Coloring Fuzzy Circular Interval Graphs

Friedrich Eisenbrand[a], Martin Niemeier[a,*]

[a]*École Polytechnique Fédérale de Lausanne, Discrete Optimization Group, Station 8, CH-1015 Lausanne*

Abstract

Given a graph G with nonnegative node labels w, a multiset of *stable sets* $S_1, \dots, S_k \subseteq V(G)$ such that each vertex $v \in V(G)$ is contained in $w(v)$ many of these stable sets is called a weighted coloring. The *weighted coloring number* $\chi_w(G)$ is the smallest k such that there exist stable sets as above.

We provide a polynomial time combinatorial algorithm that computes the weighted coloring number and the corresponding colorings for fuzzy circular interval graphs. The algorithm reduces the problem to the case of circular interval graphs, then making use of a coloring algorithm by Gijswijt.

We also show that the stable set polytopes of fuzzy circular interval graphs have the integer decomposition property.

Keywords: fuzzy circular interval graph, circular interval graph, vertex coloring, weighted coloring, integer decomposition property

1. Introduction

A *weighted k coloring* of a graph G with weights $w : V(G) \longrightarrow \mathbb{N}_0$ is a multiset of *stable sets* $S_1, \dots, S_k \subseteq V(G)$ such that each vertex $v \in V(G)$ is contained in $w(v)$ many of these stable sets. The *weighted coloring number* $\chi_w(G)$ is the smallest k such that there exist stable sets as above. The problem of bounding and computing the weighted coloring number of graphs is a classical topic in combinatorics and graph theory and, for the class of quasi-line graphs and more specifically fuzzy circular interval graphs, has received a lot of attention recently.

From a polyhedral perspective, the weighted coloring problem has an interesting connection to *integer decompositions* in the stable set polytopes of graphs. A polyhedron $P \subseteq \mathbb{R}^n$ has the *integer decomposition property*, if each integer vector $z \in \mathbb{Z}^n$ that is contained in $k \cdot P$ for some $k \in \mathbb{N}$ can be decomposed into k integer vectors of P, i.e. there exist integer vectors $z_1, \dots, z_k \in P$ such that

$$z = \sum_{i=1}^{k} z_i.$$

*Corresponding author (Phone: +41 (0)21 693 2739, Fax: +41 (0)21 693 5840)

Email addresses: `friedrich.eisenbrand@epfl.ch` (Friedrich Eisenbrand), `martin.niemeier@epfl.ch` (Martin Niemeier)

The vectors $z_1, \ldots, z_k$ are called a *k integer decomposition of z in P*. There is a one to one correspondence between weighted colorings of a graph with weights w and integer decompositions of w in its stable set polytope. Moreover, if a stable set polytope has the integer decomposition property, and the maximum weighted stable set (MWSS) problem can be solved in polynomial time, then the weighted coloring number can be computed in polynomial time, via the equivalence of separation and optimization [1].

A graph is *quasi-line* if the neighborhood of each of its vertices is the union of two cliques. Chudnovsky and Seymor [2] provided a structural result that states that a connected quasi-line graph is a *fuzzy circular interval graph* or it is the composition of fuzzy linear interval strips with a collection of disjoint cliques. In particular line-graphs are quasi-line, and thus the weighted coloring problem for quasi-line graphs subsumes the NP-complete edge-coloring problem, see, e.g. [3].

In this paper we consider the subclass of fuzzy circular interval graphs and show that the weighted coloring problem can be solved in polynomial time. We will present two approaches to the problem: A purely combinatorial approach and a polyhedral approach based on linear programming. Both approaches work by reduction to circular interval graphs, exploiting their properties [4, 5].

Our contribution. We present an efficient combinatorial algorithm to not only compute the coloring number, but also an optimal weighted coloring for fuzzy circular interval graphs. For a fuzzy circular interval graph G, it computes the weighted coloring number alone in time

$$\mathcal{O}\left(|V(G)|^2 \operatorname{size}(w)\right).$$

Given the coloring number, it computes an optimal weighted coloring in time

$$\mathcal{O}\left(|V(G)|^4 + \operatorname{size}(w)\right).$$

Here $\operatorname{size}(w)$ denotes the binary encoding length of w. The algorithm is based on a reduction to circular interval graphs using an algorithm for maximum b-matching and an algorithm of Gijswijt [5] to solve the weighted coloring problem on circular interval graphs. Our algorithm requires a so called *representation* of the fuzzy circular interval graph as input. Such a representation can be computed in time $\mathcal{O}\left(|V(G)|^2 \cdot |E(G)|\right) = \mathcal{O}\left(|V(G)|^4\right)$, see [6].

We also show that the stable set polytopes of fuzzy circular interval graphs have the integer decomposition property, which leads to a linear programming based approach to compute the weighted coloring number.

The organization of this paper is as follows. In Section 1.1, we review important structural properties of fuzzy circular interval graphs that will be exploited by our algorithm. In Section 2 we present our combinatorial coloring algorithm. Finally in Section 3 we elaborate in more detail on the relation between weighted colorings and integer decompositions and prove the integer decomposition property for the stable set polytope of fuzzy circular interval graphs.

1.1. The structure of circular interval graphs

We will review some definitions and structural properties concerning fuzzy circular interval graphs due to Chudnovsky and Seymour [2] useful throughout the paper. We start with some general notion. Given a graph G and a set of nodes $S \subseteq V(G)$, a node $v \in V(G)$ is said to be S-*complete*, if v is adjacent to every node of S. If v is adjacent to none of the nodes of S, v is said to be S-*anticomplete*. Given a node $v \in V(G)$, we define the *neighborhood* of v as

$$\mathcal{N}_G(v) := \{u \in V(G) \ : \ \{u, v\} \in E(G)\} .$$

Circular interval graphs are graphs G that can be obtained with the following construction. Let $V(G)$ be a subset of a circle $\mathcal{C}$. Further take a set $\mathcal{I}$ of intervals of the circle $\mathcal{C}$. The set of edges $E(G)$ is defined as follows: Two vertices are adjacent if and only if they are contained in a common interval of $\mathcal{I}$.

The pair $(V, \mathcal{I})$ completely describes a circular interval graph and is called a *representation* of G. These representations can be computed in linear time [7, 8, 9]. Figure 1 shows an example for a circular interval graph.

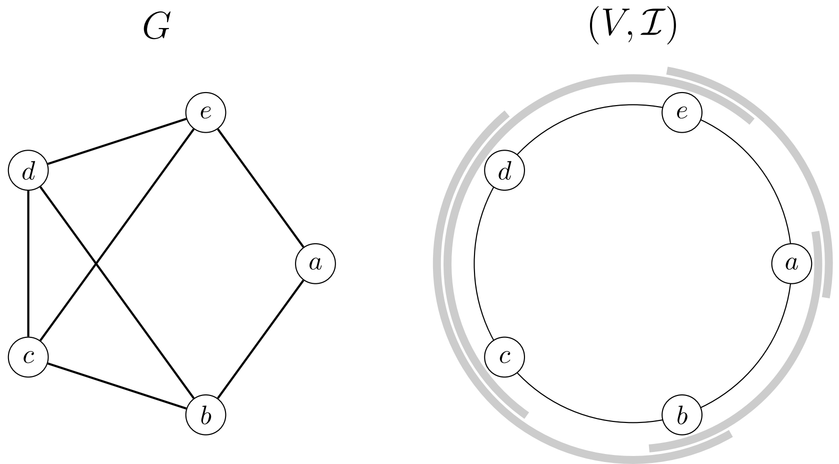

Figure 1: A circular interval graph with its representation.

Circular interval graphs can be colored efficiently. There is a combinatorial algorithm by Gijswijt [5] via integer decompositions for the stable set polytope of circular interval graphs.His result is the following.

Theorem 1 ([5]). *Given a circular interval graph G with weights w, for every $k \in \mathbb{N}$ we can decide if a weighted k coloring of (G, w) exists in time $\mathcal{O}\left(|V(G)|^2\right)$.*

A weighted k coloring can be computed in time $\mathcal{O}\left(|V(G)|^2 + \text{size}(w)\right)$. The number of different stable sets in the coloring is bounded by $\mathcal{O}\left(|V(G)|\right)$.

Fuzzy circular interval graphs [2] provide a generalization of the former class. They can be characterized as follows. A graph G is a fuzzy circular interval graph if there is a map Φ from $V(G)$ to a circle $\mathcal{C}$ and a set $\mathcal{I}$ of intervals of $\mathcal{C}$, none including another, such that no point of $\mathcal{C}$ is an endpoint of more than one interval so that:

- If two vertices u and v are adjacent, then $\Phi(u)$ and $\Phi(v)$ belong to a common interval.

- If two vertices u and v belong to the same interval, which is not an interval with endpoints $\Phi(u)$ and $\Phi(v)$, then they are adjacent.

Every fuzzy circular interval graph is quasi-line, and every circular interval graph is a fuzzy circular interval graph. The generalization provided by fuzzy circular interval graphs is regarding the adjacencies between nodes mapped to the different endpoints of an interval. While the definition of circular interval graphs requires these vertices to be adjacent, the definition of fuzzy circular interval graphs does not. For an interval $[p, q] \in \mathcal{I}$, if both sets of preimages $A := \Phi^{-1}(p)$ and $B := \Phi^{-1}(q)$ are nonempty, the pair (A, B) is called a *fuzzy pair*. Note that by definition, both A and B are cliques, but adjacencies between nodes of A and B can be arbitrary. Fuzzy pairs have another structural property that will be crucial later for our construction: Every node $v \in V(G) \setminus (A \cup B)$ is either A-*complete* or A-*anticomplete*. Similarly, v is either B-*complete* or B-*anticomplete*.

Analogous to circular interval graphs, the pair $(\Phi, \mathcal{I})$ is called a *representation* of G. It completely defines all adjacencies, except for those of fuzzy pairs. Figure 2 shows an example for a fuzzy circular interval graph and its representation. We remark that

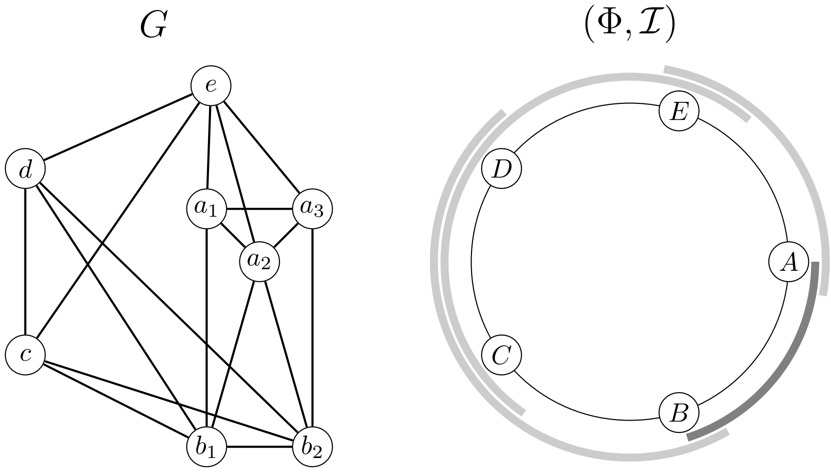

Figure 2: A fuzzy circular interval graph with its representation. Here $\Phi(a_1) = \Phi(a_2) = \Phi(a_3) = A$, $\Phi(b_1) = \Phi(b_2) = B$, $\Phi(c) = C$, $\Phi(d) = D$ and $\Phi(e) = E$

the definition of fuzzy pairs relies on the interval set $\mathcal{I}$, and hence are dependent on a representation. Given two different representations of the same graph, the fuzzy pairs might differ. In the sequel when we speak of fuzzy pairs, we implicitly assume that a representation is given. Every fuzzy circular interval graph has a representation whose number of intervals is bounded by $\mathcal{O}(|V(G)|)$: The fact that no interval is allowed to include another limits the number of irredundant intervals. From now on we assume that the number of intervals is limited by $\mathcal{O}(|V(G)|)$. Representations for fuzzy circular interval graphs can be computed efficiently:

Theorem 2 ([6]). *Given a graph G, one can decide whether G is a fuzzy circular interval graph and compute a suitable representation in time $\mathcal{O}\left(|V(G)|^2 \cdot |E(G)|\right)$.*

The coloring algorithm presented later will reduce to the case of circular interval graphs to make use of Gijswijt's coloring algorithm. As fuzzy pairs are what distinguishes circular interval graphs from fuzzy circular interval graphs, they play an essential role in the transformation. A fuzzy pair (A, B) is called *nontrivial* if $A \cup B$ contains an induced C_4 subgraph, i.e. there are four nodes such that their induced subgraph is a cycle. It is called *trivial* otherwise. A crucial observation is that fuzzy circular interval graphs whose fuzzy pairs are all trivial are actually circular interval graphs, see, e.g. [10].

Lemma 1. *Given a fuzzy circular interval graph G and a representation, if every fuzzy pair of G w.r.t. that representation is trivial, then G is a circular interval graph.*

2. The coloring algorithm

Our coloring algorithm for fuzzy circular interval graphs reduces to the case of circular interval graphs by transforming the input graph G and its weights w to a circular interval graph G^* with weights w^* such that the coloring number is preserved, i.e. $\chi_w(G) = \chi_{w^*}(G^*)$. Then it applies Gijswijt's algorithm, see Theorem 1, to obtain a coloring of G^*, which finally is transformed to a coloring of G.

The results of Chudnovsky and Ovetsky [11] and of King and Reed [12] also give algorithms to reduce fuzzy circular interval graphs to circular interval graphs preserving the coloring number. However their constructions do not consider weighted colorings. One can reduce weighted colorings to colorings by replacing each node $v \in V(G)$ of G with a clique of size $w(v)$, thereby generalizing their results. However, the size of the resulting graph is exponential in the encoding length of the weights. Our reduction instead works directly for weighted colorings and achieves polynomial running time.

Lemma 1 suggests the following approach for the reduction of a fuzzy circular interval graph G: Replace every nontrivial fuzzy pair in G with a trivial one in such a way that the weighted coloring number is preserved. This is done in several iterations, replacing the nontrivial fuzzy pairs one by one.

2.1. Fuzzy pair reduction

We now describe a single iteration, i.e. show how to replace a single fuzzy pair. Recall that fuzzy pairs (A, B) have the structural property that every node $v \notin A \cup B$ is either adjacent to all the nodes of A (of B) or to none of them. Thus as far as the stable sets of a coloring are concerned it is important to know whether a node of A (of B) is contained in a stable set whereas knowing the exact node itself is less important. Nodes in A and B can be re-distributed among the stable sets as long as they do not become adjacent in the sub-graph induced by $A \cup B$. This is reflected in the following construction to compact a fuzzy pair.

Consider a fuzzy circular interval graph G with weights w and a fuzzy pair (A, B) in G. Let $V^\circ := V(G) \backslash (A \cup B)$. For a subset $S \subseteq V$ we define $w(S) := \sum_{v \in S} w(v)$. The *reduced graph* (G', w') is defined as follows:

$$V(G') := V^\circ \cup \{a_0, a_1, b_0, b_1\},$$

$$\begin{aligned} E(G') := E(G)|_{V^\circ} &\cup \{\{v, a_0\},\ \{v, a_1\}\ :\ v \in V^\circ A\text{-complete}\} \\ &\cup \{\{v, b_0\}, \{v, b_1\}\ :\ v \in V^\circ B\text{-complete}\} \\ &\cup \{\{a_0, a_1\}, \{b_0, b_1\}, \{a_0, b_1\}, \{a_1, b_1\}, \{a_1, b_0\}\} \end{aligned}$$

$$w'(v) := \begin{cases} \alpha & \text{if } v = a_0 \text{ or } v = b_0 \\ w(A) - \alpha & \text{if } v = a_1 \\ w(B) - \alpha & \text{if } v = b_1 \\ w(v) & \text{else.} \end{cases}$$

Notice that a similar construction is used in the independent work of Oriolo, Pietropaoli and Stauffer [6] who designed an efficient recognition algorithm for fuzzy circular interval graphs.

We next specify α. The sets A and B together with the complement of the edges of $G[A \cup B]$ define a bipartite graph H. If a stable set S of G has two vertices in $A \cup B$, then those two vertices are connected by an edge in H. Furthermore the set $(S \setminus (A \cup B)) \cup \{a_0, b_0\}$ is a stable set of G'. Writing a weighted k-coloring of G as the sum of characteristic vectors of stable sets $w = \chi^{S_1} + \cdots + \chi^{S_k}$, how many of the S_i can contain two vertices of $A \cup B$? This number can be expresses as the size of a largest b-*matching* in H. Given node labels $b : V(H) \to \mathbb{N}_0$, a b-matching is a multiset of edges of $E(H)$ such that each node $v \in V(H)$ is covered by at most $b(v)$ of those edges. Alternatively one can define a maximum b-matching as optimal solution to the linear program

$$\max \left\{ \sum_{e \in E(H)} x_e \ : \ \sum_{e \in \delta(v)} x_e \leq b(v) \ \forall v \in V(H); \ x_e \geq 0 \ \forall e \in E(H) \right\}. \tag{1}$$

Since H is bipartite, the vertices of the linear program (1) are integral, see, e.g. [3].

Now setting $b := w$, the number of stable sets S_i that can contain two vertices of $A \cup B$ is clearly bounded by the size of a largest b-matching, as a coloring with ℓ many of those stable sets directly gives rise to a b-matching of size ℓ. The number α from the reduction above is the size of a largest b-matching or equivalently the optimum value of the linear program (1). We remark that this number can be computed efficiently using a combinatorial max $s - t$–flow algorithm, e.g. Karzanovs preflow push algorithm [3]. Figure 3 illustrates an example for the reduction.

In order for the reduced graph to be useful for our reduction, we need to prove that it satisfies the following three properties.

- The reduction preserves the structure of the graph, i.e. G' is still a fuzzy circular interval graph.
- If (A, B) was nontrivial, the number of nontrivial fuzzy pairs has been reduced by one.
- We have $\chi_w(G) = \chi_{w'}(G')$.

We start with showing that the construction preserves the weighted coloring number.

Lemma 2. *With the construction above one has* $\chi_w(G) = \chi_{w'}(G')$.

Proof. We show that every weighted k coloring of (G', w') gives rise to a weighted k coloring of (G, w) and vice versa.

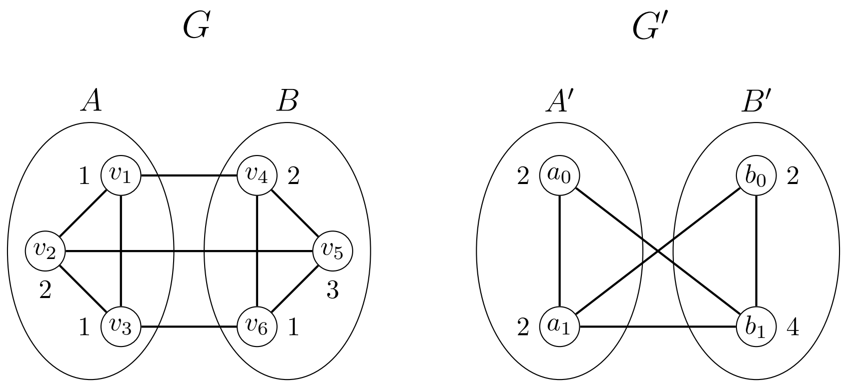

Figure 3: A fuzzy pair (A, B) and its replacement (A', B') in the reduction. The numbers next to the nodes denote their weights.

Let

$$w = \chi^{S_1} + \cdots + \chi^{S_k} \tag{2}$$

be a weighted coloring of G with k not necessarily different stable sets S_i, $i = 1, \ldots, k$. We transform this coloring into a weighted coloring of G' with weights w' using k stable sets. Each stable set S_i is replaced with a stable set S'_i of G' by the following procedure.

First we consider those stable sets S_i that contain two nodes a and b of $A \cup B$. Each such S_i is replaced by $(S_i \setminus (A \cup B)) \cup \{a_0, b_0\}$. Observe that by construction, the number μ of such stable sets S_i is bounded by α. We further replace $\alpha - \mu$ stable sets S_i containing a node of A but not a node of B by $(S_i \setminus A) \cup \{a_0\}$ and $\alpha - \mu$ stable sets S_i containing a node of B but not a node of A by $(S_i \setminus B) \cup \{b_0\}$. Now, there are $w(A) - \alpha$ stable sets left containing a node of A but not of B. Since $w'(a_1) = w(A) - \alpha$, we can replace these stable sets by $(S_i \setminus A) \cup \{a_1\}$. Similarly, we can replace the remaining $w(B) - \alpha$ stable sets containing a node of B but not a node of A by $(S_i \setminus B) \cup \{b_1\}$. The stable sets S_i which do not contain a vertex of $A \cup B$ remain the same. All-together, this yields a collection $S'_1, \ldots, S'_k$ of stable sets of G' with

$$w' = \chi^{S'_1} + \cdots + \chi^{S'_k},$$

a k coloring of (G', w').

To show that a weighted k coloring of G' can be transformed to a weighted k coloring of G, let

$$w' = \chi^{S'_1} + \cdots + \chi^{S'_k}. \tag{3}$$

be a weighted coloring of G' with k not necessarily different stable sets S'_i, $i = 1, \ldots, k$.

Let μ be the number of stable sets S'_i of (3) containing a_0 and b_0 and suppose that these stable sets are $S'_1, \ldots, S'_\mu$. By construction we have $\mu \leq \alpha$ and we can compute a b-matching M of H of size μ. This b-matching is a multiset consisting of μ edges $\{u_i, v_i\}$ with $u_i \in A$ and $v_i \in B$, $i = 1, \ldots, \mu$. Now we can replace each of these μ stable sets with

$$(S'_i \setminus \{a_0, b_0\}) \cup \{u_i, v_i\}.$$

For $v \in A \cup B$, let $d(v)$ be the number of edges of the b-matching M which are incident to v, i.e., $d(v) = |\{i \colon v \cap \{u_i, v_i\} \neq \emptyset, \{u_i, v_i\} \in M\}|$. Note that $d(v) \leq w(v)$.

Suppose that the stable sets S'_i containing a_0 or a_1 but not b_0 or b_1 are then

$$S'_{\mu+1}, \dots, S'_{w(A)} \tag{4}$$

and those containing b_0 or b_1 but not a_0 or a_1 are

$$S'_{w(A)+1}, \dots, S'_{w(A)+w(B)-\mu}. \tag{5}$$

The multiset of stable sets (4) can be partitioned into multisets $\mathcal{S}'_a$, $a \in A$ with $|\mathcal{S}'_a| = w(a) - d(a)$. Likewise the multiset of stable sets (5) can be partitioned into multisets $\mathcal{S}'_b$, $b \in B$ with $|\mathcal{S}'_b| = w(b) - d(b)$. Each $S' \in \mathcal{S}'_a$ is replaced by $(S' \setminus \{a_0, a_1\}) \cup \{a\}$ and each $S' \in \mathcal{S}'_b$ is replaced by $(S' \setminus \{b_0, b_1\}) \cup \{b\}$. The stable sets S'_i of (3) containing no vertex of $\{a_0, a_1, b_0, b_1\}$ remain unchanged. In this way, we obtain a weighted coloring of G with weights w. □

We end the section with the proof of the other two properties we require from the reduction:

Lemma 3. *If G is a fuzzy circular interval graph, then the output G' of the reduction above is a fuzzy circular interval graph as well. Moreover, if the fuzzy pair (A, B) is nontrivial, then the number of nontrivial fuzzy pairs in G' is reduced by one.*

Proof. Consider a representation $(\Phi, \mathcal{I})$ of the graph G. As (A, B) is a fuzzy pair, we have $\Phi(a) = \Phi(a')$ for all $a, a' \in A$ and $\Phi(b) = \Phi(b')$ for all $b, b' \in B$.

We define the map $\Phi' : V(G') \to \mathcal{C}$ as

$$\Phi'(v) := \begin{cases} \Phi(v), & \text{if } v \in V^\circ \\ \Phi(A), & \text{if } v = a_0 \text{ or } v = a_1 \\ \Phi(B), & \text{if } v = b_0 \text{ or } v = b_1. \end{cases}$$

It is straightforward to verify that $(\Phi', \mathcal{I})$ is a representation of the graph G'.

Note that $(\{a_0, a_1\}, \{b_0, b_1\})$ is a fuzzy pair w.r.t. this representation, in replacement for the fuzzy pair (A, B). All other fuzzy pairs remain unchanged in the new representation. Clearly $(\{a_0, a_1\}, \{b_0, b_1\})$ is trivial as it does not contain an induced C_4. Thus if (A, B) is nontrivial, the number of nontrivial fuzzy pairs is reduced by one. □

2.2. The algorithm

Using the reduction from above, we can now describe the full algorithm: Given a fuzzy circular interval graph G with weights w and a representation $(\Phi, \mathcal{I})$. Assume that there are ℓ many fuzzy pairs in that representation denoted by $(A_1, B_1), \dots, (A_\ell, B_\ell)$.

The coloring algorithm replaces the fuzzy pairs one by one with trivial fuzzy pairs, preserving the structure of the graph and the weighted coloring number by computing a sequence

$$(G, w) = (G_0, w_0), (G_1, w_1), \dots, (G_\ell, w_\ell) = (G^*, w^*),$$

where (G_i, w_i) is the reduced graph of (G_{i-1}, w_{i-1}), the fuzzy pair (A_i, B_i) being reduced.

A simple proof by induction using Lemma 3 in the induction step show that G^* is fuzzy circular interval graph without nontrivial fuzzy pair, hence by Lemma 1 a circular interval graph. Similarly, with Lemma 2 we get that

$$\chi_w(G) = \chi_{w^*}(G^*).$$

Now Gijswitj's algorithm can be used to compute the weighted coloring number for G^* and hence G. Note that the proof of Lemma 2 is constructive, i.e. it shows how to transform a weighted coloring of a reduced graph to a weighted coloring of its preimage. Hence we can transform a coloring of (G^*, w^*) to a coloring of (G, w) by successively transforming a coloring from (G_i, w_i) to (G_{i-1}, w_{i-1}) for each $i = \ell, \dots, 1$.

However, the construction in the proof of Lemma 2 does not have polynomial running time as the number k of stable sets might be exponential in the input size. For that reason, we will give a revised algorithm in Section 2.3 which uses a compressed representation of the stable sets, only taking into account the *different* stable sets. This algorithm will be efficient, i.e. its running time is polynomial in the input size.

A formal description of the algorithm looks as follows:

Algorithm 1 The coloring algorithm for fuzzy circular interval graphs

function $\textsc{ColorFCIG}(G = (V, E),\ w,\ \{(A_1, B_1), \dots, (A_\ell, B_\ell)\})$
    $G_0 \leftarrow G$, $w_0 \leftarrow w$.
    for $i = 1, \dots, \ell$ **do**
        $(G_i, w_i) \leftarrow \textsc{Reduce}(G_{i-1}, w_{i-1}, (A_i, B_i))$
    end for
    $G^* \leftarrow G_l$, $w^* \leftarrow w_\ell$
    $\langle \chi^{S_1}, \dots, \chi^{S_k} \rangle \leftarrow \textsc{ColorCIG}(G^*, w^*)$
    for $i = \ell, \dots, 1$ **do**
        $\langle \chi^{S_1}, \dots, \chi^{S_k} \rangle \leftarrow \textsc{ColorTransform}((G_i, w_i),\ (G_{i-1}, w_{i-1}),\ (A_i, B_i), \langle \chi^{S_1}, \dots, \chi^{S_k} \rangle)$
    end for
    return $\langle \chi^{S_1}, \dots, \chi^{S_k} \rangle$
end function

2.3. Running time analysis

This section is devoted to the analysis of the running time of Algorithm 1. We will show that the algorithm has a running time that is polynomial in the input. We start with the running time analysis for the reduction.

Lemma 4. *Given a circular interval graph G with weights w, and a representation $(\Phi, \mathcal{I})$ of the graph, in time $\mathcal{O}\left(|V(G)|^3\right)$ we can compute the circular interval graph G^* with weights w^* as in Algorithm 1.*

Proof. Before reducing the fuzzy pairs, we need to identify them. This can be done in time $\mathcal{O}\left(|V(G)|^2\right)$: For each interval $I \in \mathcal{I}$, compute the set of nodes that are mapped to the two endpoints of the interval (Recall that we assume that $|\mathcal{I}| = \mathcal{O}\left(|V(G)|\right)$). If both endpoints have at least one node, we have a fuzzy pair. Let ℓ be the number of fuzzy pairs. Note that $\ell \leq |V(G)|$ as fuzzy pairs are disjoint.

The running time for computing a single reduced graph graph G_i and w_i from its predecessor (G_{i-1}, w_{i-1}) is determined by the time needed to generate the helper graph H, to compute the size of a maximum b-matching in H and to output (G_i, w_i). The running time to generate H and output (G', w') is $\mathcal{O}\left(|V(G)|^2\right)$. Since the number of nodes in H is $|A_i \cup B_i|$, a maximum size b-matching can be computed in time $\mathcal{O}\left(|A_i \cup B_i|^3\right)$ using a max $s-t$-flow algorithm, e.g. Karzanovs preflow push algorithm [3]. This gives an overall running time of $\mathcal{O}\left(|V(G)|^2 + |A_i \cup B_i|^3\right)$ to compute one reduced graph. Thus, all ℓ iterations together amount a running time of

$$\mathcal{O}\left(\ell \cdot |V(G)|^2 + \sum_{i=1}^{\ell} |A_i \cup B_i|^3\right).$$

Since $\ell \leq |V(G)|$ and the fuzzy pairs are disjoint, this simplifies to $\mathcal{O}\left(|V(G)|^3\right)$. □

To compute the coloring number alone, as $\chi_w(G) = \chi_{w^*}(G^*)$, it is sufficient to determine the weighted coloring number of G^*. Note that $|V(G^*)| = \mathcal{O}\left(|V(G)|\right)$. Hence, deciding if the weighted coloring number of G^* exceeds some given k using Gijswit's algorithm takes time $\mathcal{O}\left(|V(G)|^2\right)$ as stated in Theorem 1. The weighted coloring number of G can be determined by binary search on the interval $[1, w(G)]$. Hence with Lemma 4 and using the fact that size(w), the binary encoding length of w, is at least $|V(G)|$, we get the following result:

Theorem 3. *Given a fuzzy circular interval graph G with weights w and a representation, the weighted coloring number can be computed in time $\mathcal{O}\left(|V(G)|^2 \cdot \text{size}(w)\right)$.*

To get a coloring of (G, w), a weighted coloring of G^* has to be transformed. In the proof of Lemma 2 we have seen an exponential time algorithm. The drawback of that algorithm is that every stable set of the coloring is treated individually, i.e. we consider $\chi_w(G)$ many stable sets. The weighted coloring number might not be bounded polynomially in the input size, resulting in an exponential time algorithm. However, only a polynomial number of *different* stable sets is needed to describe an optimal weighted coloring. This fact can be exploited by using a more compact formulation of the colorings: Only the *different* stable sets are written down explicitly, together with integers that specify how often the sets should be used. In fact the output of Gijswijts algorithm is of that form as well.

The following lemma proves that a revised version of the transformation algorithm that uses the compressed representation has polynomial running time.

Lemma 5. *Let (G, w) be a weighted fuzzy circular interval graph with a fuzzy pair (A, B), and let (G', w') be its reduced graph. Given a k coloring of (G', w') using β different stable sets, a k coloring of (G, w) using $\mathcal{O}\left(\beta + |A \cup B|^2\right)$ different stable sets can be computed in time*

$$\mathcal{O}\left((\beta + |A \cup B|^2) \cdot |V(G)| + |V(G)|^2 + |A \cup B|^3\right).$$

Proof. Suppose that the k coloring of (G', w') is given in a compact formulation

$$w' = \lambda_1 \chi^{S'_1} + \cdots + \lambda_\beta \chi^{S'_\beta}. \tag{6}$$

with $\lambda_i \in \mathbb{N}$ and $\sum_{i=1}^{\beta} \lambda_i = k$. Our goal is to transform it into a weighted k coloring of G.

Let $J \subseteq \{1, \ldots, \beta\}$ be the index set of stable sets S_i' of (6) containing a_0 and b_0, and let $\mu := \sum_{i \in J} \lambda_i$. Clearly we have $\mu \leq w'(a_0) = w'(b_0)$. Without loss of generality assume that $J = \{1, \ldots, |J|\}$. Let H be the complement graph of $G[A \cup B]$ as defined as in Section 2.1. Recall that by construction of reduced graphs, we have $\alpha = w'(a_0) = w'(b_0)$, where α is the value of a maximum b-matching in H with $b(v) := w(v)$ for all $v \in A \cup B$. Hence there is a b-matching of H of size μ. Let γ be the number of *different* edges in the b-matching. Note that this number is bounded by $|A \cup B|^2$, the maximum number of edges in H.

We can assume that the b-matching is given in the following compact formulation: $(\rho_j, \{u_j, v_j\})$ with $\rho_j \in \mathbb{N}$, $u_j \in A$ and $v_j \in B$ for $j = 1, \ldots, \gamma$, where ρ_j denotes the multiplicity of the edge $\{u_j, v_j\}$ in the matching. Hence $\sum_{j=1}^{\gamma} \rho_j = \mu$. We assume that the b-matching has the following property: There are indexes $0 =: k(0),\ k(1), \ldots, k(|J|) := \gamma$ such that $\sum_{j=k(i-1)+1}^{k(i)} \rho_j = \lambda_i$ for each $i \in J$. Note that this property can be established by splitting some $(\rho_j, \{u_j, v_j\})$. Also note that at most one split is needed for each $i \in J$. Thus even if we require that property, we have $\gamma \leq |J| + |A \cup B|^2$.

We use the b-matching to transform our coloring as follows. We partition the matching into $|J|$ multisets $P_1, \ldots, P_{|J|}$ in a natural way: For each $i \in J$ we set

$$P_i = \{(\rho_j, \{u_j, v_j\}) \ : \ j = k(i-1)+1, \ldots, k(i)\} =: \{(\rho_j', \{u_j', v_j'\}) \ : \ j = 1, \ldots, \gamma_i\}.$$

Note that by construction we have $\sum_{j=1}^{\gamma_i} \rho_j' = \lambda_i$. Now we can transform the λ_i many sets S_i' as follows: For each $j = 1, \ldots, \gamma_i$ output

$$(S_i' \backslash \{a_0, b_0\}) \cup \{u_j', v_j'\}$$

with multiplicity ρ_j'. Let β_0 be the number of *different* stable sets created in this transformation. Clearly the number is given by the number of different sets in the P_i, i.e. $\sum_{i \in J} \gamma_i = \gamma \leq |J| + |A \cup B|^2$.

For $v \in A \cup B$, let $d(v)$ be the number of edges of the b-matching M which are incident to v, i.e.

$$d(v) = \sum_{i=1,\ldots,\gamma,\ v \in \{u_i, v_i\}} \rho_i.$$

Each node $v \in A \cup B$ is now already contained in $d(v)$ many stable sets that we transformed already. Hence we have to put it into $w(v) - d(v)$ more stable sets to obtain a weighted coloring of G. This is done similarly to what we did before. Let J' denote the index set of the S_i' containing a node of $\{a_0, a_1\}$ but not of $\{b_0, b_1\}$. These sets are transformed in a similar manner. Consider the multiset $\{(w(a) - d(a), a) \ : \ a \in A\}$, and partition it into $|J'|$ many multisets, such that for each $i \in J'$ we have a set P_i' of the form

$$P_i' = \{(\rho_j', a_j') \ : \ j = 1, \ldots, \gamma_i\}.$$

For each $j = 1, \ldots, \gamma_i$ output

$$(S_i' \backslash \{a_0, a_1\}) \cup \{u_j', v_j'\}$$

with multiplicity of ρ_j'. This yields at most $|J'| + |A|$ different sets. The stable sets S_i' containing a node of $\{b_0, b_1\}$ but not of $\{a_0, a_1\}$ are transformed in a similar manner, covering each node $b \in B$ exactly $w(b) - d(b)$ times and increasing the number of different stable sets by at most $|B|$. The remaining sets are left unchanged.

This gives a weighted k coloring of G. The number of different sets has increased by

$$\mathcal{O}\left(\gamma + |A \cup B|^2\right) = \mathcal{O}\left(|A \cup B|^2\right),$$

and thus the number of different sets in the coloring is bounded by $\beta + \mathcal{O}\left(|A \cup B|^2\right)$. The running time needed for the classification of the sets of the coloring is $\mathcal{O}(\beta \cdot |V(G)|)$. Computing the graph H takes time $\mathcal{O}\left(|V(G)|^2\right)$, and the computation time for the b-matching is $\mathcal{O}\left(|A \cup B|^3\right)$. Performing the transformations takes time $(\beta + \mathcal{O}\left(|A \cup B|^2\right)) \cdot |V(G)|$. We conclude that the total time of the transformation is bounded by

$$\mathcal{O}\left((\beta + |A \cup B|^2) \cdot |V(G)| + |V(G)|^2 + |A \cup B|^3\right).$$

□

To transform a coloring of G^* with weights w^* to a coloring of G with weights w, we apply the transformation repeatedly for each fuzzy pair that has been reduced before. Observe that the number β of different stable sets of the coloring never exceeds $\mathcal{O}\left(|V(G)|^2\right)$: The weighted coloring of the graph G^* has only $\mathcal{O}(|V(G)|)$ different sets, as stated in Theorem 1. In each iteration, this number grows only by $|A_i \cup B_i|^2$ during the transformation for fuzzy pair A_i, B_i, as seen in Lemma 5.

There are $\ell \leq |V(G)|$ iterations of the transformation needed, and thus the total running time can be bounded with $\mathcal{O}\left(|V(G)|^4\right)$. With Theorem 1 we get the following.

Theorem 4. *Given a fuzzy circular interval graph G with weights w and a representation. For every $k \in \mathbb{N}$, in time $\mathcal{O}\left(|V(G)|^4 + \text{size}(w)\right)$ one can compute a weighted k coloring of (G, w), if such a coloring exists.*

A weighted coloring using a minimum number of colors can be computed in time

$$\mathcal{O}\left(|V(G)|^4 + |V(G)|^2 \cdot \text{size}(w)\right).$$

We remark that our algorithm requires a suitable representation of the fuzzy circular interval graph as part of the input. As stated in Theorem 2, such a representation can be computed in time $\mathcal{O}\left(|V(G)|^2 \cdot |E(G)|\right) = \mathcal{O}\left(|V(G)|^4\right)$.

3. Integer decomposition property

We will now consider the weighted coloring problem for fuzzy circular interval graphs from a polyhedral perspective. There is a strong relation between (weighted) vertex coloring of graphs and integer decompositions in their stable set polytopes. It turns out that the following two properties of a graph class imply that the weighed coloring number can be computed in polynomial time:

- The maximum weighted stable set problem (MWSS) can be solved in polynomial time.
- The stable set polytope has the integer decomposition property.

We will elaborate on this in more detail in Section 3.1. This is of particular interest for the class of fuzzy circular interval graphs, because the two properties are satisfied. An algorithm by Minty [13], revised by Nakamura and Tamura [14] solves the MWSS problem for the superclass of claw-free graphs. We will show in Section 3.2 that the stable set polytopes of fuzzy circular interval graphs have the integer decomposition property.

3.1. Vertex coloring and integer decomposition

Let G be a graph. With $\mathcal{I}(G)$ we denote the family of stable sets of G. For a subset $S \subseteq V(G)$ of vertices, the *incidence vector* $\chi^S \in \mathbb{Z}^{V(G)}$ is the $\{0,1\}$-vector defined by $\chi^S(v) = 1 \Leftrightarrow v \in S$. The *stable set polytope* $\mathrm{STAB}(G)$ is defined as the convex hull of the incidence vectors of the stable sets of G:

$$\mathrm{STAB}(G) := \mathrm{conv.\,hull}\left(\{\chi^S \;:\; S \in \mathcal{I}(G)\}\right).$$

Note that a k integer decomposition of a vector w in $\mathrm{STAB}(G)$ directly corresponds to a weighted k coloring G with weights w and vice versa.

This makes the integer decomposition property a useful tool to compute the weighted coloring number: The latter can be expressed with the following integer program

$$\chi_w(G) = \min\left\{\sum_{S\in\mathcal{I}(G)} \lambda_S \;:\; \sum_{S\in\mathcal{I}(G)} \lambda_S \chi^S = w,\ \lambda \in \mathbb{N}_0\right\}. \tag{7}$$

The *fractional* weighted coloring number is defined as the optimum of the linear relaxation:

$$\chi_w^*(G) = \min\left\{\sum_{S\in\mathcal{I}(G)} \lambda_S \;:\; \sum_{S\in\mathcal{I}(G)} \lambda_S \chi^S = w,\ \lambda \geq 0\right\}. \tag{8}$$

This number is interesting for the following reason: If the stable set polytope has the integer decomposition property, then one obtains the weighted coloring number by rounding up the fractional coloring number. This can be seen as follows. Let λ an optimal fractional solution of (8). Hence $\sum_{S\in\mathcal{I}(G)} \lambda_S = \sum \chi_w^*(G)$, which implies that $\frac{1}{\chi_w^*(G)}\lambda$ defines a convex combination of the vector $\frac{1}{\chi_w^*(G)}w$. Thus $\frac{1}{\chi_w^*(G)}w \in \mathrm{STAB}(G)$. Since $0 \in \mathrm{STAB}(G)$, this implies $\frac{1}{\lceil\chi_w^*(G)\rceil}w \in \mathrm{STAB}(G)$. In other words we have $w \in \lceil\chi_w^*(G)\rceil\,\mathrm{STAB}(G)$. Now, if $\mathrm{STAB}(G)$ has the integer decomposition property, this implies that there is a $\lceil\chi_w^*(G)\rceil$ integer decomposition of w in $\mathrm{STAB}(G)$. In other words, there is a weighted coloring of G using $\lceil\chi_w^*(G)\rceil$ many colors. We conclude:

Lemma 6. *If the stable set polytope* $\mathrm{STAB}(G)$ *has the integer decomposition property, then* $\chi_w(G) = \lceil\chi_w^*(G)\rceil$.

Hence, in order to compute the weighted coloring number, it is sufficient to compute the fractional coloring number, i.e. to solve the linear program (8). Although its dimension is of exponential size, under some circumstances one can still solve it efficiently considering the dual LP:

$$\chi_w^*(G) = \max\left\{\sum_{v\in V} w_v y_v \;:\; \sum_{v\in S} y_v \leq 1 \forall S \in \mathcal{I}(G)\right\}. \tag{9}$$

The number of constraints of this linear program is exponential. However, to solve it in polynomial time, it is sufficient to solve the separation problem efficiently [1]: Given a vector y, decide whether y is a feasible solution. If it is infeasible, provide a hyperplane that separates y from the set of feasible solutions. Note that in this case, the separation problem can be solved by using the maximum weighted stable set (MWSS) problem on G, using the solution y as weights: If the maximum weight of a stable set exceeds 1, it yields a separating hyperplane. Otherwise, if the maximum weight is bounded by 1, this asserts that y is feasible.

In general, the MWSS problem is NP-hard. However, for claw-free graphs, a superclass of quasi-line and fuzzy circular interval graphs, there is a polynomial time algorithm [14, 13]. Note that this allows only to compute the weighted coloring number, but not an optimal weighted coloring. There is a result by Orlin [15] to compute optimal solutions for covering integer programs that have the round up property as (7) and (8) in polynomial time. However, its running time is polynomial in the size of the integer program, and thus not applicable here.

3.2. Integer decomposition property of Fuzzy circular interval graphs

We will now show that the stable set polytopes of fuzzy circular interval graphs have the integer decomposition property. The result will be established by reduction to the case of circular interval graphs, using the fact that stable set polytopes of circular interval graphs have the integer decomposition property [4, 5].

We use our construction from Section 2.1 for the reduction. We remark that the construction of Chudnovsky and Ovetsky [11] or King and Reed [12] could be used as well, using a boosting argument. We have seen in Section 2.1 that our construction preserves the weighted coloring number. A key observation is that the fractional weighted coloring number is preserved as well:

Lemma 7. *Given a fuzzy circular interval graph G with weights $w \in \mathbb{N}^{V(G)}$, there is a circular interval graph G^* with weights $w^* \in \mathbb{N}^{V(G)}$ such that*

1. $\chi_w(G) = \chi_{w^*}(G^*)$,
2. $\chi^*_w(G) = \chi^*_{w^*}(G^*)$.

Proof. Let (G^*, w^*) be the circular interval graph computed by our coloring algorithm. We have seen previously that $\chi_w(G) = \chi_{w^*}(G^*)$ holds. We now show that the fractional coloring numbers are preserved as well. A straightforward observation is as follows: If we scale the weight vector w by some integer $M \in \mathbb{N}$, then the fractional coloring number is scaled by the same factor:

$$M \cdot \chi^*_w(G) = \chi^*_{M\cdot w}(G).$$

The same holds for the fractional weighted coloring number of G^*. Moreover we claim that

$$\chi_{M\cdot w}(G) = \chi_{M\cdot w^*}(G^*)$$

holds. This can be shown completely analogous to the proof of Lemma 2. The key argument here is that if the weight vector w is scaled by M, then the optimum value of the linear program (1) is scaled by M as well. Now to prove (2.), consider an optimal fractional coloring of G, i.e.

$$w = \sum_{S\in \mathrm{STAB}(G)} \lambda_S \chi^S$$

and $\sum_{S\in\mathrm{STAB}(G)} \lambda_S = \chi^*_w(G)$. We can assume that λ is rational, hence there is a number $M \in \mathbb{N}$ such that $M \cdot \lambda$ is integer. We conclude that

$$M \cdot \chi^*_w(G) = \chi^*_{M\cdot w}(G) = \chi_{M\cdot w}(G) = \chi_{M\cdot w^*}(G^*) \geq M \cdot \chi^*_{w^*}(G^*)$$

using the fact that $M \cdot \lambda$ is an optimal *integer* coloring for the second equality. This shows $\chi^*_w(G) \geq \chi^*_{w^*}(G^*)$. The converse direction is shown analogous starting with an optimal fractional coloring of G^*. □

Using this lemma, we can show the integer decomposition property:

Theorem 5. *Let G be a fuzzy circular interval graph. Then* $\mathrm{STAB}(G)$ *has the integer decomposition property.*

Proof. Let $w \in \mathbb{Z}^{V(G)}$ be a vector and $k \in \mathbb{N}$ such that $w \in k \cdot \mathrm{STAB}(G)$. We need to show that there is a k integer decomposition of w in $\mathrm{STAB}(G)$. As $w \in k \cdot \mathrm{STAB}(G)$, there is a convex combination of $\frac{1}{k}w$ using the characteristic vectors of stable sets. This implies that for the fractional weighted chromatic number we have $\chi^*_w(G) \leq \lceil\chi^*_w(G)\rceil \leq k$. Applying Lemma 7, we get a fuzzy circular interval graph G^* with weights w^* such that $\chi_w(G) = \chi_{w^*}(G^*)$ and $\chi^*_w(G) = \chi^*_{w^*}(G^*)$. Since the stable set polytope of G^* has the integer decomposition property, with Lemma 6 we have $\lceil\chi^*_{w^*}(G^*)\rceil = \chi_{w^*}(G^*)$. We conclude that $\lceil\chi^*_w(G)\rceil = \chi_w(G) \leq k$, which asserts that there is a k integer decomposition as desired. □

References

[1] M. Grötschel, L. Lovász, A. Schrijver, Geometric algorithms and combinatorial optimization, volume 2 of *Algorithms and Combinatorics*, Springer-Verlag, Berlin, second edition, 1993.

[2] M. Chudnovsky, P. D. Seymour, The structure of claw-free graphs, in: Surveys in Combinatorics, pp. 153–171.

[3] A. Schrijver, Combinatorial Optimization - Polyhedra and Efficiency, Springer, 2003.

[4] T. Niessen, J. Kind, The round-up property of the fractional chromatic number for proper circular arc graphs, J. Graph Theory 33 (2000) 256–267.

[5] D. Gijswijt, Integer decomposition for polyhedra defined by nearly totally unimodular matrices, SIAM J. Discret. Math. 19 (2005) 798–806.

[6] G. Oriolo, U. Pietropaoli, G. Stauffer, On the recognition of fuzzy circular interval graphs, manuscript, to appear (2010).

[7] X. Deng, P. Hell, J. Huang, Linear-time representation algorithms for proper circular-arc graphs and proper interval graphs, SIAM J. Comput. 25 (1996) 390–403.

[8] R. M. McConnell, Linear-time recognition of circular-arc graphs, Algorithmica 37 (2003) 93–147.

[9] H. Kaplan, Y. Nussbaum, A simpler linear-time recognition of circular-arc graphs, in: Algorithm theory—SWAT 2006, volume 4059 of *Lecture Notes in Comput. Sci.*, Springer, Berlin, 2006, pp. 41–52.

[10] F. Eisenbrand, G. Oriolo, G. Stauffer, P. Ventura, The stable set polytope of quasi-line graphs, Combinatorica 28 (2008) 45–67.

[11] M. Chudnovsky, A. Ovetsky, Coloring quasi-line graphs, J. Graph Theory 54 (2007) 41–50.

[12] A. D. King, B. Reed, Asymptotics of the chromatic number for quasi-line graphs, Electronic Notes in Discrete Mathematics 29 (2007) 327 – 331. European Conference on Combinatorics, Graph Theory and Applications.

[13] G. Minty, On maximal independent sets of vertices in claw-free graphs, J. Combin. Theory (B) 28 (1980) 284–304.

[14] D. Nakamura, A. Tamura, A revision of Minty's algorithm for finding a maximum weight stable set of a claw-free graph, 1999.

[15] J. B. Orlin, A polynomial algorithm for integer programming covering problems satisfying the integer round-up property, Math. Programming 22 (1982) 231–235.