules given as follows:

$$Proj = Proj(M_{p^n}) \subsetneq Proj(M_{p^{n-1}}) \subsetneq \cdots \subsetneq Proj(M_{p^2}) \subsetneq Proj(M_p) \subsetneq Proj(k) = \mathsf{mod}(kG)\,.$$

## 5.2. Structure of the groups of relatively endotrivial modules

Since there are exactly $n$ different proper subcategories of relatively projective modules in $\mathsf{mod}(kC_{p^n})$, given by $Proj(M_{p^r})$ for $0 \leq r \leq n$, there are also $n$ different groups of relatively endotrivial modules $T_{M_{p^r}}(C_{p^n})$ for $0 \leq r \leq n$. Besides, since there is a unique indecomposable $kC_{p^n}$-module for each $k$-dimension between 1 and $p^n$, it is clear that every such module is self-dual. Therefore, Corollary 3.6.2 ensures that any group $T_{M_{p^r}}(C_{p^n})$ is an elementary abelian 2-group (or trivial). It remains to figure out their respective ranks.

In order to figure out which indecomposable $kC_{p^n}$-modules are relative endotrivial modules, we shall divide them into 3 parts of interest as pictured below:

$$\underbrace{M_1 \quad \cdots \quad M_{p^{n-1}}}_{part\ A}\ \underbrace{\qquad \cdots \qquad}_{part\ B}\ \underbrace{M_{p^n-p^{n-1}} \quad \cdots \quad M_{p^n}}_{part\,\Omega A}$$

*Part A :* is made of the indecomposable modules $M_i$ such that $1 \leq \dim_k(M_i) \leq p^{n-1}$. The subgroup $C_p$ of $C_{p^n}$ acts trivially on any such module, therefore these modules can be seen as inflationed from the subgroup $C_{p^{n-1}} \cong C_{p^n}/C_p$: $M_i = Inf_{C_{p^n-1}}^{C_{p^n}}(M_i)$ for all $1 \leq i \leq p^{n-1}$.

*Part B :* $p^{n-1} < \dim_k(M_i) < p^n - p^{n-1}$.

*Part ΩA :* $p^n - p^{n-1} \leq \dim_k(M_i) \leq p^n$. Furthermore, any such module is the Heller translate of some module in part A, or 0: $M_i \cong \Omega(M_{p^n-i})$ for all $p^n - p^{n-1} \leq i \leq p^n$. This symmetry shall allow us to reduce the problem to the case of modules with dimensions less or equal to $p^{n-1}$ and proceed by induction.

To start with, we show that there is no relative endotrivial $kC_{p^n}$-module in part B.

LEMMA 5.2.1.
*Let $C_{p^n}$ be a cyclic $p$-group with $n \geq 2$ and let $1 \leq r \leq n$ be an integer. Then, there is no indecomposable $kC_{p^n}$-module, whose $k$-dimension lies between $p^{n-1}$ and $p^n - p^{n-1}$, which is $M_{p^r}$-endotrivial.*

The main idea of the proof is based on the following restriction formula (see [Thé95, Exercise 28.3 (a)]):

$$M_i \downarrow_Z^{C_{p^n}} \cong sM_{a+1} \oplus (p^{n-1} - s)M_a \tag{1}$$

with $i = ap^{n-1} + s$, $0 \leq s < p^{n-1}$ and $0 \leq a < p$, for all $1 \leq i \leq p^n$.

PROOF. The case $p = 2$ is trivial since $2^{n-1} = 2^n - 2^{n-1}$, therefore, we may assume that $p$ is odd. Furthermore, an $M_{p^r}$-endotrivial module is necessarily $M_p$-endotrivial since $Proj(M_{p^r}) \subseteq Proj(M_p)$, hence we may also asume that $r = 1$. The indecomposable modules, candidates to be

$M_p$-endotrivial are the indecomposable modules of the form $M_{\beta p \pm 1}$ for some $1 \leq \beta \leq p^{n-1}$. We claim that, if $p^{n-1} < \beta p \pm 1 < p^n - p^{n-1}$, then $M_{\beta p \pm 1}$ is not $M_p$-endotrivial.

First note that the symmetry given by the Heller operator $\Omega$ allows us to consider only the case $\beta p + 1$. Then, the proof proceeds ab absurdo: let us assume that $M_{\beta p+1}$ is $M_p$-endotrivial and compute $\mathrm{End}_k(M_{\beta p+1})\downarrow_Z^{C_{p^n}}$. Since $p^{n-1} < \beta p + 1 < p^n - p^{n-1}$, we have $p^{n-2} \leq \beta < p^{n-1}$ and we can write $\beta := \gamma p^{n-2} + \sigma$ with $\gamma$ and $\sigma$ integers such that $1 \leq \gamma < p - 1$ and $0 \leq \sigma < p^{n-2}$. So that

$$\beta p + 1 = (\gamma p^{n-2} + \sigma)p + 1 = \gamma p^{n-1} + \sigma p + 1.$$

Now, $M_{\beta p+1}$ is $M_p$-endotrivial, thus

$$\mathrm{End}_k(M_{\beta p+1}) \cong k \oplus (M_p - proj)$$

and

$$\mathrm{End}_k(M_{\beta p+1}) \downarrow_Z^{C_{p^n}} \cong k \oplus (M_p \downarrow_Z^{C_{p^n}} -proj)\,.$$

Let us count the number of trivial summands on both sides of this isomorphism. On the right-hand side, there is one modulo $p$ by formula (1) (this easily follows from the fact that $M_p$-projective modules have dimension divisible by $p$ by 2.8.4). On the left-hand side formula (1) yields:

$$\begin{aligned}\mathrm{End}_k(M_{\beta p+1}) \downarrow_Z^{C_{p^n}} &\cong (M_{\beta p+1} \downarrow_Z^{C_{p^n}}) \otimes (M_{\beta p+1} \downarrow_Z^{C_{p^n}}) \\ &\cong ((\sigma p + 1)M_{\gamma+1} \oplus (p^{n-1} - \sigma p - 1)M_\gamma)^{\otimes 2} \\ &\cong (\sigma p + 1)^2 (M_{\gamma+1})^{\otimes 2} \oplus 2(p^{n-1} - \sigma p - 1)(\sigma p + 1)(M_{\gamma+1} \otimes M_\gamma) \\ &\quad \oplus (p^{n-1} - \sigma p - 1)^2 (M_\gamma)^{\otimes 2}\end{aligned}$$

Since $1 \leq \gamma < p - 1$, $p \nmid \dim_k M_\gamma$ and $p \nmid \dim_k M_{\gamma+1}$, but by 2.8.1 there is exactly one trivial summand $k$ in $M_\gamma \otimes M_\gamma$ as well as in $M_{\gamma+1} \otimes M_{\gamma+1}$ and, moreover, $k$ is not a direct summand of $M_{\gamma+1} \otimes M_\gamma$. Therefore, altogether there are $(\sigma p + 1)^2 + (p^{n-1} - \sigma p - 1)^2 \equiv 2 \mod p$ trivial summands in $\mathrm{End}_k(M_{\beta p+1}) \downarrow_Z^{C_{p^n}}$, which is a contradiction. Hence the result. □

For simplicity of notation, we shall, from now on, denote by $\Omega_{M_{p^s}}$ the class of the relative syzygy module $\Omega_{M_{p^s}}(k)$ in $T_{M_{p^r}}(C_{p^n})$ for each $r \leq s \leq n$ and simply use $\Omega := \Omega_{M_{p^n}}$. The classification theorem is the following.

Theorem 5.2.2.
*Let $G := C_{p^n}$ with $n \geq 1$ be a cyclic $p$-group and $M_{p^r}$ with $1 \leq r \leq n$ be an absolutely $p$-divisible $kC_{p^n}$-module.*

(a) *If $p$ is odd, or if $p = 2$ and $r \geq 2$, then*

$$T_{M_{p^r}}(C_{p^n}) = <\{\Omega_{M_{p^s}} \,|\, r \leq s \leq n\}> \cong \prod_{j=1}^{n-(r-1)} C_2\,.$$

(b) *If $p = 2$ and $r = 1$, then*

$$T_{M_2}(C_{2^n}) = <\{\Omega_{M_{p^s}} \,|\, 1 < s \leq n\}> \cong \prod_{j=1}^{n-1} C_2\,.$$

To begin with, the following lemma on the structure of $T_{M_{p^r}}(C_{p^n})$ shall enable us to prove the theorem by induction on the integer $n$.

Lemma 5.2.3.
*Assume $G = C_{p^n}$ with $n \geq 2$ and write $C_{p^{n-1}} = C_{p^n}/Z$. Then for every integer $1 \leq r \leq n$,*

$$T_{M_{p^r}}(C_{p^n}) = \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}})) \times <\Omega> \cong T_{M_{p^r}}(C_{p^{n-1}}) \times C_2\,.$$

PROOF. Inflation induces an injective group homomorphism

$$\mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}} : T_{M_{p^r}}(C_{p^{n-1}}) \hookrightarrow T_{M_{p^r}}(C_{p^n}) \,.$$

The indecomposable representatives for the classes in the image subgroup $\mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}}))$ are $kC_{p^n}$-modules whose $k$-dimension is less than or equal to $p^{n-1}$. Moreover, as inflation commutes with direct sums, it is clear that $M_i = \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(M_i)$ is $M_{p^r}$-endotrivial if and only if $M_i$, seen as a $kC_{p^{n-1}}$-module, is an $M_{p^r}$-endotrivial $kC_{p^{n-1}}$-module. As seen in Lemma 5.2.1 there is no indecomposable $M_{p^r}$-endotrivial module with $k$-dimension between $p^{n-1}$ and $p^n - p^{n-1}$. Furthermore, for all $p^n - p^{n-1} \leq i \leq p^n$, $M_i \cong \Omega(M_{p^n-i})$ is $M_{p^r}$-endotrivial if and only if $M_{p^n-i}$ is and for such a module in $T_{M_{p^r}}(C_{p^n})$ we have $[M_i] = [\Omega(M_{p^n-i})] = \Omega + [M_{p^n-i}]$. Whence the direct product

$$T_{M_{p^r}}(C_{p^n}) = \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}})) \times <\Omega> \;.$$

□

PROOF OF THEOREM 5.2.2.

(a) The proof proceeds by induction on $n$. First, the cyclic $p$-group of smallest order for which projectivity relative to an indecomposable module of dimension $p^r$ can be considered is $C_{p^r}$, in which case $Proj(M_{p^r}) = Proj$ as $M_{p^r} \cong kC_{p^r}$. Therefore, using the assumption that $p^r > 2$ we get $T_{M_{p^r}}(C_{p^r}) = T(C_{p^r}) =< \Omega >\cong C_2$ by the classification made in [Dad78b]. Then by the lemma and the induction hypothesis we get:

$$\begin{aligned} T_{M_{p^r}}(C_{p^n}) &= \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(T_{M_{p^r}}(C_{p^{n-1}})) \times <\Omega> \\ &= \mathrm{Inf}_{C_{p^{n-1}}}^{C_{p^n}}(<\{\Omega_{M_{p^s}} \,|\, r \leq s \leq n-1\}>) \times <\Omega> \\ &= <\{\Omega_{M_{p^s}} \,|\, r \leq s \leq n-1\}> \times <\Omega> = <\{\Omega_{M_{p^s}} \,|\, r \leq s \leq n\}> \\ &\cong \prod_{j=1}^{n-(r-1)} C_2 \end{aligned}$$

since by Corollary 3.6.2 any element of $T_{M_{p^r}}(C_{p^n})$ has order 2.

(b) If $p = 2$ and $r = 1$, then $T_{M_2}(C_2) = T(C_2) = \{[k]\}$ is trivial. Hence the missing factor $C_2$ in the product. Notice that $\Omega_{M_2}(k) \cong k$. Nonetheless, the set of generators is obtained in like manner as it was in part (a).

□

COROLLARY 5.2.4.
*Let $C_{p^n}$ with $n \geq 1$ be a cyclic $p$-group. Then the Dade group $D(C_{p^n}) \cong T_{M_p}(C_{p^n})$.*

PROOF. By the description of the Dade group for cyclic $p$-groups made in [Dad78b], any indecomposable $M_p$-endotrivial $kC_{p^n}$-module is an endo-permutation module, therefore there is an isomorphism $\psi : D(C_{p^n}) \longrightarrow T_{M_p}(C_{p^n}) : [M] \longmapsto [Cap(M)]$. (The injectivity of $\psi$ will be proved in general in Theorem 7.4.1 for an arbitrary $p$-group $P$ and a module $V$ dependending on $P$ such that $Proj(V) = Proj(M_p)$ when $P = C_{p^n}$.) □

REMARK 5.2.5.
Even though we showed that for cyclic $p$-groups projectivity relative to modules is reduced to projectivity relative to subgroups, we kept notation using modules rather than subgroups because it was more manageable firstly in the description of relatively projective modules, secondly in computations and thirdly in arguments involving inflation. Nevertheless in the next sections,

treating the case of groups having a cyclic Sylow $p$-subgroup, it will be easier to think in terms of subgroups. In this system of notation the groups of relatively endotrivial modules are generated as follows:

$$T_{M_{p^r}}(C_{p^n}) = T_{k\uparrow_{Z_{n-r}}^{C_{p^n}}}(C_{p^n}) = <\{\Omega_{k\uparrow_{Z_s}^{C_{p^n}}} \mid 0 \leq s \leq n-r\}> \text{ if } M_{p^r} \neq M_2\,,$$
$$T_{M_2}(C_{2^n}) = T_{k\uparrow_{Z_{n-1}}^{C_{p^n}}}(C_{2^n}) = <\{\Omega_{k\uparrow_{Z_s}^{C_{2^n}}} \mid 0 \leq s < n-1\}> \text{ otherwise.}$$

## Part B: Groups with cyclic Sylow $p$-subgroups

For the remainder of this chapter $G$ shall denote a finite group having a non-trivial cyclic Sylow $p$-subgroup $P \cong C_{p^n}$, $n \geq 1$. We shall sometimes refer, in all generality, to this kind of groups as the "cyclic case". Analogously to Part A, for all $0 \leq r \leq n$, $Z_r$ shall denote the unique cyclic subgroup of $P$ of order $p^r$. Now, before setting foot in detailed computations, it can be worth keeping in mind that the general situation we are about to work with is best sketched by the following chain of inclusions of subgroups of $G$ (for $0 \leq r \leq n-1$):

$$\begin{array}{c} G \\ | \\ N_G(Z_r) \\ | \\ N_G(Z_{r+1}) \\ | \\ N_G(P) \\ | \\ P \\ | \\ Z_{r+1} \\ | \\ Z_r \\ | \\ 1 \end{array}$$

### 5.3. Determination of the different types of $V$-projectivities

To begin with we show that the only types of relative projectivity that occur are again the projectivities relative to subgroups.

Proposition 5.3.1.
*Let $G$ be a finite group with a cyclic Sylow $p$-subgroup $P \cong C_{p^n}$ with $n \geq 1$.*

(a) *Let $V \in \mathsf{mod}(kG)$. Then $Proj(V) = Proj(k\uparrow_Q^G)$ for some subgroup $Q$ of $P$. In particular, $V$ is absolutely $p$-divisible if and only if $Q$ is a proper subgroup of $P$.*

(b) *There is a unique chain of proper inclusions of subcategories of relatively projective $kG$-modules:*

$$Proj \subsetneq Proj(k\uparrow_{Z_1}^G) \subsetneq Proj(k\uparrow_{Z_2}^G) \subsetneq \cdots \subsetneq Proj(k\uparrow_{Z_{n-1}}^G) \subsetneq Proj(k\uparrow_P^G) = \mathsf{mod}(kG)$$

PROOF.

(a) Recall that, by Lemma 2.6.5, subcategories of relatively projective modules are determined upon restriction to $P$ in the sense that for all $U, W \in \mathsf{mod}(kG)$, $Proj(U\downarrow^G_P) = Proj(W\downarrow^G_P)$ if and only if $Proj(U) = Proj(W)$. First of all, by 5.1.4 there exists a subgroup $Q$ of $P$ such that

$$Proj(V\downarrow^G_P) = Proj(k\uparrow^P_Q)\,.$$

($Q = P$ in case $V\downarrow^G_P$ is not absolutely $p$-divisible.) Therefore, by the above remark, in order to show that $Proj(V) = Proj(k\uparrow^G_Q)$, it is enough to check that $Proj(k\uparrow^G_Q\downarrow^G_P) = Proj(k\uparrow^P_Q)$. Indeed, applying the Mackey formula yields

$$k\uparrow^G_Q\downarrow^G_P \cong \bigoplus_{x\in[P\backslash G/Q]} k\uparrow^P_{{}^xQ\cap P}$$

where the subgroups ${}^xQ\cap P$ form a chain of subgroups of $Q = {}^1Q\cap P$, since $P$ is cyclic. Hence $Proj(k\uparrow^P_{{}^xQ\cap P}) \subseteq Proj(k\uparrow^P_Q)$ for all $x \in [P\backslash G/Q]$ so that

$$Proj(k\uparrow^G_Q\downarrow^G_P) = \bigoplus_{x\in[P\backslash G/Q]} Proj(k\uparrow^P_{{}^xQ\cap P}) = Proj(k\uparrow^P_Q)\,.$$

Now, the module $V$ is absolutely $p$-divisible if and only if $Proj(V) \neq \mathsf{mod}(kG) = Proj(k)$, if and only if $Proj(V\downarrow^G_P) = Proj(k\uparrow^P_Q) \neq Proj(k\downarrow^G_P) = Proj(k)$. Thus by the characterization given in 5.1.4, $V$ is absolutely $p$-divisible if and only if $Q$ is a proper subgroup of $P$.

(b) For $G = P$, we have shown in a previous section that there is a unique chain of inclusions of subcategories of relatively projective $kP$-modules given by

$$Proj \subsetneq Proj(k\uparrow^P_{Z_1}) \subsetneq Proj(k\uparrow^P_{Z_2}) \subsetneq \cdots \subsetneq Proj(k\uparrow^P_{Z_{n-1}}) \subsetneq Proj(k\uparrow^P_P) = \mathsf{mod}(kP)\,.$$

But we proved in (a) that $Proj(k\uparrow^G_Q\downarrow^G_P) = Proj(k\uparrow^P_Q)$ for all subgroup $Q \leq P$, therefore another application of Lemma 2.6.5 yields the result.

□

SCHOLIUM 5.3.2.
*For all $0 \leq r \leq n$, projectivity relative to the $p$-subgroup $Z_r$ of $G$ restricted to a subgroup $H$ of $G$ such that either $P \leq H$ or $Z_r \lneq H \leq P$ remains projectivity relative to $Z_r$, i.e.*

$$Proj(k\uparrow^G_{Z_r}\downarrow^G_H) = Proj(k\uparrow^H_{Z_r})\,.$$

PROOF. For, we showed in the proof of the proposition that $Proj(k\uparrow^G_{Z_r}\downarrow^G_P) = Proj(k\uparrow^P_{Z_r})$, but the argument remains true if $P$ is replaced with a subgroup $H$ as given above. □

Groups of relatively endotrivial modules are defined only for absolutely $p$-divisible modules $V$, in consequence and in view of Proposition 5.3.1, we shall assume for the remainder of the chapter that $V = k\uparrow^G_{Z_r}$ for some proper subgroup $Z_r$ of $P$. The remainder of the section is devoted to the determination of the structure of the groups $T_{k\uparrow^G_{Z_r}}(G)$ with $0 \leq r < n$.

REMARK 5.3.3.
By Proposition 5.3.1 the restriction of an absolutely $p$-divisible $kG$-module $V$ remains absolutely $p$-divisible whenever either $P \leq H$ or $Z_r \lneq H \leq P$. Indeed, we have $Proj(V) = Proj(k\uparrow^G_{Z_r})$ for some $Z_r \lneq P$ then, by Lemma 2.6.3 and the above remarks

$$Proj(V\downarrow^G_H) = Proj(k\uparrow^G_{Z_r}\downarrow^G_H) = Proj(k\uparrow^H_{Z_r}) \neq \mathsf{mod}kH\,.$$

Hence $V{\downarrow}_H^G$ is an absolutely $p$-divisible $kH$-module. In consequence, for all subgroups $H$ as above, the restriction maps $\mathrm{Res}^G_H$ from the group $T_{k\uparrow^G_{Z_r}}(G)$ are well-defined and all have the form

$$\mathrm{Res}^G_H : T_{k\uparrow^G_{Z_r}}(G) \longrightarrow T_{k\uparrow^H_{Z_r}}(H)\,.$$

## 5.4. The structure theorem

First we develop a few more properties of the restriction maps. We shall then use them to deduce the structure of the groups of relatively endotrivial modules $T_{k\uparrow^G_{Z_r}}(G)$ from our knowledge of the structure of $T_{k\uparrow^P_{Z_r}}(P)$.

In order to ease up notation we simply use the symbole $\Omega_V$ to denote the class $[\Omega_V(k)]$ in $T_V(G)$ and $k{\uparrow}_Q$ instead of $k{\uparrow}^H_Q$ in indices when it is clear to which subgroup $H \leq G$ induction goes. We avoid to use a simpler notation like $\Omega_Q$ because it has been widely used to denote the class of the ordinary syzygy $\Omega(k)$ in $\mathsf{mod}(kQ)$ in articles concerned with endotrivial and endo-permutation modules.

Lemma 5.4.1.
*Let $H \leq G$ be a subgroup such that either $P \leq H$ or $Z_r \lneq H \leq P$. Then, the restriction map* $\mathrm{Res}^G_H : T_{k\uparrow_{Z_r}}(G) \longrightarrow T_{k\uparrow_{Z_r}}(H)$ *has the following properties:*

(a) $\mathrm{Res}^G_H(\Omega_{k\uparrow_{Z_s}}) = \Omega_{k\uparrow_{Z_s}}$ *for all $Z_s \leq Z_r$ so that* $\langle\{\Omega_{k\uparrow_{Z_s}} \,|\, 0 \leq s \leq r\}\rangle \leq \mathrm{Im}(\mathrm{Res}^G_H)$ *;*

(b) *if $Z_r \lneq H \leq P$, then $\mathrm{Res}^G_H$ is surjective.*

Proof.

(a) By 2.12.2, $\Omega_{k\uparrow_{Z_s}}(k){\downarrow}^G_H \cong \Omega_{k\uparrow_{Z_s}}(k) \oplus (k{\uparrow}^H_{Z_s} -proj)$. Hence $\mathrm{Res}^G_H(\Omega_{k\uparrow_{Z_s}}) = \Omega_{k\uparrow_{Z_s}}$.

(b) Follows from (a) since by 5.2.2 the group $T_{k\uparrow_{Z_r}}(H)$ is generated by the set of all relative syzygies $\Omega_{k\uparrow_{Z_s}}(k)$ such that $Z_s \leq Z_r$.

□

Corollary 5.4.2.
*Let $P$ be a cyclic $p$-group and $Z_r$ a proper subgroup of $P$. Then, the restriction maps* $\mathrm{Res}^P_H : T_{k\uparrow^P_{Z_r}}(P) \longrightarrow T_{k\uparrow^H_{Z_r}}(H)$ *are isomorphisms for all $Z_r \lneq H \leq P$.*

Proof. By the previous lemma $\mathrm{Res}^P_H$ is surjective and, by Theorem 5.2.2 and Remark 5.3.3, $|T_{k\uparrow_{Z_r}}(P)| = |T_{k\uparrow_{Z_r}}(H)|$. □

Using the criterion described in Lemma 4.3.1, we can show that the group $T_{k\uparrow_{Z_r}}(G)$ is indeed entirely determined by restriction to $N_G(Z_{r+1})$.

Proposition 5.4.3.
*Let $G$ be a finite group with a non-trivial cyclic Sylow $p$-subgroup $P$ and $Z_r$ be a proper subgroup of $P$. Then, the restriction map*

$$\mathrm{Res}^G_{N_G(Z_{r+1})} : T_{k\uparrow_{Z_r}}(G) \longrightarrow T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$$

*is an isomorphism, with inverse map induced by Green correspondence or alternatively by induction:*

$$\begin{aligned} T_{k\uparrow_{Z_r}}(G) &= \{[\Gamma(M)] \,|\, M \text{ is an indecomposable } k\uparrow_{Z_r}\text{-endotrivial } kN_G(Z_{r+1})\text{-module}\} \\ &= \{[M\uparrow^G_{N_G(Z_{r+1})}] \mid [M] \in T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))\} \end{aligned}$$

Proof. The isomorphism and both the descriptions of $T_{k\uparrow_{Z_r}}(G)$ using Green correspondence and induction follow from Lemma 4.3.1. In fact, in order to invoke 4.3.1 it suffices to check that

$$Proj(k\uparrow^{N_G(Z_{r+1})}_{Z_r}) \supseteq Proj(\mathcal{Y}),$$

where $\mathcal{Y} = \{\, {}^gP \cap N_G(Z_{r+1}) \,|\, g \in G \setminus N_G(Z_{r+1})\}$. For all $g \in G \setminus N_G(Z_{r+1})$, the subgroup ${}^gP \cap N_G(Z_{r+1})$ is a $p$-subgroup of ${}^gP$, hence of the form ${}^gZ_l$ for some $l \in \mathbb{N}_n$ since $P \cong C_{p^n}$ is cyclic. Besides, ${}^gZ_l \leq N_G(Z_{r+1})$ as well, thus contained in some Sylow $p$-subgroup of $N_G(Z_{r+1})$, say ${}^hP$ with $h \in N_G(Z_{r+1})$, so that by uniqueness of the subgroup of order $p^l$ in ${}^hP$, we have ${}^gZ_l = {}^hZ_l$. Hence $h^{-1}g$ normalizes $Z_l$ and

$$g \in hN_G(Z_l) \subseteq N_G(Z_{r+1})N_G(Z_l) \gneq N_G(Z_{r+1})$$

since $g$ does not normalize $Z_{r+1}$. This forces $N_G(Z_l)$ to contain strictly $N_G(Z_{r+1})$, because the subgroups $N_G(Z_i)$ are totally ordered by inclusion, hence $Z_l \leq Z_r$. As a consequence,

$$Proj({}^gP \cap N_G(Z_{r+1})) = Proj(Z_l) \subseteq Proj(k\uparrow^{N_G(Z_{r+1})}_{Z_r})$$

and as required:

$$Proj(\mathcal{Y}) = \bigoplus_{g \in G \setminus N_G(Z_{r+1})} Proj({}^gP \cap N_G(Z_{r+1})) \subseteq Proj(k\uparrow^{N_G(Z_{r+1})}_{Z_r})$$

□

In view of the proposition we can restrict our attention to the groups $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$. First of all, computing the kernel of the restriction map $\mathrm{Res}^{N_G(Z_{r+1})}_P : T_{k\uparrow_{Z_r}}(N_G(Z_{r+1})) \longrightarrow T_{k\uparrow_{Z_r}}(P)$ provides us with a set of generators.

Lemma 5.4.4.

(a) *There is an exact sequence*

$$0 \longrightarrow X(N_G(Z_{r+1})) \hookrightarrow T_{k\uparrow_{Z_r}}(N_G(Z_{r+1})) \xrightarrow{\mathrm{Res}^{N_G(Z_{r+1})}_P} T_{k\uparrow_{Z_r}}(P) \longrightarrow 0\,.$$

(b) *The group $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ is a finite abelian group generated by $X(N_G(Z_{r+1}))$ and the $r+1$ relative syzygy modules $\Omega = \Omega_{k\uparrow_1}, \Omega_{k\uparrow_Z}, \ldots, \Omega_{k\uparrow_{Z_r}}$.*

Proof.

(a) The map $\mathrm{Res}^{N_G(Z_{r+1})}_P$ is surjective by 5.4.1. In addition, $V := k\uparrow^{N_G(Z_{r+1})}_{Z_r}$ fulfills the hypotheses of Lemma 4.4.1. Indeed, recall from the proof of Proposition 5.3.1 that

$$k\uparrow^{N_G(Z_{r+1})}_{Z_r}\downarrow^{N_G(Z_{r+1})}_P \cong \bigoplus_{g \in [P \setminus N_G(Z_{r+1})/Z_r]} k\uparrow^P_{{}^gZ_r \cap P}$$

where each indecomposable summand $k\uparrow^P_{{}^gZ_r \cap P}$ has a vertex equal to ${}^gZ_r \cap P \leq Z_r$, which is strictly contained in ${}^xP \cap P$ for all $x \in N_G(Z_{r+1}) \setminus N_G(P)$. Indeed, any such $x$ normalizes $Z_{r+1}$, thus $Z_r \lneq Z_{r+1} \leq {}^xP \cap P$. Therefore 4.4.1 yields

$$\ker(\mathrm{Res}^{N_G(Z_{r+1})}_P) = X(N_G(Z_{r+1}))\,.$$

(b) By Theorem 5.2.2, $T_{k\uparrow_{Z_r}}(P) = \langle\{\Omega_{k\uparrow_{Z_s}} \,|\, 0 \leq s \leq r\}\rangle$. Now, by 5.4.1, $\Omega_{k\uparrow_{Z_s}}$ is a preimage by $\mathrm{Res}_P^{N_G(Z_{r+1})}$ for the generator $\Omega_{k\uparrow_{Z_s}}$ of $T_{k\uparrow_{Z_r}}(P)$ for all $0 \leq s \leq r$. Thus $X(N_G(Z_{r+1})) \cup \{\Omega_{k\uparrow_{Z_s}} \,|\, 0 \leq s \leq r\}$ is a set of generators for $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$. Lastly, the finiteness of $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ follows from both that of $T_{k\uparrow_{Z_r}}(N_G(P))$ and of $X(N_H(Z_{r+1}))$.

□

Our main purpose is to work out the structure of the groups $T_{k\uparrow_{Z_r}^G}(G)$. At this stage, we know that there is a group isomorphism $T_{k\uparrow_{Z_r}}(G) \cong T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$. Besides, we have a surjective group morphism induced by restriction

$$\mathrm{Res}_P^{N_G(Z_{r+1})} : T_{k\uparrow_{Z_r}}(N_G(Z_{r+1})) \twoheadrightarrow T_{k\uparrow_{Z_r}}(P)$$

which provides us with a set of generators for the group $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ made up of the kernel $\ker(\mathrm{Res}_P^{N_G(Z_{r+1})}) = X(N_G(Z_{r+1}))$ and the classes of the $r+1$ relative syzygy modules of the trivial module: $\Omega, \Omega_{k\uparrow_{Z_1}}, \ldots, \Omega_{k\uparrow_{Z_r}}$. The latter being preimages for the generators $\Omega_{k\uparrow_{Z_s}^P}, 0 \leq s \leq r$, of $T_{k\uparrow_{Z_r}}(P)$ which all have order 2, it remains to identify $2\Omega_{k\uparrow_{Z_s}}$, for all $0 \leq s \leq r$, with an element of the kernel, that is a one-dimensional representation of $N_G(Z_{r+1})$.

These identifications will follow from an induction argument and use the structure of the group of endotrivial modules $T(G)$ described in [MT07, Thm. 3.2]. This result makes use of a distinguished element of $X(N_G(Z))$, which we need to describe and understand before use.

For $Z$ the unique subgroup of $P$ of order $p$, let $H := N_G(Z)$ be its normaliser in $G$. As $H$ acts by conjugation on $Z$, the quotient $H/C_G(Z)$ embeds as a subgroup of $\mathrm{Aut}(Z) \cong (\mathbb{Z}/p\mathbb{Z})^\times$, thus given $c \in H$, for all $u \in Z$ we have

$${}^{c}u = u^{\nu(c)} \text{ for some } \nu(c) \in (\mathbb{Z}/p\mathbb{Z})^\times$$

where in addition $\nu(c)$ can be considered as an element of $k^\times$ via the canonical embedding $\mathbb{Z}/p\mathbb{Z} \hookrightarrow k$. In consequence, the composition $H \longrightarrow H/C_G(Z) \longrightarrow \mathrm{Aut}(Z) \cong \mathbb{Z}/p\mathbb{Z}^\times \longrightarrow k^\times$ defines a linear character of $H$. For simplicity, $\nu$ is then identified with a one-dimensional module in $X(H)$.
In fact, a similar construction can be applied to any subgroup of $G$ which normalizes $Z$. Furthermore, a Frattini argument applied to $H$ and its normal subgroup $C_G(Z)$ yields the decomposition $H = N_H(P)C_G(Z) = N_G(P)C_G(Z)$, therefore as $C_G(Z)$ acts trivially on $Z$, $\nu$ is entirely defined by its value on $N_G(P)$. In other words, $\nu$ can be viewed as a $kN_G(P)$-module which can be extended in a $k\widetilde{H}$-module for all subgroup $\widetilde{H}$ such that $H \geq \widetilde{H} \geq N_G(P)$, and for ease of notation, we also denote these modules by $\nu$, so that:

$$\mathrm{Res}_{\widetilde{H}_2}^{\widetilde{H}_1}(\nu) = \nu \quad \text{whenever } H \geq \widetilde{H}_1 \geq \widetilde{H}_2 \geq N_G(P)\,.$$

In particular, we are interested in the subgroup $H_{r+1} := N_G(Z_{r+1})$. Our aim will be to apply an induction argument to its quotient $H_{r+1}/Z_r$. In this respect, note that $P/Z_r$ is a cyclic Sylow $p$-subgroup of $H_{r+1}/Z_r$, $Z_{r+1}/Z_r$ its unique cyclic $p$-subgroup of order $p$, moreover $H_{r+1}/Z_r = N_{H_{r+1}/Z_r}(Z_{r+1}/Z_r)$ and $N_{H_{r+1}/Z_r}(P/Z_r) = N_G(P)/Z_r$. Moreover, a Frattini argument yields more precisely

$$H_{r+1}/Z_r = N_G(P)/Z_r \cdot C_{H_{r+1}/Z_r}(Z_{r+1}/Z_r)\,.$$

Therefore, there is also a corresponding $kN_G(P)/Z_r$-module $\nu = \nu_{N_G(P)/Z_r}$ which extends to $H_{r+1}/Z_r$. Finally, the following technical result computes the inflation of $\nu_{N_G(P)/Z_r}$ to a $kN_G(P)$-module.

Lemma 5.4.5.
*With the notation above we have* $\mathrm{Inf}_{N_G(P)/Z_r}^{N_G(P)}(\nu_{N_G(P)/Z_r}) = \nu_{N_G(P)}$*, that is, by abuse of notation,* $\mathrm{Inf}_{N_G(P)/Z_r}^{N_G(P)}(\nu) = \nu$.

Proof. Write $P :=< u \,|\, u^{p^n} = 1 >$. Then $Z =< u^{p^{n-1}} >$, $Z_r =< u^{p^{n-r}} >$ and thus $Z_{r+1}/Z_r =< u^{p^{n-r-1}} Z_r >$. Let $d \in N_G(P)$, we have ${}^d u = u^j$ for some integer $j$ such that $1 \leq j \leq p^n$. Then ${}^d(u^x) = (u^x)^j$ for all $1 \leq x \leq n$. Therefore ${}^d(u^{p^{n-1}}) = (u^{p^{n-1}})^j$ so that $\nu(d) \equiv j \mod p$. Likewise ${}^{dZ_r}(u^{p^{n-r-1}} Z_r) = (({}^d u) Z_r)^{p^{n-r-1}} = (u^j Z_r)^{p^{n-r-1}} = (u^{p^{n-r-1}} Z_r)^j$, hence $\mathrm{Inf}_{N_G(P)/Z_r}^{N_G(P)}(\nu)(d) \equiv j \mod p$. Hence the result. □

Theorem 5.4.6.
*Let $G$ be a finite group with a non trivial cyclic Sylow $p$-subgroup $P \cong C_{p^n}$. For all $0 \leq r \lneq n$, let $Z_r$ be the unique proper $p$-subgroup of $P$ of order $p^r$ and $H_{r+1}$ be its normaliser in $G$. Let $\nu$ be as above. Then*

$$\begin{aligned} T_{k\uparrow_{Z_r}}(H_{r+1}) &= \langle\, X(H_{r+1}), \{\Omega_{k\uparrow_{Z_s}} \,|\, 0 \leq s \leq r\}\,\rangle \\ &\cong \Big(X(H_{r+1}) \oplus \langle\, \{\Omega_{k\uparrow_{Z_s}} \,|\, 0 \leq s \leq r\}\,\rangle\Big) \Big/ \Big([\nu] - 2\Omega_{k\uparrow_{Z_s}}, 0 \leq s \leq r\Big)\,. \end{aligned}$$

Proof. We need to identify each class $2\Omega_{k\uparrow_{Z_s}}$ with an element of $X(H_{r+1})$. We claim that $2\Omega_{k\uparrow_{Z_s}} = [\nu]$ for all $0 \leq s \leq r$. The proof proceeds by induction on $r$. The case $r = 0$ holds by [MT07, Thm 3.2], because projectivity relative to $Z_0 = \{1\}$ is ordinary projectivity, thus $r = 0$ is the ordinary endotrivial case. So we may assume that $r > 0$ and as $T_{k\uparrow_{Z_{r-1}}}(H_{r+1})$ can be seen as a subgroup of $T_{k\uparrow_{Z_r}}(H_{r+1})$, by induction hypothesis, we may assume that the relations $2\Omega_{k\uparrow_{Z_s}} = [\nu]$ hold for all $0 \leq s \leq r-1$. Thus it remains to show that $2\Omega_{k\uparrow_{Z_r}} = [\nu]$.

Factoring out $H_{r+1} = N_G(Z_{r+1})$ by its normal subgroup $Z_r$ enables us to apply the induction hypothesis again to the group

$$T_{k\uparrow_{Z_r/Z_r}}(H_{r+1}/Z_r) = T(H_{r+1}/Z_r)\,,$$

for which [MT07, Thm. 3.2] provides the relation

$$2\Omega = [\nu] \text{ in } T(H_{r+1}/Z_r), \text{ that is, } 2\Omega_{k\uparrow_{Z_r/Z_r}} = [\nu] \text{ in } T_{k\uparrow_{Z_r/Z_r}}(H_{r+1}/Z_r)\,.$$

The following commutative square yields the desired relation for $T_{k\uparrow_{Z_r}}(H_{r+1})$:

$$\begin{array}{ccc} T_{k\uparrow_{Z_r}}(H_{r+1}) & \xhookrightarrow{\ \mathrm{Res}\ } & T_{k\uparrow_{Z_r}}(N_G(P)) \\ \uparrow{\scriptstyle \mathrm{Inf}_{H_{r+1}/Z_r}^{H_{r+1}}} & \circlearrowleft & \uparrow{\scriptstyle \mathrm{Inf}_{N_G(P)/Z_r}^{N_G(P)}} \\ T_{k\uparrow_{Z_r/Z_r}}(H_{r+1}/Z_r) & \xhookrightarrow[\ \mathrm{Res}\ ]{} & T_{k\uparrow_{Z_r/Z_r}}(N_G(P)/Z_r) \end{array}$$

By Lemma 2.12.2, $\mathrm{Inf}_{H_{r+1}/Z_r}^{H_{r+1}}(\Omega_{k\uparrow_{Z_r/Z_r}}) = \Omega_{k\uparrow_{Z_r}}$, therefore, inflationing the above relation to $T_{k\uparrow_{Z_r}}(H_{r+1})$ yields

$$2\Omega_{k\uparrow_{Z_r}} = [\mathrm{Inf}_{H_{r+1}/Z_r}^{H_{r+1}}(\nu)] \text{ in } T_{k\uparrow_{Z_r}}(H_{r+1})\,.$$

By the previous lemma $[\mathrm{Inf}_{N_G(P)/Z_r}^{N_G(P)}(\nu)] = [\nu]$, so that the result follows from the injectivity of $\mathrm{Res}_{N_G(P)}^{H_{r+1}}$ (Lemma 4.2.1). □

Remark 5.4.7.
Since $2\Omega_{k\uparrow_{Z_s}} = [\nu] = 2\Omega$ for all $0 \leq s \leq r$ the generators $\Omega_{k\uparrow_Z}, \ldots, \Omega_{k\uparrow_{Z_r}}$ can be replaced with

the generators $\Omega - \Omega_{k\uparrow_Z}, \ldots, \Omega - \Omega_{k\uparrow_{Z_r}}$, all of which have order 2. Thus the abelian group $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ contains a direct sum of $r$ copies of $\mathbb{Z}/2\mathbb{Z}$.

Finally, using the isomorphism of Proposition 5.4.3, the description by generators and relations of $T_{k\uparrow_{Z_r}}(N_G(Z_{r+1}))$ extends to $T_{k\uparrow_{Z_r}}(G)$ which is a finite abelian group generated by the relative syzygy modules $\Omega = \Omega_{k\uparrow_1^G}, \Omega_{k\uparrow_Z^G}, \ldots, \Omega_{k\uparrow_{Z_r}^G}$ and an isomorphic copy of $X(N_G(Z_{r+1}))$, made up of all the classes of the Green correspondents of the one-dimensional $kN_G(Z_{r+1})$-modules, with the relations $2\Omega_{k\uparrow_{Z_s}^G} = [\Gamma(\nu)]$ for all $0 \leq s \leq r$.

# CHAPTER 6

# About $p$-Nilpotent Groups

The aim of this short chapter is mainly to generalise part of the results of [CMT11a, Sect. 3] concerning the group of endotrivial modules of a $p$-nilpotent group. Throughout the chapter $G$ is a $p$-nilpotent group, that is, $G$ has a normal $p$-complement, or in other words, there is a normal subgroup $N$ of $G$ of order coprime to $p$ such that $G/N$ is a $p$-group. Equivalently, $G$ is a semidirect product $G = N \rtimes P$, with $N = O_{p'}(G)$ and $P$ a Sylow $p$-subgroup of $G$. Moreover $G = NP$, $N \cap P = \{1\}$ and we shall denote by $\varphi : P = P/N \cap P \xrightarrow{\cong} NP/N = G/N$ the isomorphism provided by the second isomorphism Theorem. It follows that any $kP$-module $L$ can be seen as a $kG$-module via $\varphi$ followed by inflation from $G/N$ to $G$, which we shall still denote by $L$ instead of $\mathrm{Inf}_{G/N}^{G} \circ \mathrm{Iso}(\varphi)(L)$. We shall keep this notation throughout the chapter and the following ones.

## 6.1. Preliminaries

To begin with, relative projectivity for $p$-nilpotent groups is well-behaved with respect to group operations in the following sense:

Lemma 6.1.1.
*Let $G = N \rtimes P$ be a $p$-nilpotent group.*

(a) $\mathrm{Res}_{P}^{G} \circ \mathrm{Inf}_{G/N}^{G} \circ \mathrm{Iso}(\varphi)(M) \cong M$ *for every $kP$-module $M$.*
(b) $Proj(V) = Proj(\mathrm{Inf}_{G/N}^{G} \circ \mathrm{Iso}(\varphi) \circ \mathrm{Res}_{P}^{G}(V))$ *for every $kG$-module $V$.*

Proof.

(a) Let $L$ be a $k[G/N]$-module. Using formula (f) of Proposition 1.1.2 with $H = P$ we get:
$$\mathrm{Res}_{P}^{G} \circ \mathrm{Inf}_{G/N}^{G}(L) \cong \mathrm{Inf}_{P/P\cap N}^{P} \circ \mathrm{Iso}(\varphi^{-1}) \circ \mathrm{Res}_{PN/N}^{G/N}(L)$$
Since $G$ is $p$-nilpotent this formula reads $\mathrm{Res}_{P}^{G} \circ \mathrm{Inf}_{G/N}^{G}(L) \cong \mathrm{Iso}(\varphi^{-1})(L)$ . Replacing $L$ with $\mathrm{Iso}(\varphi)(M)$ where $M$ is a $kP$-module yields the result.
(b) It follows from part (a) that
$$\mathrm{Res}_{P}^{G} \circ \mathrm{Inf}_{G/N}^{G} \circ \mathrm{Iso}(\varphi) \circ \mathrm{Res}_{P}^{G}(V) = \mathrm{Res}_{P}^{G}(V)\,.$$

Therefore $Proj(\mathrm{Res}^G_P \circ \mathrm{Inf}^G_{G/N} \circ \mathrm{Iso}(\varphi) \circ \mathrm{Res}^G_P(V)) = Proj(\mathrm{Res}^G_P(V))$ and Lemma 2.6.5 yields

$$Proj(V) = Proj(\mathrm{Inf}^G_{G/N} \circ \mathrm{Iso}(\varphi) \circ \mathrm{Res}^G_P(V))\,.$$

□

Most of the results in this chapter rely upon the following facts concerning blocks of $p$-nilpotent groups, recalled and proved in [CMT11a].

PROPOSITION 6.1.2 ([CMT11a], Prop. 3.1 ).
*Let $G = N \rtimes P$ be a $p$-nilpotent group. Let $Z$ be a simple $kN$-module, let $e$ be the central primitive idempotent of $kN$ corresponding to $Z$, and $H$ be the inertial subgroup of $Z$. Then:*

(a) *The idempotent $e$ is a block idempotent of $kH$ and $f := \sum_{g\in[G/H]} geg^{-1}$ is a block idempotent of $kG$. Moreover induction induces a Morita equivalence between $\mathsf{mod}(kHe)$ and $\mathsf{mod}(kGf)$.*
(b) *If $H \lneq G$, then every module in $\mathsf{mod}(kGf)$ has dimension divisible by $p$.*
(c) *If $H = G$, then there is a $kG$-module structure on $Z$, still written $Z$, that extends the initial $kN$-module structure, and this extension is unique. Moreover, $Z\downarrow^G_P$ is an endo-permutation module and $Z$ has vertex $P$.*
(d) *If $H = G$, then $Z$ is the unique simple module in the block $kGe$ and, for any $kGe$-module $Y$, the restriction $Y\downarrow^G_N$ is isomorphic to a direct sum of copies of the simple module $Z\downarrow^G_N$.*
(e) *If $H = G$, then $\mathsf{mod}(kGe)$ is Morita equivalent to $\mathsf{mod}(kP)$ via the functor*

$$\begin{array}{rccc}\Phi: & \mathsf{mod}(kP) & \longrightarrow & \mathsf{mod}(kGe)\\ & X & \longmapsto & Z\otimes X\end{array}$$

*where $X$ is seen as a $kG$-module via $\varphi$ and inflation, and where $G$ acts diagonally on the tensor product.*

### 6.2. First Properties of the groups of relative endotrivial modules

LEMMA 6.2.1 (Generalisation of [CMT11a], Lem. 3.2).
*Let $G = N \rtimes P$ be a $p$-nilpotent group, let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module, and let $M \in \mathsf{mod}(kG)$ be an indecomposable $V$-endotrivial module. Let $Z$ be the unique simple module in the block containing $M$. Then $Z\downarrow^G_N$ is a simple $kN$-module.*

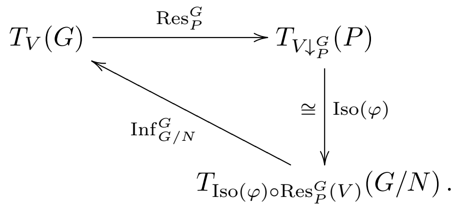
PROOF. Let $W$ be a simple summand $Z\downarrow^G_N$ and let $H$ be its inertial subgroup. Since the block containing $Z$ also contains the $V$-endotrivial $M$, which has $k$-dimension coprime to $p$, part (b) of Lemma 6.1.2 implies that $H = G$, so that by part (c) of the same lemma, $W$ extends uniquely to $G$. This forces $W = Z\downarrow^G_N$. □

Next we state two first results about the structure of the groups $T_V(N \rtimes P)$. They generalise [CMT11a, Thm. 3.3, Cor. 3.4]. From now on, if $V$ is an absolutely $p$-divisible $kG$-module, we let $K_V(G)$ denote the kernel of the restriction map $\mathrm{Res}^G_P : T_V(G) \longrightarrow T_{V\downarrow^G_P}(P)$.

THEOREM 6.2.2.
*Let $G = N \rtimes P$ be a $p$-nilpotent group and $V$ be an absolutely $p$-divisible $kG$-module. Then, the restriction map $\mathrm{Res}^G_P : T_V(G) \longrightarrow T_{V\downarrow^G_P}(P)$ is split surjective. A section is provided by the map $\mathrm{Inf}^G_{G/N} \circ \mathrm{Iso}(\varphi)$. In other words, there is an isomorphism*

$$T_V(G) \cong K_V(G) \oplus T_{V\downarrow^G_P}(P)\,.$$

Proof. By part (b) of Lemma 6.1.1 there is a diagram




furthermore, part (a) of Proposition 6.1.1 implies that the homomorphism $\mathrm{Inf}^G_{G/N}\circ\mathrm{Iso}(\varphi)$ is a section for the restriction map $\mathrm{Res}^G_P$. □

Proposition 6.2.3.
*Let $G = N \rtimes P$ be a $p$-nilpotent group and let $V$ be an absolutely $p$-divisible $kG$-module. Let $M$ be an indecomposable $V$-endotrivial $kG$-module. Let $Z$ be the unique simple module in the block $B$ containing $M$. Then:*

(a) *$M$ admits a decomposition $M \cong Z \otimes U$, where $U$ is an indecomposable $kP$-module. In particular, $Z$ is $V$-endotrivial and $U$ is $V\downarrow^G_P$-endotrivial.*
(b) *$[M] \in K_V(G)$ if and only if $M$ has a decomposition $M \cong Z \otimes S^*$, where $S \in \mathsf{mod}(kP)$ is a source for $Z$.*

Proof.
(a) Because the functor $\Phi : \mathsf{mod}(kP) \longrightarrow \mathsf{mod}B$ defines a Morita equivalence, there exists an indecomposable $kP$-module $U$ such that $M = Z \otimes U$ (with $U$ seen as a $kG$-module). By Lemma 3.3.3, both $Z$ and $U$ are $V$-endotrivial modules, so that $Z\downarrow^G_P$ and $U$, regarded as a $kP$-module, are both $V\downarrow^G_P$-endotrivial ($U\downarrow^G_P = U$ by Lemma 6.1.1).
(b) Let $S \in \mathsf{mod}(kP)$ be a source for $Z$. By Lemma 3.5.1 $S$ is also $V\downarrow^G_P$-endotrivial and we can write $Z\downarrow^G_P = S \oplus (V\downarrow^G_P -proj)$. Then

$$\begin{aligned} M\downarrow^G_P &= (Z\otimes U)\downarrow^G_P \cong Z\downarrow^G_P \otimes U \\ &\cong (S \oplus (V\downarrow^G_P -proj)) \otimes U \cong (S\otimes U) \oplus (V\downarrow^G_P -proj) \end{aligned}$$

Now, $[M] \in K_V(G)$ if and only if $[M\downarrow^G_P] = [k]$, if and only if $k \,|\, S \otimes U$. But, by the Benson-Carlson Theorem (2.8.1), this happens if and only if $U \cong S^*$. (For $U$ being $V\downarrow^G_P$-endotrivial, its $k$-dimension is coprime to $p$.)

□

Corollary 6.2.4.
*Let $G = N \rtimes P$ be a $p$-nilpotent group with $N$ abelian. Let $V$ be an absolutely $p$-divisible $kG$-module. Then $K_V(G) = X(G)$.*

Proof. The inclusion $K_V(G) \supseteq X(G)$ is always true. We need to show that $K_V(G) \subseteq X(G)$. Let $[M] \in K_V(G)$ with $M$ indecomposable. Then by Proposition 6.2.3, $M$ has a decomposition $M \cong Z \otimes S^*$, where $Z$ is the unique simple module in the block $B$ containing $M$ and $S \in \mathsf{mod}(kP)$ is a source for $Z$. In addition $Z$ is $V$-endotrivial, so that $\dim_k(Z)$ is coprime to $p$, thus its inertial subgroup is $G$ by part (b) of Lemma 6.1.2. Moreover by part (c) and (d) of the same lemma $Z\downarrow^G_N$ is simple, hence one-dimensional since $N$ is abelian. It follows that $S$ is trivial and that $M \cong Z \in X(G)$. □

# CHAPTER 7

# The Dade Group of a Finite Group

We come in this chapter to a chief reason of interest in relative endotrivial modules: it provides a way to define a group structure on collections of representations of an arbitrary finite group $G$ which gives a natural generalisation for the Dade Group $D(P)$ of a finite $p$-group $P$. In particular, the purpose of the chapter is to shed new light on the class of endo-$p$-permutation modules studied in [Urf06] and [Urf07]. We shall use endotrivial modules relatively to a well-chosen family of subgroups to provide a definition of the Dade group of a finite group $G$ in general, which coincides with the Dade group when $G$ is a $p$-group. To achieve this goal, our first task is to understand better the links between permutation, $p$-permutation modules and relative projectivity to modules and subgroups. Likewise, it is necessary to understand endo-permutation and endo-$p$-permutation modules in terms of relatively endotrivial modules.

Unless otherwise specified, throughout the chapter $G$ shall denote a finite group, $P$ a Sylow $p$-subgroup of $G$ and $N_G(P)$ its normaliser in $G$. Furthermore, we assume that the reader is acquainted with endo-permutation modules, the Dade group and endo-$p$-permutation modules. We refer to the first chapter for basic definitions and further references.

### 7.1. Preliminaries on projectivity relative to families of subgroups

A natural module to use to manufacture relative endotrivial modules would be the module $V(\mathcal{H}) = \bigoplus_{H\in\mathcal{H}} k{\uparrow}_H^G$ associated with the family $\mathcal{H} := \{H \leq G\}$ of *all* subgroups of a given group $G$. However, this module is not absolutely $p$-divisible and in consequence there is no well-defined group of $V(\mathcal{H})$-endotrivial modules in the sense of definition 3.6. Hence arises the question of finding the largest possible family $\mathcal{F}$ of subgroups of $G$ which would provide us with a well-defined group structure $T_{V(\mathcal{F})}(G)$. Remembering Remark 2.5.3, it becomes easy to see that the right family to look at is the family of all proper subgroups of a given Sylow $p$-subgroup $P$ of the group $G$.

Lemma 7.1.1.
*Let $P$ be a Sylow $p$-subgroup of $G$. Let $H$ be a subgroup of $G$ and $Q$ be a Sylow $p$-subgroup of $H$. Let $\mathcal{H} = \{X \leq G\}$. Then,*

(a) $Proj(k{\uparrow}_H^G) = Proj(k{\uparrow}_Q^G)$;
(b) *The permutation module $k{\uparrow}_H^G$ is absolutely $p$-divisible if and only if $Q \lneq_G P$.*

(c) *An indecomposable p-permutation module $L \in \mathsf{mod}(kG)$ is absolutely p-divisible if and only if it has a vertex $R \lneq_G P$.*
(d) $Proj(\mathcal{H}) = Proj(\{R \leq P\}) = \mathsf{mod}(kG)$.

Proof.

(a) Since $Q$ is a Sylow $p$-subgroup of $H$, $k \in Proj(k\uparrow_Q^H)$, so that by Lemma 2.6.1

$$k\uparrow_H^G \in Proj(k\uparrow_Q^H\uparrow_H^G) = Proj(k\uparrow_Q^G)$$

and therefore $Proj(k\uparrow_H^G) \subseteq Proj(k\uparrow_Q^G)$. Moreover, $k\uparrow_Q^H \in \mathsf{mod}(kH) = Proj(k)$, hence, again by Lemma 2.6.1,

$$k\uparrow_Q^G = k\uparrow_Q^H\uparrow_H^G \in Proj(k\uparrow_H^G)\,,$$

which proves the reverse inclusion $Proj(k\uparrow_Q^G) \subseteq Proj(k\uparrow_H^G)$.

(b) By (a), $Proj(k\uparrow_H^G) = Proj(k\uparrow_Q^G)$. Therefore by Proposition 2.8.3 the module $k\uparrow_H^G$ is absolutely $p$-divisible if and only if $Proj(k\uparrow_Q^G) \neq \mathsf{mod}(kG)$, that is if and only if $Q \lneq_G P$. (This is well-known from the theory of vertices and sources.)

(c) The sufficient condition is a particular case of Lemma 3.5.1. The necessary condition is easier proven by contraposition. Indeed, if $R =_G P$, the trivial $kP$-module is a source of $L$, therefore $k \,|\, L\downarrow_P^G$. In consequence $\mathsf{mod}(kP) = Proj(k) \subseteq Proj(L\downarrow_P^G)$ and so $Proj(L\downarrow_P^G) = \mathsf{mod}(kP)$. Therefore $L\downarrow_P^G$ is not absolutely $p$-divisble and by 2.9.3 neither is $L$.

(d) First by the omnibus properties of relative projectivity we have:

$$Proj(\mathcal{H}) = Proj(\bigoplus_{X\in\mathcal{H}} k\uparrow_X^G) = \bigoplus_{X\in\mathcal{H}} Proj(k\uparrow_X^G) = \bigoplus_{X\in\mathcal{H}} Proj(k\uparrow_{Q(X)}^G)$$

where for all $X \in \mathcal{H}$, $Q(X)$ is a Sylow $p$-subgroup of $X$, and where the penultimate equality follows from part (a). In consequence $Proj(\mathcal{H}) = \mathsf{mod}(kG)$ by Proposition 2.8.3 because $Proj(k\uparrow_P^G) = \mathsf{mod}(kG)$ by (b). Moreover $Proj(\mathcal{H}) = Proj(\{R \leq P\})$ by Remark 2.5.3.

□

As a consequence we can restrict our attention to the family of subgroups of a fixed Sylow $p$-subgroup $P$ of $G$. According to the above proof, in order to obtain an absolutely $p$-divisible family of subgroups, it is necessary to remove $P$ itself from this family $\{Q \leq P\}$.

Notation.
For $G$ a finite group, fix a Sylow $p$-subgroup $P$. Then set $\mathcal{F}_G := \{Q \lneq P\}$ and $V(\mathcal{F}_G) := \bigoplus_{Q\in\mathcal{F}_G} k\uparrow_Q^G$. Then $Proj(V(\mathcal{F}_G))$ corresponds to projectivity relative to the family of all non maximal $p$-subgroups of $G$ and it does not depend on the choice of the Sylow $p$-subgroup $P$.

Lemma 7.1.2.
*Let $P$ be a Sylow $p$-subgroup of $G$, $H$ be a subgroup of $G$ such that $P \leq H \leq G$. Then:*

(a) $Proj(V(\mathcal{F}_G)\downarrow_H^G) = Proj(V(\mathcal{F}_H))$.
(b) *$V(\mathcal{F}_H)$ is absolutely p-divisible.*

Proof.

(a) The Mackey formula yields

$$V(\mathcal{F}_G){\downarrow}_H^G = \bigoplus_{Q\in\mathcal{F}_G} k{\uparrow}_Q^G{\downarrow}_H^G \cong \bigoplus_{Q\in\mathcal{F}_G}\bigoplus_{x\in[H\backslash G/Q]} k{\uparrow}_{{}^xQ\cap H}^H = \underbrace{\bigoplus_{Q\lneq P} k{\uparrow}_Q^H}_{V(\mathcal{F}_H)} \oplus X$$

where $X$ is a direct sum of modules of the form $k{\uparrow}_S^H$ with $S \lneq P$. Thus by Proposition 2.2.2 (b), (c) and (h) we conclude that $Proj(V(\mathcal{F}_G){\downarrow}_H^G) = Proj(V(\mathcal{F}_H))$.

(b) By Green's indecomposability Criterion, the direct sum decomposition

$$V(\mathcal{F}_P) = \bigoplus_{Q\in\mathcal{F}_P} k{\uparrow}_Q^P$$

is a decomposition into indecomposable modules, all of which have dimension divisible by $p$. In other words, $V(\mathcal{F}_P)$ is absolutely $p$-divisible and, therefore by part (a) and Lemma 2.9.1 so are the modules $V(\mathcal{F}_H)$ for every $P \leq H \leq G$.

(c)/(d) A module $L$ is a $p$-permutation $kG$-module if and only if there exists a subgroup $R \leq G$ such that $L \,|\, k{\uparrow}_R^G$. Therefore by the previous lemma

$$Proj(L) \subseteq Proj(k{\uparrow}_R^G) = Proj(k{\uparrow}_Q^G) = Proj(k{\uparrow}_{{}^gQ}^G)$$

where $Q$ is a Sylow $p$-subgroup of $R$ and $g \in G$ such that ${}^gQ \lneq P$. Then (c) and (d) are consequences of Theorem 2.9.3 and the fact that $Proj(k{\uparrow}_{{}^gQ}^G) = \mathsf{mod}(kG)$ if and only if ${}^gQ = P$. Indeed, in case ${}^gQ = P$ then any $kG$-module is $P$-projective and conversely, if ${}^gQ \lneq P$ then $k{\uparrow}_{{}^gQ}^G \,|\, V(\mathcal{F}_G)$ which is absolutely $p$-divisible by part (b).

□

LEMMA 7.1.3.
*Let $N$ be a normal subgroup of the group $G$ such that $p \,|\, |G/N|$. Then*

$$Proj(\mathrm{Inf}_{G/N}^G(V(\mathcal{F}_{G/N}))) \subseteq Proj(V(\mathcal{F}_G))\,.$$

PROOF. Let $P$ be a Sylow $p$-subgroup of $G$ and $PN/N$ the corresponding Sylow $p$-subgroup of $G/N$. By definition,

$$V(\mathcal{F}_{G/N}) = \bigoplus_{R\lneq PN/N} k{\uparrow}_R^{G/N}\,.$$

Moreover, if $R \lneq PN/N$, there exists a subgroup $Q$ such that $P\cap N \leq Q \lneq P$ and $R = QN/N$. Whence

$$\mathrm{Inf}_{G/N}^G(V(\mathcal{F}_{G/N})) = \bigoplus_{P\cap N\leq Q\lneq P} \mathrm{Inf}_{G/N}^G(k{\uparrow}_{QN/N}^{G/N}) = \bigoplus_{P\cap N\leq Q\lneq P} k{\uparrow}_{QN}^G\,.$$

Now, since $Q$ is a Sylow $p$-subgroup of $QN$, by Lemma 7.1.1 $Proj(k{\uparrow}_{QN}^G) = Proj(k{\uparrow}_Q^G)$. Whence

$$\begin{aligned} Proj(\mathrm{Inf}_{G/N}^G(V(\mathcal{F}_{G/N}))) &= Proj(\bigoplus_{P\cap N\leq Q\lneq P} k{\uparrow}_{QN}^G) \\ &= \bigoplus_{P\cap N\leq Q\lneq P} Proj(k{\uparrow}_Q^G) \subseteq Proj(V(\mathcal{F}_G))\,. \end{aligned}$$

□

## 7.2. $V(\mathcal{F}_G)$-endotrivial modules

Having proven that the module $V(\mathcal{F}_G)$ is absolutely $p$-divisible, it makes sense to consider $V(\mathcal{F}_G)$-endotrivial modules since then we obtain a well-defined associated group structure $T_{V(\mathcal{F}_G)}(G)$. To start with, here is a short summary of elementary properties of this group that can easily be deduced from the general theory of relative endotrivial modules that has been developed in the preceding chapters.

Lemma 7.2.1.
*Let $P$ be a Sylow $p$-subgroup of $G$ and $H$ be a subgroup of $G$ such that $P \leq H \leq G$.*

(a) *There is a well-defined restriction map*

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & T_{V(\mathcal{F}_G)}(G) & \longrightarrow & T_{V(\mathcal{F}_H)}(H) \\ & [M] & \longmapsto & [M{\downarrow}^G_H]\,. \end{array}$$

(b) *If $H$ contains $N_G(P)$, then the restriction map $\mathrm{Res}^G_H : T_{V(\mathcal{F}_G)}(G) \xrightarrow{\cong} T_{V(\mathcal{F}_H)}(H)$ is an isomorphism, whose inverse is induced by the Green correspondence on the indecomposable $V(\mathcal{F}_H)$-endotrivial modules.*

(c) $\ker(\mathrm{Res}^{N_G(P)}_P) = X(N_G(P))$.

(d) *If $\Gamma$ denotes the Green correspondence and $\Gamma(X(N_G(P)))$ denotes the subgroup of $T_{V(\mathcal{F}_G)}(G)$ made up of the classes of the $kG$-Green correspondents of the modules in $X(N_G(P))$, then*

$$\ker(\mathrm{Res}^G_P) = \Gamma(X(N_G(P)))\,.$$

*Moreover $\Gamma(X(N_G(P))) \cong X(N_G(P))$ via restriction and is a finite group.*

Proof.

(a) This follows from the definition of a restriction map and parts (a) and (b) of Lemma 7.1.2.

(b) This is a consequence of Lemma 4.3.1 and the remarks in section 4.3.

(c) This follows from part (a) of Lemma 7.1.2 and Lemma 4.4.1.

(d) Since $\mathrm{Res}^G_{N_G(P)}$ is an isomorphism, (d) is a straightforward consequence of (c). The group is finite because $X(N_G(P))$ is the $p'$-part of the abelianization of $N_G(P)$.

□

Occurences of $V(\mathcal{F}_G)$-endotrivial modules.

Thus far, we have essentially two families of examples of $V(\mathcal{F}_G)$-endotrivial modules at our disposal.

**1.** The $kG$-Green correspondents of the one-dimensional representations of the normaliser $N_G(P)$, provided by part (d) of the above lemma.

**2.** The relative syzygies $\Omega^n_W(M)$ with $W \in \mathit{Proj}(V(\mathcal{F}_G))$, $n \in \mathbb{Z}$ and $M$ a $V(\mathcal{F}_G)$-endotrivial module as described in Lemma 3.3.1.

Anyhow, a profusion of examples are given by the relative syzygies $\Omega^n_{\mathcal{H}}(k)$ of the trivial module, where $\mathcal{H}$ is a family of subgroups of $G$ such that the associated module $V(\mathcal{H})$ is absolutely $p$-divisible. (According to the previous section, $\mathcal{H}$ might as well be assumed to be a family of proper $p$-subgroups of a given Sylow $p$-subgroup $P$ of $G$.) Then $\mathit{Proj}(\mathcal{H}) \subseteq \mathit{Proj}(V(\mathcal{F}_G))$ and so

$$T_{V(\mathcal{H})}(G) \leq T_{V(\mathcal{F}_G)}(G)\,.$$

Furthermore, $T_{V(\mathcal{F}_G)}(G)$ admits the subgroups $T^{\Omega}(G) := \langle \Omega_V(k) \,|\, V \in Proj(\mathcal{F}_G) \rangle$ and $D^{\Omega}(G) := \langle \Omega_{\mathcal{H}}(k) \,|\, \mathcal{H} \subseteq \mathcal{F}_G \rangle$.

It is known from Alperin [Alp01] that the relative syzygies $\Omega^n_{\mathcal{H}}(k)$ are endo-permutation modules in the case that $G$ is a $p$-group. In similar manner, it can be seen that they are endo-$p$-permutation modules when $G$ is arbitrary (see [Urf06, Prop. 5.8]). We shall show in section 7.7 that the same is true for the modules in $\Gamma(X(N_G(P)))$. Therefore there are strong connections between $V(\mathcal{F}_G)$-endotrivial modules and endo-permutation modules as well as endo-$p$-permutation modules. Our task for the next sections is to understand more precisely these connections.

## 7.3. Preliminaries on endo-permutation modules

Let $P$ be a $p$-group. The Dade Group of $P$ is constructed by E. Dade in [Dad78a] as described below. It relies essentially on the following proposition:

Proposition 7.3.1 ([Dad78a], Thm. 3.8).
*If $M$ is an endo-permutation $kP$-module, then any two indecomposable direct summands of $M$ with vertex $P$ are isomorphic.*

Thus, if $M$ is an arbitrary endo-permutation $kP$-module, it has at most one isomorphism class of indecomposable direct summands with vertex $P$. If such direct summands exist, then their isomorphism class is denoted by $Cap(M)$ and $M$ is called *capped*. We note that the class of all capped endo-permutation $kP$-modules is closed under taking direct summands, duals and tensor products.

Furthermore, this class can be endowed with the following equivalence relation: two capped endo-permutation $kP$-modules are said to be equivalent if their caps are isomorphic. We shall write $[M]$ for the equivalence class of the endo-permutation module $M$ and let $D(P)$ denote the resulting set of equivalence classes. It is an abelian group for the composition law:

$$([M],[N]) \longmapsto [M]+[N] := [M \otimes N]$$

The zero element is the class $[k]$ of the trivial module, while the opposite of a class $[M]$ is the class $[M^*]$ of the dual module. The group $D(P)$ is called the *Dade group* of $P$.

As Dade notices, there is another useful way of thinking of the group $D(P)$. In every equivalence class, there is, up to isomorphism, a unique indecomposable module, namely the cap of any module in the class. Therefore, if $\widehat{D}(P)$ is the set of isomorphism classes of indecomposable endo-permutation $kP$-modules with vertex $P$, then there is a bijection

$$\begin{array}{ccc} D(P) & \longrightarrow & \widehat{D}(P) \\ [M] & \longmapsto & [Cap(M)] \end{array} .$$

In consequence, $\widehat{D}(P)$ is an abelian group for the operation $[M]+[N] := [Cap(M\otimes N)]$, where the square brackets are also used to denote the isomorphism class of a module.

Yet another way of thinking about the group $D(P)$, the one we are interested in, is an intermediary version of the two previous ones. An endo-permutation $kP$-module $M$ is called *capped in the strong sense* if $Cap(M)$ has multiplicity one as a direct summand of $M$. The class of all such $kP$-modules is a subclass of that of capped endo-permutation $kP$-modules to which the above equivalence relation can be restricted and following the above process it induces a third group $\widehat{D}(P)$ isomorphic to $D(P)$.

### 7.4. Relative endotrivial modules as a generalisation for the Dade group

The above construction of the Dade group $D(P)$ is valid only in case the group $P$ is a $p$-group. This is linked to the facts that $kP$-permutation modules are indecomposable, whereas $kG$-permutation modules are not in general for an arbitrary group $G$, and moreover that their direct summands are not permutation modules.

Notwithstanding, one way to obtain a similar notion to that of the Dade Group for arbitrary groups is to consider endo-$p$-permutation modules as described by Urfer in [Urf06]. He shows that if $P$ is a $p$-subgroup of a group $G$, this notion induces a group structure, denoted by $D_P(G)$, on a set of equivalence classes of indecomposable endo-$p$-permutation $kG$-modules with vertex $P$. (The equivalence relation is a generalisation of Dade's compatibility relation.) However, the main drawback of this approach resides in the fact that there is not a unique indecomposable representative, up to isomorphism, for the classes in $D_P(G)$.

Our notion of relative endotrivial modules can generalize the Dade group in a more natural way. Let us fix $P$ a finite $p$-group. The first observation to make is that an indecomposable capped endo-permutation $kP$-module $M$ (i.e. with vertex $P$) is always a relative endotrivial module, that is relatively to some intrinsincally defined $kP$-module $V_M$. Indeed, it is an elementary fact about capped endo-permutation modules that the trivial $kP$-module $k$ has to be a direct summand of $End_k(M)$ ([Dad78a, Prop. 3.7]), while by 2.8.1 it is clear that the multiplicity of $k$ is exactly one. It yields the characterization:

$$\mathrm{End}_k(M) = (\text{permutation module}) \cong k \oplus k{\uparrow}_{Q_1}^P \oplus \cdots \oplus k{\uparrow}_{Q_s}^P$$

for some subgroups $Q_1, \ldots, Q_s \lneq P$, $s \in \mathbb{N}$. Thus one can set $V_M := \bigoplus_{i=1}^s k{\uparrow}_{Q_i}^P$, which is clearly absolutely $p$-divisible by 7.1.2. Then, by very definition, $M$ becomes a $V_M$-endotrivial module. Besides, $V_M \mid V(\mathcal{F}_G)$ and so $M$ is also $V(\mathcal{F}_P)$-endotrivial.

In other words, the module $V(\mathcal{F}_P)$ is a universal module relatively to which any indecomposable capped endo-permutation module is endotrivial. This construction leads to the following natural embedding of $D(P)$ in $T_{V(\mathcal{F}_G)}(P)$, in which the equivalence classes do have a unique indecomposable representative, up to isomorphism.

Theorem 7.4.1.
*The Dade group $D(P)$ can be identified with a subgroup of $T_{V(\mathcal{F}_P)}(P)$ via the canonical injective homomorphism*

$$\begin{array}{ccc} D(P) & \longrightarrow & T_{V(\mathcal{F}_P)}(P) \\ [M] & \longmapsto & [Cap(M)] \end{array} .$$

Proof. If $[M]$ is a class in $D(P)$, then $Cap(M)$ is the unique indecomposable representative of this class. Moreover, according to the above construction, any indecomposable capped endo-permutation module is $V(\mathcal{F}_P)$-endotrivial. Hence, since both in $D(P)$ and in $T_{V(\mathcal{F}_P)}(P)$ there is a unique indecomposable representative for the classes, the map $[M] \longmapsto [Cap(M)]$ of the statement is a well-defined, injective group morphism. □

Remark 7.4.2.
We shall see that if $P = C_{p^n}$, a cyclic $p$-group, or if $p = 2$ and $P = C_2 \times C_2$, then $D(P) \cong T_{V(\mathcal{F}_G)}(P)$.

## 7.5. Endo-$p$-permutation modules and the Dade group of a finite group

With the classical case of endo-permutation modules in mind we can pass to the case of endo-$p$-permutation modules. To begin with, let us recall a few facts about this class of modules. An *endo-$p$-permutation $kG$-module* is defined to be a module $M \in \mathsf{mod}(kG)$ whose endomorphism algebra $\mathrm{End}_k(M)$ is a $p$-permutation[1] $kG$-module. In other words, if $\mathrm{End}_k(M) \cong \bigoplus_{i\in I} N_i$ where each $N_i$ is indecomposable, then for every $i \in I$, $N_i \mid k{\uparrow}_{Q_i}^G$ for some $p$-subgroup $Q_i$ of $G$. Equivalently, $M$ is endo-$p$-permutation if $M{\downarrow}_Q^G$ is an endo-permutation $kQ$-module for every $p$-subgroup $Q$ of $G$. In fact, since $p$-permutation modules are preserved under conjugation and restriction, it is enough to check that $M{\downarrow}_P^G$ is an endo-permutation $kP$-module for $P$ a fixed Sylow $p$-subgroup of $G$. A few other elementary properties of this class of modules are the following:

Lemma 7.5.1.
*Let $M \in \mathsf{mod}(kG)$ be an indecomposable endo-$p$-permutation module with vertex $P$. Then:*

(a) *$M{\downarrow}_P^G$ is capped endo-permutation.*
(b) *$p \nmid \dim_k M$.*
(c) *$k \mid \mathrm{End}_k(M)$ with multiplicity 1.*

Proof.

(a) It is easy to see that $M{\downarrow}_P^G$ is forced to have a summand with vertex $P$, thus it is capped endo-permutation. See [Urf06, Chapter 2] for details.
(b) Assume $M$ were an indecomposable $kG$-module with $k$-dimension divisible by $p$, that is absolutely $p$-divisible. Then, by Theorem 2.9.3, so would be $M{\downarrow}_P^G$, which contradicts statement (a). Indeed, $M{\downarrow}_P^G$ being capped, it has got at least one direct summand with $k$-dimension not divisible by $p$, for according to the previous section, $Cap(M{\downarrow}_P^G)$ is an indecomposable endo-permutation module, hence $V(\mathcal{F}_P)$-endotrivial and thus by Lemma 3.1.3 we have $\dim_k Cap(M{\downarrow}_P^G) \equiv \pm 1 \mod (p)$.
(c) This is a consequence of (b) and the Benson-Carlson Theorem (2.8.1).

□

It can be seen from the work of [Urf06] that setting an equivalence relation on the whole class of endo-$p$-permutation modules with vertex $P$ given by a generalisation of Dade's compatibility relation (cf [Dad78a]) does not lead to a group structure induced by tensor product on the set of isomorphism classes of indecomposable endo-$p$-permutation modules with vertex $P$. The idea is then to find a subclass of this class which has more similarities with that of *capped endo-permutation* modules for a $p$-group and obtain a group structure induced by tensor product which embeds naturally in $T_{V(\mathcal{F}_G)}(G)$, generalising the embedding $D(P) \leq T_{V(\mathcal{F}_P)}(P)$ of Theorem 7.4.1. In this respect we shall focus on endo-$p$-permutation modules which are $V(\mathcal{F}_G)$-endotrivial at the same time.

Proposition 7.5.2.
*Let $M \in \mathsf{mod}(kG)$ be an endo-$p$-permutation module. The following conditions are equivalent:*

(a) *$M$ is $V(\mathcal{F}_G)$-endotrivial;*
(b) *$M{\downarrow}_P^G$ is $V(\mathcal{F}_P)$-endotrivial;*
(c) *$M$ has a unique indecomposable summand with vertex $P$, say $M_0$ and, in addition, if $S \in \mathsf{mod}(kP)$ is a source for $M_0$, then the multiplicity of $S$ as a direct summand of $M{\downarrow}_P^G$ is one;*

[1] In English, a $p$-permutation module is also often termed a *trivial source module*.

(d) $\mathrm{End}_k(M) \cong k \oplus N$ *where* $N$ *is a* $p$*-permutation* $kG$*-module, all of whose indecomposable summands have a vertex strictly contained in* $P$*.*

PROOF.
(a)⇔(b): By Lemma 7.1.2, $Proj(V(\mathcal{F}_G))\downarrow^G_P) = Proj(V(\mathcal{F}_P))$, therefore statements (a) and (b) are equivalent by Lemmas 3.3.4 and 4.1.1. As a matter of fact, this equivalence is independent of the initial assumption that $M$ is endo-$p$-permutation.

(a)⇒(c): This implication does not need either the assumption that $M$ is endo-$p$-permutation. Indeed, assuming (a) yields a decomposition:

$$M \cong M_0 \oplus (V(\mathcal{F}_G) - proj)$$

where $M_0$ is the unique indecomposable $V(\mathcal{F}_G)$-endotrivial summand of $M$. By Lemma 3.5.1, $M_0$ has vertex $P$, whereas all the other summands of $M$ have a vertex strictly smaller than $P$ by definition of $Proj(V(\mathcal{F}_G))$. Furthermore, still by Lemma 3.5.1, if $S \in \mathsf{mod}(kP)$ is a source for $M_0$, then $S$ has multiplicity one in $M_0{\downarrow^G_P}$. In consequence, since $M{\downarrow^G_P}$ is $V(\mathcal{F}_P)$-endotrivial we have

$$M{\downarrow^G_P} \cong M_0{\downarrow^G_P} \oplus (V(\mathcal{F}_P) - proj) \cong S \oplus (V(\mathcal{F}_P) - proj)$$

where the Krull-Schmidt Theorem forces $S$ to be isomorphic to the unique $V(\mathcal{F}_P)$-endotrivial summand of $M{\downarrow^G_P}$. Thus $S$ has multiplicity one in $M{\downarrow^G_P}$ as well.

(c)⇒(b): Write $M = M_0 \oplus L$ with $M_0$ indecomposable with vertex $P$ and $L$ a module all of whose indecomposable summands have a vertex strictly smaller than $P$. Thus $L \in Proj(V(\mathcal{F}_G))$ and restricting $M$ to $P$ yields

$$M{\downarrow^G_P} \cong M_0{\downarrow^G_P} \oplus (V(\mathcal{F}_P) - proj)\,.$$

Now $M_0$ is endo-$p$-permutation as a direct summand of an endo-$p$-permutation module, therefore $M_0{\downarrow^G_P}$ is capped endo-permutation by Lemma 7.5.1. Moreover $S \mid M_0{\downarrow^G_P}$ and because $S$ has vertex $P$ too, we must have $S \cong Cap(M_0{\downarrow^G_P})$, so that the fact that the multiplicity of $S$ is one forces all the remaining direct summands of $M_0{\downarrow^G_P}$ to have a vertex strictly smaller than $P$, that is to be $V(\mathcal{F}_P)$-ptojective. Hence $M{\downarrow^G_P}$ is $V(\mathcal{F}_P)$-endotrivial.

(a)⇔(d): Given that $M$ is endo-$p$-permutation, then $End_k(M)$ is a $p$-permutation module. Thus $M$ satisfies condition (d) if and only if it is $V(\mathcal{F}_G)$-endotrivial, by definition of the family $\mathcal{F}_G$. □

DEFINITION 7.5.3.
An endo-$p$-permutation $kG$-module $M$ is said to be *strongly capped* if it satisfies the equivalent conditions of Proposition 7.5.2. Moreover, the unique summand of $M$ with vertex $P$ given by condition (c) is called the *cap* of $M$ and denoted by $Cap(M)$.

REMARKS 7.5.4.

(a) The cap of a strongly capped endo-$p$-permutation module is its unique indecomposable direct summand which is itself strongly capped.

(b) A strongly capped endo-$p$-permutation $kG$-module has a direct sum decomposition of the form

$$M \cong Cap(M) \oplus (V(\mathcal{F}_G) - proj)\,.$$

where the $V(\mathcal{F}_G)$-projective part is also an endo-$p$-permutation module.

LEMMA 7.5.5.
*The class of strongly capped endo-*$p$*-permutation* $kG$*-modules is closed under taking duals, tensor products and restrictions to a subgroup containing a Sylow* $p$*-subgroup of* $G$*.*

PROOF. Taking duals and tensor products are stable operations for both the classes of endo-$p$-permutation modules and of $V(\mathcal{F}_G)$-endotrivial modules, therefore they are stable for strongly capped endo-$p$-permutation modules too. Moreover, the restriction of an endo-$p$-permutation module to a subgroup containing a Sylow $p$-subgroup is an endo-$p$-permutation module and the restriction of a $V(\mathcal{F}_G)$-endotrivial module to a subgroup $H$ containing a Sylow $p$-subgroup is a $V(\mathcal{F}_H)$-endotrivial module by Lemma 7.2.1. Thus the restriction to $H$ of a strongly capped endo-$p$-permutation module is strongly capped. □

Using a similar approach to that used by Dade for endo-permutation modules, one can define an equivalence relation $\sim$ on the class of all strongly capped endo-$p$-permutation modules by setting:

$$M \sim N \Leftrightarrow Cap(M) \cong Cap(N)$$

We shall write $[M]$ for the equivalence class of the module $M$ and let $D(G)$ denote the resulting set of equivalence classes.

Observe that this equivalence relation is the restriction to the class of strongly capped endo-$p$-permutation of the equivalence relation $\sim_{V(\mathcal{F}_G)}$ on $V(\mathcal{F}_G)$-endotrivial modules of Definition 3.6. In consequence, if $M$ and $N$ are two strongly capped endo-$p$-permutation $kG$-modules, then $M \sim N$ if and only if $M \sim_{V(\mathcal{F}_G)} N$. Therefore, the reader should be aware that the classes do not mean the same thing in $T_{V(\mathcal{F}_G)}(G)$ and in $D(G)$, and moreover that in general there are more representatives for a given class in $T_{V(\mathcal{F}_G)}(G)$ than in $D(G)$.

COROLLARY-DEFINITION 7.5.6.
*The set $D(G)$ is an abelian group for the composition law*

$$([M],[N]) \longmapsto [M]+[N] := [M \otimes N]\,,$$

*called the* generalized Dade group of $G$, *or simply the* Dade group of $G$. *Moreover, $D(G)$ can be identified with a subgroup of $T_{V(\mathcal{F}_G)}(G)$ through the natural embedding*

$$\begin{array}{rccc} \imath : & D(G) & \longrightarrow & T_{V(\mathcal{F}_G)}(G) \\ & [M] & \longmapsto & [M]\ . \end{array}$$

PROOF. Lemma 7.5.5 and the uniqueness of the caps ensure that the assignment

$$([M],[N]) \longmapsto [M \otimes N]$$

of the statement is a well-defined composition law for $D(G)$. The zero element is the class $[k]$ of the trivial module, while the opposite of a class $[M]$ is the class $[M^*]$ of the dual module.

Now $\imath$ is well-defined by the above observation on $\sim$ and $\sim_{V(\mathcal{F}_G)}$ and it is a homomorphism because the addition is induced by $\otimes_k$ on both sides. Finally, it is injective because $\ker(\imath) = \{[k]\}$. Indeed, if $\imath([M]) = [k]$, then $M \sim_{V(\mathcal{F}_G)} k$ which is equivalent to $M \sim k$ because both $M$ and $k$ are strongly capped endo-$p$-permutation modules. □

For the sake of simplicity, we shall from now on identify $D(G)$ with its image $\imath(D(G))$ and therefore view $D(G)$ as a subgroup of $T_{V(\mathcal{F}_G)}(G)$.

REMARK 7.5.7.
Notice that any ordinary endotrivial module is strongly capped, and in particular, so is any one-dimensional $kG$-module. Therefore, up to identifications, the groups $T(G)$ and $X(G)$ can also be viewed as subgroups of $D(G)$ and we have a series of subgroup inclusions:

$$X(G) \leq T(G) \leq D(G) \leq T_{V(\mathcal{F}_G)}(G)$$

The group $D^{\Omega}(G) = \langle \Omega_{\mathcal{H}}(k) \,|\, \mathcal{H} \subseteq \mathcal{F}_G \rangle$ is also a subgroup of $D(G)$ because of the next Lemma.

Lemma 7.5.8.
*Let $\mathcal{H}$ be a subfamily of $\mathcal{F}_G$. If $M$ is a strongly capped endo-p-permutation module, then $\Omega_{V(\mathcal{H})}(M)$ is a strongly capped endo-p-permutation $kG$-module.*

Proof. Since $M$ is assumed to be strongly capped, it is both endo-$p$-permutation and $V(\mathcal{F}_G)$-endotrivial. In consequence, on the one hand $\Omega_{V(\mathcal{H})}(M)$ is $V(\mathcal{F}_G)$-endotrivial by Lemma 3.3.1, hence $V(\mathcal{H})$-endotrivial and on the second hand, it is shown in [Urf06, Proposition 5.8] that it is endo-$p$-permutation, hence strongly capped, as required. □

To end up the section we note that $D(G)$ can be identified with two other groups of endo-$p$-permutation modules.

The first one is the set of isomorphism classes of indecomposable strongly capped endo-$p$-permutation $kG$-modules endowed with an abelian group structure induced by tensor product in the same way as the group $\widehat{D}(P)$ is for a $p$-group $P$. More accurately, observe that in every equivalence class in $D(G)$ there is a unique indecomposable module, namely the cap of any module in the class, therefore, if $\widehat{D}(G)$ denotes the set of isomorphism classes of strongly capped endo-$p$-permutation modules, then there is a bijection

$$\begin{array}{ccc} D(G) & \longrightarrow & \widehat{D}(G) \\ [M] & \longmapsto & [Cap(M)] \end{array}$$

and $\widehat{D}(G)$ endowed with the operation $[M] + [N] := [Cap(M \otimes N)]$ becomes an abelian group (where the square brackets are also used, in a non-misleading way, to denote the isomorphism class of a module).

The second one is based on Dade's construction for endo-permutation modules and the idea of allowing endo-$p$-permutation modules to have caps with a multiplicity. We shall momentarily say that an endo-$p$-permutation module $M$ is *weakly capped* if it has, up to isomorphism a unique indecomposable summand with vertex $P$, denoted by $Cap(M)$, and which is moreover strongly capped. Such a module $M$ has a direct sum decomposition of the form

$$M \cong Cap(M)^{\oplus r} \oplus (V(\mathcal{F}_G) - proj)$$

with $r \geq 1$ an integer. Then we can set an equivalence relation on the class of weakly capped endo-$p$-permutation $kG$-modules by setting $M \sim N \Leftrightarrow Cap(M) \cong Cap(N)$. Then, this class is closed under taking duals and tensor products. Indeed, for tensor products, consider two modules $M = Cap(M)^{\oplus r(M)} \oplus (V(\mathcal{F}_G) - proj)$ and $N = Cap(N)^{\oplus r(N)} \oplus (V(\mathcal{F}_G) - proj)$ with $r(M), r(N) \in \mathbb{N}$. Then

$$M \otimes N \cong (Cap(Cap(M) \otimes Cap(N)))^{\oplus r(M) \cdot r(N)} \oplus (V(\mathcal{F}_G) - proj)\,.$$

In consequence, if $\widetilde{D}(G)$ denotes the resulting set of equivalence classes, it can be endowed with an abelian group structure for the operation $([M], [N]) \longmapsto [M] + [N] := [M \otimes N]$. Furthermore, the map

$$\begin{array}{ccc} \widetilde{D}(G) & \longrightarrow & D(G) \\ [M] & \longmapsto & [Cap(M)] \end{array}$$

is a group isomorphism.

The three groups $D(G)$, $\widehat{D}(G)$ and $\widetilde{D}(G)$ all have the property to have a unique indecomposable representative in their classes. The main drawback of $\widetilde{D}(G)$ in our vision resides in the fact the representatives in the classes are not relatively endotrivial modules as soon as their cap has a multiplicity greater or equal to 2. In consequence, we favour the approach of $D(G)$ and we may also identify $D(G)$ and $\widehat{D}(G)$ without further mention.

## 7.6. Group operations

The operations of restriction and inflation induce group homomorphisms between the generalised Dade groups, whereas we will provide a counterexample to show that tensor induction does not.

### 1. Restriction.

Lemma 7.6.1.
*Let $P$ be a Sylow $p$-subgroup of $G$ and let $H$ be a subgroup of $G$ such that $P \leq H \leq G$. Then restriction induces a group homomorphism*

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & D(G) & \longrightarrow & D(H) \\ & [M] & \longmapsto & [M\!\downarrow^G_H] \end{array}.$$

*Furthermore, if $H$ contains the normaliser $N_G(P)$, then the map $\mathrm{Res}^G_H$ is injective.*

Proof. As seen in 7.2.1, there is a restriction homomorphism for groups of relative endotrivial modules

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & T_{V(\mathcal{F}_G)}(G) & \longrightarrow & T_{V(\mathcal{F}_H)}(H) \\ & [M] & \longmapsto & [M\!\downarrow^G_H] \end{array},$$

which is, furthermore, an isomorphism if $H$ contains $N_G(P)$. In consequence, it suffices to check that this maps $D(G)$ to a subgroup of $D(H)$. In fact, if $[M] \in D(G)$, then, $M\!\downarrow^G_H$ is clearly both $V(\mathcal{F}_H)$-endotrivial and endo-$p$-permutation, hence it is strongly capped and $[M\!\downarrow^G_H] \in D(H)$, as required. Consequently, set $\mathrm{Res}^G_H : D(G) \longrightarrow D(H)$ to be the restriction of map

$$\mathrm{Res}^G_H : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_H)}(H)$$

to $D(G)$. It is injective if $H \geq N_G(P)$. □

Remark 7.6.2.
In particular, the injectivity of the restriction map $\mathrm{Res}^G_{N_G(P)} : D(G) \longrightarrow D(N_G(P))$ allows us to identify the Dade group $D(G)$ of a group $G$ with a subgroup of the Dade group $D(N_G(P))$ of the normaliser $N_G(P)$ of its Sylow $p$-subgroup $P$.

### 2. Inflation.

Lemma 7.6.3.
*Let $N$ be a normal subgroup of the group $G$ such that $G/N$ has order divisible by $p$. Then inflation induces a group homomorphism*

$$\begin{array}{rccc} \mathrm{Inf}^G_{G/N}\colon & D(G/N) & \longrightarrow & D(G) \\ & [M] & \longmapsto & [\mathrm{Inf}^G_{G/N}(M)] \end{array}.$$

Proof. Let $M$ be a strongly capped endo-$p$-permutation $k[G/N]$-module. Then so is $\mathrm{Inf}^G_{G/N}(M)$. Indeed, on the one hand it is $V(\mathcal{F}_G)$-endotrivial, because it is $\mathrm{Inf}^G_{G/N}(V(\mathcal{F}_{(G/N)}))$-endotrivial by Section 3.9 and moreover $\mathit{Proj}(\mathrm{Inf}^G_{G/N}(V(\mathcal{F}_{G/N}))) \subseteq \mathit{Proj}(V(\mathcal{F}_G))$ by Lemma 7.1.3. On the second hand it is an endo-$p$-permutation module as well, for if $\varphi : P/P\cap N \xrightarrow{\cong} PN/N$ denotes the canonical group morphism, then

$$\mathrm{Res}^G_P \circ \mathrm{Inf}^G_{G/N}(M) = \mathrm{Inf}^P_{P/P\cap N} \circ \mathrm{Iso}(\varphi^{-1}) \circ \mathrm{Res}^{G/N}_{PN/N}(M)$$

is endo-permutation because both isomorphism and inflation preserve endo-permutation modules. In consequence, there is a group homomorphism $\mathrm{Inf}_{G/N}^{G} : D(G/N) \longrightarrow D(G)$ defined by the following diagram

$$\begin{array}{ccccc} T_{V(\mathcal{F}_{G/N})}(G/N) & \xrightarrow{\mathrm{Inf}_{G/N}^{G}} & T_{\mathrm{Inf}_{G/N}^{G} V(\mathcal{F}_{G/N})}(G) & \xrightarrow{i} & T_{V(\mathcal{F}_{G})}(G) \\ \uparrow{\scriptstyle i} & & & & \uparrow{\scriptstyle i} \\ D(G/N) & & \overset{\mathrm{Inf}_{G/N}^{G}}{\dashrightarrow} & & D(G) \end{array}$$

as the restriction of the top map $i \circ \mathrm{Inf}_{G/N}^{G}$. □

## 3. Tensor induction.

Although the tensor induction of an endo-$p$-permutation module is an endo-$p$-permutation module (see [Urf06, Prop. 2.5]), the tensor induction of a strongly capped endo-$p$-permutation module is not necessarily a strongly capped endo-$p$-permutation module. We provide here an even more interesting counterexample in which the tensor induction of an endotrivial module is not even endotrivial relatively to any $kG$-module. It will show at the same time that the tensor induction of a $V$-endotrivial module is not necessarily a $V\!\uparrow_{\otimes H}^{G}$-endotrivial module. We refer to Section 1.1 and [CR90, §13] for a careful construction and description of the tensor induced module.

Counterexample 7.6.4.
Consider the 3-nilpotent group $G := C_7 \rtimes C_3$ in characteristic 3. Write $C_7 :=< a >$ and $C_3 :=< u >$. The action of $C_3$ on $C_7$ is given by $uau^{-1} = a^2$. Consider the module $\Omega(k) \in \mathsf{mod}(kC_3)$. It is an endotrivial module of dimension 2. However the module $\Omega(k)\!\uparrow_{\otimes C_3}^{G}$ is neither an endotrivial module nor a strongly capped endo-3-permutation module. In fact, we have the following:

Lemma 7.6.5.
*There exists no absolutely 3-divisible $kG$-module $V$ such that the tensor induced module $\Omega(k)\!\uparrow_{\otimes C_3}^{G}$ is $V$-endotrivial.*

Proof. Write $M := \Omega(k)$. Then $M^* \otimes M \cong k \oplus kC_3$, so that

$$(M\!\uparrow_{\otimes C_3}^{G})^* \otimes M\!\uparrow_{\otimes C_3}^{G} \cong (k \oplus kC_3)\!\uparrow_{\otimes C_3}^{G} \cong k \oplus N$$

for some $kG$-module $N$ and we need to show that $N$ is not an absolutely 3-divisible module. The module $L := k \oplus kC_3$ is a permutation $kC_3$-module, so let $X := \{x_0, x_1, x_2, x_3\}$ with $x_0 \in k$ and $x_1, x_2, x_3 \in kC_3$ form a $C_3$-invariant $k$-basis for $L$. Without loss of generality we may assume that $ux_1 = x_2$, $ux_2 = x_3$ and $ux_3 = x_1$. Moreover, choose $\{a^0, a^1, a^2, a^3, a^4, a^5, a^6\}$ as a set of coset representatives for the subgroup $C_3$ in $G$. Then

$$L\!\uparrow_{\otimes C_3}^{G} \cong \bigotimes_{i=0}^{6} (a^i \otimes L)$$

is also a permutation module with a $G$-invariant $k$-basis given by

$$\{(a^0 \otimes x_{j_0}) \otimes \cdots \otimes (a^6 \otimes x_{j_6}) \,|\, x_{j_0}, \ldots, x_{j_6} \in X\}\,.$$

Let us compute fixed points under the action of the Sylow 3-subgroup $C_3$ of $G$. Of course, $(a^0 \otimes x_0) \otimes \cdots \otimes (a^6 \otimes x_0)$ is fixed under the action of $C_3$. This basis element corresponds to the trivial summand in $(k \oplus kC_3)\uparrow^G_{\otimes C_3} \cong k \oplus N$. Moreover, the element

$$b := (a^0 \otimes x_0) \otimes (a^1 \otimes x_1) \otimes (a^2 \otimes x_2) \otimes (a^3 \otimes x_3) \otimes (a^4 \otimes x_3) \otimes (a^5 \otimes x_2) \otimes (a^6 \otimes x_1)$$

is also fixed under the action of $C_3$, for

$$\begin{aligned} u \cdot b = \; & u(a^0 \otimes x_0) \otimes u(a^1 \otimes x_1) \otimes u(a^2 \otimes x_2) \otimes u(a^3 \otimes x_3) \otimes u(a^4 \otimes x_3) \otimes u(a^5 \otimes x_2) \otimes u(a^6 \otimes x_1) \\ = \; & (a^0 \otimes ux_0) \otimes (a^2 \otimes ux_1) \otimes (a^4 \otimes ux_2) \otimes (a^6 \otimes ux_3) \otimes (a^1 \otimes ux_3) \otimes (a^3 \otimes ux_2) \otimes (a^5 \otimes ux_1) \\ = \; & (a^0 \otimes x_0) \otimes (a^2 \otimes x_2) \otimes (a^4 \otimes x_3) \otimes (a^6 \otimes x_1) \otimes (a^1 \otimes x_1) \otimes (a^3 \otimes x_3) \otimes (a^5 \otimes x_2) \\ = \; & (a^0 \otimes x_0) \otimes (a^1 \otimes x_1) \otimes (a^2 \otimes x_2) \otimes (a^3 \otimes x_3) \otimes (a^4 \otimes x_3) \otimes (a^5 \otimes x_2) \otimes (a^6 \otimes x_1) = b \end{aligned}$$

after identification of $(a^0 \otimes L) \otimes (a^2 \otimes L) \otimes (a^4 \otimes L) \otimes (a^6 \otimes L) \otimes (a^1 \otimes L) \otimes (a^3 \otimes L) \otimes (a^5 \otimes L)$ with $\bigotimes_{i=0}^6 (a^i \otimes L)$.
In consequence, $(k \oplus kC_3)\uparrow^G_{\otimes C_3} \cong k \oplus N$ must contain at least a second direct summand, apart from $k$, with vertex the Sylow 3-subgroup $C_3$ of $G$. Such a summand is then a 3-permutation module, but is not absolutely $p$-divisible by part (d) of Lemma 7.1.2. This prevents $(M\uparrow^G_{\otimes C_3})^* \otimes M\uparrow^G_{\otimes C_3}$ from being of the form $k \oplus (V - proj)$ for any absolutely 3-divisible $kG$-module $V$. $\square$

This counterexample easily generalises to groups of the form $G := C_q \rtimes C_p$ with $p, q$ odd primes and $p \,|\, q-1$, $char(k) = p$. In characteristic 2, it generalises to groups of the form $G := C_q \rtimes C_4$ with $q$ a prime such that $4 \,|\, q-1$.

## 7.7. Towards the structure of $D(G)$

We aim to determine the structure of the group $D(G)$. To begin with we recall some more results on endo-$p$-permutation modules. A first key tool is provided by the following theorem proven by Dade and never published.

THEOREM 7.7.1 (Theorem 7.1, [Dad82]).
*Let $G$ be a finite group having a normal Sylow $p$-subgroup $P$. Let $M$ be an endo-permutation $kP$-module. Then $M$ extends to a $kG$-module if and only if $M$ is $G$-stable.*

In order to set up notation, let us recall that if $\mathcal{M}$ is a Mackey functor, $H$ a subgroup of a group $G$ and if Res denotes the restriction and $c_g$ denotes the conjugation by $g \in G$, then an element $m \in \mathcal{M}(H)$ is called **$G$-stable** iff:

$$(\mathrm{Res}^{\,{}^gH}_{{}^gH\cap H} \circ c_g)(m) = \mathrm{Res}^H_{{}^gH\cap H}(m), \quad \forall\, g \in G.$$

Moreover, $\mathcal{M}(H)^{G-st}$ denotes the subgroup of all $G$-stable elements of $\mathcal{M}(H)$. In particular, it follows easily from this definition that $D(P)^{N_G(P)-st} = D(P)^{N_G(P)}$, that is the subgroup of fixed points of $D(P)$ under the action of the normaliser $N_G(P)$ by conjugation.

A second key tool is provided by the following characterisation of endo-$p$-permutation modules by J.-M. Urfer:

THEOREM 7.7.2 (Theorem 1.5, [Urf07]).
*Let $G$ be a finite group. Let $M \in \mathsf{mod}(kG)$ be an indecomposable module with vertex $P$ and source $S \in \mathsf{mod}(kP)$. Then $M$ is an endo-$p$-permutation module if and only if $S$ is an endo-permutation module whose class $[S]$ in the Dade group $D(P)$ belongs to $D(P)^{G-st}$.*

Notation.
For ease of notation, we write $X := X(N_G(P))$ for the group of one-dimensional representations of $N_G(P)$, identified with a subgroup of $D(N_G(P))$ as noticed in Remark 7.5.7. Likewise we write $\Gamma(X) := \Gamma(X(N_G(P)))$ for the subgroup of $T_{V(\mathcal{F}_G)}(G)$ made up of the classes of the kG-Green correspondents of the modules in $X(N_G(P))$ defined in Lemma 7.2.1.

Theorem 7.7.3.
*Let $G$ be a finite group with a non-trivial Sylow $p$-subgroup $P$. Then,*

(a) *restriction from $N_G(P)$ to $P$ yields an exact sequence*

$$0 \longrightarrow X \hookrightarrow D(N_G(P)) \xrightarrow{\mathrm{Res}_P^{N_G(P)}} D(P)^{N_G(P)} \longrightarrow 0\,;$$

(b) *restriction from $G$ to $P$ yields an exact sequence*

$$0 \longrightarrow \Gamma(X) \hookrightarrow D(G) \xrightarrow{\mathrm{Res}_P^G} D(P)^{G-st} \longrightarrow 0\,.$$

In order to avoid confusion, and for the purpose of the following proof only, we shall momentarily denote by $R_H^G$ the restriction maps $\mathrm{Res}_H^G : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_H)}(H)$ at the level of $V(\mathcal{F}_G)$-endotrivial modules and keep the notation $\mathrm{Res}_H^G : D(G) \longrightarrow D(H)$ for the restriction maps at the level of the Dade groups.

Proof.

(1) First, it follows from Theorem 7.7.2 that $\mathrm{Im}(\mathrm{Res}_P^G) \leq D(P)^{G-st}$. In particular,
$$\mathrm{Im}(\mathrm{Res}_P^{N_G(P)}) \leq D(P)^{N_G(P)-st} = D(P)^{N_G(P)}\,.$$

(2) *We claim that indeed* $\mathrm{Im}(\mathrm{Res}_P^G) = D(P)^{G-st}$.
Let $[S] \in D(P)^{G-st}$ with $S$ indecomposable. Notice that $D(P)^{G-st} \subseteq D(P)^{N_G(P)}$, so that by Dade's Theorem $S \in \mathsf{mod}(kP)$ extends to a $kN_G(P)$-module $\widetilde{S}$. In other words, $\widetilde{S}\downarrow_P^{N_G(P)} \cong S$ and $S$ is a source for $\widetilde{S}$. By construction $\widetilde{S}$ is strongly capped endo-$p$-permutation because its source is endo-permutation and has multiplicity 1 in its restriction. Hence $[\widetilde{S}] \in D(N_G(P))$ and $\mathrm{Res}_P^{N_G(P)}([\widetilde{S}]) = [S]$. In particular, this argument proves the surjectivity of the map $Res_P^{N_G(P)}$ onto $D(P)^{N_G(P)}$.
Now if $\Gamma(\widetilde{S})$ is the $kG$-Green correspondent of $\widetilde{S}$, then it has source $S$ as well. Therefore $\Gamma(\widetilde{S})$ is endo-$p$-permutation by Theorem 7.7.2. It is moreover $V(\mathcal{F}_G)$-endotrivial by Lemma 7.2.1 because the restriction map $R_{N_G(P)}^G$ is an isomorphism whose inverse is induced by Green correspondence on indecomposable $kN_G(P)$-modules. Consequently $[\Gamma(\widetilde{S})] \in D(G)$ and $\mathrm{Res}_P^G([\Gamma(\widetilde{S})]) = [S] \in D(P)^{G-st}$, as required.

(3) *We claim that the kernel of the restriction map* $\mathrm{Res}_P^G : D(G) \longrightarrow D(P)$ *is* $\Gamma(X)$.
First, it was established in Lemma 7.2.1 that $\ker(R_P^{N_G(P)}) = X$. Therefore
$$\ker(\mathrm{Res}_P^{N_G(P)}) = \ker(R_P^{N_G(P)}) \cap D(N_G(P)) = X \cap D(N_G(P)) = X$$
because $X \leq D(N_G(P))$ as noticed in Remark 7.5.7. Furthermore,
$$\begin{aligned}\ker(\mathrm{Res}_P^G) &= (\mathrm{Res}_{N_G(P)}^G)^{-1}\left(\ker(\mathrm{Res}_P^{N_G(P)})\right)\\ &= (\mathrm{Res}_{N_G(P)}^G)^{-1}(X)\\ &= (R_{N_G(P)}^G)^{-1}(X) \cap D(G) = \Gamma(X) \cap D(G)\,.\end{aligned}$$
It remains to show that $\Gamma(X) \leq D(G)$. By the very definition of $\Gamma(X)$, the indecomposable representatives of the classes in $\Gamma(X)$ are $V(\mathcal{F}_G)$-endotrivial modules. Besides, they

are also endo-$p$-permutation modules, thus strongly capped. Indeed, if $\chi \in X$, then its $kG$-Green correspondent $\Gamma(\chi)$ has the same source as $\chi$, that is the trivial module $k \in \mathsf{mod}(kP)$. Therefore $\Gamma(\chi) \mid k{\uparrow}_P^G$, or in other words, it is a $p$-permutation module and thus an endo-$p$-permutation module.

□

Corollary 7.7.4.
*The generalized Dade group $D(G)$ of a finite group $G$ is finitely generated.*

Proof. The group $\Gamma(X) \cong X$ is finite. The group $D(P)^{G-st}$ is finitely generated as a subgroup of $D(P)$, which is finitely generated by [Pui90]. Thus the exact sequence

$$0 \longrightarrow \Gamma(X) \hookrightarrow D(G) \xrightarrow{\mathrm{Res}_P^G} D(P)^{G-st} \longrightarrow 0\,.$$

of the Theorem implies that $D(G)$ is finitely generated too. □

## 7.8. The generalized Dade group and control of $p$-fusion

The Dade group $D(G)$ may always be identified with a subgroup of the Dade group $D(N_G(P))$ of the normaliser of a Sylow $p$-subgroup $P$ of $G$. Then we may naturally ask when these groups are equal. The control of $p$-fusion in $G$ by a subgroup $H$ gives a partial answer to this question.

Proposition 7.8.1.
*Let $H$ be a subgroup of $G$ such that $N_G(P) \leq H \leq G$. Then, up to identification via restriction, $D(G) \cong D(H)$ if and only if $D(P)^{G-st} = D(P)^{H-st}$.*

Proof. Since $H \leq G$, $D(P)^{G-st} \leq D(P)^{H-st}$. Thus, there is a commutative diagram with exact rows given by Theorem 7.7.3

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_G(X) & \longrightarrow & D(G) & \xrightarrow{\mathrm{Res}_P^G} & D(P)^{G-st} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\mathrm{Res}_H^G} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \Gamma_H(X) & \longrightarrow & D(H) & \xrightarrow{\mathrm{Res}_P^H} & D(P)^{H-st} & \longrightarrow & 0
\end{array}$$

where $\Gamma_G(X)$ and $\Gamma_H(X)$ denote the subgroups made of the classes of the Green correspondents of the one-dimensional $kN_G(P)$-modules for $G$ and $H$ respectively. Furthermore, $\Gamma_G(X) \cong X \cong \Gamma_H(X)$. In consequence, it follows from the five-lemma (or the 2/3-lemma) that the map $\mathrm{Res}_H^G$ is surjective if and only if the map $i$ is. Thus, up to identification, $D(G) = D(H)$ if and only if $D(P)^{G-st} = D(P)^{H-st}$. □

Then, we recall that Urfer has already established links between control of $p$-fusion and the $G$-stable points of the Dade group of a $p$-group:

Proposition 7.8.2 ([Urf06], Prop. 2.22).
*Let $P$ be a $p$-subgroup of $G$ and assume that $p$-fusion in $G$ is controlled by $H \leq G$. Then $D(P)^{G-st} = D(P)^{H-st}$. In particular, if $H = N_G(P)$, then $D(P)^{G-st} = D(P)^{N_G(P)}$ is the subgroup of fixed points of $D(P)$ under the action of $N_G(P)$ by conjugation.*

COROLLARY 7.8.3.
*Assume that the $p$-fusion of $G$ is controlled by a subgroup $H \leq G$.*

(a) *If $G \geq H \geq N_G(P)$, then $D(G) = D(H)$.*
(b) *If $N_G(P) \geq H \geq P$, then $D(G) = D(N_G(P))$.*

PROOF.
(a) is a straightforward consequence of the above Propositions 7.8.1 and 7.8.2.
(b) If $N_G(P) \geq H \geq P$, then $N_G(P)$ certainly controls $p$-fusion as well and part (a) yields the result.

□

EXAMPLE 7.8.4.
For instance, it follows from the corollary that any finite group $G$ belonging to one of the following families of groups is such that $D(G) \cong D(N_G(P))$ via restriction:

- $G$ is a group with an abelian Sylow $p$-subgroup $P$. Indeed, in this case $N_G(P)$ controls $p$-fusion in $G$ by Burnside's Theorem.
- $G$ is $p$-nilpotent. Indeed, in this case $P$ controls $p$-fusion.
- $p$ is odd and $G$ is a group with a metacyclic Sylow $p$-subgroup $P$. Then $N_G(P)$ controls $p$-fusion in $G$ because such $p$-groups are resistant to fusion. (See [Sta02].)
- $G$ is a group with a generalised extraspecial Sylow $p$-subgroup $P$, excepting the case when $P = E \times A$ where $A$ is elementary abelian and $E$ is dihedral of order 8 (when $p = 2$) or extraspecial of order $p^3$ and exponent $p$ (when $p$ is odd). Such $p$-groups are also resistant by [Sta02], therefore $N_G(P)$ controls $p$-fusion in $G$.

EXAMPLE 7.8.5.
An example in which $D(G) \lneq D(N_G(P))$ is provided by $G := GL_3(\mathbb{F}_3)$ and its extraspecial Sylow 3-subgroup $P$ of order 27 which consists of the upper unitriangular matrices. This subgroup $P$ is generated by the matrices:

$$x := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; y := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } z := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then it is proven in [Urf07, Section 4] that the class in $D(P)$ of the relative syzygy module $\Omega_{k\uparrow^P_{<x>}}(k)$ is $N_G(P)$-stable but not $G$-stable. Thus $D(P)^{G-st} \lneq D(P)^{N_G(P)}$ and it follows from Proposition 7.8.1 that $D(G) \lneq D(N_G(P))$.

## 7.9. The $p$-nilpotent case

Let $G = N \rtimes P$ be a $p$-nilpotent group as in Chapter 6, so that $P$ controls $p$-fusion in $G$. Thus Corollary 7.8.3 yields $D(G) = D(N_G(P))$, up to identification via restriction.

THEOREM 7.9.1.
*Let $G = N \rtimes P$ be a $p$-nilpotent group. The restriction map $\mathrm{Res}^G_P : D(G) \longrightarrow D(P)$ is split surjective. In consequence there is an isomorphism*

$$D(G) \cong X(N_G(P)) \oplus D(P)\,.$$

PROOF. Theorem 6.2.2 states that the restriction map $\mathrm{Res}^G_P : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_P)}(P)$ is split surjective and moreover that a section is provided by the map

$$T_{V(\mathcal{F}_P)}(P) \xrightarrow{\mathrm{Iso}(\varphi)} T_{V(\mathcal{F}_{G/N})}(G/N) \xrightarrow{\mathrm{Inf}^G_{G/N}} T_{V(\mathcal{F}_G)}(G)\,.$$

Restricting these maps to the Dade groups imply that $\mathrm{Inf}^G_{G/N} \circ \mathrm{Iso}(\varphi) : D(P) \longrightarrow D(G)$ is a section for $\mathrm{Res}^G_P : D(G) \longrightarrow D(P)$. In other words, $\mathrm{Res}^G_P$ is surjective with kernel is $\Gamma(X)$ (by Theorem 7.7.3) so that $D(G)$ decomposes as a direct sum $D(G) \cong \Gamma(X) \oplus D(P)$. Moreover, $\Gamma(X) \cong X(N_G(P))$ by 7.2.1. The result follows. □

## 7.10. The cyclic case

Before going back to more general considerations, we shortly investigate the case of a group $G$ with a non-trivial cyclic Sylow $p$-subgroup $P \cong C_{p^n}$, $n \geq 1$. In this case the classification provided in Chapter 5 for the groups of relative endotrivial modules allows us to determine the generalised Dade group $D(G)$ with ease.

PROPOSITION 7.10.1.
*Let $G$ be a group with a non-trivial cyclic Sylow $p$-subgroup $P \cong C_{p^n}$, $n \geq 1$. Then*

$$D(G) = T_{V(\mathcal{F}_G)}(G)\,.$$

PROOF. Since $P$ is cyclic, it is abelian, thus $N_G(P) =: N$ controls $p$-fusion as noticed in 7.8.4. Therefore $D(G) \cong D(N)$. Next we show that $D(N) = T_{V(\mathcal{F}_N)}(N)$. Write $Z_r$ for the unique cyclic subgroup of $P$ of order $p^r$, then $V(\mathcal{F}_N) = \bigoplus_{s=0}^{n-1} k{\uparrow}^N_{Z_s}$ so that

$$Proj(V(\mathcal{F}_N)) = \bigoplus_{s=0}^{n-1} Proj(k{\uparrow}^N_{Z_s}) = Proj(k{\uparrow}^N_{Z_{n-1}})$$

because $Proj(k{\uparrow}^N_{Z_s}) \subseteq Proj(k{\uparrow}^N_{Z_{n-1}})$ for every $s \leq n-1$ as is well-known from the theory of vertices and sources (otherwise see Proposition 5.3.1). Therefore

$$T_{V(\mathcal{F}_N)}(N) = T_{k{\uparrow}^N_{Z_{n-1}}}(N)\,.$$

In addition, by Theorem 5.4.6, we have

$$T_{k{\uparrow}^N_{Z_{n-1}}}(N) = <X(N), \{\Omega_{k{\uparrow}^N_{Z_s}} \,|\, 0 \leq s \leq n-1\}>$$

where all the generators, that is more precisely their indecomposable representatives, are not only $k{\uparrow}^N_{Z_{n-1}}$-endotrivial modules but also endo-$p$-permutation modules. Indeed $X(N) \leq D(N)$ and the relative syzygy modules $\Omega_{k{\uparrow}^N_{Z_s}}(k)$ are endo-$p$-permutation modules by Lemma 7.5.8. Whence $D(N) = T_{V(\mathcal{F}_N)}(N)$. Finally, recall that , $T_{V(\mathcal{F}_G)}(G) \cong T_{V(\mathcal{F}_N)}(N)$ via restriction, by Lemma 7.2.1. Consequently, we are in the following situation:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{V(\mathcal{F}_G)}(G) & \xrightarrow[\cong]{\mathrm{Res}^G_N} & T_{V(\mathcal{F}_N)}(N) & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \\
0 & \longrightarrow & D(G) & \xrightarrow[\cong]{\mathrm{Res}^G_N} & D(N) & \longrightarrow & 0
\end{array}$$

where the left-hand side vertical arrow is the inclusion of $D(G)$ as a subgroup of $T_{V(\mathcal{F}_G)}(G)$. Thus, the equality $D(G) = T_{V(\mathcal{F}_G)}(G)$ follows by diagram chasing. □

Remark 7.10.2.
In characteristic 2, if the Sylow $p$-subgroup is not cyclic but isomorphic to a Klein group $C_2 \times C_2$, then the situation is similar and it is also easy to show that $D(G) = T_{V(\mathcal{F}_G)}(G)$. We shall treat this case in details in Chapter 8.

## 7.11. The group $D^{\Omega}(G)$

A main source of examples of strongly capped endo-$p$-permutation modules is provided by the $\mathcal{H}$-relative syzygy modules $\Omega_{\mathcal{H}}(k)$ of the trivial module, where $\mathcal{H}$ is a family of subgroups of $G$ such that $Proj(\mathcal{H}) \subseteq Proj(V(\mathcal{F}_G))$. Indeed, as already noticed in section 7.2, these modules are both $V(\mathcal{F}_G)$-endotrivial and endo-$p$-permutation, hence strongly capped.

If $P$ is a $p$-group, with $p$ odd, then one of the main results of the classification of endo-permutation modules asserts that $D(P) = D^{\Omega}(P)$. Therefore one might naturally ask whether or not a similar result holds for the generalised Dade group.

Notation.
Set $D^{\Omega}(G)$ to be the subgroup of $D(G)$ generated by all the relative syzygies $\Omega_{V(\mathcal{H})}$ where $\mathcal{H}$ runs over all the subfamilies of subgroups of $\mathcal{F}_G$.

Lemma 7.11.1.
*The group $D^{\Omega}(G)$ is generated by the relative syzygies $\Omega_{k\uparrow_Q^G}$, where $Q$ runs over the proper subgroups of $P$:*

$$D^{\Omega}(G) = \langle \{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle \,.$$

Proof. If $\mathcal{H} \subseteq \mathcal{F}_G$ is a family of subgroups, set $n_{\mathcal{H}} := \max\{|H| \,|\, H \in \mathcal{H}\}$. We claim that $\Omega_{\mathcal{H}} \in \langle \{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$ for every $\mathcal{H} \subseteq \mathcal{F}_G$ and the proof proceeds by induction on the natural number $n_{\mathcal{H}}$.
First, if $n_{\mathcal{H}} = 1$, then $Proj(\mathcal{H}) = Proj$ is ordinary projectivity, so that

$$\Omega_{\mathcal{H}} = \Omega = \Omega_{\{1_G\}} \in \langle \{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle \,.$$

Then, let $\mathcal{H} := \{H_1, \ldots, H_n\}$ be a subfamily of $\mathcal{F}_G$ such that $n_{\mathcal{H}} \geq 2$ and assume as induction hypothesis that $\Omega_{\mathcal{F}} \in \langle \{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$ for every subfamily $\mathcal{F} \subseteq \mathcal{F}_G$ such that $1 \leq n_{\mathcal{F}} < n_{\mathcal{H}}$. Furthermore, we may assume that $H_i \nleq_G H_j \;\; \forall\, i \neq j, 1 \leq i, j \leq n$ since conjugate subgroups generate the same relative projectivity (cf. Remark 2.5.3). Then, according to Remark 3.8.2 we can write

$$\Omega_{\mathcal{H}} = \sum_{i=1}^{n} \Omega_{\{H_i\}} - \sum_{j=2}^{n} \Omega_{\,{}^G\{H_1,\ldots,H_{j-1}\}\cap\{H_j\}} \text{ in } T_{V(\mathcal{H})}(G)\,.$$

The sum $\sum_{i=1}^{n} \Omega_{\{H_i\}} \in \langle \{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$, clearly. Besides, for every $2 \leq j \leq n$, the family of subgroups ${}^G\{H_1, \ldots, H_{j-1}\} \cap \{H_j\}$ is made up of the subgroups of the form ${}^gH_i \cap H_j$ with $g \in G$ and $1 \leq i \leq j-1$, which all satisfy ${}^gH_i \cap H_j \lneq H_j$ by the above assumption. In consequence, the sum $\sum_{j=2}^{n} \Omega_{\,{}^G\{H_1,\ldots,H_{j-1}\}\cap\{H_j\}}$ belongs to $\langle \{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$ by induction hypothesis, and the result follows. $\square$

Scholium 7.11.2.
*Let $\mathcal{H}$ be a subfamily of $\mathcal{F}_G$. Then $\Omega_{\mathcal{H}} \in \langle \{\Omega_{k\uparrow_Q^G} \,|\, Q \leq H$ for some $H \in \mathcal{H}\}\rangle$.*

We recall that in case $G = P$ is a $p$-group and $p$ is odd, then the Dade group of $P$ is $D(P) = D^{\Omega}(P)$ (see [Bou06]).

**Question**: Does a similar result hold in general for $D(G)$ when $G$ is an arbitrary finite group?

**Non-answer**: Because the one-dimensional representations are always in $D(G)$ this result obviously has to be adapted when $G$ is not a $p$-group anymore. Nonetheless, we claim that in the forthcoming cases, $D(G)$ is $D^{\Omega}(G)$ **modulo** the group of Green correspondents $\Gamma(X)$ of one-dimensional representations of $N_G(P)$ (with $P$ a Sylow $p$-subgroup of G):
(a) when $G$ has a cyclic Sylow $p$-subgroup;
(b) when $p$ is odd, then we shall prove that $D(N_G(P))$ is $D^{\Omega}(N_G(P))$ **modulo** $X(N_G(P))$;
(c) when $N_G(P)$ controls $p$-fusion in the Sylow $p$-subgroup $P$;
(d) it is also true for $G = GL_3(\mathbb{F}_p)$ with $p$ odd.
We leave open the question of determining if this result holds in general.

**(a) The cyclic case.** In case the group $G$ has a cyclic Sylow $p$-subgroup $P$, then it was proven in Proposition 7.10.1 that $D(G) \cong T_{V(\mathcal{F}_G)}(G)$. In addition, the structure Theorem 5.4.6 implies that

$$T_{V(\mathcal{F}_G)}(G) = T_{k\uparrow^G_{Z_{n-1}}}(G) = <\Gamma(X(N_G(P))), \{\Omega_{k\uparrow^{N_G(P)}_{Z_s}} \,|\, 0 \le s \le n-1\}> \ .$$

Hence $D(G)$ is indeed $D^{\Omega}(G)$ modulo $\Gamma(X)$.

**(b) The normal odd case (!):** In order to prove (b), we first recall that a set of generators for $D(P)^{G-st}$ is provided by Urfer in [Urf06] in the following form:

PROPOSITION 7.11.3 ([Urf06], Cor. 3.7).
*Suppose that $p$ is an odd prime and $P$ is a Sylow $p$-subgroup of the group $G$. Then the abelian group $D(P)^{N_G(P)}$ is spanned by the elements*

$$f_Q := \sum_{g\in[N_G(P)/PN_G(P,Q)]} \Omega_{k\uparrow^P_{{}^gQ}}$$

*where $N_G(P,Q) = \{g \in N_G(P) \,|\, {}^gQ = Q\}$ and $Q$ runs over $\mathcal{F}_G$.*

In what follows, we shall consider that $P \trianglelefteq G$, then we note that in this case $N_G(P,Q) = N_G(Q)$ for every subgroup $Q \le P$. We still need another technical result on projectivity relative to $p$-subgroups.

LEMMA 7.11.4.
*Let $G$ be a group with a normal Sylow $p$-subgroup $P$ and $R$ be a proper subgroup of $P$. Then*

$$Proj(k\uparrow^G_R\downarrow^G_P) = Proj(\bigoplus_{x\in[G/PN_G(R)]} k\uparrow^P_{{}^xR})\,.$$

PROOF. The Mackey formula yields

$$Proj(k\uparrow^G_Q\downarrow^G_P) = Proj(\bigoplus_{x\in[G/P]} k\uparrow^P_{{}^xQ})\,.$$

Now, in order to obtain the equality of the statement, recall from the omnibus properties of relative projectivity (Proposition 2.2.2) that if $V, W \in \mathsf{mod}(kG)$ and $Proj(V) = Proj(W)$ then

$Proj(V \oplus W) = Proj(V)$. Therefore, in

$$Proj(\bigoplus_{x\in[G/P]} k\uparrow^P_{{}^xQ}) = \bigoplus_{x\in[G/P]} Proj(k\uparrow^P_{{}^xQ})$$

it is enough to keep only one copy of the summands generating the same relative projectivity. Let $x, y \in G$ and compute:

$$\begin{aligned} Proj(k\uparrow^P_{{}^xQ}) = Proj(k\uparrow^P_{{}^yQ}) &\iff \exists p \in P \text{ such that } {}^{px}Q = {}^{y}Q \\ &\iff y^{-1}x \in PN_G(Q) \text{ since } P \lhd G \\ &\iff y \equiv x \mod PN_G(Q). \end{aligned}$$

Whence $Proj(k\uparrow^G_Q\downarrow^G_P) = Proj(\bigoplus_{x\in[G/PN_G(Q)]} k\uparrow^P_{{}^xQ})$. □

Proposition 7.11.5.
*Let $p$ be an odd prime and $P$ be a normal Sylow $p$-subgroup of $G$. Then the restriction map*

$$\mathrm{Res}^G_P : D^\Omega(G) \twoheadrightarrow D(P)^G$$

*is surjective.*
*More accurately, if $Q \lneq P$, then any generator $f_Q$ of $D(P)^G$ described in proposition 7.11.3 can be expressed as*

$$f_Q = \sum_{g\in[G/PN_G(Q)]} \Omega_{k\uparrow^P_{{}^xQ}} = \mathrm{Res}^G_P(\Omega_{k\uparrow^G_Q}) + X$$

*where $X \in \langle\{f_R \in D(P)^G \mid R \lneq P \text{ and } |R| < |Q|\}\rangle$.*

Proof. The proof proceeds by induction on the order of the subgroup $Q$.

**Case** $|Q| = 1$: $\Omega(k)\downarrow^G_P = \Omega(k) \oplus (proj)$ (by 2.12.2), hence $f_{\{1\}} = \Omega = \mathrm{Res}^G_P(\Omega) \in \mathrm{Res}^G_P(D^\Omega(G))$.

**Induction step**: Let $Q \lneq P$ such that $|Q| > 1$ and assume as induction hypothesis that for every subgroup $S \lneq P$ such that $|S| < |Q|$, the generator $f_S = \sum_{x\in[G/PN_G(S)]} \Omega_{k\uparrow^G_{{}^xS}}$ of $D(P)^G$ belongs to $\mathrm{Res}^G_P(D^\Omega(G))$. Now, in $D(P)$, we have

$$\mathrm{Res}^G_P(\Omega_{k\uparrow^G_Q}) = \Omega_{k\uparrow^G_Q\downarrow^G_P} = \Omega_V$$

where $V := \bigoplus_{x\in[G/P]} k\uparrow^P_{{}^xQ}$, so that the second equality follows from the Mackey formula whereas the first equality follows from Lemma 2.12.2. In fact, in this situation it shall be more fruitful to take the vision of $P$-sets in which $\Omega_V = \Omega_Y$ for $Y$ the $P$-set defined by $Y := \bigsqcup_{x\in[G/P]} P/{}^xQ$. Then [Bou00, Lem. 5.2.3] provides us with the formula

$$\Omega_Y = \sum_{\substack{U,V\in[s_P] \\ U\leq_P V \\ Y^V\neq\emptyset}} \mu_P(U,V)\Omega_{P/U}$$

where $[s_P]$ is a set of representatives of conjugacy classes, under the action of $P$, of subgroups in $P$ and $\mu_P$ is the Möbius function of the poset $([s_P], \leq_P)$. Translating this in terms of $kP$-modules yields:

$$\begin{aligned}
\Omega_V &= \sum_{\substack{U\in[s_P]\\ U\le_G Q}} \Big(\sum_{\substack{V\in[s_P]\\ U\le_P V\le_G Q}} \mu_P(U,V)\Big)\Omega_{k\Uparrow_U^P} \\
&= \sum_{\substack{U\in[s_P]\\ U=_G Q}} \Omega_{k\Uparrow_U^P} + \sum_{\substack{U\in[s_P]\\ U\lneq_G Q}} \Big(\sum_{\substack{V\in[s_P]\\ U\le_P V\le_G Q}} \mu_P(U,V)\Big)\Omega_{k\Uparrow_U^P} \\
&= \underbrace{\sum_{x\in[G/PN_G(Q)]} \Omega_{k\Uparrow_{{}^xQ}^P}}_{f_Q} + \sum_{\substack{U\in[G\backslash[s_P]]\\ U<_G Q}} \Bigg(\Big(\sum_{\substack{V\in[s_P]\\ U\le_P V\le_G Q}} \mu_P(U,V)\Big) \underbrace{\sum_{x\in[G/PN_G(U)]} \Omega_{k\Uparrow_{{}^xU}^P}}_{f_U}\Bigg) \\
&= f_Q + \sum_{\substack{U\in[G\backslash[s_P]]\\ U<_G Q}} \Big(\sum_{\substack{V\in[s_P]\\ U\le_P V\le_G Q}} \mu_P(U,V)\Big) f_U
\end{aligned}$$

where $[G\backslash[s_P]]$ denotes a set of representatives of conjugacy classes of classes of subgroups in $[s_P]$ under the left action of $G$. Then set

$$-X := \sum_{\substack{U\in[G\backslash[s_P]]\\ U<_G Q}} \Big(\sum_{\substack{V\in[s_P]\\ U\le_P V\le_G Q}} \mu_P(U,V)\Big) f_U \;\in\; \langle\{f_R\in D(P)^G \mid R\lneq P, |R|<|Q|\}\rangle\,.$$

Thus $X\in \mathrm{Res}^G_P(D^{\Omega}(G))$ by induction hypothesis and so $f_Q = \mathrm{Res}^G_P(\Omega_{k\Uparrow_Q^G}) + X \in \mathrm{Res}^G_P(D^{\Omega}(G))$ as required. □

THEOREM 7.11.6.
*Let $p$ be an odd prime and $G$ a finite group having a normal Sylow $p$-subgroup. Then*

$$D(G) = X(G) + D^{\Omega}(G)\,.$$

PROOF. This is a direct consequence of the previous proposition together with Theorem 7.7.3 since the latter states that the map $\mathrm{Res}^G_P$ induces an isomorphism $D(G)/X(G)\cong D(P)^G$ and the former that its restriction to $D^{\Omega}(G)$ is surjective. □

REMARK 7.11.7.
Notice that the sum $D(G) = X(G) + D^{\Omega}(G)$ of Theorem 7.11.6 need not be direct. A counterexample is provided by taking $G$ to be a group with a normal Sylow $p$-subgroup isomorphic to a cyclic $p$-group $C_{p^n}$ with $p,n\ge 3$. Indeed, we have shown, on the one hand, in Theorem 7.10.1 that $D(G) = T_{V(\mathcal{F}_G)}(G) = T_{k\Uparrow^G_{Z_{n-1}}}(G)$, and, on the the other hand, in Theorem 5.4.6 that there are relations

$$2\Omega_{k\Uparrow^G_{Z_s}} = [\nu]\ \forall\ 0\le s\le r\,.$$

Thus $[\nu]$ belongs to both $X(G)$ and $D^{\Omega}(G)$. (Notation is that of Chapter 5.)

### (c) $D^{\Omega}$ and control of fusion.

LEMMA 7.11.8.
*Let $H$ be a subgroup of $G$ containing the Sylow $p$-subgroup $P$ and assume moreover that $H$ controls $p$-fusion. Then:*

(a) *The restriction map* $\mathrm{Res}^G_H : D^\Omega(G) \longrightarrow D^\Omega(H)$ *is surjective.*
(b) *If* $N_G(P) \le H \le G$*, then restriction induces an isomorphism* $D^\Omega(G) \cong D^\Omega(H)$*.*

Proof.
(a) The proof is similar to that of Proposition 7.11.5. We claim that for every $Q \lneq P$,

$$\mathrm{Res}^G_H(\Omega_{k\uparrow^G_Q}) = \Omega_{k\uparrow^H_Q} + X$$

with $X \in \langle\{\Omega_R \in D^\Omega(H) \,|\, R \lneq P, |R| < |Q|\}\rangle$. Again we proceed by induction on the order of the subgroup $Q$.

**Case** $|Q| = 1$: $\Omega_{k\uparrow^G_Q} = \Omega$ so that $\mathrm{Res}^G_H(\Omega_{k\uparrow^G_Q}) = \Omega = \Omega_{k\uparrow^H_{\{1\}}} \in \mathrm{Res}^G_H(D^\Omega(G))$.

**Induction step**: Let $Q \lneq P$ be a subgroup such that $|Q| \ge 2$ and assume that $\mathrm{Res}^G_H(\Omega_{k\uparrow^G_S})$ has the required form for every subgroup $S \lneq P$ such that $|S| < |Q|$. Compute, by 2.12.2, that

$$\mathrm{Res}^G_H(\Omega_{k\uparrow^G_Q}) = \Omega_{k\uparrow^G_Q\downarrow^G_H} = \Omega_V\,,$$

where by the Mackey Formula one can set $V := \bigoplus_{x\in[H\backslash G/Q]} k\uparrow^H_{{}^xQ\cap H}$. Then decompose

$$V = \bigoplus_{x\in[H\backslash G/Q]} k\uparrow^H_{{}^xQ\cap H} = \underbrace{\bigoplus_{\substack{x\in[H\backslash G/Q]\\ {}^xQ\le H}} k\uparrow^H_{{}^xQ}}_{=:V_1} \oplus \underbrace{\bigoplus_{\substack{x\in[H\backslash G/Q]\\ {}^xQ\not\le H}} k\uparrow^H_{{}^xQ\cap H}}_{=:V_2}\,.$$

Then, by formula (c) of Lemma 3.8.1, $\Omega_V = \Omega_{V_1} + \Omega_{V_2} - \Omega_{V_1\otimes V_2}$.
Now, firstly, since $H$ controls fusion, for every $x \in [H\backslash G/Q]$ such that ${}^xQ \le H$, there exists $h \in H$, such that ${}^xQ = {}^hQ$. As a consequence $Proj(V_1) = Proj(k\uparrow^H_Q) = Proj({}^x\{Q\})$ by 2.2.2 and thus, by 2.11.14, $\Omega_{V_1} = \Omega_{k\uparrow^H_Q}$.
Secondly, $Proj(V_2)$ corresponds to projectivity relative to the family of subgroups $\mathcal{H} := \{{}^xQ \cap H \,|\, x \in [H\backslash G/Q], {}^xQ \not\le H\}$, all of whose elements have order stricltly smaller than $|Q|$. Therefore Scholium 7.11.2 states that

$$\Omega_{V_2} = \Omega_{\mathcal{H}} \in \langle\{\Omega_{k\uparrow^G_S} \,|\, S \lneq P, |S| < |Q|\}\rangle\,.$$

Thirdly, according to Corollary 2.12.6, $\Omega_{V_1\otimes V_2} = \Omega_{\mathcal{H}\cap {}^H\{Q\}}$. Since $\mathcal{H}$ consists of subgroups all of order strictly smaller than $|Q|$, so does the family $\mathcal{H}\cap {}^H\{Q\}$. Thus, the same argument as above yields

$$\Omega_{V_1\otimes V_2} = \Omega_{\mathcal{H}\cap {}^H\{Q\}} \in \langle\{\Omega_{k\uparrow^G_S} \,|\, S \lneq P, |S| < |Q|\}\rangle\,.$$

Therefore, to sum up, if we set $X := -\Omega_{V_2} + \Omega_{V_1\otimes V_2}$, we get

$$\Omega_{k\uparrow^H_Q} = \mathrm{Res}^G_H(\Omega_{k\uparrow^G_Q}) + X$$

with $X \in \langle\{\Omega_R \in D^\Omega(H) \,|\, R \lneq P, |R| < |Q|\}\rangle$, as required. Then, by induction hypothesis, $X \in \mathrm{Res}^G_H(D^\Omega(G))$ and thus so does $\Omega_{k\uparrow^H_Q}$. Therefore, all the generators of $D^\Omega(H)$ are in $\mathrm{Res}^G_H(D^\Omega(G))$ and the surjectivity of $\mathrm{Res}^G_H : D^\Omega(G) \longrightarrow D^\Omega(H)$ follows.
(b) Since $N_G(P) \le H \le G$, the map $\mathrm{Res}^G_H : D(G) \longrightarrow D(H)$ is injective by 7.6.1. Thus part (a) yields the required isomorphism.

□

Corollary 7.11.9.
*Let* $p$ *be an odd prime. If* $N_G(P)$ *controls* $p$*-fusion, then the Dade group decomposes as*

$$D(G) = D^\Omega(G) + \Gamma(X)\,.$$

Proof. Theorem 7.7.3 provides us with the exact sequence

$$0 \longrightarrow \Gamma(X) \hookrightarrow D(G) \xrightarrow{\mathrm{Res}^G_P} D(P)^{G-st} \longrightarrow 0\,.$$

Thus it suffices to prove that the map $\mathrm{Res}^G_P : D^{\Omega}(G) \longrightarrow D(P)^{G-st}$ is surjective. Indeed, since $N_G(P)$ controls $p$-fusion, $D(P)^{N_G(P)} = D(P)^{G-st}$ by 7.8.2. Therefore,
$\mathrm{Res}^G_P : D^{\Omega}(G) \longrightarrow D(P)^{G-st}$ is equal to the composition

$$D^{\Omega}(G) \xrightarrow{\mathrm{Res}^G_{N_G(P)}} D^{\Omega}(N_G(P)) \xrightarrow{\mathrm{Res}^{N_G(P)}_P} D(P)^{N_G(P)} = D(P)^{G-st}$$

where $\mathrm{Res}^G_{N_G(P)}$ is surjective by Lemma 7.11.8 and $\mathrm{Res}^{N_G(P)}_P$ is surjective by Proposition 7.11.5. Hence the result. □

**(d) The example of $GL_3(\mathbb{F}_p)$.** In this subsection, we let $G := GL_3(\mathbb{F}_p)$ for an odd prime $p$. This group has a Sylow $p$-subgroup $P$ isomorphic to an extraspecial group of order $p^3$ and consisting of the upper unitriangular matrices. Let

$$x := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ y := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } z := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the subgroup $P$ is generated by $x$, $y$ and $z$.

The $G$-stable points of $D(P)$ were computed in [LM09, Example 6.6], and in what follows we use their notation. As a matter of fact, the computation is made for $PSL_3(\mathbb{F}_p)$. However, since the fusion in $P$ is the same under the action of $GL_3(\mathbb{F}_p)$ or under the action of $PSL_3(\mathbb{F}_p)$, $D(P)^{GL_3(\mathbb{F}_p)-st} = D(P)^{PSL_3(\mathbb{F}_p)-st}$.

First, by [BM04], $D(P) = \langle\, \Omega_{k\uparrow^P_Q} \,|\, 1 \le Q \lneq P \,\rangle \cong \mathbb{Z}^{p+2} \oplus (\mathbb{Z}/2\mathbb{Z})^{p+2}$. To simplify let
$e_0 := \Omega$,
$e_1 := \Omega_{k\uparrow^P_{<x>}}$, $e_2 := \Omega_{k\uparrow^P_{<y>}}$,
$e_3 := \Omega_{k\uparrow^P_{<xy>}}, \cdots, e_{p+1} := \Omega_{k\uparrow^P_{<xy^{p-1}>}}$,
$e_{p+2} := \Omega_{k\uparrow^P_{<z>}}$,
$e_{p+3} := \Omega_{k\uparrow^P_{<x,z>}}$, $e_{p+4} := \Omega_{k\uparrow^P_{<y,z>}}$,
$e_{p+5} := \Omega_{k\uparrow^P_{<xy,z>}}, \cdots, e_{2p+3} := \Omega_{k\uparrow^P_{<xy^{p-1},z>}}$ .
In addition, these generators are subject to the following relations: $2e_0 = \sum_{i=1}^{p+1} 2e_i$ and $2e_i = 0$, for all $p+3 \le i \le 2p+3$.

Then $D(P)^{G-st}$ is described in [LM09, Example 6.6] as the group generated by the following elements:

$$f_0 := e_0,\ f_1 := e_{p+4} + e_1,\ f_2 := e_{p+3} + e_2,\ f_3 := \sum_{i=3}^{p+1} e_i,\ f_4 := \sum_{i=p+5}^{2p+3} e_i$$

We claim that this list is not a set of generators for $D(P)^{G-st}$, or more precisely that it misses one element to be a set of generators. Set $f_5 := e_2 + e_{p+2} + e_{p+4} = \Omega_{k\uparrow^P_{<y>}} + \Omega_{k\uparrow^P_{<z>}} + \Omega_{k\uparrow^P_{<y,z>}} \in D(P)$. We claim that

$$D(P)^{G-st} = \langle\, f_0, f_1, f_2, f_3, f_4, f_5 \,\rangle\,.$$

We will not show in details that this list forms a complete set of generators, because this is quite tedious. The method is explained in [LM09]:

$$D(P)^{G-st} = D(P)^{N_G(P)} \cap \bigcap_{\substack{E \leq P \\ E\ p\text{-essential}}} D(P)^{N_G(E)-st}.$$

In the current case $GL_3(\mathbb{F}_p)$ has exactly two $p$-essential subgroups, namely $E_1 :=< x, z >$ and $E_2 :=< y, z >$, both of which contain $N_G(P)$.

$$D(P)^{G-st} = D(P)^{N_G(E_1)-st} \cap D(P)^{N_G(E_2)-st}$$

Now let

$$H_p := N_G(E_1) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$GL_2(\mathbb{F}_p)$}} & * \\ & & * \\ \hline 0 & 0 & \mathbb{F}_p^* \end{array}\right) \text{ and } K_p := N_G(E_2) = \left(\begin{array}{c|cc} \mathbb{F}_p^* & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$GL_2(\mathbb{F}_p)$}} \\ 0 & & \end{array}\right).$$

which are the two maximal parabolic subgroups in $GL_3(\mathbb{F}_p)$. The group $D(P)^{H_p-st}$ is generated by:

$$\Omega_P, \Omega_{P/<y>}, \Omega_{P/<xy>} + \ldots + \Omega_{P/<xy^{p-1}>}$$
$$\Omega_{P/<x,z>}, \Omega_{P/<xy,z>} + \ldots + \Omega_{P/<xy^{p-1},z>}$$
$$\Omega_{P/<z>} + \Omega_{P/<y,z>}$$
$$\Omega_{P/<x>} + \Omega_{P/<y,z>}$$

And the group $D(P)^{K_p-st}$ is generated by:

$$\Omega_P, \Omega_{P/<x>}, \Omega_{P/<xy>} + \ldots + \Omega_{P/<xy^{p-1}>},$$
$$\Omega_{P/<y,z>}, \Omega_{P/<xy,z>} + \ldots + \Omega_{P/<xy^{p-1},z>},$$
$$\Omega_{P/<z>} + \Omega_{P/<x,z>},$$
$$\Omega_{P/<y>} + \Omega_{P/<x,z>},$$

Taking the intersection yields that $D(P)^{G-st}$ is generated by:

$$\Omega_P = f_0$$
$$\Omega_{P/<x>} + \Omega_{P/<y,z>} = f_1,$$
$$\Omega_{P/<y>} + \Omega_{P/<x,z>} = f_2,$$
$$\Omega_{P/<xy>} + \ldots + \Omega_{P/<xy^{p-1}>} = f_3,$$
$$\Omega_{P/<xy,z>} + \ldots + \Omega_{P/<xy^{p-1},z>} = f_4,$$
$$\Omega_{P/<y>} + \Omega_{P/<z>} + \Omega_{P/<y,z>} = f_5,$$

The fact that $f_5$ is $G$-stable will follow from the result we are interested in:

Lemma 7.11.10.
*Let $p$ be an odd prime and let $G := GL_3(\mathbb{F}_3)$. Then* $\mathrm{Res}^G_P(D^\Omega(G)) = D(P)^{G-st}$.

Proof. Exempla gratia, we compute explicitly $\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<x>}})$. First by Lemma 2.12.2, we have $\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<x>}}) = \Omega_{k\uparrow^G_{<x>}\downarrow^G_P}$. By the Mackey Formula

$$k\uparrow^G_{<x>}\downarrow^G_P \cong \bigoplus_{g \in [P\backslash G/<x>]} k\uparrow^P_{{}^g<x>\cap P}.$$

Therefore $Proj(k\uparrow^G_{<x>}\downarrow^G_P) = Proj(kP \oplus k\uparrow^P_{<x>} \oplus k\uparrow^P_{<y>} \oplus k\uparrow^P_{<z>})$ by Proposition 2.2.2. Thus using 2.11.14 and applying formula (c) of Lemma 3.8.1 we get

$$\begin{aligned}\Omega_{k\uparrow^G_{<x>}\downarrow^G_P} &= \Omega_{kP\oplus k\uparrow^P_{<x>}\oplus k\uparrow^P_{<y>}\oplus k\uparrow^P_{<z>}}\\ &= \Omega_{k\uparrow^P_{<x>}\oplus k\uparrow^P_{<y>}\oplus k\uparrow^P_{<z>}} + \Omega - \Omega\\ &= \Omega_{k\uparrow^P_{<x>}} + \Omega_{k\uparrow^P_{<y>}} + \Omega_{k\uparrow^P_{<z>}} - \Omega - \Omega = f_1 + f_5 - 2f_0\,.\end{aligned}$$

The last equality follows from the fact that $\Omega_{P/<y,z>} = e_{p+4}$ has order 2 in $D(P)$.

Similarly, using the Mackey formula, the formulae of Lemma 3.8.1, the fact that $Proj(V) = Proj(V\oplus V)$ (Proposition 2.2.2), the fact that $\Omega_V = \Omega_W$ if $Proj(V) = Proj(W)$ (Lemma 2.11.14), it is easy to compute the following restrictions. Details of the computations are left to the reader.

$$\begin{aligned}&\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{\{1\}}}) = \Omega_{k\uparrow^P_{\{1\}}} = f_0\\ &\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<x,z>}}) = f_2 - f_0\\ &\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<y,z>}}) = f_1 - f_0\\ &\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<xy>}}) = f_3 - (p-2)f_0\\ &\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<xy,z>}}) = f_4 + f_1 + f_5 - \tfrac{p-1}{2}(2\Omega_{k\uparrow^P_{<z>}}) = f_4 + f_1 + f_5 - \tfrac{p-1}{2}(2f_5 - 2f_2)\end{aligned}$$

It follows that all the generators $f_0, f_1, f_2, f_3, f_4, f_5$ belong to $\mathrm{Res}^G_P(D^{\Omega}(G))$. Hence the result.

We leave the details of the computations to the reader. □

Corollary 7.11.11.
$D(G) = D^{\Omega}(G) + \Gamma(X)\,.$

Proof. Similarly to 7.11.9, this is a straightforward consequence of the surjectivity of the restriction map $\mathrm{Res}^G_P : D^{\Omega}(G) \longrightarrow D(P)^{G-st}$. □

Corollary 7.11.12.
*The element $f_5 \in D(P)$ is $G$-stable but does not belong to $\langle\, f_0, f_1, f_2, f_3, f_4 \,\rangle$.*

Proof. Since $f_5$ has $e_{p+2} = \Omega_{k\uparrow^P_{<z>}}$ as a summand, and in view of the relations in $D(P)$, it is clear that $f_5$ is not spanned by $f_0, f_1, f_2, f_3$ and $f_4$. However, according to the proof of Lemma 7.11.10:

$$\begin{aligned}f_5 &= \mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<x>}}) - f_1 + f_0 = \mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<x>}}) - (\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<y,z>}}) + f_0) + f_0\\ &= \mathrm{Res}^G_P(\Omega_{k\uparrow^G_{<x>}} - \Omega_{k\uparrow^G_{<y,z>}}) \in \mathrm{Res}^G_P(D^{\Omega}(G)) = D(P)^{G-st}\end{aligned}$$

by the previous corollary. □

Remark 7.11.13.
Using the above data, it is also easy to compute that the maps $\mathrm{Res}^{H_p}_P : D^{\Omega}(H_p) \longrightarrow D(P)^{H_p-st}$ and $\mathrm{Res}^{K_p}_P : D^{\Omega}(H_p) \longrightarrow D(P)^{K_p-st}$ are surjective.

# CHAPTER 8

# The Klein Case

The aim of this short chapter is to compute the groups of relative endotrivial modules and the Dade group for the very particular case of groups $G$ having a Sylow 2-subgroup $P$ isomorphic to the Klein group $C_2 \times C_2$. Throughout the chapter $k$ denotes an algebraically closed field of characteristic 2.

As a matter of fact, we shall soon realise that nothing much exciting happens and that relative endotrivial modules are not of much interest in this case. For a first intuition, consider the case $G = C_2 \times C_2$, then everyone knows that the complete list of odd-dimensional indecomposable $k[C_2 \times C_2]$-modules is given by the relative syzygies $\Omega^n(k)$ $(n \in \mathbb{Z})$ of the trivial module. In other words, the only candidates to be $V$-endotrivial are indeed already ordinary endotrivial modules. This fact allows us to deduce with ease that in case $P$ is normal in $G$, any group of relative endotrivial modules turns out to be isomorphic to the group $T(G)$ of ordinary endotrivial modules, whose structure is made explicit in [Maz07]. Nonetheless the Klein case is still worth considering because it is a nice source of examples and counter-examples for general behaviours of the groups of relative endotrivial modules.

In case $P$ is not normal in $G$, we do not have in this text a general answer to provide the reader with for the structure of the group $T_V(G)$ for an arbitrary $kG$-module $V$, nevertheless it is not very difficult to show that any such group can be identified with a subgroup of the group $T(N_G(P))$. Moreover, some special cases can be deduced from other properties we have developed in the previous chapters.

## 8.1. Relative endotrivial modules

Theorem 8.1.1.
*Let $G$ be a finite group with a normal Sylow 2-subgroup $P \cong C_2 \times C_2$. Let $V$ be any absolutely 2-divisible $kG$-module. Then there is a group isomorphism*

$$\begin{array}{rccc} \varphi\colon & T_V(G) & \longrightarrow & T(G) \\ & [M]_V & \longmapsto & [M_0] \end{array}$$

*where $M \cong M_0 \oplus (V-proj)$ with $M_0$ the unique indecomposable and $V$-endotrivial summand of $M$. In particular, if $G = C_2 \times C_2$, then $T_V(G) = <[\Omega(k)]_V> \cong \mathbb{Z}$.*

Proof. To begin with, consider the case $G = C_2 \times C_2$ itself. The Klein group is a 2-group, therefore the indecomposable modules that bear chances to be $V$-endotrivial must have odd $k$-dimension. By the classification of indecomposable $k[C_2 \times C_2]$-modules, the odd-dimensional indecomposable modules are precisely the modules $\Omega^n(k)$, $n \in \mathbb{Z}$, which are all endotrivial modules in the usual sense. In consequence, on the one hand, $<[\Omega(k)]_V> \cong <[\Omega(k)]> = T(G) \cong \mathbb{Z}$ and on the other hand, $T_V(G) \cong T(G)$ via $\varphi$. Hence

$$T_V(G) \ = <[\Omega(k)]_V> \cong \mathbb{Z}$$

although the classes in $T_V(G)$ may contain more modules than the classes in $T(G)$. Now, let $G$ be an arbitrary group with a normal Sylow 2-subgroup isomorphic to $C_2 \times C_2$. By 4.1.2, a $kG$-module $M$ is indecomposable $V$-endotrivial if and only if its restriction $M\downarrow^G_{C_2\times C_2}$ is indecomposable and $V\downarrow^G_{C_2\times C_2}$-endotrivial. But we have just shown that any such $k[C_2 \times C_2]$-module is in fact an ordinary endotrivial module hence, by the same criterion, $M$ is endotrivial. In consequence, $\varphi$ is a well-defined group homomorphism. Then, the uniqueness of the summand $M_0$ yields the bijection. $\square$

Remark 8.1.2.
Note that in the normal case, that is $G \trianglerighteq P$, the structure of $T(G)$ is described more accurately in [Maz07, Thm. 2.6] as follows

$$T(G) = X(G) \oplus <\Omega(k)> \cong X(G) \oplus \mathbb{Z}$$

with $X(G)$ denoting the group of one-dimensional representations of $G$. In terms of modules Corollary 4.1.3 tells us that the indecomposable endotrivial $kG$-modules consist of all the extensions to $G$ of the $k[C_2 \times C_2]$-modules $\Omega^n(k)$, $n \in \mathbb{Z}$, which are given by the family of modules $\Omega^n(k) \otimes k_\omega$ such that $n \in \mathbb{Z}$ and $k_\omega$ is a one-dimensional $kG$-module.

Corollary 8.1.3.
*Let $G$ be a finite group with a Sylow 2-subgroup $P \cong C_2 \times C_2$.*

(a) *For any absolutely 2-divisible $kG$-module $V$, the group $T_V(G)$ identifies with a subgroup of $T_{V(\mathcal{F}_G)}(G) \cong T(N_G(P))$.*
(b) *Moreover $D(G) = T_{V(\mathcal{F}_G)}(G)$, up to identification. And, in particular, in the normal case $D(N_G(P)) = T(N_G(P))$.*

Proof. Set $N := N_G(P)$.

(a) Let $V \in \mathsf{mod}(kG)$ be absolutely 2-divisible. The map $\mathrm{Res}^G_N : T_V(G) \longrightarrow T_{V\downarrow^G_N}(N)$ is injective by Lemma 4.2.1. By Lemma 7.2.1, the map $\mathrm{Res}^G_N : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_N)}(N)$ is an isomorphism whose inverse is induced by Green correspondence on the indecomposable $V(\mathcal{F}_N)$-endotrivial modules. Furthermore $T_{V(\mathcal{F}_N)}(N) \cong T(N) \cong T_{V\downarrow^G_N}(N)$ by the preceding theorem. Therefore, the situation is as described in the following diagram:

$$\begin{array}{ccccc} T_{V(\mathcal{F}_G)}(G) & \longleftarrow & - - - - - - - - - - & \hookleftarrow & T_V(G) \\ {\scriptstyle \text{Green corresp.}} \uparrow \ \cong \ \downarrow {\scriptstyle \mathrm{Res}^G_N} & & & & \downarrow {\scriptstyle \mathrm{Res}^G_N} \\ T_{V(\mathcal{F}_N)}(N) & \overset{\cong}{\longleftrightarrow} & T(N) & \overset{\cong}{\longleftrightarrow} & T_{V\downarrow^G_N}(N) \end{array}$$

Thus, if $L$ denotes an indecomposable $V$-endotrivial module, then we can define an injective group homomorphism $T_V(G) \longrightarrow T_{V(\mathcal{F}_G)}(G) : [L]_V \longmapsto [L]_{V(\mathcal{F}_G)}$.
(b) We treat first the normal case. The series of embeddings

$$T(N) \leq D(N) \leq T_{V(\mathcal{F}_N)}(N)$$

and Theorem 8.1.1, which identifies $T(N)$ with $T_{V(\mathcal{F}_N)}(N)$, allow us to conclude that

$$T(N) = D(N) = T_{V(\mathcal{F}_N)}(N)\,.$$

Secondly, for the general case, the fact that $C_2 \times C_2$ is abelian implies that $N$ controls $p$-fusion in $G$ so that $D(G) \cong D(N)$ via restriction, by Corollary 7.8.3. Therefore we are in a similar situation to that of the proof of Proposition 7.10.1 in the cyclic case and have again a commutative diagram of the form:

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_{V(\mathcal{F}_G)}(G) & \xrightarrow[\cong]{\mathrm{Res}^G_N} & T_{V(\mathcal{F}_N)}(N) & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \\
0 & \longrightarrow & D(G) & \xrightarrow[\cong]{\mathrm{Res}^G_N} & D(N) & \longrightarrow & 0
\end{array}$$

Whence $D(G) = T_{V(\mathcal{F}_G)}(G)$.

□

Remarks 8.1.4.

(a) The theorem and its corollary could have been merged in one unique statement saying that the group $T_V(G)$ identifies with a subgroup of the group $T_{V(\mathcal{F}_G)}(G)$, which, moreover, is isomorphic to the Dade group $D(G)$. Nevertheless, we thought it was worth separating the normal and non normal cases, since in the normal case the description of the structure of $T_V(G)$ is complete however the module $V$, but not in the non normal case.

(b) As shows the proof of the corollary, in the non normal case, the only remaining problem to provide a complete description of the structure of the group $T_V(G)$ for an arbitrary module $V$, is the question of determining whether or not the Green correspondents of the ordinary endotrivial $kN_G(P)$-modules are $V$-endotrivial modules. Indeed, since $T(N_G(P)) = X(N_G(P)) \oplus <\Omega(k)>$ and the $kG$-Green correspondent of $\Omega(k)$ is $\Omega(k)$, the real question is to determine which one-dimensional $kN_G(P)$-modules have $V$-endotrivial $kG$-Green correspondents.

(c) Nevertheless, if the Sylow 2-subgroup $C_2 \times C_2$ is strongly 2-embedded in $G$, as for instance if $G = A_5$, then Corollary 4.3.2 ensures that the restriction map

$$\mathrm{Res}^G_{N_G(P)} : T_V(G) \longrightarrow T_{V\downarrow^G_{N_G(P)}}(N_G(P))$$

is an isomorphism for any absolutely 2-divisible $V$. Thus in this case, again there are only endotrivial modules, that is: $T_V(G) \cong T(G) \cong T(N_G(P))$.

(d) If $G$ is a 2-nilpotent group, then Theorem 6.2.2 establishes that there is an isomorphism

$$T_V(G) \cong K_V(G) \oplus T_{V\downarrow^G_P}(P)\,.$$

Furthermore, as just proven $T_{V\downarrow^G_P}(P) \cong T(P) \cong \mathbb{Z}$ and by Corollary 6.2.4 $K_V(G) = X(G)$. Thus in conclusion,

$$T_V(G) \cong X(G) \oplus T(C_2 \times C_2) \cong X(G) \oplus \mathbb{Z}$$

for any absolutely 2-divisible $kG$-module $V$. As a result $T_V(G) \cong T(G)$ for any absolutely 2-divisible $kG$-module $V$, again.

## 8.2. Relative projectivity to modules

Although, in the "normal Klein case", there is, up to isomorphism, only one group of relatively endotrivial $kG$-modules, there are infinitely many different subcategories of $V$-projective modules, which, in particular, do not correspond to projectivity relative to a subgroup.

We give here a complete description of these subcategories for $G = C_2 \times C_2$. Recall that the algebra $k[C_2 \times C_2]$ has domestic representation type and, moreover, that a complete set of representatives of isomorphism classes of indecomposable $k[C_2 \times C_2]$-modules was provided by Bašev, Heller and Reiner. We refer to [Ben98a, section 4.3] for a detailed description. The Green ring structure on $k[C_2 \times C_2]$ is also known and was computed by Bašev [Baš61] and later corrected by Conlon [Con65].

First notice that an indecomposable $k[C_2 \times C_2]$-module is absolutely 2-divisible if and only if it is even-dimensional. Furthermore, these modules are parametrised by $\mathbb{P}^1(k)$ in the following sense: let $\lambda \in \mathbb{P}^1(k)$ and $n \geq 1$ be an integer, then there is a unique $2n$-dimensional indecomposable $k[C_2 \times C_2]$-module with projective variety $\{\lambda\}$, which we denote by $M_{2n}(\lambda)$. (cf. [Ben98a, Thm. 4.3.3] and [Ben98b, Section 5.13].)

LEMMA 8.2.1.
*The indecomposable modules projective relative to $M_{2n}(\lambda)$ are:*

(a) $IProj(M_{2n}(\lambda)) = \{M_{2m}(\lambda) \,|\, 1 \leq m \leq n\} \cup \{k[C_2 \times C_2]\}$ *if* $\lambda = 0, 1, \infty$;
(b) $IProj(M_2(\lambda)) = \{M_2(\lambda), M_4(\lambda), k[C_2 \times C_2]\}$ *if* $\lambda \neq 0, 1, \infty$;
(c) $IProj(M_{2n}(\lambda)) = \{M_{2m}(\lambda) \,|\, 1 \leq m \leq n\} \cup \{k[C_2 \times C_2]\}$ *if* $\lambda \neq 0, 1, \infty$ *and* $n \geq 2$.

PROOF. This follows from the Green ring structure on $k[C_2 \times C_2]$ and this proof only consists in reading the $\otimes_k$-multiplication table for $k[C_2 \times C_2]$-indecomposable modules given in [Con65]. $\square$

# CHAPTER 9

# More Endotrivial-like Modules

In this closing chapter we introduce more *endotrivial-like modules*, which we have not deeply investigated, but that we introduce as possible developments of the research presented in this text and could lead to a different approach to the subject.

## 9.1. A giant group

Due to the facts that, but for exceptions, the group algebra $kG$ has wild representation type and that almost nothing is known about the structure of the Green ring $A(kG)$, it is, at this stage, virtually impossible to compute the structure of the group $T_V(G)$ for an arbitrary finite group $G$ and an arbitrary absolutely $p$-divisible $kG$-module $V$. Hence a need for new methods to treat the subject.

One approach, differing from the ones used thus far is the following: instead of fixing the module $V$, one could start with an indecomposable $kG$-module $M$ such that $\dim_k(M) \equiv \pm 1 \pmod{p}$, consider its endomorphism algebra, which has the form $\mathrm{End}_k(M) \cong k \oplus X$ for some $X \in \mathsf{mod}(kG)$. If $X$ is absolutely $p$-divisible then $M$ is $X$-endotrivial and gives rise to an element of the group $T_X(G)$. Similarly, starting with another indecomposable module $M' \in \mathsf{mod}(kG)$, with dimension equal to $\pm 1 \pmod{p}$, such that $\mathrm{End}_k(M) \cong k \oplus X'$ with $X'$ absolutely $p$-divisible, yields an element $[M'] \in T_{X'}(G)$. The groups $T_X(G)$ and $T_{X'}(G)$ might not have much in common, but one possibility to connect them is to view them as subgroups of the larger group of relative endotrivial modules $T_{X \oplus X'}(G)$.

This procedure leads to the idea of stopping to specify a module $V$, but building a much larger group that contains all the groups $T_V(G)$ for every absolutely $p$-divisible module $V \in \mathsf{mod}(kG)$.

Let $\mathcal{I}$ denote the set of isomorphism classes of indecomposable modules with $k$-dimension divisible by $p$ and let $\mathcal{P}(\mathcal{I})_{<\infty} := \{I \subset \mathcal{I} \mid |I| \leq \infty\}$ denote the set of finite families of modules in $\mathcal{I}$. The set $\mathcal{P}(\mathcal{I})_{<\infty}$ is in bijection with the set of isomorphism classes of finite-dimensional absolutely $p$-divisible $kG$-modules whose indecomposable direct summands all have multiplicity 1. Then, let $\mathbf{E} := \bigoplus_{V \in \mathcal{I}} V \in \mathsf{Mod}(kG)$ and if $I \in \mathcal{P}(\mathcal{I})_{<\infty}$, let $V_I := \bigoplus_{V \in I} V$. Notice that if $I, J \in \mathcal{P}(\mathcal{I})_{<\infty}$, then $Proj(V_{I \cup J}) = Proj(V_I \oplus V_J)$ by the omnibus properties of relative projectivity.

Definition 9.1.1.
A $kG$-module $M \in \mathsf{mod}(kG)$ is called **E**-*endotrivial* if and only if its endomorphism algebra has the form

$$\mathrm{End}_k(M) \cong k \oplus X$$

where $X$ is an absolutely $p$-divisible $kG$-module.

If $M$ and $N$ are $V$-endotrivial and $V'$-endotrivial, respectively, for two absolutely $p$-divisible $kG$-modules $V$ and $V'$, then $M \otimes N$ is certainly a $V \oplus V'$-endotrivial module. Thus, if $T_{\mathbf{E}}(G)$ denotes the set of isomorphism classes of **indecomposable E**-endotrivial modules, it can be endowed with the following group law:

$$[M] + [N] := [(M \otimes N)_0]$$

where $(M \otimes N)_0$ is the unique $(V \oplus V')$-endotrivial summand of $M \otimes N$. In other words, if we consider $T_V(G)$ and $T_{V'}(G)$ as subgroups of the group $T_{V\oplus V'}(G)$, then the class $[M] \in T_V(G)$ and the class $[N] \in T_{V'}(G)$ can be added in $T_{V\oplus V'}(G)$:

$$[M] + [N] = [M \otimes N] \in T_{V\oplus V'}(G)$$

For $I, J \in \mathcal{P}(\mathcal{I})_{<\infty}$, $I \subseteq J$, set $f_{IJ} : T_{V_I}(G) \hookrightarrow T_{V_J}(G)$ to be the canonical inclusions of Lemma 3.7.1. Then $\mathcal{P}(\mathcal{I})_{<\infty}$ ordered by inclusion is a directed set and $(T_{V_I}(G))_{I\in\mathcal{I}}$ and the morphisms $(f_{IJ})_{I,J\in\mathcal{P}(\mathcal{I})_{<\infty}}$ form a directed system of groups. For every $I \in \mathcal{P}(\mathcal{I})_{<\infty}$, define an injective homomorphism

$$\begin{array}{rccc} \varphi_I : & T_{V_I} & \longrightarrow & T_{\mathbf{E}}(G) \\ & [M] & \longmapsto & [M_0] \end{array}$$

where $M_0$ is the unique $V_I$-endotrivial indecomposable summand of $M$. This leads to the following description of $T_{\mathbf{E}}(G)$ as a direct limit:

Proposition 9.1.2.
*Let* $\mathbf{E}$ *and* $\mathcal{I}$ *be as above. Then*

$$T_{\mathbf{E}}(G) \;\cong\; \varinjlim_{I\in\mathcal{P}(\mathcal{I})_{<\infty}} T_{V_I}(G)\,.$$

Proof. By the above construction $T_{\mathbf{E}}$ and the homomorphisms $\varphi_I$, $I \in \mathcal{P}(\mathcal{I})_{<\infty}$ satisfy the universal property of the direct limit . □

Example 9.1.3.
Two first examples, not so interesting, are given by the Klein and cyclic cases:

(a) If $G$ is a finite group having a cyclic Sylow $p$-subgroup, then

$$T_{\mathbf{E}}(G) \cong D(G)\,.$$

For, by Proposition 7.10.1, $T_{V(\mathcal{F}_G)}(G) \cong D(G)$ and moreover, by Proposition 5.3.1, for any absolutely $p$-divisible module $V$, $Proj(V) = Proj(k\uparrow_{Q(V)}^G)$ for some $p$-subgroup $Q(V) < G$ so that $T_V(G) \le T_{V(\mathcal{F}_G)}(G)$ by Lemma 3.7.1.

(b) A similar isomorphism holds in characteristic 2 if $G$ has a Sylow 2-subgroup isomorphic to $C_2 \times C_2$. Indeed, then by Corollary 8.1.3 for any absolutely $p$-divisble module $V$, $T_V(G) \le T_{V(\mathcal{F}_G)}(G) \cong D(G)$. Whence

$$T_{\mathbf{E}}(G) \cong T_{V(\mathcal{F}_G)}(G) \cong D(G)\,.$$

## 9.2. Inspiration from the dihedral 2-groups $D_{4q}$

The reason why we became interested in a group such as $T_{\mathbf{E}}(G)$ is linked to the dihedral 2-groups $D_{4q}$, with $q \geq 2$ a power of 2. For a complete description of the indecomposable $kD_{4q}$-modules, we refer the reader to [Ben98a, Section 4.11].

In contrast with the cyclic and Klein cases, very little is known about the structure of the Green ring $A(kD_{4q})$, and in particular, there is no description of how tensor products of $kD_{4q}$-modules decompose, which makes it difficult to compute the subcategories of relative projective modules $Proj(V)$ for absolutely 2-divisible $V$'s and in consequence to compute the associated groups $T_V(D_{4q})$. However, indecomposable odd-dimensional $kD_{4q}$-modules have remarkable properties proved by L. Archer in [Arc08].

First, for dimensional reasons, the candidates to be indecomposable relative endotrivial modules (that is relatively to a non specified absolutely 2-divisible module) are all the odd-dimensional indecomposable $kD_{4q}$-modules (which are all string modules). Moreover:

Lemma 9.2.1 ([Arc08], Lem. 3.1).
*Let $M, N$ be two odd-dimensional indecomposable $kD_{4q}$-modules. Then, the tensor product $M \otimes N$ decomposes into a direct sum of exactly one odd-dimensional indecomposable summand and even-dimensional summands.*

As a consequence the set of isomorphism classes of indecomposable odd-dimensional $kD_{4q}$-modules can be endowed with an abelian group structure induced by the tensor product by setting:

$$[M] + [N] \mapsto [\textit{the unique odd-dimensional summand of } M \otimes N]$$

This group is denoted $\Gamma(kD_{4q})$ by Archer and it is isomorphic to the group $T_{\mathbf{E}}(D_{4q})$.

Furthermore $\Gamma(kD_{4q})$ is torsion-free by [Arc08, Thm. 3.4] and it is not finitely generated by [Arc08, Thm. 3.5], whence an indication that the group $T_{\mathbf{E}}(G)$ is not finitely generated in general.

## 9.3. Endotrivial modules relative to module varieties

Let $G$ be a finite group and let $\mathcal{V}$ be a closed homogeneous **proper** subvariety of $\mathcal{V}_G$.

Call $\mathcal{V}$-*endotrivial* a module $M \in \mathsf{mod}(kG)$ such that

$$\mathrm{End}_k(M) \cong k \oplus X$$

with $X \in \mathsf{mod}(kG)$ a module such that $\mathcal{V}_G(X) \subseteq \mathcal{V}$.

It follows at once that any $\mathcal{V}$-endotrivial has $k$-dimension coprime to $p$ (by the Benson-Carlson Theorem) and also that its variety is $\mathcal{V}_G$. Also, any module $X$ such that $\mathcal{V}_G(X) \subseteq \mathcal{V}$ has to be absolutely $p$-divisible (otherwise its variety would be $\mathcal{V}_G$.)

Adapting the proofs of Chapter 3, sections 3.3 and 3.6, it is easy to show that:
- the tensor product of two $\mathcal{V}$-endotrivial modules is again $\mathcal{V}$-endotrivial ;
- any $\mathcal{V}$-endotrivial module $M$ decomposes as $M \cong M_0 \oplus Y$, where $M_0$ is indecomposable and $\mathcal{V}$-endotrivial, and $Y$ is such that $\mathcal{V}_G(Y) \subseteq \mathcal{V}$.
(Here the condition that $\mathcal{V}$ is a **proper** subvariety of $\mathcal{V}_G$ plays the role that absolute $p$-divisibility played for $V$-endotrivial modules.)

Thus there is an equivalence relation $\sim_{\mathcal{V}}$ on the class of $\mathcal{V}$-endotrivial $kG$-modules given by

$$M \sim_{\mathcal{V}} N \text{ if and only if } M_0 \cong N_0\,,$$

where $M_0$ and $N_0$ are the unique $\mathcal{V}$-endotrivial indecomposable summands of $M$ and $N$, respectively. Then let $T_{\mathcal{V}}(G)$ denote the resulting set of equivalence classes. It follows that the tensor product $\otimes_k$ induces an abelian group structure on the set $T_{\mathcal{V}}(G)$ defined as follows:

$$[M] + [N] := [M \otimes_k N]$$

The zero element is $[k]$ and the opposite of a class $[M]$ is the class $[M^*]$.
Let $\mathcal{I}(\mathcal{V})$ denote the set of isomorphism classes of modules $V$ in $\mathsf{mod}(kG)$ such that $\mathcal{V}_G(V) \subseteq \mathcal{V}$. If $V \in \mathcal{I}(\mathcal{V})$, then all the modules in $Proj(V)$ also have a variety contained in $\mathcal{V}$ (see Lemma 2.10.1) and thus there is a canonical inclusion $T_V(G) \longrightarrow T_{\mathcal{V}}(G) : [M] \longrightarrow [M]$. In consequence, it is also possible to describe the group $T_{\mathcal{V}}(G)$ as a direct limit:

Lemma 9.3.1.
*Let $\mathcal{V}$ be as above. Then*

$$T_{\mathcal{V}}(G) \;\cong \varinjlim_{\substack{V \in \mathsf{mod}(kG) \\ \mathcal{V}_G(V) \subseteq \mathcal{V}}} T_V(G) \;\cong T_{\mathbf{E}(\mathcal{V})}(G)$$

*where $\mathbf{E}(\mathcal{V}) := \bigoplus_{V \in \mathcal{I}(\mathcal{V})} V \in \mathsf{Mod}(kG)$ and $T_{\mathbf{E}(\mathcal{V})}(G)$ is defined analogously to $T_{\mathbf{E}}(G)$.*

Remark 9.3.2.
As noticed in part (c) of Lemma 2.10.1, there exist indecomposable absolutely $p$-divisible modules $V$ such that $\mathcal{V}_G(V) = \mathcal{V}_G$. Therefore there are also groups of relative endotrivial modules $T_V(G)$ which cannot be seen as subgroups of a group $T_{\mathcal{V}}(G)$ for some variety $\mathcal{V}$.

*To be continued...*

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# Curriculum Vitæ — Caroline Lassueur

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**Spoken languages:** French: mother tongue. English: fluent (CPE with A-pass). German: level C1. Italian: level B1. Maori: basic. Japenese: level A2.

## Education.

*2008-2011*: Preparation of Ph.D. Thesis [EPFL — Advisor: Prof. J. Thévenaz]
*2006-2007*: Master Thesis [Academic exchange, Hamburg University]
*2005-2007*: M.Sc. in Pure Mathematics [EPFL]
*2002-2005*: B.Sc in Mathematics [University of Lausanne + EPFL]
*2001-2002*: Certificate of Proficiency in English (CPE), passed with grade A [Nelson, NZ]
*1998-2001*: Maturité Fédérale (Opt.: Physics and Math., Chemistry, Latin) [Gymnase d'Yverdon (VD)]
*1989-1998*: Ecoles primaires et secondaires inférieures [Vallorbe (VD)]

**Master Thesis**: *The p-local splitting of the infinite complex projective space and co-H-structures.* Written in algebraic topology at Hamburg Univeristy, Germany, under the supervision of Prof. Dr. Birgit Richter and co-supervision of Prof. Kathryn Hess (EPFL).

## Articles.

- *Relative Projectivity and relative endotrivial modules*, J. Algebra, 337: 285-317, 2011.
- In preparation: *The Dade group of a finite group.*

## Some talks.

- *The Dade group of a finite group* [Nov. 2011, Group Theory Seminar, EPFL].
- *From relative projectivity to the Dade group of a finite group* [Jul. 2011, Geometric presentations of finite and infinite groups, Birmingham].
- *From relative projectivity to the Dade group of a finite group* [Jun. 2011, 13th Postgraduate Group Theory Conference, Aberdeen].
- *Relative endotrivial modules: the Klein and cyclic cases* [Sept. 2010, Group Theory Seminar, EPFL].
- *How to recover a group scheme from its representations* [Jul. 2010, Workshop on Tannakian Categories, EPFL].
- *Endotrivial modules and relatives* [Jun. 2010, Group Theory Seminar, EPFL].
- *Endotrivial modules and relatives* [Feb. 2010, 6th Swiss Graduate Colloquium, Geneva].
- *$p$-localizations of the infinite complex projective space and co-H-structures* [Mar. 2007 Topology Seminar, EPFL].

## Teaching/Assistant work at EPFL.

1st and 2nd year Bachelor courses:
- Algebra I & II [Mathematics, 2005-06, 2007-08-09]
- Analysis I [Computer & Communication Sciences, 2009]
- Descriptive Geometry [Architecture, 2010]
- General Mathematics [Forensic Sciences, 2008]
- General Mathematics [HEP, 2008]
- Linear Algebra [Mathematics, 2010, 2010-2011]
- Linear Algebra [Civil Engineering, 2008]
- Mathematics [Architechure, 2010]
- Theoretical Computer Sciences [Computer Sciences, 2004-05]
- Topology II [Mathematics, 2011]

Bachelor and Master courses for mathematicians:
- Group Theory [2009]
- Rings and Modules [2010]
- Groups and Cohomology [2011]
- Semester project coaching [2009, 2011]

**Awards**: EPFL annual award for exceptional teaching received thrice, in 2009, 2010 and 2011.

## Extracurricular activities.

*2006-2010*: Organisation of the *Séminaire Burlet*. Yearly seminar aimed at $2^{nd}$ year B.Sc. students in Mathematics at EPFL, consisting in a week revision work in the Swiss Alps with qualified teaching assistants.

*2003*: Creation of the *Bachelor Seminar* - math talks for and by EPFL B.Sc. math students.

*2008*: Creation of the *Master Seminar* - math talks for and by EPFL Ph.D./M.Sc. math students.

*2005*: Creation of *CQFD* - EPFL's Association of Mathematics Student.

*2005-2010*: Committee member of *CQFD*.

*2010-*$\infty$: Honorary member of *CQFD*.

*2005-2006*: Representative for M. Sc. students at the EPFL *Mathematics Section*.

*2005*: Member of the EPFL *School of Basic Sciences Council*.

*2008-2011*: Representative for Ph.D. students at the EPFL *Mathematics Section*.

*2009-2011*: Representative for EPFL in the *Swiss Doctoral Program in Mathematics Council*.

*2009-2011*: Participation in the organisation of the Swiss semi-finals and finals of the *Championnat International de Jeux Mathématiques et Logiques*.

*2010*: Participation in the organisation of the Conference *Group Representation Theory and Related Topics* in honour of Prof. J. Thévenaz on his 60th birthday

*2011*: Organisation of the *7th Swiss Graduate Colloquium* of the Swiss Doctoral Program in Math.

*2012*: Organisation of the *Young Algebraists' Conference 2012* - Summer School and Conference for Ph.D students in representation theory.