The Facet-to-Facet Property of Solutions to Convex Parametric Quadratic Programs and a new Exploration Strategy

Jørgen Spjøtvold*‡, Eric C. Kerrigan†§, Colin N. Jones†, Petter Tøndel* and Tor A. Johansen*

***Abstract*— In some of the recently-developed algorithms for convex parametric quadratic programs it is implicitly assumed that the intersection of the closures of two adjacent critical regions is a facet of both closures; this will be referred to as the facet-to-facet property. It is shown by an example, whose solution is unique, that the facet-to-facet property does not hold in general, and consequently, some existing algorithms cannot guarantee that the entire parameter space will be explored. A simple method applicable to several existing algorithms is presented for the purpose of overcoming this problem.**

***Index Terms*— Parametric programming. Quadratic programming. Explicit model predictive control.**

I. Introduction

Several algorithms for solving a convex parametric quadratic program (pQP) [1], [2], [3], [4], [5] and a parametric linear program (pLP) [6] have recently been developed. The growing interest in parametric programming is due to the observation that explicit solutions to model predictive control (MPC) problems can be obtained by solving parametric programs [7], [2], [3]. Parametric linear and quadratic programs are also used as tools in constrained control allocation [8], in the computation of non-conservative penalty weights for the soft constrained linear MPC problem [9], in prioritized infeasibility handling in MPC [10] and for solving sub-problems in parametric nonlinear programming algorithms [11].

The algorithms proposed in [2] and [6] introduce artificial cuts in the parameter space in the search for the solution, while in [3] an algorithm based on considering all combinations of constraints is presented. In [1] and [12] the authors propose a method for exploring the parameter space, which is conceptually and computationally more efficient than in [2], [6] and [3]; by stepping a sufficiently small distance over the boundary of a so-called critical region[1] and solving an LP or QP for the resulting parameter, a new critical region is defined. This procedure looks promising, but implicitly relies on the assumption that the facets of the closures of adjacent critical regions satisfy a certain property, namely that their intersection is a facet of both regions. We will refer to this as the facet-to-facet property.

In [4] and [5] the authors propose a method in which each facet of the critical region is examined and, depending on whether the facet ensures feasibility or optimality, the active set in the neighboring critical region is found by adding or removing a constraint from the current active set. The examination of each facet relies on a number of assumptions and in cases where these assumptions are not satisfied, the algorithm assumes that the facet-to-facet property holds when stepping a small distance over a facet to determine the active set in the adjacent region.

The algorithms presented in [1], [2], [12], [3] and [4] are applied to strictly convex pQPs and utilized to obtain explicit solutions to model predictive control problems. We show by an example that for this problem class a critical region may have more than one adjacent critical region for each facet. Consequently, the facet-to-facet property does not generally hold. Finally, we present a simple and efficient modification of the algorithm in [4], based on results from [2], such that it does not rely on the facet-to-facet property.

II. Preliminaries

If A is a matrix or column vector, then A_i denotes the i^{th} row of A and $A_{\mathcal{I}}$ denotes the sub-matrix of the rows of A corresponding to the index set $\mathcal{I}$. Recall that the set of affine combinations of points in a set $S \subset \mathbb{R}^n$ is called the *affine hull* of S, and is denoted $\operatorname{aff}(S)$. The *dimension of a set* $S \subset \mathbb{R}^n$ is the dimension of $\operatorname{aff}(S)$, and is denoted $\dim(S)$; if $\dim(S) = n$, then S is said to be full-dimensional. The *closure* and *interior* of a set S is denoted $\operatorname{cl}(S)$ and $\operatorname{int}(S)$, respectively. The *relative interior* of a set S is the interior relative to $\operatorname{aff}(S)$, i.e. $\operatorname{relint}(S) := \{x \in S \,|\, B(x,r) \cap \operatorname{aff}(S) \subseteq S \text{ for some } r > 0\}$, where the ball $B(x,r) := \{y \,|\, \|y - x\| \leq r\}$ and $\|\cdot\|$ is any norm. A *polyhedron* is the intersection of a finite number of closed halfspaces. A non-empty set F is a *face* of the polyhedron $P \subset \mathbb{R}^n$ if there exists a hyperplane $\{z \in \mathbb{R}^n \mid a^T z = b\}$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, such that $F = P \cap \{z \in \mathbb{R}^n \mid a^T z = b\}$ and $a^T z \leq b$ for all $z \in P$. Given an s-dimensional polyhedron $P \subset \mathbb{R}^n$, where $s \leq n$, the *facets* of P are the $(s-1)$-dimensional faces of P.

Consider the following strictly convex parametric quadratic program:

$$V^*(\theta) := \min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x^T H x \,\middle|\, Ax \leq b + S\theta \right\}, \tag{1}$$

* Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway. E-mail: {jorgesp,pettert,torj}@itk.ntnu.no

† Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, United Kingdom.

‡ Corresponding author.

§ Royal Academy of Engineering Research Fellow.

[1] A critical region is defined as the set of parameters for which some fixed set of constraints are fulfilled with equality at all solutions of an optimization problem.

where $\theta \in \mathbb{R}^s$ is the *parameter* of the optimization problem, and the vector $x \in \mathbb{R}^n$ is to be optimized for all values of $\theta \in \Theta$, where $\Theta \subseteq \mathbb{R}^s$ is some polyhedral set. Moreover, $H = H^T \in \mathbb{R}^{n\times n}$, $H > 0$, $A \in \mathbb{R}^{q\times n}$, $b \in \mathbb{R}^{q\times 1}$, and $S \in \mathbb{R}^{q\times s}$. For a given parameter θ, the minimizer to (1) is denoted by $x^*(\theta)$. Without loss of generality, the following standing assumption is made [2], [6]:

Assumption 1: The set of admissible parameters Θ is full-dimensional, and for all $\theta \in \Theta$, the set of feasible points $X(\theta) := \{x \in \mathbb{R}^n \mid Ax \leq b + S\theta\}$ is non-empty.

Definition 1 (Optimal active set): Let x be a feasible point of (1) for a given θ. The active constraints are the constraints that fulfill $A_i x - b_i - S_i\theta = 0$. The indices of the constraints that are active at the solution $x^*(\theta)$ is referred to as the *optimal active set* and it is denoted by $\mathcal{A}^*(\theta)$, i.e.

$$\mathcal{A}^*(\theta) := \{i \in \{1, 2, \ldots, q\} \,|A_i x^*(\theta) - b_i - S_i\theta = 0\}.$$

Definition 2 (Critical region): Given an index set $\mathcal{A}$, the critical region $\Theta_{\mathcal{A}}$ associated with $\mathcal{A}$ is the set of parameters for which the optimal active set is equal to $\mathcal{A}$, i.e.

$$\Theta_{\mathcal{A}} := \{\theta \in \Theta \mid \mathcal{A}^*(\theta) = \mathcal{A}\}.$$

In the above definition, note that if $\mathcal{A}$ is not an optimal active set for some parameter, then $\Theta_{\mathcal{A}}$ is the empty set. Hence, when referring to a critical region $\Theta_{\mathcal{A}}$, we will assume that $\mathcal{A}$ is an optimal active set for some $\theta \in \Theta$.

Definition 3 (LICQ): For an index set $\mathcal{A}$, we say that the linear independence constraint qualification (LICQ) holds for $\mathcal{A}$ if the gradients of the set of constraints indexed by $\mathcal{A}$ are linearly independent, i.e. $A_{\mathcal{A}}$ has full row rank.

Theorem 1 (Solution properties [2]): Consider the pQP in (1). The value function $V^* : \Theta \to \mathbb{R}$ is convex and continuous. The minimizer function $x^* : \Theta \to \mathbb{R}^n$ is continuous and piecewise affine in the sense that there exists a finite set of full-dimensional polyhedra $\mathcal{R} := \{R_1, \ldots, R_K\}$ such that $\Theta = \cup_{k=1}^K R_k$, $\operatorname{int}(R_i) \cap \operatorname{int}(R_j) = \emptyset$ for all $i \neq j$ and the restriction $x^*|_{R_k} : R_k \to \mathbb{R}^n$ is affine for all $k \in \{1, \ldots, K\}$.

A method for computing the expression for the restriction (affine function) $x^*|_{R_k}$ and its polyhedral domain R_k is summarized below. The KKT conditions for (1) are:

$$Hx + A^T\lambda = 0, \quad \lambda \in \mathbb{R}^q, \tag{2a}$$
$$\lambda_i\left(A_i x - b_i - S_i\theta\right) = 0, \quad \forall i \in \{1, \ldots, q\}, \tag{2b}$$
$$Ax - b - S\theta \leq 0, \tag{2c}$$
$$\lambda_i \geq 0, \quad \forall i \in \{1, \ldots, q\} \tag{2d}$$

where λ are the Lagrange multipliers. Assume that an index set $\mathcal{A}$ is given such that it is an optimal active set for some parameter $\theta \in \Theta$ and let $\mathcal{N} := \{1, 2, \ldots, q\}\backslash\mathcal{A}$. If LICQ holds for $\mathcal{A}$, then the KKT conditions can be manipulated [2] to obtain the following two affine functions:

$$x^*_{\mathcal{A}}(\theta) := -H^{-1}A^T_{\mathcal{A}}\lambda^*_{\mathcal{A}}(\theta), \tag{3a}$$
$$\lambda^*_{\mathcal{A}}(\theta) := -(A_{\mathcal{A}}H^{-1}A^T_{\mathcal{A}})^{-1}(b_{\mathcal{A}} + S_{\mathcal{A}}\theta). \tag{3b}$$

If R_k is the closure of the critical region associated with $\mathcal{A}$, i.e.

$$R_k := \operatorname{cl}(\Theta_{\mathcal{A}}) = \left\{\theta \in \Theta \,\middle|\, \begin{array}{l} A_{\mathcal{N}}x^*_{\mathcal{A}}(\theta) \leq b_{\mathcal{N}} + S_{\mathcal{N}}\theta \\ \lambda^*_{\mathcal{A}}(\theta) \geq 0 \end{array}\right\}, \tag{4}$$

then the restriction of the minimizer function x^* to the polyhedron R_k is given by $x^*|_{R_k}(\theta) = x^*_{\mathcal{A}}(\theta)$. If LICQ does not hold, then closure of a critical region associated with an optimal active set can be found by projecting a polyhedron in the (x, λ)-space onto the parameter space [2], [5].

Algorithm 1 Exploring the parameter space.

Input: Data to problem (1).
Output: Set of closures of full-dimensional critical regions $\mathcal{R}$.

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Find a θ ∈ Θ such that dim(cl(Θ_{A*(θ)})) = s.
R ← {cl(Θ_{A*(θ)})} and U ← {cl(Θ_{A*(θ)})}.
while U ≠ ∅ do
    Choose any element U ∈ U.
    U ← U\{U}.
    for all facets f of U do
        Find the set S of full-dimensional critical regions
        adjacent to U along the facet f.
        U ← U ∪ (S\R).
        R ← R ∪ S.
    end for
end while
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In the sequel, the closure of a critical region will be written in the more compact form

$$\operatorname{cl}(\Theta_{\mathcal{A}}) =: \{\theta \in \Theta \,|C_i\theta \leq d_i,\ i = 1, \ldots, J\}, \tag{5}$$

where (5) is obtained from (4) or by projection. An inequality $C_i\theta \leq d_i$ in the description of $\operatorname{cl}(\Theta_{\mathcal{A}})$ is said to be *facet-defining* if $\{\theta \,|C_i\theta = d_i\}$ equals the affine hull of one of the facets of $\operatorname{cl}(\Theta_{\mathcal{A}})$. If there exists more than one facet-defining inequality for a given facet, these inequalities are referred to as *coinciding inequalities*. A representation of $\operatorname{cl}(\Theta_{\mathcal{A}})$ where every redundant inequality has been removed is referred to as an *irredundant* representation (note that an irredundant representation does not have any coinciding inequalities).

III. Algorithms for exploring the parameter space

The goal of most algorithms for solving pQPs is to identify only the closures of the full-dimensional critical regions [1], [2], [6], [12], [4], [5]. For this purpose we introduce the notion of *adjacent critical regions*.

Definition 4 (Adjacent critical regions): Two full-dimensional critical regions $\Theta_{\mathcal{A}}$ and $\Theta_{\mathcal{B}}$ are said to be adjacent if $\dim\left(\operatorname{cl}(\Theta_{\mathcal{A}}) \cap \operatorname{cl}(\Theta_{\mathcal{B}})\right) = s - 1$.

The framework for studying the various algorithms is given in Algorithm 1, where the auxiliary set $\mathcal{U}$ is defined as the set of closures of identified regions whose adjacent regions have not been found. The output of Algorithm 1 is a collection $\mathcal{R}$ of closures of full-dimensional critical regions for (1). From this point on, we will let K denote the number of sets in $\mathcal{R}$. Where it is clear from context, R_k will refer to

Procedure 1 Finding an adjacent full-dimensional critical region along a given facet.

Input: Irredundant representation of the closure of a full-dimensional critical region $U =: \{\theta \,|C_i\theta \le d_i,\ i = 1, \dots, J\}$ and the index j whose corresponding inequality defines facet f.

Output: Closure of a full-dimensional critical region $\mathcal{S}$ adjacent to U along the facet f.

$\mathcal{S} \leftarrow \emptyset$.
Choose any $\hat{\theta} \in \operatorname{relint}(f)$.
if the facet f is not on the boundary of Θ **then**
  Choose any scalar $\varepsilon > 0$ such that $\theta := \hat{\theta} + \varepsilon C_j^T \in \Theta$ and θ is in a full-dimensional critical region adjacent to U.
  Compute $\mathcal{A}^*(\theta)$ by solving the QP (1).
  $\mathcal{S} \leftarrow \left\{\operatorname{cl}\left(\Theta_{\mathcal{A}^*(\theta)}\right)\right\}$.
end if

the k^{th} set in $\mathcal{R}$ and $R_{\mathcal{A}}$ will refer to the set in $\mathcal{R}$ associated with the optimal active set $\mathcal{A}$.

We will consider the algorithms in [4], [1], [12] and [5]. It should be noted that, on a conceptual level, these algorithms differ only in step 7 in Algorithm 1 and that the different strategies may not always yield a satisfactory result. This will be addressed in the rest of this section.

A. Identifying adjacent regions from a QP

The procedure used in [1] and [12] as step 7 of Algorithm 1 is given in Procedure 1. This method is also used in The Multi Parametric Toolbox (MPT) [13]. Note that at most one adjacent critical region is identified for each facet of the region under consideration. The implementation of the procedure will not be discussed.

B. Identifying adjacent regions from inequalities

Let $\mathcal{A}$ be a given optimal active set for some $\theta \in \Theta$. The objective is to identify a critical region adjacent to $\Theta_{\mathcal{A}}$ along a given facet f of its closure. Consider the following conditions [4]:

1) LICQ holds for $\mathcal{A}$.
2) There are no coinciding inequalities for facet f in (4), where redundant constraints have not yet been removed.
3) There are no weakly active constraints at $x^*(\theta)$ for all $\theta \in \operatorname{cl}(\Theta_{\mathcal{A}})$, that is, $\nexists i \in \mathcal{A} \Rightarrow \lambda_i^*(\theta) = 0,\ \forall\theta \in \operatorname{cl}(\Theta_{\mathcal{A}})$.

If these conditions hold, then [4] proves that there is only one critical region adjacent to $\Theta_{\mathcal{A}}$ along facet f and that the corresponding optimal active set can be found by determining what type of inequality that defines f. If the inequality that defines f is of the type $\lambda_i \ge 0$, then i is removed from $\mathcal{A}$, hence $\mathcal{S} = \left\{\operatorname{cl}\left(\Theta_{\mathcal{A}\setminus\{i\}}\right)\right\}$. On the other hand, if the inequality is of the type $A_i x^*(\theta) \le b_i + S_i\theta$, then i is added to $\mathcal{A}$, hence $\mathcal{S} = \left\{\operatorname{cl}\left(\Theta_{\mathcal{A}\cup\{i\}}\right)\right\}$. If the conditions do not hold, then Procedure 1 is used. Clearly, as in Section III-A, only one adjacent critical region is identified for each facet with this strategy.

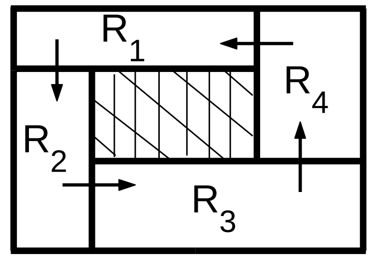

Fig. 1. Illustration that Algorithm 1 may fail to identify all the critical regions if the facet-to-facet property does not hold, the strategies in Section III-A or III-B are employed at step 7 of Algorithm 1 and no additional assumptions on the problem are given. The shaded region is unexplored.

C. Required solution properties

Consider now the question: What conditions must the solution to (1) satisfy in order to ensure that the strategies in Section III-A or III-B guarantee that $\bigcup_{k=1}^{K} R_k = \Theta$? For this purpose, we introduce the following definition:

Definition 5 (Facet-to-facet): Let $\mathcal{P} := \{P_i \mid i \in \mathcal{I}\}$ be a finite collection of full-dimensional polyhedra in $\mathbb{R}^s$, where $\operatorname{int}(P_i) \cap \operatorname{int}(P_j) = \emptyset$ for all (i,j), $i \ne j$. We say that the *facet-to-facet* property holds for $\mathcal{P}$ if $F_{(i,j)} := P_i \cap P_j$ is a facet of both P_i and P_j for all $(s-1)$-dimensional intersections $F_{(i,j)}$.

It is clear that the facet-to-facet property is important when referring to the set of full-dimensional critical regions of (1). If the set of closures of the full-dimensional critical regions do not satisfy the facet-to-facet property, then it may be insufficient to only identify one adjacent region for each facet, as illustrated in Figure 1.

The following example illustrates that the facet-to-facet property does not generally hold for strictly convex pQPs. Hence, the algorithms in [1], [12], [4] and [5] cannot guarantee that the entire parameter space will be explored.

Example 1: Consider the problem:

$$V^*(\theta) := \min_{x\in\mathbb{R}^3} \left\{ \frac{1}{2}x^T x \,\middle|\, x \in \mathcal{P}(\theta) \right\},\ \theta \in \Theta,$$

$$\mathcal{P}(\theta) := \left\{ x \in \mathbb{R}^3 \,\middle|\, \begin{array}{rcl} x_1 - x_3 & \le & -1 + \theta_1 \\ -x_1 - x_3 & \le & -1 - \theta_1 \\ x_2 - x_3 & \le & -1 - \theta_2 \\ -x_2 - x_3 & \le & -1 + \theta_2 \\ \frac{3}{4}x_1 + \frac{16}{25}x_2 - x_3 & \le & -1 + \theta_1 \\ -\frac{3}{4}x_1 - \frac{16}{25}x_2 - x_3 & \le & -1 - \theta_1 \end{array} \right\},$$

$$\Theta := \left\{ \theta \in \mathbb{R}^2 \,\middle|\, -\tfrac{3}{2} \le \ \theta_i \ \le \tfrac{3}{2},\ i = 1, 2 \right\}.$$

The unique set of full-dimensional critical regions is depicted in Figure 2, where we have indexed the critical regions with the optimal active sets. The critical regions $R_{\{1,4,5\}}$, $R_{\{1,3,6\}}$, $R_{\{2,4,5\}}$, and $R_{\{2,3,6\}}$ have more than one adjacent critical region along one of their facets, hence the facet-to-facet property is violated for the set of closures of full-dimensional critical regions.

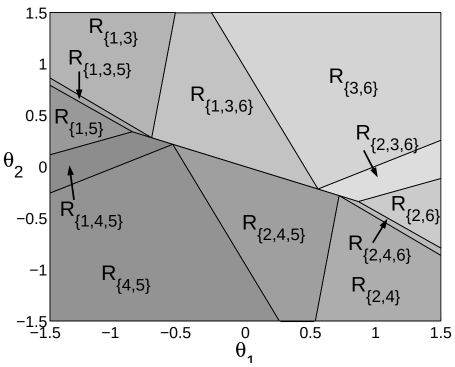

Fig. 2. Facet-to-facet property violated.

In [14] it is verified *analytically* that LICQ holds for all optimal active sets, that the KKT conditions hold for $(x^*(\theta), \lambda^*(\theta))$ for a parameter in the interior of each full-dimensional critical region, and numerically verified that every other combination of active constraints yield empty or lower-dimensional critical regions. Thus, the violation of the facet-to-facet property is not a consequence of numerical inaccuracies. However, there is a lower-dimensional critical region of particular interest, namely the critical region defined by $\mathcal{A} = \{1, \dots, 6\}$, which is analytically computed in [14] as

$$\mathrm{cl}(\Theta_{\{1,\dots,6\}}) = \left\{ \theta \,\middle|\, \theta_1 = -\frac{64}{25}\theta_2, -\frac{1600}{4721} \leq \theta_2 \leq \frac{1600}{4721} \right\}.$$

The representations of $R_{\{1,4,5\}}$, $R_{\{1,3,6\}}$, $R_{\{2,4,5\}}$, $R_{\{2,3,6\}}$, $R_{\{1,3,5\}}$, and $R_{\{2,4,6\}}$ obtained from (4) all have three coinciding inequalities along the line $\theta_1 = -\frac{64}{25}\theta_2$. This suggests that, due to the statements in Section III-B, coinciding inequalities in the description of the critical regions may be the reason for the violation of the facet-to-facet property. Empirical examination also shows that the presented example is not an isolated incident of the facet-to-facet property being violated. By letting the constant values on the right hand side be written as $-[1, 1, 1, 1+\alpha, 1+\alpha, 1+\alpha]^T$, the facet-to-facet property is violated for any $\alpha \in [-\frac{1}{10}, \frac{2}{5}]$.

IV. A NEW EXPLORATION STRATEGY

The algorithm in [2] does not rely on the facet-to-facet property but, as mentioned in the introduction, introduces a number of artificial cuts in the parameter space as it searches for the solution. As a consequence the performance degrades as the number of critical regions become large. In [4] the authors propose a more efficient way of exploring the parameter space, but it relies on the facet-to-facet property. We aim at modifying the algorithm in [4] in order to ensure it's correctness. The proposed method finds all critical regions adjacent to a critical region along a given facet and in order to preserve the efficiency of the algorithm in [4] the modification is to be utilized *only* when the conditions in Section III-B do not hold. We use the algorithm in [2] to explore the parameter space in a small polyhedral subset $M \subset \Theta$ and discard the artificial cuts once the solution has been found. For a given optimal active set $\mathcal{A}$, if the goal is to identify the critical regions adjacent to $\Theta_{\mathcal{A}}$ along a given facet f of its closure, then the polyhedron M must be full-dimensional and satisfy the property:

$$\mathrm{cl}(\Theta_{\mathcal{A}}) \cap M = f.$$

For use in the proposed method, the set of optimal active sets associated with the polyhedron M is defined as:

$$\mathcal{C}(M) := \{ \mathcal{A} \subseteq \{1, 2, \dots, q\} | \dim (M \cap \mathrm{cl}(\Theta_{\mathcal{A}})) = s \}.$$

A method for obtaining all adjacent regions is given in Procedure 2. The choice of ε in step 6 is arbitrary from a theoretical point of view, but it is important to note that too small a value will cause numerical problems and too large a value may result in an unnecessary increase in the computational effort. This issue will be further discussed in Section V. Note that $\mathcal{C}(M_j)$ may define additional critical regions that are not adjacent to the critical region considered and/or critical regions that have already been discovered. However, this is not a problem since one can either choose to keep them as identified regions or discard them. In Procedure 2 we have chosen to return all those critical regions

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Procedure 2 Identifying all adjacent full-dimensional critical regions along a given facet.

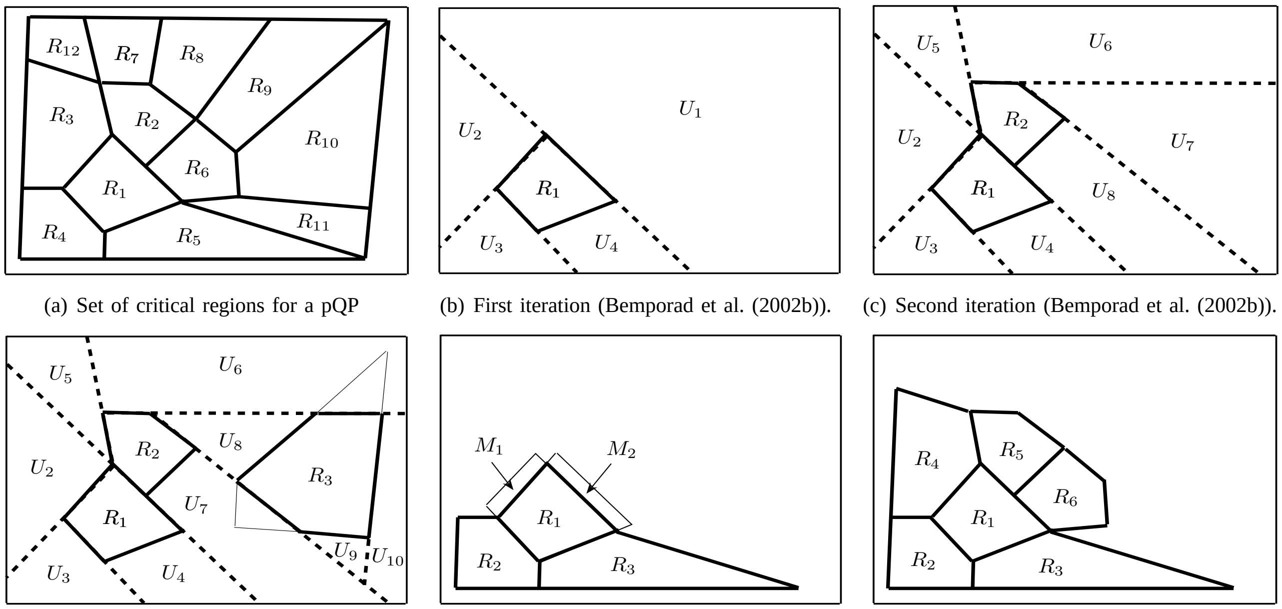
Input: Irredundant representation of the closure of a full-dimensional critical region $U =: \{\theta \,|C_i\theta \leq d_i,\ i = 1, \dots, J\}$ and the index j whose corresponding inequality defines facet f.

Output: Set $\mathcal{S}$ of closures of full-dimensional critical regions adjacent to U along the facet f, and set $\mathcal{T}$ which is a subset of the full-dimensional critical regions not adjacent to U.

1: $\mathcal{S} \leftarrow \emptyset$ and $\mathcal{T} \leftarrow \emptyset$.
2: **if** the facet f is not on the boundary of Θ **then**
3: **if** the conditions in Section III-B hold **then**
4: Find the optimal active set as described in Section III-B and let $\mathcal{T} \leftarrow \mathcal{T} \cup \mathrm{cl}(\Theta_{\mathcal{A}})$.
5: **else**
6: Choose any scalar $\varepsilon > 0$ and construct the polyhedron

$$M_j := \left\{ \theta \in \Theta \,\middle|\, \begin{array}{lcl} C_i\theta & \leq & d_i,\ \forall i \in \{1, \dots, J\} \backslash \{j\} \\ C_j\theta & \geq & d_j \\ C_j\theta & \leq & d_j + \varepsilon \end{array} \right\}.$$

7: Compute the set $\mathcal{C}(M_j)$ by solving the pQP (1) for all $\theta \in M_j$ using the algorithm in [2].
8: **for** each $\mathcal{A} \in \mathcal{C}(M_j)$ **do**
9: **if** $\dim(\mathrm{cl}(\Theta_{\mathcal{A}}) \cap U) = s - 1$ **then**
10: $\mathcal{S} \leftarrow \mathcal{S} \cup \{\mathrm{cl}(\Theta_{\mathcal{A}})\}$. {Adjacent critical region}
11: **else**
12: $\mathcal{T} \leftarrow \mathcal{T} \cup \{\mathrm{cl}(\Theta_{\mathcal{A}})\}$.
13: **end if**
14: **end for**
15: **end if**
16: **end if**



(a) Set of critical regions for a pQP (b) First iteration (Bemporad et al. (2002b)). (c) Second iteration (Bemporad et al. (2002b)).

(d) Third iteration (Bemporad et al. (2002b)). (e) First iteration. proposed method (f) Regions after artificial cuts are discarded

Fig. 3. Illustration of different exploration strategies

which are not adjacent to U and those that have already been discovered; step 8 of Algorithm 1 can be replaced by $\mathcal{U} \leftarrow \mathcal{U} \cup (\mathcal{S} \backslash \mathcal{R}) \cup (\mathcal{T} \backslash \mathcal{R})$ and step 9 by $\mathcal{R} \leftarrow \mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$. We illustrate the difference between the exploration strategy in [2] and the proposed method with an example.

Example 2: Assume that the set of closures of full-dimensional critical regions for a pQP is as depicted in Figure 3(a). The first step of the algorithm in [2] is to find an initial critical region R_1 and then partition the rest of the parameter space into a set of unexplored polyhedra $\mathcal{U}$, see Figure 3(b). It then continues by exploring one of these polyhedra, for instance U_1, finds a new region R_2 and partitions the space again, see Figure 3(c). A possible third iteration is depicted in Figure 3(d). In Figure 3(e) we have shown a possible first iteration of the proposed method. Note that for two facets of R_1 the conditions in Section III-B do not hold, and hence, the sets M_1 and M_2 are constructed. After identifying the optimal active sets in M_j, the set of critical regions is as illustrated in Figure 3(f).

The two key issues we want to illustrate with the above example is that $i)$ for the algorithm in [2] the artificial cuts affect the exploration strategy in parts of the parameter space where the cuts are unnecessary, causing the performance to degrade for large K, and $ii)$ the proposed method discards the artificial partitioning once a set M_j has been fully explored. Since the number of regions intersecting M is expected to be small, the algorithm in [2] is well suited to explore inside M_j.

The efficiency of the algorithm in [4] compared to the one in [2] is well documented, so the performance of the proposed procedure relies on how often the conditions in Section III-B do not hold. Numerical results will be given in the next section. The correctness of the proposed algorithm is proven in Theorem 2.

Theorem 2 (Correctness of the Algorithm): Algorithm 1 combined with Procedure 2 ensures that $\cup_{k=1}^{K} R_k = \Theta$.

Proof: Let $(\mathcal{P}, \mathcal{R})$ be a partition of

$$\{\mathrm{cl}\,(\Theta_{\mathcal{A}})\,|\dim(\Theta_{\mathcal{A}}) = s \text{ for (1)}\},$$

and M_j^R denote the set in Procedure 2 associated with the j^{th} facet of $R \in \mathcal{R}$. Moreover, assume the proposed method terminates with $\cup_{R\in\mathcal{R}} R \subset \Theta$. By the correctness of the algorithm in [2] and the fact that $\dim\left(\mathrm{cl}\,(\Theta_{\mathcal{A}}) \cap M_j^R\right) = s$ if R and $\Theta_{\mathcal{A}}$ are adjacent along the j^{th} facet of R, all full-dimensional critical regions adjacent to R have been identified. Hence, for any pair $(R, P) \in \mathcal{R} \times \mathcal{P}$ we must have $\dim(R \cap P) < s-1$, otherwise P would be a member of $\mathcal{R}$. Moreover, we have $\Theta = (\cup_{R\in\mathcal{R}} R) \bigcup (\cup_{P\in\mathcal{P}} P)$. Hence, by Lemma 1 in [14], a contradiction is reached since Θ cannot be convex when the dimension of the intersection of $\cup_{R\in\mathcal{R}} R$ and $\cup_{P\in\mathcal{P}} P$ is less than $s-1$. ■

V. Numerical example

In this section we make a quantitative comparison of the following exploration strategies: (i) the algorithm in [2], and (ii) the proposed algorithm of combining Algorithm 1 and Procedure 2. The algorithms are tested on an MPC problem for a linear time invariant system

$$z(k+1) = \Phi z(k) + \Gamma u(k),\; z(0) = z_0, \tag{6}$$

where $z(k) \in \mathbb{R}^4$ and $u(k) \in \mathbb{R}^2$ are the state and input at time k, respectively, and Φ and Γ are matrices with suitable dimensions. The objective is to minimize the following cost function

$$J(z_0) := \sum_{k=1}^{N} \left(z(k)^T Q z(k) + u(k-1)^T R u(k-1) \right)$$

	LP^1	LP^2	QP	Times found
$\varepsilon = 10^{-4}$	$5.7 \cdot 10^4$	$4.5 \cdot 10^3$	$1.8 \cdot 10^3$	7.6
$\varepsilon = 10^{-2}$	$7.9 \cdot 10^4$	$5.8 \cdot 10^3$	$2.4 \cdot 10^3$	8.3
$\varepsilon = 0.5$	$2.5 \cdot 10^5$	$1.4 \cdot 10^4$	$6.6 \cdot 10^3$	9.9
Bemporad et al.	$2.8 \cdot 10^5$	$3.7 \cdot 10^4$	$6.8 \cdot 10^3$	17.2

TABLE I

SIMULATION RESULT FOR RANDOM DATA.

where $Q = Q^T \geq 0$ and $R = R^T > 0$, subject to the system equation (6), state constraints $z \in \mathcal{Z} := \{z \,|\, \underline{z} \leq z \leq \overline{z}\}$, and input constraints $u \in \mathcal{U} := \{u \,|\, \underline{u} \leq u \leq \overline{u}\}$. This problem is recast as a pQP as described in [2] and the algorithms are tested on 80 random instances of $(\Phi, \Gamma, Q, R, \mathcal{Z}, \mathcal{U})$ with a prediction horizon $N \in [3, 5]$. For simplicity, all systems are stable, controllable and observable. The results are given in Table I, where we have also tried different values for ε, and used the following abbreviations: LP^1: Average number of LPs solved to obtain irredundant representations of polyhedra, LP^2: Average number of LPs solved to find an interior-point of a polyhedron, QP: Average number of QPs solved obtain optimal active sets, and Times found: Average number of times a critical region is discovered. The solutions have an average of 317 critical regions. In Figure 4 the total number of optimization problems solved by the algorithms are compared.

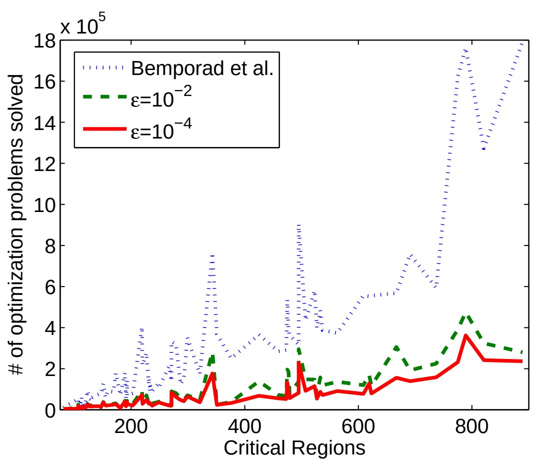


Fig. 4. Comparison of the number of optimization problems solved by the algorithm, that is, the sum of LP^1, LP^2 and QP.

As expected, the computational effort used to find an explicit solution is on average lowest for alternative (ii). This shows that alternative (ii) is preferable also in practice. Note that although the performance of the proposed method relies on the choice of ε, it is not difficult to chose a value such that the proposed method is more efficient than the algorithm in [2]. Even for the inappropriate choice of $\epsilon = 0.5$, the computational effort is lower. Also, from Figure 4 it is apparent that the difference in the computational effort is expected to grow as K increases.

VI. Conclusion

It has been shown by an example that, for strictly convex parametric quadratic programs, a critical region may have more than one adjacent critical region for each facet. This renders some of the recently developed algorithms for this problem class without guarantees that the entire parameter space will be explored. A simple and efficient method based on the algorithms in [2] and [4] was proposed such that the completeness of the exploration strategy is guaranteed. Numerical results also show that the proposed method is computationally more efficient than the algorithm in [2].

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