Conjectures on an algorithm for convex parametric quadratic programs

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Abstract

An algorithm for convex parametric QPs is studied. The algorithm explores the parameter space by stepping a sufficiently small distance over the facets of each critical region and thereby identifying the neighboring regions. Some conjectures concerning this algorithm and the structure of the solution of a parametric QP are presented.

Key words: Parametric programming, Quadratic programming, Linear programming.

1 Introduction

Algorithms for solving parametric quadratic programs [2,6] and parametric linear programs [3] have been developed recently. The algorithms proposed in [2] and [3] introduce artificial cuts in the parameter space in the search for the solution, while in [6] an algorithm based on considering all faces of the constraint polyhedron is presented. In [1] and [4] the authors propose a method for exploring the parameter space, which is conceptually more efficient than in [2,3,6]; by stepping a suffi-

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ciently small distance over the boundary of a so-called critical region [1] and solving an LP/QP for the resulting parameter, a new critical region is defined. This procedure looks promising, but seems to implicitly rely on the assumption that the facets of neighboring regions satisfy a certain property, namely that their intersection is a facet of both regions. We will refer to this as the facet-to-facet property. It seems intuitively correct that *if* the facet-to-facet property holds, an algorithm based on stepping over the facets will explore the whole parameter space; however, to the best of our knowledge, a proof that the critical regions satisfy the facet-to-facet property has not been presented in the literature.

In [7,8] the authors propose a method in which each facet of the critical region is examined and depending on whether the facet ensures feasibility or optimality, the active set in the neighboring is found by adding or removing a constraint from the current active set. This algorithm relies on the LICQ assumption and must, in some cases, also step an ϵ-distance over a facet to determine the active set in the adjacent region.

The algorithms presented in [1,2,4,6] are applied to parametric QPs with a positive definite Hessian. We will, in addition to the strictly convex problem, consider the more general formulation given in [8] where the Hessian is allowed to be positive semidefinite, the objective function can be linear and/or include a bilinear term. We state some conjectures that need to be proven before an algorithm based on stepping over the facets will guarantee that the critical regions cover the part of the parameter space that renders the optimization problem feasible.

2 Notation, definitions, problem setup and assumptions

If A is a matrix, then A_i denotes the i^{th} row of A and $A_{\mathcal{I}}$ denotes the rows of A corresponding to the index set $\mathcal{I}$.

Recall that the set of affine combinations of points in a set $S \subset \mathbb{R}^n$ is called the *affine hull* of S. The *dimension of a set* $S \subset \mathbb{R}^n$ is the dimension of the affine hull of S, and is denoted $\dim(S)$; if $\dim(S) = n$, then S is said to be full-dimensional (note that a set is full-dimensional if and only if its interior is non-empty). A *polyhedron* is the intersection of a finite number of closed halfspaces. F is a *face* of the polyhedron $P \subset \mathbb{R}^n$ if there exists a hyperplane $\{z \in \mathbb{R}^n \mid a^T z = b\}$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, such that $F = P \cap \{z \in \mathbb{R}^n \mid a^T z = b\}$ and $a^T z \leq b$, $\forall z \in P$. Given an s-dimensional polyhedron $P \subset \mathbb{R}^n$, where $s \leq n$, the *facets* of P are the $(s-1)$-dimensional faces of P.

[1] A critical region is the set of parameters for which some fixed set of constraints are fulfilled with equality at all solutions of an optimization problem.

Consider the following parametric quadratic program (QP):

$$J^*(\theta) \triangleq \min_{x\in\mathbb{R}^n} \left\{ f(x,\theta) \triangleq \frac{1}{2} x^T H x + \theta^T F^T x + c^T x \mid Ax \leq b + S\theta \right\}, \qquad (1)$$

where $\theta \in \mathbb{R}^s$ is a parameter of the optimization problem, and the vector $x \in \mathbb{R}^n$ is to be optimized for all values of $\theta \in \Theta$, where $\Theta \subseteq \mathbb{R}^s$ is some polyhedral set. Moreover, $H = H^T \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times s}$, $A \in \mathbb{R}^{q\times n}$, $b \in \mathbb{R}^{q\times 1}$, $S \in \mathbb{R}^{q\times s}$ and $c \in \mathbb{R}^{n\times 1}$. If, in addition $H \geq 0$ or $H > 0$, then the parametric QP is convex or strictly convex, respectively. If $H = 0$, then we refer to (1) as a parametric linear program (LP).

The set of parameters for which the minimum in (1) exists, denoted Θ^*, is generally a subset of Θ. If Θ^* is a strict subset of Θ, the set of parameters for which we seek the solution is redefined, i.e. $\Theta \triangleq \Theta^*$. If Θ^* is lower-dimensional, problem (1) can be re-parameterized [3] and one can consider a reduced parameter vector $\bar{\theta}$ such that $\bar{\Theta}^*$ is full-dimensional. Consequently, in the sequel we will make the following assumption:

Assumption 1 *The set of admissible parameters Θ is full-dimensional. We also assume that for all $\theta \in \Theta$, the set of feasible points $X(\theta) \triangleq \{x \in \mathbb{R}^n \mid Ax \leq b + S\theta\}$ is non-empty and the minimum in* (1) *exists.*

Definition 1 (Active set) *Let x be a feasible point of (1) for a given θ. We define the active constraints as the constraints that fulfill $A_i x - b_i - S_i\theta = 0$, and the inactive constraints as those that fulfill $A_i x - b_i - S_i\theta < 0$. The active set $\mathcal{A}(x,\theta)$ is the set of indices of the active constraints, that is,*

$$\mathcal{A}(x,\theta) \triangleq \{i \in \{1,\ldots,q\} \mid A_i x - b_i - S_i\theta = 0\}.$$

Moreover, let $\mathcal{N}(x,\theta)$ denote the set of inactive constraints, that is,

$$\mathcal{N}(x,\theta) \triangleq \{1,\ldots,q\}\backslash\mathcal{A}(x,\theta).$$

Definition 2 (Solution set) *The set of solutions to (1) for a given θ is defined as*

$$X^*(\theta) \triangleq \{x \in \mathbb{R}^n \mid Ax \leq b + S\theta,\ f(x,\theta) = J^*(\theta)\}.$$

Definition 3 (Optimal active set) *Let θ be given, then the optimal active set $\mathcal{A}^*(\theta)$ is the set of constraints that are active for all $x \in X^*(\theta)$, that is*

$$\mathcal{A}^*(\theta) \triangleq \{i \mid i \in \mathcal{A}(x,\theta),\ \forall x \in X^*(\theta)\} = \bigcap_{x\in X^*(\theta)} \mathcal{A}(x,\theta).$$

Let $\mathcal{N}^*(\theta) \triangleq \{1,\ldots,q\}\backslash\mathcal{A}^*(\theta)$.

Definition 4 (Critical region) *Given an optimal active set $\mathcal{A}^*$, the critical region $\Theta_{\mathcal{A}^*}$ is the set of parameters for which the optimal active set remains unchanged, that* *is,*

$$\Theta_{\mathcal{A}^*} \triangleq \{\theta \in \Theta \mid \mathcal{A}^*(\theta) = \mathcal{A}^*\}. \qquad (2)$$

Definition 5 (LICQ) *For an active set $\mathcal{A}$, we say that the linear independence constraint qualification (LICQ) holds if the set of active constraint gradients are linearly independent, i.e. $A_{\mathcal{A}}$ has full row rank.*

3 An algorithm for exploring the parameter space

We will consider the performance of a conceptual algorithm based on stepping a small distance over all facets of a critical region and identifying the optimal active set in all (or some) of the neighboring regions. In [1,4] this algorithm is utilized to solve parametric QPs with a positive definite Hessian. The focal point of this document is to establish the properties (1) must fulfill in order to ensure that the algorithm is well behaved. Before the algorithm is presented, the following properties for the parametric QP (1) should be noted [2,3,7]:

- Critical regions are convex and their closures are polyhedral.
- Θ is convex and polyhedral.
- The optimal active set is unique for all $\theta \in \Theta$.
- Since the optimal active set is unique, critical regions cannot intersect. However, though the intersection of any two full-dimensional critical regions is empty, the intersection of their closures may be non-empty.
- Since the set of admissible parameters Θ is assumed to be full-dimensional and the number of optimal active sets is finite, there exists a finite number of full-dimensional critical regions such that the union of their closures is equal to Θ.

In the light of the properties above the goal of the algorithm considered here is to identify only the full-dimensional critical regions. Since we are only identifying the full-dimensional regions, we will, in conformity with [1–4,6–8], only work with the *closure* of each critical region instead of the region itself. In the sequel, we will therefore abbreviate *closure of the/a full-dimensional critical region* to *critical region*.

The procedure for exploring the parameter space is given in Algorithm 1. The output of Algorithm 1 is a collection $\mathcal{R}$ of closures of full-dimensional critical regions for (1). From this point on, we will let K denote the number of sets in $\mathcal{R}$ and $\mathcal{R}_k$ refer to the k^{th} set in $\mathcal{R}$.

Consider now the question: Under which assumptions on the problem data of (1) will Algorithm 1 guarantee that $\bigcup_{k=1}^{K} \mathcal{R}_k = \Theta$? For this purpose, we introduce the following definition:

Algorithm 1 Exploring the parameter space.

Input: A parameter θ in the interior of a critical region.
Output: Set of critical regions $\mathcal{R}$.

1: Identify $\mathcal{A}^*(\theta)$.
2: Construct the irredundant representation of the critical region $\mathrm{cl}(\Theta_{\mathcal{A}^*(\theta)}) = \{\theta \mid C_i\theta \leq d_i,\ i = 1, \ldots, m\}$.
3: Add $\mathrm{cl}(\Theta_{\mathcal{A}^*(\theta)})$ to the set $\mathcal{R}$ of discovered regions.
4: **for** each facet i in the description of $\mathrm{cl}(\Theta_{\mathcal{A}^*(\theta)})$ **do**
5: Let $\theta_0 = \hat{\theta} + \epsilon C_i$, where $\hat{\theta}$ is such that $C_i\hat{\theta} = d_i$ and $C_j\hat{\theta} < d_j$, for all $j \neq i$, and $\epsilon > 0$ is a sufficiently small scalar such that the resulting parameter point θ_0 is in the interior of a neighboring, critical region $\Theta_{\mathcal{A}^*(\theta_0)}$ in the sense that $\mathrm{cl}(\Theta_{\mathcal{A}^*(\theta)}) \cap \mathrm{cl}(\Theta_{\mathcal{A}^*(\theta_0)}) \neq \emptyset$.
6: If θ_0 is not in a previously discovered critical region, make a recursive call to Algorithm 1 with θ_0 as the new parameter.
7: **end for**

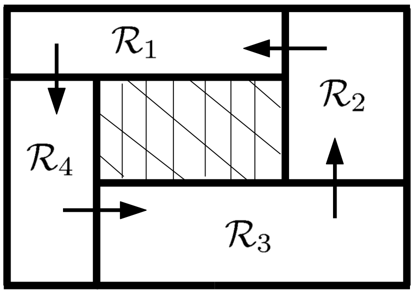


Fig. 1. Illustration of Algorithm 1 failing to identify all the critical regions if the facet-to–facet property does not hold. The shaded region is unexplored.

Definition 6 (Facet-to-facet) *Let* $P \triangleq \{P_i \mid i \in \mathcal{I}\}$ *be a finite collection of full-dimensional polyhedra in* $\mathbb{R}^s$*, where* $\mathrm{int}(P_i) \cap \mathrm{int}(P_j) = \emptyset$ *for all* (i, j)*,* $i \neq j$*. We say that the* facet-to-facet *property holds if* $F_{(i,j)} \triangleq P_i \cap P_j$ *is a facet of both* P_i *and* P_j *for all* $(s-1)$*-dimensional intersections* $F_{(i,j)}$*,* $i \neq j$*.*

It is clear that the facet-to-facet property is directly related to the full-dimensional critical regions for (1). If the closures of the full-dimensional critical regions do not satisfy the facet-to-facet property, then Algorithm 1 may fail to identify all the critical regions, as illustrated in Figure 1.

Figure 2 illustrates that additional assumptions must be made on the collection of full-dimensional critical regions in order to ensure that Algorithm 1 is well behaved. This is because Definition 6 allows for any two members of P to only have intersections of dimension strictly less than $s - 1$. Before we introduce a definition in order to ensure the correctness of Algorithm 1, recall that if $\mathcal{G} \triangleq (N, E)$ is a graph, where N denotes the nodes and E denotes the edges, the graph is said to be *connected* if there is a path from any node N_i to any other node N_j.

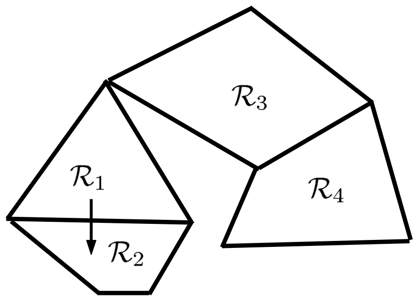


Fig. 2. The collection $\mathcal{R} = \{\mathcal{R}_k \mid k = 1, \ldots, 4\}$ satisfies the facet-to-facet property, but Algorithm 1 fails to identify $\mathcal{R}_3$ and $\mathcal{R}_4$. Note that the situation depicted in this figure cannot happen for (1) since Θ is non-convex in this example.

Definition 7 *Let $P \triangleq \{P_i \mid i \in \mathcal{I}\}$ be a finite collection of full-dimensional polyhedra in $\mathbb{R}^s$, where $\mathrm{int}(P_i) \cap \mathrm{int}(P_j) = \emptyset$ for all (i, j), $i \neq j$. We define the graph $\mathcal{G}(P)$ as*

$$\mathcal{G}(P) \triangleq (P, E(P)),$$

where a pair $(P_i, P_j) \in P \times P$ is in $E(P)$ if and only if $\dim(P_i \cap P_j) = s - 1$.

The next result follows immediately from the above definitions:

Propostion 1 *Let P be the set of closures of the full-dimensional critical regions of (1). If the facet-to-facet property holds for P and $\mathcal{G}(P)$ is a connected graph, then Algorithm 1 will guarantee that $\bigcup_{k=1}^{K} \mathcal{R}_k = \Theta$.*

4 Strictly convex parametric QP

If $H > 0$ in (1), then the problem can be reformulated such that only a quadratic term remains in the objective function [2]. Without loss of generality we use the following formulation for strictly convex parametric QPs:

$$J^*(\theta) \triangleq \min_{x \in \mathbb{R}^n} \left\{ f(x) \triangleq \frac{1}{2} x^T H x \mid Ax \leq b + S\theta \right\}. \tag{3}$$

The KKT conditions for (3) are:

$$Hx + A^T \lambda = 0, \quad \lambda \in \mathbb{R}^q, \tag{4a}$$
$$\lambda_i \left(A_i x - b_i - S_i \theta\right) = 0, \quad i \in \{1, \ldots, q\}, \tag{4b}$$
$$Ax - b - S\theta \leq 0, \tag{4c}$$
$$\lambda_i \geq 0, \quad i \in \{1, \ldots, q\} \tag{4d}$$

where λ are the Lagrangian multipliers. Given an optimal active set $\mathcal{A}^*$ and assuming that LICQ holds, the KKT conditions can be manipulated to obtain [2]

$$x^* = -H^{-1}A_{\mathcal{A}^*}^T \lambda_{\mathcal{A}^*}, \tag{5a}$$
$$\lambda_{\mathcal{A}^*} = -(A_{\mathcal{A}^*}H^{-1}A_{\mathcal{A}^*}^T)^{-1}(b_{\mathcal{A}^*} + S_{\mathcal{A}^*}\theta), \tag{5b}$$

and the closure of the critical region becomes

$$\mathrm{cl}(\Theta_{\mathcal{A}^*}) = \{\theta \in \Theta \mid A_{\mathcal{N}^*}x^*(\theta) \leq b_{\mathcal{N}^*} + S_{\mathcal{N}^*}\theta,\ \lambda_{\mathcal{A}^*}(\theta) \geq 0\}. \tag{6}$$

Conjecture 1 *Let $H > 0$ in (3). If LICQ holds for $A_{\mathcal{A}^*}$ for all optimal active sets that define full-dimensional critical regions for (3), then Algorithm 1 guarantees that $\bigcup_{k=1}^{K} \mathcal{R}_k = \Theta$.*

4.1 Non-unique Lagrangian multipliers

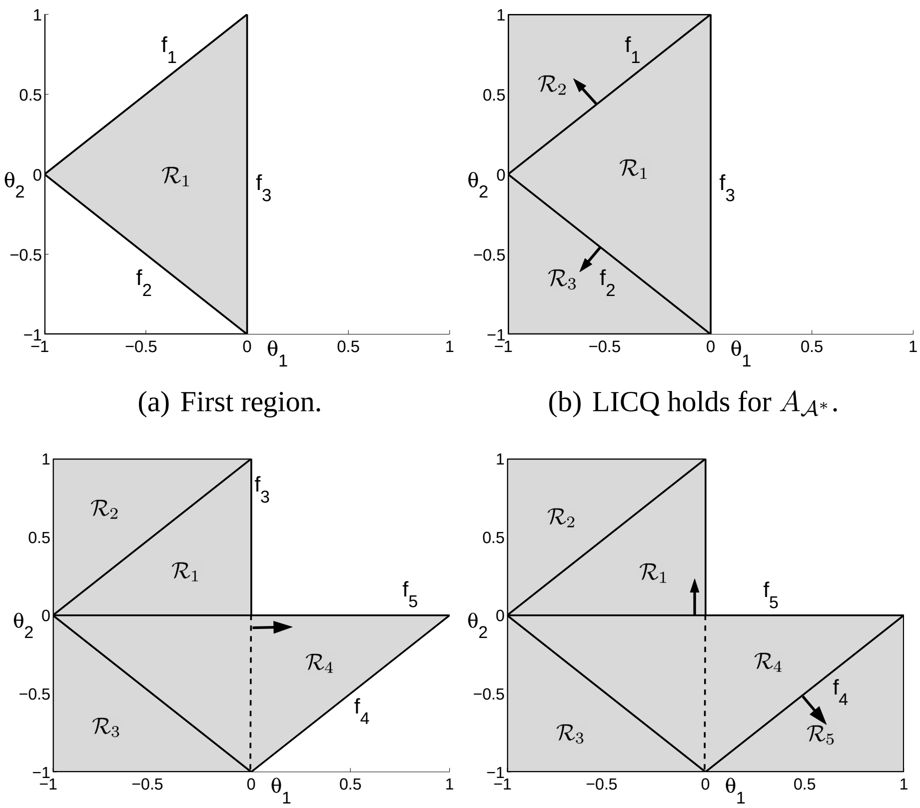
If LICQ is violated for $A_{\mathcal{A}^*}$ then one cannot define $\lambda_{\mathcal{A}^*}$ by (5b). In [2] this is solved simply by selecting a subset of the active constraints such that the resulting system of equalities has full rank. The region is then characterized using (5a) and (5b) on the reduced system. The resulting region is not a critical region in the sense of Definition 4; that is, $\Theta_{\mathcal{A}^*}$ is partitioned into subregions.

Consider the question: Will Algorithm 1 guarantee that $\bigcup_{k=1}^{K} \mathcal{R}_k = \Theta$ for (3) if regions are constructed by using a reduced active set whenever LICQ is violated?

Example 1 Consider the following problem [8]:

$$J^*(\theta) \triangleq \min_{x\in\mathbb{R}^3} \left\{ \frac{1}{2}x^T x \mid x \in X(\theta),\ \theta \in \Theta \right\},$$
$$X(\theta) \triangleq \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{rrcl} x_1 & -x_3 & \leq & -1 + \theta_1 \\ -x_1 & -x_3 & \leq & -1 - \theta_1 \\ x_2 & -x_3 & \leq & -1 - \theta_2 \\ -x_2 & -x_3 & \leq & -1 + \theta_2 \end{array} \right\}, \Theta \triangleq \left\{ \theta \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -1 \leq \theta_1 \leq 1 \\ -1 \leq \theta_2 \leq 1 \end{array} \right\}.$$

Let $\theta_0 = \begin{bmatrix} -0.5 & -0.2 \end{bmatrix}^T$, which results in $\mathcal{A}^*(\theta_0) = \{1, 2, 3, 4\}$ and LICQ is violated for $A_{\mathcal{A}^*}$. If the method in [2] is used, one may choose $\mathcal{A} = \{1, 3, 4\}$ as the reduced active set, and the resulting region is depicted in Figure 3(a). We iterate Algorithm 1 for the closure $\mathcal{R}_1$ of the first critical region. Note that the order in which the facets are stepped over differs slightly from Algorithm 1, however, the concept is the same.



(a) First region.

(b) LICQ holds for $A_{\mathcal{A}^*}$.

(c) LICQ violated for $A_{\mathcal{A}^*}$.

(d) Termination with $\bigcup_{k=1}^{5} \mathcal{R}_k \neq \Theta$.

Fig. 3. Execution of Algorithm 1 for Example 1.

(1) Facet f_1: Crossing this facet at the point indicated in Figure 3(b) yields $\mathcal{A}^* = \{1, 3\}$ and the region $\mathcal{R}_2$.

(2) Facet f_2: Crossing this facet at the point indicated in Figure 3(b) yields $\mathcal{A}^* = \{1, 4\}$ and the region $\mathcal{R}_3$.

(3) Facet f_3: Crossing this facet at the point indicated in Figure 3(c) yields $\mathcal{A}^* = \{1, 2, 3, 4\}$. A valid choice for a reduced active set is $\mathcal{A} = \{1, 2, 4\}$ and region $\mathcal{R}_4$ is constructed. Note that $\mathcal{R}_1$ and $\mathcal{R}_4$ have mutually intersecting interiors.

Since there are no facets to step over for $\mathcal{R}_2$ and $\mathcal{R}_3$, $\mathcal{R}_4$ is considered.

(1) Facet f_4: Crossing this facet at the point indicated in Figure 3(d) yields $\mathcal{A}^* = \{2, 4\}$ and the region $\mathcal{R}_5$.

(2) Facet f_5: Crossing this facet at the point indicated in Figure 3(d) yields a point in $\mathcal{R}_1$, hence no new region is constructed.

The algorithm terminates with $\bigcup_{k=1}^{5} \mathcal{R}_k \neq \Theta$ since there are no more facets to explore.

It is clear that Algorithm 1 may fail to guarantee $\bigcup_{k=1}^{K} \mathcal{R}_k = \Theta$ if a reduced set of active constraints is used to define the regions. By defining regions as in Defi-

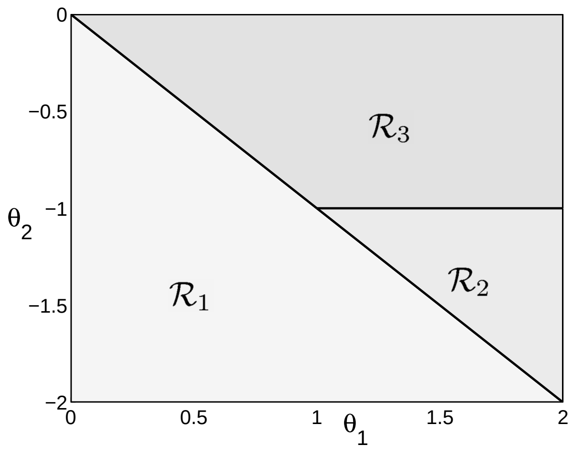


Fig. 4. Full-dimensional critical regions for Example 2.

nition 4, neighboring regions cannot have mutually intersecting interiors. Thus, we state the following conjecture:

Conjecture 2 *Let $H > 0$ in (3). If critical regions are defined as in Definition 4, then Algorithm 1 will guarantee that $\bigcup_{k=1}^{K} \mathcal{R}_k = \Theta$ for (3).*

5 Convex parametric QP

Consider (1) and let $H = H^T \geq 0$. Note that $H \geq 0$ includes the case where $H = 0$. An example will illustrate that the facet-to-facet property may not hold for this problem class.

Example 2

$$J^*(\theta) \triangleq \min_{x \in \mathbb{R}^2} \left\{ \theta_2 x_1 + \theta_1 x_2 \mid x \in X(\theta), \theta \in \Theta \right\},$$

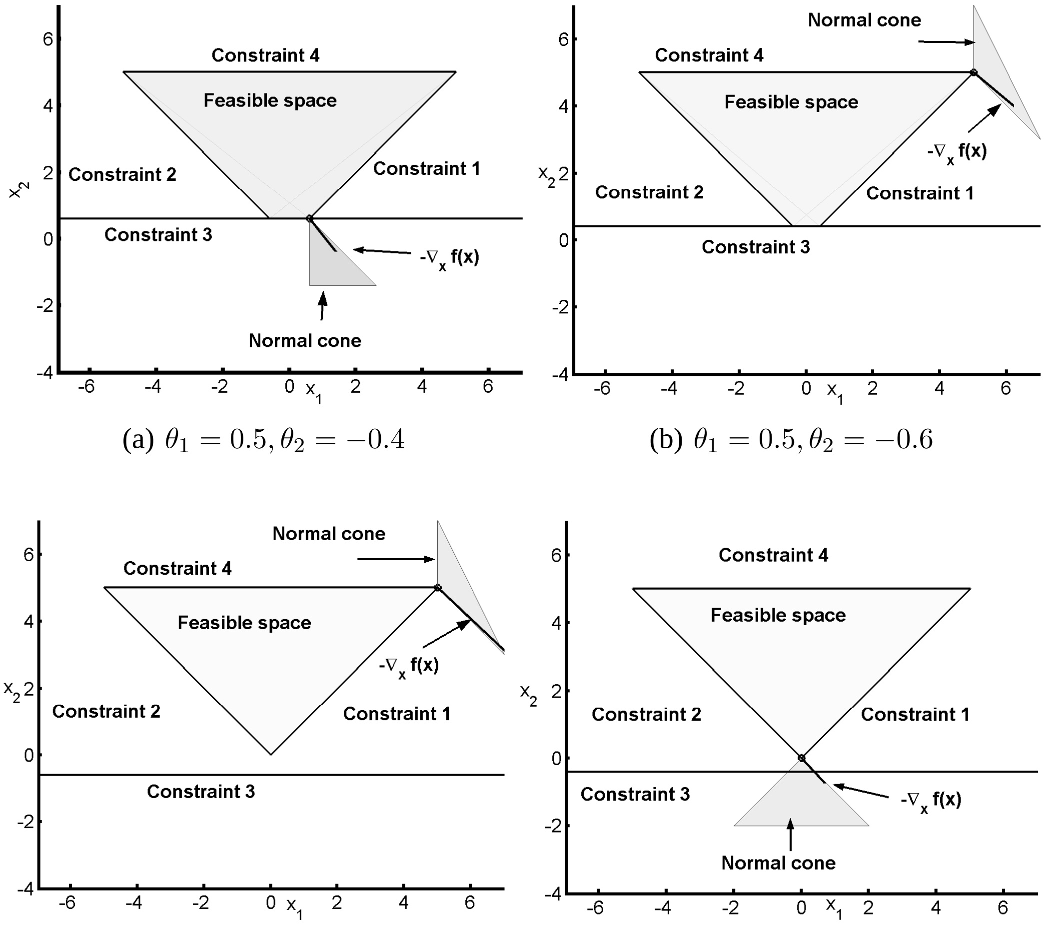
$$X(\theta) \triangleq \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{rrl} x_1 & -x_2 \leq & 0 \\ -x_1 & -x_2 \leq & 0 \\ & -x_2 \leq & -1-\theta_2 \\ & x_2 \leq & 5 \end{array} \right\}, \Theta \triangleq \left\{ \theta \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} 0 \leq \theta_1 \leq 2 \\ -2 \leq \theta_2 \leq 0 \end{array} \right\}.$$

The unique solution for this problem is depicted in Figure 4 and the active sets and optimizers are given in Table 1.

The solution to the problem for some fixed parameter vectors are depicted in Figures 5(a)–5(d). It is clear that by stepping over the diagonal facet of $\mathcal{R}_1$, two regions can be defined, depending on the value of $\hat{\theta}$. Readers that are unfamiliar with the normal cone optimality condition are referred to the appendix. It should

Table 1
Active sets and optimizers for Example (2).

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$
$\mathcal{A}^*(\theta)$	$\{1,4\}$	$\{1,2\}$	$\{1,3\}$
$x^*(\theta)$	$x_1^*(\theta)=5$	$x_1^*(\theta)=0$	$x_1^*(\theta)=\theta_2+1$
	$x_2^*(\theta)=5$	$x_2^*(\theta)=0$	$x_2^*(\theta)=\theta_2+1$



(a) $\theta_1 = 0.5, \theta_2 = -0.4$ (b) $\theta_1 = 0.5, \theta_2 = -0.6$

(c) $\theta_1 = 1.5, \theta_2 = -1.6$ (d) $\theta_1 = 1.5, \theta_2 = -1.4$

Fig. 5. Illustration of Example 2.

be noted that the violation of the facet-to-facet property can just as easily occur if $H \neq 0$. Adding an additional variable x_3 to the problem and modifying the cost to $\frac{1}{2}x_3^2 + \theta_2 x_1 + \theta_1 x_2$ yields the same collection of full-dimensional critical regions.

The violation of the facet-to-facet property seems to be a consequence of the bilinear term in the objective function in combination with parameters on the right hand side of the constraints. In Example 2 the solution changes with the parameter in a discontinuous fashion due to the presence of the bilinear term, and as a result the facet-to-facet property is violated. In the absence of a bilinear term, the point-to-set map $X^*(\cdot)$ is continuous and the authors have not been able to find an example for which the facet-to-facet property is violated. On the other hand, if $S = 0$, but $F \neq 0$, then the solution may change in a discontinuous fashion, but then we are minimizing over a fixed polyhedron and it seems reasonable that the facet-to-facet property holds. We therefore state the following conjecture:

Conjecture 3 *Let $H \geq 0$ in (1). If critical regions are defined as in Definition 4, then Algorithm 1 will guarantee that $\bigcup_{k=1}^{K} \mathcal{R}_k = \Theta$ for (1) if either $F = 0$ or $S = 0$.*

6 Conclusions

We presented some conjectures that need to be proven before an algorithm for parametric QPs based on stepping over each facet of a critical region will guarantee that the whole parameter space is explored. An example showed that one needs to ensure that critical regions are uniquely defined for each parameter vector. A simple example also illustrated that the facet-to-facet property does not hold for a special class of parametric QPs. Current research is devoted to proving the conjectures.

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A Normal cone optimality condition

Consider the following problem

$$\min_x \; f(x) \text{ such that } x \in \Omega. \tag{A.1}$$

where

$$\Omega = \{x \in \mathbb{R}^n \mid g_i(x) = 0, \; i \in \mathcal{E}; \; g_j(x) \leq 0, \; j \in \mathcal{I}\}, \tag{A.2}$$

where $\mathcal{E}$ and $\mathcal{I}$ are finite index sets, f, g_i and g_j are smooth, real-valued functions on a subset of $\mathbb{R}^n$.

The following are taken from [5]:

Definition 8 (Tangent Vector) *A vector $w \in \mathbb{R}^n$ is tangent to Ω at x if for all vector sequences $\{x_i\}$ with $x_i \to x$ and $x_i \in \Omega$, and all positive scalar sequences $t_i \downarrow 0$, there is a sequence $w_i \to w$ such that $x_i + t_i w_i \in \Omega$ for all i.*

Definition 9 (Tangent Cone) *The tangent cone $T_\Omega(x)$ is the collection of all tangent vectors to Ω at x.*

Definition 10 (Normal Cone) *The normal cone to Ω at x, $N_\Omega(x)$, is the orthogonal complement of the tangent cone, that is*

$$N_\Omega(x) = \left\{ v \mid v^T w \leq 0, \quad \forall w \in T_\Omega(x) \right\}. \tag{A.3}$$

Theorem 1 (First order necessary optimality condition) *If x^* is a local minimizer of f in Ω, then*

$$-\nabla_x f(x^*) \in N_\Omega(x^*). \tag{A.4}$$