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Unitary descent properties

Eva Bayer–Fluckiger*

Ecole Polytechnique Fédérale de Lausanne, Mathématiques,
EPFL–FSB–IMB–CSAG, Station 8, 1015 Lausanne, Switzerland
e-mail: eva.bayer@epfl.ch.

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Abstract. Let k be a field of characteristic not 2, and let L be an odd degree Galois extension of k. A theorem of Rosenberg and Ware [6] states that $W(L)^{\mathrm{Gal}(L/k)} = W(k)$. The present paper contains a generalization of the Rosenberg–Ware theorem to Witt groups of division algebras with involution. It also extends a descent result of Rost [7] concerning Witt groups in arbitrary odd degree extensions. Descent questions for hermitian forms and their relations to isotropy properties are also discussed, as well as descent in Galois cohomology. Finally, an application is given to bilinear forms invariant by finite groups.

Introduction

Let k be a field of characteristic $\neq 2$, and let us denote by $W(k)$ the Witt ring of k. For any field extension L of k, there is a canonical ring homomorphism $r_{L/k} : W(k) \to W(L)$. It is a consequence of a theorem of Springer [12] that if L/k is a finite extension of odd degree, then $r_{L/k}$ is injective.

If L/k is a Galois extension, then the image of $r_{L/k}$ is contained in $W(L)^{\mathrm{Gal}(L/k)}$. A result of Rosenberg and Ware [6] states that if L is a finite Galois extension of odd degree of k, then $r_{L/k} : W(k) \to W(L)^{\mathrm{Gal}(L/k)}$ is an isomorphism.

Let A be a finite dimensional k-algebra, and let $\sigma : A \to A$ be a k-linear involution. Then one can consider non-degenerate hermitian forms over free (A, σ)-modules of finite rank, and obtain a Witt group $W(A, \sigma)$ (cf §1). For any field extension L of k, there exists a canonical group homomorphism $r_{L/k} : W(A, \sigma) \to W(A_L, \sigma_L)$, where $A_L = A \otimes_k L$ and σ_L is the extension of σ to A_L. If L/k is a finite extension of odd degree, then $r_{L/k}$ is injective (cf. [1]). For Galois extensions, we have (cf. §1, 1.2)

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Theorem. *Let L be a finite Galois extension of odd degree of k. Then the canonical homomorphism $r_{L/k} : W(A, \sigma) \to W(A_L, \sigma_L)^{\mathrm{Gal}(L/k)}$ is an isomorphism.*

The Rosenberg–Ware theorem is generalized by Rost in [7], where he obtains a result for arbitrary odd degree extensions. The same method gives the following result for Witt groups of hermitian forms. Let L be a finite separable extension of k, and set $R = L \otimes_k L$. Let us denote by i and i' the two natural inclusions of L in R, and by $j, j' : W(A, \sigma) \to W(A_R, \sigma_R)$ the induced maps at the level of Witt groups (cf. §3). Then we have (cf. 3.2)

Theorem. *Suppose that L is a finite separable extension of k. Then the sequence of $W(k)$-modules*

$$0 \to W(A, \sigma) \xrightarrow{r_{L/k}} W(A_L, \sigma_L) \xrightarrow{j-j'} W(A_R, \sigma_R)$$

is exact.

The Rosenberg–Ware theorem implies that if L is a finite Galois extension of odd degree of k and if q is a quadratic form over L that is invariant by $\mathrm{Gal}(L/k)$, then q is extended from k. In order to derive this from the Rosenberg–Ware theorem, one needs a theorem of Springer stating that if a quadratic form becomes isotropic over an odd degree extension, then it is isotropic over the ground field (see §4 for details). The analog of this result of Springer does not hold in general for hermitian forms over algebras with involution, as shown by an example of Parimala (cf. [4], §4). On the other hand, Parimala, Sridharan and Suresh proved that it does hold for hermitian forms over quaternion fields with an orthogonal involution. It is a very interesting open question to decide over which algebras with involution one does have such an analogue. In §4, we discuss the implications of this question to the descent of hermitian forms.

Classification of hermitian forms can be reformulated in terms of Galois cohomology of classical groups. The aim of §5 is to give some descent properties in Galois cohomology based on the results of the previous sections. As an application, §6 gives a descent result for bilinear forms invariant under the action of a finite group.

1. Ground field extension and transfer for hermitian forms

All quadratic forms are supposed to be non-degenerate. We denote by $W(k)$ the Witt ring of k (see for instance [8]). If L/k is a field extension, then we have a canonical homomorphism

$$r_{L/k} : W(k) \to W(L).$$

If L/k is an extension of finite odd degree, then Springer's theorem states that this map is injective (see [12,8], 2.5.4.).

Let A be a finite dimensional k-algebra. An *involution* of A is a k-linear anti–automorphism $\sigma : A \to A$ such that σ^2 is the identity. A *hermitian form* over (A, σ) is by definition a pair (V, h), where V is a finite dimensional free right A-module, and $h : V \times V \to A$ is hermitian with respect to σ. All hermitian forms are supposed to be non-degenerate. We say that (V, h) is a *hyperbolic plane* if V is a free A-module of rank 2 that has a basis (e, f) such that $h(e, e) = h(f, f) = 0$ and $h(e, f) = h(f, e) = 1$. A *hyperbolic form* is by definition an orthogonal sum of hyperbolic planes. This leads to a notion of Witt group $W(A, \sigma)$ (cf. for instance [8], Chap. 7, §2). Note that $W(A, \sigma)$ is a $W(k)$-module.

If L/k is a field extension, then we have a canonical homomorphism

$$r_{L/k} : W(A, \sigma) \to W(A_L, \sigma_L).$$

For extensions of finite odd degree, this map is injective (cf. [1]), thereby generalizing Springer's theorem.

If L/k is a Galois extension, then $r_{L/k}(W(A, \sigma)) \subset W(A_L, \sigma_L)^{\mathrm{Gal}(L/k)}$. The theorem of Rosenberg and Ware [6] states that we have equality if $D = k$, in other words

Theorem 1.1 (Rosenberg–Ware). *Let L be a Galois extension of odd degree of k. Then $r_{L/k}(W(k)) = W(L)^{\mathrm{Gal}(L/k)}$.*

This result can be generalized as follows :

Theorem 1.2. *Let L/k be a Galois extension of odd degree. Then the canonical map*

$$r_{L/k} : W(A, \sigma) \to W(A_L, \sigma_L)^{\mathrm{Gal}(L/k)}$$

is an isomorphism.

The original proof of the theorem of Rosenberg and Ware uses the Feit–Thompson theorem (see [6]). More recently, two simpler proofs have been published : one by Knebusch and Scharlau ([8], 2.8.3), and another by Rost [7]. Both of these can be extended to the case of finite dimensional k-algebras with involution, thereby providing two proofs of th. 1.2, see §2 and §3.

Both methods use Scharlau's transfer. Let $s : L \to k$ be a non-zero k-linear map. Then there is an associated transfer map $s_* : W(L) \to W(k)$, cf. [7], 2.5. If L/k has odd degree, then s can be chosen so that $s_*(\langle 1 \rangle) = \langle 1 \rangle$ (see [7], 2.5.8.). From now on, we suppose that s has this property.

The map $s : L \to k$ induces a k-linear map $s_A : A_L \to A$, and we get a transfer map $s_* : W(A_L, \sigma_L) \to W(A, \sigma)$, cf. [8], or [1], §1. We have the following

Proposition 1.3. *We have*

$$s_*(q \otimes r_{L/k}(h)) = s_*(q) \otimes h$$

for all $q \in W(L)$ and all $h \in W(A,\sigma)$.

Proof. See [1], §1. □

Proposition 1.4. *Let L/k be an odd degree extension, and let $s : L \to k$ be a k-linear map such that $s_* : W(L) \to W(k)$ satisfies $s_*(\langle 1 \rangle) = \langle 1 \rangle$. Then the composition*

$$s_* r_{L/k} : W(A,\sigma) \to W(A,\sigma)$$

is the identity.

Proof. Let $h \in W(A,\sigma)$. By prop. 1.3, we have $s_* r_{L/k}(h) = s_*(\langle 1 \rangle) \otimes h = \langle 1 \rangle \otimes h = h$. □

Corollary 1.5. *Let L/k be an odd degree extension, and let $s : L \to k$ be a k-linear map such that $s_* : W(L) \to W(k)$ has the property $s_*(\langle 1 \rangle) = \langle 1 \rangle$. Then the map $s_* : W(A,\sigma) \to W(A_L,\sigma_L)$ satisfies $s_*(\langle 1 \rangle) = \langle 1 \rangle$.*

Proof. This follows from prop. 1.4. □

2. The method of Knebusch and Scharlau

The aim of this section is to give a proof of th. 1.2 based on a method of Knebusch and Scharlau (see [8], 2.8.3). We keep the notation of §1. Throughout this section, L is a finite Galois extension of k.

The k-linear map $\mathrm{Tr} = \mathrm{Tr}_{L/k} : L \to k$ induces

$$\mathrm{Tr}_* : W(A_L,\sigma) \to W(A,\sigma).$$

Proposition 2.1. *Let (V,h) be a hermitian form over (A,σ). Then we have a canonical isomorphism*

$$r_{L/k}\mathrm{Tr}_*(V,h) \simeq \bigoplus_{g \in \mathrm{Gal}(L/k)} (V,h)^g.$$

Proof. Following [7], 2.8.1, let us define a map $f : V \otimes_k L \to \bigoplus_{g \in \mathrm{Gal}(L/k)} V^g$ by

$$f(x \otimes \lambda) = \Sigma_{g \in \mathrm{Gal}(L/k)} x(g^{-1}\lambda).$$

It is straigthforward to check that this gives us the desired isomorphism, cf. [7], 2.8.1. □

Proof of Theorem 1.2. We construct an inverse map to

$$r_{L/k} : W(A,\sigma) \to W(A_L,\sigma_L)^{\mathrm{Gal}(L/k)}.$$

Let $s : L \to k$ be a k-linear map such that $s_*(\langle 1\rangle) = \langle 1\rangle$. This map induces $s_* : W(A_L,\sigma_L) \to W(A,\sigma)$. By 1.4. we know that the composition

$$s_* r_{L/k} : W(A,\sigma) \to W(A_L,\sigma_L)^{\mathrm{Gal}(L/k)} \to W(A,\sigma)$$

is the identity. It remains to show that

$$r_{L/k} s_* : W(A_L,\sigma_L)^{\mathrm{Gal}(L/k)} \to W(A,\sigma) \to W(A_L,\sigma_L)^{\mathrm{Gal}(L/k)}$$

is also the identity.

Since L/k is Galois, Tr is a non-zero map, hence there exists $\lambda \in L$ such that $s(x) = \mathrm{Tr}(\lambda x)$ for all $x \in L$ (cf. [7], 2.5.7). Therefore $s_*(h) = \mathrm{Tr}_*(\lambda h)$ for all $h \in W(D_L,\sigma_L)$. Using this and 2.1, we have

$$r_{L/k}(s_*(h)) \simeq r_{L/k}\mathrm{Tr}_*(\lambda h) \simeq \bigoplus_{g\in \mathrm{Gal}(L/k)} (\lambda h)^g \simeq \bigoplus_{g\in \mathrm{Gal}(L/k)} (\langle g\lambda\rangle \otimes h^g).$$

Set $\psi = \bigoplus_{g\in\mathrm{Gal}(L/k)} \langle g\lambda\rangle$. If $h \in W(A_L,\sigma_L)^{\mathrm{Gal}(L/k)}$, then $h^g \simeq h$, hence the above expression is isomorphic to $\psi \otimes h$. Hence

$$r_{L/k} s_* : W(A_L,\sigma_L)^{\mathrm{Gal}(L/k)} \to W(A,\sigma) \to W(A_L,\sigma_L)^{\mathrm{Gal}(L/k)}$$

is multiplication by ψ. We have $\psi = \langle 1\rangle$ in $W(L)$. Indeed, we ave the following equalities in $W(L)$ (cf. [7], 2.8.1):

$$\langle 1\rangle_L = r_{L/k}\langle 1\rangle_k = r_{L/k}s_* r_{L/k}\langle 1\rangle_k = r_{L/k}s_*\langle 1\rangle_L = \psi \otimes \langle 1\rangle_L = \psi.$$

This proves that $r_{L/k}s_*$ is the identity, hence the theorem is proved.

3. The method of Rost

The aim of this section is to generalize a descent result of Rost [7] to hermitian forms over division algebras. This will lead to another proof of th. 1.2.

We keep the notation of §1. Let L and L' be two finite degree separable extensions of k. Set $R = L \otimes_k L'$, and let $i : L \to R$, $i' : L' \to R$ be defined by $i(x) = x \otimes 1$ and $i'(x') = 1 \otimes x'$. These maps induce

$$j : W(A_L,\sigma_L) \to W(A_R,\sigma_R),$$

and

$$j' : W(A_{L'},\sigma_{L'}) \to W(A_R,\sigma_R).$$

In the special case where $L = L'$, we get two maps

$$j,\ \ j' : W(A_L,\sigma_L) \to W(A_R,\sigma_R).$$

Theorem 3.1. *Let L be a finite separable extension of odd degree of k, and let $h \in W(A_L, \sigma_L)$. If $j(h) = j'(h)$, then $h \in r_{L/k}(W(A, \sigma))$.*

Proof. Let $s : L \to k$ be a k-linear map such that $s(\langle 1 \rangle) = \langle 1 \rangle$ (this is possible by §1). Let L' be a finite separable extension of odd degree of k, and let $S : R \to L'$ be defined by $S(x \otimes x') = s(x)x'$. Then the composition

$$W(A_{L'}, \sigma_{L'}) \xrightarrow{j'} W(A_R, \sigma_R) \xrightarrow{S_*} W(A_{L'}, \sigma_{L'})$$

is the identity. On the other hand, the composition

$$W(A_L, \sigma_L) \xrightarrow{j} W(A_R, \sigma_R) \xrightarrow{S_*} W(A_{L'}, \sigma_{L'})$$

is equal to the composition

$$W(A_L, \sigma_L) \xrightarrow{s_*} W(A, \sigma) \xrightarrow{r_{L'/k}} W(A_{L'}, \sigma_{L'}).$$

□

Set now $L' = L$, and let $h \in W(A_L, \sigma_L)$ such that $j(h) = j'(h)$. We have

$$h = S_* j'(h) = S_* j(h) = r_{L/k}(s * (h)) \in \mathrm{Im}(r_{L/k}).$$

This concludes the proof of the theorem.

Corollary 3.2. *Suppose that L is a finite separable extension of k. Then the sequence of $W(k)$-modules*

$$0 \to W(A, \sigma) \xrightarrow{r_{L/k}} W(A_L, \sigma_L) \xrightarrow{j - j'} W(A_R, \sigma_R)$$

is exact.

Proof. This is an immediate consequence of 3.1. □

The above result gives another proof of th. 1.2:

Proof of Theorem 1.2. Let L be a Galois extension of finite odd degree of k, and let

$$h \in W(A_L, \sigma_L)^{\mathrm{Gal}(L/k)}.$$

Let us show that this implies $j(h) = j'(h)$. Indeed, $R = L \otimes L_k \simeq L \times \cdots \times L$, the product of $n = [L : k]$ copies of L. Therefore $W(A_R, \sigma_R)$ is isomorphic to the product of n copies of $W(A_L, \sigma_L)$. The map j sends h diagonally to $(h, \dots, h)$, whereas the map j' sends h diagonally to the element with components h^g, for $g \in G$. By hypothesis $h^g \simeq h$ for all $g \in G$, hence these two elements are equal.

Therefore we have $j(h) = j'(h)$, hence by 3.1. h is in the image of

$$r_{L/k} : W(A, \sigma) \to W(A_L, \sigma_L)^{\mathrm{Gal}(L/k)}.$$

4. Isotropy properties and descent of hermitian forms

The above results give us descent properties for Witt classes, but not for the hermitian forms themselves. More precisely, let us consider the following definition

Definition. Let h be a hermitian form over (A_L, σ_L). We say that h is *extended from* (A, σ) if there exists a hermitian form h' over (A, σ) such that $h'_L \simeq h$ as hermitian forms over (A_L, σ_L).

The Rosenberg–Ware theorem implies that if L/k is an odd degree Galois extension, then every quadratic form over L that is stable by $\mathrm{Gal}(L/k)$ is extended from k. In order to deduce this from the Rosenberg–Ware result, one uses Springer's theorem for isotropy:

Theorem 4.1 (Springer, [12] or [8]). *Let L/k be an odd degree extension. If a quadratic form over k becomes isotropic over L, then it is isotropic over k.*

This result does not extend to hermitian forms over arbitrary algebras with involution. Indeed, Parimala gave examples of division algebras with involution for which the isotropy property below does not hold (cf. [4], §4).

Definition. Let h be a hermitian form over (A, σ). We say that h is *isotropic* if there exists a hyperbolic plane H over (A, σ) such that $h \simeq H \oplus h'$ for some hermitian form h' over (A, σ). Otherwise, h is said to be *anisotropic*.

Definition. We say that (A, σ) has the *isotropy property* if for all odd degree extensions L/k and all hermitian forms h over (A, σ) we have : if h_L is isotropic, then h is isotropic.

Corollary 4.2. *Suppose that (A, σ) has the isotropy property. Let L be a finite Galois extension of odd degree of k. Then every hermitian form over (A_L, σ_L) that is invariant by* $\mathrm{Gal}(L/k)$ *is extended from* (A, σ).

Proof. Let h be a hermitian form over (A_L, σ_L) that is invariant by $\mathrm{Gal}(L/k)$. We can assume that h is anisotropic. By 1.2, there exists a hermitian form h' over (A, σ) and a hyperbolic form H over (A_L, σ_L) such that $h'_L \simeq h \oplus H$. If $H = 0$, then there is nothing to prove. If not, then this implies that h'_L is isotropic, hence by the isotropy property h' is isotropic. Therefore $h' \simeq H' \oplus h''$ where H' is a hyperbolic plane over (A, σ) and where h'' is some hermitian form over (A, σ). This implies that $h''_L \simeq h \oplus H''$ for some hyperbolic form H'' over (A_L, σ_L) of rank less than the rank of H. Continuing inductively, we see that h is extended from (A, σ). □

Deciding for which algebras with involution the isotropy property holds is an interesting open question. Springer's theorem (see 4.1 above) means that the isotropy property holds for commutative fields endowed with the trivial involution. It is easy to see that the same proof yields the result for commutative fields with a non-trivial involution, as well as for quaternion fields endowed with the canonical (symplectic) involution. Hence the first non-trivial case is the one of quaternion fields with an orthogonal involution. This question is settled by the following

Theorem 4.3 (Parimala, Sridharan, Suresh, [5]). *The isotropy property holds for quaternion fields with an orthogonal involution.*

This has the following consequence

Corollary 4.4. *Let D be a quaternion field, and let $\sigma : D \to D$ be an involution of the first kind of D. Let L be a finite Galois extension of odd degree of k. Then every hermitian form over (D_L, σ_L) that is invariant by $\mathrm{Gal}(L/k)$ is extended from (D, σ).*

Proof. This follows from 4.2. and 4.3. □

One can also refine the isotropy property as follows.

Definition. Let n be a positive integer. We say that (A, σ) has the *n-isotropy property* if for all extensions L/k of degree prime to n and all hermitian forms h over (A, σ) we have : if h_L is isotropic, then h is isotropic.

Corollary 4.5. *Let n be an even integer. Suppose that (A, σ) has the n-isotropy property, and let L be a finite Galois extension of degree prime to n of k. Then every hermitian form over (A_L, σ_L) that is invariant by $\mathrm{Gal}(L/k)$ is extended from (A, σ).*

Proof. This is proved by the same method as 4.2. □

Question. Let d be the degree of A, and let $n = 2d$. When does (A, σ) have the n-isotropy property?

Note that similar questions were raised in [3] and [5].

5. Forms in odd degree Galois extensions

Springer's theorem implies that if two quadratic forms become isomorphic over a finite odd degree extension, then they are isomorphic over the ground field k. This can be reformulated in Galois cohomology terms, as follows. Let q be a non-degenerate quadratic form defined over k, and let $O(q)$ be its orthogonal group. If L is a finite extension of odd degree of k, then the canonical map $H^1(k, O(q)) \to H^1(L, O(q))$ is injective.

This result is extended in [1] to classical groups. Let A be a finite dimensional k-algebra with a k-linear involution $\sigma : A \to A$. Let U_A be the linear algebraic group over k defined by

$$U_A(E) = \{x \in A_E \mid x\sigma(x) = 1\}$$

for any commutative k-algebra E. The group U_A is called the *unitary group* of (A, σ). If L is a finite extension of odd degree of k, then the canonical map $H^1(k, U_A) \to H^1(L, U_A)$ is injective (cf. [1]).

Suppose now that L/k is a Galois extension of odd degree. The image of $H^1(k, U_A) \to H^1(L, U_A)$ is then contained in $H^1(L, U_A)^{\mathrm{Gal}(L/k)}$, and it is natural to ask whether one has equality. In the case of orthogonal groups, this is a consequence of the theorem of Rosenberg and Ware and Springer's theorem. Parimala's example [4], §4 shows that it does not hold for arbitrary algebras with involution. However, it does hold for algebras with involution all simple involution-invariant components of which have the isotropy property (see 5.1).

Let k_s be a separable closure of k, and set $\Gamma_k = \mathrm{Gal}(k_s/k)$. Let $H^1(k, U_A) = H^1(\Gamma_k, U_A(k_s))$ (see [10,11] for basic facts concerning non-abelian Galois cohomology). For any field extension L/k, we have a canonical map

$$r_{L/k} : H^1(k, U_A) \to H^1(L, U_A).$$

Let R_A be the radical of the algebra A, and set $\bar{A} = A/R$. We have

$$\bar{A} \simeq A_1 \times \cdots \times A_s \times (A_{s+1} \times A'_{s+1}) \times \cdots \times (A_m \times A'_m),$$

where A_i is a simple algebra for all $i = 1, \ldots, m$, with $\sigma(A_i) = A_i$ for $i = 1, \ldots, s$ and $\sigma(A_i) = A'_i$ for $i = s+1, \ldots, m$. Then $A_i = M_{n_i}(D_i)$ for some division algebra D_i central over a finite extension F_i of k.

Theorem 5.1. *Let L be a Galois extension of finite odd degree of k. Suppose that (D_i, τ_i) has the isotropy property for all $i = 1, \ldots, s$ and for any involution τ_i of D_i. Then the canonical map $r_{L/k} : H^1(k, U_A) \to H^1(L, U_A)^{\mathrm{Gal}(L/k)}$ is bijective.*

Proof. Recall that R_A be the radical of the algebra A, and that $\bar{A} = A/R$. Then the projection $A \to \bar{A}$ induces a bijection of pointed sets $H^1(k, U_A) \to H^1(k, U_{\bar{A}})$. Recall that

$$\bar{A} \simeq A_1 \times \cdots \times A_s \times (A_{s+1} \times A'_{s+1}) \times \cdots \times (A_m \times A'_m),$$

where A_i is a simple algebra for all $i = 1, \ldots, m$, with $\sigma(A_i) = A_i$ for $i = 1, \ldots, s$ and $\sigma(A_i) = A'_i$ for $i = s+1, \ldots, m$. Let $\sigma_i : A_i \to A_i$ be the restriction of σ_A to A_i for $i = 1, \ldots, s$, and let us denote by $\sigma_i : A_i \times A'_i \to$

$A_i \times A_i'$ the restriction of σ_A to $A_i \times A_i'$ if $i = s+1, \dots, m$. Let F_i be the maximal subfield of the center of A_i such that σ_i is F_i-linear if $i = 1, \dots, s$, and let U_i be the unitary group of (A_i, σ_i). For $i = s+1, \dots, m$, let F_i be the center of A_i, and let U_i be the unitary group of $((A_i \times A_i), \sigma_i)$. Then U_i is a linear algebraic group defined over F_i for all $i = 1, \dots, m$. We have a bijection of pointed sets

$$H^1(k, U_A) \to \prod_{i=1,\dots,m} H^1(F_i, U_i).$$

If $i = s+1, \dots, m$, then U_i is a general linear group, hence $H^1(F_i, U_i) = 0$. Hence we have a bijection of pointed sets

$$H^1(k, U_A) \to \prod_{i=1,\dots,s} H^1(F_i, U_i).$$

For all $i = 1, \dots, s$, we have $A_i = M_{n_i}(D_i)$ for some division algebra D_i central over F_i, and σ_i is induced by some hermitian form h_i over D_i. Let $L_i = LF_i$. Recall that $H^1(F_i, U_i)$ is in bijection with the isomorphism classes of hermitian forms over D_i that become isomorphic to h_i over a separable closure of F_i. □

Let $L_i = LF_i$. Then L_i is a finite extension of odd degree of F_i for all $i = 1, \dots, s$. Hence 4.2. implies that $r_{L_i/F_i} : H^1(F_i, U_i) \to H^1(L_i, U_i)^{\mathrm{Gal}(L_i/F_i)}$ is bijective. Therefore $r_{L/k}$: $H^1(k, U_A) \to H^1(L, U_A)^{\mathrm{Gal}(L/k)}$ is bijective as well, which proves the theorem.

We also have an analog of 3.1. If E is an étale algebra over k, $E = E_1 \times \cdots \times E_r$ where E_i is a separable extension of k for all $i = 1, \dots, r$, then set

$$H^1(E, U) = H^1(E_1, U_A) \times \cdots \times H^1(E_r, U_A).$$

Let L be a finite separable extension of k, and set $R = L \otimes L$. Let $i : L \to R$ and $i' : L \to R$ be the two natural inclusions of L in R, defined by $j(x) = x \otimes 1$ and $j'(x) = 1 \otimes x$. These maps induce $j : H^1(L, U_A) \to H^1(R, U_A)$ and $j' : H^1(L, U_A) \to H^1(R, U_A)$. We have the following

Theorem 5.2. *Let L be a separable extension of finite odd degree of k. Suppose that (D_i, τ_i) has the isotropy property for all $i = 1, \dots, s$ and for any involution τ_i of D_i. Let $x \in H^1(L, U_A)$ such that $j(x) = j'(x)$. Then*

$$x \in r_{L/k}(H^1(k, U_A)).$$

Proof. This follows from 3.1 and 4.2 using the method of the proof of 5.1. □

The same methods give the following

Theorem 5.3. *Let n be an even integer, and let L be a Galois extension of degree prime to n of k. Suppose that (D_i, τ_i) has the n-isotropy property for all $i = 1, \ldots, s$ and for any involution τ_i of D_i. Then the canonical map $r_{L/k} : H^1(k, U_A) \to H^1(L, U_A)^{\mathrm{Gal}(L/k)}$ is bijective.*

Theorem 5.4. *Let n be an even integer, and let L be a separable extension of degree prime to n of k. Suppose that (D_i, τ_i) has the n-isotropy property for all $i = 1, \ldots, s$ and for any involution τ_i of D_i. Let $x \in H^1(L, U_A)$ such that $i(x) = j'(x)$. Then*

$$x \in r_{L/k}(H^1(k, U_A)).$$

6. Bilinear forms invariant by the action of a finite group

Let k be a field of characteristic $\neq 2$, and let G be a finite group. A *G-form* is by definition a pair (V, b), where V is a $k[G]$-module that is a finite dimensional k-vector space, and $b : V \times V \to k$ is a bilinear form such that $b(gx, gy) = b(x, y)$ for all $x, y \in V$ and all $g \in G$. An *isomorphism* between two G-forms (V, b) and (V', b') is an isomorphism $f : V \to V'$ of $k[G]$-modules such that $b'(fx, fy) = b(x, y)$ for all $x, y \in V$.

Theorem 6.5. *Let L be a finite Galois extension of odd degree of k. Suppose that G has no unitary characters of Schur index greater than one. Then every G-form over L that is invariant by $\mathrm{Gal}(L/k)$ is extended from k.*

Proof. Let (V, b) be a G-form. Let us consider the k-algebra A_b defined by $A_b = \{(e, f) \in \mathrm{End}_G(V) \times \mathrm{End}_G(V)^{\mathrm{op}} \mid b(ex, y) = b(x, fx),\ b(x, ey) = b(fx, y), \text{for all } x, y \in V\}$. Let $\sigma : A_b \to A_b$ be defined by $\sigma(e, f) = (f, e)$. Then the group of automorphisms of (V, b) can be identified with the unitary group of A_b. Let us denote this group by U_b. Then the set of isomorphism classes of G-forms over k that become isomorphic to b over k_s is in bijection with the Galois cohomology set $H^1(k, U_b)$. □

The group algebra $k[G]$ carries the canonical k-linear involution $\sigma_G : k[G] \to k[G]$ characterized by $g \mapsto g^{-1}$. Let R_G be the radical of $k[G]$, and set $\overline{k[G]} = k[G]/R_G$. Then σ_G induces the involution $\bar{\sigma}_G : \overline{k[G]} \to \overline{k[G]}$. Let (B, σ) be an involution invariant simple component of $(\overline{k[G]}, \bar{\sigma}_G)$, and let $B = M_r(D)$ for some division algebra D with center F. If σ is orthogonal or symplectic, then either $D = F$ or D is a quaternion algebra (see for instance [8], 8.13.5). By 4.3, the isotropy property holds for hermitian forms over quaternion algebras with an orthogonal involution. On the other hand, an easy generalization of Springer's theorem shows that the isotropy property holds for hermitian forms over quaternion algebras with the (unique) symplectic involution. If σ is unitary, then by the assumption on G we have $D = F$, and a simple modification Springer's theorem implies that the isotropy property holds for hermitian forms over commutative fields. Hence the hypothesis of 5.1 are satisfied, and therefore the canonical map $r_{L/k} : H^1(k, U_b) \to H^1(L, U_b)^{\mathrm{Gal}(L/k)}$ is bijective. This implies the desired result.

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