Guaranteed recovery of a low-rank and joint-sparse matrix from incomplete and noisy measurements

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I. Problem statement

Suppose you are given a matrix $X \in \mathbb{R}^{n_1 \times n_2}$ with rank $r \ll \min(n_1, n_2)$. Moreover, assume this matrix has sparse nonzero elements so that, due to the column-wise dependencies, they are all supported on $k \ll n_1$ number of rows (it can also be column-wise supported). This matrix wont have many degrees of freedom; If one knows the position of those k nonzero rows, the corresponding sub-matrix contains only $(k + n_2 - r)r$ degrees of freedom.

Provided by the enormous developments in areas of compressed sensing and low rank-matrix recovery [1][2][3][4], one may wonder if it is possible to acquire the whole matrix elements from a very few number of non-adaptive linear measurements. In this regard, three questions immediately follow; what should be those measurements? How to design a computationally tractable algorithm to recover this matrix from those possibly noisy measurements? And finally, how to evaluate the performance i.e., how many measurements do we need to recover exact low-rank and sparse matrix, and does the algorithm performs stable with respect to matrices that are approximately low-rank or not exactly joint-sparse but *compressible*? This paper attempts to answer the questions above.

II. Prior arts

Recently a few papers consider *rank awareness* in data joint-recovery from multiple measurement vectors (MMV) [5] [6]. More precisely, sparse MMV inverse problem (also known as simultaneous sparse approximation), focuses on recovering a joint-spase matrix X from a set of measurements $Y \in \mathbb{R}^{\overline{m} \times n_2}$ acquired as $Y = AX$. There, $A \in \mathbb{R}^{\overline{m} \times n2}$ is the measurement matrix that is unique for compressive sampling signals of all the n_2 channels (columns of X). Davis *et al.* [5] proposed a specific rank-aware *greedy* algorithm, that in case of using a random i.i.d. Gaussian A, is able to recover (with high probability) an *exact* k-joint-sparse and rank-r X from its noiseless MMV, if the total number of measurements scales as,

$$m = n_2 \overline{m} \gtrsim \mathcal{O}\big(n_2 k (\log n_1 / r + 1)\big). \tag{1}$$

III. Originality of our work

Our work contrasts with prior arts in three main aspects:

1- Let us define the linear map $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ and model our sampling mechanism by $y = \mathcal{A}(X) + z$, for a noise vector $z \in \mathbb{R}^m$. As we can see, this measurement scheme is able to model more general cases than a uniform sampling matrix for all the channels e.g., in *distributed* compressed sensing scenarios, each channel can be sampled by an independent measurement matrix (rather than a unique one), or even in non-distributed cases where the sampling matrix is designed so that each measurement reflects a global average behavior of the whole matrix rather than a local specific channel.

2- Our recovery algorithm is different and is based on the following convex minimization,

$$\begin{aligned} \arg\min_X \quad & \|X\|_{2,1} + \lambda \|X\|_* \\ \text{subject to} \quad & \|y - \mathcal{A}(X)\|_2 \leq \epsilon. \end{aligned} \tag{2}$$

The $l_{2,1}$ mixed-norm is defined as $\|X\|_{2,1} := \sum_i (\sum_j X_{i,j}^2)^{1/2}$ and the nuclear norm $\|X\|_*$ is the sum of the singular values of X.

3- Our performance analysis, guarantees *stability* of our recovery approach against noisy measurements, non-exact sparse and approximately low-rank data matrices. We prove that, if our measurement system satisfies a *specific* restricted isometry property (RIP), the solution of (2), stably recovers *all* joint-sparse and low-rank matrices. In particular, we show that, for certain random measurement schemes, the number of measurements m sufficient for stable recovery scales as,

$$m \geq \mathcal{O}\Big(k\big(r + \log(n_1/k)\big) + n_2 r\Big). \tag{3}$$

Regarding rank of the data matrix, our bound is of a different nature than (1) i.e., the lower the rank, less measurements are required. Indeed, in many multichannel signal applications, where (due to the structure behind) a huge data matrix turns out to have a low-rank ($r \ll k \ll n_2$), our approach outperforms those in the state-of-the-art, reflecting the importance of a good design for the measurements $\mathcal{A}$ together with the recovery approach benefiting those structures (i.e., joint-sparse and low-rank).

In the rest of this paper, we develop an algorithm to solve (2) using proximal splitting methods [7]. A number of simulations on synthetic data as well as an interesting important application in Hyperspectral imaging, demonstrate a massive saving of the number of measurements required to recover data, compared to the existing methods.

References

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