Tackling the Supersymmetric Flavour Problem in String Models

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Abstract

In this work we address one of the phenomenological issues of beyond the Standard Model scenarios which embed Supersymmetry, namely the Supersymmetric Flavour Problem, in the context of String Theory. Indeed, the addition of new interactions to the Standard Model generically spoils its flavour structure which is one of its major achievements since it for example leads to a very elegant understanding of the absence of flavour changing neutral currents in the leptonic sector and of the stability of the proton, thanks to accidental symmetries. We focus on a subset of the phenomenologically dangerous operators, namely the soft scalar masses.

One way out of the Supersymmetric Flavour Problem is to geographically separate the observable and hidden sectors along a fifth dimension, gravity being the only interaction propagating in the bulk. In such scenarios, the soft scalar masses are vanishing at the classical level since there is no direct contact term between the observable and hidden multiplets and tend to be universal at the loop-level. However such setups hardly ever come about in String Theory, which is one of the most promising candidates of quantum gravity. In order to make contact with the five-dimensional picture, we focus on the prototypical case of the $E_8 \times E_8$ Heterotic $\mathcal{M}$-Theory which, in a certain regime, effectively looks five-dimensional and embeds matter fields on two end-of-the-world branes. In these scenarios, not only gravity but also vector multiplets propagate in the five-dimensional bulk, effectively spoiling the sequestered picture.

However, since the contact terms responsible for the appearance of soft scalar masses arise due to the exchange of heavy vectors, they do enjoy a current-current structure which can be exploited to inhibit the emergence of soft scalar masses by postulating a global symmetry in the hidden sector. In order to assess the possibility of realising such a mechanism, we first study the full dependence of the Kähler potential on both the moduli and the matter fields in the case of orbifold and Calabi-Yau compactifications. We then determine whether an effective sequestering may be achieved thanks to a global symmetry and argue that whereas for orbifold models our strategy can naturally be put at work, it can only be implemented in a subset of Calabi-Yau models.

Keywords : Beyond the Standard Model, Flavour Structure, Supersymmetry, Hidden Sector, Soft Terms, Supergravity, String Theory, Heterotic Superstring, $\mathcal{M}$-Theory, Sequestering, Orbifold, Calabi-Yau.

Résumé

Dans ce travail nous adressons l'une des problématiques phénoménologiques des scénarios allant au delà du Modèle Standard qui englobent la supersymétrie, dans le cadre de la théorie des cordes. En effet, l'adjonction de nouvelles interactions au Modèle Standard a génériquement pour effet de compliquer sa structure de saveur qui est l'un des succès de ce modèle puisqu'elle explique notamment de façon très élégante la stabilité du proton et l'absence de courants neutres dans le secteur leptonique grâce à des symétries accidentelles. Nous nous intéresserons plus spécifiquement à un sous-ensemble des opérateurs dangereux d'un point de vue phénoménologique : les masses scalaires dites soft.

Une des solutions au problème de la saveur supersymétrique est de séparer géographiquement le secteur visible du secteur caché le long d'une cinquième dimension, la gravitation étant la seule force capable de propager dans la cinquième dimension. Dans de tels scénarios, les masses scalaires soft sont absentes au niveau classique puisqu'il n'y a pas d'interaction directe couplant les champs du secteur visible et du secteur caché et tendent à être universelles au niveau quantique. Afin de faire contact avec la configuration cinq-dimensionnelle, nous nous concentrons sur la théorie $\mathcal{M}$ hétérotique $E_8 \times E_8$ qui, dans un certain régime, est effectivement cinq-dimensionnelle et contient des champs de matière sur deux branes se situant aux frontières de la cinquième dimension. Dans de tels scénarios, la gravitation n'est plus la seule interaction présente dans la cinquième dimension. Un certain nombre de multiplet vectoriels y propagent aussi, rendant caduque l'analyse des termes soft faite précédemment.

Néanmoins, puisque les termes de contacts responsables de l'émergence de masses soft sont dus à l'échange de multiplets vectoriels lourds, ils ont une structure de type courant-courant qui peut être exploitée afin de supprimer les termes soft au niveau classique en postulant une symétrie globale dans le secteur caché. Afin d'évaluer la possibilité d'implémenter un tel mécanisme, nous étudions tout d'abord la dépendance du potentiel de Kähler des modules et des champs de matière, à la fois dans le contexte des orbifolds et des Calabi-Yau. Nous déterminons ensuite si ce potentiel peut admettre une symétrie conforme à nos besoins et trouvons qu'alors que dans le cas des orbifolds notre stratégie peut naturellement être mise en œuvre, elle n'est applicable que dans un sous-ensemble des compactifications Calabi-Yau.

Mots clefs : Au delà du Modèle Standard, Structure de saveur, Supersymétrie, Secteur caché, Termes soft, Supergravité, Théorie des cordes, Corde hétérotique, Théorie $\mathcal{M}$, Séquestration, Orbifold, Calabi-Yau.

À peine exprimons-nous quelque chose qu'étrangement nous le dévaluons. Nous pensons avoir plongé au plus profond des abîmes, et quand nous revenons à la surface, la goutte d'eau ramenée à la pointe pâle de nos doigts ne ressemble plus à la mer dont elle provient. Nous nous figurons avoir découvert une mine de trésors inestimables, et la lumière du jour ne nous montre plus que des pierres fausses et des tessons de verre; et le trésor, inaltéré, n'en continue pas moins à briller dans l'obscur.

MAETERLINCK

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Contents

Chapter 1

Introduction

All models are wrong but some are useful.

George Box and Norman Draper

1.1 From Experimental Clashes to Gedankenexperimenten

Before the appearance of rational science during the Middle Age, the world was thought to be best described by the *Elements* : Fire, Earth, Air and Water. From the pre-Socratic point of view, the Elements were enough to answer the fundamental question *How did the ordered cosmos in which we live come to be?* Many competing theories were elaborated, some based on Water, others on Air. Most of the proposals were based on the assumption of continuous matter until Democritus (460-370) proposed the first atomist theory and introduced the concept of void as the place where atoms are located.

Aristotle (384-322) later argued[1] that the Elements have to be supplemented with a more divine one, the quintessence also known as the Æther, in order to account for the apparent perfection of stellar movement opposed to the corrupt human sublunar world.

We may now rise, with all the respect due to both Democritus and Aristotle, the following question : when introducing atoms or quintessence, are they doing *science*? The modern point of view on determining whether a theory stands within the scope of science is certainly close to the one of Waissman[2] : 'If there is no possible way to determine whether a statement is true then that statement has no meaning whatsoever. For the meaning of a statement is the method of its verification.'

However at the times of Democritus and Aristotle no method could be used to falsify their views on Nature. Nonetheless, the Waismann criterion only asks for the logical possibility of falsification, without specifying *when* such experiments have to be performed. From this point of view, the questions raised by Democritus and Aristotle are genuine scientific questions which, in the meantime, have been answered.

[1] Traité du Ciel, Livre I, Chapitre 2, §10

[2] Waismann, "Logische Analyse des Wahrscheinlichkeitsbegriffs", Erkenntnis 1, 1903, p. 229.

It seems that, after decades of evolution driven by experimental clashes[1], Science has taken us back to the times of Aristotle, leaving us with questions that seem not to have immediate falsification methods. Indeed, most of the motivations for going beyond the well-established theories are not coming from experimental evidences but rather from abstract principles or from *Gedankenexperimenten*. The answers to such questions are incredibly sophisticated and there is no clear path to their falsification. So, again, the question arises, are we doing science? Since the logical possibility of falsifying such theories exists, the answer should be positive. But whether we, as a society, want to devote people, time and money to falsify these theories is another question.

1.2 Towards a Completion of Okun's Cube

The twentieth century witnessed two major breakthroughs which have revolutionised our understanding of Nature : *Quantum Mechanics* (QM) and *General Relativity* (GR). The former consists of a description of microscopic physics such as the discrete energy levels in atoms phrased in a probabilistic language, leading to endless debates on its philosophical implications. Quantum Mechanics' major success is the removal of the $r = 0$ singularity in the Coulomb law, achieved thanks to the fuzziness it introduces. The basic ingredient of General Relativity is Special Relativity whose primary concern was to unveil the consequences of theories having a maximal speed, the speed of light, based on the principle that an observer cannot determine its speed by any experiment if moving at constant speed relative to another observer, *i.e.* being at rest is a relative statement. The extension of the principle of relativity to situations in which the observers' relative speed is unconstrained leads to General Relativity.

All three theories we have introduced are characterised by an expansion parameter which measures the deviation from Newtonian mechanics. These respectively are the reduced Planck constant $\hbar$, the speed of light c and Newton's gravitational constant G_N which are measured to be [2] :

$$\begin{aligned} \hbar &\simeq 1.05 \cdot 10^{-34}\ \mathrm{m}^2\ \mathrm{kg}\ \mathrm{s}^{-1}, \\ c &\simeq 2.99 \cdot 10^{8}\ \mathrm{m}\ \mathrm{s}^{-1}, \\ G_N &\simeq 6.64 \cdot 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}. \end{aligned} \tag{1.1}$$

Newtonian mechanics is the limit in which both $\hbar$ and G_N are sent to zero while c is sent to infinity. The theory in which effects from all the three quantities in (1.1) are taken into account is not yet settled, String Theory certainly being the most promising candidate, as summarised in Okun's Cube displayed in Figure 1.1.

Throughout the history of Physics, the quest for unified theories has led to a much better understanding of the phenomena under consideration since such a mother theory contains the

[1] See the first chapters of the admirable book by Gian Giudice, *A Zeptospace Odyssey*, for a historical perspective. [1]

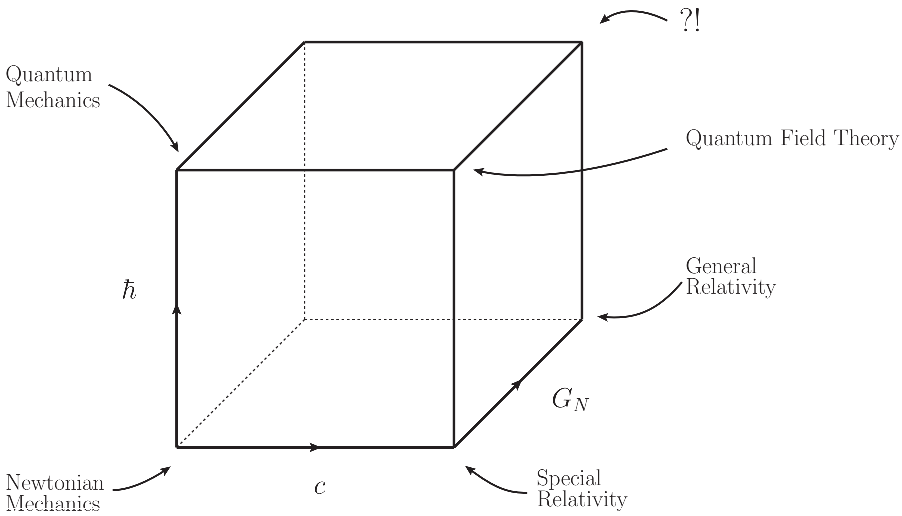


Figure 1.1: Okun's Cube

theories it was constructed upon as different limits and thus relates the parameters of those theories. Let us now briefly describe which are the basic building blocks the assumptive theory unifying General Relativity and Quantum Mechanics has to incorporate.

1.3 The Building Blocks

1.3.1 The Particle Zoo and Gauge Theories

In order to understand the outcome of present experiments, only a relatively small number of degrees of freedom have to be introduced. These are arranged in three generations, *i.e.* three copies, of the following pattern of fermions :

$$\begin{array}{cc} \begin{pmatrix} \ell_L \\ \nu_L \end{pmatrix} & \ell_R \\ \begin{pmatrix} u_L & u_L & u_L \\ d_L & d_L & d_L \end{pmatrix} & \begin{array}{c} \begin{pmatrix} u_R & u_R & u_R \end{pmatrix} \\ \begin{pmatrix} d_R & d_R & d_R \end{pmatrix} \end{array} \end{array} \tag{1.2}$$

where the leptons and quarks, which come in three colors, have both left and right chiralities and are massive while the neutrino is left-handed and massless. The particle mediating the electromagnetic, weak and strong forces are bosons. These respectively are the photon γ, the massive $W^{\pm}$ and Z^0 vector bosons and the massless gluons G^a. The fact that the electro-weak (EW) bosons are massive is understood as coming from a spontaneously broken gauge symmetry triggered by a Higgs mechanism [3]. The Standard Model of particle physics (SM) is thus defined as a spontaneously broken gauge theory [4–6], see [7] for a bottom-top reconstruction, with the matter content given by three copies of (1.2). Since, depending on the matter content, non-Abelian gauge theories may admit strong coupling at low energy (IR) while being free at high energy (UV), *i.e.* they may enjoy *asymptotic freedom* [8, 9], one can understand the pattern of observed resonances as different bound-states of quarks. Gauge theories are thus essential ingredients that have to be present in any unified theory.

1.3.2 General Relativity

General Relativity [10, 11] describes the dynamics of the metric field fluctuations. It was developed in the same spirit as Aristotle's Æther, *i.e.* starting from principles rather than from some experimental clash with theory. Its action is given by the Hilbert-Einstein term :

$$\mathcal{S} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R \tag{1.3}$$

where G_N is the above introduced Newton's constant which has for dimension GeV^{-2} in natural units. One may couple the Standard Model fields to GR by covariantising all derivatives and contracting all Lorentz indices by using the metric. However the dimensionality of G_N makes it impossible for GR to be power-counting renormalisable thus effectively forbidding General Relativity to be quantised at least in the context of perturbative quantum field theory. Note that Gravity may still be quantised in the context of effective field theories, see [12] for a state of the art review.

Moreover, and certainly more importantly, the simultaneous use of Quantum Mechanics and of General Relativity leads to problematic situations not only in the high-energy range but also in the infrared. Indeed, Hawking has shown that black holes radiate their mass away due to quantum-mechanical effects [13]. This phenomenon can be understood to take place when a particle-antiparticle pair is created, one of the particles then falling into the black hole whilst the other radiates away, thereby effectively reducing the black hole mass. The black hole information paradox [14] states that when a black hole evaporates, pure states are turned into mixed states, *i.e.* information is lost during the process, which is at odds with Quantum Mechanics.

1.4 So, what's next? Strings?

A complete change in paradigm is now invoked to tackle both the issue of renormalisability of gravity and that of the information paradox. The Quantum Field Theory sitting at the $(c = 1, \hbar = 1, G_N = 0)$ corner of Okun's cube treats the particles as point-like entities. Instead if one considers the fundamental objects of the theory to have a one dimensional extension, *i.e.* to be strings, the loops cannot be shrunk to zero-size anymore leading the string size ℓ_s to play the rôle of a UV cutoff. Note that the natural value of the string length may be estimated by requiring that at energies of order ℓ_s^{-1}, the strength of gravitational interactions is of the same order as the one of gauge interactions, *i.e.* $G_N \ell_s^{-2} \sim \alpha_{\mathrm{GUT}}$, resulting in ℓ_s to be roughly given by $\ell_s \sim \ell_{\mathrm{Pl}}$ where :

$$\ell_{\mathrm{Pl}} = \sqrt{\frac{\hbar\, G_N}{c^3}} \simeq 1.61 \cdot 10^{-35}\ \mathrm{m} \tag{1.4}$$

is the Planck length, thus making it impossible for current experiments to resolve strings which effectively appear point-like. We will see that the String action is almost unique, so that one could hope String Theory to be highly predictive, but we will argue that this is unfortunately not the case in realistic scenarios.

Moreover, there are hints that String Theory may solve the black hole information paradox. Indeed the very description of black holes changes in String Theory. The singularity sitting at $r = 0$ is replaced by a *fuzzball* made of vibrating strings and which spreads all the way to the black hole horizon. The crucial difference with GR black holes is that there is no void between the matter inside the black hole and the horizon, leading to the possible escape of information stored in the fuzzball due to the black hole evaporation and thus solving the black hole information paradox [15].

From the low-energy point of view the point-like particles are identified with string harmonics among which one always find a symmetric tensor, *i.e.* the metric field. GR is thus embedded in String Theory from the very beginning. In order to reproduce the Standard Model as its low-energy effective theory, String Theory also has to encapsulate gauge theories, *i.e.* non-Abelian Yang-Mills theories. It turns out that chiral matter with gauge symmetries does naturally arise in String Theory, for example in the $E_8 \times E_8$ Heterotic Superstring we will consider in this work, but no clear mechanism selecting the Standard Model gauge group has yet emerged.

Moreover the five known String Theories effective actions embed *Supersymmetry* (SUSY), *i.e.* the bosonic and fermionic degrees of freedom appear in a very constrained fashion. The appearance of Supersymmetry may be seen as a positive feature of String Theory since it may cure the Naturalness Problem from which the Standard Model suffers depending on the energy range at which it is broken, see Chapter 2. The Naturalness Problem, or Hierarchy Problem, motivates the introduction of Supersymmetry from a bottom-up approach, which is the one we will follow in the main part of this work.

However String Theory comes with its drawbacks, the first one being that, since String Theory predicts the number of space-time dimensions to be ten, one has to choose a proper compactification manifold. The choice of manifold does moreover influence the spectrum. In the Heterotic String, for example, the net number of generations depends on the topology of the compactification manifold and there is no clear reason why the number three should be singled-out.

A second problem introduced by String Theory is that Supersymmetry has to be broken. Indeed the boson and fermion masses in supersymmetric theories are degenerate which is not the case in Nature leading to the necessity of engineering a supersymmetry-breaking sector. Again this introduces many parameters in the four-dimensional effective action since different supersymmetry-breaking schemes predict different couplings among the effective degrees of freedom given in (1.2) and their superpartners, for example spoiling the experimentally well verified flavour structure enjoyed by the Standard Model.

1.5 The String Zoo and $\mathcal{M}$-Theory

In the previous section, we anticipated that the String action is almost unique. In fact, it is given by the integral over the area, the *worldsheet*, that is spanned by the string over time. However when one introduces worldsheet fermions to account for space-time fermions, one can choose among different boundary conditions (Ramond or Neveu-Schwarz) and different consistent projections among sectors of the theory. Nevertheless consistency reduces the number of independent String Theories to a handful : type IIA, IIB, I, $SO(32)$ Heterotic and $E_8 \times E_8$ Heterotic. The type II theories are $\mathcal{N} = 2$ theories in ten dimensions while the other three are $\mathcal{N} = 1$ theories. As soon as it was argued that the various String Theories are related among each other by a complex web of dualities, see [16], the idea of a mother theory of which the five known String Theories are limits has been put forward and is illustrated in Figure 1.2. This mother theory, called $\mathcal{M}$-Theory, is an eleven-dimensional quantum theory that interpolates between the five known String Theories and which has eleven-dimensional Supergravity as its low-energy effective theory.

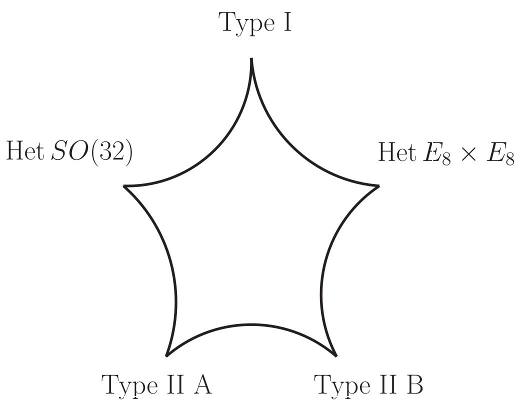


Figure 1.2: The Unavoidable $\mathcal{M}$-Theory Graph

The seven extra-dimensions of $\mathcal{M}$-Theory naturally split as $6 + 1$: the six extra-dimensions which were already present at the String level are supplemented by a seventh one. Since their size need not be the same, there are two orders in which the compactification to four space-time dimensions can be performed, the smallest dimensions being compactified first :

$$11 \to 10 \to 4 \qquad \text{or} \qquad 11 \to 5 \to 4. \tag{1.5}$$

By using the phenomenological values of the four-dimensional Newton's constant and of the gauge couplings, one may show that the second path should be chosen, leading to some energy range in which the universe effectively looks five-dimensional [17, 18].

1.6 From $\mathcal{M}$-Theory down to the Standard Model

The reconciliation of General Relativity with Quantum Mechanics has led us to introduce many exotic features in our description of Nature, namely extra-dimensions, Supersymmetry and large gauge groups. The standard lore for unveiling the Standard Model as an effective four-dimensional theory is the following. First one has to identify the relevant light fields in the String Theory spectrum and to write down a Lagrangian density describing their dynamics. Then the extra-dimensions have to be compactified, it will be argued in Chapter 6 that the manifold on which the compactification is to be performed has to be such that it allows for a minimal amount of Supersymmetry to remain unbroken in four dimensions. The compactification process generates towers of massive modes as is explained in Chapter 6 of which only the lightest are relevant to describe Nature at accessible energies. At this point the gauge group is still large and therefore unifies strong, weak and electromagnetic interactions and is given the name of Grand Unified Theory (GUT) group which needs to be broken at low energy in order to recover the Standard Model gauge group. Moreover the theory still exhibits Supersymmetry. Many mechanisms are available on the market to break the latter, two of them being discussed at the end of Chapter 3. In order to break the GUT gauge group one may invoke either perturbative effects, like the Higgs mechanism, or non-perturbative effects, like the breakdown of chiral symmetry associated with the pions.

When looked at from a bottom-top perspective the route we have pursued is seen as follows. First Supersymmetry manifests itself and pretty remarkably leads to the unification of the three gauge couplings at the GUT scale, *i.e.* around 10^{15} GeV. At higher scales extra-dimensions begin to unfold. By pushing the energy further and further one will meet all the string harmonics.

1.7 The Supersymmetric Flavour Problem

1.7.1 Top-Bottom Perspective

Let us pause a moment to look back at what was achieved. We started from the Standard Model which is in a wonderful agreement with experimental data and from General Relativity whose agreement with data is not less impressive. The introduction of String Theory permits to solve the apparent dichotomy among the quantum and gravitational worlds but at the price of spoiling the impressive predictivity of the Standard Model. In this work we propose to examine a particular aspect of this generic loss of agreement with data, the so-called *Supersymmetric Flavour Problem*. More precisely we will focus on soft scalar mass operators, which are potentially dangerous since they can induce flavour-flipping.

The flavour structure of the Standard Model enjoys some accidental symmetries which are the remnant of a $U(3)^5$ symmetry broken by the Yukawa terms, see Chapter 2. These symmetries are among other things responsible for the absence of proton decay and the absence of flavour changing processes in the lepton sector. The accidental symmetries are however generically lost when adding new particles to the spectrum, as is the case in supersymmetric extensions of the Standard Model for example. More precisely, the $U(3)^5$ symmetry is broken not only by the Yukawa couplings but also by other operators, thereby it generically has no remnant. One of the issues encountered in extending the Standard Model is thus to devise a mechanism to control the Standard Model loop-suppressed processes (as $b \to s + \gamma$ protected by the GIM mechanism [19]) and absent ones (as $\mu \to e + \gamma$ protected by individual lepton number conservation).

1.7.2 Bottom-Top Perspective

The Supersymmetric Flavour Problem also arises when trying to solve the Standard Model Naturalness Problem by introducing Supersymmetry. Since Supersymmetry has to be broken if it is to provide a realistic theory, one has to devise both a supersymmetry-breaking sector and a way to ensure its transmission to the Standard Model fields. Randall and Sundrum proposed in [20] a five-dimensional setup where the Standard Model is located on a end-of-the-world 3-brane while Supersymmetry is broken on the another one. Such a strategy goes under the name of *sequestering*. Gravity is given the rôle of transmitting supersymmetry-breaking from one brane to the other. Such five-dimensional theories with gravity being the only interaction capable of joining the two branes, *i.e.* which propagates in the bulk, have a sequestered Kähler potential which forbids the appearance of soft scalar masses at the classical level. From the four-dimensional low-energy effective theory point of view, this emerges in the same way as in the so-called no-scale models [21].

However the top-bottom perspective hardly ever generates five-dimensional models in which gravity is the only force propagating in the bulk. Indeed as noted by [22, 23], the eleven-dimensional gravity multiplet is rearranged in $\mathcal{N} = 2$ vectors and hypermultiplets which couple the two branes, spoiling the sequestered picture as indicated by Figure 1.3.

We will argue in Chapter 3 that Supersymmetry has to be broken in a sector distinct from the observable sector, that is in the so-called *hidden sector*. In Figure 1.3, the observable sector is located on one of the 3-branes while the hidden sector consists of the matter fields on the other brane together with the light fields surviving from the $\mathcal{N} = 2$ vectors and hypermultiplets, which collectively go under the name of *moduli*.

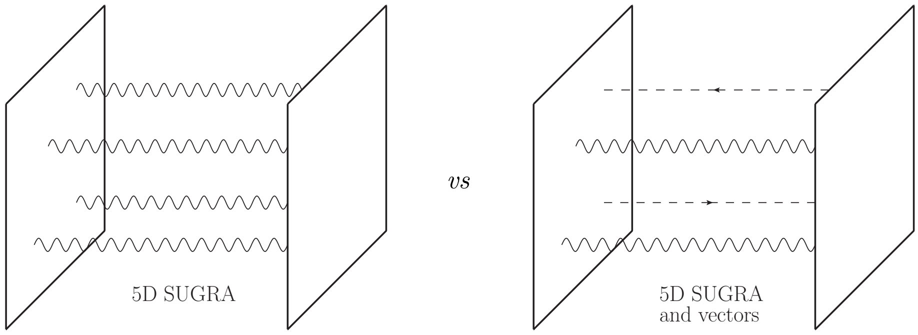


Figure 1.3: Pure 5D SUGRA *vs* Heterotic $\mathcal{M}$-Theory Compactified on a Calabi-Yau

1.8 Tackling the SUSY Flavour Problem, a Strategy

In order to tackle the Supersymmetric Flavour Problem we will adopt the following strategy. First we choose the $E_8 \times E_8$ Superstring which naturally embeds end-of-the-world branes supporting charged fields in the eleven-dimensional picture and, as previously argued, effectively looks five-dimensional within some energy range when considered as coming from $\mathcal{M}$-Theory, leading to a natural comparison with the Randall and Sundrum proposal. We will then compactify this theory to four dimensions and compute the resulting soft scalar masses. Since they arise due to the exchange of heavy vector fields which are in a one-to-one correspondence with the non-minimal Kähler moduli, the terms responsible for the soft scalar masses in the effective theory will be of the current-current-type mimicking the four-Fermi interaction below the electro-weak scale.

We will then try to exploit this very peculiar form of interaction and to engineer a mechanism which effectively forbids the appearance of soft scalar masses at the classical level. Generically soft scalar masses will then be generated at the quantum level, but, thanks to the geographic separation among the visible and hidden sectors, loops cannot be shrunk to zero-size leading to a relative insensitivity to far UV physics. In particular, one may certainly devise situations in which quantum effects are only sensitive to scales below the one breaking flavour [24–27], thus effectively leading to universal soft scalar masses.

Fortunately, mechanisms devised to suppress tree-level current-current operators have already been proposed in the literature. Indeed in the context of conformal sequestering in which the soft masses are suppressed by large running effects one cannot suppress conserved currents since they are characterised by a vanishing anomalous dimension. However it was noted [28] that the supersymmetric version of Noether's theorem not only implies the conservation of the associated vector current but also leads to the vanishing of the current's auxiliary fields. Since the most relevant operators giving rise to soft scalar masses are higher-dimensional operators mixing two visible and two hidden Superfields in the effective Kähler potential, and more in general are of the form :

$$K \sim Z(X, X^\dagger)\Phi^\dagger\Phi \tag{1.6}$$

where both the F and D components of Z give rise to soft scalar masses, it is sufficient to ask for Z to be the conserved current of a global symmetry of the hidden sector to suppress the soft scalar masses at the classical level [29]. Such a mechanism is given the name of *mild-sequestering*. Since the hidden sector generically involves two subsectors, see *e.g.* [30], we will consider the general case in which both the moduli and the hidden brane matter fields participate to supersymmetry-breaking. Moreover, since we are interested in dimension-6 operators, we will need to determine the Kähler potential at the fourth order in the matter fields. In the case of orbifolds, this is well-known [31, 32] but in the more general context of Calabi-Yau compactifications only the leading quadratic order is currently under control [33]. An interesting claim of the all-orders structure of the Kähler potential in the matter fields has recently appeared in the literature [34]. In this work we present a direct and systematic derivation of the full Kähler potential and argue that the result proposed in [34] is valid only under rather strong assumptions we will discuss. Under these assumptions, the full dependence of the Kähler potential in the matter fields is known and the question we have to assess is whether it allows for mild-sequestering to be implemented.

We will show that in the context of orbifold compactifications our strategy can naturally be put to work whereas in the context of Calabi-Yau models, only a subset of the compactification manifolds provided with a stable holomorphic gauge bundle admit the possibility for such a mechanism to be implemented.

1.9 Outline of the Thesis

The Thesis is presented in the bottom-up perspective. In Chapter 2 we introduce the Standard Model of particle physics, describe its action and focus on its accidental symmetries. Limits on the effective cutoff of the theory are then presented. Finally the Naturalness Problem is exposed together with some of its solutions. In Chapter 3, we investigate one of these solutions, namely Supersymmetry and discuss its breaking by introducing soft terms among which the soft scalar mass we will be focussing on. We then explain the necessity of introducing a distinct sector in which supersymmetry-breaking occurs from sum-rules arguments. In Chapter 4 we introduce Supergravity which is the supersymmetrised version of General Relativity and its coupling to matter fields. We then describe the general structure of soft scalar masses in generic Supergravity theories. In Chapter 5 we introduce String Theory particularly focusing on Heterotic String models and sketch how they are related to $\mathcal{M}$-Theory. In Chapter 6 we explain the compactification process from both the ten-dimensional Heterotic Supergravity and the eleven-dimensional Heterotic $\mathcal{M}$-Theory on both orbifolds and smooth manifolds. In Chapter 7 we describe the computation the effective Kähler potential in four dimensions. In Chapter 8 we first derive the structure of the soft masses from the Kähler potential resulting from the previous Chapter and then discuss the possibility of implementing a symmetry to cancel the non-universal soft scalar masses at the classical level. Finally in Chapter 9 we present our conclusions.

This work is based on the following two research papers :

- ◇ C. Andrey and C. A. Scrucca, *Mildly Sequestered Supergravity Models and their Realization in String Theory*, Nuclear Physics B **834** 363–389, 2010. `arXiv:1002.3764` [35]
- ◇ C. Andrey and C. A. Scrucca, *Sequestering by Global Symmetries in Calabi-Yau String Models*, Nuclear Physics B **851** 245-288, 2011. `arXiv:1104.4061` [36]

Chapter 2

The Standard Model and Beyond

In this Chapter we first give a short review of the Standard Model of particle physics of which both quantum and relativistic effects are in agreement with present laboratory experiments with remarkable accuracy. The only yet unseen Standard Model degree of freedom is the Higgs boson which, if it exists, is constrained by precision experiments to be very light compared with, say, the Planck mass [2] but still above the current experimental exclusion bounds. However light scalar degrees of freedom will be shown to be quite unnatural in quantum field theory since their masses are not stable under radiative corrections. This is known as the *Hierarchy Problem*. Several ways out have been engineered and are briefly described at the end of the present Chapter.

2.1 The Standard Model

2.1.1 Particle Content

The Standard Model of particle physics is an effective field theory based on the following gauge group :

$$\text{SM}: \qquad SU(3)_C \times SU(2)_L \times U(1)_Y \tag{2.1}$$

with respective couplings g_s, g and g'. The matter content of the SM consists of three copies, or generations, of the fields given in (1.2) whose quantum numbers are recorded in Table 2.1. The three generations correspond to the following fields which were introduced in Chapter 1, where we do not repeat the color structure :

	ℓ_R	ν_L	ℓ_L	u_R	d_R	u_L	d_L
First generation	e^-_R	$\nu_{e\,L}$	e^-_L	u_R	d_R	u_L	d_L
Second generation	μ^-_R	$\nu_{\mu\,L}$	μ^-_L	c_R	s_R	c_L	s_L
Third generation	τ^-_R	$\nu_{\tau\,L}$	τ^-_L	t_R	b_R	t_L	b_L

(2.2)

Field	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
l_R	$\mathbf{1}$	$\mathbf{1}$	-1
$L_L = \begin{pmatrix} \nu_L \\ l_L \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
u_R	$\mathbf{3}$	$\mathbf{1}$	$2/3$
d_R	$\mathbf{3}$	$\mathbf{1}$	$-1/3$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\mathbf{3}$	$\mathbf{2}$	$1/6$

Table 2.1: Standard Model matter fields

A complex scalar field H which is responsible for the Electro-Weak Symmetry Breaking (EWSB) is added and transforms in the $(\mathbf{1}, \mathbf{2}, 1/2)$. The gauge fields are respectively given by the gluons G_μ in the $(\mathbf{8}, \mathbf{1}, 0)$, the weak gauge fields W_μ in the $(\mathbf{1}, \mathbf{3}, 0)$ and the hypercharge gauge field B_μ in the $(\mathbf{1}, \mathbf{1}, 0)$.

2.1.2 Standard Model Lagrangian

The Standard Model Lagrangian is given by the most general renormalisable Lagrangian compatible with both the gauge and the Poincaré symmetry. The covariant derivative entering the kinetic part of the Lagrangian acts as follows :

$$D_\mu = \partial_\mu + ig_s G_\mu + igW_\mu + ig'YB_\mu \tag{2.3}$$

where the G_μ, W_μ and B_μ action on the fields may be read from Table 2.1. The Lagrangian describing the gauge fields dynamics is as usual given by their field-strength squared. Let us record for later use the action of the covariant derivative on a $SU(2)_L$ doublet :

$$D_\mu = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 2g'YB_\mu + gW^3_\mu & g(W^1_\mu - iW^2_\mu) \\ g(W^1_\mu + iW^2_\mu) & 2g'YB_\mu - gW^3_\mu \end{pmatrix}. \tag{2.4}$$

The diagonal entries are coupling fields of the same species and are called neutral currents while the off-diagonal entries couple the two elements of the doublet leading to the so-called charged currents for reasons that will shortly become clear. Let us start with the kinetic terms for the leptons. The allowed terms are :

$$\mathcal{L}_{\mathrm{SM}} \ni i\alpha_{ij} \bar{L}^i_L \not{D} L^j_L + i\beta_{ij} \bar{l}^i_R \not{D} l^j_R + \lambda^L_{ij} \bar{L}^i_L H l^j_R. \tag{2.5}$$

By suitable field redefinitions one may diagonalise all three terms. First one diagonalises both the matrices α and β by redefining the L_L and l_R fields, *i.e.* by choosing U and V such that both $U^\dagger \alpha U$ and $V^\dagger \beta V$ are diagonal. Then by rescaling the fields one can achieve a situation where the kinetic term is diagonal, *i.e.* where the α and β matrices are both given by the identity matrix :

$$\mathcal{L}_{\mathrm{SM}} \ni i\bar{L}^i_L \not{D} L^i_L + i\bar{l}^i_R \not{D} l^i_R + \lambda^L_{ij} \bar{L}^i_L H l^j_R. \tag{2.6}$$

The structure of the kinetic terms still allows for a further unitary redefinition of the fields under which $L_L \to U L_L$ and $l_R \to V l_R$ with $U^\dagger U = V^\dagger V = \mathbb{1}$. By appropriately choosing U and V one can diagonalise the λ matrix without however being able to bring it to the unit matrix since it would spoil the normalisation of the kinetic terms. The final form is thus :

$$\mathcal{L}_{\mathrm{SM}} \ni i\bar{L}^i_L \not{D} L^i_L + i\bar{l}^i_R \not{D} l^i_R + \lambda^L_i \bar{L}^i_L H l^i_R. \tag{2.7}$$

For the quarks the situation is slightly different since there is a further term one can add to the Lagrangian :

$$\mathcal{L}_{\mathrm{SM}} \ni iA_{ij}\bar{Q}^i_L \not{D} Q^j_L + iB_{ij}\bar{d}^i_R \not{D} d^j_R + iC_{ij}\bar{u}^i_R \not{D} u^j_R + \lambda^D_{ij}\bar{Q}^i_L H d^j_R + \lambda^U_{ij}\bar{Q}^i_L \tilde{H} u^j_R \tag{2.8}$$

where we have introduced $\tilde{H} = i\sigma^2 H^* = \epsilon H^*$ which is easily shown to transform in the $(\mathbf{1}, \mathbf{2}, -1/2)$. Applying the same strategy we have used for the leptonic part of the Lagrangian yields $A = B = C = \mathbb{1}$. Let us for the moment leave the Yukawa matrices untouched :

$$\mathcal{L}_{\mathrm{SM}} \ni i\bar{Q}^i_L \not{D} Q^i_L + i\bar{d}^i_R \not{D} d^i_R + i\bar{u}^i_R \not{D} u^i_R + \lambda^D_{ij}\bar{Q}^i_L H d^j_R + \lambda^U_{ij}\bar{Q}^i_L \tilde{H} u^j_R. \tag{2.9}$$

The last part of the SM Lagrangian is related to the Higgs field H. Since it is a scalar field, the following terms can enter the Lagrangian :

$$\mathcal{L}_{\mathrm{SM}} \ni D_\mu H (D^\mu H)^\dagger + \mu^2 |H|^2 - \lambda |H|^4 \tag{2.10}$$

where λ is restricted to be positive in order for the potential to be bounded from below. The sign of μ^2 does not suffer from any restriction and thus defines two phases. For $\mu^2 < 0$, the global minimum of the Higgs potential sits at $\langle H \rangle = 0$ and the whole SM spectrum remains massless while for $\mu^2 > 0$ the Higgs field acquires a vacuum expectation value (VEV) $\langle H \rangle = \mu/2\lambda$ which triggers the partial breaking of the gauge group :

$$SU(3)_C \times SU(2)_L \times U(1)_Y \quad \to \quad SU(3)_C \times U(1)_{\mathrm{QED}} \tag{2.11}$$

and induces masses for all fermions but the neutrinos through the Yukawa couplings and to most of the gauge fields. The gauge field corresponding to $U(1)_{\mathrm{QED}}$ is identified with the photon which is a linear combination of W^3_μ and B_μ. The remaining three $SU(2)_L \times U(1)_Y$ gauge fields acquire a mass term through the covariant derivative of the Higgs field and are identified with the Z^0 and $W^\pm$ fields. Indeed, acting with the covariant derivative (2.4) on the Higgs field and only retaining the $\mathcal{O}(v^2)$ terms yields, in the unitary gauge where $\langle H\rangle = (0,v)^T$:

$$D_\mu H(D^\mu H)^\dagger \ni \frac{1}{4}g^2v^2|W^1_\mu - iW^\mu_2|^2 + \frac{1}{4}v^2(g'B_\mu - gW^3_\mu)^2. \tag{2.12}$$

We thus conclude that the complex vector field $W_\mu \propto W^1_\mu - iW^\mu_2$ gets a mass equal to g^2v^2. After diagonalising the second term by a rotation of angle θ_W, the field $Z_\mu \propto \cos\theta_W W^3_\mu - \sin\theta_W B_\mu$ gets a mass equal to $v^2(g^2+g'^2)$ while its orthogonal partner $A_\mu \propto \sin\theta_W W^3_\mu + \cos\theta_W B_\mu$ remains massless. The value of the Weinberg angle θ_W is given by :

$$\theta_W = \mathrm{atan}\left(\frac{g'}{g}\right). \tag{2.13}$$

When rewriting the covariant derivative in terms of A_μ, Z^0_μ and $W^\pm_\mu$, one easily identifies the electric charge, *i.e.* the charge under $U(1)_{\text{QED}}$, as being given by :

$$Q = T^3 + Y \tag{2.14}$$

where T^3 is the eigenvalue of $\sigma^3/2$. The charge assignment thus corresponds to leptons with charge -1, neutral neutrinos, up-type quarks with charge $2/3$ and down-type quarks with charge $-1/3$. The denomination of neutral and charged currents should now have become clear.

2.1.3 Flavour Changing Currents

Let us now investigate interactions mixing different generations. These are said to violate flavour. In the leptonic sector we were able by suitable field redefinitions to diagonalise both the kinetic terms and the Yukawa matrices as shown by (2.7). There are thus no flavour violations in the leptonic sector of the Standard Model.

In the quark sector however the situation changes. Let us restart from (2.9) in the unitary gauge where $H = (0, v+h)^T$ and make the Q_L doublets explicit :

$$\begin{aligned}\mathcal{L}_{\text{SM}} \ni\; & i\begin{pmatrix}\bar{u}^i_L & \bar{d}^i_L\end{pmatrix}\begin{pmatrix}\not{D}_N & \not{D}_C \\ \not{D}^\dagger_C & \not{D}_N\end{pmatrix}\begin{pmatrix}u^i_L \\ d^i_L\end{pmatrix} + i\bar{d}^i_R\not{D}d^i_R + i\bar{u}^i_R\not{D}u^i_R \\ & + \lambda^D_{ij}\bar{d}^i_L(v+h)d^j_R + \lambda^U_{ij}\bar{u}^i_L(v+h)u^j_R\end{aligned} \tag{2.15}$$

where the $\not{D}_{N/C}$ denote the neutral-current and charged-current entries of the covariant derivative and can be read from (2.4). We can now achieve diagonal mass matrices by rotating the fields with unitary matrices :

$$u_L \to U_Lu_L, \quad u_R \to U_Ru_R, \quad d_L \to D_Ld_L, \quad d_R \to D_Rd_R. \tag{2.16}$$

This rotation is not a symmetry of the Lagrangian and has the net effect of modifying the charged-current entries of the covariant derivative, namely :

$$\not{D}_C \to \not{D}_C U^\dagger_L D_L \equiv V\not{D}_C. \tag{2.17}$$

The charged-current interactions are thus non-diagonal in flavour space. They are parametrised by a unitary matrix V known as the Cabbibo-Kobayashi-Maskawa (CKM) matrix [37, 38]. To conclude, let us stress the results we have obtained at tree-level, *i.e.* without quantum corrections :

- ◇ No flavour violation in the leptonic sector,
- ◇ No flavour changing neutral current in the quark sector,
- ◇ Flavour changing charged currents in the quark sector, parametrised by the CKM matrix.

2.1.4 Accidental Symmetries

In the previous subsection we have written down all renormalisable terms compatible with the gauge symmetry. Since we restricted ourselves to renormalisable interactions, the symmetry enjoyed by the Lagrangian density is enhanced by some accidental global symmetries. Since the SM should ultimately be considered as an effective field theory, it has to be supplemented with non-renormalisable terms suppressed by a certain scale M which are to respect the gauge symmetry, but not the accidental ones. Let us assign the following charges to the SM fields under new global $U(1)$'s L_i and B where the i index is in flavour space :

$$L_i(L_L^j) = \delta_{ij}, \qquad L_i(l_R^j) = \delta_{ij}, \qquad L_i(\text{others}) = 0 \tag{2.18}$$

and :

$$B(Q_L) = \frac{1}{3}, \qquad B(u_R) = \frac{1}{3}, \qquad B(d_R) = \frac{1}{3}, \qquad B(\text{others}) = 0. \tag{2.19}$$

These operators respectively correspond to electron number, muon number, tau number and to baryon number and are symmetries of the renormalisable SM Lagrangian. Such symmetries prevent $\mu \to e + \gamma$ from happening since such a process would violate both L_e and L_μ. Proton decay is also understood to be forbidden by these accidental symmetries. One possible channel would be $p^+ \to e^+ + \pi^0$, violating both L_e and B.

Since we observe neither proton decay nor flavour changing neutral current (FCNC) processes like $\mu \to e + \gamma$ (see [2]), the proton lifetime and the branching ratio of FCNC processes can be used to put bounds on the energy scale M at which the operators violating the Standard Model accidental symmetries are generated.

Another lesson these accidental symmetries teach us is that we have to be very careful when going beyond the SM. Indeed by introducing new degrees of freedom one may generate operators which would spoil these accidental symmetries which however seem not to be violated by Nature. To be more precise, the individual lepton numbers seem to be violated by neutrino oscillations. However, the experimental facilities aimed at answering the question of whether the total lepton number, denoted by L, is conserved or not have not yet reached sufficient a precision in order to discriminate among the Dirac or Majorana nature of the neutrinos.

One will have to remember this lesson when introducing Supersymmetry. Indeed many of its parameters are not flavour-universal, *i.e.* they mix different flavours, and thus give rise to phenomena like FCNC in the leptonic sector which, again, have not yet been observed.

2.1.5 Experimental Success of the Standard Model

The Standard Model Higgs phase, that is when $\mu^2 > 0$ in (2.10), leads to a spectrum that is in a very broad accordance with the one observed in present experiments. The Higgs boson has however not yet been observed but its mass is greatly constrained by EW precision tests and its non-observation at LEP 2 [2] :

$$114.4 \text{ GeV} \lesssim m_H \lesssim 149 \text{ GeV} \qquad \text{both bounds at 95\% C.L.} \tag{2.20}$$

As anticipated in the introduction to this Chapter, the Standard Model quantum effects account for what has been observed so far. The observables which are best suited to test the quantum structure are those whose tree-level predictions vanish. The standard example is the kaon oscillation $K_0 - \bar{K}_0$ which cannot occur in the tree approximation since FCNC vanish at the classical level but which are allowed by quantum effects, *i.e.* at the loop level. Notice that this process does not violate any symmetry at the quantum level, all four accidental charges being preserved by quantum effects. At the quantum level, the kaon oscillation is obtained by a double flavour-violating loop diagram which involve four powers of the CKM matrix and would be vanishing in the absence of mass-splittings in the quark sector thanks to the unitarity of the CKM matrix. This almost-cancellation is known as the *GIM mechanism* [19] which is also at work to suppress the $b \to s + \gamma$ transition for example. Other great successes of the Standard Model are for example the agreement on the EW gauge bosons masses, the unitarity of the CKM matrix, etc...

2.1.6 However...

...the Standard Model is not without imperfections. Indeed in the case where the Standard Model is minimally coupled to gravity, the predictions do not agree with our observations of Nature since the matter-antimatter asymmetry cannot be explained, the leptogenesis and baryogenesis mechanisms remain largely unknown, there is no dark matter candidate, no particle to drive inflation (except if one were to add a non-minimal coupling between the Higgs field and the Ricci scalar [39]).

Moreover since General Relativity does not seem to be renormalisable, it has to be interpreted as the effective theory of a yet unknown fundamental microscopic theory. Many attempts have been made towards a quantum theory of gravity among which String Theory, asymptotically safe theories, etc.

Finally, the Standard Model contains quite a number of parameters. A careful counting leads to 19 parameters : the three gauge couplings, the two parameters of the Higgs potential, the nine fermion masses, the four independent parameters (three angles and one CP-violating phase) of the CKM matrix and the $SU(3)$ gauge group θ-angle which appears multiplied by $F\tilde{F}$ and which leads to non-trivial consequences since the $SU(3)$ vacuum structure itself exhibits a non-trivial pattern. A great step towards a better understanding of Nature would be the construction of a model with only a handful of parameters, ideally none, leading, for example, to the comprehension of the pattern of the fermion masses.

Yet another motivation for going beyond the Standard Model is provided by the so-called Hierarchy Problem to which we devote the next subsection.

2.1.7 Small Parameters, Naturalness and the Hierarchy Problem

It turns out that some of the 19 Standard Model parameters are small compared to the relevant scales of the model. Small parameters are understood to be *natural* if in the limit where they are set to zero one unveils a new symmetry. The idea of naturalness is due to 't Hooft and was formalised in [40]. Let us systematically review the parameters and check if they satisfy the 't Hooft criterion :

- ◇ According to the criterion the gauge coupling smallness is natural since when these are set to zero the species which were interacting via gauge interactions decouple.

- ◇ When setting the Yukawa couplings to zero an $U(3)^5$ symmetry acting in flavour space emerges, the smallness of the Yukawa is thus also understood to be natural. Note that the Yukawa couplings do not totally break $U(3)^5$:

$$U(3)^5 \quad \rightarrow \quad L_e, \quad L_\mu, \quad L_\tau \quad \text{and} \quad B. \tag{2.21}$$

Indeed, a close inspection of equations (2.7) and (2.9) reveals that the accidental symmetries previously discussed are the remnant of the $U(3)^5$ symmetry which is broken by the Yukawa couplings : λ^L breaks $U(3)_L \times U(3)_l$ to L_e, L_μ and L_τ when combined with the hypercharge and λ^U and λ^D break $U(3)_Q \times U(3)_u \times U(3)_d$ to B.

- ◇ Setting a fermion mass to zero also unveils a new symmetry called the chiral symmetry which contains a discrete subgroup acting as :

$$\psi \rightarrow \gamma^5\psi \qquad \text{leading to} \qquad \bar{\psi}\partial\!\!\!/\psi \rightarrow \bar{\psi}\partial\!\!\!/\psi \qquad \text{and} \qquad \bar{\psi}\psi \rightarrow -\bar{\psi}\psi. \tag{2.22}$$

From this observation we conclude that since quantum effects will not spoil the symmetry, the renormalisation of the fermion mass will be proportional to the mass itself. Roughly speaking, if a fermion is coupled to a boson of mass m_B with strength λ, we have :

$$\delta m_\psi \sim \frac{\lambda^2}{16\pi^2} m_\psi \log\left(\frac{\Lambda}{m_B}\right). \tag{2.23}$$

Vector fields enjoy the same protection against radiative corrections thanks to the gauge symmetry which is recovered when setting the vector mass to zero.

- ⋄ Now, what about the Higgs mass? Data suggests that the Higgs should be found in the interval given in (2.20) which is much smaller than the cutoff of the theory, say $M_{\text{Pl}} \sim 10^{19}$ GeV. Can we understand this situation from our perspective? In other words, do we recover a symmetry when a scalar field mass is set to zero? The answer is no. This is roughly speaking the Hierarchy Problem, *i.e.* we do not understand why the Higgs has a small mass compared to the cutoff of the theory. Let us now investigate the consequences of Higgs mass failure to satisfy the 't Hooft condition.

If we seriously consider the SM as being an effective theory meaning that there exist new degrees of freedom at higher scales, then the Hierarchy Problem can be rephrased in a more convincing way. Let us consider the following toy model of a scalar field with mass m, playing the rôle of the Higgs field, coupled to a heavier fermion of mass M which would describe the microscopic theory of which the SM is an effective theory :

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - g\phi\bar{\psi}\psi. \tag{2.24}$$

Here ψ plays the rôle of a field whose mass is larger than the EW scale. The ϕ mass in the SM, *i.e.* in the effective theory, is to be understood as its mass once ψ has been integrated out. A quick computation leads to the following result :

$$m_{\text{eff}}^2 = m^2 - \frac{g^2}{16\pi^2}M^2 \tag{2.25}$$

where both m^2 and m_{eff}^2 are understood to be the renormalised masses at the scale $\mu = M$ [41]. Then having a small effective mass m_{eff}^2 compared to M^2 leads to a fine-tuning problem. Indeed we would have to adjust the mass m of the microscopic theory in such a way that the right-hand side of the previous equation is of the order of the EW scale. The amount of fine-tuning may be evaluated as :

$$\text{Fine-tuning} \quad \sim \quad \frac{16\pi^2}{g^2}\frac{m_{\text{eff}}^2}{M^2}. \tag{2.26}$$

The question of determining whether there is a fine-tuning problem has been translated into the evaluation of the scale M at which new degrees of freedom are to be taken into account. Adding d-dimensional irrelevant operators $\mathcal{O}_i^{(d)}$, *i.e.* suppressed by the scale M :

$$\delta\mathcal{L}_{\text{SM}} = \sum_i \sum_{d>4} c_i^{(d)} \frac{\mathcal{O}_i^{(d)}}{M^{d-4}} \tag{2.27}$$

of course modifies the SM predictions. A lower bound on M is thus given by the lowest value M can take without spoiling the SM predictions which are tightly constrained since they are in almost perfect agreement with observations. Let us give two examples :

- ⋄ As argued in subsection 2.1.6, the Standard Model predicts the neutrinos to be massless. However neutrino oscillations favour a tiny mass which may be described by the following dimension-five operator which could emerge as an effective effect of heavy right-handed neutrinos ν_R in the $(\mathbf{1},\mathbf{1},0)$ coupled to the SM via $\lambda^N_{ij}\bar{L}^i_L\tilde{H}\nu^j_R - M\bar{\nu}^c_R\nu_R$:

$$\delta\mathcal{L}_{\rm SM} \sim (\lambda^N)^2_{ij}\frac{1}{M}\bar{L}^i_L\tilde{H}\bar{H}\tilde{L}^j_L \qquad \rightarrow \qquad m_\nu \sim (\lambda^N)^2_{ij}\frac{v^2}{M} \tag{2.28}$$

In order to recover the correct amplitude for the neutrino masses, M should be of order $M \sim 10^{13}$ GeV [2] assuming the couplings are of order one.

- Since the main subject of this work is to try to devise a mechanism which solves the SUSY flavour problem, it is certainly interesting to give an example of one of the operators which would lead to tensions with flavour physics observations. $K^0 - \bar{K}^0$ oscillations are for example generated by the following gauge-invariant irrelevant four-Fermi operator :

$$\delta\mathcal{L}_{\rm SM} \sim \frac{1}{M^2}(\bar{s}_R\gamma^\mu d_R)(\bar{s}_R\gamma_\mu d_R). \tag{2.29}$$

Current data suggests that those operators could enter the SM Lagrangian without spoiling their agreement with the SM contributions given that their defining scale M is bigger than 10^7 GeV [2].

All those scales are well above the EW scale leading to a very precise fine-tuning condition (2.26) and thus to the unnaturalness of the Standard Model.

2.2 Beyond the Standard Model

As pointed out in the previous section, the Standard Model is not without imperfections. It fails to explain neutrino oscillations, baryogenesis, leptogenesis, inflation, dark matter and when minimally coupled to the Einstein-Hilbert action it does not lead to a consistent theory of gravity at the quantum level. Moreover it suffers from the Hierarchy Problem. Going beyond the SM thus seems to be unavoidable.

There are many ways to introduce alternatives to the Standard Model. Some of them consist in small modifications, others ones in a complete change of paradigm.

2.2.1 Minimal Approaches

The νMSM One of the minimal modifications of the SM is the νMSM which aims at solving neutrino oscillations, leptogenesis, baryogenesis, dark matter and inflation when coupling the Higgs to the Ricci scalar in a non-minimal way. The Hierarchy problem is however not addressed. See [42] for a state of the art review.

Technicolor In order to evade the Hierarchy Problem, one may devise a strategy towards an enforcement of a protection of the Higgs mass against radiative corrections. One of the ideas on the market consists in trading the Higgs as an elementary particle for a fermion condensate. In such scenarios EWSB is achieved via strong-dynamics effects in the newly introduced gauge sector, the EW scale being generated by dimensional transmutation in the same way as Λ_{QCD}, see [43, 44] for the original papers.

2.2.2 Non-Minimal Approaches

A very non-minimal, but highly ambitious, model towards a theory of Nature is String Theory to which we will devote most of Chapter 5. String Theory aims at a consistent theory of gravitation and gauge interactions at the quantum level with as few parameters as possible. However since String Theory constrains the number of space-time dimensions to be ten, many parameters emerge from the choice of the compactification manifold. Many light fields emerge from the compactification process, their stabilisation being one of the major challenges of String Theory together with the fact that like all extensions of the Standard Model, the accidental flavour structure is generically lost. String Theory is also known to admit many vacua collectively named *the landscape*, most of them not resembling Nature. Nevertheless String Theory certainly is the most promising candidate to describe the quantum regime of gravity and is *per se* a fascinating human endeavour.

2.2.3 Supersymmetry

The attentive reader may have noticed that we did not include Supersymmetry [45–47] neither in the Minimal nor in the Non-Minimal approaches to modifying the SM. Supersymmetry's aim is to solve the Hierarchy problem. The deepest roots of the Hierarchy Problem lie in the fact that a scalar field mass is not protected against radiative corrections which attract it towards the theory cutoff. Indeed, we have seen that if a heavy fermion ψ with mass M is coupled to the Higgs fields via $g_F H\bar{\psi}\psi$, it generates a quantum correction to its mass given at leading order by :

$$\Delta m_H^2 = -\frac{g_F^2}{16\pi^2}M^2. \tag{2.30}$$

A possible way out of the Hierarchy Problem would be to introduce a complex scalar field of mass M coupled to the Higgs via $g_B H^2|\phi|^2$ which would induce a quantum correction to the Higgs mass given at leading order by :

$$\Delta m_H^2 = +\frac{g_B}{16\pi^2}M^2. \tag{2.31}$$

A theory which relates bosons and fermions and thereby arranges a conspiracy such that $g_B = g_F^2$ is realised would thus solve the Hierarchy Problem. Supersymmetry is such a theory and is the subject of the next Chapter.

Chapter 3

Supersymmetry and its Breaking

In this Chapter we begin by reviewing the basics of Supersymmetry. We then motivate its introduction as a solution to the Hierarchy Problem which has been discussed in the last Chapter. We then argue that SUSY has to be broken in order to be compatible with present experiments and thus parametrise its breaking pattern. Finally we review two common proposals for the SUSY-breaking mechanism and the mediation of its effects to the Standard Model.

3.1 A Non-Technical Overview

Supersymmetry is a symmetry relating bosons and fermions. Since it has to change the statistics of the field acted upon, its parameter, denoted by ϵ, has to be a fermion. Schematically SUSY acts as :

$$\delta_\epsilon \varphi = \epsilon \psi \qquad \text{and} \qquad \delta_\epsilon \psi = \bar{\epsilon} \partial\!\!\!/ \varphi \tag{3.1}$$

where φ and ψ are respectively a boson and a fermion. Since the operators realising this symmetry have to be fermionic, they do carry a half-integer spin [48] and thus act non trivially on the Poincaré generators. In other words, SUSY is extending the space-time symmetry. However, Coleman and Mandula proved in [49] that under rather reasonable assumptions the symmetry of a Quantum Field Theory is restricted to take the form of the direct product of Poincaré symmetry with an internal symmetry. Indeed the addition of space-time symmetries translates into new constraints the observables have to satisfy. As an example, let us consider a non-relativistic $2 \to 2$ scattering of same-mass particles. Energy and momentum conservation are respectively expressed as :

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 \qquad \text{and} \qquad p_1^2 + p_2^2 = p_3^2 + p_4^2 \qquad\qquad p_i \equiv |\vec{p}_i|. \tag{3.2}$$

This in particular implies that $\vec{p}_1 \cdot \vec{p}_2 = \vec{p}_3 \cdot \vec{p}_4$. Now let us imagine adding a space-time symmetry to this system, for example $p_1^4 + p_2^4 = p_3^4 + p_4^4$. Together with the conservation of energy, this last condition leads to $p_1 p_2 = p_3 p_4$, and thus the angle between the initial particles and the final ones is predicted to be the same which is to say that the $\mathcal{S}$-matrix is not analytic in the kinematical variables. Another example where too restrictive conservation laws lead to uninteresting physics can be found in [50].

However, the Coleman and Mandula theorem can be evaded by introducing the concept of graded Lie algebra. Indeed the Haag-Łopuszański-Sohnius theorem [51] states that if allowing for generators to anticommute then one can construct a non-trivial extension of the space-time symmetries which realises SUSY. The algebra is restricted to the following structure in four dimensions :

$$\begin{aligned} &\left[P^\mu, Q^i_\alpha\right] = 0 \qquad &&\left\{Q^i_\alpha, \bar{Q}^j_{\dot{\alpha}}\right\} = \delta^{ij}\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \qquad &&\left[Q^i_\alpha, M_{\mu\nu}\right] = \frac{1}{2}\left(\sigma_{\mu\nu}\right)_\alpha{}^\beta Q^i_\beta \\ & &&\left\{Q^i_\alpha, Q^j_\beta\right\} = 0 &&\left\{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\right\} = 0 \end{aligned} \tag{3.3}$$

where the Q^i_α are the SUSY generators of which the i index labels the generation, *i.e.* the number of supersymmetries, α the spinorial index and where $\bar{Q}^i_{\dot{\alpha}} \equiv Q^{i\dagger}_\alpha$. The number of supersymmetries is usually denoted by $\mathcal{N}$. The first equation implies that all the particles related by Supersymmetry share the same mass while the third expresses that the SUSY generators have spin $1/2$. Let us explore the second equation in the massless case. By orienting the axes such that the particle moves along the third axis one gets for each generation of Supersymmetry generators :

$$\left\{Q^i_1, \bar{Q}^i_{\dot{1}}\right\} = 4E \qquad \text{and} \qquad \left\{Q^i_2, \bar{Q}^i_{\dot{2}}\right\} = 0. \tag{3.4}$$

Rescaling the Q^i_1 generators by $\left(2\sqrt{E}\right)^{-1}$ provides us with a typical $\mathcal{N}$-dimensional fermionic algebra :

$$\left\{a^i, a^{j\dagger}\right\} = \delta^{ij} \qquad \left\{a^i, a^j\right\} = 0 \qquad \left\{a^{i\dagger}, a^{j\dagger}\right\} = 0 \tag{3.5}$$

while the Q^i_2 algebra, which is totally anticommuting, has to be represented by zero. If $|\lambda\rangle$ denotes a state of helicity λ satisfying the Clifford vacuum condition $a^i|\lambda\rangle = 0$, then $a^{i\dagger}|\lambda\rangle$ will have helicity $\lambda + 1/2$. For a $\mathcal{N} = 1$ theory, the massless multiplet contains $|\lambda\rangle$ and $a^{1\dagger}|\lambda\rangle$ whose helicities are respectively given by λ and $\lambda + 1/2$. In a $\mathcal{N} = 2$ theory, the different states related by Supersymmetry are : $|\lambda\rangle$, $a^{1\dagger}|\lambda\rangle$, $a^{2\dagger}|\lambda\rangle$ and $a^{1\dagger}a^{2\dagger}|\lambda\rangle$. Note that in order to achieve a CPT-complete theory, one usually has to double the spectrum. Indeed if we take an $\mathcal{N} = 1$ theory with $\lambda = 0$ then the spectrum would be $\left(0^1, 1/2^1\right)$ where the superscript indicates the number of states of a given helicity. Its CPT-completion is thus given by $\left(-1/2^1, 0^2, 1/2^1\right)$ and is called the chiral multiplet. Another representation of SUSY we will often encounter is the vector multiplet which consists of $\left(-1^1, -1/2^1, 1/2^1, 1^1\right)$.

Since the application of a creation operator increases the helicity by $1/2$, the range of helicity contained in a theory is $\mathcal{N}/2$. This fixes a limit on the value of $\mathcal{N}$. Indeed massless particles can only be consistently coupled if their helicity is smaller or equal to two. A very nice discussion based on soft massless particles can be found in [52]. Note that in an $\mathcal{N} = 1$ theory the graviton partner is thus found to be a spin $3/2$ particle, the gravitino. We have thus established that :

$$\mathcal{N}_{\max} = 8. \tag{3.6}$$

From Table (C.1), we can read that in four dimensions an $\mathcal{N} = 1$ Supersymmetry is specified by four real parameters and thus has four generators, also called supercharges, which we identify with Q_1, Q_2, $\bar{Q}_{\dot{1}}$ and $\bar{Q}_{\dot{2}}$. The maximal number of supercharges is thus given by $4 \times \mathcal{N}_{\max} = 32$. By going back to Table (C.1), one can read that the maximal number of space-time dimensions consistent with Supersymmetry is :

$$d_{\max} = 11. \tag{3.7}$$

3.2 A Technical Overview

Let us now briefly review SUSY in four space-time dimensions. Many very good reviews on this topic are available, among which [53–57]. We refer the reader to [54] for the conventions used throughout this work. The essential notations are settled in Appendix A. The basics of Supersymmetry are moreover given in Appendix C.3, in particular Superspace, which extends the Minkowski space-time to include fermionic directions labelled by θ and $\bar{\theta}$. A Supersymmetry transformation can be shown to take the form of a translation in Superspace.

3.2.1 Chiral Models

Let us begin by reviewing the non-linear sigma model describing chiral Superfields [58, 59]. The most general two-derivative supersymmetric Lagrangian density describing the dynamics of chiral Superfields can be written as [54] :

$$\mathcal{L} = \int d^4\theta \, K(\Phi, \bar{\Phi}) + \left[\int d^2\theta \, W(\Phi) + \text{h.c.} \right] \tag{3.8}$$

where K and W respectively are the Kähler potential which control the kinetic terms and the superpotential which is the analogue of the potential in usual field theories. Chiral fields satisfy the constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$, *i.e.* they are functions of the sole $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and θ_α and therefore can be expanded as :

$$\begin{aligned}\Phi(x, \theta, \bar{\theta}) = \Phi(y, \theta) &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \\ &= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2 F(x)\end{aligned} \tag{3.9}$$

The Lagrangian density is easily computed by replacing the $F(x)$'s by their algebraic equation of motion and reads :

$$\begin{aligned}\mathcal{L} = &- K_{i\bar{j}}\partial_\mu\phi^i\partial^\mu\bar{\phi}^j - iK_{i\bar{j}}\bar{\psi}^j\bar{\sigma}^\mu D_\mu\psi^i + \frac{1}{4}R_{i\bar{j}m\bar{n}}\psi^i\psi^m\bar{\psi}^j\bar{\psi}^n \\ &- \frac{1}{2}\nabla_i W_j\psi^i\psi^j - \frac{1}{2}\nabla_{\bar{i}}W_{\bar{j}}\bar{\psi}^i\bar{\psi}^j - V_S\end{aligned} \tag{3.10}$$

where the scalar potential V_S is given by :

$$V_S = K^{i\bar{j}}W_iW_{\bar{j}} = K_{i\bar{j}}F^i\bar{F}^j. \tag{3.11}$$

Since the Lagrangian density is constructed out of Superfields, it is automatically SUSY-invariant. However it will prove useful in the following to know the transformation laws of the fields $\phi(x)$, $\psi(x)$ and $F(x)$ under a SUSY transformation of parameter ϵ :

$$\begin{array}{lcll}\delta_\epsilon\phi &=& \sqrt{2}\epsilon\psi & \rightarrow\ 0 \\ \delta_\epsilon\psi &=& i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\phi + \sqrt{2}\epsilon F & \rightarrow\ \sqrt{2}\epsilon F \\ \delta_\epsilon F &=& i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi & \rightarrow\ 0\end{array} \tag{3.12}$$

where in the last column we have indicated the vacuum expectation value of the SUSY-variation. From the last equation we read that a spontaneously broken Supersymmetry manifests itself by a non-vanishing expectation value of $F(x)$, leading to a non-zero value of the vacuum energy as may be noticed by examination of (3.11).

3.2.2 Gauge Models

Let us now continue by reviewing the gauge-invariant non-linear sigma model describing both chiral and vector Superfields [60, 61]. The most general two-derivative gauge-invariant Lagrangian density is entirely specified by three functions : the Kähler potential K, the superpotential W and the gauge kinetic function H_{ab} :

$$\mathcal{L} = \int d^4\theta\, K(\Phi, \bar{\Phi}, V) + \left[\int d^2\theta \left(W(\Phi) + \frac{1}{16g^2}H_{ab}(\Phi)W^aW^b\right) + \text{h.c.}\right] \tag{3.13}$$

where W^a is the supersymmetric field-strength. The chiral Superfield Φ and vector Superfield V transform under the action of the gauge group as :

$$\delta\Phi^i = g\Lambda^a X^i_a(\Phi) \qquad \delta V = -\frac{i}{2}\mathcal{L}_{-gV}\left[(\Lambda + \bar{\Lambda}) + \coth(\mathcal{L}_{-gV})(\Lambda - \bar{\Lambda})\right] + \mathcal{O}(\Lambda^2) \tag{3.14}$$

where $\mathcal{L}_{-gV}$ denotes the Lie derivative along $-gV$ [54, 62]. Decomposing V and Λ on the gauge group generators which satisfy $[T_a, T_b] = if^c{}_{ab}T_c$ yields :

$$\delta V^a = -\frac{i}{2}(\Lambda - \bar{\Lambda})^a + \frac{g}{2}f^a{}_{bc}(\Lambda + \bar{\Lambda})^bV^c + \mathcal{O}(\Lambda^2, V^2). \tag{3.15}$$

Under such a transformation, the field-strength transforms in the adjoint : $\delta W^a_\alpha = gf^a{}_{bc}\Lambda^bW^c_\alpha$.

The Lagrangian density invariance requires δK to be at most a Kähler transformation and both δW and $\delta(H_{ab}W^aW^b)$ to vanish. These three conditions respectively imply :

$$\begin{aligned} &K_iX_a^i - \frac{i}{2g}K_a + \frac{1}{2}K_bf^b{}_{ac}V^c = f_a + \mathcal{O}(\Lambda^2, V^2), \\ &W_iX_a^i = 0, \\ &H_{abi}X_c^i = -2f^d{}_{c(b}H_{a)d}. \end{aligned} \tag{3.16}$$

By taking two successive derivatives of the first equation and setting V to zero on recovers the Killing equation :

$$\nabla_iX_{a\bar{\jmath}} + \nabla_{\bar{\jmath}}X_{ai} = 0. \tag{3.17}$$

Finally by taking the derivative of the imaginary part of the first equation and setting V to zero, one easily finds :

$$K_{ab} = 4g^2K_{i\bar{\jmath}}X^i_{(a}\bar{X}^{\bar{\jmath}}_{b)}. \tag{3.18}$$

For isometries characterised by a vanishing Kähler transformation, the Kähler potential in the Wess-Zumino gauge assumes the following form :

$$K(\Phi,\bar{\Phi},V) = K(\Phi,\bar{\Phi}) - 2igK_iX_a^iV^a + 2g^2K_{i\bar{\jmath}}X_a^i\bar{X}_b^{\bar{\jmath}}V^aV^b. \tag{3.19}$$

In the case of a linearly realised symmetry, *i.e.* the Killing fields are given by $X_a^i = -i(T^a)^i{}_j\Phi^j$, and starting with $K(\Phi,\bar{\Phi}) = \bar{\Phi}^i\Phi^i$, one easily gets :

$$K(\Phi,\bar{\Phi},V) = \bar{\Phi}\left(1 - 2gV + 2g^2V^2\right)\Phi = \bar{\Phi}\,e^{-2gV}\,\Phi. \tag{3.20}$$

The equation (3.19) thus consists of a recipe to promote global symmetries to local ones and moreover allows for a more direct computation of the Lagrangian density. In the case of a trivial gauge-kinetic function, which is the case in the supersymmetrisation of the Standard Model, by first replacing the auxiliary fields by their algebraic equation of motion :

$$F^i = -K^{i\bar{\jmath}}W_{\bar{\jmath}} + \frac{1}{2}\Gamma^i_{jk}\psi^j\psi^k \qquad \text{and} \qquad D^a = -\frac{1}{2}K_a \tag{3.21}$$

one finds the following Lagrangian density :

$$\mathcal{L} = -K_{i\bar{\jmath}}D_\mu\phi^iD^\mu\bar{\phi}^{\bar{\jmath}} - iK_{i\bar{\jmath}}\psi^i\not{D}\bar{\psi}^{\bar{\jmath}} - i\lambda^a\not{D}\bar{\lambda}^a - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - V_S - V_F \tag{3.22}$$

where :

$$\begin{aligned} V_S &= K^{i\bar{\jmath}}W_iW_{\bar{\jmath}} + \frac{1}{8}K_aK_a, \\ V_F &= \frac{1}{2}\left(\nabla_iW_j\psi^i\psi^j + \text{h.c.}\right) - \frac{1}{4}R_{i\bar{\jmath}m\bar{n}}\psi^i\psi^m\bar{\psi}^{\bar{\jmath}}\bar{\psi}^{\bar{m}} - \sqrt{2}gK_{i\bar{\jmath}}(X_a^i\bar{\psi}^{\bar{\jmath}}\bar{\lambda}^a + \text{h.c.}) \end{aligned} \tag{3.23}$$

and where the covariant derivative acts as $D_\mu\phi^i = \partial_\mu\phi^i - gA^a_\mu X^i_a$ leading to a mass term for A^a_μ in the case of broken gauge symmetry. One can easily identify the origin of the different terms of (3.22) and (3.23) from the expression (3.19) : the first term of (3.19) together with the superpotential and the HWW term generate the standard kinetic terms for the scalars, fermions, gauge bosons and gauginos, the scalar potential V_S and the two first terms of V_F. The second and third terms of (3.19) covariantise all derivatives and generate a fermion-gaugino mixing in V_F.

The SUSY variations are given by :

$$\begin{array}{lclcl}
\delta_\epsilon\phi & = & \sqrt{2}\epsilon\psi & \to & 0 \\
\delta_\epsilon\psi & = & i\sqrt{2}\sigma^\mu\bar{\epsilon}D_\mu\phi + \sqrt{2}\epsilon F & \to & \sqrt{2}\epsilon F \\
\delta_\epsilon F & = & i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu D_\mu\psi - 2gX_a\bar{\epsilon}\bar{\lambda}^a & \to & 0 \\
\delta_\epsilon A_\mu^a & = & i\bar{\epsilon}\bar{\sigma}_\mu\lambda^a - i\bar{\lambda}^a\bar{\sigma}_\mu\epsilon & \to & 0 \\
\delta_\epsilon\lambda^a & = & \sigma^{\mu\nu}\epsilon F_{\mu\nu}^a + i\epsilon D^a & \to & i\epsilon D^a \\
\delta_\epsilon D^a & = & -\epsilon\sigma^\mu D_\mu\bar{\lambda}^a - D_\mu\lambda^a\sigma^\mu\bar{\epsilon} & \to & 0.
\end{array} \tag{3.24}$$

From the last equation we read that a spontaneously broken SUSY gauge theory manifests itself either by $\langle F^i\rangle \neq 0$ or $\langle D^a\rangle \neq 0$ which are respectively referred to as F*-breaking* and D*-breaking*. In other words, a theory is supersymmetric *if and only if* the vacuum energy vanishes.

This statement could have been derived directly from the SUSY algebra. Indeed, the energy of a particle, *i.e.* P^0, is found by taking the trace of the $\{Q, \bar{Q}\}$ anticommutator :

$$P^0 = \frac{1}{4}\mathrm{Tr}\left(\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}\right) = \frac{1}{4}\sum_\alpha \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \tag{3.25}$$

which is the sum of positive definite operators, in particular leading the vacuum energy to be non-negative :

$$E_\Omega = \langle\Omega|H|\Omega\rangle \geq 0. \tag{3.26}$$

The vacuum energy is non-vanishing *if and only if* the supercharges fail to annihilate $|\Omega\rangle$, *i.e.* when SUSY is spontaneously broken.

3.3 Supersymmetry as a Solution to the Hierarchy Problem

As argued in the previous Chapter, Supersymmetry happens to be an appealing solution to the Hierarchy Problem since it extends the chiral symmetry protecting the fermion masses from large UV contributions to scalar fields. Indeed, let us consider to following simple case of a trivial Kähler potential $K = \bar{\Phi}\Phi$ provided with the superpotential $W = \frac{1}{3}g\Phi^3$. The interaction among two fermions and one boson is given by the first two terms of the second line of (3.10) : $g\phi\psi\psi$ while the scalar self-interaction is found in the scalar potential (3.11) : $g^2\phi^4$. Supersymmetry thus realises the announced conspiracy : $g_B = g_F^2$.

In order to enforce Supersymmetry to the Standard Model, one first has to recast all its fields in either chiral or vector Superfields. Chiral Superfields contain a Weyl fermion and

Superfield	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$\bar{l}_R$	$\mathbf{1}$	$\mathbf{1}$	1
$L_L = \begin{pmatrix} \nu_L \\ l_L \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
$\bar{u}_R$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$-2/3$
$\bar{d}_R$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$1/3$
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\mathbf{3}$	$\mathbf{2}$	$1/6$

Table 3.1: MSSM chiral matter Superfields

a complex scalar field in which the SM matter fields and the Higgs fields will be embedded. Vector Superfields contain a spin-one vector boson and a Weyl fermion, both transforming in the adjoint of the gauge group. The SM gauge fields are thus to be embedded in such representations.

As will be explained in the next section, the minimal supersymmetric version of the Standard Model (MSSM) spectrum consists of a Superfield associated with each SM particle. No two SM particles are to be found in the same Superfield since Supersymmetry commutes with the gauge symmetry thereby forces the two fields of a multiplet to share the same quantum numbers [57]. By convention all Superfields are named after the SM particle they contain. The leptons and quarks spin-zero partners are respectively called sleptons and squarks and denoted by the same symbol as their SM partner with a tilde, *e.g.* $\tilde{e}_L^-$ is the left-handed electron partner. The Higgs field also defines a chiral Superfield, its fermionic partner being called the Higgsino. The structure of SUSY-invariant theories is such that one is forced to introduce a second Higgs Superfield in order to generate masses for both the up-type and down-type quarks. Finally the SM vector bosons fit in vector Superfields together with their partners, the gauginos.

The final step towards a supersymmetric realisation of the Standard Model is to specify both the Kähler potential and the superpotential, which are of course to be compatible with the gauge group.

3.4 The MSSM

A convenient way to label the matter chiral fields is found in Table 3.1 which slightly differs from the notation we adopted when discussing the SM in order to accommodate with the restriction of holomorphicity of the superpotential.

Since the Higgs field enters the SM Lagrangian both in the form H and $\tilde{H} = i\sigma^2 H^*$, a single Higgs Superfield will not be able to generate masses for both the up-type and down-type quarks since the superpotential has to be holomorphic. Two Higgs Superfields having the same gauge quantum numbers as H and $\tilde{H}$ are thus introduced : H_u in the $(\mathbf{1}, \mathbf{2}, {}^1\!/_2)$ and H_d in the $(\mathbf{1}, \mathbf{2}, -{}^1\!/_2)$. It should be noted that H_d and L_L share the same gauge quantum numbers.

In order to determine the MSSM Lagrangian, one has to specify the Kähler potential K, the superpotential W and the gauge kinetic function H. The requirement of renormalisibility constrains the Kähler potential to be quadratic, the superpotential to be at most cubic and the gauge kinetic function to be trivial. The MSSM superpotential is then given by :

$$W_{\text{MSSM}} = \lambda^L_{ij} \bar{l}^i_R H_d L^j_L + \lambda^U_{ij} \bar{u}^i_R H_u Q^j_L + \lambda^D_{ij} \bar{d}^i_R H_d Q^j_L + \mu H_u H_d. \tag{3.27}$$

As already noticed, H_d and L_L share the same transformation properties under the gauge group. One can thus generate the following gauge-invariant terms :

$$W_L = \alpha_{ijk} \bar{l}^i_R L^j_L L^k_L + \beta_{ijk} \bar{d}^i_R L^j_L Q^k_L + \gamma_i H_u L^i_L. \tag{3.28}$$

Moreover, one further gauge-invariant contribution to the superpotential should be added. Indeed, since the mass-dimension of a chiral Superfield is one, the following term is power-counting renormalisable :

$$W_B = \zeta_{ijk} \bar{u}^i_R \bar{d}^j_R \bar{d}^k_R. \tag{3.29}$$

If, inspired by the discussion of subsection 2.1.4, one wants the superpotential to be given by the sole W_{MSSM}, one then has to impose a further $\mathbb{Z}_2$ symmetry known as matter-parity defined as :

$$P_M = (-1)^{3(B-L)} \tag{3.30}$$

under which $P_M(H_u) = P_M(H_d) = 1$ while all Superfields recorded in Table 3.1 have $P_M = -1$, leading to :

$$P_M(W_{\text{MSSM}}) = +1 \qquad P_M(W_L) = P_M(W_B) = -1 \tag{3.31}$$

and thus effectively forbidding the appearance of both W_L and W_B which violate respectively the lepton and baryon numbers. Note that in the literature it is often made usage of R-parity instead of matter-parity. These are related through :

$$P_R = (-1)^{2s} P_M \tag{3.32}$$

where s is the spin of the particle. Particles in the same multiplet thus carry different R-parities : all Standard Model particles and the Higgs bosons carry a positive R-parity whilst the squarks, sleptons, Higgsinos and gauginos have a negative charge under R-parity.

As was the case in the construction of the Standard Model Lagrangian, one always can make a field redefinition in order to bring the Kähler potential in a diagonal form in its flavour indices.

3.5 Softly Broken Supersymmetry

If Supersymmetry were a symmetry of Nature, then she would have chosen a SUSY-breaking vacuum. Indeed a feature of SUSY-invariant theories is that they force the fields appearing in the same Superfield to share the same mass. Experiments however have neither detected any scalar particle with the same mass as the SM leptons and quarks nor the massless partners of the gluons and of the photon. Supersymmetry thus has to be broken.

Indeed once SUSY is broken, a mass-splitting is generated and situations where all the superpartners are above the detection threshold may be engineered. In order not to spoil the goal for which SUSY was introduced, the extension of chiral symmetry to scalars, Supersymmetry has to be *softly* broken. Supersymmetry breaking is said to be soft provided the SUSY-breaking terms appearing in the Lagrangian have parameters of positive mass dimension. Such a requirement ensures that the Higgs mass does not suffer from any quadratic divergences even when Supersymmetry is broken. This will be shown to be satisfied by spontaneously broken SUSY.

The soft SUSY-breaking terms compatible with both the gauge group and matter-parity consist of several parts as shown in [63] by a very nice spurion analysis. In the gauge sector, gaugino masses for each gauge group have to be introduced. In the scalar sector, linear, bilinear and trilinear terms compatible with the gauge symmetry appear. Applied to the MSSM, this procedure yields :

$$\begin{aligned}\mathcal{L}_{\text{soft}} = {} & M_C \tilde{G}^a \tilde{G}^a + M_L \tilde{W}^a \tilde{W}^a + M_Y \tilde{B}\tilde{B} \\ & + m^2_{Qij} \tilde{Q}^{\dagger i}_L \tilde{Q}^j_L + m^2_{\bar{u}ij} \tilde{\bar{u}}^{\dagger i}_R \tilde{\bar{u}}^j_R + m^2_{\bar{d}ij} \tilde{\bar{d}}^{\dagger i}_R \tilde{\bar{d}}^j_R + m^2_{Lij} \tilde{L}^{\dagger i}_L \tilde{L}^j_L + m^2_{\bar{l}ij} \tilde{\bar{l}}^{\dagger i}_R \tilde{\bar{l}}^j_R \\ & + m^2_u H^\dagger_u H_u + m^2_d H^\dagger_d H_d + B_\mu H_u H_d \\ & + A^U_{ij} \tilde{\bar{u}}^i_R H_u \tilde{Q}^j_L + A^D_{ij} \tilde{\bar{d}}^i_R H_d \tilde{Q}^j_L + A^L_{ij} \tilde{\bar{l}}^i_R H_d \tilde{L}^j_L .\end{aligned} \tag{3.33}$$

Since most of the contributions to $\mathcal{L}_{\text{soft}}$ have so far generic flavour structure, soft Supersymmetry breaking leads to many flavour-violating processes which were either absent or very tightly constrained in the Standard Model. In order to be compatible with experimental flavour searches, such as $\mu \to e + \gamma$ and $K_0 - \bar{K}_0$ oscillations, which are compatible with the SM flavour structure, the softly broken MSSM has to obey severe constraints. The departure from universality should be small for all sleptons and squarks masses, the A-terms should be dominantly proportional to the corresponding Yukawa couplings and the CP-violating phases should be small [57].

These further requirements call for a mechanism to enforce them. Indeed there are no reasons they should be satisfied within the softly broken MSSM : it indeed has $\mathcal{O}(100)$ parameters which spoil the nice accidental flavour structure of the Standard Model [64]. In order to tackle this issue, one may hope that explicit models of spontaneous Supersymmetry-breaking will induce relations among the parameters of (3.33) and thus render the MSSM compatible with experimental data. Let us now argue that Supersymmetry breaking is forced to happen in a distinct sector.

3.6 Evading the Supertrace Formula

In this section we will review a sum rule, known as the Supertrace formula, which is valid both for the unbroken and the spontaneously broken phases of supersymmetric theories. The Supertrace is defined as a weighted sum over spin-j contributions :

$$\mathrm{STr}(m^2) = \sum_j (-1)^{2j}(2j+1)\mathrm{Tr}(m_j^2). \tag{3.34}$$

We are now in a position to compute the masses for the scalars, fermions and vectors. The third term of (3.19) is a mass term for the gauge fields A_μ^a : $m_{ab}^2 = 2g^2 K_{i\bar{j}} X_a^i \bar{X}_b^{\bar{j}}$. The fermion mass matrix in the (ψ^i, λ^a) basis is given by :

$$m_F = \begin{pmatrix} \nabla_i W_j & \sqrt{2} g K_{i\bar{n}} \bar{X}_b^{\bar{n}} \\ \sqrt{2} g K_{j\bar{n}} \bar{X}_a^{\bar{n}} & 0 \end{pmatrix} \to m_F^\dagger m_F = \begin{pmatrix} \nabla_i W_k \nabla^j W^k + 2g^2 \bar{X}_{ci} X_c^j & \sqrt{2} g \nabla_{\bar{i}} W_{\bar{j}} \bar{X}_a^{\bar{j}} \\ \sqrt{2} g \nabla_i W_j X_a^j & 2g^2 \bar{X}_{ai} X_b^i \end{pmatrix}. \tag{3.35}$$

Let us finally compute the scalar masses by taking two successive derivatives of V_S :

$$\begin{aligned} m_{i\bar{j}}^2 &= \nabla_i W_k \nabla_{\bar{j}} W^k - R_{i\bar{j}m\bar{n}} W^m W^{\bar{n}} + g^2 \bar{X}_{ai} X_{a\bar{j}} - \frac{1}{2} i g K_a \nabla_i X_{a\bar{j}}, \\ m_{ij}^2 &= W^k \nabla_i \nabla_j W_k - g^2 \bar{X}_{ai} \bar{X}_{aj} - \Gamma_{ij}^k \partial_k V_S. \end{aligned} \tag{3.36}$$

The Supertrace is thus given by :

$$\mathrm{STr}(m^2) = -2R_{i\bar{j}} W^i W^{\bar{j}} - i g K_a \nabla_i X_a^i. \tag{3.37}$$

In the case where the gauge-kinetic function is kept unspecified, the computation is slightly more involved [65], the net result being that the RHS of (3.37) is sourced by terms involving the gauge-kinetic function derivatives.

Application to the MSSM Let us apply the Supertrace formula to the case of the MSSM. Since the gauge symmetries are linearly realised, *i.e.* $X_a^i = -i(T^a)^i{}_j \Phi^j$, and the Kähler manifold is flat, *i.e.* $K_{i\bar{j}} = \delta_{ij}$, one finds [66] :

$$\mathrm{STr}(m_{\mathrm{MSSM}}^2) = 2g\mathrm{Tr}(T^a) D^a. \tag{3.38}$$

The non-Abelian groups $SU(3)_C$ and $SU(2)_L$ generators all have vanishing traces, the only remaining concern is about the $U(1)_Y$ generator trace. By going back to (3.1), one may check that the trace vanishes individually for leptons and quarks. We have thus obtained that, in the case of the MSSM, the Supertrace vanishes both when SUSY is unbroken and when it is spontaneously broken :

$$\mathrm{STr}(m_{\mathrm{MSSM}}^2) = 0. \tag{3.39}$$

Note that this relation holds separately for all conserved quantum numbers since mass insertions cannot relate particles having different gauge transformation properties and that this result is actually valid for any renormalisable supersymmetric theory whose gauge group is free from gravitational anomalies. The Supertrace formula thus puts very stringent constraints on the way SUSY is to be broken in renormalisable models such as the MSSM. Indeed, let us in turn consider F- and D-breaking :

- In the situation in which $\langle F^i \rangle \neq 0$ and $\langle D^a \rangle = 0$, the only term in the scalar mass matrix due to SUSY-breaking is the first one on the second line of (3.36). Since it is an off-diagonal entry in the scalar mass matrix, the scalar masses will be shifted proportionally to this term leading to a situation where, in order to satisfy the Supertrace constraint, the fermion keeps its supersymmetric mass m while the two scalars masses are shifted around it by an equal and opposite amount : $m \pm \Delta$, *i.e.* they are subject to level-repulsion. F-term SUSY-breaking is thus phenomenologically not viable since it predicts one sfermion mass to be smaller than the known lepton and quark masses. Such particles have experimentally been ruled out.

- In the opposite situation, where $\langle F^i \rangle = 0$ and $\langle D^a \rangle \neq 0$, the only term originating from SUSY-breaking in the scalar masses is the last one of the first line of (3.36). One may hope that such a term could lift the scalar masses, but the MSSM charge assignment leads to both positive and negative shifts of the scalar masses and thus to an unacceptable spectrum.

3.7 The Hidden Sector Paradigm

According to above mentioned criteria neither F-type nor D-type SUSY-breaking can occur inside the MSSM since they would lead to an unacceptable spectrum. Supersymmetry-breaking is therefore assumed to occur in another sector, the hidden sector, by an unspecified mechanism and mediated to the MSSM Superfields, in the visible sector, by non-renormalisable effective interactions. When the hidden sector is integrated-out, the effective theory Supertrace should be non-vanishing. Moreover since SUSY is assumed to be mediated by suppressed interactions, it has to be broken at scales well above the EW scale.

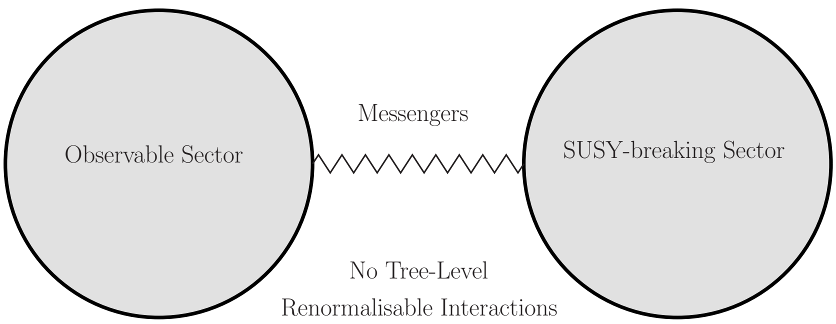


Figure 3.1: SUSY-breaking mediated *via* Messengers Fields

The breaking of SUSY in a disjoint sector and its mediation to the observable sector by messengers may be compared to the situation of EWSB in which the EW symmetry is broken in the Higgs sector and then mediated to the observable sector via the Yukawa couplings.

The necessity of introducing a hidden sector responsible for SUSY-breaking represents an opportunity to tackle the supersymmetric flavour problem. If the interaction mediating SUSY-breaking is flavour-blind, the soft terms introduced in (3.33) will tend to be universal and will thus not spoil the flavour structure of the P_M-invariant MSSM.

The precise mechanism of SUSY-breaking in the hidden sector is an open issue and may be quite complicated. We will thus parametrise the SUSY-breaking by assuming that a chiral Superfield's auxiliary field obtains a vacuum expectation value $\langle F \rangle$. The order of magnitude of the soft terms will then roughly be :

$$m_{\rm soft} \sim \frac{\langle F \rangle}{M} \tag{3.40}$$

where M is the scale suppressing the effective interactions mediating SUSY-breaking from the hidden sector to the visible one.

The structure of soft terms will thus depend on the mediating interactions and not only on the precise way Supersymmetry is broken in the hidden sector. The Supertrace constraint can be traced back to the renormalisibility of the theory, in particular to the fact that the kinetic terms have a minimal structure. When the hidden sector and the messengers are integrated-out, the effective theory is a non-linear sigma-model characterised by a non-trivial metric in front of the kinetic terms which will induce gaugino and scalar soft masses.

Two flavour-blind candidates generating a non-renormalisable effective theory naturally emerge : gauge interactions and gravity. Let us now roughly describe both of these possibilities.

3.7.1 Gauge Mediation

In gauge-mediated SUSY-breaking [67], one introduces a set of chiral messenger Superfields Φ, $\tilde{\Phi}$ charged under the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group and coupled to the source of SUSY-breaking, parametrised by a gauge-singlet S, in a renormalisable way :

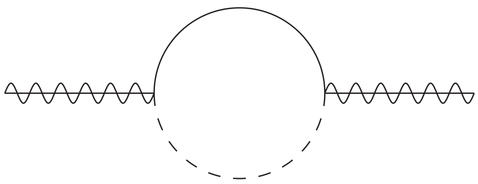
$$W_{\rm GM} = \alpha S \Phi \tilde{\Phi} \tag{3.41}$$

where $\tilde{\Phi}$'s quantum numbers are conjugated with respect those of Φ. Since the microscopic Lagrangian is renormalisable, the Supertrace does vanish at tree-level. However, at the quantum level, the effective Lagrangian describing the observable sector will have non-renormalisable kinetic terms induced by gauge interactions and thus a non-vanishing Supertrace. The renormalisation of the Superspace wave-function leads to the appearance of soft terms [68]. Let us briefly sketch how this mechanism may be realised.

When S breaks SUSY by, say, an O'Raifeartaigh mechanism both its scalar and auxiliary components are assumed to get a VEV. One may then replace αS by $M_S + \theta^2 F_S$ in the superpotential (3.41), leading to both fermion and scalar masses for the messengers. After having integrated out the auxiliary fields of the messengers fields, the potential for the messenger's scalar and fermion fields contains :

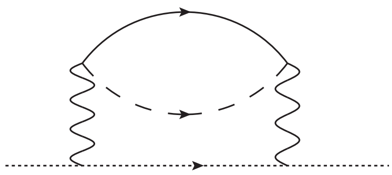
$$V \ni M_S \psi \tilde{\psi} - F_S \phi \tilde{\phi} + M_S^2 \phi^\dagger \phi + M_S^2 \tilde{\phi}^\dagger \tilde{\phi}. \tag{3.42}$$

The fermions thus get a supersymmetric mass term $m_F^2 = M_S^2$ and the bosons masses are shifted around m_F by an equal and opposite quantity : $m_B^2 = m_F^2 \pm F_S$. Supersymmetry-breaking has thus been transferred from the S singlet to the messengers. Since the messengers are charged under the MSSM gauge group, Supersymmetry-breaking will further be communicated to the MSSM Superfields at loop level. At the one-loop level gaugino masses are generated, as illustrated by Figure 3.2, while flavour-blind scalar masses are generated at the two-loop level as shown by Figure 3.3. Note that there are many more diagrams contributing to both these masses, see [57] for the complete set.



$$\rightarrow \qquad M_a \sim \frac{g_a^2}{16\pi^2} \frac{F_S}{M_S}$$

Figure 3.2: Soft gaugino mass term



$$\rightarrow \qquad m^2 \sim \left(\frac{g_a^2}{16\pi^2}\right)^2 \left(\frac{F_S}{M_S}\right)^2$$

Figure 3.3: Soft scalar mass term

Both effects lead to the following qualitative order of magnitude for soft terms :

$$m_{\text{soft}} \sim \frac{g^2}{16\pi^2} \frac{F_S}{M_S}. \tag{3.43}$$

Since the A-terms are also generated at the two-loop level, they give suppressed effects compared to the other soft masses and may roughly be neglected. Gauge-mediation of SUSY-breaking proves to be very attractive since the squark and slepton masses only depend on their gauge quantum numbers, automatically leading to the suppression of FCNC. A very complete review of gauge mediation is the one of Giudice and Rattazzi [69].

3.7.2 Gravity Mediation

To the contrary of gauge mediation for which the microscopic theory is renormalisable and thus has a vanishing Supertrace, gravity mediation [70–72] occurs in a non-renormalisable theory having a non-vanishing Supertrace. The effective low-energy theory leads to the appearance of soft terms.

Let us again parametrise the SUSY-breaking hidden sector by a chiral gauge-singlet Superfield X whose F-component F^X gets a vacuum expectation value. The interactions between X and the visible sector are schematically given by :

$$\begin{aligned}\mathcal{L}_{\text{soft}} = &\sum_{\text{gauge}} \int d^2\theta \frac{\alpha}{M_{\text{Pl}}} X W^a W^a + \sum_{\text{Yukawa}} \int d^2\theta \frac{\zeta_{ij}}{M_{\text{Pl}}} X \bar{\Phi}^i_R H \Phi^j_L \\ &+ \sum_{\text{matter}} \int d^4\theta \frac{\beta_{ij}}{M^2_{\text{Pl}}} X^\dagger X \Phi^{i\dagger} \Phi^j + \int d^2\theta \frac{\gamma}{M^2_{\text{Pl}}} X^\dagger X H_u H_d \\ &+ \int d^4\theta \frac{\kappa}{M_{\text{Pl}}} X^\dagger H_u H_d\end{aligned} \tag{3.44}$$

which lead to :

$$m_{\text{soft}} \sim \frac{F^X}{M_{\text{Pl}}} \qquad i.e. \qquad \sqrt{F^X} \sim 10^{11}\text{GeV} \tag{3.45}$$

if taking the soft parameters to be of the 1 TeV order. The detailed structure of the soft terms arising in gravity mediation has been worked out in [73–75]. In contradistinction to gauge mediation, gravity mediation does not constrain the above parameters to yield universal soft scalar masses and thus generically generates FCNC despite the fact that gravity is flavour-blind in the IR. Indeed the term controlled by β_{ij} leads to soft scalar masses when the X auxiliary field F^X takes its vacuum expectation value :

$$\int d^4\theta \frac{\beta_{ij}}{M^2_{\text{Pl}}} X^\dagger X \Phi^{i\dagger} \Phi^j = \beta_{ij} \frac{|F^X|^2}{M^2_{\text{Pl}}} \phi^{\dagger i} \phi^j \quad \rightarrow \quad m^2_{i\bar{\jmath}} = -\beta_{ji} \frac{|F^X|^2}{M^2_{\text{Pl}}} \tag{3.46}$$

where β_{ij} and all other parameters entering (3.44) are determined by the details of the UV theory.

The relevant term for our purpose, *i.e.* computing the soft scalar masses, is thus a dimension-6 operator whose structure consists of two chiral Superfields belonging to the observable sector together with two chiral Superfields of the hidden sector. Let us investigate a slightly more general form of interaction :

$$\int d^4\theta Z^{ij}(X, \bar{X}) \Phi^i \Phi^{j\dagger} \tag{3.47}$$

where the Superspace wave-function Z^{ij} may depend on several hidden Superfields X^α and on their conjugates. Note that the indices on the wave function may be interpreted as derivatives with respect to Φ^i and $\Phi^{j\dagger}$ of a Kähler potential $Z(X, \bar{X}, \Phi, \bar{\Phi})$, (3.47) being the second term in the Taylor series. When replacing the Φ's auxiliary fields by the solution to their algebraic equation of motion, one easily gets the following expression for the soft masses :

$$m^2_{i\bar{j}} = -\left[Z^{ij}_{\alpha\bar{\beta}} - (Z^{-1})^{mn} Z^{im}_{\alpha} Z^{nj}_{\bar{\beta}}\right] F^{\alpha}\bar{F}^{\beta} \tag{3.48}$$

where the first term is an effect of the D-term of Z^{ij} while the second captures F-term effects. In the case at hand, the relevant part of Z^{ij} is given by :

$$Z^{ij} = \frac{\beta_{ji}}{M^2_{\rm Pl}} X^{\dagger} X. \tag{3.49}$$

Assuming that the X scalar does not take any vacuum expectation value, *i.e.* only the first term in (3.48) contributes, one indeed recovers (3.46) :

$$m^2_{i\bar{j}} = -Z^{ij}_{X\bar{X}} |F^X|^2 = -\beta_{ji} \frac{|F^X|^2}{M^2_{\rm Pl}}. \tag{3.50}$$

These masses are generically not flavour-universal and depend both on the Kähler potential of the microscopic theory through β_{ij} and on its superpotential which fixes the direction of F^X. A possible way out of this problem is to impose flavour-universality at the Planck scale, the resulting theory going under the name of mSUGRA. Of course one should then explain how such a conspiracy emerges at the Planck scale.

Chapter 4

Supergravity

In this Chapter we will introduce Supergravity (SUGRA) which is the supersymmetrisation of General Relativity describing the dynamics of the graviton and the gravitino and their coupling to matter. Several approaches are available in the literature to derive the SUGRA Lagrangian and its coupling to matter. However since our interest lies in the knowledge of the scalar potential we may discard terms involving direct couplings between the gravitational multiplet and matter fields. Such a formulation fortunately exists and allows for a very simple and direct computation of the scalar potential. Once the scalar potential has been derived, determining the scalar masses is straightforward.

We will first closely follow the procedure of [76] in order to derive the field content of superconformal SUGRA and then briefly discuss the vierbein procedure of [54] from which we will extract the relevant terms for the computation of the scalar potential. Let us however briefly review the different known approaches to SUGRA [77] :

Noether Procedure The first approach consists in defining SUGRA to be the theory obtained by extending SUSY to local transformations, *i.e.* to promote SUSY to a local symmetry. The gravitino then emerges as the gauge field of this particular Yang-Mills theory [78–80]. However the derivation is rather lengthy and not very transparent. A detailed calculation of the SUGRA Lagrangian and its coupling to matter may be found in [81].

Superspace Approach The second procedure consists in using the Superspace technology. Vierbein E_A^M are introduced for the whole of Superspace together with their associated torsions. However one has to find the right constraints to impose to those torsions in order to recover minimal SUGRA, which *a priori* is not an easy task. Moreover once constraints have been introduced, the Bianchi identities are not identities anymore and have to be solved, which again is rather an unpleasant work. This approach is extensively discussed in [82] and [54].

Superconformal Approach The third possibility to establishing the SUGRA Lagrangian is to introduce more symmetry than needed and then to gauge-fix them [83–86]. The advantage of such a procedure being that a high degree of symmetry puts severe constraints on the Lagrangian thus effectively reducing the number of independent parameters. We adopt this last strategy in the following.

4.1 Constraints versus New Symmetries

In this first section, we briefly illustrate the procedure we will use to derive Supergravity. When promoting a global symmetry to a local one, new degrees of freedom are introduced in order to covariantise the Lagrangian density [87]. These degrees of freedom are arranged in a vector representation of the Lorentz group known as the *gauge field*. However since a gauge field transforms under the gauge symmetry, not all of its components are physical. In other words, one could use the gauge symmetry to gauge away some of the components, *i.e.* to set them to zero. In the case of Electro-Dynamics (QED), the field $A^\mu_{\mathrm{T}}(x)$, which is identified with the photon, has two degrees of freedom, identified with the two transverse polarisations of the photon. However it proves useful to reintroduce a longitudinal component $A^\mu_{\mathrm{L}}(x)$ and a gauge symmetry in the context of the path integral formulation of QED. The gauge symmetry is said to be *compensated* by the longitudinal component of the photon.

Another use of compensating fields is to effectively reduce the symmetry of a theory. This exactly corresponds to the case we will meet in the context of deriving the SUGRA Lagrangian. Let us for example consider a theory invariant under general change of coordinates :

$$\delta x^\mu = -K^\mu(x) \quad \rightarrow \quad \delta g_{\mu\nu} = \nabla_\mu K_\nu + \nabla_\nu K_\mu \tag{4.1}$$

and under the following local Weyl symmetry :

$$\delta g_{\mu\nu} = -2\sigma g_{\mu\nu} \qquad \delta\phi = \sigma\phi \tag{4.2}$$

where $g_{\mu\nu}$ is a real symmetric spin-2 field, ∇_μ its compatible covariant derivative and ϕ a real scalar field. The following action is invariant under all symmetries :

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}\left(\frac{1}{6}R\phi^2 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \tag{4.3}$$

where R and g respectively are the Ricci scalar and the determinant constructed out of $g_{\mu\nu}$. Under (4.2), the Ricci scalar and $\sqrt{-g}$ can be shown to transform as :

$$\delta R = 6\Box\sigma + 2\sigma R \qquad \text{and} \qquad \delta\sqrt{-g} = -4\sigma\sqrt{-g} \tag{4.4}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. One can now use the Weyl symmetry to gauge-fix the field ϕ to $\phi_0 = \sqrt{3/4\pi G_N}$ to recover the Einstein-Hilbert action of General Relativity :

$$\mathcal{S}|_{\phi\to\phi_0} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R. \tag{4.5}$$

Inspired by this example, one may wonder if it is possible to write the SUGRA Lagrangian as a gauge-fixed superconformal theory.

4.2 Superconformal Formulation

4.2.1 What are we looking for?

The SUGRA Lagrangian is required to describe the dynamics of the multiplet containing the graviton and the gravitino. A simple counting of the number of off-shell degrees of freedom leads to the introduction of six auxiliary fields. The SUGRA Superfields should thus contain the following set of fields :

$$g_{\mu\nu}, \quad \Psi_{\alpha\mu}, \quad R_\mu \quad \text{and} \quad F_\varphi. \tag{4.6}$$

We will now see that these fields are split among a gravitational Superfield and a compensator Superfield. Inspired by equation (4.1), one may wonder which object plays the rôle of the metric when the Superspace coordinates are varied. To answer this question one first needs to introduce the notion of complex Superspace.

4.2.2 Complex Superspace

A point in Superspace as we have introduced it in Appendix C.3 is labelled by x^μ, θ_α and $\bar{\theta}_{\dot{\alpha}}$. Four of the labels are bosonic and four of them are fermionic which motivates to denote Superspace as $\mathbb{R}^{4|4}$. It is useful in the context of Supergravity to interpret Superspace as a section of $\mathbb{C}^{4|2}$. A point in $\mathbb{C}^{4|2}$ is labelled by y^μ and θ_α where both y and θ are understood to be complex *i.e.* if viewed as a point of the real Superspace $\mathbb{R}^{8|4}$ it is labelled by y^μ, $\bar{y}^\mu$, θ_α and $\bar{\theta}_{\dot{\alpha}}$.

Let us now introduce surfaces in $\mathbb{C}^{4|2}$ defined by real Superfields $H^\mu(x, \theta, \bar{\theta})$ on $\mathbb{R}^{4|4}$ where $x^\mu = {}^1\!/\!{}_2(y^\mu + \bar{y}^\mu)$:

$$y^\mu - \bar{y}^\mu = 2iH^\mu. \tag{4.7}$$

Since each set of H^μ's fixes the imaginary part of the y^μ's, the equation (4.7) defines a real Superspace which we will denote by $\mathbb{R}^{4|4}(H)$. It can easily be shown that the real Superspace constructed in the Appendix C.3 is obtained by choosing :

$$H^\mu = \theta\sigma^\mu\bar{\theta}. \tag{4.8}$$

Indeed the surfaces defined by the previous equation are stable under Super-Poincaré transformations. We thus have $\mathbb{R}^{4|4} = \mathbb{R}^{4|4}(\theta\sigma\bar{\theta})$, or in different words the $\mathbb{C}^{4|2}$ surface constrained by $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ corresponds to the real flat Superspace.

A space-time is said to be flat if there exists a coordinate system $\bar{x}^\mu$ in which the metric reduces to the Minkowski metric, *i.e.* when $g_{\mu\nu}(\bar{x}) = \eta_{\mu\nu}$. The equivalent equation in the case of Superspace defining a flat Superspace is given by :

$$H^\mu = \theta\sigma^\mu\bar{\theta}. \tag{4.9}$$

The complex Superspace is also very well suited to describe chiral Superfields. Indeed instead of having a complicated constraint depending on a combination of the coordinates $\bar{D}_{\dot{\alpha}}\Phi = 0$, chiral Superfields can be viewed as holomorphic Superfields in $\mathbb{C}^{4|2}$: $\Phi = \Phi(y, \theta)$. It is indeed trivial to show that $\bar{D}_{\dot{\alpha}}\Phi(y,\theta) = 0$ since $\bar{D}_{\dot{\alpha}}y^\mu = 0$ and $\bar{D}_{\dot{\alpha}}\theta^\alpha = 0$.

4.2.3 Superconformal Supergravity

We now make the following observation : when flat space-time coordinates are allowed to vary in an arbitrary way the metric changes according to equation (4.1) and the theory obtained from the principle of general covariance is GR. We now mimic this procedure in Superspace and allow for the coordinates to vary in an arbitrary way, thus spoiling the property (4.9), and interpret the Superfields H^μ as the dynamical object of this theory. We thus have the following correspondences :

$$\begin{array}{llcl}
\text{Flatness} & g_{\mu\nu} = \eta_{\mu\nu} & \longleftrightarrow & H^\mu = \theta\sigma^\mu\bar{\theta} \\
\text{Variations} & \delta x & \longleftrightarrow & \delta y,\ \delta\theta \\
\text{Dynamical field} & g_{\mu\nu}(x) & \longleftrightarrow & H^\mu(x,\theta,\bar{\theta}) \\
\text{Theory} & \text{GR} & \longleftrightarrow & \text{SUGRA?}
\end{array} \tag{4.10}$$

To assess if the obtained theory really is Supergravity, *i.e.* if it has the spectrum discussed in subsection 4.2.1, we again take advantage of the comparison with GR. In General Relativity, one reduces the number of independent degrees of freedom of the metric to two by choosing an appropriate shift K^μ in the space-time coordinates. This is a two-step procedure, first one may go in the Lorentz gauge and then, using the remaining gauge freedom, one may go in the TT gauge. In other words, eight of the ten components of the metric are gauge-fixed leading to the theory of the two remaining degrees of freedom identified with the two polarisations of gravitational waves.

Let us now apply this scheme to $\mathbb{C}^{4|2}$. The coordinates are allowed to vary in an arbitrary fashion :

$$y^\mu \to y^\mu - k^\mu(y,\theta) \qquad \text{and} \qquad \theta_\alpha \to \theta_\alpha - k_\alpha(y,\theta). \tag{4.11}$$

In order to be an allowed change of variables, the k^μ and k_α should be such that the Berezinian, also known as the Superdeterminant, of the transformation is non-vanishing. Under (4.11), x^μ

and H^μ which respectively are the real and imaginary parts of y^μ transform as :

$$x^\mu \to x^\mu - \frac{1}{2}\left(k^\mu + \bar{k}^\mu\right) \qquad \text{and} \qquad H^\mu \to H^\mu + \frac{i}{2}\left(k^\mu - \bar{k}^\mu\right). \tag{4.12}$$

The variation of H^μ is thus given by :

$$\delta H^\mu = H'^\mu - H^\mu = \frac{i}{2}\left(k^\mu - \bar{k}^\mu\right) + \left[\frac{1}{2}\left(k^\nu + \bar{k}^\nu\right)\partial_\nu + k^\alpha\partial_\alpha + \bar{k}^{\dot{\alpha}}\partial_{\dot{\alpha}}\right] H^\mu. \tag{4.13}$$

It is a straightforward exercise to verify that $\mathbb{R}^{4|4}(\theta\sigma\bar{\theta})$ is left invariant under Super-Poincaré transformations. As an example, let us illustrate this in the case of a SUSY variation which is generated by choosing $k^\mu = 2i\theta\sigma^\mu\bar{\xi}$ and $k^\alpha = \xi^\alpha$. Such parameters do indeed not generate any change in $H^\mu = \theta\sigma^\mu\bar{\theta}$:

$$\delta H^\mu = \frac{i}{2}\left(2i\theta\sigma^\mu\bar{\xi} + 2i\xi\sigma^\mu\bar{\theta}\right) + \xi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}}(-\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha) = 0. \tag{4.14}$$

Since H^μ is a Superfield, one can expand it in an exact Taylor series in order to investigate its content. Then using (4.12) one may gauge-fix some of its components to zero, in analogy with the Lorentz gauge in GR or with the Wess-Zumino gauge in SUSY gauge theories. By decomposing the variation k^μ and k^α as exact Taylor series in θ, one may choose them such that H^μ takes the following form :

$$H^\mu = \theta\sigma^a\bar{\theta}e_a{}^\mu + i\bar{\theta}^2\theta\Psi^\mu - i\theta^2\bar{\theta}\bar{\Psi}^\mu + \theta^2\bar{\theta}^2 R^\mu. \tag{4.15}$$

The field content of the theory is thus given by the vierbein $e_a{}^\mu$, the gravitino Ψ^μ_α and a gauge field R^μ. However when choosing the gauge, *i.e.* k^μ and k^α, to bring H^μ in the form (4.15), a certain gauge freedom is leftover which is the analogue of the residual gauge symmetry one finds when going in the Wess-Zumino gauge in the context of SUSY gauge theories. By properly choosing field-dependent parameters k^μ and k^α, one can generate three more transformations which are a Weyl transformation, a chiral transformation and a second Supersymmetry transformation which do not spoil the gauge (4.15).

We have thus established that the theory generated from the analogy (4.10) is a conformal Supergravity when the gauge parameters k^μ and k_α are arbitrary. As noticed in [88], it turns out that if one restricts them to obey an unimodular restriction :

$$\partial_\mu k^\mu = \partial_\alpha k^\alpha \qquad \leftrightarrow \qquad \mathrm{Ber}(\delta y, \delta\theta) = 1 \tag{4.16}$$

then the obtained theory is Supergravity. Indeed the constraint (4.16) puts a restriction on the gauge-fixing and as a result one can no longer put H^μ in the form (4.15). Indeed, taking into account (4.16), one may choose the parameters k^μ and k_α such that H^μ takes the following form :

$$H^\mu = \theta^2 B^\mu + \bar{\theta}^2\bar{B}^\mu + \theta\sigma^a\bar{\theta}e_a{}^\mu + i\bar{\theta}^2\theta\Psi^\mu - i\theta^2\bar{\theta}\bar{\Psi}^\mu + \theta^2\bar{\theta}^2 R^\mu. \tag{4.17}$$

However, thanks to a suitable redefinition of the fields, the B^μ field only appears in the action in the combination $F_\varphi \equiv \partial_\mu B^\mu$. The field content is thus indeed the one of SUGRA as advertised.

As argued in section 4.1, in order to accommodate theories with constraints, the most efficient path is to introduce a new degree of freedom together with a symmetry whose gauge-fixing will restrict the theory to obey the constraint. The multiplet introduced to take care of the constraint (4.16) is called the conformal compensator and will be denoted by φ. If we impose the following transformation property to the compensator under coordinate transformation on $\mathbb{C}^{4|2}$:

$$\varphi \to \varphi + \frac{1}{3}\left(\partial_\mu k^\mu - \partial_\alpha k^\alpha\right)\varphi \tag{4.18}$$

then gauge-fixing φ to one selects unimodular coordinate transformations and leads to minimal Supergravity. Let us now sketch the more general construction of [89] which will lead to the construction of a Lagrangian density describing the dynamics of the Superfields H^μ and φ and their coupling to matter. The description of Ogievetsky and Sokatchev will appear to be a particular case of the Siegel and Gates one.

In [89], Siegel and Gates determine Supergravity as the gauge theory of the Superspace translation group under which :

$$x^\mu \to x^\mu - k^\mu \qquad \theta^\alpha \to \theta^\alpha - k^\alpha \qquad \bar{\theta}^{\dot{\alpha}} \to \bar{\theta}^{\dot{\alpha}} - k^{\dot{\alpha}}. \tag{4.19}$$

This approach is more general in the sense that all the Superspace coordinates are treated on the same footing which was not the case in the procedure presented above where the flat Superspace condition (4.9) was only carrying Lorentz indices. In the Siegel and Gates procedure, three sorts of gauge fields are introduced : U_α, $U_{\dot{\alpha}}$ and U_μ. The superconformal gauge group they find only admits a subgroup which breaks the Weyl symmetry and include the SUSY transformations if there exists a complex number n satisfying the following relation :

$$(3n+1)\partial_{\dot{\alpha}} k^{\dot{\alpha}} = (n+1)\left(\partial_\mu k^\mu - \partial_\alpha k^\alpha\right). \tag{4.20}$$

For $n = -1/3$ this relation implies the same condition as the constraint (4.16) and leads to the possibility of achieving $U_\alpha = 0$, $U_{\dot{\alpha}} = 0$ and $U^\mu = H^\mu$ with H^μ given by the equation (4.17). In order to write down an action, one introduces supervierbein in close analogy with the gauge covariant derivatives of SUSY gauge theories : $\hat{E}_\alpha = e^{-2U}\partial_\alpha e^{2U}$, $\hat{E}_{\dot{\alpha}} = \partial_{\dot{\alpha}}$ and $\hat{E}_a = \frac{i}{4}\sigma^{a\alpha\dot{\beta}}\{\hat{E}_\alpha, \hat{E}_{\dot{\beta}}\}$ where $U = i(U^\mu\partial_\mu + U^\alpha\partial_\alpha + U^{\dot{\alpha}}\partial_{\dot{\alpha}})$. It then can be shown that the action :

$$\mathcal{S} = \int d^8 z \left(1 \cdot e^{-2\overleftarrow{U}}\right)^{(n+1)/2} \hat{E}^n \qquad\qquad \hat{E} = \mathrm{Ber}\left(\hat{E}_A{}^M\right) \tag{4.21}$$

where, using z to collectively denote x, θ and $\bar{\theta}$, [56] :

$$\left(1 \cdot e^{-2\overleftarrow{U}}\right) = \mathrm{Ber}\left[\partial_M (e^{-2U} z^N)\right] \tag{4.22}$$

is invariant whenever (4.20) is satisfied. In the particular case $n = -1/3$, using the compensator φ permits to recast the previous action into the following form :

$$\mathcal{S} = \int d^8z \left(1 \cdot e^{-2\overleftarrow{U}}\right)^{1/3} \hat{E}^{-1/3} \left(e^{2U}\varphi\right)^\dagger \varphi + \left(\int d^6z\, \varphi^3 + \text{c.c.}\right) \tag{4.23}$$

where no constraint has to be applied and where the second term has been added since it is allowed by the symmetries, as can be checked from (4.18). The action (4.21) with $n = -1/3$ is then recovered when gauge-fixing φ to one.

The coupling of matter fields Φ to Supergravity is done by assuming that they transform like scalars. The action then takes the following form :

$$\mathcal{S} = \int d^8z \left(1 \cdot e^{-2\overleftarrow{U}}\right)^{1/3} \hat{E}^{-1/3} \left(e^{2U}\varphi\right)^\dagger \varphi\, \Omega(\Phi,\bar{\Phi}) + \left(\int d^6z\, \varphi^3 W(\Phi) + \text{c.c.}\right). \tag{4.24}$$

A very common gauge-fixing choice is to transfer the F_φ field from U^μ to φ, a posteriori justifying its name. In this gauge, $U^\mu = H^\mu$ with H^μ given by equation (4.15) and :

$$\varphi = e(1 - 2\theta\sigma^\mu\bar{\Psi}_\mu + \theta^2 F_\varphi) \tag{4.25}$$

where e is the vierbein determinant.

4.3 Scalar Potential

If one is only interested in the scalar potential for the matter scalar fields in the context of Supergravity, one may discard all interaction terms among the graviton, the gravitino and the matter fields. The action then takes the following very simple form :

$$\mathcal{S} = \int d^8z \bar{\varphi}\varphi\Omega(\Phi,\bar{\Phi}) + \left(\int d^6z\, \varphi^3 W(\Phi) + \text{c.c.}\right) \tag{4.26}$$

with $\varphi = 1+\theta^2 F_\varphi$. In order to recover a nice flat space-time limit, *i.e.* when taking the $G_N \to 0$ limit, one usually writes :

$$\Omega = -3e^{-K/3}. \tag{4.27}$$

The action (4.26) enjoys the following symmetry :

$$K \to K + X + \bar{X}, \quad W \to e^{-X}W \quad \text{and} \quad \varphi \to e^{X/3}\varphi \tag{4.28}$$

which can be used to reach the point $K \to G \equiv K + \log W + \log \bar{W}$ and $W \to 1$:

$$\mathcal{S} = \int d^8z \bar{\varphi}\varphi \left(-3e^{-G/3}\right) + \left(\int d^6z\, \varphi^3 + \text{c.c.}\right) \tag{4.29}$$

with $\varphi = e^{G/6}(1 + \theta^2 F_\varphi) \equiv \eta + \theta^2 F$. The overall factor of φ has been chosen in order for the action to be in the Einstein frame as noticed by [89]. Let us now extract the scalar potential from the action (4.29). As explained in subsection 4.2.2, chiral Superfields are functions of the sole y^μ and θ^α. A chiral Superfield is thus given by :

$$
\begin{aligned}
\Phi(x,\theta,\bar\theta) &= \Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) \\
&= \phi(x) + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar\theta^2\Box\phi(x) \\
&\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar\theta + \theta^2 F(x).
\end{aligned}
\tag{4.30}
$$

Since the scalar potential does depend neither on the derivatives of the scalar fields nor on the fermionic fields, one can safely replace the Superfields $\Phi^i(x,\theta,\bar\theta)$ by :

$$
\Phi^i(x,\theta,\bar\theta) = \phi^i(x) + \theta^2 F^i(x). \tag{4.31}
$$

The relevant terms for the computation of the scalar potential extracted from the action (4.29) are the following :

$$
\mathcal{L} \ni V \equiv |F|^2\Omega + \bar F\eta\Omega_F + \bar\eta F\Omega_{\bar F} + |\eta|^2\Omega_D + 3\eta^2 F + 3\bar\eta^2\bar F \tag{4.32}
$$

where :

$$
\begin{aligned}
\Omega_F &\equiv \Omega_i F^i = e^{-G/3}G_i F^i, \\
\Omega_{\bar F} &\equiv \Omega_{\bar\jmath} F^{\bar\jmath} = e^{-G/3}G_{\bar\jmath} F^{\bar\jmath}, \\
\Omega_D &\equiv \Omega_{i\bar\jmath} F^i F^{\bar\jmath} = e^{-G/3}\left(G_{i\bar\jmath} - \frac{1}{3}G_i G_{\bar\jmath}\right) F^i F^{\bar\jmath}.
\end{aligned}
\tag{4.33}
$$

The algebraic equations of motion for F and F^i are easily solved by :

$$
F = e^{G/3}\bar\eta^2\left(1 - \frac{1}{3}G_i G^i\right) \qquad \text{and} \qquad F^i = -e^{G/3}G^i\frac{\bar\eta^2}{\eta} \tag{4.34}
$$

which when plugged back into (4.32) yield the following scalar potential :

$$
V = e^G\left(G_i G^i - 3\right). \tag{4.35}
$$

The expression of V in terms of K and W is simply recovered using the definition $G = K + \log W + \log\bar W$:

$$
V = e^K\left[K^{i\bar\jmath}D_i W D_{\bar\jmath}\bar W - 3|W|^2\right] = K_{i\bar\jmath}F^i F^{\bar\jmath} - 3e^K|W|^2 \tag{4.36}
$$

where we have introduced the Kähler covariant derivative : $D_i W = W_i + K_i W$. The scalar potential thus depends both on the Kähler potential K and on the superpotential W. However, there exist the possibility that a non-trivial superpotential is generated by non-perturbative effects and would thus not be grasped by our procedure. We will thus take the following point of view : we will keep W unspecified and consider it as a parameter of the theory, *i.e.* we can for example tune W to achieve $\langle V\rangle = 0$.

4.4 Scalar Masses

In the last section we have derived the relevant part of the Supergravity Lagrangian for the computation of the scalar fields masses. What is left to do is to take the second derivative of the scalar potential (4.35) and to evaluate it at the minimum of the potential which is defined by the field configuration ϕ^* :

$$\partial_i V(\phi^*, \bar{\phi}^*) = \nabla_i V(\phi^*, \bar{\phi}^*) = 0. \tag{4.37}$$

We then expand V around this point in field space :

$$V = V(\phi^*, \bar{\phi}^*) + \frac{1}{2} \begin{pmatrix} \phi^i & \bar{\phi}^i \end{pmatrix} \begin{pmatrix} \partial_i \partial_{\bar{\jmath}} V & \partial_i \partial_j V \\ \partial_{\bar{\imath}} \partial_{\bar{\jmath}} V & \partial_{\bar{\imath}} \partial_j V \end{pmatrix} \Bigg|_{\phi = \phi^*} \begin{pmatrix} \bar{\phi}^j \\ \phi^j \end{pmatrix} + \mathcal{O}(\phi^3). \tag{4.38}$$

We thus identify the matrix of squared masses with :

$$\begin{pmatrix} m^2_{i\bar{\jmath}} & m^2_{ij} \\ m^2_{\bar{\imath}\bar{\jmath}} & m^2_{\bar{\imath}j} \end{pmatrix} \equiv \begin{pmatrix} \partial_i \partial_{\bar{\jmath}} V & \partial_i \partial_j V \\ \partial_{\bar{\imath}} \partial_{\bar{\jmath}} V & \partial_{\bar{\imath}} \partial_j V \end{pmatrix} \Bigg|_{\phi = \phi^*} = \begin{pmatrix} \nabla_i \nabla_{\bar{\jmath}} V & \nabla_i \nabla_j V \\ \nabla_{\bar{\imath}} \nabla_{\bar{\jmath}} V & \nabla_{\bar{\imath}} \nabla_j V \end{pmatrix} \Bigg|_{\phi = \phi^*} \tag{4.39}$$

where we were able to replace all ordinary derivatives by covariant derivatives. In the mixed indices case we first used that $\partial_{\bar{\jmath}} V = \nabla_{\bar{\jmath}} V$ thanks to the scalar nature of V and then that $\nabla_i \nabla_{\bar{\jmath}} V = \partial_i \nabla_{\bar{\jmath}} V$ since the Christoffel symbols with mixed indices vanish as we chose the covariant derivative to be compatible with the complex structure. When the indices are both holomorphic or antiholomorphic, the Christoffel symbol does not vanish and we have $\nabla_i \nabla_j V = \partial_i \nabla_j V - \Gamma^k_{ij} \nabla_k V$. However as we evaluate this quantity at the minimum of the scalar potential, the connection term cancels out.

Let us now evaluate the matrix of second covariant derivatives of V. The first covariant derivative gives :

$$\begin{aligned} \nabla_{\bar{\jmath}} V &= \nabla_{\bar{\jmath}} \left[e^G \left(G^{m\bar{n}} G_m G_{\bar{n}} - 3 \right) \right] \\ &= G_{\bar{\jmath}} V + e^G \left(G^m \nabla_{\bar{\jmath}} G_m + G^{\bar{n}} \nabla_{\bar{\jmath}} G_{\bar{n}} \right) \\ &= G_{\bar{\jmath}} V + e^G \left(G_{\bar{\jmath}} + G^{\bar{n}} \nabla_{\bar{\jmath}} G_{\bar{n}} \right) \end{aligned} \tag{4.40}$$

which vanishes on the vacuum. Let us now apply the operator ∇_i on this result :

$$\begin{aligned} \nabla_i \nabla_{\bar{\jmath}} V &= \nabla_i \left[G_{\bar{\jmath}} V + e^G \left(G_{\bar{\jmath}} + G^{\bar{n}} \nabla_{\bar{\jmath}} G_{\bar{n}} \right) \right] \\ &= G_{i\bar{\jmath}} V + G_{\bar{\jmath}} \nabla_i V + G_i e^G \left(G_{\bar{\jmath}} + G^{\bar{n}} \nabla_{\bar{\jmath}} G_{\bar{n}} \right) + e^G \left[G_{i\bar{\jmath}} + \nabla_i (G^{\bar{n}} \nabla_{\bar{\jmath}} G_{\bar{n}}) \right]. \end{aligned} \tag{4.41}$$

If we now choose W such that the vacuum energy is compatible with a small positive cosmological constant (see [90] for the actual number) then only the last bracket of (4.41) contributes to the scalar squared mass matrix element :

$$\begin{aligned} m^2_{i\bar{\jmath}} &= e^G \left[G_{i\bar{\jmath}} + (\nabla_i G^{\bar{n}})(\nabla_{\bar{\jmath}} G_{\bar{n}}) + G^{\bar{n}} \nabla_i \nabla_{\bar{\jmath}} G_{\bar{n}} \right] \\ &= e^G \left[G_{i\bar{\jmath}} + (\nabla_i G^{\bar{n}})(\nabla_{\bar{\jmath}} G_{\bar{n}}) \right] - R_{i\bar{\jmath}m\bar{n}} F^m F^{\bar{n}}. \end{aligned} \tag{4.42}$$

By again using the vanishing of the cosmological constant condition, we find :

$$e^G = \frac{1}{3} e^G G_i G^i = \frac{1}{3} G_{i\bar{\jmath}} F^i F^{\bar{\jmath}} \tag{4.43}$$

which permits to rewrite the masses as :

$$m^2_{i\bar{\jmath}} = e^G(\nabla_i G^{\bar{n}})(\nabla_{\bar{\jmath}} G_{\bar{n}}) - \left(R_{i\bar{\jmath}m\bar{n}} - \frac{1}{3}G_{i\bar{\jmath}}G_{m\bar{n}}\right) F^m F^{\bar{n}}. \tag{4.44}$$

The same procedure can be applied to compute the off-diagonal elements of the squared masses matrix. The result is found to be given by the following formula :

$$m^2_{ij} = e^G\left(\nabla_i G_j + \nabla_j G_i + \frac{1}{2}G^m\{\nabla_i, \nabla_j\}G_m\right). \tag{4.45}$$

4.5 Scalar Masses in Hidden Sector Scenarios

Since our primary concern lays in the determination of the soft masses appearing in (3.33), let us specialise the equation (4.44) to the case where according to the discussion of section 3.7 the fields are split among the visible and hidden sectors :

$$\Phi^i \quad \rightarrow \quad \begin{cases} Q^\alpha & \text{Visible Sector} \\ \Phi^\Theta & \text{Hidden Sector} \end{cases} \tag{4.46}$$

Since the visible fields are characterised by a vanishing vacuum expectation value, one has :

$$G_\alpha = G_{\alpha\bar{\Theta}} = \nabla_\Theta G_\alpha = 0 \tag{4.47}$$

on the vacuum. Moreover in all the cases we will be focusing on in the following, matter fields in the visible sector do not admit holomorphic quadratic invariants of the gauge symmetry, and thus :

$$\nabla_\alpha G_\beta = 0 \tag{4.48}$$

on the vacuum. We are thus able to rewrite the equation (4.44) under the hypothesis (4.47) and (4.48) as :

$$m^2_{\alpha\bar{\beta}} = -\left(R_{\alpha\bar{\beta}\Theta\bar{\Gamma}} - \frac{1}{3}K_{\alpha\bar{\beta}}K_{\Theta\bar{\Gamma}}\right) F^\Theta \bar{F}^\Gamma \tag{4.49}$$

in accordance with [73] where we have replaced the mixed derivatives of G with those of K since they coincide. Note that the expression inside the brackets of (4.49) only depends on the Kähler potential and is thus a purely geometric object with no dependence on the superpotential, except for the selection of the vacuum point. The superpotential W only affects the direction of F^Θ. The crucial ingredient of the soft scalar masses computation is thus the Kähler potential to which we will devote Chapter 7.

Relation with the Gravity-Mediated Soft Masses In subsection 3.7.2 we have determined that, in the context of gravity-mediation, the soft scalar masses are given by :

$$m^2_{\alpha\bar{\beta}} = -\left[Z^{\alpha\beta}_{\Theta\bar{\Gamma}} - (Z^{-1})^{mn} Z^{\alpha m}_{\Theta} Z^{n\beta}_{\bar{\Gamma}}\right] F^{\Theta} \bar{F}^{\Gamma} \tag{4.50}$$

where Z^{mn} is the Superspace wave-function. In order to make contact with (4.49), one has to take into account that in Supergravity the term appearing in the integral over Superspace is not the Kähler potential K but Ω which has been defined in (4.27). Therefore, in order to compare the two expressions for the soft scalar masses we first have to express (4.49) with respect to Ω. Using the technology developed in Appendix B.2.4.2, one easily finds :

$$R_{\alpha\bar{\beta}\Theta\bar{\Gamma}} = \frac{1}{3} K_{\alpha\bar{\beta}} K_{\Theta\bar{\Gamma}} - \frac{3}{\Omega}\left[\Omega_{\alpha\bar{\beta}\Theta\bar{\Gamma}} - \Omega_{\alpha\Theta\bar{\gamma}}(\Omega^{-1})^{\rho\bar{\gamma}}\Omega_{\rho\bar{\beta}\bar{\Gamma}}\right] \tag{4.51}$$

which when replaced in (4.49) gives :

$$m^2_{\alpha\bar{\beta}} = \frac{3}{\Omega}\left[\Omega_{\alpha\bar{\beta}\Theta\bar{\Gamma}} - \Omega_{\alpha\Theta\bar{\gamma}}(\Omega^{-1})^{\rho\bar{\gamma}}\Omega_{\rho\bar{\beta}\bar{\Gamma}}\right] F^{\Theta} \bar{F}^{\Gamma} \tag{4.52}$$

which manifestly has the same structure as (3.48) provided we interpret the upper indices as derivatives as has been argued in subsection 3.7.2.

Remark that the computation of masses we have performed has been done by only considering chiral fields, despite the fact that the models we are interested in are gauge models. The formula (4.49) is thus valid only in those situations where the chiral multiplets dominate SUSY-breaking in the hidden sector. For record, the full dependence of $m^2_{i\bar{j}}$ and m^2_{ij} on the gauge-kinetic function and on D-terms may for example be found in [91].

Chapter 5

Heterotic $\mathcal{M}$-Theory

In this Chapter we are introducing Heterotic $\mathcal{M}$-Theory as a prototype theory in the context of which the idea of sequestering can be put at work. We first review the basics of Superstring theory and in particular $E_8 \times E_8$ Heterotic String Theory. There exist many good reviews of this subject among which [92–95]. We will then introduce $\mathcal{M}$-Theory as a conjectured eleven-dimensional mother theory of which the five known string theories represent particular limits. The effective theory of the $E_8 \times E_8$ Heterotic $\mathcal{M}$-Theory is thus defined as an eleven dimensional Supergravity theory where one set of E_8 gauge fields lives on each end-of-the-world brane. Such a setup contains natural candidates for both the hidden and the visible sectors : the observable sector consists in the fields living on one of the brane while the hidden sector contains the fields living on the distant brane together with the moduli which are the internal components of the Supergravity multiplet.

5.1 A New Paradigm

In a first attempt to unify all known forces of Nature in a quantum theory, one may try to couple the Standard Model, which unifies the strong and electro-weak forces, to General Relativity. However, General Relativity does not seem to be renormalisable. Indeed as the coupling controlling the strength of gravitational interactions, Newton's constant G_N, has a mass dimension GeV^{-2}, the ratio of a one-graviton correction to the zero-graviton amplitude is roughly given by $\sqrt{G_N}E$ where E is the characteristic energy scale of the process. Gravitational interactions as described by General Relativity are thus understood to be irrelevant.

Non-renormalisability of General Relativity may be taken as a hint for the need of a new paradigm just as the non-renormalisability of Fermi theory led to the introduction of gauge bosons mediating the electro-weak force. The way String Theory solves the UV divergence issue is by postulating that the fundamental objects of the theory, strings, have a characteristic length denoted by ℓ_s. The string length acts as a regulator for UV divergences since it is not possible to shrink loops below the ℓ_s scale.

The area spanned by a string moving in space-time, the string worldsheet, seen from a distance much greater than ℓ_s, or equivalently at a low enough energy, looks like a one-dimensional worldline. The low energy effective action of String Theory is thus a theory of point-like particles.

The consistency of String Theory at the quantum level requires the strings to evolve in a ten-dimensional space-time. The four-dimensional effective theory thus crucially depends on the manifold on which String Theory is to be compactified. We will draw our attention on this topic in Chapter 6. The four-dimensional effective theory spectrum will consist of the lowest excitations of the strings, which are massless in the ten-dimensional picture, the infinite tower of massive higher harmonics being integrated out.

5.2 Introduction to String Theory

The Poincaré-invariant action describing the dynamics of a point-like particle in a flat space-time is given by :

$$\mathcal{S} = -m \int ds = -m \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \tag{5.1}$$

where the integral is to be performed along the particle's trajectory parametrised by τ. The appearance of a square-root renders this action not very well-suited for a path-integral treatment. The introduction of an auxiliary field permits the rewriting of the action in the following way :

$$\mathcal{S} = \frac{1}{2} \int d\tau \left(e^{-1} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - e m^2 \right). \tag{5.2}$$

When the auxiliary field e, the einbien, is replaced by its algebraic equation of motion, one recovers the action (5.1). The action is invariant both under the Poincaré group and reparametrisation $\tau \to \tau'(\tau)$ under which $e(\tau) \to e'(\tau') = e(\tau) d\tau / d\tau'$. Using the reparametrisation invariance, one may reach the gauge $e = 1$ in which the action (5.2) is easy to handle. Note that the conjugated momentum defined out of (5.1) suffers from a mass-shell condition :

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} = m \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}} \quad \rightarrow \quad \Pi_\mu \Pi^\mu = -m^2 \tag{5.3}$$

or, equivalently, one may show that the Hamiltonian is vanishing. When using the action (5.2) with e gauge-fixed to one, one has to impose by hand the vanishing of the Hamiltonian.

Bosonic String Theory The so-called bosonic string action is constructed in a similar fashion. It is written as the integral over the area spanned by the string. A position on the worldsheet is specified by two parameters, $X^\mu = X^\mu(\tau, \sigma)$, and the action reads :

$$\mathcal{S} = -\frac{1}{2\pi \ell_s^2} \int d\tau d\sigma \sqrt{-\det\left(\eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu\right)}. \tag{5.4}$$

In order to get rid of the square root, one introduces a worldsheet metric γ_{ab} in terms of which the string action may be written as :

$$\mathcal{S} = -\frac{1}{4\pi\ell_s^2}\int d\tau d\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int d\tau d\sigma \sqrt{-\gamma}R \tag{5.5}$$

where we have added a term proportional to the Ricci scalar defined out of γ_{ab} since it is compatible with all the symmetries of the first term which are :

- ◇ Space-time Poincaré invariance,
- ◇ Worldsheet diffeomorphisms, under which the X^μ's transform like scalars,
- ◇ Weyl rescaling acting as $\gamma_{ab} \to e^{\omega(\tau,\sigma)}\gamma_{ab}$ for arbitrary $\omega(\tau,\sigma)$.

In two dimensions the second term is a total derivative and thus only depends on the topology of the worldsheet, corresponding to its Euler characteristic χ. Note that the action (5.5) can be interpreted as the action describing the dynamics of bosons X^μ living in a two-dimensional world, the worldsheet, the number of bosons being given by the dimensionality of space-time.

One may then use these symmetries to choose a gauge in which the calculations are easy to handle, in analogy with the case of the point particle in which we chose the $e = 1$ gauge. By using the worldsheet diffeomorphisms and the Weyl rescaling, one can bring the γ_{ab} metric to the Minkowski metric η_{ab}. The equation of motion for $X^\mu(\tau,\sigma)$ then resembles a wave equation :

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu(\tau,\sigma) = 0 \tag{5.6}$$

with two constraints : $(\dot{X} \pm X')^2 = 0$ where $\dot{X}^\mu = \partial_\tau X^\mu$ and $X^{\mu'} = \partial_\sigma X^\mu$. These constraints are the string-equivalent of the vanishing of the Hamiltonian in the point-particle context. The solution is thus to be expanded in left-moving and right-moving modes. Imposing canonical commutation relations among X^μ and its conjugated momentum generates a bosonic algebra for the modes coefficients from which a Fock space is constructed. The spectrum is then obtained by acting with the creation operators on the Fock vacuum. The masses are shown to increase by steps of the inverse string length. In this scheme, the critical dimension of space-time in which strings propagate emerges as being the only one compatible with a physical interpretation of the spectrum [92]. For the bosonic string, one finds the critical dimension to be 26.

When considering open strings, the left and right-movers are related by the boundary conditions and we are only left with one set of creation and annihilation operators. Acting on the vacuum $|\Omega\rangle$ generates the spectrum. The first few levels are thus : $|\Omega\rangle$, $a_m^{\mu\dagger}|\Omega\rangle$, $a_m^{\mu\dagger}a_n^{\nu\dagger}|\Omega\rangle$, ... where m labels the harmonic. $|\Omega\rangle$ can be shown to be a tachyon, $a_1^{\mu\dagger}|\Omega\rangle$ a massless vector field and all other excitations massive fields.

In the case of closed strings, the constraints translate into a level-matching condition : only an equal number of left-moving and right-moving creation operators are allowed to act on the vacuum which, again, is a tachyon. The massless states are shown to be obtained by acting once with a left-moving creation operator and once with a right-moving one : $a_1^{\mu\dagger}\tilde{a}_1^{\nu\dagger}|\Omega\rangle$. The result is a transverse two-tensor which may be decomposed into a symmetric traceless tensor, an antisymmetric tensor and a scalar which are interpreted as respectively being the metric $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and the dilaton Φ.

In order to obtain a space-time Lagrangian density for the massless excitations of String Theory, ignoring the tachyon, one first writes down the String Theory action in presence of a background for $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ :

$$\mathcal{S} = -\frac{1}{4\pi\ell_s^2}\int d\tau d\sigma\sqrt{-\gamma}\left[\left(\gamma^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \ell_s^2 R\Phi(X)\right]. \tag{5.7}$$

The last step towards the construction of an action for the massless fields is to enforce the Weyl anomaly to vanish by imposing the tracelessness of the energy-momentum tensor constructed out of the γ_{ab} metric. Indeed, at the classical level the energy-momentum tensor has a vanishing trace thanks to the Weyl rescaling symmetry. However, this does not hold anymore at the quantum level, leading to an anomaly. One thus needs to impose that the theory is anomaly-free by requiring that the trace of the energy-momentum tensor vanishes not only classically but also at the quantum level. This condition will depend on a combination of the derivatives of the $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ fields which are interpreted as the equations of motion deriving from a space-time action. In the case at hand the corresponding space-time action is given by [92] :

$$\mathcal{S} = \frac{1}{2\kappa_0^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\left[R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi + \mathcal{O}(\ell_s^2)\right] \tag{5.8}$$

where κ_0 is a free parameter since the equations of motion do not depend on the overall scale of the action.

Superstring Theory In order for String Theory to play a rôle in describing Nature, it not only has to make sense of the tachyons appearing in the Fock space but it should also definitely cope with fermions. Having in mind the interpretation of the String Theory action as the action of bosonic fields X^μ living on the worldsheet leads to the natural introduction of worldsheet fermions ψ_α^μ which has the desired effect since it permits to generate space-time fermions. The resulting theory is called *Superstring Theory* and is only consistent in ten space-time dimensions. Note that at this stage it is far from being obvious that the fields ψ_α^μ will describe space-time fermions since they transform as vectors under the space-time Lorentz group.

The superstring action is obtained by adding the following piece to (5.5) :

$$\begin{aligned}\Delta\mathcal{S} &= -\frac{i}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}\bar{\psi}^\mu\Gamma^a\partial_a\psi_\mu \\ &= -\frac{i}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}\left[\psi_1^\mu(\partial_\tau + \partial_\sigma)\psi_{1\mu} + \psi_2^\mu(\partial_\tau - \partial_\sigma)\psi_{2\mu}\right]\end{aligned} \tag{5.9}$$

where the Γ^a matrices satisfy the two-dimensional Clifford algebra, for example by choosing $\Gamma^0 = \sigma^2$ and $\Gamma^1 = i\sigma^1$. The boundary term appearing when computing the equations of motion may be set to zero by either the two following choices of boundary conditions :

$$\begin{aligned} \psi_1^\mu(\pi,\tau) &= +\,\psi_2^\mu(\pi,\tau) \qquad \text{Ramond sector,} \\ \psi_1^\mu(\pi,\tau) &= -\,\psi_2^\mu(\pi,\tau) \qquad \text{Neveu-Schwarz sector.} \end{aligned} \tag{5.10}$$

Open Strings Let us now investigate the consequences of these two boundary conditions. More precisely we wish to classify the string excitations in terms of the little group of the ten-dimensional Lorentz group $SO(1,9)$ which is $SO(8)$ for massless representations :

- ⋄ In the Neveu-Schwarz (NS) sector, the solution to the ψ_α^μ equations of motion is to be expanded with half-integrally moded exponentials with coefficients having to satisfy the quantisation condition taking the form of a fermionic algebra. It can be shown [16] that the spectrum starts with a $SO(8)$-singlet tachyon. The massless spectrum is obtained by acting with one of the creation operators on the Fock vacuum, leading to a space-time massless vector, *i.e.* an $\mathbf{8}_v$ under $SO(8)$. The NS sector massless states are thus space-time bosons.

- ⋄ In the Ramond (R) sector, the solution to the ψ_α^μ equations of motion is to be expanded with integrally moded exponentials with coefficients having to satisfy the quantisation condition taking the form of a fermionic algebra. The most important difference between the Ramond and the NS sectors is that the Ramond sector contains zero modes which do not contribute to the mass of the states and whose anticommutation relations are nothing but the space-time Clifford algebra. One may then construct ground states $|\Omega_R\rangle$ which in ten space-time dimensions form a massless 32-dimensional Dirac representation $\mathbf{32}_\mathrm{D}$ of the Clifford algebra on which the creation operators can act. Since the $\mathbf{32}_\mathrm{D}$ decomposes as follows under $SO(1,9) \to SO(1,1) \times SO(8)$:

$$\mathbf{32}_\mathrm{D} \quad \to \quad (1/2, \mathbf{8}) \oplus (-1/2, \mathbf{8}') \oplus (1/2, \mathbf{8}') \oplus (-1/2, \mathbf{8}) \tag{5.11}$$

  where the two first and two last factors respectively come from the two inequivalent 16-dimensional Weyl representation of 10-dimensional spinors : $\mathbf{32}_\mathrm{D} = \mathbf{16} \oplus \mathbf{16}'$, when going on-shell half the degrees of freedom are killed by the Dirac equation and we are left with two inequivalent massless Weyl representations of $SO(8)$: $\mathbf{8}$ and $\mathbf{8}'$ which are space-time fermions.

Closed Strings For closed strings, the left and right-moving fermions are independent and can be chosen to be either in the Ramond sector or in the Neveu-Schwarz sector giving rise to the following possibilities :

$$(\mathrm{R},\mathrm{R}) \qquad (\mathrm{NS},\mathrm{NS}) \qquad (\mathrm{R},\mathrm{NS}) \qquad \text{and} \qquad (\mathrm{NS},\mathrm{R}) \tag{5.12}$$

where the first two options describe space-time bosons while the two last ones are space-time fermions since a product of two fermionic representations is a bosonic one. In the first two sectors, one finds, at the massless level :

$$(\mathbf{8} \oplus \mathbf{8}')^2 = [0]_8^2 \oplus \ldots \oplus [3]_8^2 \oplus [4]_8 \qquad \text{and} \qquad \mathbf{8}_v \otimes \mathbf{8}_v = [0]_8 \oplus [2]_8 \oplus (2)_8 \tag{5.13}$$

where $[n]_d$ is a totally antisymmetric n-tensor in d dimensions, *i.e.* of dimension :

$$C_n^d = \frac{d!}{n!(d-n)!} \tag{5.14}$$

and where $(n)_d$ is a symmetric traceless n-tensor in d dimensions. In the two last sectors, ignoring the dilaton, one finds :

$$(\mathbf{8} \oplus \mathbf{8}') \otimes \mathbf{8}_v = \mathbf{8} \oplus \mathbf{8}' \oplus \mathbf{56} \oplus \mathbf{56}' \tag{5.15}$$

which are two spin-1/2 fermions and two gravitinos. In order to get rid of the tachyon, let us now introduce a way to project it out.

GSO Projection In order to get rid of the tachyon in the NS sector and to enforce space-time Supersymmetry, one may try to devise a consistent projection on the spectrum. To do so, let us first introduce the worldsheet fermion number F which can only take the values zero and one, *i.e.* it determines whether the state is a worldsheet fermion or not by counting how many fermionic creation operators have been applied on the Fock vacuum. Then the operator :

$$(-1)^F = \pm 1 \tag{5.16}$$

anticommutes with the fermionic creation operators and defines two sectors. The R and NS sectors are thus further subdivided into NS± and R± where R± are the $\mathbf{8}$ and $\mathbf{8}'$, NS+ the $\mathbf{8}_v$ and NS− the tachyon. The combinations of right and left-moving sectors leading to a massless spectrum are found in Table 5.1. The projection onto $(-1)^F$ eigensectors is called the Gliozzi-Scherk-Olive (GSO) projection [96].

A Superstring theory is thus specified by the sectors it contains. One can build 16 of them (NS or R, + or −) *a priori* leading to 2^{16} different String theories, but since the NS− contains a tachyon it is usually discarded leaving us with a choice of 9 sectors to include or not, *i.e.* to 2^9 different theories. The IIA and IIB Superstring theories correspond to choosing :

$$\text{IIA}: \quad \left\{\begin{matrix}\text{NS}+\\ \text{R}+\end{matrix}\right\}_L \otimes \left\{\begin{matrix}\text{NS}+\\ \text{R}-\end{matrix}\right\}_R = (\text{NS}+,\text{NS}+) \quad (\text{R}+,\text{NS}+) \quad (\text{NS}+,\text{R}-) \quad (\text{R}+,\text{R}-),$$

$$\text{IIB}: \quad \left\{\begin{matrix}\text{NS}+\\ \text{R}+\end{matrix}\right\}_L \otimes \left\{\begin{matrix}\text{NS}+\\ \text{R}+\end{matrix}\right\}_R = (\text{NS}+,\text{NS}+) \quad (\text{R}+,\text{NS}+) \quad (\text{NS}+,\text{R}+) \quad (\text{R}+,\text{R}+).$$

Sector	Under $SO(8)$	Massless spectrum
(R+, R+)	$\mathbf{8} \otimes \mathbf{8}$	$[0]_8 \oplus [2]_8 \oplus [4]_8^{\rm sd}$
(R−, R−)	$\mathbf{8}' \otimes \mathbf{8}'$	$[0]_8 \oplus [2]_8 \oplus [4]_8^{\rm asd}$
(R+, R−)	$\mathbf{8} \otimes \mathbf{8}'$	$[1]_8 \oplus [3]_8$
(NS+, NS+)	$\mathbf{8}_v \otimes \mathbf{8}_v$	$[0]_8 \oplus [2]_8 \oplus (2)_8$
(NS+, R+)	$\mathbf{8}_v \otimes \mathbf{8}$	$\mathbf{8}' \oplus \mathbf{56}$
(NS+, R−)	$\mathbf{8}_v \otimes \mathbf{8}'$	$\mathbf{8} \oplus \mathbf{56}'$

Table 5.1: Combination of left and right-moving sectors

(5.17)

The other combinations either lead to inconsistent theories or to theories containing no fermion or a tachyon. Note that both the IIA and IIB theories contain two gravitinos : two $\mathbf{56}$'s for the IIB, one $\mathbf{56}$ and one $\mathbf{56}'$ for the IIA. Combining those with the graviton in the (NS+, NS+) sector leads to $\mathcal{N} = 2$ space-time Supersymmetry.

Open + Closed Strings The IIB Superstring is left-right symmetric, *i.e.* it is invariant under Ω which acts as $\sigma \to \pi - \sigma$. Since $\Omega^2 = 1$, its eigenvalues are ± 1. By applying Ω to X^μ or ψ^μ one can find the parity eigenvalue of the creation operators. Consistent string theories may be obtained by only keeping the $\Omega = +1$ sector, *i.e.* the *unoriented* sector, thereby restricting the spectrum. The action of Ω on a closed string state $|L^i R^j\rangle$ is given by :

$$\Omega |L^i R^j\rangle = |R^i L^j\rangle = \pm |L^j R^i\rangle \tag{5.18}$$

where $|L^i R^j\rangle$ stands for the state obtained by successively acting with the right-handed j-th and the left-handed i-th creation operator on the Fock vacuum. The sign in the second equality is determined by the statistics obeyed by the L^i and R^j states. Let us now derive the spectrum of IIB/Ω. In the (NS+, NS+) sector, the positive sign is selected since NS+ is a boson. Then the $\Omega = +1$ eigenstate is given by $|L^i R^j\rangle + |L^j R^i\rangle$, the antisymmetric $[2]_8$ being killed by the projection. The two sectors (NS+, R+) and (R+, NS+) together lead to the symmetric $\mathbf{8}' \oplus \mathbf{56}$. Finally, the (R+, R+) selects the minus sign in (5.18) since $R+$ is a spinor. The $\Omega = +1$ eigenstate is given by $|L^i R^j\rangle - |L^j R^i\rangle$, *i.e.* it is the $[2]_8$. Summarising, the IIB massless $\Omega = +1$ states are :

$$\mathbf{8}' \oplus \mathbf{56} \oplus [0]_8 \oplus [2]_8 \oplus (2)_8. \tag{5.19}$$

However the above spectrum does not lead to the cancellation of the gravitational anomaly. The consistency condition required to ensure that the effective theory is anomaly-free is the so-called RR tadpole cancellation which is solved by the addition of open unoriented strings with Chan-Paton indices, which describe the gauge group, belonging to $SO(32)$, see [94] for a nice discussion. The massless spectrum of the $SO(32)$ Type I Superstring is thus found to be given by :

$$\mathbf{8}' \oplus \mathbf{56} \oplus [0]_8 \oplus [2]_8 \oplus (2)_8 \oplus (\mathbf{8}_v \oplus \mathbf{8})_{SO(32)} \tag{5.20}$$

which respectively are the dilatino, the gravitino, the dilaton, a two-form, the graviton and $SO(32)$ gauge bosons and gauginos. The type I Superstring thus is a $\mathcal{N} = 1$ SUSY theory with $SO(32)$ gauge group which embeds both open and closed strings.

5.3 The Heterotic String

Yet another closed Superstring theory can be constructed by combining the left-moving sector of the 26-dimensional bosonic string with the right-moving sector of the 10-dimensional Superstring [97, 98]. The right-moving sector consists of ten X^μ and ten ψ^μ_α while the left-moving one contains twenty-six X^M which are divided into ten X^μ and sixteen transverse X^I which are traded by fermionisation for thirty-two space-time singlets worldsheet fermions λ^A [98]. The worldsheet action is then given by :

$$\mathcal{S} = -\frac{1}{4\pi\ell_s^2}\int d\tau d\sigma\sqrt{-\gamma}\left[\gamma^{ab}\partial_a X^\mu\partial_b X_\mu + i\ell_s^2\psi^\mu(\partial_0+\partial_1)\psi_\mu + i\ell_s^2\lambda^A(\partial_0-\partial_1)\lambda^A\right]. \tag{5.21}$$

The Heterotic String is consistent in a ten-dimensional space-time. Note that the left-moving worldsheet fermions enjoy an $SO(32)$ symmetry under which they transform in the fundamental representation.

$SO(32)$ **Heterotic String** Since the Heterotic String right-moving sector is the same as the type II one, it consists of an $(\mathbf{8},\mathbf{1}) \oplus (\mathbf{8}_v,\mathbf{1})$ under $SO(8)\times SO(32)$ at the massless level. A GSO projection is defined on the left-moving sector in order to remove the tachyon from the spectrum. Furthermore the λ^A's have to satisfy Neveu-Schwarz boundary conditions if they are to produce massless states [16]. The massless states are thus found by acting on the NS vacuum either with a bosonic creation operator $a_1^{\mu\dagger}|\Omega\rangle_{\rm NS}$ or with two half-moded fermionic creation operators $\lambda^{A\dagger}_{1/2}\lambda^{B\dagger}_{1/2}|\Omega\rangle_{\rm NS}$. These states respectively transform as $(\mathbf{8}_v,\mathbf{1})$ and $(\mathbf{1},[2]_{32})$ under $SO(8)\times SO(32)$. The massless spectrum is given by the product of the left-moving and right-moving massless states :

$$\text{Heterotic } SO(32): \qquad \Big[(\mathbf{8}_v,\mathbf{1}) \oplus (\mathbf{1},[2]_{32})\Big] \otimes \Big[(\mathbf{8},\mathbf{1}) \oplus (\mathbf{8}_v,\mathbf{1})\Big] \tag{5.22}$$

which corresponds to an $\mathcal{N} = 1$ $SO(32)$ gauge theory since $\dim([2]_{32}) = 496$ is the dimension of the $SO(32)$ adjoint. The $(\mathbf{1},[2]_{32}) \otimes (\mathbf{8}_v,\mathbf{1})$ are thus identified with $SO(32)$ gauge bosons and while $(\mathbf{1},[2]_{32}) \otimes (\mathbf{8},\mathbf{1})$ are their supersymmetric partners, the gauginos.

$E_8 \times E_8$ Heterotic String In the $SO(32)$ heterotic string theory construction we have chosen to maintain the $SO(32)$ symmetry in the λ^A's sector. One may consider how the situation is changed given the first n λ^A to have NS boundary conditions and the remaining $32-n$ to have R boundary conditions thus allowing the possibility of constructing a Clifford algebra from their zero-modes. It turns out that the only consistent theory has $n=16$, which will lead to a 256-dimensional Dirac representation of $SO(16)$: $\mathbf{256}_{\mathrm{D}}$. Let us now go through the different subsectors the left-moving states contain. In the NS-NS sector, the result is almost the same than in the $SO(32)$ case : $a_1^{\mu\dagger}|\Omega\rangle_{\mathrm{NS}}$ and $\lambda^{A\dagger}_{1/2}\lambda^{B\dagger}_{1/2}|\Omega\rangle_{\mathrm{NS}}$ where due to the GSO projection the two labels A and B should belong to the same set of 16. The NS-NS states thus transform as $(\mathbf{8}_v,\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{120},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{120})$ under $SO(8)\times SO(16)\times SO(16)$. The NS-R sector produces the announced Dirac representation of $SO(16)$ which is the sum of two inequivalent Weyl representations $\mathbf{256}_{\mathrm{D}}=\mathbf{128}\oplus\mathbf{128}'$ of which one is killed when going on-shell. The NS-R thus produces $(\mathbf{1},\mathbf{1},\mathbf{128})$ while the R-NS sector gives a $(\mathbf{1},\mathbf{128},\mathbf{1})$. The R-R sector does not contain any massless states. The massless spectrum consists of the product of the left-moving and right-moving massless states :

$$\begin{aligned}\text{Heterotic } E_8\times E_8 : \quad & \Big[(\mathbf{8}_v,\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{120},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{120})\oplus(\mathbf{1},\mathbf{1},\mathbf{128})\oplus(\mathbf{1},\mathbf{128},\mathbf{1})\Big]\\ & \otimes\Big[(\mathbf{8},\mathbf{1},\mathbf{1})+(\mathbf{8}_v,\mathbf{1},\mathbf{1})\Big].\end{aligned} \tag{5.23}$$

The massless vectors in the $(\mathbf{8}_v,\mathbf{120},\mathbf{1})$ and $(\mathbf{8}_v,\mathbf{128},\mathbf{1})$ should transform in the adjoint of the gauge group. One is thus led to look for a group G whose adjoint splits into $\mathbf{120}\oplus\mathbf{128}$ under $SO(16)$. The only group having this property is the exceptional group E_8. The gauge group of the second Heterotic superstring is thus $E_8\times E_8$. The massless spectrum of the $E_8\times E_8$ Heterotic string theory is recorded in Table 5.2 in which the transformation properties under $SO(8)\times E_8\times E_8$ are indicated.

5.4 The $E_8 \times E_8$ Heterotic Effective Action

In order to derive the effective action describing the dynamics of the above-mentioned spectrum, one may proceed as in the bosonic case, *i.e.* one computes the trace of the energy momentum on the worldsheet and imposes that it vanishes. The emerging dynamical relations among the space-time fields are then interpreted as their equation of motion from which on reconstructs the action. However in the case at hand, since the spectrum exhibits $\mathcal{N}=1$ SUSY, the action is pretty constrained and can be shown to be the following in which only the bosonic fields are recorded [99] :

$$\mathcal{S}=\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\,e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}|H|^2+\frac{\kappa_{10}^2}{g_{10}^2}\mathrm{Tr}(|F|^2)+\mathcal{O}(\ell_s^6)\right] \tag{5.24}$$

$d=10$, $\mathcal{N}=1$ SUGRA	$(\mathbf{1},\mathbf{1},\mathbf{1})$	Dilaton	Φ
	$(\mathbf{28},\mathbf{1},\mathbf{1})$	Antisymmetric Tensor	B_{MN}
	$(\mathbf{35},\mathbf{1},\mathbf{1})$	Metric Tensor	G_{MN}
	$(\mathbf{8}',\mathbf{1},\mathbf{1})$	Dilatino	χ_α
	$(\mathbf{56},\mathbf{1},\mathbf{1})$	Gravitino	Ψ_α^M
$E_8\times E_8$ gauge sector	$(\mathbf{8}_v,\mathbf{248},\mathbf{1})$	Gauge bosons	A_M^X
	$(\mathbf{8}_v,\mathbf{1},\mathbf{248})$		
	$(\mathbf{8},\mathbf{248},\mathbf{1})$	Gauginos	λ_α^X
	$(\mathbf{8},\mathbf{1},\mathbf{248})$		

Table 5.2: $E_8\times E_8$ Heterotic Massless Spectrum

where H is a modified field-strength for B_{MN} :

$$H=\mathrm{d}B-\frac{\kappa_{10}^2}{g_{10}^2}\mathrm{Tr}\left(A\wedge\mathrm{d}A-\frac{2}{3}A\wedge A\wedge A\right)-\frac{\kappa_{10}^2}{g_{10}^2}\mathrm{Tr}\left(\omega\wedge\mathrm{d}\omega-\frac{2}{3}\omega\wedge\omega\wedge\omega\right) \tag{5.25}$$

where the second Chern-Simons term is a higher-derivative effect which nevertheless is important for the consistency of the theory. The fact that B has a shifted field-strength implies it has to satisfy a non-standard Bianchi identity :

$$\mathrm{d}H=\frac{\kappa_{10}^2}{g_{10}^2}\Big(\mathrm{Tr}\,(R\wedge R)-\mathrm{Tr}\,(F\wedge F)\Big). \tag{5.26}$$

The non-standard field-strength for B can also be seen as coming from an anomaly-cancellation effect on the worldsheet which further constrains the gauge coupling g_{10}^2 to satisfy [97, 98] :

$$g_{10}^2=4\frac{\kappa_{10}^2}{\ell_s^2}. \tag{5.27}$$

The Heterotic String effective action (5.24) has an $\mathcal{N}=1$ local Supersymmetry which acts as follows on the fermionic fields [99] :

$$\begin{aligned}
\delta_\epsilon\Psi_M &= \nabla_M\epsilon-\frac{1}{8}H_{MNP}\Gamma^{NP}\epsilon\\
\delta_\epsilon\chi &= -\frac{1}{2}\Gamma^M\partial_M\Phi\epsilon+\frac{1}{24}H_{MNP}\Gamma^{MNP}\epsilon\\
\delta_\epsilon\lambda^A &= -\frac{1}{2}F^A_{MN}\Gamma^{MN}\epsilon
\end{aligned} \tag{5.28}$$

up to terms involving fermions.

5.5 $\mathcal{M}$-Theory

The IIA effective theory spectrum is easily shown to result from the dimensional reduction of the 11-dimensional $\mathcal{N} = 1$ SUGRA on a circle which indeed generates an $\mathcal{N} = 2$ theory since $\mathcal{N} = 1$ theories in eleven dimensions have 32 supercharges, see Table C.1. Moreover the five known Superstring theories are believed to be related by various dualities, suggesting that they are different limits of a greater theory. Witten then conjectured [100] that the IIA strong coupling limit consists in an 11-dimensional yet to be specified $\mathcal{M}$-Theory of which 11-dimensional SUGRA is to be the effective theory. The web of dualities relating the different Superstring theories leads to the identification of the strong coupling regime of the $E_8 \times E_8$ Heterotic String with $\mathcal{M}$-Theory compactified on $\mathbb{S}^1/\mathbb{Z}_2$, which is nothing but a segment. The fact that the gauge group is a product of two E_8's is then understood as coming from an anomaly cancellation argument in the eleven-dimensional picture and is interpreted as the localisation of the Yang-Mills fields on two ten-dimensional branes located at each of the segment ends [101, 102], which are commonly called end-of-the-world branes.

The setup consisting of an 11-dimensional theory bounded by two 10-dimensional branes, each supporting E_8 gauge fields is called Heterotic $\mathcal{M}$-Theory. Such a theory has seven extra-dimensions which are to be compactified to give rise to the four-dimensional effective theory. Note that not all extra-dimensions are on equal footing since the eleventh dimension is related to the string coupling which has no relation to the six extra-dimensions on the branes. The order in which the compactification is to be performed thus depends on the relative size of the extra-dimensions.

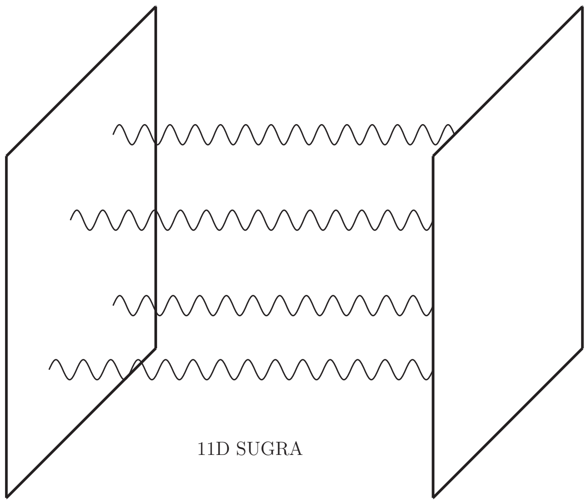


Figure 5.1: Heterotic $\mathcal{M}$-theory Setup

In the situation where the eleventh dimension is the first to be compactified, the eleven-dimensional SUGRA bosonic spectrum which, as shown in the Appendix C.2, consists of the metric G_{AB} and of the 3-form C_{ABC} generates the ten-dimensional SUGRA spectrum given the following parity assignments :

$$\mathbb{Z}_2 \,:\, x^{11} \rightarrow -x^{11} \qquad G \rightarrow G \qquad C \rightarrow -C. \tag{5.29}$$

The surviving components then are :

$$\begin{array}{lcll} G_{AB} & : & G_{MN} & \rightarrow & G_{MN} \\ & & G_{M11} & \rightarrow & \text{nothing} \\ & & G_{11\,11} & \rightarrow & \Phi \\ \\ C_{ABC} & : & C_{MNP} & \rightarrow & \text{nothing} \\ & & C_{MN11} & \rightarrow & B_{MN} \end{array} \tag{5.30}$$

which effectively coincides with the ten-dimensional SUGRA multiplet derived in Appendix C.2. Moreover, since the gauge fields A_M^X live on the two ten-dimensional branes located at the orbifold's fixed points, they are unaffected by the projection and we recover the Heterotic effective theory bosonic spectrum, see Table 5.2.

The further compactification of the six remaining internal dimensions is the subject of the next Chapter but we can already anticipate the fact that if we desire that the four-dimensional effective theory is an $\mathcal{N} = 1$ theory then the manifold X on which the compactification is to be performed has to be chosen such that it kills three-quarters of the supercharges.

Let us briefly sketch what the situation would be if we first had to compactify the six dimensions on the branes. This situation will be investigated in Section 7.5 since this requires some knowledge about the manifold upon which the compactification is to be performed. The result will be shown to be a $\mathcal{N} = 1$ five-dimensional theory if the compactification manifold is chosen to be X, which is an $\mathcal{N} = 2$ theory from the four-dimensional point of view. Finally the $\mathbb{Z}_2$ projection kills half the supercharges leading to an $\mathcal{N} = 1$ theory in four dimensions. We can also already anticipate the fact that in the resulting five-dimensional theory, in contradistinction to pure five-dimensional SUGRA, not only the metric and the graviphoton propagate in the bulk but also a number of vector multiplets and hypermultiplets coming from the internal components of both the metric and the 3-form C, as pictured on Figure 1.3.

Chapter 6

Compactification

In Chapter 5 we have introduced Superstring Theory and in particular the Heterotic $E_8 \times E_8$ Theory to which we will now devote all our attention since it represents a plausible framework in which sequestering can be put at work. In this Chapter we will first revisit the Kaluza-Klein compactification of a single extra-dimension on a circle. Extending this to more dimensions will lead us to discuss toroidal compactifications.

The manifold on which the compactification is to be performed may be used to reduce the high degree of Supersymmetry of the microscopic theory. Since the Heterotic String is a $\mathcal{N} = 1$ theory in ten dimensions, it contains 16 supercharges which would generate an extended $\mathcal{N} = 4$ theory in the effective four-dimensional theory if the compactification manifold is chosen to be flat, *e.g.* in the case of a toroidal compactification.

We start this Chapter by a brief discussion of Kaluza-Klein compactifications, we will then look for manifolds which when the Heterotic 10-dimensional action is compactified upon have the effect of killing some or all of the Supersymmetry. Both singular and smooth manifolds are discussed.

6.1 A Kaluza-Klein Warm-Up

6.1.1 Quantum Mechanical Example

In order to grasp the essential features of the compactification procedure, let us introduce a two-dimensional quantum-mechanical example. A particle is assumed to be moving on a cylinder of length L and radius $R \ll L$. The solution to the Schrödinger equation is given by :

$$\psi(x,y)_{m,n} \propto \sin\left(\frac{mx}{L}\right)\left[\sin\left(\frac{ny}{R}\right) + \alpha\cos\left(\frac{ny}{R}\right)\right] \tag{6.1}$$

where $m \geq 1$ and $n \in \mathbb{Z}$. The energy levels are given by :

$$E^{\text{2D}}_{m,n} = \frac{\hbar^2}{2m}\left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n}{R}\right)^2\right]. \tag{6.2}$$

On the other hand, the energy levels of a particle on a segment of length L are given by :

$$E_m^{1\mathrm{D}} = \frac{\hbar^2}{2m}\left(\frac{m\pi}{L}\right)^2 \tag{6.3}$$

with $m \geq 1$. The smallest new energy level compared to the case of a particle on a segment is thus :

$$E_{1,1}^{2\mathrm{D}} = \frac{\hbar^2}{2m}\left[\left(\frac{\pi}{L}\right)^2 + \left(\frac{1}{R}\right)^2\right] \simeq \frac{\hbar^2}{2m}\left(\frac{\bar{m}\pi}{L}\right)^2 = E_{\bar{m}}^{1\mathrm{D}} \tag{6.4}$$

which corresponds to the level $\bar{m} \sim L/(\pi R) \gg 1$ of the particle on a segment. We thus conclude from this simple example that a small compact extra-dimension can be hidden provided its size is such that the energy level characterised by $\bar{m}$ is not accessible to present experiments.

6.1.2 Implementation in Quantum Field Theory

Let us now consider a massive scalar field propagating in five space-time dimensions where the fifth dimension is compact, *i.e.* we identify $y \sim y + 2\pi R$. The scalar field $\phi(x, y)$ may be expanded in Fourier modes compatible with the boundary condition :

$$\phi(x,y) = \frac{1}{\sqrt{2\pi R}} \sum_{n\in\mathbb{Z}} \phi_n(x) \exp\left(i\frac{n}{R}y\right) \tag{6.5}$$

which when acted upon with the five-dimensional Klein-Gordon operator $(\Box_5 - m^2)$ yields :

$$\Box_4 \phi_n(x) = \left[m^2 + \left(\frac{n}{R}\right)^2\right]\phi_n(x). \tag{6.6}$$

A compact dimension thus manifests itself by a tower of excitations with increasing masses, called the Kaluza-Klein (KK) tower. One may also directly replace (6.5) in the action :

$$\mathcal{S} = \frac{1}{2}\int d^5x \left(-\partial_M\phi\partial^M\phi - m^2\phi^2\right) = \frac{1}{2}\sum_n \int d^4x \left(-\partial_\mu\phi_n\partial^\mu\phi_n - \left(m^2 + \frac{n^2}{R^2}\right)\phi_n^2\right). \tag{6.7}$$

In the context of String Theory, we have already discarded the massive microscopic excitations since their mass is proportional to the inverse string length and have thus effectively set m^2 to zero in the previous two equations. The massless spectrum of String Theory thus appears as pure KK towers in four dimensions. The effective four-dimensional massless spectrum thus consists in the $n = 0$ mode, *i.e.* the mode which is annihilated by the internal part of the Klein-Gordon operator. In this example, this procedure amounts to replacing $\phi(x, y)$ by $\phi_0(x) = \phi(x, 0)$ in the action.

6.1.3 Kaluza-Klein Mechanism

We have seen in the previous subsection that the effective massless spectrum is the one annihilated by the internal part of the wave operator. Now let us investigate with another example what happens in a situation where the field carries a Lorentz structure.

Kaluza proposed in [103] that the four observed space-time dimensions be supplemented by a fifth one in order to unify the description of General Relativity and Quantum Electro-Dynamics (QED). Kaluza's idea was to identify the vector which appears in the decomposition of the 5-dimensional metric g_{MN} into $g_{\mu\nu}$, $g_{5\mu}$ and g_{55} to the Abelian vector potential of QED. More precisely the 5-dimensional metric is written as :

$$g_{MN} = (-\phi)^{-1/3} \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix} \tag{6.8}$$

This parametrisation permits to identify $g_{\mu\nu}$ as the 4-dimensional metric, see [104] for a review. The gravitational action for g_{MN} is the Einstein-Hilbert action. When replacing the components of the metric by (6.8), the action truncated to its zero-modes reads :

$$\mathcal{S} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(R + \frac{1}{4} \phi F_{\mu\nu} F^{\mu\nu} - \frac{1}{6\phi^2} \partial_\mu \phi \partial^\mu \phi \right) \tag{6.9}$$

which when ϕ is set to -1 precisely boils down to QED minimally coupled to GR.

6.2 Orbifolds

In last section examples, the y coordinate has been integrated on the $\mathbb{S}^1$ covering space which consists of the closed interval $[-\pi R, \pi R]$ with identified boundaries. In order to further reduce the symmetries, and therefore the field content, of a compactified theory, one may impose a set of discrete symmetries by defining a discrete subgroup Γ of the Poincaré group and then only retain the part of the spectrum which is invariant under Γ. As an example, let us consider the Kaluza setup with $\Gamma = \mathbb{Z}_2$ acting on the fifth coordinate. The parity assignments should be such that the Lagrangian density is invariant under Γ, which in our case acts as reflections on the $\mathbb{S}^1$ covering space. A direct inspection of the five-dimensional ds^2 line element yields :

$$P(g_{\mu\nu}) = +1 \qquad P(A_\mu) = -1 \qquad P(\phi) = +1 \tag{6.10}$$

where P denotes the parity of the argument under $\mathbb{Z}_2$. The physical states are the ones which are even under $\mathbb{Z}_2$, the orbifold thus projects out the A_μ gauge field and yields the following effective four-dimensional action :

$$\mathcal{S} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(R - \frac{1}{6\phi^2} \partial_\mu \phi \partial^\mu \phi \right) \tag{6.11}$$

which describes the dynamics of a scalar field $\sigma \equiv (6\kappa)^{-1/2} \ln(-\phi)$ minimally coupled to gravity. The orbifold projection has thus effectively killed the gauge symmetry of (6.9) and its associated gauge field. The same $\mathbb{S}^1/\mathbb{Z}_2$ projection is responsible for the fact that $\mathcal{M}$-Theory compactified on a circle results in an $\mathcal{N} = 2$ theory (IIA Superstring) while when compactified on an orbifold it leads to an $\mathcal{N} = 1$ theory ($E_8 \times E_8$ Heterotic). Inspired by these examples, one may devise orbifold projections responsible for leaving the four-dimensional effective theory with only one Supersymmetry instead of the four it would get were it to be compactified on a six-dimensional torus. To achieve this scenario, we not only have to know how the discrete Γ group acts on tensors but also on the spinors generating SUSY.

6.2.1 Orbifold Construction

The action of the discrete group Γ, called the space group, on the six extra-dimensions collected in a vector X is given by :

$$g = (\theta, v) \in \Gamma \qquad \Gamma : X \mapsto gX = \theta X + v. \tag{6.12}$$

We restrict the discussion to Abelian orbifolds, *i.e.* the θ which should a priori belong to $SO(6)$ are restricted to obey trivial commutation relations. We can thus pick J_{45}, J_{67} and J_{89} to be the orthonormal generators of the Cartan group of $SO(6)$ in terms of which the θ's assume the following form [105] :

$$\theta = \theta(\phi_1, \phi_2, \phi_3) = \exp[2\pi i(\phi_1 J_{45} + \phi_2 J_{67} + \phi_3 J_{89})]. \tag{6.13}$$

The action of θ on the vector X is more conveniently written in terms of complexified variables $Z^i \equiv X^{2i+2} + iX^{2i+3}$, with $i \in \{1, 2, 3\}$, as can be checked by using the explicit form for the J_{ij} generators :

$$\theta Z^i = \exp(2\pi i\phi_i)Z^i. \tag{6.14}$$

The orbifold Ω is defined by the coset of Euclidean space $\mathbb{R}^6$ divided by the space group Γ :

$$\Omega = \mathbb{R}^6/\Gamma \qquad i.e. \qquad X \sim gX \tag{6.15}$$

leading to identified points along the orbit generated by the discrete group Γ. Another way to form the orbifold is to first divide the Euclidean space by the subgroup Λ containing the shifts $(\mathbb{1}, v)$ of Γ leading to a six-dimensional torus $T^6 = \mathbb{R}^6/\Lambda$ and then to identify points related through the action of $\bar{P} = \Gamma/\Lambda$. Note that $\bar{P}$ differs from the point group P containing the elements of the form $(\theta, 0)$ since elements of $\bar{P}$ may also involve shifts. The point group is easily shown to be the orbifold holonomy group. The orbifold is thus obtained as :

$$\Omega = T^6/\bar{P}. \tag{6.16}$$

The orbifold action is lifted to act on the worldsheet fermions λ^A which, in the $E_8 \times E_8$ Heterotic String, are divided in two sets of 16 which enjoy different boundary conditions. The orbifold action can be taken to be diagonal when applied on the complexified worldsheet fermions $\lambda_+^A \equiv \lambda^{2A-1} + i\lambda^{2A}$:

$$g\lambda_+^A = \exp(2\pi i\alpha_A)\lambda_+^A. \tag{6.17}$$

The orbifold is said to be of order N if the latter is the smallest integer such that $g^N = 1$ and is, in such a case, denoted by $\mathbb{Z}_N$. The transformations (6.14) and (6.17) imply that the ϕ_i's and α_A's can be written as integers divided by N. The transformation of spinors further constrains the charges under the orbifold action. Indeed the spinors transform as :

$$\Psi_{\vec{s}} \to \exp(2\pi i\vec{s} \cdot \vec{\phi})\Psi_{\vec{s}} \tag{6.18}$$

where $\vec{\phi}$ is equal to $(0, \phi_1, \phi_2, \phi_3)$ for the $SO(8)$ fermions while it is a 16-component vector for the $SO(16) \times SO(16)$ fermions and where the $\vec{s}$ vector is the spinor weight vector which indicates which creation operators created from the Clifford Γ-matrices are to be applied on the Clifford vacuum to generate the spinor under consideration [106]. The weight vector is thus of the form :

$$\vec{s} = \frac{1}{2}(\pm 1, \ldots, \pm 1) = \frac{1}{2}\vec{\eta}. \tag{6.19}$$

If we now take the N-th power of g acting on spinors and setting $\vec{\phi} = \vec{n}/N$ where the n's are integers, we get the following condition :

$$g^N = \exp(\pi i \vec{\eta} \cdot \vec{n}) = 1 \qquad \rightarrow \qquad \vec{\eta} \cdot \vec{n} = 2m \tag{6.20}$$

where m is an integer. Since this last condition has to be fulfilled by all combinations of vectors $\vec{\eta}$, it is sufficient to impose it for $\vec{\eta} = (+1, \ldots, +1)$ since flipping one sign will change the sum by two units and thus also satisfy the condition. We have thus found that we must impose :

$$\sum_i n_i = 0 \mod 2 \tag{6.21}$$

where $\vec{n}$ generically stands for the twist vectors of the $SO(8)$ fermions and the ones of each of the two sets of $SO(16)$ fermions. Another condition based on modular invariance has to be imposed [105] and when combined with the mod 2 constraint we have derived yields :

$$\sum_{i=1}^{3} a_i^2 - \sum_{i=1}^{16} b_i^2 = 0 \mod 2N \tag{6.22}$$

where $\vec{a}$ and $\vec{b}$ respectively are the $\vec{n}$ vectors of the $SO(8)$ fermions and of the $SO(16) \times SO(16)$ fermions. Furthermore the equation (6.18) indicates that if the $\vec{a}$ vector is chosen to satisfy :

$$\sum_{i=1}^{3} a_i = 0 \qquad i.e. \qquad P \subset SU(3) \subset SO(6), \quad P \not\subset SU(2) \tag{6.23}$$

then twelve of the sixteen supercharges will be broken by the orbifold action. The fact that P belongs to $SU(3)$ is most easily seen when considering its action on the complexified coordinates Z. This choice leads to an $\mathcal{N} = 1$ theory in four dimensions. If P is chosen to belong to $SU(2)$, only half the supercharges are broken by the orbifold action, leading to $\mathcal{N} = 2$ in the compactified theory.

To break or not to break, or what to break to? The choice of the $\vec{a}$ vector, and thereby of the point group, determines the amount of unbroken Supersymmetry resulting from the orbifold compactification. If the point group is chosen to be trivial, the compactification will result in an $\mathcal{N} = 4$ theory in four dimensions since $\vec{a} = 0$ generates nothing but a toroidal compactification. If one of the $\vec{a}$ component is chosen to be zero while the two others generate an $SU(2)$ point group, then the resulting theory will have two Supersymmetries in four dimensions since eight of the supercharges would be invariant under the point group. Finally, the choice (6.23) ensures that the resulting four-dimensional theory is characterised by a single Supersymmetry. The fact that SUSY has not been observed in any experiment should be an indication towards a choice of $\vec{a}$ leaving no supercharge unbroken. However, space-time SUSY is desirable to hold at energies above the TeV scale and then to be spontaneously broken since in that way it may solve the Hierarchy Problem and furthermore ensures that String Theory is both finite and tachyon-free since it leads to an enhanced worldsheet symmetry [16]. One is thus tempted to leave some amount of unbroken SUSY. The question of how much of it is solved by noticing that only $\mathcal{N} = 1$ theories admit chiral couplings. We will thus concentrate on orbifold compactifications whose point group is a maximal-rank subgroup of $SU(3)$.

6.2.2 Twisted and Untwisted Sectors

A peculiarity of orbifold compactifications is the emergence of a new kind of closed strings : the *twisted* sector. These strings are open strings before the orbifold identification is performed and close only as a result of the identification of X with its g-induced orbit. Let us illustrate this in the simple orbifold $\mathbb{C}/\mathbb{Z}_2$ in which the $\mathbb{Z}_2$ acts as $z \sim -z$. Since points in the lower half-plane are identified with points in the upper half-plane, the orbifold consists of the latter where the points on the real axis are identified according to $x \sim -x$, *i.e.* it forms a cone with a singularity located at the fixed-point of the group action, that is at $z = 0$.

Untwisted Sector The untwisted sector consists of strings which are invariant under the group action. These are constructed by linearly combining strings which are already closed in $\mathbb{C}$.

Twisted Sector Let us now imagine a open string solution to the equations of motion. Of course, if one considers a theory of closed strings, such a state will not be admitted in the spectrum. However if the ends of the open string sit at, say, a and $-a$ on the real axis then the string will be closed once the orbifold identification is performed, *i.e.* when the cone is folded.

Importance of the twisted sector The twisted sector may first be thought of as a peculiarity of orbifold compactifications. However this sector proves to be essential in order to preserve modular invariance and thus the consistency of the theory [93]. In different words, the states arising from the twisted sector are necessary if one is to recover the spectrum obtained from smooth compactifications when *blowing up* the orbifold singularities, see [107]. For simplicity, we will concentrate on the untwisted sector.

Case	Point Group $P = \mathbb{Z}_N$	$\vec{a}$ twist	Commutant H
(**a**)	$\mathbb{Z}_3$	$(1, 1, -2)$	$SU(3)$
(**b**)	$\mathbb{Z}_6$	$(1, 1, -2)$	$SU(2) \times U(1)$
(**c**)	$\mathbb{Z}_7$	$(1, 2, -3)$	$U(1) \times U(1)$

Table 6.1: Point groups and their commutants in $SU(3)$

6.2.3 Spectrum

The spectrum determination is straightforward for the fields related to the metric G_{MN}, the antisymmetric tensor B_{MN} and dilaton Φ. We will consider three possibilities for the point group for which we indicate both the corresponding $\vec{a}$ vector and their commutant H in $SU(3)$ in Table 6.1.

Other choices are possible but lead to non-hermitian metrics in Z-space [32] which we choose to discard for simplicity. The twist vectors satisfy the mod 2 condition (6.21) and ensure that the effective theory will have $\mathcal{N} = 1$ SUSY since the corresponding θ matrices all are elements of $SU(3)$. Under the various transformations compiled in Table 6.1 the complexified coordinates transform as :

$$Z^i \to \exp\left(2\pi i \frac{a_i}{N}\right) Z^i \tag{6.24}$$

which respectively lead to the following bilinear invariants :

$$\begin{array}{lll} (\mathbf{a}) & Z^i \bar{Z}^j & \forall\, i, j \in \{1, 2, 3\} \\ (\mathbf{b}) & Z^i \bar{Z}^j, Z^3 \bar{Z}^3 & \forall\, i, j \in \{1, 2\} \\ (\mathbf{c}) & Z^i \bar{Z}^i & \forall\, i \in \{1, 2, 3\} \end{array} \tag{6.25}$$

leading to the following spectrum :

$$\begin{array}{lll} (\mathbf{a}) & G_{i\bar{\jmath}},\, B_{i\bar{\jmath}} & \forall\, i, j \in \{1, 2, 3\} \\ (\mathbf{b}) & G_{i\bar{\jmath}},\, B_{i\bar{\jmath}},\, G_{3\bar{3}},\, B_{3\bar{3}} & \forall\, i, j \in \{1, 2\} \\ (\mathbf{c}) & G_{i\bar{\imath}},\, B_{i\bar{\imath}} & \forall\, i \in \{1, 2, 3\} \end{array} \tag{6.26}$$

to which should be added the dilaton Φ, and the four-dimensional space-time components of both G and B : $G_{\mu\nu}$ and $B_{\mu\nu}$.

Let us at this point introduce the following notation which will be used in the more general context of Calabi-Yau compactifications too. The number of bilinears of the form $Z^i \bar{Z}^j$ which are preserved by the orbifold projection defines the $h^{1,1}$ Hodge number while the number of bilinears of the form $Z^i Z^j$ which are preserved by the orbifold projection is called the $h^{2,1}$ Hodge number. The Hodge numbers of the three cases under consideration are thus respectively given by $h^{1,1} = 9,\, 5,\, 3$ and $h^{2,1} = 0$.[1]

In order to determine which of the gauge fields survive the orbifold projection, one has to specify its action encoded by the $\vec{b}$ vector. The choice of $\vec{b}$ is restricted by both the mod 2 condition (6.21) and the level matching condition (6.22). The most common choice in the literature is to choose the three first components of $\vec{b}$ to be equal to the three components of $\vec{a}$. This is known as the *Standard Embedding* :

$$\vec{b} = (a_1, a_2, a_3, 0^5; 0^8). \tag{6.27}$$

Since this procedure treats the first three components of $\vec{b}$ on another level, it is useful to decompose the vector representation of $E_8 \times E_8$ with respect to $SU(3) \times E_6 \times E_8$:

$$\mathbf{496} \to (\mathbf{8}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{27}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \overline{\mathbf{27}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{248}). \tag{6.28}$$

The vector fields A_M^X of the first E_8 factor is accordingly decomposed as :

$$A_M^X = \{A_M^a,\, A_M^\alpha,\, A_M^{ix},\, A_M^{\bar{\imath}\bar{x}}\} \tag{6.29}$$

where a is an adjoint $SU(3)$ index, α an adjoint E_6 index and (ix) a bi-fundamental $SU(3) \times E_6$ index. The fields surviving the projection are those which are left invariant under the combined action of $\vec{a}$ and $\vec{b}$. In the **(a)** case, one finds :

$$\begin{array}{cccccc} A_M^X \to & A_M^a & A_M^\alpha & A_M^{ix} & A_M^{\bar{\imath}\bar{x}} \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & A_\mu^a & A_\mu^\alpha & A_{\bar{\jmath}}^{ix} & A_j^{\bar{\imath}\bar{x}} \end{array} \tag{6.30}$$

while for the (**b**) and (**c**) cases, the $SU(3)$ indices suffer the same restriction as in (6.25) and the a index is respectively restricted to the commutant of $\mathbb{Z}_6$ and $\mathbb{Z}_7$ in $SU(3)$. The four-dimensional gauge group is thus found to be given by $H \times E_6 \times E_8$. The last remaining task to obtain the effective four-dimensional theory is to compactify the Heterotic Superstring action (5.24) on a six-torus throwing away the fields which are killed by the orbifold projection. This will be the subject of Chapter 7.

6.3 Calabi-Yau Manifolds

In the previous section we have considered orbifold compactifications which are nothing but toroidal compactifications only retaining a restricted spectrum determined by the orbifold point group. The orbifold in fact represents a subset of the possible manifolds on which String Theory

[1]Strictly speaking, the Hodge numbers are defined on smooth manifolds which can be obtained by blowing up orbifolds. The Hodge numbers we encounter in this section are thus understood as being given by the true Hodge number minus the Hodge number given by the blowing up moduli, *i.e.* we ignore the twisted sector moduli.

can be compactified. We now introduce smooth manifolds which when compactified upon only permits $\mathcal{N} = 1$ SUSY to remain unbroken in four dimensions. The ten-dimensional SUSY is generated by a **16** Majorana-Weyl fermion of $SO(1,9)$ which decomposes as follows under $SO(1,9) \to SO(1,3) \times SO(6)$:

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \tag{6.31}$$

which generate $\mathcal{N} = 4$ in four space-time dimensions, since $\mathcal{N} = 1$ is generated by a pair of Weyl spinors Q_α and $\bar{Q}_{\dot{\alpha}}$, respectively in the **2** and $\bar{\mathbf{2}}$ of $SO(1,3)$. Moreover in order for some Supersymmetry to be preserved, the corresponding SUSY variation of the fermionic fields given by (5.28) should vanish as has been argued in Chapter 3. The variation of the bosonic fields automatically vanishes since fermions cannot handle taking vacuum expectation values.

6.3.1 Zero Torsion

Let us first investigate the simple case in which the 3-form H vanishes and the dilaton Φ is constant, *i.e.* $H = \mathrm{d}\Phi = 0$, following [108]. The background metric is assumed to take the following form :

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + G_{mn}(y) dy^m dy^n \tag{6.32}$$

where η is the Minkowski metric. Under these assumptions, the Killing equation coming from the gravitino variation in (5.28) is given by :

$$\delta_\epsilon \Psi_M = \nabla_M \epsilon = 0. \tag{6.33}$$

Unbroken $\mathcal{N} = 1$ SUSY implies that one and only one such spinor exists. Since $\epsilon = \alpha(x) \otimes \beta(y)$ where x and y respectively are the space-time and internal coordinates, the previous equation implies that both $\nabla_\mu \alpha$ and $\nabla_m \beta$ vanish. The β spinor thus has to be covariantly constant, *i.e.* it has to remain unchanged after being parallel transported around a closed curve on the internal manifold. In other words, β has to be a singlet under the holonomy group $\mathcal{H}$ of the six-dimensional manifold which is to be contained in $SO(6) \simeq SU(4)$. Since under $SU(3)$ the **4** decomposes into a triplet and a singlet : $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$, a natural candidate for the holonomy is $\mathcal{H} = SU(3)$. On such manifolds there is one covariantly constant spinor of positive chirality and one of negative chirality, which we denote by $\beta_\pm$ and which transform as $(\mathbf{2}, \mathbf{1})$ and $(\bar{\mathbf{2}}, \mathbf{1})$ under $SO(1,3) \times SU(3)$. Note that the same mechanism is at work in the orbifold case where the point group P, which is the orbifold holonomy group, has to belong to $SU(3)$ in order to ensure the breaking of twelve of the sixteen supercharges.

Had we chosen $\mathcal{H}$ to be $SU(2)$ there would have been two right-handed and two left-handed covariantly constant spinors since under $SU(2)$ the **4** decomposes into a doublet and two singlets leading to $\mathcal{N} = 2$. There could be as many as four covariantly constant spinors of each chirality as occurs when the manifold is a flat six-dimensional torus T^6 which has a trivial holonomy. The ϵ spinor generating SUSY may finally be written as :

$$\epsilon(x,y) = \alpha_+(x) \otimes \beta_+(y) + \alpha_-(x) \otimes \beta_-(y) \tag{6.34}$$

where $\alpha_\pm(x)$ are two two-component Weyl spinors in $SO(1,3)$. Note that since ϵ is a Majorana spinor, we have $\alpha_-^* = \alpha_+$ and $\beta_-^* = \beta_+$. The $SO(6)$ spinors may be used to define an almost complex structure $\mathcal{J}$:

$$\mathcal{J}_m{}^n = i\beta_+^\dagger \Gamma_m{}^n \beta_+ \tag{6.35}$$

which can be checked to indeed obey $\mathcal{J}^2 = -1$ and which shares the covariant constancy of β_+ implying that the associated Nijenhuis tensor :

$$N^k_{mn} = \mathcal{J}_p{}^k \partial_{[m}\mathcal{J}_{n]}{}^p + \mathcal{J}_n{}^p \partial_p \mathcal{J}_m{}^k - \mathcal{J}_m{}^p \partial_p \mathcal{J}_n{}^k \tag{6.36}$$

vanishes which in turn leads to the fact that the compactification manifold is complex [62] and thus admits an Hermitian metric. Furthermore the fact that the almost complex structure is covariantly constant implies that the Kähler form whose components are given by :

$$J_{i\bar{j}} = \mathcal{J}_i{}^k g_{k\bar{j}} = i g_{i\bar{j}} \tag{6.37}$$

is closed : $\mathrm{d}J = 0$. The compactification manifold is thus not only complex but also Kähler. Note that Kähler manifolds do not admit torsion.

Moreover the covariant constancy condition may be iterated, leading to the integrability condition :

$$[\nabla_m, \nabla_n]\beta = \frac{1}{4} R_{mnpq} \Gamma^{pq} \beta = 0 \tag{6.38}$$

where Γ^{pq} is the antisymmetrised product of Clifford Γ matrices. By using the symmetry properties of R_{mnpq} and the Clifford algebra, one can show that this condition is equivalent to imposing that the internal manifold should be Ricci-flat : $R_{mn} = 0$, which is equivalent to say that the internal manifold has a vanishing first Chern class [62]. An alternative route to reach this conclusion is by considering the following $(3,0)$-form :

$$\Omega_{mnp} = \beta_+^T \Gamma_{mnp} \beta_+ . \tag{6.39}$$

It is easily shown that Ω is closed and holomorphic, *i.e.* $\mathrm{d}\Omega = 0$, while it is not exact. As it will become clear in the following sections, the compactification manifolds only admit one $(3,0)$-form cohomology, Ω is thus its representative. Since the compactification manifold has complex dimension three, Ω can be written as :

$$\Omega = \frac{1}{3!}\Omega(z)\epsilon_{mnp} dz^m \wedge dz^n \wedge dz^p \quad \rightarrow \quad ||\Omega||^2 \equiv \frac{1}{3!}\Omega_{mnp}\bar{\Omega}^{mnp} = |\Omega(z)|^2 \det(g^{m\bar{n}}) \tag{6.40}$$

leading to the following expression for the Ricci form, see Appendix B :

$$R = -i\partial\bar{\partial}\log\det(g_{m\bar{n}}) = i\partial\bar{\partial}\log||\Omega||^2 = -\frac{i}{2}\mathrm{d}(\partial - \bar{\partial})\log||\Omega||^2 \tag{6.41}$$

from which we conclude that, since $||\Omega||$ is globally defined, R is exact leading to a vanishing first Chern class as advertised. Kähler manifolds with vanishing first Chern class are called *Calabi-Yau manifolds*.

A final restriction on the way the compactification is to be performed comes from the gaugino variation in (5.28) which in order to vanish when acting on the spinor generating $\mathcal{N} = 1$ in four dimensions imposes that the vector bundle should be both holomorphic and stable :

$$F^A_{ij} = F^A_{\bar{\imath}\bar{\jmath}} = 0 \qquad \text{and} \qquad G^{i\bar{\jmath}}F^A_{i\bar{\jmath}} = 0 \tag{6.42}$$

where we have used internal complex indices $m \to i, \bar{\imath}$ since the manifold upon which the compactification is to be performed is a complex one. The existence of such vector bundles is guaranteed by the Uhlenbeck-Yau theorem [109].

To summarise, imposing the fermion's SUSY-variations to vanish for a single spinor under the assumptions that $H = \mathrm{d}\Phi = 0$ has two consequences :

- ◇ The internal manifold X has to be a Calabi-Yau manifold with $SU(3)$ holonomy,
- ◇ The vector bundle has to be stable and holomorphic.

Note that since $H = 0$, the Bianchi identity (5.26) implies that $\mathrm{Tr}\,(R \wedge R) = \mathrm{Tr}\,(F \wedge F)$ which is obeyed if the spin connection is embedded in the gauge group, *i.e.* the gauge and tangent bundles are identified, as will be discussed in the next sections. The embedding of the spin connection in the gauge connection goes under the name of *Standard Embedding*.

6.3.2 Non-Zero Torsion

We may now wonder how the situation is changed if the simplifying assumptions $H = \mathrm{d}\Phi = 0$ which were discussed in the previous subsection are abandoned [110]. First the Standard Embedding does not solve the Bianchi identity (5.26) anymore. Second the compactification manifold is not Kähler anymore but rather semi-Kähler [108] which seems to forbid the Calabi-Yau solution. Indeed since the variation of the gravitino in (5.28) now contains a contribution coming from the non-zero background value of H, which is identified with a Bismut torsion term [111] :

$$\delta_\epsilon \Psi_M = \nabla_M \epsilon - \frac{1}{8} H_{MNP}\Gamma^{NP}\epsilon \equiv \nabla^{(T)}\epsilon \tag{6.43}$$

the almost complex structure defined out of the spinor which are covariantly constant with respect to $\nabla^{(T)}$ satisfies $\nabla^{(T)}_m \mathcal{J}_n{}^p = 0$ but the 2-form defined out of it by lowering an index with the metric does not obey $\mathrm{d}J = 0$ anymore, precisely because of the torsion term, leading to a non-Kähler but still Hermitian internal manifold since the Nijenhuis tensor again vanishes.

Moreover even though the background is no longer Kähler, it still satisfies $\mathrm{d}(e^{-2\Phi}J\wedge J)=0$, *i.e.* it is conformally balanced [112]. As a consequence of $\nabla^{(T)}J=0$, the 3-form H is expressed as :

$$H = i(\partial - \bar{\partial})J. \tag{6.44}$$

Since J is not closed anymore, one calls it the fundamental form instead of the Kähler form is the torsionless case. Moreover since it is again possible to define a covariantly constant holomorphic 3-form Ω_{mnp} thanks to the dilatino condition in (5.28) [110] :

$$\Omega_{mnp} = e^{-2\Phi}\beta_+^T\Gamma_{mnp}\beta_+ \tag{6.45}$$

the internal manifold has vanishing first Chern class and thus has $SU(3)$ holonomy with respect to $\nabla^{(T)}$.

On the other hand, the gaugino variation in (5.28) depends neither on H nor on $\mathrm{d}\Phi$ and thus leads to the same requirement of a stable and holomorphic gauge bundle.

To summarise, imposing the fermion's SUSY-variations to vanish for a single spinor without the assumptions that $H=\mathrm{d}\Phi=0$ has two consequences :

- ⋄ The internal manifold X has to be a conformally balanced Hermitian manifold with $SU(3)$ holonomy,
- ⋄ The vector bundle has to be stable and holomorphic.

Using (6.44), the Bianchi identity (5.26) takes the following form :

$$i\partial\bar{\partial}J = \frac{\ell_s^2}{8}\Big(\mathrm{Tr}\,(R\wedge R) - \mathrm{Tr}\,(F\wedge F)\Big). \tag{6.46}$$

The compactification with torsion, also called flux compactifications, are a very active area of research since they induce a non-trivial superpotential which allows for the stabilisation of some of the moduli fields present in String effective theories. However the question of determining the low-energy spectrum is by far a more involved and not fully settled procedure compared to the torsionless case. We will thus focus on the latter leaving the case with fluxes for further investigations. See [113–116] for recent discussions of flux compactifications of the Heterotic Superstring.

6.3.3 Standard and General Embeddings

Let us investigate the consequences of the $H=\mathrm{d}\Phi=0$ assumptions, focusing on how one may solve the Bianchi identity which under the mentioned assumptions reads :

$$\mathrm{d}H = \frac{\ell_s^2}{4}\Big(\mathrm{Tr}\,(R\wedge R) - \mathrm{Tr}\,(F\wedge F)\Big) = 0 \tag{6.47}$$

where R is the field-strength associated with the spin connection ω on the tangent bundle TX and F is the field-strength associated with the gauge connection A on the vector bundle V.

Standard Embedding An economic way to solve both the above equation and the requirement for the gauge bundle to be stable is to identify a subset of the gauge connection A to the spin connection viewed as a gauge field of the holonomy group $\mathcal{H} = SU(3)$ of X. This procedures embeds the spin connection in the gauge group, hence its name. Let us pick the gauge group $SU(3)$ factor in the first E_8, leading to the following gauge group decomposition :

$$\mathbf{496} \to (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{78},\mathbf{1}) \oplus (\mathbf{3},\mathbf{27},\mathbf{1}) \oplus (\overline{\mathbf{3}},\overline{\mathbf{27}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{248}). \tag{6.48}$$

The four-dimensional gauge group is thus identified with the subgroup which commutes with $\mathcal{H}$, *i.e.* $E_6 \times E_8$ which are respectively the second and fifth term of the above decomposition. In terms of bundles, the bundle V for which the gauge fields are connections is taken to be the tangent bundle of X denoted by TX. We will investigate the spectrum descending from the Standard Embedding choice in the following subsections. Let us before consider the perturbative stability of the Calabi-Yau solution, leading to the introduction of more general embeddings, *i.e.* embeddings which do not limit the effective four-dimensional gauge group to be $E_6 \times E_8$.

Perturbative Stability As already mentioned, the String action (5.24), the SUSY variations (5.28) and the Bianchi identity (5.26) will receive further stringy corrections controlled by the string length ℓ_s. One may then worry about the stability of the above-mentioned assumptions, namely $H = \mathrm{d}\Phi = 0$, when those effects are taken into account. In particular, one should ask whether the ℓ_s corrections still allow the background to be a Calabi-Yau manifold or not.

This question has been studied in a slightly different context [117]. Indeed the requirement for the gauge connection to be identified with the spin connection leads the unbroken four-dimensional gauge group to be E_6, *i.e.* to be the commutant of the holonomy of the Calabi-Yau manifold since the latter is identified with the structure group of the vector bundle. One may then wonder whether there exist different embeddings which lead to other gauge groups such as $SO(10)$ or $SU(5)$ which are desirable GUT groups which can be broken to the Standard Model gauge group by Wilson lines [93].

The first order deviations in ℓ_s^2 from the Calabi-Yau solution are related among each other since the perturbed quantities have to satisfy the Killing equations. They may thus be written as [118] :

$$\delta G_{i\bar{\jmath}} = \ell_s^2 h_{i\bar{\jmath}} \qquad \delta H_{ij\bar{k}} = -\ell_s^2 \nabla_{[i} h_{j]\bar{k}} \qquad \delta A = \ell_s^2 a \tag{6.49}$$

while the dilaton deviation depends on the gauge choice (diffeomorphisms) for $h_{i\bar{\jmath}}$ and may be set to zero at this order [119]. The δH equation is easily seen to be implied by (6.44). In order for h_{mn} to solve the equations of motion descending from (5.24) at first order in ℓ_s^2 it has to satisfy :

$$\Delta_L h_{mn} = \frac{1}{4}\bigg(\mathrm{Tr}\,(F_{mp}F_n{}^p) - R_{mpqr}R_n{}^{pqr}\bigg) \tag{6.50}$$

where the right-hand side depends on the unperturbed quantities and where Δ_L is the Lichnerowicz operator defined by :

$$R_{mn}(g+\epsilon h) = R_{mn}(g) + \epsilon \Delta_L h_{mn} + \mathcal{O}(\epsilon^2). \tag{6.51}$$

The corrections to the Calabi-Yau metric can thus be expressed as functions of the unperturbed solutions as long as Δ_L is invertible which is indeed the case since all the zero-modes of the Lichnerowicz operator can be recast in the definition of the Calabi-Yau metric [120]. Note that in the Standard Embedding case, $\Delta_L h_{mn}$ vanishes indicating that the Calabi-Yau solution does not receive any perturbative correction. This was already known from Witten's work [117] in which he has shown that the sigma-model beta function, whose vanishing dictates the target-space fields (*i.e.* space-time fields) equations of motion, remains zero to all orders in perturbation theory given that one embeds the spin connection in the gauge connection.

General Embeddings In the case where the spin connection is not embedded in the gauge connection, the ℓ_s corrections do destabilise the zeroth order Calabi-Yau solution precisely by modifying the metric in a way that cannot be recast in the zeroth order metric, leading to a non-Kähler compactification manifold. This in turn induces a non-trivial H through (6.44). Since H is non-vanishing, the gauge bundle structure group is not constrained to be equal to the $SU(3)$ holonomy of the compactification manifold anymore, see [117, 121, 122] (see also more recently [123]). One can for example imagine the vector bundle structure group S to be $SU(4)$ or $SU(5)$ which respectively lead the four-dimensional gauge group G to be $SO(10)$ and $SU(5)$. The $E_8 \times E_8$ adjoint representation then splits as :

$$\mathbf{496} \to (\mathbf{Adj}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{Adj}) \oplus \bigoplus_i (\mathbf{R_i}, \mathbf{r_i}) \tag{6.52}$$

where the $(\mathbf{R_i}, \mathbf{r_i})$ are representations of $G \times S$. For the above mentioned possibilities, one chooses the structure group in the first E_8 and finds [124] :

$$\begin{array}{ccc} G & S & \oplus_i(\mathbf{R_i},\mathbf{r_i}) \\ \hline E_6 \times E_8 & SU(3) & (\mathbf{3},\mathbf{27}) \oplus (\overline{\mathbf{3}},\overline{\mathbf{27}}) \\ SO(10)\times E_8 & SU(4) & (\mathbf{4},\mathbf{16}) \oplus (\overline{\mathbf{4}},\overline{\mathbf{16}}) \oplus (\mathbf{6},\mathbf{10}) \\ SU(5)\times E_8 & SU(5) & (\mathbf{5},\mathbf{10}) \oplus (\overline{\mathbf{5}},\overline{\mathbf{10}}) \oplus (\mathbf{10},\overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}},\mathbf{5}) \end{array} \tag{6.53}$$

6.3.4 Zero-Modes

In the context of orbifold compactification the effective theory massless spectrum did coincide with the dimensional reduction of the ten-dimensional fields : the massless field emerging from $G_{i\bar{\jmath}}(x,y)$ in the untwisted sector was simply given by $G_{i\bar{\jmath}}(x,0)$. In the more general context of Calabi-Yau compactification this will no longer hold true. However the logic remains : the four-dimensional massless spectrum consists of the fields which are annihilated by the internal wave operator. Indeed, the equations of motion admit the following generic form :

$$\hat{\mathcal{O}}\chi = \hat{\mathcal{O}}_{\text{ext}}\chi + \hat{\mathcal{O}}_{\text{int}}\chi = 0. \tag{6.54}$$

Decomposing χ on the $\hat{\mathcal{O}}_{\text{int}}$ eigenbasis ω^A as $\chi = \chi_A\omega^A$ yields :

$$\omega^A\hat{\mathcal{O}}_{\text{ext}}\chi_A + \chi_A\hat{\mathcal{O}}_{\text{int}}\omega^A = \omega^A\left(\hat{\mathcal{O}}_{\text{ext}}\chi_A + \chi_A\lambda^A\right) = 0 \tag{6.55}$$

meaning that the effective theory massless spectrum corresponds to the $\lambda^A = 0$ modes, *i.e.* the modes which are annihilated by $\hat{\mathcal{O}}_{\text{int}}$ which takes the form of a Laplacian when acting on forms. On Kähler manifolds the Laplacian is expressed as $\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}$ where the exterior derivative d may be decomposed as the sum of the holomorphic and antiholomorphic derivatives : $\mathrm{d} = \partial + \bar{\partial}$ and where $\mathrm{d}^\dagger$ is defined as the dual of d using the following scalar product :

$$\langle A, B\rangle = \int A \wedge *\bar{B}. \tag{6.56}$$

Since the Calabi-Yau is in particular a Kähler manifold, the Laplacians constructed from d, ∂ and $\bar{\partial}$ share the same zero-modes [125, 126] :

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{6.57}$$

The determination of the number of massless fields translates into finding the number of independent zero modes the internal Laplacian $\Delta_{\bar{\partial}}$ admits. This is a cohomology problem whose solution depends on the topology of the Calabi-Yau manifold. The number $h^{r,s}$ of independent harmonic forms of bidegree (r, s), which are the Laplacian's zero modes, are given by the following Hodge diamond :

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{6.58}$$

where, since the internal manifold is Kähler, one can relate the various Hodge numbers through complex-conjugation and Hodge-duality [62, 126] :

$$h^{r,s} = h^{s,r} \qquad \text{and} \qquad h^{r,s} = h^{3-r,3-s} \tag{6.59}$$

which, by using the fact that the manifold is Ricci flat, are enough to determine all but two Hodge numbers : $h^{1,1}$ and $h^{2,1}$. Note that basics about complex spaces and complex differential geometry are recorded in Appendix B.

Since general embeddings do not lead Kähler compactification manifolds but rather to conformally balanced Hermitian manifolds, it should be investigated whether the above mentioned procedure used to find the light fields is still valid. The corrections we have considered are perturbative deviations from the Calabi-Yau solution which cannot change the whole picture dramatically since the superpotential can be shown to be unaffected by ℓ_s^2-corrections at the perturbative level [117]. In particular, light fields remain light.

6.3.5 Spectrum

Let us now determine the effective four-dimensional massless spectrum in the case of Calabi-Yau compactifications. In the orbifold case we first determined which components of the metric G, the antisymmetric tensor B and the dilaton Φ were surviving the orbifold projection. In order to derive the spectrum emerging from the ten-dimensional gauge fields we had to specify how the level-matching condition was implemented. Once such a realisation is chosen the untwisted spectrum is easily found. Let us now follow the same strategy in order to determine the effective massless spectrum emerging from the Heterotic Superstring action (5.24).

Let us first split the ten-dimensional Lorentz index M into the four-dimensional Lorentz index μ and the complex internal coordinates i and $\bar{\imath}$. The dilaton Φ is a zero-form and thus gives rise to a single scalar field since $h^{0,0} = 1$. The same is true for each component of both $G_{\mu\nu}$ and $B_{\mu\nu}$ respectively leading to a symmetric tensor and a scalar by dualisation in four dimensions. The mixed components $G_{\mu i}$, $G_{\mu\bar{\jmath}}$, $B_{\mu i}$ and $B_{\mu\bar{\jmath}}$ do not generate any massless four-dimensional field since $h^{1,0} = h^{0,1} = 0$. The B_{ij} and $B_{\bar{\imath}\bar{\jmath}}$ components are $(2,0)$-forms which do not have zero-modes since $h^{2,0} = 0$, whereas G_{ij} and $G_{\bar{\imath}\bar{\jmath}}$ lead to $h^{2,1}$ complex scalar fields since they can be combined with the holomorphic 3-form Ω into $G_{ij}G^{j\bar{m}}\Omega_{\bar{m}\bar{n}\bar{p}}$ which is a $(1,2)$-form. Finally the $G_{i\bar{\jmath}}$ and $B_{i\bar{\jmath}}$ combine into $h^{1,1}$ complex scalar fields. Let us now turn to the gauge fields A_M^X considering both the standard and general embeddings.

Standard Embedding Recall that the $E_8 \times E_8$ adjoint splits into the following when the first E_8 factor is decomposed as $E_8 \to SU(3) \times E_6$:

$$\mathbf{496} \to (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{78},\mathbf{1}) \oplus (\mathbf{3},\mathbf{27},\mathbf{1}) \oplus (\bar{\mathbf{3}},\overline{\mathbf{27}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{248}). \tag{6.60}$$

We will denote the $SU(3)$ adjoint index by a, the E_6 ajoint index by α, and the bifundamental index by ix. In such a case the A_μ^α combine with the last component of the previous sum to generate the four-dimensional $E_6 \times E_8$ gauge fields since they commute with the holonomy group. Note that the gauge group is further enhanced in the orbifold context, where since the holonomy is a discrete subgroup of $SU(3)$ a part of the $SU(3)$ gauge component has a trivial commutator with it and is thus part of the gauge group. This leads to the so-called gauge group enhancement $H \times E_6 \times E_8$ where H is $SU(3)$, $SU(2) \times U(1)$ and $U(1) \times U(1)$ for the $\mathbb{Z}_3$, $\mathbb{Z}_6$ and $\mathbb{Z}_7$ orbifolds we have considered.

The other components of the ten-dimensional gauge field are organised as follows. A_i^a does not lead to $(1,0)$-forms when we consider the Standard Embedding since the a index is an $SU(3)$ adjoint index which qualitatively is the same as having a pair of fundamental times anti-fundamental indices. These fields define the $H^1(\mathrm{End}\, V)$ cohomology which generates E_6 singlets known as gauge bundle moduli which we will ignore for simplicity. Their possible rôle in producing neutrino masses is discussed in [117]. The A_i^α are $(1,0)$-forms taking their values in the E_6 adjoint which do not generate zero-modes since $h^{1,0}=0$. The A_i^{jx} may be seen as $(1,2)$-forms taking their values in the **27** using the same trick we used for the metric components and thus lead to $h^{2,1}$ zero-modes. Finally the $A_i^{\bar{j}\bar{x}}$ are $(1,1)$-forms taking their values in the $\overline{\mathbf{27}}$ which lead to $h^{1,1}$ zero-modes. Note that in this case the net number of generations is given by half the Euler number of the Calabi-Yau X :

$$\chi = \sum_{p,q} (-1)^{p+q} h^{p,q} = 2(h^{1,1} - h^{2,1}). \tag{6.61}$$

General Embeddings In the case of general embeddings, the $E_8 \times E_8$ adjoint is shown to decompose as :

$$\mathbf{496} \rightarrow (\mathbf{Adj}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{Adj}) \oplus \bigoplus_i (\mathbf{R_i}, \mathbf{r_i}) \tag{6.62}$$

under $G \times S$ where the latter is the structure group of V. We may now define a family of vector bundles $V_{\mathbf{r_i}}$ associated with any representation $\mathbf{r_i}$ of S by promoting the transition functions of V, which are matrices in the fundamental representation of S, to the corresponding matrices in the representation $\mathbf{r_i}$ of S. The number of fields transforming in the representation $\mathbf{R_i}$ of the four-dimensional gauge group G is then given by :

$$n_{\mathbf{R_i}} = h^1(X, V_{\mathbf{r_i}}) \tag{6.63}$$

where $h^1(X, V_{\mathbf{r_i}})$ denotes the dimension of the corresponding bundle-valued cohomology group $H^1(X, V_{\mathbf{r_i}})$ [93]. When the structure group is taken to be $SU(3)$ one should recover the Standard Embedding spectrum by taking $V \simeq TX$. According to (6.63), $n_{\mathbf{27}} = h^1(X, V)$ and $n_{\overline{\mathbf{27}}} = h^1(X, V^*)$, leading to $n_{\mathbf{27}} = h^1(X, TX) = h^{2,1}$ and $n_{\overline{\mathbf{27}}} = h^1(X, T^*X) = h^{1,1}$ in the Standard Embedding case, in agreement with the previous paragraph.

Chapter 7

Effective Kähler Potential Calculation

In Chapter 6, we have exposed both singular and smooth compactification manifolds. In order to compute the soft scalar masses, we now have to derive the Kähler potential on which they crucially depend through (4.49). More precisely, the most relevant terms in the Kähler potential are those mixing two visible matter fields and moduli and those mixing two visible matter fields and two hidden matter fields, *i.e.* we have to compute the dependence of the Superspace wave-function Z on the hidden fields. As has been argued in subsection 3.7.2, soft scalar masses are indeed generated whenever Z has a non-vanishing F or D term.

In general the hidden sector tends to contain two subsectors : one to effectively break SUSY and the other to allow a small cosmological constant [30, 127–130]. This subdivision of the hidden sector is a further motivation for the inclusion not only of the moduli but also of the matter fields in our analysis. The soft scalar masses will thus be found to be fed by two contributions : the moduli-mediated effect and the brane-to-brane effect.

The effective Kähler potential describing the low-mass modes coming from the Heterotic Superstring action is thus needed at all orders in the moduli fields since these have sizeable VEVs and at fourth order in the matter fields, since these are assumed to have small VEVs. For orbifolds, the Kähler potential for the untwisted sector is well known and was first derived in [31, 32]. In the case of Calabi-Yau compactifications, the Kähler potential neglecting the matter fields has first been derived in [120, 131] which also contains the Kähler potential for the complex structure moduli which we do not discuss in this work for simplicity. The leading corrections to the Kähler potential that are quadratic in the matter fields were first discussed in [33, 132]. The subleading corrections that are quartic or higher-order in the matter fields are instead more difficult to compute since they correspond to kinetic interactions mixing matter fields and Kähler moduli. The only case in which the full result is known is the case of a single Kähler modulus [133], *i.e.* the $h^{1,1} = 1$ case. A proposal for the all-orders dependence of the Kähler potential in the matter fields for models with arbitrary $h^{1,1}$ has recently appeared in the literature [34]. Here we extend the result of [133] to an arbitrary number of Kähler moduli by performing a direct and systematic computation, following [35, 36]. This computation is done under some assumptions we will discuss and confirms the claim of [34] but clarifies important restrictions on the range of validity of the result.

7.1 Orbifold Compactification of the Heterotic String

In order to derive the effective four-dimensional action from the ten-dimensional Heterotic Superstring action one follows a two step procedure. Since the orbifold is not only described as being the result of dividing the Euclidean space $\mathbb{R}^6$ by the space group Γ but also as the division of the six-torus T^6 by the point group $\bar{P}$, one should first compactify the Heterotic Superstring on a flat six-torus and then identify which of the fields do not survive the orbifold projection and thus eliminate them from the effective action.

Compactification on a Torus The compactification on a torus consists of a simple generalisation of the compactification on a circle which was carried out for pure five-dimensional gravity in subsection 6.1.3. Following the same procedure yields :

$$\begin{aligned} \mathcal{S} = \frac{(2\pi R)^6}{2\kappa_{10}^2} \int d^4x \sqrt{-g} \Big[& R - 2\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} e^{-4\Phi} |H|^2 - \frac{\ell_s^2}{8} e^{-2\Phi} \mathrm{Tr}\left(|F|^2\right) - \frac{1}{4} G^{i\bar{j}} G^{p\bar{q}} \partial_\mu G_{i\bar{q}} \partial^\mu G_{p\bar{j}} \\ & + \frac{1}{4} G^{i\bar{j}} G^{p\bar{q}} \left[\partial_\mu B_{i\bar{q}} + \frac{\ell_s^2}{4} \mathrm{Tr}\left(A_i \overset{\leftrightarrow}{\partial_\mu} \bar{A}_{\bar{q}} \right) \right] \left[\partial^\mu B_{p\bar{j}} + \frac{\ell_s^2}{4} \mathrm{Tr}\left(A_p \overset{\leftrightarrow}{\partial_\mu} \bar{A}_{\bar{j}} \right) \right] \\ & - \frac{\ell_s^2}{4} G^{i\bar{j}} \mathrm{Tr}\left(\partial_\mu A_i \partial^\mu \bar{A}_{\bar{j}} \right) \Big] + \dots \end{aligned} \tag{7.1}$$

where the ellipsis stand for terms which do not involve four-dimensional space-time derivatives. Since H satisfies a non-trivial Bianchi identity, the dualisation of the $B_{\mu\nu}$ term will not only produce a kinetic term for the axion a but also a coupling $aF\tilde{F}$ which is related by SUSY to the $e^{-2\Phi}FF$ term. The dilaton and axion are assembled in a new complex field S which will appear in the gauge-kinetic function. The relevant terms for the computation of the Kähler potential are :

$$\begin{aligned} \mathcal{S} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \Big[& -2 \frac{\partial_\mu S \partial^\mu \bar{S}}{(S+\bar{S})^2} - \frac{1}{4} G^{i\bar{j}} G^{p\bar{q}} \partial_\mu G_{i\bar{q}} \partial^\mu G_{p\bar{j}} \\ & + \frac{1}{4} G^{i\bar{j}} G^{p\bar{q}} \left[\partial_\mu B_{i\bar{q}} + \mathrm{Tr}\left(A_i \overset{\leftrightarrow}{\partial_\mu} \bar{A}_{\bar{q}} \right) \right] \left[\partial^\mu B_{p\bar{j}} + \mathrm{Tr}\left(A_p \overset{\leftrightarrow}{\partial_\mu} \bar{A}_{\bar{j}} \right) \right] \\ & - G^{i\bar{j}} \mathrm{Tr}\left(\partial_\mu A_i \partial^\mu \bar{A}_{\bar{j}} \right) \Big] \end{aligned} \tag{7.2}$$

where $\kappa_4^2 = \kappa_{10}^2/(2\pi R)^6$ and where we have rescaled the matter fields A_i in such a way to absorb the $\ell_s^2/4$ factor, in other words this amounts to set $\ell_s^2 = 4$.

Without Matter Fields Let us first ignore the A matter fields. Under such an assumption the quest for the Kähler potential simplifies a lot. We first notice that the $G_{i\bar{\jmath}}$ and $B_{i\bar{\jmath}}$ kinetic terms may be assembled in a single term by defining $T_{i\bar{\jmath}} = G_{i\bar{\jmath}} + B_{i\bar{\jmath}}$. These fields are called *Kähler moduli fields*. It is a straightforward exercise to verify that :

$$G^{i\bar{\jmath}}G^{p\bar{q}}\partial_\mu T_{i\bar{q}}\partial^\mu T_{\bar{\jmath}p} = G^{i\bar{\jmath}}G^{p\bar{q}}\left(\partial_\mu G_{i\bar{q}}\partial^\mu G_{p\bar{\jmath}} - \partial_\mu B_{i\bar{q}}\partial^\mu B_{p\bar{\jmath}}\right) \tag{7.3}$$

where we have used $T_{\bar{\jmath}i} = G_{\bar{\jmath}i} + B_{\bar{\jmath}i} = G_{i\bar{\jmath}} - B_{i\bar{\jmath}}$. Since the second derivative of the Kähler potential defines the sigma-model metric, K has to be such that its second derivative with respect to T gives G^{-2}. Recalling that since invertible square matrices M admit the following identity :

$$\left(M^{-1}\right)_{ij} = \partial_{M_{ji}} \log\det M \quad \rightarrow \quad \left(M^{-1}\right)_{ij}\left(M^{-1}\right)_{pq} = -\partial_{M_{jp}}\partial_{M_{qi}} \log\det M, \tag{7.4}$$

the structure of K is found to be given by [31, 32] :

$$\begin{aligned} K &= -\log(S+\bar{S}) - \log\det(T_{i\bar{\jmath}} + T_{\bar{\jmath}i}) \\ &= -\log(S+\bar{S}) - \log\det(T+T^\dagger). \end{aligned} \tag{7.5}$$

Restoring Matter Fields Let us now restore the A matter fields. Since the sigma-model metric for the A fields involves two powers of A, the argument of the determinant has to be shifted by A^2 in order to reproduce the above action. It turns out that the Kähler potential is given by [31] :

$$\begin{aligned} K &= -\log(S+\bar{S}) - \log\det\left[T_{i\bar{\jmath}} + T_{\bar{\jmath}i} - \mathrm{Tr}\left(A_i\bar{A}_{\bar{\jmath}}\right)\right] \\ &= -\log(S+\bar{S}) - \log\det\left[T + T^\dagger - \mathrm{Tr}\left(A\otimes\bar{A}\right)\right] \\ &\equiv -\log(S+\bar{S}) - \log V \end{aligned} \tag{7.6}$$

where the definition of $T_{i\bar{\jmath}}$ is now :

$$T_{i\bar{\jmath}} = \frac{1}{2}\left(G_{i\bar{\jmath}} + B_{i\bar{\jmath}} + \mathrm{Tr}\left(A_i\bar{A}_{\bar{\jmath}}\right)\right). \tag{7.7}$$

For later comparison with the case of Calabi-Yau models, it is instructive to rewrite this result in a slightly different form [35]. Since the indices carried by the gauge fields belong to $SU(3)$, one may write V as :

$$V \equiv \det(J_{i\bar{\jmath}}) = \det(\lambda^A_{i\bar{\jmath}}J^A) \tag{7.8}$$

with :

$$\begin{aligned} J_{i\bar{\jmath}} &= T_{i\bar{\jmath}} + T_{\bar{\jmath}i} - \lambda^A_{i\bar{\jmath}}\bar{A}^s_m\lambda^A_{mn}A^s_n \\ J^A &= T^A + \bar{T}^A - \bar{A}^s_m\lambda^A_{mn}A^s_n \end{aligned} \tag{7.9}$$

where the λ^A's, with $A = \{0, a\}$, $a = \{1, \ldots, 8\}$ are the $U(3)$ generators which we choose to be normalised as $\mathrm{Tr}\left(\lambda^A \lambda^B\right) = \delta^{AB}$:

$$
\begin{aligned}
&\lambda^0 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \qquad \lambda^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} \qquad \lambda^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\\
&\lambda^3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad \lambda^4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix} \qquad \lambda^5 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\\
&\lambda^6 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \qquad \lambda^7 = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.
\end{aligned} \tag{7.10}
$$

Moreover the following completeness relation holds true :

$$
\lambda^A_{ij}\lambda^A_{pq} = \delta_{iq}\delta_{jp} \quad \to \quad \lambda^a_{ij}\lambda^a_{pq} = \delta_{iq}\delta_{jp} - \frac{1}{3}\delta_{ij}\delta_{pq}. \tag{7.11}
$$

In order to make contact with the Calabi-Yau compactification we will perform later in this Chapter, the cubic polynomial V is written as :

$$
V = \frac{1}{6} d_{ijpqrs} J_{ij} J_{pq} J_{rs} = \frac{1}{6} d^{ABC} J^A J^B J^C \tag{7.12}
$$

where the d_{ijpqrs} and d^{ABC} numbers which are related by $d_{ijpqrs} = \lambda^A_{ji}\lambda^B_{qp}\lambda^C_{sr} d^{ABC}$ are given by :

$$
\begin{aligned}
&d_{ijpqrs} = \epsilon_{ipr}\epsilon_{jqs},\\
&d^{ABC} = 2\mathrm{Tr}\left(\lambda^{(A}\lambda^B\lambda^{C)}\right) - 3\mathrm{Tr}\left(\lambda^{(A}\right)\mathrm{Tr}\left(\lambda^B\lambda^{C)}\right) + \mathrm{Tr}\left(\lambda^{(A}\right)\mathrm{Tr}\left(\lambda^B\right)\mathrm{Tr}\left(\lambda^{C)}\right).
\end{aligned} \tag{7.13}
$$

All the above formulae are valid in all the three cases listed in Table 6.1, with the understanding that the number of Kähler moduli and the allowed values for the a and i indices should be suitably restricted. In case **(a)**, one has $h^{1,1} = 9$ and thus all the 9 Kähler moduli $T_{i\bar{j}}$, corresponding to T^A with $A = 0, \ldots, 8$. In case **(b)**, $h^{1,1} = 5$ leading to restricted spectrum $T_{1\bar{1}}$, $T_{1\bar{2}}$, $T_{2\bar{1}}$, $T_{2\bar{2}}$ and $T_{3\bar{3}}$, corresponding to T^A with $A = 0, 1, 2, 3, 8$. Finally in the **(c)** case, $h^{1,1} = 3$ and thus the spectrum consists of $T_{1\bar{1}}$, $T_{2\bar{2}}$ and $T_{3\bar{3}}$ corresponding to T^A with $A = 0, 3, 8$. It will prove convenient in the following to distinguish between the $A = 0$ and $A = a$ $U(3)$ generators. From the Gell-Mann matrices properties, one has :

$$
d^{000} = \frac{2}{\sqrt{3}}, \qquad d^{00a} = 0, \qquad d^{0ab} = -\frac{1}{\sqrt{3}}\delta^{ab} \qquad \text{and} \qquad d^{abc} = 2\mathrm{Tr}\left(\lambda^{(a}\lambda^b\lambda^{c)}\right). \tag{7.14}
$$

In this section we have thus shown the effective Kähler potential for the untwisted sector of orbifold models derived in [31, 32] may be rewritten as :

$$
K = -\log\left[\frac{1}{6} d^{ABC} J^A J^B J^C\right] \quad \text{where} \quad J^A = T^A + \bar{T}^A - \bar{A}^s_{\bar{j}}\lambda^A_{ji}A^s_i \tag{7.15}
$$

where the numbers d^{ABC} and $\lambda^A_{i\bar{\jmath}}$ both have group-theoretical interpretations. Let us now turn to the more general case of Calabi-Yau compactifications and see if such a structure arises for smooth manifolds and if not, under which assumptions this structure emerges. Since orbifolds are singular limits of Calabi-Yau manifolds, such restrictions have to exist.

7.2 Calabi-Yau Compactification of the Heterotic Superstring

The effective Kähler potential for Calabi-Yau models can be determined by performing the reduction of the ten-dimensional bosonic kinetic terms by integrating over the compact Calabi-Yau X and comparing the result with the standard general form of the Lagrangian of four-dimensional SUGRA theories. To perform this computation, we will closely follow [36] and make two approximations which are commonly done and which crucially simplify the task :

- The first approximation is that we will ignore the higher-derivative corrections to the ten-dimensional effective action and the deformations of the background, and therefore simply consider the reduction of the action (5.24) on a generic Calabi-Yau manifold X with a generic stable holomorphic vector bundle V over it. This implies that the result will only be accurate for terms involving arbitrary powers of the moduli fields and arbitrary powers of the combination of ℓ_s^2 times two matter fields, and will miss corrections involving powers of ℓ_s^2 that are not accompanied by two matter fields, but this is not a big limitation for our purposes (see *e.g.* [119] for an explicit computation of the leading ℓ_s^2 correction to the moduli Kähler potential).

- The second approximation is that we will ignore the effect of properly integrating out massive Kaluza-Klein modes and restrict to the truncation of the action to the four-dimensional low-energy massless zero-modes. This would generically imply that the result is accurate only for terms involving an arbitrary number of moduli but at most two matter fields, since terms with four and more matter fields can receive corrections induced by the exchange of heavy neutral modes, and this would represent a dramatic limitation for our purposes. We will therefore imagine to restrict ourselves to those models for which these effects happen to be absent, at least for the term involving four matter fields in which we are primarily interested. This is guaranteed to happen if there is no cubic coupling between two light matter modes and one heavy moduli mode (see *e.g.* [134]).

Finally, we shall for simplicity restrict our attention to the dilaton, the $h^{1,1}$ Kähler moduli and $n_{\mathbf{R}}$ families of charged matter fields in the representation $\mathbf{R}$, and instead completely discard the $h^{1,2}$ complex structure moduli, the vector bundle moduli and the other families of matter fields.

To compute the 4D effective kinetic terms, we now proceed as follows. We start from (5.24) restricted to the modes associated to $G_{i\bar{\jmath}}$, $B_{i\bar{\jmath}}$ and A_i and integrate over the internal manifold X. We then express the result in terms of the 4D gravitational and gauge couplings. These are defined as $\kappa_4^2 = \kappa_{10}^2/V$ and $g_4^2 = g_{10}^2/V$, where V denotes the background value of the volume of the manifold X, and are again related as $\kappa_4^2/g_4^2 = \ell_s^2/4$. In the following, we shall set $\kappa_4 = 1$ by a choice of units. Moreover we shall effectively set $g_4 = 1$ in the scalar sector of the Lagrangian by suitably rescaling the charged matter fields. This corresponds to setting $\ell_s^2 = 4$. In this way, one finds the following result :

$$\begin{aligned}\mathcal{L}_4 = \frac{1}{V}\int d^6y\sqrt{G}\Big[&- \frac{1}{4}G^{i\bar{j}}G^{p\bar{q}}\partial_\mu G_{i\bar{q}}\partial^\mu G_{p\bar{j}} \\ &+ \frac{1}{4}G^{i\bar{j}}G^{p\bar{q}}\left[\partial_\mu B_{i\bar{q}} + \mathrm{Tr}\left(A_i \overset{\leftrightarrow}{\partial_\mu} \bar{A}_{\bar{q}}\right)\right]\left[\partial^\mu B_{p\bar{j}} + \mathrm{Tr}\left(A_p \overset{\leftrightarrow}{\partial_\mu} \bar{A}_{\bar{j}}\right)\right] \\ &- G^{i\bar{j}}\mathrm{Tr}\left(\partial_\mu A_i \partial^\mu \bar{A}_{\bar{j}}\right)\Big].\end{aligned} \tag{7.16}$$

Note that we have discarded the dilaton kinetic term since, as is the case in the orbifold context, it simply leads to the addition of $-\log(S+\bar{S})$ to the Kähler potential determined from (7.16). We will restore the dilaton dependence when computing the soft masses in Chapter 8. To proceed, we associate the $G_{i\bar{j}}$, $B_{i\bar{j}}$ and A_i fields to differential forms J, B and A, which are defined as follows in local complex coordinates z^i :

$$\begin{aligned}J &= iG_{i\bar{j}}\,dz^i \wedge d\bar{z}^j, \\ B &= B_{i\bar{j}}\,dz^i \wedge d\bar{z}^j, \\ A &= A_i dz^i.\end{aligned} \tag{7.17}$$

We then decompose these forms onto suitable bases of harmonic forms, with coefficients identified with the four-dimensional light fields. To define the moduli fields, we shall need to introduce a basis of harmonic $(1,1)$-forms $\omega_A = \omega_{Ai\bar{j}}\,dz^i \wedge d\bar{z}^j$ on X with $A = 0, \ldots, h^{1,1}-1$, which can also be viewed as 1 forms with values in T^*X over X. To define the matter fields, we shall also need a basis of Lie-algebra-valued harmonic 1-forms $u_P = u_{Pi}\,dz^i$ on $V_{\mathbf{r}}$ over X. We observe now that the forms constructed by taking the product of one u_P and one conjugate $\bar{u}_Q$ and tracing over the representation $\mathbf{r}$ yield $(1,1)$-forms on X. These $(1,1)$-forms are related to the description of the gauge invariant composite field that can be formed out of two charged matter fields. Since they play an important rôle in the following, we shall define a dedicated symbol for them :

$$c_{PQ} = i\,\mathrm{Tr}\left(u_P \wedge \bar{u}_Q\right). \tag{7.18}$$

A crucial observation is that these $(1,1)$-forms are however generically not harmonic. As a result, their scalar product with the non-harmonic $(1,1)$-forms describing massive neutral modes is in general non-vanishing.

It turns out that the low-energy effective Kähler potential always depends on the volume V of X, which is given by the following expression in terms of the Kähler form J :

$$V = \frac{1}{6}\int_X J \wedge J \wedge J. \tag{7.19}$$

More explicitly, when rewritten in terms of the four-dimensional fields describing the moduli and matter fields, this will depend on two quantities characterising X and V. The first one is given by the integral of three harmonic $(1,1)$-forms ω_A, which defines the intersection numbers of X :

$$d_{ABC} = \int_X \omega_A \wedge \omega_B \wedge \omega_C. \tag{7.20}$$

The second is given by the integral of the $(1,1)$-forms c_{PQ} and a dual harmonic $(2,2)$ form ω^A, which defines the component of the harmonic part of c_{PQ} along ω_A and therefore encodes the overlap between the traced product of the 1-forms u_P and $\bar{u}_Q$ with the $(1,1)$-forms ω_A :

$$c^A_{PQ} = \int_X \omega^A \wedge c_{PQ}. \tag{7.21}$$

It should be emphasised that (7.20) is a topological invariant, as a result of the fact that the forms ω_A are harmonic, whereas (7.21) is a priori not, since the forms c_{PQ} are in general not harmonic.

In the following, we shall restrict to the special case where the forms c_{PQ} are harmonic and c^A_{PQ} is a constant topological invariant, and derive the low-energy effective Kähler potential under these assumptions. We believe that this is a priori necessary to guarantee that the result obtained by truncating to the massless modes, without properly integrating out the massive modes, is reliable. But as matter of fact, we will also crucially exploit these assumptions to be able to obtain a simple result. We shall discuss in subsection 7.2.4 what may happen in the more general case where c_{PQ} is not harmonic and c^A_{PQ} is not a topological invariant. For notational simplicity, we shall from now on omit to write any trace over the representation $\mathbf{R}$ of the gauge group, since the way in which these traces appear can be reconstructed in an unambiguous way at any stage of the derivation.

7.2.1 Kähler Moduli Space

The effective Kähler potential for the Kähler moduli, ignoring matter fields, is well known [33, 120]. It can be derived in a straightforward way by only retaining the terms depending quadratically on space-time derivatives of the $G_{i\bar{\jmath}}$ and $B_{i\bar{\jmath}}$ fields in (7.16). To work out the reduction, one considers the real $(1,1)$ forms J and B associated to these two fields and decomposes the complex combination $J + iB$ onto the basis of real harmonic $(1,1)$ forms ω_A, with complex coefficients T^A defining the four-dimensional complex moduli fields :

$$J + iB = 2T^A \omega_A. \tag{7.22}$$

In components this means $G_{i\bar{\jmath}} = -i(T^A + \bar{T}^A)\omega_{Ai\bar{\jmath}}$ and $B_{i\bar{\jmath}} = -i(T^A - \bar{T}^A)\omega_{Ai\bar{\jmath}}$. Plugging these decompositions into the first two terms of (7.16), one then finds a kinetic term for the complex scalar fields T^A of the form :

$$\mathcal{L}_4 \ni -g^{\rm mod}_{A\bar{B}}\, \partial_\mu T^A \partial^\mu \bar{T}^B \tag{7.23}$$

where :

$$g^{\rm mod}_{A\bar B} = -\frac{1}{V}\int d^6y\sqrt{G}G^{i\bar j}G^{p\bar q}\omega_{Ai\bar q}\omega_{Bp\bar j} = \frac{1}{V}\int_X \omega_A \wedge *\omega_B. \tag{7.24}$$

This metric does not depend at all on the c_{PQ} forms, and the issue of whether these are harmonic or not is therefore trivially irrelevant here. Using the decomposition $J = J^A\omega_A$ with $J^A = T^A + \bar T^A$, which implies that $\partial_A J^B = \delta_A^B$, and the relation (B.80), one can rewrite (7.24) in the following form :

$$g^{\rm mod}_{A\bar B} = -\partial_A\partial_{\bar B}\log V. \tag{7.25}$$

From this expression we deduce that the Kähler potential is given, up to a Kähler transformation, by $K = -\log V$. This can finally be rewritten more explicitly in terms of the chiral multiplets T^A and the intersection numbers d_{ABC} as :

$$K = -\log\left[\frac{1}{6}d_{ABC}J^AJ^BJ^C\right] \qquad \text{where} \qquad J^A = T^A + \bar T^A. \tag{7.26}$$

This result has the property of being special-Kähler and also of the no-scale type, *i.e.* it satisfies :

$$K_AK^A = 3. \tag{7.27}$$

Notice finally that in geometrical terms the quantities K_A and K^A have the following simple expressions :

$$K_A = -\frac{1}{V}\int_X \omega_A\wedge *J \qquad \text{and} \qquad K^A = -\int_X \omega^A \wedge J. \tag{7.28}$$

7.2.2 Matter Field Metric

Let us next consider the addition of matter fields, under the simplifying assumption that their background value vanishes. In this situation, all the terms involving the fields A_i without space-time derivatives can be neglected in (7.16), and the only term to be considered is therefore the last one. In this limit the matter sector can be seen as a small perturbation to the moduli sector, and one can neglect the interference between these two sectors. To work out the reduction, one decomposes the 1-forms A on the basis of harmonic 1-forms u_P taking values in the representation $\mathbf{r}$ of S with complex coefficients Φ^P taking values in the representation $\mathbf{R}$ of G and defining the four-dimensional matter fields : $A = \Phi^P u_P$. In components this means $A_i = \Phi^P u_{Pi}$. Plugging this decomposition into the last term of (7.16), one finds a kinetic term for the complex scalar fields Φ^P of the form :

$$\mathcal{L}_4 \ni -g^{\rm mat}_{P\bar Q}\partial_\mu\Phi^P\partial^\mu\bar\Phi^Q \tag{7.29}$$

where :

$$g^{\rm mat}_{P\bar Q} = -\frac{i}{V}\int d^6y\sqrt{G}G^{i\bar j}c_{PQi\bar j} = \frac{1}{V}\int_X c_{PQ}\wedge *J. \tag{7.30}$$

This metric depends on the forms c_{PQ}, but only through their scalar product with the Kähler form J, which is harmonic. As a result, only the harmonic component of the Hodge decomposition of c_{PQ} matters, and the issue of whether the whole forms c_{PQ} are harmonic or not is therefore again irrelevant. Using the decomposition $J = J^A \omega_A$ with $J^A = T^A + \bar{T}^A$, which as before implies that $\partial_A J^B = \delta_A^B$, as well as the decomposition of $*J$ on the dual basis ω^A and the relation (B.80), one may rewrite (7.30) in the following form :

$$g^{\rm mat}_{P\bar{Q}} = \partial_A \log V c^A_{PQ}. \tag{7.31}$$

This means that the matter metric is related to the moduli Kähler potential by $g^{\rm mat}_{P\bar{Q}} = -K_A c^A_{PQ}$ [34, 132]. This in turn implies that the leading matter-dependent correction to the Kähler potential is given by this metric contracted with two matter fields :

$$\Delta K = -K_A c^A_{PQ} \Phi^P \bar{\Phi}^Q. \tag{7.32}$$

Notice finally that one can write simple geometric expressions for the following contractions :

$$K_A c^A_{PQ} = -\frac{1}{V}\int_X c_{PQ} \wedge *J \qquad \text{and} \qquad K_{AB} c^B_{PQ} = \frac{1}{V}\int_X \omega_A \wedge *c_{PQ}. \tag{7.33}$$

7.2.3 Full Scalar Manifold

Let us finally consider the full dependence on both the Kähler moduli and the matter fields, which is relevant when the matter fields have a non-vanishing VEV. In this case, one has to consider all the terms in (7.16). The relevant fields are as before $G_{i\bar{\jmath}}$, $B_{i\bar{\jmath}}$ and A_i. The first two again can be combined to form a complex $(1,1)$-form $J + iB$, and decomposed onto the basis of harmonic $(1,1)$-forms ω_A. The second can be viewed as matrix-valued 1-forms A, and decomposed onto the basis of harmonic 1-forms u_P.

It however turns out that that the precise definition of the four-dimensional moduli fields T^A and matter fields Φ^S that allows to recast the action in a manifestly supersymmetric form involves a non-trivial shift. The form of this shift may be guessed by generalising the results applying in the two special cases of Calabi-Yau manifolds with a single modulus and of orbifolds, which are also the only two cases where a derivation of the full effective Kähler potential is already known, respectively from [133] and [31]. The only quantity that can possibly enter in the non-trivial shift is c^A_{PQ}, and the appropriate definitions turn out to be :

$$J + iB = 2\Big(T^A - \frac{1}{2} c^A_{PQ} \Phi^P \bar{\Phi}^Q\Big)\omega_A \qquad \text{and} \qquad A = \Phi^P u_P. \tag{7.34}$$

In components this means $G_{i\bar{\jmath}} = -i(T^A + \bar{T}^A - c^A_{PQ}\Phi^P\bar{\Phi}^Q)\omega_{Ai\bar{\jmath}}$, $B_{i\bar{\jmath}} = -i(T^A - \bar{T}^A)\omega_{Ai\bar{\jmath}}$ and $A_i = \Phi^P u_{Pi}$. By plugging these decompositions into (7.16), one finds kinetic terms for the complex scalar fields T^A and Φ^P of the form :

$$\mathcal{L}_4 \ni -g^{\rm mod}_{A\bar{B}} \partial_\mu T^A \partial^\mu \bar{T}^B - g^{\rm mat}_{P\bar{Q}} \partial_\mu \Phi^P \partial^\mu \bar{\Phi}^Q - \Big(g^{\rm mix}_{A\bar{Q}} \partial_\mu T^A \partial^\mu \bar{\Phi}^Q + \text{c.c.}\Big) \tag{7.35}$$

where :

$$
\begin{aligned}
g^{\mathrm{mod}}_{A\bar{B}} &= -\frac{1}{V}\int d^6y\sqrt{G}G^{i\bar{\jmath}}G^{p\bar{q}}\omega_{Ai\bar{q}}\omega_{Bp\bar{\jmath}} \\
&= \frac{1}{V}\int_X \omega_A\wedge *\omega_B, \\
\\
g^{\mathrm{mat}}_{P\bar{Q}} &= -\frac{i}{V}\int d^6y\sqrt{G}G^{i\bar{\jmath}}c_{PQi\bar{\jmath}} - \frac{1}{V}\int d^6y\sqrt{G}G^{i\bar{\jmath}}G^{m\bar{n}}c_{PSi\bar{n}}c_{RQm\bar{\jmath}}\Phi^R\bar{\Phi}^S \\
&= \frac{1}{V}\int_X c_{PQ}\wedge *J + \frac{1}{V}\left(\int_X c_{PS}\wedge *c_{RQ}\right)\Phi^R\bar{\Phi}^S, \\
\\
g^{\mathrm{mix}}_{A\bar{Q}} &= \frac{1}{V}\int d^6y\sqrt{G}G^{i\bar{\jmath}}G^{m\bar{n}}\omega_{Ai\bar{n}}c_{RQm\bar{\jmath}}\Phi^R \\
&= -\frac{1}{V}\left(\int_X \omega_A\wedge *c_{RQ}\right)\Phi^R.
\end{aligned}
\tag{7.36}
$$

This metric now significantly depends on the forms c_{PQ}, not only through their scalar product with the Kähler form J or the basis forms ω_A, which are harmonic, but also through their scalar products among themselves. As a result, not only the harmonic part but also the exact and coexact parts of the Hodge decomposition of c_{PQ} matter, and the issue of whether c_{PQ} is harmonic or not is therefore crucial in this case. As already said, we will assume that c_{PQ} is harmonic and c^A_{PQ} is constant, so that one can use the decomposition $c_{PQ} = c^A_{PQ}\omega_A$. Taking into account the decomposition $J = J^A\omega_A$ with $J^A = T^A + \bar{T}^A - c^A_{PQ}\Phi^P\bar{\Phi}^Q$, which still implies that $\partial_A J^B = \delta^B_A$ since c^A_{PQ} is constant, and using the relation (B.80), the metric components (7.36) can be rewritten as :

$$
\begin{aligned}
g^{\mathrm{mod}}_{A\bar{B}} &= -\partial_A\partial_{\bar{B}}\log V, \\
g^{\mathrm{mat}}_{P\bar{Q}} &= \partial_A\log V c^A_{PQ} - \partial_A\partial_{\bar{B}}\log V c^A_{PS}c^B_{RQ}\Phi^R\bar{\Phi}^S = -\partial_P\partial_{\bar{Q}}\log V, \\
g^{\mathrm{mix}}_{A\bar{Q}} &= \partial_A\partial_{\bar{B}}\log V c^B_{RQ}\Phi^R = -\partial_A\partial_{\bar{Q}}\log V.
\end{aligned}
\tag{7.37}
$$

From these expressions we see that, modulo an arbitrary Kähler transformation, the Kähler potential is simply given by $K = -\log V$. More explicitly, this reads in this case :

$$
K = -\log\left[\frac{1}{6}d_{ABC}J^AJ^BJ^C\right] \qquad \text{where} \qquad J^A = T^A + \bar{T}^A - c^A_{PQ}\Phi^P\bar{\Phi}^Q. \tag{7.38}
$$

This result coincides with the one proposed in [34] on the basis of an $\mathcal{M}$-Theory argumentation. It manifestly reproduces the result (7.26) for the moduli and the leading order correction (7.32) at quadratic order in the matter fields. Moreover its satisfies a no-scale property generalising the one found when only considering moduli fields, *i.e.* (7.27). In order to demonstrate this assertion we introduce the Z^X symbol as a generic field, *i.e.* it takes all values in T^A and Φ^P. Since V is homogeneous of degree three in the currents J^A, we have :

$$
\frac{\partial V}{\partial J^A}J^A = V_AJ^A = 3V \tag{7.39}
$$

where V_A denotes the derivative of V with respect to T^A. Taking a derivative with respect to $\bar{Z}^Y$ leads to :

$$V_{A\bar{Y}} J^A + V_A \frac{\partial J^A}{\partial \bar{Z}^Y} = 3V_{\bar{Y}} \tag{7.40}$$

where the second term on the left-hand-side is shown to be nothing but $V_{\bar{Y}}$ by using the Leibniz rule. Multiplying both sides by $V^{-1}V^{X\bar{Y}}V_X$ and using $V^{X\bar{Y}}V_{A\bar{Y}} = \delta^X_A$ and (7.39) leads to :

$$\theta \equiv \frac{V^{X\bar{Y}}V_X V_{\bar{Y}}}{V} = \frac{3}{2} \tag{7.41}$$

which by using the technology developed in the Appendix B.2.4.2 yields $K_X K^X = 3$. When including the dilaton whose Kähler potential is $\tilde{K} = -\log(S + \bar{S})$, one finds in accordance with [34] :

$$K_X K^X + \tilde{K}_S \tilde{K}^S = 4. \tag{7.42}$$

Notice finally that K_A, K_P, K^A and K^P can be written in the following simple geometrical terms by using the relations (B.65) :

$$\begin{aligned} K_A &= -\frac{1}{V}\int_X \omega_A \wedge *J, \qquad & K^A &= -\int_X \omega^A \wedge J, \\ K_P &= \frac{1}{V}\int_X c_{PS}\bar{\Phi}^S \wedge *J, \qquad & K^P &= 0. \end{aligned} \tag{7.43}$$

Moreover, from the assumption that the forms c_{PQ} are harmonic it follows that also the contraction $K_{AB}c^A_{PQ}c^B_{RS}$ admits a simple geometrical expression :

$$K_{AB}c^A_{PQ}c^B_{RS} = \frac{1}{V}\int_X c_{PQ} \wedge *c_{RS}. \tag{7.44}$$

Similarly one also finds that :

$$d_{ABC}c^A_{PQ}c^B_{RS}c^C_{MN} = \int_X c_{PQ} \wedge c_{RS} \wedge c_{MN}. \tag{7.45}$$

7.2.4 Range of Validity

The simple derivation presented in last subsection is manifestly valid in those cases where the forms c_{PQ} are harmonic and the quantities c^A_{PQ} are constant topological invariants. One special situation in which this is certainly true is when all the involved forms ω_A and u_P are actually not only harmonic but actually covariantly constant. As we shall see more explicitly in next section, this is for instance the case for toroidal orbifold models. But we believe that it could be true also in a less trivial fashion. We will imagine that this is indeed the case for some subset of smooth Calabi-Yau models. For further use, let us then explore a few simple consequences of the above assumptions. Recall that $A = 0, \cdots, h^{1,1} - 1$ labels the different Kähler moduli and $P, Q = 1, \cdots, n_{\mathbf{R}}$ label the different matter fields. By definition, for each of the $h^{1,1}$ values of A the quantity c^A_{PQ} is a Hermitian $n_{\mathbf{R}} \times n_{\mathbf{R}}$ matrix. This means that even when $h^{1,1} > n^2_{\mathbf{R}}$, the number of these matrices that are linearly independent cannot exceed $n^2_{\mathbf{R}}$. In fact, the $h^{1,1}$ matrices c^A_{PQ} can always be rewritten as linear combinations of the $n^2_{\mathbf{R}}$ independent transposed Hermitian matrices $\lambda^{A'}_{QP}$, with $A' = 0, \cdots, n^2_{\mathbf{R}} - 1$ and where the transposition is included for later convenience. Notice that whereas the matrices c^A_{PQ} do a priori not satisfy any completeness relation and do not generate any closed algebra, the matrices $\lambda^{A'}_{PQ}$ do instead satisfy an obvious completeness relation since they form a basis of Hermitian matrices and generate a closed algebra, which is that of $U(n_{\mathbf{R}})$. We therefore know that under the assumptions that we made :

- ⋄ The c^A_{PQ} are linear combinations of $\lambda^{A'}_{QP}$,
- ⋄ The $\lambda^{A'}_{PQ}$ are $n_{\mathbf{R}} \times n_{\mathbf{R}}$ matrices representing $U(n_{\mathbf{R}})$.

The extension to more general situations where instead c_{PQ} is not harmonic and the quantities c^A_{PQ} are not constant topological invariants is clearly more challenging, and one may wonder whether a result similar to (7.38) could hold true. One first major change arising for a non-harmonic c_{PQ} is that since its Hodge decomposition contains now not only a harmonic piece but also an exact piece and a coexact piece, the relation (7.44) does no longer hold true. More precisely, its left-hand side acquires extra terms matching the contributions to the right-hand side coming from the non-harmonic parts of c_{PQ}, which are clearly more difficult to deal with.

In particular, when going from (7.36) to (7.37), one would get additional terms that clearly have to do with the effect of heavy non-zero modes. In fact, these heavy modes must be related to the 10D B field. Indeed, using a democratic formulation of the original 10D theory involving not only the 2-form B but also its magnetic dual 6-form $\tilde{B}$, the contact term from which the problem originates can be deconstructed and the seed for its origin is then reduced to a linear coupling between $\tilde{B}$ and $\mathrm{d}\Gamma = \mathrm{Tr}\,(F \wedge F)$. When reducing on X, one then gets a direct coupling between two light matter modes coming from A and one heavy mode coming from $\tilde{B}$ whenever c_{PQ} is not harmonic, and this must be responsible form the extra contributions to the contact terms.

A second source of difficulty arising for a non-constant c^A_{PQ} is that this quantity may then be expected to depend on continuous deformations of both the vector bundle V and the manifold X. The first of these dependences, which was already mentioned in [34], does not concern us since it would be related to vector bundle moduli, which we have ignored from the beginning. But the second of these dependences, which we believe should also be a source of concern, is instead directly relevant for our derivation, since it is related to the Kähler moduli that we want to keep in the effective theory. Now, a moduli dependence c^A_{PQ} would imply additional terms in (7.36). Moreover, it would also affect the simple relation $\partial_A J^B = \delta^B_A$ that was used to rewrite these metric in the form (7.37). At first one might hope that these two sources of complications could compensate each other, but things do not seem to be so simple. One may then perhaps have to generalise the decomposition (7.34) through a more complicated and implicit definition of the moduli and matter fields. We were however not able to reach a conclusive assessment of this possibility.

We believe that subtleties very similar to those explained here for Heterotic models may actually arise also for orientifold models. More precisely, it seems to us that the results derived in [135, 136] concerning the higher-order dependence of the Kähler potential on the matter fields arising from D-brane sectors should a priori also be correct and reliable only for those special models in which massive non-zero modes do not induce non-trivial corrections. We attribute the fact that this is not directly signaled by a technical difficulty in the derivation of [135, 136] to the use of a democratic formulation in terms of all the Ramond-Ramond forms, which deconstructs the original 10D contact term and hides the subtlety.

7.2.5 Standard Embedding

The concerns raised in previous subsection may be illustrated more concretely by considering in some detail the special case of Calabi-Yau manifolds X with a generic number of moduli but Standard Embedding for the vector bundle V. In this case the situation is somewhat simpler and there exists an alternative way of performing the dimensional reduction for the matter fields. Indeed, recall that in this case V is identified with TX, so that $S = SU(3)$ and $G = E_6 \times E_8$. As a consequence, the additional index in the representation $\mathbf{r} = \bar{\mathbf{3}}$ can be reinterpreted as a cotangent space index, and one may exploit this to construct the $SU(3)$-valued harmonic 1-forms u_A in terms of the harmonic $(1,1)$-forms ω_A.

In the approximation where one works at leading order in the matter fields and neglects the interference between moduli and matter fields, as in subsection 7.2.2, the way in which this decomposition can be done has been explained in [116, 137]. In the end, it essentially amounts to describe the matter modes in terms of a standard $(1,1)$-form A and decompose it on the basis of harmonic $(1,1)$ forms ω_A with $h^{1,1}$ complex coefficients Φ^A taking values in the representation $\mathbf{R} = (\overline{\mathbf{27}}, \mathbf{1})$ of $E_6 \times E_8$ and defining the 4D matter fields. It has been shown in [137] that one must however include a suitable power of the norm of the covariantly constant holomorphic $(3,0)$-form of X in this decomposition, in order to be able to express the potential coming from the non-derivative part of the action in terms of a holomorphic superpotential. Here, since we are considering the case of absent or frozen complex structure moduli, this simply implies some extra power of the volume V, and the correct definition turns out to be :

$$A = V^{1/6} \Phi^A \omega_A. \tag{7.46}$$

One then finds a kinetic term of the form :

$$\mathcal{L}_4 \ni -g^{\mathrm{mat}}_{A\bar{B}} \partial_\mu \Phi^A \partial^\mu \bar{\Phi}^B \tag{7.47}$$

where :

$$\begin{aligned} g^{\text{mat}}_{A\bar{B}} &= -\frac{1}{V^{2/3}} \int d^6y \sqrt{G} G^{i\bar{j}} G^{p\bar{q}} \omega_{Ai\bar{q}} \omega_{Bp\bar{j}} \\ &= \frac{1}{V^{2/3}} \int_X \omega_A \wedge *\omega_B. \end{aligned} \tag{7.48}$$

Through the usual manipulations, this metric can be rewritten as :

$$g^{\text{mat}}_{A\bar{B}} = -V^{1/3} \partial_A \partial_{\bar{B}} \log V. \tag{7.49}$$

This implies that the matter metric is in this case linked to the moduli metric by the relation $g^{\text{mat}}_{A\bar{B}} = e^{-K/3} g^{\text{mod}}_{A\bar{B}}$, which was first derived in [33] by matching an actual string scattering amplitude computation. The leading matter-dependent correction to the moduli Kähler potential must then have the form :

$$\Delta K = e^{-K/3} K_{A\bar{B}} \Phi^A \bar{\Phi}^B. \tag{7.50}$$

Comparing the result (7.50) with the general expression (7.32) and requiring them to be equal, we deduce that in the case of Standard Embedding the matrices c^A_{BC} must have a special form. Indeed, the components of the $(1,1)$-form c_{AB} are found to be given by :

$$c_{ABi\bar{j}} = -iV^{1/3} G^{p\bar{q}} \omega_{Ai\bar{q}} \omega_{Bp\bar{j}}. \tag{7.51}$$

It is a straightforward exercise to verify that the forms c_{AB} defined by these components are generically not harmonic, except for the particular case where ω_A and/or ω_B is identified with the Kähler form J or happen more in general to be a covariantly constant $(1,1)$-form. Since K^A is given by (7.28), one has $K^A \omega_A = -J$, meaning that the c_{AB} forms are not harmonic but $K^A c_{AB}$ and $K^B c_{AB}$ are.

One may nevertheless compute the quantity c^A_{BC} by using the expression (7.51) for the components of c_{PQ}. The result depends on the metric and is thus a function of $T^A + \bar{T}^A$. It might be possible to express this function in terms of derivatives of the Kähler potential K for the moduli. But even without writing an explicit expression, one can observe that the factor $V^{1/3} G^{p\bar{q}}$ appearing in the expression (7.51) is a homogenous function of degree 0 in the components of the metric, and therefore in the geometric moduli fields. More precisely, one finds that $c^0_{00} = 1$ when $h^{1,1} = 1$ and there is a single modulus T^0, whereas $c^A_{BC} = c^A_{BC}((T^D + \bar{T}^D)/(T^E + \bar{T}^E))$ when $h^{1,1} > 1$ and there are several moduli T^A. Since by (7.28) one has $K^D = -(T^D + \bar{T}^D)$, this means that the c^A_{BC}'s are not constant but that $K^D \partial_D c^A_{BC} = 0$.

Finally, one easily verifies that c^A_{BC} does indeed satisfy an identity ensuring that the two expressions (7.32) and (7.50) are identical :

$$-K_A c^A_{BC} = e^{-K/3} K_{BC}. \tag{7.52}$$

One can easily demonstrate that the above relation forces c^A_{BC} to be constant in the special case $h^{1,1} = 1$ and non-constant when instead $h^{1,1} > 1$. To do so, one starts by assuming that (7.52) is satisfied with a constant c^A_{BC}. One may then take a derivative of (7.52), use $\partial_D c^A_{BC} = 0$ and act with the inverse of the moduli metric to derive the expression $c^A_{BC} = -e^{-K/3}K^{AD}\big(K_{BCD} - \frac{1}{3}K_D K_{BC}\big)$. Finally, one may compute the derivative of this expression to check whether it is really zero, as assumed. In particular, using the identity $\partial_A K^B = -\delta^B_A$ one finds rather easily that $\partial_A c^A_{BC} K^B K^C = -3\,e^{-K/3}\big(h^{1,1}-1\big)$, which vanishes when $h^{1,1}=1$ but not when $h^{1,1}>1$, contradicting in this last case the hypothesis that c^A_{BC} was constant.

When attempting to go on and work out the result at higher orders in the matter fields, one can no longer neglect the interference between matter and moduli fields. One then needs to properly change the definition of the moduli fields. The natural guess based on our general derivation is that the definition of the moduli fields should be shifted by a term that is quadratic in the matter fields and involves c^A_{BC}. Indirect evidence in favour of this has been found in [137] (whose quantity σ_{ABC} is seen to be proportional to our c^A_{BC} specified by (7.51) with the upper index lowered with the moduli metric) by studying the interference of this redefinition and the possible emergence of a non-trivial superpotential. It is however not obvious how one should proceed to work out the full result, as both of the subtleties discussed in subsection 7.2.4, namely the non-harmonicity of c_{BC} and the non-constancy of c^A_{BC}, have been manifestly shown to arise in this case, except for the particular situations where $h^{1,1}=1$, for which the result (7.38) holds true and reduces to the result derived in [133].

7.3 The Heterotic String on an Orbifold Revisited

It is interesting to compare the general situation occurring for compactifications on a smooth Calabi-Yau manifold X with that of compactifications on orbifolds of the type $T^6/\mathbb{Z}_N$ [105, 138], which represent singular limits of them from the geometrical point of view. We shall as before focus on the Kähler moduli and the matter fields, restricting to the untwisted sector for which a simple derivation based on dimensional reduction was presented in 7.1, and show how the known exact results for the dependence of the Kähler potential on the Kähler moduli and matter fields can be rephrased in the same language as in the previous section.

Moreover the condition (6.22) is understood to be the analogue of the Bianchi identity (5.26) that must be imposed for smooth Calabi-Yau compactifications and which constrains the choice of vector bundle V for a given tangent bundle TX. The states arising in the untwisted sector are now associated to the subset of harmonic forms on T^6 that are left invariant by the $\mathbb{Z}_N$ twist. The restriction indicated in Table 6.1 to the prototypical cases based on $N=3$, 6 and 7 which lead to $h^{1,1}=9$, 5 and 3 and $h^{1,2}=0$, is chosen so that the comparison with the case of smooth Calabi-Yau compactification, where we neglected the complex structure moduli, is more transparent.

7.3.1 Effective Kähler Potential

The results of subsection 7.1 may also be obtained by proceeding exactly as we did for compactifications on smooth Calabi-Yau manifolds. We shall briefly summarise how this is done for the three different kind of models under consideration. As before, for notational simplicity we shall omit to write explicitly the traces over the representation $\mathbf{R}$ of the gauge group G. We also omit any detail about the trace over the representation $\mathbf{r}$ of the structure group S, since this is discrete.

Models with $H = SU(3)$: Let us first consider the case of the $\mathbb{Z}_3$ orbifold, where $H = SU(3)$. In this case, $n_{\overline{\mathbf{27}}} = 9$. There are 9 harmonic $(1,1)$-forms ω_{ij} and 3 $\mathbb{Z}_3$-valued harmonic 1-forms u_i :

$$\omega_{ij} = idz^i \wedge d\bar{z}^j \qquad \text{and} \qquad u_i = dz^i. \tag{7.53}$$

The intersection numbers are found to be :

$$d_{ijpqrs} = \epsilon_{ipr}\epsilon_{jqr}. \tag{7.54}$$

The forms $c_{ij} = iu_i \wedge \bar{u}_j$ are found to be given by $c_{ij} = \omega_{ij}$, and their components on the ω_{mn} basis read :

$$c_{ij}^{mn} = \delta_i^m \delta_j^n. \tag{7.55}$$

The moduli fields T^{ij} and the matter fields Φ^i are defined by the following expressions :

$$J + iB = 2\Big(T^{ij} - \frac{1}{2}\Phi^i\bar{\Phi}^j\Big)\omega_{ij} \qquad \text{and} \qquad A = \Phi^i u_i. \tag{7.56}$$

The Kähler potential is finally found to be given by [31] :

$$K = -\log\left[\det\left(T^{ij} + \bar{T}^{ij} - \Phi^i\bar{\Phi}^j\right)\right]. \tag{7.57}$$

Models with $H = SU(2) \times U(1)$: Let us next consider the case of the $\mathbb{Z}_6$ orbifold, where $H = SU(2) \times U(1)$. In this case, $h^{1,1} = 5$ and thus $n_{\overline{\mathbf{27}}} = 5$. There are 5 harmonic $(1,1)$-forms ω_{ij}, ω_{33} and 3 $\mathbb{Z}_6$-valued harmonic 1-forms u_i, u_3, with $i = 1, 2$:

$$\begin{aligned} \omega_{ij} &= idz^i \wedge d\bar{z}^j, & \omega_{33} &= idz^3 \wedge d\bar{z}^3, \\ u_i &= dz^i, & u_3 &= dz^3. \end{aligned} \tag{7.58}$$

The non-vanishing entries of the intersection numbers are :

$$d_{ijpq33} = \epsilon_{ip3}\epsilon_{jq3}. \tag{7.59}$$

The forms $c_{ij} = i\, u_i \wedge \bar{u}_j$ are easily computed and one finds $c_{ij} = \omega_{ij}$, $c_{33} = \omega_{33}$, while the other vanish. The non-vanishing components of these forms on the ω_{mn} basis are :

$$c_{ij}^{mn} = \delta_i^m \delta_j^n \qquad \text{and} \qquad c_{33}^{33} = 1. \tag{7.60}$$

In this case, the moduli fields T^{ij}, T^{33} and the matter fields Φ^i, Φ^3 are defined by the following expressions :

$$\begin{aligned} &J + iB = 2\left(T^{ij} - \frac{1}{2}\Phi^i\bar{\Phi}^j\right)\omega_{ij} + 2\left(T^{33} - \frac{1}{2}\Phi^3\bar{\Phi}^3\right)\omega_{33}, \\ &A = \Phi^i u_i + \Phi^3 u_3. \end{aligned} \tag{7.61}$$

The Kähler potential is finally found to be given by [31] :

$$K = -\log\left[\det\left(T^{ij} + \bar{T}^{ij} - \Phi^i\bar{\Phi}^j\right)\left(T^{33} + \bar{T}^{33} - \Phi^3\bar{\Phi}^3\right)\right]. \tag{7.62}$$

Models with $H = U(1) \times U(1)$: Let us finally consider the case of the $\mathbb{Z}_7$ orbifold, where $H = U(1) \times U(1)$. In this case, $h^{1,1} = 3$ and thus $n_{\mathbf{\overline{27}}} = 3$. There are 3 harmonic $(1,1)$-forms ω_{11}, ω_{22}, ω_{33} and 3 $\mathbb{Z}_7$-valued harmonic 1-forms u_1, u_2, u_3 :

$$\begin{aligned} &\omega_{11} = idz^1 \wedge d\bar{z}^1, \qquad \omega_{22} = idz^2 \wedge d\bar{z}^2, \qquad \omega_{33} = idz^3 \wedge d\bar{z}^3, \\ &u_1 = dz^1, \qquad\qquad\quad u_2 = dz^2, \qquad\qquad\quad u_3 = dz^3. \end{aligned} \tag{7.63}$$

The non-vanishing entries of the intersection numbers are found to be :

$$d_{112233} = 1. \tag{7.64}$$

The forms $c_{ij} = i\, u_i \wedge \bar{u}_j$ are found to be given by $c_{11} = \omega_{11}$, $c_{22} = \omega_{22}$, $c_{33} = \omega_{33}$, while the others vanish. The non-vanishing components of these c_{ij} on the ω_{mn} basis read :

$$c_{11}^{11} = 1, \qquad c_{22}^{22} = 1 \qquad \text{and} \qquad c_{33}^{33} = 1. \tag{7.65}$$

The moduli fields T^{11}, T^{22}, T^{33} and the matter fields Φ^1, Φ^2, Φ^3 are defined by the following expressions :

$$J + iB = 2\Big(T^{ii} - \frac{1}{2}\Phi^i\bar{\Phi}^i\Big)\omega_{ii} \qquad \text{and} \qquad A = \Phi^i u_i. \tag{7.66}$$

The Kähler potential is finally found to be given by [31] :

$$K = -\log\left[\left(T^{11} + \bar{T}^{11} - \Phi^1\bar{\Phi}^1\right)\left(T^{22} + \bar{T}^{22} - \Phi^2\bar{\Phi}^2\right)\left(T^{33} + \bar{T}^{33} - \Phi^3\bar{\Phi}^3\right)\right]. \tag{7.67}$$

7.3.2 General Structure

The above results can be rewritten in a more convenient and unified way by performing a suitable change of basis for the harmonic $(1,1)$-forms, which clarifies their similarity with the results derived for Calabi-Yau compactifications, as already explained in section 7.1. To perform this change of basis, we can proceed in parallel for all the three models considered above and introduce the 3×3 Hermitian matrices λ^A representing the generators of $U(1) \times H$ and normalised in such a way that $\mathrm{Tr}\left(\lambda^A\lambda^B\right) = \delta^{AB}$. More precisely, λ^0 denotes the generator of $U(1)$ proportional to the identity matrix and λ^a the generators of H associated to a subset of the Gell-Mann matrices spanning the fundamental representation of $SU(3)$ ($a = 1, \cdots, 8$ for $H = SU(3)$, $a = 1, 2, 3, 8$ for $H = SU(2) \times U(1)$, $a = 3, 8$ for $H = U(1) \times U(1)$).

We then define the new basis of harmonic $(1,1)$-forms $\omega_A = \lambda^A_{ij}\omega_{ij}$. The corresponding new moduli fields then read $T^A = \lambda^A_{ji} T^{ij}$, and since the matrices λ^A are Hermitian, one finds $\bar{T}^A = \lambda^A_{ji}\bar{T}^{ij}$, where $\bar{T}^{ij}$ denotes as in the previous formulae the Hermitian conjugate of T^{ij} as a matrix. In this new basis, the intersection numbers are given by $d_{ABC} = \lambda^A_{ij}\lambda^B_{pq}\lambda^C_{rs} d_{ijpqrs}$, and the components c^A_{ij} of c_{ij} are given by $c^A_{ij} = \lambda^A_{nm} c^{mn}_{ij}$ which simply gives :

$$c^A_{ij} = \lambda^A_{ji}. \tag{7.68}$$

In this basis, the fields are defined as :

$$J + iB = 2\left(T^A - \frac{1}{2}c^A_{ij}\Phi^i\bar{\Phi}^j\right)\omega_A \qquad \text{and} \qquad A = \Phi^i u_i \tag{7.69}$$

and the Kähler potential takes the form (7.15), namely :

$$K = -\log\left[\frac{1}{6}d_{ABC}J^A J^B J^C\right] \qquad \text{where} \qquad J^A = T^A + \bar{T}^A - c^A_{ij}\Phi^i\bar{\Phi}^j. \tag{7.70}$$

For the untwisted sector of these orbifolds, one thus finds exactly the same kind of result as for smooth Calabi-Yau manifolds, with the peculiarity, however, that the intersection numbers d_{ABC} and the quantities c^A_{ij} admit a group-theoretical interpretation. This corresponds to the fact that the scalar manifold becomes a symmetric space. More precisely, in the three kinds of models under consideration the scalar manifolds are given by :

$$\begin{aligned}
\mathcal{M}_{SU(3)} &= \frac{SU(3,3+n)}{U(1)\times SU(3)\times SU(3+n)}, \\
\mathcal{M}_{SU(2)\times U(1)} &= \frac{SU(2,2+n)}{U(1)\times SU(2)\times SU(2+n)} \times \frac{SU(1,1+n)}{U(1)\times SU(1+n)}, \\
\mathcal{M}_{U(1)\times U(1)} &= \left(\frac{SU(1,1+n)}{U(1)\times SU(1+n)}\right)^3.
\end{aligned} \tag{7.71}$$

7.3.3 Range of Validity

For the untwisted sector of orbifold models, we see that the low-energy effective Kähler potential can always be derived in an exact way, without any limitation. From the perspective of the more general study that we performed for smooth Calabi-Yau manifolds, this reflects the fact that untwisted orbifold sectors automatically satisfy the assumptions that we made in subsection 7.2. More specifically, we see that the forms c_{ij} are harmonic and the quantities c^A_{ij} are constants. This can be traced back to the fact that in this case the forms ω_A and u_i are not only harmonic, but actually covariantly constant, which is a much stronger property.

7.4 General Structure of the Scalar Manifold

We have seen that for compactifications on both smooth Calabi-Yau manifolds and singular orbifolds the Kähler potential for the Kähler moduli and matter fields takes the same general form, at least under the already explained assumptions. We will now study in some more detail the general geometric features of this scalar manifold, which will be relevant for the structure of the soft scalar masses induced in the presence of a non-trivial superpotential. We will introduce for this purpose a new parametrisation of the scalar manifold, which will turn out to be very convenient at some special reference point.

7.4.1 Canonical Parametrisation

The general class of scalar manifolds we want to study is defined by the following Kähler potential, which only depends on the two symmetric and Hermitian but otherwise arbitrary constants d_{ABC} and c^A_{PQ} :

$$K = -\log\left[\frac{1}{6}d_{ABC}J^A J^B J^C\right] \qquad \text{where} \qquad J^A = T^A + \bar{T}^A - c^A_{PQ}\Phi^P\bar{\Phi}^Q. \tag{7.72}$$

The fields T^A and Φ^P define a specific parametrisation of the scalar manifold defined by this Kähler potential, which naturally emerges from string theory. We are however free to make holomorphic changes of coordinates as well as Kähler transformations to define other equivalent parametrisations. It turns out that this freedom can be used to define a particularly convenient kind of parametrisation. We shall call this the *canonical parametrisation*, because it is a natural generalisation including the $\mathcal{N} = 1$ matter sector of the one that was introduced in [139, 140] for the very special Kähler manifolds describing the $\mathcal{N} = 2$ moduli sector.

The main idea is to think of some reference point of particular interest on the scalar manifold, and then to perform a field redefinition that allows to simplify things as much as possible around that point. This reference point can for instance be thought of as the one defined by the VEVs $\langle T^A\rangle$ and $\langle\Phi^P\rangle$ that the scalar fields would eventually acquire in the presence of a non-trivial superpotential. Since our primary goal is to study situations where the moduli have sizeable VEVs whereas the matter fields have small VEVs, we shall start by considering the situation where :

$$\langle T^A\rangle \neq 0 \qquad \text{and} \qquad \langle\Phi^P\rangle = 0. \tag{7.73}$$

We may now reparametrise the fields in such a way to simplify the metric and the curvature tensor at such a point. To this aim, we shall consider the following linear field redefinitions :

$$\hat{T}^A = U^A{}_B T^B \qquad \text{and} \qquad \hat{\Phi}^P = V^P{}_Q\Phi^Q. \tag{7.74}$$

In addition, we may also perform a Kähler transformation on K. In particular, we may perform a trivial constant shift of the type :

$$\hat{K} = K - \log|\alpha|^2. \tag{7.75}$$

For our purposes, it will be enough to take $U^A{}_B$ to be a real matrix, $V^P{}_Q$ to be a complex matrix and α to be a real number. Under such transformations, the new Kähler potential in terms of the new fields has the same form as the original Kähler potential in terms of the original fields, but with new numerical coefficients given by :

$$\hat{d}_{ABC} = \alpha^2 (U^{-1})^D{}_A (U^{-1})^E{}_B (U^{-1})^F{}_C d_{DEF} \quad \text{and} \quad \hat{c}^A_{PQ} = U^A{}_B (V^{-1})^R{}_P (\bar{V}^{-1})^S{}_Q c^B_{RS}. \tag{7.76}$$

At this point, we may choose $U^A{}_B$ and $V^P{}_Q$ in such a way that the VEVs of the fields are aligned along one direction, the VEV of the metric becomes diagonal, and the overall scale of one of these two quantities (but not both) is set to some reference value. We may furthermore choose α to set the overall scale of the intersection numbers to a convenient value. More specifically, we shall require that in the new basis the reference point should be at :

$$\langle \hat{T}^A \rangle = \frac{\sqrt{3}}{2}\delta^A_0 \quad \text{and} \quad \langle \hat{\Phi}^P \rangle = 0. \tag{7.77}$$

The metric at that point should take the form :

$$\langle \hat{g}_{AB} \rangle = \delta_{AB} \qquad \langle \hat{g}_{PQ} \rangle = \delta_{PQ} \qquad \langle \hat{g}_{AQ} \rangle = 0 \tag{7.78}$$

and finally the Kähler frame should be such that at that point :

$$\langle \hat{K} \rangle = 0. \tag{7.79}$$

It is easy to get convinced by a counting of parameters that it is indeed always possible to impose this kind of conditions. Moreover, by comparing the transformed expressions for the VEVs of the fields, the metric and the Kähler potential with the values required in the previous equations, we deduce that the new values of the numerical coefficients $\hat{d}_{ABC}$ and $\hat{c}^A_{PQ}$ must satisfy the following properties :

$$\begin{aligned} &\hat{d}_{000} = \frac{2}{\sqrt{3}}, \qquad \hat{d}_{00a} = 0, \qquad \hat{d}_{0ab} = -\frac{1}{\sqrt{3}}\delta_{ab}, \\ &\hat{c}^0_{PQ} = \frac{1}{\sqrt{3}}\delta_{PQ} \end{aligned} \tag{7.80}$$

while $\hat{d}_{abc}$ and $\hat{c}^a_{PQ}$ are not constrained. The Kähler potential after the change of basis is then given by :

$$\hat{K} = -\log\left[\frac{1}{6}\left(\frac{2}{\sqrt{3}}\hat{\jmath}^0\hat{\jmath}^0\hat{\jmath}^0 - \sqrt{3}\hat{\jmath}^0\hat{\jmath}^a\hat{\jmath}^a + \hat{d}_{abc}\hat{\jmath}^a\hat{\jmath}^b\hat{\jmath}^c\right)\right] \tag{7.81}$$

where now :

$$\begin{aligned} \hat{\jmath}^0 &= \hat{T}^0 + \hat{\bar{T}}^0 - \frac{1}{\sqrt{3}}\delta_{PQ}\hat{\Phi}^P\hat{\bar{\Phi}}^Q, \\ \hat{\jmath}^a &= \hat{T}^a + \hat{\bar{T}}^a - \hat{c}^a_{PQ}\hat{\Phi}^P\hat{\bar{\Phi}}^Q. \end{aligned} \tag{7.82}$$

The above canonical parametrisation has a nice interpretation from the point of view of the properties of the Calabi-Yau manifold X and the holomorphic vector bundle V over it, on which the model is based. It essentially corresponds to a particular choice of bases for the harmonic forms $\hat{\omega}_A$ and $\hat{u}_P$ at the reference point defined by the VEVs. More specifically, the sets of harmonic forms $\hat{\omega}_A$ and $\hat{u}_P$ can be chosen to be orthonormal with respect to the natural positive definite metrics defined by $\hat{g}_{AB} = V^{-1} \int_X \hat{\omega}_A \wedge *\hat{\omega}_B$ and $\hat{g}_{PQ} = V^{-1} \int_X \hat{c}_{PQ} \wedge *J$, and one can moreover orient them in such a way that $\hat{\omega}_0$ is aligned with the Kähler form J. In this way the multiplets $\hat{T}^0$ and $\hat{T}^a$ describe respectively the overall volume and the relative Kähler moduli, and the fields $\hat{\Phi}^P$ are canonically defined. In this new basis, the VEV of the metric is the identity matrix, with $\hat{g}_{AB} = \delta_{AB}$ and $\hat{g}_{PQ} = \delta_{PQ}$, and as shown in Appendix B.3 the intersection numbers $\hat{d}_{ABC}$ and the quantities $\hat{c}^A_{PQ}$ do indeed take the structure of (7.80), after effectively setting the volume V to unity by a rescaling. It is worth remarking that if the traceful part of $\hat{c}_{PQ}$ were parallel to J and thus proportional to $\hat{\omega}^0$, whereas the remaining traceless part of $\hat{c}_{PQ}$ were orthogonal to J and thus a linear combination of the $\hat{\omega}^a$'s, all the matrices $\hat{c}^a_{PQ}$ would be traceless. This turns out to be the case for orbifolds, and it is not inconceivable that it might actually also hold true for most if not all of the Calabi-Yau's subject to the stringent restriction that the $(1,1)$-forms c_{PQ} are harmonic. We were not able to verify this, but we find it rather suggestive that the trace part of $\hat{c}_{PQ}$ indeed has positive-definite components, like J.

Notice that the new coordinates that have been introduced do not exactly coincide with normal coordinates at the reference point. Indeed, some of the components of the Christoffel connection have non-trivial values :

$$\begin{aligned} &\langle \Gamma_{00\bar{0}} \rangle = -\frac{2}{\sqrt{3}}, \quad \langle \Gamma_{0a\bar{b}} \rangle = -\frac{2}{\sqrt{3}}\delta_{ab}, \quad \langle \Gamma_{ab\bar{0}} \rangle = -\frac{2}{\sqrt{3}}\delta_{ab}, \quad \langle \Gamma_{ab\bar{c}} \rangle = -\hat{d}_{abc}, \\ &\langle \Gamma_{AP\bar{Q}} \rangle = -\hat{c}^A_{PQ}. \end{aligned} \tag{7.83}$$

Nevertheless, they turn out to lead to rather simple expressions for the Riemann curvature tensor at the reference point.

7.4.2 Curvature for Calabi-Yau Models

In the general case of compactifications on a smooth Calabi-Yau manifold, the scalar manifold $\mathcal{M}$ on which the low-energy effective theory is based is a generic Kähler manifold. The curvature of such a manifold depends on the point. Let us then consider the special reference point introduced above, assuming that it is dynamically selected by the superpotential, and let us switch to the canonical parametrisation. After a simple computation, one finds the following results for the VEV of the Riemann tensor :

$$
\begin{aligned}
&\langle R_{A\bar{B}C\bar{D}}\rangle = \delta_{AB}\delta_{CD} + \delta_{AD}\delta_{BC} - \hat{d}_{ACE}\hat{d}_{BDE}, \\
&\langle R_{P\bar{Q}R\bar{S}}\rangle = \frac{1}{3}\left(\delta_{PQ}\delta_{RS} + \delta_{PS}\delta_{RQ}\right) + \hat{c}^a_{PQ}\hat{c}^a_{RS} + \hat{c}^a_{PS}\hat{c}^a_{RQ}, \\
&\langle R_{P\bar{Q}0\bar{0}}\rangle = \frac{1}{3}\delta_{PQ}, \qquad \langle R_{P\bar{Q}a\bar{b}}\rangle = \frac{2}{3}\delta_{PQ}\delta_{ab} + (\hat{d}_{abc}\hat{c}^c - \hat{c}^a\hat{c}^b)_{PQ}, \\
&\langle R_{P\bar{Q}0\bar{b}}\rangle = \frac{1}{\sqrt{3}}\hat{c}^b_{PQ}.
\end{aligned}
\tag{7.84}
$$

These expressions are valid only around the point under consideration. In particular, they get deformed if one switches to a non-vanishing VEV for the matter fields.

7.4.3 Curvature for Orbifold Models

In the special case of orbifold compactifications, the scalar manifold $\mathcal{M}$ on which the low-energy effective theory is based is a symmetric Kähler manifold. The curvature of such a manifold does not depend on the point. Let us nevertheless consider the special reference point introduced above and switch as before to the canonical parametrisation. It is straightforward to verify that the new parametrisation described in subsection 7.3.2 actually coincides with the canonical one. To do so, one simply needs to recall that c^0 is equal to $\mathbb{1}/\sqrt{3}$, whereas the c^a are a subset of the transposed of the Gell-Mann matrices λ^a. One then gets :

$$
\hat{d}_{abc} = 2\mathrm{Tr}\left(\lambda^{(a}\lambda^b\lambda^{c)}\right) \quad \text{and} \quad \hat{c}^a_{ij} = \lambda^a_{ji}. \tag{7.85}
$$

We see that in this case $\hat{d}_{abc}$ is the symmetric invariant symbol of the group H, whereas the $\hat{c}^a_{ij}$ are the transposed of the generators of H in the representation $\mathbf{h}$ descending from the $\mathbf{3}$ of $SU(3)$ in terms of 3×3 matrices. In this case the transposed of the matrices $\hat{c}^a_{ij}$ possess the non-trivial property of being traceless and generating the Lie algebra of H, whose structure constants can be written as :

$$
f_{abc} = -2i\mathrm{Tr}\left(\lambda^{[a}\lambda^b\lambda^{c]}\right). \tag{7.86}
$$

Moreover, for all the three kinds of models one finds :

$$
[\lambda^a,\lambda^b] = if_{abc}\lambda^c \qquad \text{and} \qquad \{\lambda^a,\lambda^b\} = d_{abc}\lambda^c + \frac{2}{3}\delta_{ab}\mathbb{1}. \tag{7.87}
$$

Using these properties of the matrices λ^a, the components of the Riemann tensor are then seen to simplify and can entirely be rewritten in terms of these matrices :

$$
\begin{aligned}
&\langle R_{A\bar{B}C\bar{D}}\rangle = \mathrm{Tr}\left(\hat{c}^A\hat{c}^B\hat{c}^C\hat{c}^D\right) + \mathrm{Tr}\left(\hat{c}^A\hat{c}^D\hat{c}^C\hat{c}^B\right), \\
&\langle R_{P\bar{Q}R\bar{S}}\rangle = \hat{c}^A_{PQ}\hat{c}^A_{RS} + \hat{c}^A_{PS}\hat{c}^A_{RQ}, \\
&\langle R_{P\bar{Q}C\bar{D}}\rangle = (\hat{c}^D\hat{c}^C)_{PQ}.
\end{aligned}
\tag{7.88}
$$

These expressions are actually valid at any point of the scalar manifold, as already said. Their simple form reflects the fact that the curvature of symmetric manifolds is completely determined by the structure constants of their isometry group as is shown in the Appendix B.4.

7.5 $\mathcal{M}$-Theory Interpretation

In Chapter 5 we have introduced the $\mathcal{M}$-Theory conjecture according to which there exist an eleven-dimensional theory which when compactified on $\mathbb{S}^1/\mathbb{Z}_2$ leads to the $E_8 \times E_8$ Heterotic Superstring. The low-energy $\mathcal{M}$-Theory consists of eleven-dimensional SUGRA with E_8 matter fields located at each of the orbifold's fixed points. In the case where the six dimensions on the branes are first compactified on a Calabi-Yau with $SU(3)$ holonomy characterised by the two independent Hodge numbers $h^{1,1}$ and $h^{2,1}$, the resulting theory is an $\mathcal{N} = 1$ five-dimensional SUGRA with $h^{1,1} - 1$ vector multiplets and $h^{2,1} + 1$ hypermultiplets [141–144]. Indeed the eleventh-dimensional SUGRA bosonic sector consists of, as shown in Appendix C.2, the metric G_{AB} and a 3-form C_{ABC} which when splitting the eleven-dimensional indices A, B into five-dimensional indices M, N and internal indices i, $\bar{\jmath}$ gives :

$$
\begin{aligned}
G_{AB} \quad &: \quad G_{MN} && \rightarrow \quad \text{Graviton,} \\
& \quad\;\; G_{ij} && \rightarrow \quad h^{2,1}\,\text{complex scalar fields,} \\
& \quad\;\; G_{i\bar{\jmath}} && \rightarrow \quad h^{1,1}\,\text{real scalar fields,} \\
\\
C_{ABC} \quad &: \quad C_{MNP} && \rightarrow \quad 1\,\text{real scalar field,} \\
& \quad\;\; C_{Mi\bar{\jmath}} && \rightarrow \quad h^{1,1}\,\text{real vector fields,} \\
& \quad\;\; C_{ijk} && \rightarrow \quad 1\,\text{complex scalar field,} \\
& \quad\;\; C_{ij\bar{k}} && \rightarrow \quad h^{2,1}\,\text{complex scalar fields.}
\end{aligned}
\tag{7.89}
$$

The other components do not give rise to light modes since the corresponding number of zero-modes vanishes as indicated by (6.58). In five dimensions, the SUGRA multiplet $\mathcal{G}$ contains the graviton G_{MN}, the gravitino $\Psi_{M\alpha}$ and the graviphoton A_M which is identified with one of the $h^{1,1}$ vectors coming from $C_{Mi\bar{\jmath}}$. The other five-dimensional SUSY representations are the hypermultiplet $\mathcal{H}$ whose bosonic spectrum consists of two complex scalar fields and which can be recast as two four-dimensional chiral multiplets and the vector multiplet $\mathcal{V}$ whose bosonic spectrum consists of a real scalar field and a vector field which, in terms of four-dimensional multiplets, can be recast as the sum of a chiral multiplet T and a vector multiplet V. Let us spend a few words on this.

Superfield Formulation for five-dimensional Vector Multiplets A five-dimensional vector A_M may be split among a chiral Superfield T and a vector Superfield V as follows :

$$
T \;\ni\; \frac{i}{2} A_5(x^5, y) \qquad \text{and} \qquad V \;\ni\; -\theta\sigma^\mu\bar{\theta} A_\mu(x^5, y) \tag{7.90}
$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. The five-dimensional gauge-invariance translates into :

$$
T \rightarrow T + \partial_5\Lambda \qquad \text{and} \qquad V \rightarrow V + \Lambda + \bar{\Lambda} \tag{7.91}
$$

where Λ is an arbitrary chiral Superfield. One can then form the following two gauge-invariant combinations :

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V \qquad \text{and} \qquad -\partial_5 V + T + \bar{T} \ \ni\ \theta\sigma^\mu\bar{\theta}\left(\partial_5 A_\mu - \partial_\mu A_5\right). \tag{7.92}$$

The rigid SUSY five-dimensional Lagrangian density for $\mathcal{V} = (T, V)$ would then be written as follows :

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\int d^2\theta W^\alpha W_\alpha + \int d^4\theta \left(-\partial_5 V + T + \bar{T}\right)^2 + \text{h.c.} \\ &\ni -\frac{1}{4}F_{MN}F^{MN}\end{aligned} \tag{7.93}$$

where the second line is obtained after the auxiliary component of V is replaced with the solution to its algebraic equation of motion, see [145–147] and [148, 149] for the generalisation to the non-Abelian case. If allowing for a Chern-Simons term, *i.e.* a term of the form $A_5 F\tilde{F}$, the following gauge-variant quantity may be added to the Lagrangian density [146] :

$$\begin{aligned}\mathcal{L} &= \int d^2\theta T W^\alpha W_\alpha - \frac{1}{3}\int d^4\theta \left(\partial_5 V \overleftrightarrow{D}_\alpha V W^\alpha\right) + \frac{2}{3}\int d^4\theta \left(-\partial_5 V + T + \bar{T}\right)^3 + \text{h.c.} \\ &\ni -\frac{1}{2}\epsilon^{MNOPQ}A_M F_{NO} F_{PQ}.\end{aligned} \tag{7.94}$$

Both the expressions (7.93) and (7.94) will be relevant when discussing the rigid effects in the next subsection.

Parity Assignments The spectrum (7.89) can thus be arranged in one SUGRA multiplet $\mathcal{G} = (G_{MN}, \Psi_{M\alpha}, A^0_M)$, $h^{1,1} - 1$ vector multiplets $\mathcal{V}^a = (T^a, V^a)$ and $h^{2,1} + 1$ hypermultiplets split into $h^{2,1}$ $\mathcal{H}^x = (Z^x, Z^{x\prime})$ and one universal hypermultiplet $\mathcal{S} = (S, S')$. In order to recover the four-dimensional theory derived from the ten-dimensional Heterotic effective theory compactified on a Calabi-Yau, the following charge assignment emerges from section 5.5 :

$$\begin{aligned}&(T^a, V^a) \to (T^a, -V^a), \qquad (S, S') \to (S, -S'), \qquad (Z^x, Z^{x\prime}) \to (Z^x, -Z^{x\prime}), \\ &G \to G \qquad \text{and} \qquad A^0 \to -A^0\end{aligned} \tag{7.95}$$

whereas for $\mathcal{G}$ only the four-dimensional SUGRA multiplet $\mathcal{E}$ and one chiral multiplet T^0 formed out of A^0_5 and G_{55}, called the universal Kähler modulus, are preserved by the projection. The even $\mathcal{N} = 1$ multiplets leading to light modes in four dimensions thus consist in the gravitational multiplet $\mathcal{E}$, the dilaton S, the $h^{1,1}$ Kähler moduli T^A and the $h^{2,1}$ complex structure moduli Z^x.

The structure of the Kähler potential characterising the four-dimensional low-energy effective theories of heterotic string models admits a simple interpretation in terms of the intermediate five-dimensional effective theory emerging from the Calabi-Yau compactification of $\mathcal{M}$-Theory. In particular, the definition of the chiral multiplets and the Kähler potential structure can be understood quite naturally and intuitively within this framework.

As we shall now see, this is a consequence of the fact that the matter contact terms arising from the non-trivial shift in the field-strength of the 2-form B in the heterotic picture arises in the $\mathcal{M}$-Theory picture from the non-trivial Bianchi identity for the field-strength associated to C since C couples to the fields on the end-of-the-world branes :

$$(\mathrm{d}G)_{11IJKL} = -\mathrm{Tr}\,(F \wedge F)_{IJKL}\,\delta(y - y_0) \tag{7.96}$$

where G is the field-strength associated to the 3-form C and the indices are running on both the space-time and the Calabi-Yau manifold, excluding the $\mathbb{S}^1/\mathbb{Z}_2$ segment. Here and in the following, we shall implicitly understand the splitting of the charged fields over the two brane sectors located at different positions y_0, but for notational simplicity we shall not display this explicitly in the formulae.

Note that the Bianchi identity (7.96) does imply that the field-strength associated to the $h^{1,1}$ five-dimensional gauge fields A^A_M emerging from $C_{Mi\bar{\jmath}}$ as $C_{Mi\bar{\jmath}} = A^A_M \omega_{Ai\bar{\jmath}}$ has to satisfy a non-trivial Bianchi identity :

$$(\mathrm{d}F^A)_{5\mu\nu} = -ic^A_{PQ}\left(\partial_\mu \Phi^P \partial_\nu \bar{\Phi}^Q - \partial_\mu \bar{\Phi}^Q \partial_\mu \Phi^P\right)\delta(y - y^0) \tag{7.97}$$

which is solved by :

$$F^A_{5\mu} = \partial_5 A^A_\mu - \partial_\mu A^A_5 + ic^A_{PQ}\Phi^P \overset{\leftrightarrow}{\partial_\mu} \bar{\Phi}^Q \delta(y - y^0). \tag{7.98}$$

This suggests that the second relation in (7.92) has to be modified in order to take into account the non-trivial Bianchi identity. It will indeed prove useful to define the following quantity of which the $\theta\sigma^\mu\bar{\theta}$-component is easily shown to coincide with $F^A_{5\mu}$:

$$J^A_5 \equiv -\partial_5 V^A + T^A + \bar{T}^A - c^A_{PQ}\Phi^P \bar{\Phi}^Q \delta(y - y^0). \tag{7.99}$$

7.5.1 Effective Kähler Potential

The four-dimensional effective Kähler potential can be determined by performing the reduction of the eleven-dimensional theory on the Calabi-Yau manifold X, and then further reducing the resulting five-dimensional theory on $\mathbb{S}^1/\mathbb{Z}_2$. In this case, it is possible to do the last step by using Superfields to directly compute the Kähler potential, rather than working with the components and looking at the bosonic kinetic terms. To perform this computation, we shall do the same approximations as in section 7.2. We shall first neglect the effects of higher-derivative corrections to the eleven-dimensional effective theory and deformations of the basic background, and simply consider the reduction of the two-derivative effective theory on $X \times \mathbb{S}^1/\mathbb{Z}_2$. We shall then also discard the effects of massive Kaluza-Klein modes on X, although we will retain the effects of massive Kaluza-Klein modes on $\mathbb{S}^1/\mathbb{Z}_2$, which turn out to be crucial to understand the contact terms. Correspondingly, we will also make the same assumptions as in section 7.2, namely that the $(1,1)$-forms c_{PQ} associated to composites of two matter fields are harmonic and that the quantities c^A_{PQ} are constant topological invariants. Finally, we shall again restrict to the Kähler moduli T^A and the charged matter fields Φ^P.

The starting point is thus the 5D intermediate theory, where we retain not only the $\mathbb{Z}_2$-even submultiplets T^0, T^a, Φ^P, which contain the light four-dimensional moduli and matter modes, but also the $\mathbb{Z}_2$-odd submultiplets V^a, which contain the heavy Kaluza-Klein modes that have non-trivial linear couplings to the other fields and therefore need to be properly integrated out. It is convenient to work with $\mathcal{N}=1$ Superfields T^0, T^a, Φ^P and V^a depending also on the internal coordinate y, and integrate out the heavy modes associated to the V^a's directly at the Superfield level by solving their equations of motion, neglecting space-time derivatives, to determine their wave-function profile.

Rigid Effects In the limit where gravity is decoupled, this can be done with usual Superfields within rigid Supersymmetry along the lines of [145–150], with $T = T^0/\sqrt{3}$ playing the rôle of the radion Superfield. Indeed in such a case one may generalise (7.93) and (7.94) as the following :

$$\begin{aligned}\mathcal{L} = \frac{1}{4}\int d^2\theta \left[T\mathcal{F}_{ab}\left(\frac{T^c}{T}\right)W^aW^b - \frac{1}{12}\mathcal{F}_{abc}\bar{D}^2\left(V^a \overset{\leftrightarrow}{D^\alpha}\,\partial_5 V^b\right)W^c_\alpha\right] \\ + \int d^4\theta (T+\bar{T})\mathcal{F}\left(\frac{J^a_5}{T+\bar{T}}\right)\end{aligned} \tag{7.100}$$

where, since the five-dimensional theory has an enhanced $\mathcal{N}=2$ Supersymmetry from the four-dimensional point of view, $\mathcal{F}$ is an at-most cubic prepotential [151] of the form :

$$\mathcal{F}(Z^a) = \frac{1}{2}Z^aZ^a - \frac{1}{6}\mathfrak{d}_{abc}Z^aZ^bZ^c \tag{7.101}$$

where the first term in $\mathcal{F}$ is responsible for the generalisation of (7.93) whilst the second relates to (7.94). The effective four-dimensional theory is found by dropping the first term in (7.100) and by replacing the currents J^a_5 by their zero-mode $J^a = T^a + \bar{T}^a - c^a_{PQ}\Phi^P\bar{\Phi}^Q$. As one can notice by plugging the currents (7.99) into the norm function (7.101), this procedure is not totally straightforward since the latter expression contains powers of the brane-localising δ-function. The physical meaning of such terms has first been grasped by Mirabelli and Peskin [152] in the context of a five-dimensional Super-Yang-Mills theory coupled to chiral fields on end-of-the-world branes. It was shown that the higher powers of $\delta(x^5)$ were serving as counter-terms in the microscopic theory in order to compensate for singularities introduced by Superfields that are odd under the orbifold action. The extension of this work to five-dimensional Supergravity coupled to both chiral and vector Superfields on the branes has been performed in [26] in order to compute the loop-induced soft scalar masses in the Randall and Sundrum setup [20], which as argued in the Introduction leads to vanishing soft scalar masses at the classical level, see Section 8.1. In both [152] and [26], the higher powers of the δ-function appeared when integrating the auxiliary fields out, *i.e.* when going on-shell. In our case the situation is slightly different since the $\delta^n(0)$ terms already appear off-shell. The procedure on how to deal with such terms is however similar and has been explained in [153]. Let us exemplify it in a simple case in which the five-dimensional Lagrangian is given by :

$$\mathcal{L} = \int d^4\theta \left(-\partial_5 V + T + \bar{T} - \delta(y)C\right)^2 = \int d^4\theta J_5^2. \tag{7.102}$$

In order to obtain the effective four-dimensional theory, we first integrate V out. Its equation of motion is given by :

$$\frac{\delta \mathcal{L}}{\delta V} = 2\partial_5 \left(-\partial_5 V + T + \bar{T} - \delta(y)C\right) \stackrel{!}{=} 0 \tag{7.103}$$

which is solved by :

$$J_5 = -\partial_5 V + T + \bar{T} - \delta(y)C = \text{const}. \tag{7.104}$$

In order to determine the constant on the RHS of (7.104), one integrates both sides along the y coordinate, yielding :

$$\text{const} = T + \bar{T} - C = J \tag{7.105}$$

where T now stands for its zero-mode since all other Kaluza-Klein modes integrate to zero and where C has be renormalised by the orbifold covering-space length. The four-dimensional effective theory obtained after having integrated V out is thus found by replacing J_5 by its zero-mode J as advertised. Generalising this procedure to arbitrary powers of J_5 is straightforward. When applied to the five-dimensional Lagrangian density (7.100), this procedure yields the following four-dimensional expression :

$$\mathcal{L} = \int d^4\theta (T + \bar{T}) \mathcal{F}\left(\frac{J^a}{T + \bar{T}}\right) = \int d^4\theta \left(\frac{1}{2}\frac{J^a J^a}{T + \bar{T}} - \frac{1}{6}\mathfrak{d}_{abc}\frac{J^a J^b J^c}{(T + \bar{T})^2}\right). \tag{7.106}$$

We will show in the next paragraph that by setting $\mathfrak{d}_{abc} = d_{abc}$ one successfully reproduces the structure of the interactions involving two and three currents but misses all other orders which are thus genuine gravitational effects.

Gravitational Effects Taking into account gravitational effects is slightly more complicated, but can actually be done in a very similar way by using a superconformal Superfield formalism within Supergravity, where half of the Supersymmetry is manifestly realised off-shell. This formalism has been developed in [154–157] and further elaborated in [153, 158–161]. It has the nice feature of allowing to describe the graviphoton A_M^0 on the same footing as the other odd gauge fields A_M^a, and the volume modulus T^0 on the same footing as the other Kähler moduli T^a, through vector multiplets V^A and chiral multiplets T^A with $A = 0, a$, at the price of introducing also some constraints. The relevant 5D Lagrangian turns out to be :

$$\begin{aligned}\mathcal{L}_5 = &\int d^2\theta \left[-\frac{1}{4}\mathcal{N}_{AB}(T^A) W^A W^B + \frac{1}{48}\mathcal{N}_{ABC}\bar{D}^2\left(V^A \overleftrightarrow{D}^\alpha \partial_5 V^B\right) W_\alpha^C\right] + \text{c.c.} \\ &+ \int d^4\theta (-3)\mathcal{N}^{1/3}(J_5^A).\end{aligned} \tag{7.107}$$

In this expression, $\mathcal{N}$ is a norm function playing the rôle of a real prepotential, which is identified with the cubic polynomial defined by the intersection numbers d_{ABC} of the Calabi-Yau manifold X :

$$\mathcal{N}(Z^A) = \frac{1}{6} d_{ABC} Z^A Z^B Z^C. \tag{7.108}$$

The quantity W_α^A denotes the usual super-field-strength associated to V^A, namely :

$$W_\alpha^A = -\frac{1}{4} \bar{D}^2 D_\alpha V^A. \tag{7.109}$$

Finally, the quantity J_5^A is a current expressed in terms of the quantities c^A_{PQ} characterising the vector bundle V over X defined in (7.99).

Rigid Limit Let us now use the canonical parametrisation we have introduced in the last section. By introducing $Z = Z^0/\sqrt{3}$ and using the intersection numbers (7.80) we can rewrite $\mathcal{N}(Z^A)$ as :

$$\mathcal{N}(Z^A) = Z^3 - \frac{1}{2} Z Z^a Z^a + \frac{1}{6} d_{abc} Z^a Z^b Z^c. \tag{7.110}$$

The relevant quantity when taking into account gravitational effects is given by $-3\,\mathcal{N}^{1/3}$ as shown by (7.107). Since at the point under consideration we have $J_5 \gg J_5^a$, we may approximate it by :

$$-3\,\mathcal{N}^{1/3}(J_5^A) \simeq -3J_5 + \frac{1}{2J_5} J_5^a J_5^a - \frac{1}{6J_5^2} d_{abc} J_5^a J_5^b J_5^c \tag{7.111}$$

whose second and third term match the rigid expression in (7.100) :

$$(T+\bar{T})\mathcal{F}\left(\frac{J_5^a}{T+\bar{T}}\right) = \frac{1}{2(T+\bar{T})} J_5^a J_5^a - \frac{1}{6(T+\bar{T})^2} \mathfrak{d}_{abc} J_5^a J_5^b J_5^c \tag{7.112}$$

provided that we identify $\mathfrak{d}_{abc}$ with d_{abc} and that J_5, associated to the graviphoton, is decoupled and identified with the radion field.

Kähler Potential In the above expressions, the bosonic modes of T^A come from the decomposition of the Kähler form J and the 2-form C_5 with components $iG_{i\bar{\jmath}}$ and $C_{5i\bar{\jmath}}$ on the basis of harmonic $(1,1)$-forms ω_A, the bosonic modes of Φ^P come from the decomposition of the Lie-algebra-valued 1-forms A with components A_i on the basis of harmonic 1-forms u_P, and finally the bosonic modes of V^A come from the decomposition of the 2-forms C_μ with components $C_{\mu i\bar{\jmath}}$ on the basis ω_A. The correct definition of the chiral multiplets in terms of the above modes turns out to be [34] :

$$T^A = \frac{1}{2}\left(J^A + iC_5^A + c^A_{PQ} A^P \bar{A}^Q \delta(y-y_0)\right) \qquad \text{and} \qquad \Phi^P = A^P \tag{7.113}$$

where $C_5^A = A_5^A$. We see that these definitions reproduce the ones we have introduced in the component derivation of subsection 7.2.3 based on the weakly coupled heterotic string when averaged over the extra dimension. Here these definitions ensure that the lowest component of J_5^A simply reduces to the metric components, as required in order to reproduce an Einstein gravitational kinetic term coming entirely from the bulk and not from the branes, whereas the $\theta\sigma^\mu\bar{\theta}$ component of J_5^A correctly reproduces the modified version of the mixed components of the field strength implied by the reduction of the Bianchi identity (7.96) :

$$\begin{aligned} J_5^A| &= -\partial_5 V^A| + J^A| = J^A, \\ J_5^A|_{\theta\sigma^\mu\bar{\theta}} &= \partial_5 A_\mu^A - \partial_\mu A_5^A + ic_{PQ}^A \Phi^P \overset{\leftrightarrow}{\partial_\mu} \bar{\Phi}^Q \delta(y-y_0). \end{aligned} \tag{7.114}$$

This provides a nice Superfield interpretation on the need for the shift in the definition of the moduli chiral multiplets.

Integrating out the heavy modes of the vector multiplets V^A again effectively amounts to replacing the currents J_5^A with their zero modes in the term of the action that does not involve the vector fields. This is however more difficult to show than in the rigid limit, where only the V^a matter, since in the Supergravity regime all the V^A appear but suffer from non-trivial constraints [159–161]. One finds the following expression, written within the usual superconformal Superfield formalism :

$$\mathcal{L}_4 = \int d^4\theta(-3)\mathcal{N}^{1/3}(J^A) \tag{7.115}$$

where now :

$$J^A = T^A + \bar{T}^A - c_{PQ}^A \Phi^P \bar{\Phi}^Q. \tag{7.116}$$

The effective Kähler potential can finally be deduced by matching the integrand of this expression with $-3\,e^{-K/3}$. This gives $K = -\log \mathcal{N}(J^A) = -\log V$, which is the same result as we obtained directly from the Heterotic string :

$$K = -\log\left[\frac{1}{6} d_{ABC} J^A J^B J^C\right] \qquad \text{where} \qquad J^A = T^A + \bar{T}^A - c_{PQ}^A \Phi^P \bar{\Phi}^Q. \tag{7.117}$$

A component version of this five-dimensional derivation is also possible, and was presented in [162] for the particular case where $h^{1,1} = 1$ with Standard Embedding.

The effective Kähler potential for the untwisted sector of orbifold compactifications can be similarly derived from an $\mathcal{M}$-Theory perspective. The only changes are that the intersection numbers d_{ABC} and the quantities c_{PQ}^A acquire a simple group-theoretical interpretation. Moreover, in this case the forms c_{PQ} are automatically harmonic and the quantities c_{PQ}^A are always constant. Further details on a component version of this five-dimensional derivation can be found in [163–165].

Current-Current Structure Let us again use the canonical parametrisation in order to rewrite the four-dimensional Kähler potential and then compare it to the case of the Randall and Sundrum setup. Since we have found $K = -\log V$, we have $\Omega = -3e^{-K/3} = -3V^{1/3}$:

$$\begin{aligned}\Omega &= -3\left(J^3 - \frac{1}{2}J^a J^a + \frac{1}{6}d_{abc}J^a J^b J^c\right)^{1/3} \\ &\simeq -3J + \frac{1}{2}\frac{J^a J^a}{J} - \frac{1}{6}d_{abc}\frac{J^a J^b J^c}{J^2}\end{aligned} \tag{7.118}$$

whereas in the Randall and Sundrum setup we find $K = -\log(J^3)$ [20] and thus :

$$\Omega_{\mathrm{RS}} = -3J. \tag{7.119}$$

We have thus shown that the deviation for the sequestered picture is indeed due to current-current interactions [35], as argued in [22, 23]. More precisely, there is one such interaction for every non-minimal Kähler modulus T^a which are associated with vector multiplets in the five-dimensional picture.

7.5.2 Range of Validity

We have seen in the previous subsection that the results derived in subsection 7.2 for the low-energy effective Kähler potential admit a simple 5D interpretation, in which the non-trivial contact terms spoiling the sequestered structure arise from the exchange of heavy 4D Kaluza-Klein modes of the light 5D vector multiplets coming from the harmonic components of the $\mathcal{M}$-Theory 3-form C on X. This interpretation was however derived under the restrictive assumptions that the forms c_{PQ} are harmonic and that the quantities c^A_{PQ} are constants. It is then natural to wonder once again what would be the situation if these assumptions were to be relaxed.

The relevance of the assumptions about c_{PQ} and c^A_{PQ} within the $\mathcal{M}$-Theory perspective must obviously be very similar to that already discussed within the Heterotic perspective. But it turns out to offer a slightly sharper perspective. The harmonicity of c_{PQ} is as before needed to ensure the trivial decoupling of heavy neutral modes from pairs of light charged modes. More specifically, we see here that when c_{PQ} is not harmonic a direct danger comes from the heavy 5D vector multiplets that arise from the non-harmonic components of the 3 form C on X. Indeed, such heavy modes can be brutally truncated away only when they are not sourced by light fields, and from the reduction of the solution of the Bianchi identity (7.96) we see that this is the case only when the non-harmonic parts of C describing the heavy 5D vector modes have no overlap with the forms c_{PQ} describing the composite of two light matter modes, that is when c_{PQ} is harmonic. In the opposite case, one would have to properly integrate out these heavy 5D vector modes too, and this would give extra contributions to the contact terms in the 4D effective Kähler potential. These additional effects must correspond to the additional terms that would arise in the left-hand side of (7.37) within the Heterotic perspective. The constancy of c^A_{PQ} is again needed to ensure a simple determination of the right definition of the chiral multiplets containing the moduli. More specifically, we see here that for moduli-dependent c^A_{PQ} it is not clear how one should modify the definitions (7.113) to arrange that (7.114) holds true.

Chapter 8

Soft Scalar Masses and Sequestering

In the last Chapter we have derived the form of the effective four-dimensional Kähler potential for both orbifold and Calabi-Yau compactifications of the Heterotic $E_8 \times E_8$ Superstring Theory. Using the formulae we have derived in Chapter 4 we can now compute the contributions to the visible scalar masses from the hidden sector. Since these masses are found to be generically non-vanishing and non-universal, they induce FCNC processes which lead to the rejection of such theories since Nature seems to have chosen not to allow such processes, at least at today's accessible energies. One of the ideas towards a solution of this problem is sequestering [20], *i.e.* a setup in which the visible and hidden sectors are geographically separated along an extra-dimension thus effectively forbidding local contact terms. Soft scalar masses do then vanish at the classical level. Quantum effects will tend to generate soft scalar masses which, thanks to the geographical separation among the visible and hidden sectors, are insensitive to far UV physics and may thus lead to universal soft scalar masses [24–27]. The purpose of the present Chapter is to investigate whether a similar mechanism can apply in Heterotic string models.

8.1 Mild Sequestering

From the effective four-dimensional theory the sequestering mechanism manifests itself by restricting the form of the Kähler potential K. Indeed if gravity was turned off the two sectors would not be able to communicate forcing the Lagrangian density to be the sum of two terms, one for the visible sector and one for the hidden one :

$$\Omega = -3e^{-K/3} = \Omega_v + \Omega_h \qquad i.e. \qquad K = -3\log\left(\Omega_v + \Omega_h\right) \tag{8.1}$$

which is indeed the form taken by the Kähler potential in the Randall and Sundrum setup as can be read from (7.81) :

$$K = -3\log\left(T + \bar{T} - \frac{1}{3}\Phi^P\bar{\Phi}^P\right) \tag{8.2}$$

where $T = T^0/\sqrt{3}$ is the Kähler modulus associated to the graviphoton [20]. Following the methods developed in Chapter 4, the soft masses are straightforwardly shown to vanish reflecting the fact that the scalar manifold is maximally symmetric [21, 166]. Indeed by using the technology developed in Appendix B.2.4.2, the Riemann tensor entering the expression of the soft scalar masses (4.49) is easily shown to be given by :

$$R_{\alpha\bar{\beta}\Theta\bar{\Gamma}} = \frac{1}{3}K_{\alpha\bar{\beta}}K_{\Theta\bar{\Gamma}} \tag{8.3}$$

where α and β denote visible fields and Θ and Γ hidden fields. By plugging (8.3) in (4.49) we find that the soft scalar masses vanish :

$$m^2_{\alpha\bar{\beta}} = 0. \tag{8.4}$$

However the situation is not as satisfactory as it seems to be at first sight since when trying to apply the idea of sequestering to $\mathcal{M}$-Theory-inspired models in which the five-dimensional picture arises after $\mathcal{M}$-Theory is compactified on a Calabi-Yau manifold one is confronted with the appearance of vector multiplets propagating in the bulk which spoil the sequestered structure displayed by (8.1) as argued in [22, 23]. This phenomenon occurs in a rather clear way in the case of Heterotic $\mathcal{M}$-Theory compactified either on an orbifold or on a Calabi-Yau manifold where the appearance of Kähler moduli associated with five-dimensional vector multiplets descending from the C 3-form induce non-trivial corrections to the Kähler potential which spoil the fact that the scalar manifold is maximally symmetric. In such situations one generically finds non-vanishing non-universal soft scalar masses generated from the contact terms induced by the vector multiplets.

Albeit not being in a good position, the idea of sequestering the visible and hidden sectors along an extra-dimension may be saved by exploiting the peculiar structure the contact terms enjoy. Indeed since these terms originate from the integration of heavy Kaluza-Klein modes, they essentially are of the current-current form. As has been illustrated in subsection 3.7.2, the most relevant terms giving rise to soft scalar masses are dimension-6 operators containing two visible Superfields Φ together with two hidden Superfields X which are encoded in the Superspace wave-function $Z^{\alpha\beta}$:

$$\mathcal{L} = \int d^4\theta Z^{\alpha\beta}(X,\bar{X})\Phi^\alpha\bar{\Phi}^\beta. \tag{8.5}$$

The soft masses arising from such a Lagrangian density are given by (3.48) :

$$m^2_{\alpha\bar{\beta}} = -\left[Z^{\alpha\beta}_{\Theta\bar{\Gamma}} - (Z^{-1})^{mn}Z^{\alpha m}_{\Theta}Z^{n\beta}_{\bar{\Gamma}}\right]F^\Theta\bar{F}^\Gamma. \tag{8.6}$$

The soft scalar masses thus depend on both the F and D terms of $Z^{\alpha\beta}$. Now, if we can engineer a situation such that $Z^{\alpha\beta}$'s F and D terms vanish, the previous equation would imply the vanishing of the soft scalar masses at the classical level. Let us argue that the auxiliary components of a conserved Superfield exactly satisfies this request.

Indeed, the conservation of a current at the Superfield level corresponds not only to the conservation of the vector current but also to the vanishing of both their F and D components. Let us illustrate this mechanism in rigid SUSY by considering the following Lagrangian density where we have used the properties of the Berezin integral to rewrite it as an integral on half the Superspace :

$$\mathcal{L} = \int d^2\theta \left(-\frac{1}{4}\bar{D}^2 K + W\right) \qquad \overset{\text{EOM}}{\longrightarrow} \qquad -\frac{1}{4}\bar{D}^2 K_i + W_i = 0. \tag{8.7}$$

Let us furthermore imagine that the global symmetry is enforced by $\delta\Phi^i = \Lambda^a X_a^i$ where the X_a^i are holomorphic Killing fields. Under the global symmetry the Kähler potential is allowed to be shifted by a Kähler transformation $\delta K = \Lambda^a f_a + \bar{\Lambda}^a \bar{f}_a$ while the superpotential variation has to vanish identically $\delta W = 0$. These two conditions respectively imply :

$$\Re\text{e}(K_i X_a^i - f_a) = 0 \qquad \text{and} \qquad W_i X_a^i = 0. \tag{8.8}$$

The Noether currents :

$$J^a = \Im\text{m}(K_i X_a^i - f_a) \tag{8.9}$$

are shown to be conserved, *i.e.* $\bar{D}^2 J^a = D^2 J^a = 0$, since :

$$\bar{D}^2(K_i X_a^i - f_a) = (\bar{D}^2 K_i) X_a^i = 4 W_i X_a^i = 0. \tag{8.10}$$

The conservation equation for J straightforwardly implies that its F-component vanishes. Moreover it is easily shown that the vector field contained in J has a vanishing divergence and that the D-component of J also vanishes, see for example [167]. This means that terms which couple the conserved-current multiplet to the visible sector do not generate any soft masses. This mechanism of soft masses cancellation is called *mild sequestering* [29] and was first introduced in the context of conformal sequestering where the current operators proved to be problematic to suppress via large running effect since they are characterised by a vanishing anomalous dimension forbidding them to run [28].

The vanishing of the F and D components of J^a are summarised in the following two equations :

$$\begin{aligned} J^a|_F = 0 \qquad &\leftrightarrow \qquad X_{a\bar{j}} \bar{F}^{j} = 0, \\ J^a|_D = 0 \qquad &\leftrightarrow \qquad \nabla_i X_{a\bar{j}} F^i \bar{F}^{j} = 0. \end{aligned} \tag{8.11}$$

These equations may also be obtained by expressing the superpotential invariance for the first one and by multiplying the first one by $W^i \nabla_i$ and using the stationarity condition $W^m \nabla_i W_m =$

0 for the second one. This second procedure has the advantage of being simple to generalise when considering SUGRA theories [35]. In the latter the Kähler potential and superpotential variations under $\delta\phi^i = \Lambda^a X^i_a$ are not independent : $\delta K = \Lambda^a f_a + \bar{\Lambda}^a \bar{f}_a$ and $\delta W = e^{-\Lambda^a f_a} W$ meaning that $G = K + \log|W|^2$ has to be invariant under the symmetry.

The scalar potential in SUGRA theories has been derived in section 4.3 and reads :

$$V = e^G (G_i G^i - 3). \tag{8.12}$$

The cosmological constant is set to zero by tuning $G_i G^i = 3$ at the minimum while the stationarity condition reads $G_i + G^j \nabla_i G_j = 0$. Recall also that the auxiliary fields are given by $F^i = -e^{G/2} G^i$ and that the gravitino mass is $m_{3/2} = e^{G/2}$. The G invariance then directly leads to $\Re\mathrm{e}(G_i X^i_a) = 0$ which implies :

$$X_{a\bar{j}} \bar{F}^j = -i \Im\mathrm{m}(G_i X^i_a) m_{3/2} \tag{8.13}$$

while acting with the operator $G^i \nabla_i$ on the invariance condition and using the stationarity condition gives :

$$\nabla_i X_{a\bar{j}} F^i \bar{F}^j = 2i \Im\mathrm{m}(G_i X^i_a) m^2_{3/2}. \tag{8.14}$$

From the last two equations we conclude that the identities responsible for the mild-sequestering mechanism get modified in presence of gravity, leading to the question of the effectiveness of such a mechanism in the context of local SUSY. In Chapter 6 a careful inspection of the sigma-model metric lead us to the knowledge of the second derivatives of the function G. However such an information does not allow us to reconstruct the whole of G, indeed terms that can be written as the sum of an holomorphic with an antiholomorphic function will not be captured. We thus define the result coming from the sigma-model metric inspection to be the Kähler potential K since all the terms we miss can be recast in a superpotential. This amounts to fixing a Kähler gauge in which the variation under a symmetry are given by $\delta K = \Lambda^a f_a + \bar{\Lambda}^a \bar{f}_a$ and $\delta W = e^{-\Lambda^a f_a} W$. We can now rewrite $G_i X^i_a$ as :

$$G_i X^i_a = K_i X^i_a + \frac{W_i X^i_a}{W} = K_i X^i_a - f_a. \tag{8.15}$$

We may thus rewrite the equations (8.13) and (8.14) as :

$$X_{a\bar{j}} \bar{F}^j = i D_a m_{3/2} \qquad \text{and} \qquad \nabla_i X_{a\bar{j}} F^i \bar{F}^j = -2i D_a m^2_{3/2} \tag{8.16}$$

where we have introduced $D_a = -\Im\mathrm{m}(G_i X^i_a) = -\Im\mathrm{m}(K_i X^i_a - f_a)$. This notation reflects the fact that the D_a's are the Killing potentials for the X^i_a's :

$$i \nabla_i D_a = X_{ai} \qquad \text{and} \qquad -i \nabla_{\bar{j}} D_a = X_{a\bar{j}}. \tag{8.17}$$

In order for mild-sequestering to be at work, the D_a have to vanish or to be negligible. We will thus concentrate on symmetries which do not involve f_a shifts and which are such that $K_i X^i_a = 0$.

8.2 Soft Scalar Masses

Let us now come to the crucial question of what are the properties of soft scalar masses in the effective theories for heterotic string models compactified on a generic Calabi-Yau manifold with a generic stable holomorphic vector bundle over it, in the presence of some source of supersymmetry-breaking. We shall restrict our analysis to the Kähler moduli and matter fields, for which we know the form of the Kähler potential, and to the neighbourhood of the reference point introduced in the last section, by assuming that the superpotential that induces supersymmetry-breaking is such that the scalar VEVs of the moduli and matter scalar fields are respectively generic and small. We will first work out the general structure of the soft scalar masses and then study the possibility of ensuring the vanishing of these masses with the help of some kind of global symmetry.

8.2.1 Structure of Scalar Masses

Our starting point is the effective Kähler potential (7.72), which is characterised by the two constants d_{ABC} and c^A_{PQ}. Since we want to study soft terms at the particular reference point (7.77) introduced in Chapter 6, it will be convenient to switch to the canonical parametrisation that we defined there. From now on, we shall for simplicity drop all the hats on the redefined parameters and fields, and also the brackets denoting VEVs at the reference point. It will moreover be convenient to further redefine $T = T^0/\sqrt{3}$ and correspondingly $J = J^0/\sqrt{3}$, and to explicitly split the matter fields Φ^P into two sets Q^α and X^i respectively coming from the two E_8 factors. The visible sector is then identified with the fields Q^α and the hidden sector generically contains all the remaining fields X^i, T, T^a, and the Kähler potential becomes :

$$K = -\log\left(J^3 - \frac{1}{2}JJ^aJ^a + \frac{1}{6}d_{abc}J^aJ^bJ^c\right) \tag{8.18}$$

where :

$$\begin{aligned} J &= T + \bar{T} - \frac{1}{3}Q^\alpha\bar{Q}^\alpha - \frac{1}{3}X^i\bar{X}^i, \\ J^a &= T^a + \bar{T}^a - c^a_{\alpha\beta}Q^\alpha\bar{Q}^\beta - c^a_{ij}X^i\bar{X}^j. \end{aligned} \tag{8.19}$$

Let us now study this expression around the point under consideration given by (7.77), which when expressed in the new coordinates sits at :

$$T = \frac{1}{2}, \qquad T^a = 0, \qquad Q^\alpha = 0 \qquad \text{and} \qquad X^i = 0. \tag{8.20}$$

Note that at this point the only non-vanishing component of the first derivative of K is along the T direction, so that $K_\alpha = 0$, $K_i = 0$ and $K_a = 0$. Under the mild restriction that the considered symmetry should not act on T and should not involve Kähler shifts, meaning that both $k^T_a = 0$ and $f_a = 0$ should be satisfied, one gets $D_a = 0$. Under this assumption, one can then use the rigid version of (8.16).

At the point under consideration the metric takes a diagonal form, the only non-vanishing entries being given by :

$$g_{T\bar{T}} = 3, \qquad g_{a\bar{b}} = \delta_{ab}, \qquad g_{\alpha\bar{\beta}} = \delta_{\alpha\beta} \qquad \text{and} \qquad g_{i\bar{j}} = \delta_{ij}. \tag{8.21}$$

The Christoffel connection non-vanishing components are found to be given by :

$$\begin{aligned} &\Gamma_{TT\bar{T}} = -6, \quad \Gamma_{Ta\bar{b}} = -2\delta_{ab}, \quad \Gamma_{ab\bar{T}} = -2\,\delta_{ab}, \\ &\Gamma_{ab\bar{c}} = -d_{abc}, \quad \Gamma_{TP\bar{Q}} = -\delta_{PQ}, \quad \Gamma_{aP\bar{Q}} = -c^a_{PQ}. \end{aligned} \tag{8.22}$$

The components of the Riemann tensor that are relevant for soft scalar terms, with a pair of indices along the visible sector fields and the other pair along the hidden sector fields, are then found to be :

$$\begin{aligned} &R_{\alpha\bar{\beta}i\bar{j}} = \frac{1}{3}\delta_{\alpha\beta}\delta_{ij} + c^a_{\alpha\beta}c^a_{ij}, && R_{\alpha\bar{\beta}T\bar{T}} = \delta_{\alpha\beta}, \\ &R_{\alpha\bar{\beta}a\bar{b}} = \frac{2}{3}\delta_{\alpha\beta}\delta_{ab} + (d_{abc}c^c - c^a c^b)_{\alpha\beta}, && R_{\alpha\bar{\beta}T\bar{b}} = c^b_{\alpha\beta}. \end{aligned} \tag{8.23}$$

We are now in position to finally compute the soft scalar masses induced for the visible-sector fields Q^α when the hidden-sector fields $\Phi^\Theta = X^i, T, T^a$ get non-vanishing auxiliary fields, at the reference point under consideration. This can be done by using the geometrical expression derived in section 4.5 :

$$m^2_{\alpha\bar{\beta}} = -\left(R_{\alpha\bar{\beta}\Theta\bar{\Gamma}} - \frac{1}{3}K_{\alpha\bar{\beta}}K_{\Theta\bar{\Gamma}}\right) F^\Theta \bar{F}^{\bar{\Gamma}}. \tag{8.24}$$

Using the results (8.21) and (8.23) for the metric and the Riemann tensor at the point under consideration, this gives :

$$\begin{aligned} m^2_{\alpha\bar{\beta}} = &- c^a_{\alpha\beta}c^a_{ij}F^i\bar{F}^j - \left(\frac{1}{3}\delta_{\alpha\beta}\delta_{ab} + (d_{abc}c^c - c^a c^b)_{\alpha\beta}\right) F^a\bar{F}^b \\ &- c^a_{\alpha\beta}F^a\bar{F}^T + \text{c.c.} \end{aligned} \tag{8.25}$$

which is our most important result since it takes into account both brane-to-brane and moduli effects. As promised in Chapter 6, we now reintroduce the dilaton term whose Kähler potential reads :

$$\tilde{K} = -\log(S + \bar{S}). \tag{8.26}$$

If the dilaton is fixed by some mechanism at, say, $\langle S\rangle = 1/2$, its contribution to the masses (8.25) is simply given by :

$$\Delta m^2_{\alpha\bar{\beta}} = \frac{1}{3}\delta_{\alpha\beta}F^S\bar{F}^S \tag{8.27}$$

since the Riemann terms with both visible and dilaton indices vanishes and where we have used $\tilde{K}_{S\bar{S}} = (S+\bar{S})^{-2}$. Note that the dilaton contribution to the soft scalar masses does not induce flavour-changing neutral currents since it is diagonal in flavour space [73]. One could then wonder whether a situation in which the dilaton is the only source of supersymmetry-breaking can be engineered. It turns out that the requirement of metastability makes it impossible for the dilaton to dominate SUSY breaking as shown in [128, 168–170].

The structure of the soft scalar masses (8.25) can also be understood in terms of ordinary Superfields. To do so, one considers the kinetic function $\Omega = -3\, e^{-K/3}$, which is the gravitational analogue of the rigid Kähler potential. At the considered reference point, it is sufficient to expand it at cubic order in $J^a \ll J$. In this way one finds :

$$\Omega \simeq -3J + \frac{1}{2}\frac{J^a J^a}{J} - \frac{1}{6} d_{abc} \frac{J^a J^b J^c}{J^2} \tag{8.28}$$

The relevant terms are selected by decomposing the fields in scalar VEVs plus fluctuations, so that $J = 1 + \tilde{J}$ and $J^a = \tilde{J}^a$, and retaining up to cubic terms in an expansion in powers of the fluctuations. This yields $\Omega = -3 + \tilde{\Omega}$ with :

$$\tilde{\Omega} \simeq -3\tilde{J} + \frac{1}{2}\tilde{J}^a\tilde{J}^a - \frac{1}{2}\tilde{J}\tilde{J}^a\tilde{J}^a - \frac{1}{6} d_{abc}\tilde{J}^a\tilde{J}^b\tilde{J}^c. \tag{8.29}$$

The soft scalar masses can then be computed by looking at the quadratic part of the contribution to the scalar potential from $\tilde{\Omega}$: $\mathcal{L}_{m^2} = -\tilde{\Omega}|_{D,q^2}$. The various terms in (8.25) thus emerge as follows from $\tilde{\Omega}|_D$, after splitting the currents into visible-sector and hidden-sector parts. The term $-c^a_{\alpha\beta} c^a_{ij} F^i \bar{F}^{\bar{j}}$ comes from $\tilde{J}^a_{\rm v}|\tilde{J}^a_{\rm h}|_D$, the term $-1/3\, \delta_{\alpha\beta}\delta_{ab} F^a \bar{F}^{\bar{b}}$ comes from $-\tilde{J}_{\rm v}|\tilde{J}^a_{\rm h}|_F \tilde{J}^a_{\rm h}|_{\bar{F}}$, the term $-c^a_{\alpha\beta} F^a \bar{F}^T +$c.c. comes from $-\tilde{J}_{\rm h}|_{\bar{F}}\tilde{J}^a_{\rm v}|\tilde{J}^a_{\rm h}|_F +$c.c., the term $(c^a c^b)_{\alpha\beta} F^a \bar{F}^{\bar{b}}$ comes from the combination of $-3\,\tilde{J}_{\rm v}|_D$ and $\tilde{J}^a_{\rm v}|_F \tilde{J}^a_{\rm h}|_{\bar{F}} +$c.c., and finally the term $-d_{abc} c^a_{\alpha\beta} F^b \bar{F}^{\bar{c}}$ comes from $-d_{abc}\tilde{J}^a_{\rm v}|\tilde{J}^b_{\rm h}|_F \tilde{J}^c_{\rm h}|_{\bar{F}}$.

8.2.2 Sequestering by Global Symmetries

From the form of the expression (8.25), we can deduce the following observations. In the particular case where $h^{1,1} = 1$, the soft scalar masses vanish identically, even in the presence of generic non-vanishing values for F^T and F^i. This is the well known situation arising in sequestered models. In the general case where $h^{1,1} > 1$, on the contrary, the soft scalar masses receive non-trivial contributions in the presence of generic non-vanishing values of F^T, F^i and F^a. However, these contributions involve very special combinations of these auxiliary fields, controlled by the quantities d_{abc} and the matrices $c^a_{\alpha\beta}$ and c^a_{ij}. One may then wonder whether it is possible to ensure that these combinations of auxiliary fields vanish, so that the soft scalar masses would again vanish, by assuming that some approximate global symmetry of the Kähler potential K is extended to constrain also the superpotential W and therefore the Goldstino direction. It would also be interesting to study what constraints are put on the Goldstino direction by the requirement that there should exist a metastable supersymmetry-breaking vacuum, generalising the results derived in [171] for Kähler moduli to include also matter fields, but we shall not attempt to do this here.

From the results derived in the previous subsection, and taking into account that the scalar VEVs of the fields T^a and X^i are assumed to be negligible, we see that a simple and general possibility to get vanishing soft scalar masses is to require that :

$$c^a_{ij} F^i \bar{F}^j = 0 \qquad \text{and} \qquad F^a = 0. \tag{8.30}$$

These two relations clearly have the form of the two D and F type Ward identities that would be implied by the conservation of the following currents :

$$J^a_{\mathrm{h}} = T^a + \bar{T}^a - c^a_{ij} X^i \bar{X}^j. \tag{8.31}$$

Notice however that one might also view the two relations (8.30) as emerging from the conservation of the following two independent currents, which each lead to only one non-trivial Ward identity, respectively the D and F type one :

$$J^a_{\mathrm{h}X} = -c^a_{ij} X^i \bar{X}^j \qquad \text{and} \qquad J^a_{\mathrm{h}T} = T^a + \bar{T}^a. \tag{8.32}$$

This follows form the observation that at the considered vacuum reference point one finds $J^a_{\mathrm{h}X}|_D = J^a_{\mathrm{h}}|_D$, $J^a_{\mathrm{h}X}|_F = 0$, $J^a_{\mathrm{h}T}|_D = 0$ and $J^a_{\mathrm{h}T}|_F = J^a_{\mathrm{h}}|_F$.

The question is now whether it is possible to engineer a symmetry whose currents correspond to J^a_{h} or to the pair of currents $J^a_{\mathrm{h}X}$ and $J^a_{\mathrm{h}T}$. In order to simplify the discussion let us start by investigating the leading quadratic part of K which concerns the X^i and T^a fields :

$$K \simeq \frac{1}{2}(T^a + \bar{T}^a)(T^a + \bar{T}^a) + X^i \bar{X}^i. \tag{8.33}$$

In order to match (8.9) with the two partial currents (8.32), we would then respectively need to take $X^i_a \simeq -i c^a_{ji} X^j$ for the matter fields X^i and $X^b_a \simeq i\delta^b_a$ for the moduli fields T^a. These Killing vectors define two sets of transformations that independently leave the leading Kähler potential (8.33) invariant :

$$\delta_a X^i = X^i_a \simeq -i c^a_{ji} X^j \qquad \text{and} \qquad \delta_a T^b = X^b_a \simeq i\delta^b_a \tag{8.34}$$

since the c^a matrices are Hermitian. The next question is whether the transformations (8.34) are eligible to represent an approximate global symmetry of K around the vacuum reference point under consideration or not. A first condition is that the matrices c^a should form a closed algebra with $[c^a, c^b] = -i f_{abc} c^c$. In this way the X^i transformations would form an algebra with structure constants f_{abc} associated to a group H, while the T^a transformations automatically form an Abelian algebra associated to $U(1)^{h^{1,1}-1}$. A second condition is that higher order terms in K should have an unimportant effect and that it should somehow be meaningful to impose to W a symmetry that only leaves the leading quadratic part of K invariant. One possibility is that the corrections spoil the symmetries (8.34) but only in a parametrically suppressed way. It is however not clear whether this can robustly happen. A more appealing possibility is that (8.34) can be extended to exact symmetries of the full scalar manifold, thereby guaranteeing the existence of exactly conserved currents which reduce to (8.32) in the vicinity of the point under consideration.

We notice however that from the form of K given in (8.18) the symmetry acting on X^i may only be generalised to an exact symmetry by extending it to act linearly on the T^a's in the adjoint representation of H and only if the d_{abc} corresponds to an invariant of H, while the symmetry acting on T^a is always an exact symmetry. The exact conserved currents differ from (8.32), on one hand because of the extension in the symmetry action and on the other hand because of the Kähler potential non-linearity. However, taken together they still ensure that $c^a_{ij} F^i \bar{F}^j = 0$ and $F^a = 0$, which guarantee the vanishing of the soft scalar masses.

In addition to the general possibility that we just explored, there might also be other options that arise in specific situations. For instance, the three terms of the second piece in (8.25) may conspire to give a simpler structure, and one might try to exploit this in the search for a different global symmetry that could ensure the vanishing of soft masses by constraining the F^a's but without setting them all to zero. In such a case one would however have to assume that F^T vanishes to get rid of the last piece in (8.25).

Let us now study more specifically what are the options for both Calabi-Yau models and orbifold models, focusing for simplicity on models with a symmetric embedding in the visible and hidden sectors, for which the set of matrices $c^a_{\alpha\beta}$ and c^a_{ij} are identical.

8.2.3 Calabi-Yau Models

For generic Calabi-Yau models, the intersection numbers d_{abc} and the Hermitian matrices $c^a_{\alpha\beta}$ or equivalently c^a_{ij} are a priori generic, with $a = 1, \cdots, h^{11}-1$ and $\alpha, \beta, i, j = 1, \cdots, n_{\mathbf{R}}$ with the restriction that the matrices c^a and c^0 may always be written as transposed linear combinations of the $n_{\mathbf{R}}^2$ matrices $\lambda^{A'}$ representing the $U(n_{\mathbf{R}})$ generators in the fundamental representation. As remarked at the end of section 7.4, a further property that could conceivably arise in some situations to determine is that these matrices might be traceless. In that case they could then be expressed in terms of the $n_{\mathbf{R}}^2 - 1$ traceless generators of $SU(n_{\mathbf{R}})$. On the other hand, further restrictions leading to yet smaller subgroups H' seem less likely, and the minimal case where the matrices c^a themselves generate a group H of dimension $h^{1,1}-1$ appears to be very special.

Consider first the brane-mediated effect corresponding to the first term of (8.25). If the matrices c^a happen to be transposed linear combinations of the generators $\lambda^{a'}$ of some group $H' \subset U(n_{\mathbf{R}})$, we may ensure the vanishing of this contribution by imposing the global symmetry H' that acts as in (8.34) but with c^a_{ji} replaced by $\lambda^{a'}_{ij}$: $\delta_{a'} X^i = -i\,\lambda^{a'}_{ij} X^j$. This is still an approximate symmetry of K and leads to the conservation of the larger set of currents $J^{a'}_{\mathrm{h}X} = -\lambda^{a'}_{ji} X^i \bar{X}^j$, which implies the stronger Ward identity $\lambda^{a'}_{ji} F^i \bar{F}^j = 0$. The maximal choice $H' = U(n_{\mathbf{R}})$ is available for any generic model, but has the drawback that it would actually imply $F^i = 0$, due to the completeness relation $\lambda^{a'}_{ij} \lambda^{a'}_{pq} = \delta_{iq}\delta_{pj}$. Other non-maximal choices $H' \subset U(n_{\mathbf{R}})$ are instead available only in particular models, but have the advantage of allowing $F^i \neq 0$. Notice finally that such an approximate symmetry group H' can in general not be extended to an exact symmetry of the full scalar manifold. The only very special case where this is possible is when the c^a generate by themselves a minimal group H of dimension $h^{1,1}-1$ and the intersection numbers d_{abc} are invariant under this group H.

Consider next the moduli-mediated effect corresponding to the remaining terms of (8.25). In general one may ensure that these vanish by imposing the independent Abelian global symmetry $U(1)^{h^{1,1}-1}$ acting as in (8.34) : $\delta_a T^b = i\delta_a^b$. This symmetry leads to the conservation of the currents $J^a_{\mathrm{h}T} = T^a + \bar{T}^a$, and the corresponding F type Ward identity implies that $F^a = 0$. Moreover it always corresponds to an exact symmetry of the full scalar manifold. Notice finally that in this case it is rather unlikely that the second piece of (8.25) could simplify dramatically enough to allow for other options.

We conclude that for smooth Calabi-Yau compactifications there generically exists the possibility of ensuring the vanishing of soft scalar masses at points with negligible VEVs for X^i and T^a by imposing the approximate global symmetry $U(n_{\mathbf{R}}) \times U(1)^{h^{1,1}-1}$, where the first factor acts linearly on the X^i and the second acts as a shift on the T^a. However, this forces both the F^i and the F^a to vanish, meaning that there is actually no breaking of Supersymmetry at all. Moreover, it is not a true symmetry of the full scalar manifold. A more interesting situation may be obtained in the special cases where the matrices c^a generate some non-maximal subgroup $H \subset U(n_{\mathbf{R}})$. In such a situation, the F^i would be constrained but not forced to vanish, although the F^a would still vanish, and Supersymmetry can be broken. Moreover, this symmetry can be extended to a true symmetry of the full scalar manifold that still implies the vanishing of the scalar masses.

8.2.4 Orbifold Models

For orbifold models, the intersection numbers d_{abc} and the matrices $c^a_{\alpha\beta}$ or equivalently c^a_{ij}, with $a = 1, \cdots, h^{1,1}-1$ and $\alpha, \beta, i, j = 1, 2, 3$, are a respectively the symmetric invariant symbol and the transposed tridimensional representation of the generators of a group $H \subset SU(3)$. Moreover, one can easily verify that the second term in (8.25) simplifies to :

$$\frac{1}{3}\delta_{\alpha\beta}\delta_{ab} + (d_{abc}c^c - c^a c^b)_{\alpha\beta} = (c^b c^a)_{\alpha\beta} - \frac{1}{3}\delta_{ab}\delta_{\alpha\beta} \tag{8.35}$$

which is traceless. As a result, the mass matrix (8.25) is traceless and depends only on $h^{1,1}-1$ independent parameters, which can be taken to be $c^a_{ji} m^2_{ij}$.

Consider first the first brane-mediated term in (8.25). In this case, this can be ensured to vanish by imposing the global symmetry H acting as in (8.34) : $\delta_a X^i = -i\lambda^a_{ij} X^j$. This leads to the conservation of the currents $J^a_{\mathrm{h}X} = -\lambda^a_{ji} X^i \bar{X}^j$, which implies the D type Ward identity $\lambda^a_{ji} F^i \bar{F}^j = 0$. Moreover, this approximate symmetry can be extended to an exact symmetry of the full manifold by assigning a non-trivial linear transformation law to the fields T^a in the adjoint representation of H. Notice finally that in this case one does not have the option of enlarging the symmetry to a bigger group $H' \subset U(n_{\mathbf{R}})$, because the various generations are grouped into triplets transforming in the fundamental representation of the gauge group enhancement factor, which happens to coincide with H.

Consider next the remaining moduli-mediated terms in (8.25). In general, we may again ensure the vanishing of these terms by imposing an independent Abelian global symmetry $U(1)^{h^{1,1}-1}$ acting as in (8.34) : $\delta_a T^b = i\delta_a^b$. This leads to the conservation of the currents $J^a_{\mathrm{h}T} = T^a + \bar{T}^a$, which implies the F type Ward identity $F^a = 0$. Moreover, this symmetry is actually as before an exact symmetry of the full scalar manifold. Notice finally that in this case the second piece of (8.25) actually simplifies to $(d_{abc} + i f_{abc}) F^b \bar{F}^c$.

We conclude that for toroidal orbifold compactifications there always exists the possibility of ensuring the vanishing of soft scalar masses at points with negligible VEVs for X^i and T^a by imposing the approximate global symmetry $H \times U(1)^{h^{1,1}-1}$, where the first factor acts linearly on the X^i and the second factor acts as a shift on the T^a. In this situation, the F^i would be constrained but not forced to vanish, although the F^a would still vanish, and Supersymmetry can be broken. Moreover, this symmetry can be extended to a true symmetry of the full scalar manifold that still implies the vanishing of the scalar masses.

Chapter 9

Conclusion

Le doute est un hommage rendu à l'espoir.

Comte de Lautréamont

9.1 Summary

In this work we have shown the feasibility of implementing a mechanism to suppress the classical-level soft mass terms appearing when breaking Supersymmetry in the $E_8 \times E_8$ Heterotic $\mathcal{M}$-Theory setup. The strategy we have developed has been shown to work for singular compactification manifolds together with a subset of Calabi-Yau manifolds provided with a stable and holomorphic vector bundle. Let us briefly summarise the steps we have followed together with the assumptions we had to do.

We have first chosen to concentrate on the low energy effective theory describing the dynamics of the Kähler moduli, discarding the complex structure moduli, and of charged matter fields present on both the E_8 end-of-the-world branes. We were able to derive the Kähler potential describing the interactions among the fields of the observable sector with the ones of the hidden sector, *i.e.* with the Kähler moduli and the charged matter field present on the distant brane. We have then computed the soft masses under the assumptions that the hidden sector fields auxiliary components acquire a non-vanishing vacuum expectation value due to some unspecified superpotential.

However in order to derive the Kähler potential we had to assume that the quantities $c_{PQ} = i\mathrm{Tr}\,(u_P \wedge \bar{u}_Q)$ describing the product of matter fields are harmonic with respect to the derivative defined on the Calabi-Yau. From both the four-dimensional picture and the five-dimensional picture arising from $\mathcal{M}$-Theory compactified on a Calabi-Yau, this requirement is traced back to the condition of a proper decoupling of the heavy modes.

Under the assumption that the c_{PQ} forms are harmonic, the Kähler potential depends on two quantities : the coefficients c^A_{PQ} of c_{PQ} when developed on a basis of harmonic forms ω_A and d_{ABC} which are the intersection numbers of the Calabi-Yau. The possibility of implementing a symmetry forbidding the appearance of soft scalar mass terms at the classical level depends on properties of both these two quantities. In the particular case of singular compactification manifolds, *i.e.* orbifolds, both these quantities have a group-theoretical interpretation. Indeed, the c^A_{PQ}'s are shown to be the generators of a group $U(1)\times H \subset U(3)$ while d_{ABC} is a symmetric invariant symbol of the latter and the scalar manifold is found to be a symmetric Kähler manifold. Since the quantities c^A_{PQ} and d_{ABC} are tightly constrained in the case of the untwisted sector of orbifold compactification, there exists a global symmetry of the Kähler potential which effectively forbids the phenomenologically dangerous soft mass operators if extended to the superpotential.

In the more general case of smooth Calabi-Yau compactifications, the quantities c^A_{PQ} and d_{ABC} controlling the Kähler potential do not have any group-theoretical interpretations and the scalar manifold is not a symmetric one anymore. Only under the restrictive assumptions that c^A_{PQ} and d_{ABC} respectively are the generators and an invariant of a subgroup of $U(3)$, the strategy we devised to suppress the soft masses can be implemented.

9.2 Future Directions

As discussed in the previous section, our work has been done in the context of Heterotic $\mathcal{M}$-Theory under some assumptions. Let us review the most relevant ones and describe how our results may be generalised were these assumptions to be abandoned :

- *Background Fluxes* : In the case of smooth compactifications, we have focused on backgrounds which do not admit large fluxes, *i.e.* the compactification manifold may be taken as being a Calabi-Yau. Relaxing this assumption would lead to non-Kähler compactification manifolds for which the identification of the zero-modes of the internal wave operator is not yet a settled issue.

- *Complex Structure Moduli* : For simplicity we have discarded the $h^{2,1}$ complex structure moduli. The Kähler potential for those fields is known at zeroth order in the matter fields. A generalisation such as the one we have performed for Kähler moduli where we have included the possibility for the matter fields to take a vacuum expectation value would be a first natural extension of our work.

- *Properties of c_{PQ}* : As emphasised in the previous section, the derivation of the effective Kähler potential in a closed form strongly relies on the fact that the c_{PQ}'s are harmonic forms with constant coefficients c^A_{PQ}. The relaxation of this assumptions certainly has to do with the proper decoupling of heavy modes. A study of the feasibility of writing down the Kähler potential in closed form without this assumption would certainly be of great interest.

- *Properties of c^A_{PQ} and d_{ABC}* : Since the implementation of mild sequestering relies on some properties of both c^A_{PQ} and d_{ABC}, it would be interesting to determine how restrictive these assumptions are and whether they naturally emerge in some scenario, besides the already mentioned special cases of single-modulus Calabi-Yau and orbifolds.

Appendix A

Notations and Conventions

In this Appendix we would like to settle the notations and conventions we will be using in this work.

Metric The metric we will be using is the mostly plus one, in order to facilitate the comparison with the standard SUSY literature and with [54] :

$$\eta = \mathrm{diag}(-1, +1, \ldots, +1). \tag{A.1}$$

Pauli Matrices and Antisymmetric Symbols The Pauli Matrices are the standard ones where the spinorial indices are understood to be downstairs :

$$\sigma^i_{\alpha\dot{\beta}} = \text{Pauli Matrices}. \tag{A.2}$$

They are supplied with $\sigma^0 = -\mathbb{1}_2$. Note the sign which when taken to be the opposite changes the sign in front of the $\{Q, \bar{Q}\}$ algebra. Spinorial indices are raised and lowered using the $SL(2,\mathbb{C})$-invariant ϵ symbols :

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.3}$$

Lower Index Derivative Notation When dealing with Supersymmetry, one often abbreviates the derivatives with respect to a Superfield or to a field by putting a lower index on the derived quantity :

$$\frac{\partial V}{\partial \Phi^i} \to V_i. \tag{A.4}$$

Moreover one may define a metric out of the second derivative of the Kähler potential :

$$\frac{\partial^2 K}{\partial \Phi^i \partial \bar{\Phi}^j} \to K_{i\bar{\jmath}} \tag{A.5}$$

whose properties are worked out in Appendix B.2.4.

Appendix B

Complex Manifolds, Kähler Geometry and Calabi-Yau Manifolds

In this Appendix we will review some basic features of Riemannian Geometry and then explain how to make sense of those when dealing with complex manifolds. Some standard references are Nakahara's book [62] and [172].

B.1 Riemannian Geometry

B.1.1 Tangent Space, Cotangent Space and Forms

Let us first remind what a differentiable manifold is :

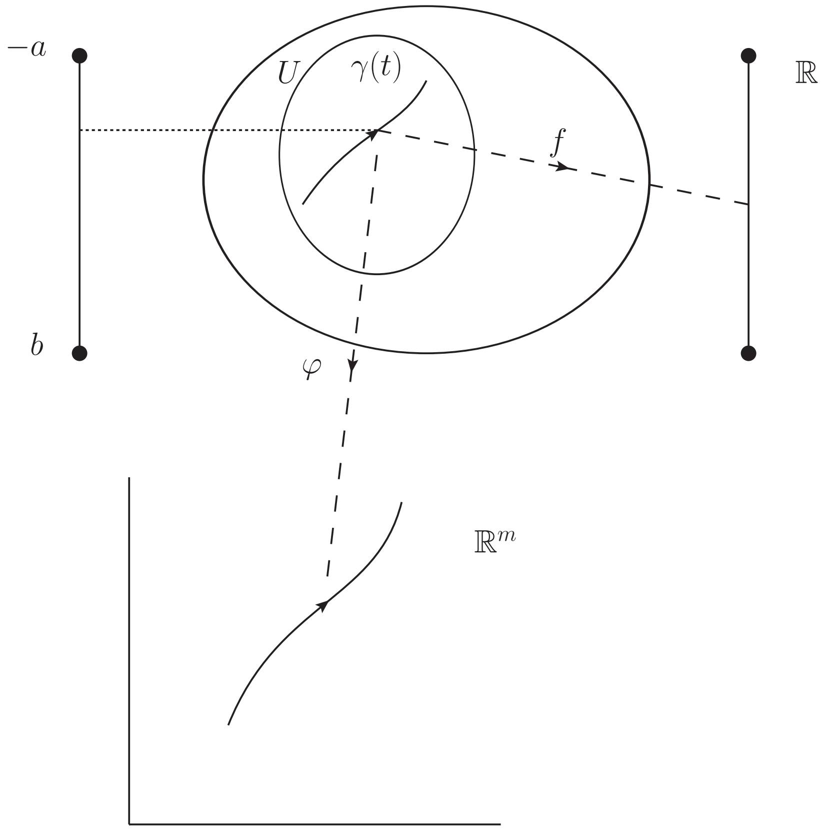
Differentiable Manifold A m-dimensional manifold is said to be differentiable if it is a topological space provided with charts (U_i, φ_i) where the $\{U_i\}$ are a family of open sets which cover the manifold and the $\{\varphi_i\}$ are homeomorphisms from U_i to an open subset of $\mathbb{R}^m$. Furthermore the transition functions $\varphi_i \circ \varphi_j^{-1}$ have to be infinitely differentiable provided that the φ_i and φ_j domains of definition do overlap.

Note that in many places we will speak about differentiability of maps between manifolds X and Y. What is to be understood is the following. Consider the charts (U_X, φ_X) and (U_Y, φ_Y) associated with the X and Y manifolds. Then to a map f from X to Y we can associate the function $F = \varphi_Y \circ f \circ \varphi_X^{-1}$ which goes from $\mathbb{R}^{\dim(X)}$ to $\mathbb{R}^{\dim(Y)}$. The notions of differentiability and continuity of f are then to be understood as the properties satisfied by F. If we now imagine that the Y manifold is given by $\mathbb{R}$, then φ_Y is simply given by the identity and $F = f \circ \varphi_X^{-1}$ is defined from $\mathbb{R}^{\dim(X)}$ to $\mathbb{R}$. This is how a *function* is defined on M.

We are now ready to define *curves* on a manifold M. Let us add an interval $(-a, b)$ with both a and b strictly positive to our setup and a map $\gamma(t)$ from this interval to the manifold M. Varying t in the $(-a, b)$ interval draws a curve on the manifold M. This curve will be called $\gamma(t)$. To the curve on the manifold one can associate a coordinate representation $\gamma \circ \varphi$ where φ

is the homeomorphism associated with the open set where the curve is traced on the manifold. Thus $\gamma \circ \varphi$ is a function of $\mathbb{R}$ (or a subset of it) and takes its value in $\mathbb{R}^m$.

We are now equipped to define the tangent space of a manifold. Intuitively the tangent space is a vector space which is fixed to the manifold M at a point p such that its normal vector coincides with the one of the manifold. Such a space is usually denoted by T_pM. More precisely, let us consider the following setup :



The aim is now to quantify how much f computed on the curve $\gamma(t)$ varies when varying the parameter t. The corresponding mathematical quantity computed at the point $p = \gamma(0) \in M$ is given by :

$$\left.\frac{df(\gamma(t))}{dt}\right|_{t=0} = \left.\frac{d(f \circ \varphi^{-1} \circ \varphi \circ \gamma)}{dt}\right|_{t=0} = \left.\frac{d(f \circ \varphi^{-1})}{dx^\mu} \frac{dx^\mu(\gamma(t))}{dt}\right|_{t=0} . \tag{B.1}$$

We now introduce the following notation where the φ^{-1} is dropped for clarity :

$$X = X^\mu \frac{\partial}{\partial x^\mu} \quad \rightarrow \quad \left.\frac{df(\gamma(t))}{dt}\right|_{t=0} = X[f] \tag{B.2}$$

which defines X^μ. X is called a *tangent vector* to M along the curve $\gamma(t)$ at point $p = \gamma(0)$.

Tangent Space T_pM The tangent space at p in determined as follows. First find all the curves on M passing at p and adjust the $(-a, b)$ parameters such that they do so at $t = 0$. The tangent space at p is the vector space spanned by all the X's corresponding to those curves.

Having defined the tangent space, it is now natural to define the cotangent space which consists of linear functions from T_pM to $\mathbb{R}$. As ususal in the context of dual spaces, the tangent space is denoted by T_p^*M. We now need to specify the action of an element of T_p^*M on an element of T_pM. To do so we introduce a basis dual to the $\partial/\partial x^\mu$ basis of T_pM which we will denote by dx^μ. Its action is defined to be :

$$\begin{aligned} dx^\mu : \quad T_pM \quad &\rightarrow \quad \mathbb{R}, \\ \frac{\partial}{\partial x^\nu} \quad &\mapsto \quad \delta^\mu_\nu. \end{aligned} \tag{B.3}$$

Let us illustrate this by an example. Let $X \in T_pM$ and $Y \in T_p^*M$. One can decompose both of them on their basis : $X = X^\mu \partial_\mu$ and $Y = Y_\mu dx^\mu$. Then the action of Y on X is given by :

$$Y(X) = Y_\mu X^\nu dx^\mu(\partial_\nu) = Y_\mu X^\mu. \tag{B.4}$$

An element of the dual of T_pM is called a *one-form*. As the name suggests it, one may define $p-$forms and tensors. A $p-$form and a (p, q)-tensor are respectively belonging to the following spaces :

$$\bigwedge_{i=1}^{p} T_p^*M \qquad \text{and} \qquad \bigotimes_{i=1}^{p} T_pM \bigotimes_{j=1}^{q} T_p^*M \tag{B.5}$$

where the $\wedge$-product antisymmetrises the $\otimes$-product. For example $dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$. A general $p-$form is conventionally written as

$$A = \frac{1}{p!} A_{i_1 \dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{B.6}$$

where $A_{i_1 \dots i_p} = A_{[i_1 \dots i_p]}$. The product of two forms is simply given by $A \wedge B$. The exterior derivative maps $p-$forms to $(p+1)-$forms and acts as :

$$\mathrm{d}A = \frac{1}{p!} \partial_a A_{i_1 \dots i_p} dx^a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{B.7}$$

Due to the antisymmetry of the $\wedge-$product, the exterior derivative is a nilpotent operator : $\mathrm{d}^2 = 0$.

B.1.2 The Metric

We are now going to introduce a $(0, 2)-$tensor which will allow us to talk about covariant derivatives, geodesics, etc...

The Metric The metric g is a symmetric and non-degenerate $(0,2)$-tensor. Let X and Y belong to T_pM, then $g(X,Y) = g(Y,X)$ (*symmetric*) and if $g(X,Y) = 0$ for all Y's then $X = 0$ (*non-degenerate*). In components, the metric is written $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$ which will always be written as $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, leaving the symbol g to denote the determinant of $g_{\mu\nu}$ seen as matrix.

In order to compare vectors belonging to two tangent spaces T_pM and T_qM one needs a method to transport a vector from T_qM to T_pM. This is of crucial importance if one wants to take the derivative of a vector, process which involves the subtraction of this vector evaluated at two different points. In order to define such a derivative, one introduces the affine connection ∇ which takes two vectors X and Y and maps them to another vector $\nabla_X Y$. This imposes that the vector basis satisfy $\nabla_{\partial_\mu}\partial_\nu = \Gamma^\rho_{\mu\nu}\partial_\rho$. In order to illustrate how ∇ acts on vectors, let us expand X and Y as $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$. Then one has :

$$\nabla_X Y = X^\mu \nabla_{\partial_\mu} (Y^\nu \partial_\nu) = X^\mu \left(\partial_\mu Y^\nu \partial_\nu + Y^\nu \nabla_{\partial_\mu}\partial_\nu\right) = X^\mu \left(\partial_\mu Y^\rho + Y^\nu \Gamma^\rho_{\mu\nu}\right)\partial_\rho. \tag{B.8}$$

A commonly adopted notation for covariant derivatives is the following :

$$\nabla_\mu Y^\nu = \partial_\mu Y^\nu + \Gamma^\nu_{\mu\rho} Y^\rho \tag{B.9}$$

where $\nabla_\mu \equiv \nabla_{\partial_\mu}$. The affine connection may be extended such that it can be applied to general tensors and obey the Leibniz rule. In such a case one can easily compute the covariant derivative of dx^μ :

$$0 = \nabla_\alpha (dx^\beta(\partial_\beta)) = \left(\nabla_\alpha dx^\beta + \Gamma^\beta_{\alpha\gamma} dx^\gamma\right)\partial_\beta \tag{B.10}$$

leading to the covariant derivative of one-forms :

$$\nabla_\mu Y_\nu = \partial_\mu Y_\nu - \Gamma^\rho_{\mu\nu} Y_\rho. \tag{B.11}$$

Up to this point the coefficients $\Gamma^\rho_{\mu\nu}$ are arbitrary.

Compatible and Levi-Civita Connections The connection is said to be compatible with the metric if the covariant derivative of the metric vanishes : $\nabla_\alpha g_{\mu\nu} = 0$. If the connection coefficients satisfy $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$, then the associated connection ∇ is called Levi-Civita connection.

The last object to be introduced is the Riemann tensor constructed out of the metric. It is defined as follows :

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{B.12}$$

where the brackets stand for the Lie bracket of X and Y : $[X,Y] = \mathcal{L}_X Y$. Note that the combination $[X,Y]$ is also a vector field :

$$\mathcal{L}_X Y = [X,Y] = XY - YX = [X^\mu(\partial_\mu Y^\nu) - Y^\mu(\partial_\mu X^\nu)]\,\partial_\nu. \tag{B.13}$$

B.1.3 Inner Product and Adjoints

Having the metric at our disposal, we define a new operations on forms.

Hodge Duality Given a $p-$form A defined on a $m-$dimensional manifold M,

$$A = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{B.14}$$

one defines its Hodge dual as

$$*A = \frac{\sqrt{g}}{p!(m-p)!} A_{\mu_1 \dots \mu_p} \epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_m} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_m} \tag{B.15}$$

where the indices on the $\epsilon-$symbol were raised with the metric.

Let us now show that the operator $**$ acts as a scalar multiplication on forms. The two following properties of the $\epsilon-$ symbol will be of great use to derive this result :

$$\begin{aligned} \epsilon^{\mu_1 \dots \mu_m} &= g^{\mu_1 \nu_1} \dots g^{\mu_m \nu_m} \epsilon_{\nu_1 \dots \nu_m} \\ &= \epsilon_{\mu_1 \dots \mu_m} g^{1 \nu_1} \dots g^{m \nu_m} \epsilon_{\nu_1 \dots \nu_m} \\ &= \epsilon_{\mu_1 \dots \mu_m} \det \left(g^{-1} \right) \end{aligned} \tag{B.16}$$

and

$$\epsilon_{\mu_1 \dots \mu_p \mu_{p+1} \dots \mu_m} \epsilon_{\mu_1 \dots \mu_p \nu_{p+1} \dots \nu_m} = p! \, \delta^{\nu_{p+1}}_{[\mu_{p+1}} \dots \delta^{\nu_m}_{\mu_m]} \tag{B.17}$$

where the antisymmetrisation is defined without numerical factors and where we have written the ν indices upstairs to ease the reading, there are no metrics involved here. We thus find :

$$\begin{aligned} **A &= \frac{g}{p!p!(m-p)!} A_{\mu_1 \dots \mu_p} \epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_m} \epsilon^{\mu_{p+1} \dots \mu_m}{}_{\nu_1 \dots \nu_p} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p} \\ &= \frac{(-1)^{p(m-p)}}{p!p!(m-p)!} A_{\mu_1 \dots \mu_p} \epsilon_{\mu_1 \dots \mu_m} \epsilon_{\nu_1 \dots \nu_p \mu_{p+1} \dots \mu_m} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p} \\ &= (-1)^{p(m-p)} A. \end{aligned} \tag{B.18}$$

We are now in a position to define a scalar product on forms. Let A and B be $p-$forms, then the following operation satisfies all properties of a scalar product :

$$\langle A, B \rangle = \int_M A \wedge *B \tag{B.19}$$

With this scalar product we can define the adjoint of the exterior derivative d by imposing the following to hold :

$$\langle \mathrm{d}A, B \rangle = \langle A, \mathrm{d}^\dagger B \rangle \tag{B.20}$$

for A a $p-$form and B a $(p-1)-$form. The operator $\mathrm{d}^\dagger$ thus maps $p-$forms to $(p-1)-$forms and, as it is the case for d, is a nilpotent operator : ${\mathrm{d}^\dagger}^2 = 0$. The operator $\mathrm{d}\mathrm{d}^\dagger$ thus maps

$p-$forms to $p-$forms. For a manifold M without boundary, using Stokes' theorem, one can easily show that $\mathrm{d}^\dagger \propto *\mathrm{d}*$. Indeed :

$$\begin{aligned} 0 = \int_M \mathrm{d}(A\wedge *B) &= \int_M \mathrm{d}A\wedge *B + (-1)^p A\wedge \mathrm{d}*B \\ &= \langle \mathrm{d}A, B\rangle + (-1)^{mp+m-1}\langle A, *\mathrm{d}*B\rangle. \end{aligned} \tag{B.21}$$

We define the *Laplacian* Δ as :

$$\Delta = \left(\mathrm{d}+\mathrm{d}^\dagger\right)^2 = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}. \tag{B.22}$$

Note that the Laplacian takes p-forms to p-forms and commutes with both d and $*$:

$$[\Delta, \mathrm{d}] = 0 \qquad \text{and} \qquad [\Delta, *] = 0. \tag{B.23}$$

From the definition of Δ and from the positivity of the scalar product, we conclude that the following quantity

$$\langle A, \Delta A\rangle = \langle \mathrm{d}A, \mathrm{d}A\rangle + \langle \mathrm{d}^\dagger A, \mathrm{d}^\dagger A\rangle \tag{B.24}$$

is always positive. This equality also tells us that a form is *harmonic* if and only if it is both *closed* and *co-closed* :

$$\Delta A = 0 \qquad \leftrightarrow \qquad \mathrm{d}A = 0 \quad \text{and} \quad \mathrm{d}^\dagger A = 0. \tag{B.25}$$

Let us examine how harmonicity restricts the components of a 1-form $A = A_\mu dx^\mu$. The fact that is it closed simply gives

$$\mathrm{d}A = 0 \qquad \to \qquad \partial_\mu A_\nu dx^\mu \wedge dx^\nu = 0 \qquad \to \qquad \partial_\mu A_\nu = \partial_\nu A_\mu. \tag{B.26}$$

The fact that A is co-closed gives after a little algebra

$$\mathrm{d}^\dagger A = 0 \qquad \to \qquad \partial_\mu\left(\sqrt{g}g^{\mu\nu}A_\nu\right) = 0 \qquad \to \qquad \nabla_\mu A^\mu = 0. \tag{B.27}$$

Note that a form A can be closed because it is itself the d of another form : $A = \mathrm{d}B$. Such forms are *exact*. One may then define the *de Rahm cohomology* as the vector space whose elements are the equivalence classes of closed over exact $p-$forms : $A' \in [A]$ if $\exists\, B \,|\, A' = A + \mathrm{d}B$. If such a $(p-1)$-form exists, A and A' are said to be cohomologous. Schematically one writes :

$$H^p = \{A\ p\text{-forms}\,|\,\mathrm{d}A = 0\}/\{A\, p\text{-forms}\,|\,\exists\, B \text{ such that } A = \mathrm{d}B\}. \tag{B.28}$$

The dimension of the H^p seen as a vector space is denoted by b^p and is called the Betti number.

B.2 Complex Manifolds

B.2.1 Tangent Space, Cotangent Space and Forms

Certain differentiable manifolds of even dimension can be viewed as complex manifolds. In order to define a complex manifold M with $\dim M = 2n$, the axioms defining a usual manifold have to be supplemented by the requirement that the map between different open sets covering the manifold has to be holomorphic in $z^\mu = x^\mu + iy^\mu$ where the x^μ's and y^μ's are coordinates of the manifold seen as a $2n-$dimensional manifold. The tangent space T_pM of a complex manifold M is spanned by the $2n$ vectors $\partial/\partial a^\mu$ where $a = \{x, y\}$. The cotangent space is then spanned by da^μ where again $a = \{x, y\}$.

The vector basis on T_pM is given by :

$$\frac{\partial}{\partial z^\mu} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}^\mu} \equiv \left(\frac{\partial}{\partial z^\mu}\right)^* . \tag{B.29}$$

Its dual basis is then simply given by $dz^\mu \equiv dx^\mu + idy^\mu$ and $d\bar{z}^\mu \equiv (dz^\mu)^*$. The action of the dual basis on the vector basis is given by : $dz^\mu(\partial/\partial z^\nu) = \delta^\mu_\nu$, $dz^\mu(\partial/\partial \bar{z}^\nu) = 0$, $d\bar{z}^\mu(\partial/\partial z^\nu) = 0$ and $d\bar{z}^\mu(\partial/\partial \bar{z}^\nu) = \delta^\mu_\nu$. Note that :

$$\sum_{\mu=1}^{m} \frac{\partial}{\partial z^\mu}\frac{\partial}{\partial \bar{z}^\mu} = \frac{\Delta}{4} \tag{B.30}$$

where Δ is the Laplacian in $\mathbb{R}^{2m}$.

The following real linear map is called the *almost complex structure*, it is defined by its action on the $\partial/\partial a^\mu$ vectors : $\mathcal{J}(\partial/\partial x^\mu) = \partial/\partial y^\mu$ and $\mathcal{J}(\partial/\partial y^\mu) = -\partial/\partial x^\mu$. This map is naturally extended to the z-basis : $\mathcal{J}(\partial/\partial z^\mu) = i\partial/\partial z^\mu$ and $\mathcal{J}(\partial/\partial \bar{z}^\mu) = -i\partial/\partial \bar{z}^\mu$. One may thus view the tangent space as the direct sum of two disjoint vector spaces depending on the eigenvalue ($\pm i$) of $\mathcal{J}$:

$$T_pM = T_pM^+ \oplus T_pM^- . \tag{B.31}$$

An element of T_pM^+ is said to be a holomorphic vector, and one of T_pM^- an anti-holomorphic vector. Note that one can further extend $\mathcal{J}$ to act on elements of T^*_pM as $\mathcal{J}(dz^\mu) = -idz^\mu$ and $\mathcal{J}(d\bar{z}^\mu) = id\bar{z}^\mu$ such that :

$$\mathcal{J}(\delta^\mu_\nu) = \mathcal{J}\left(dz^\mu \frac{\partial}{\partial z^\nu}\right) = \mathcal{J}(dz^\mu)\frac{\partial}{\partial z^\nu} + dz^\mu \mathcal{J}\left(\frac{\partial}{\partial z^\nu}\right) = 0. \tag{B.32}$$

Differential forms can also be extended to complex manifolds. Viewed as a differentiable manifold, M allows the definition of $r-$forms as $(0, r)$ antisymmetrised tensors. A complex differential q-form is then defined as $A = A_1 + iA_2$ where both A_1 and A_2 are both q-forms. The conjugate $\bar{A}$ is defined as $\bar{A} = A_1 - iA_2$. In order to track holomorphicity properties, one defines the notion of bidegree. The most direct way of doing so is to attribute bidegree $(1, 0)$ to dz^μ and bidegree $(0, 1)$ to $d\bar{z}^\mu$. In other words, the bidegree counts the number of dz^μ and $d\bar{z}^\mu$'s of a complex differential form. Then any complex q-form can be decomposed on forms of bidegree (r, s) with the constraint $r + s = q$. One thus writes :

$$A = \sum_{\substack{r,s \\ r+s=q}} A_{r,s} \qquad \text{where} \qquad A_{r,s} = \frac{1}{r!s!} A_{\mu_1\ldots\mu_r\nu_1\ldots\nu_s} dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \cdots \wedge d\bar{z}^{\nu_s}. \tag{B.33}$$

In order to distinguish between vector indices in T_pM^+ and T_pM^-, one usually bars the indices belonging to the latter. For example, one often writes :

$$B = B_{\mu\nu}\, dz^\mu \wedge d\bar{z}^\nu \qquad \rightarrow \qquad B = B_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu. \tag{B.34}$$

The exterior derivative d maps p-forms into $(p+1)$-forms. The generalisation to complex forms is straightfoward :

$$\mathrm{d} = \partial + \bar{\partial} \tag{B.35}$$

which takes an (r, s)-form to the sum of a $(r+1, s)$-form (via ∂) and of a $(r, s+1)$-form (via $\bar{\partial}$). These two operators are called Dolbeault operators. The usual property of exterior derivatives of being nilpotent is here translated in $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. The concept of closedness ($\mathrm{d}A = 0$) of a form is extended to the notions of holomorphicity ($\bar{\partial}A = 0$) and anti-holomorphicity ($\partial A = 0$). The concept of exactness ($A = \mathrm{d}B$) finds its generalisation in the notions of ∂-exactness ($A = \partial B$) and of $\bar{\partial}$-exactness ($A = \bar{\partial}B$). In the context of real forms one defines the de Rahm cohomology H^q. The generalisation to complex forms is immediate and is called Dolbeault cohomology. Of course, having two nilpotent operators at our disposal, one can define both ∂-Dolbeault cohomology and $\bar{\partial}$-Dolbeaut cohomology. In practice, the (r, s) $\bar{\partial}$-Dolbeault cohomology will prove to be useful :

$$H^{(r,s)}_{\bar{\partial}} = \{\omega\ (r, s)\text{-forms} \,|\, \bar{\partial}\omega = 0\}/\{\omega\ (r, s)\text{-forms} \,|\, \exists\, \alpha \text{ such that } \omega = \bar{\partial}\alpha\}. \tag{B.36}$$

The dimension of the $H^{(r,s)}_{\bar{\partial}}$ vector space is denoted by $h^{r,s}$ and is called the *Hodge number*.

B.2.2 The Metric

Let us now focus on the metric g on complex manifolds :

$$g = g_{\mu\nu}\, dz^\mu \otimes dz^\nu + g_{\bar{\mu}\nu}\, d\bar{z}^\mu \otimes dz^\nu + g_{\mu\bar{\nu}}\, dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\bar{\nu}}\, d\bar{z}^\mu \otimes d\bar{z}^\nu. \tag{B.37}$$

The components of g on the basis consisting of the $\partial/\partial z^\mu$'s and $\partial/\partial\bar{z}^\mu$'s are denoted by $g_{\mu\nu}$, $g_{\bar{\mu}\nu}$, $g_{\mu\bar{\nu}}$ and $g_{\bar{\mu}\bar{\nu}}$ and are symmetric by definition : $g_{\mu\nu} = g_{\nu\mu}$, $g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu}$ and $g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}}$. The metric is said to be Hermitian if it satisfies $g(A, B) = g(\mathcal{J}A, \mathcal{J}B)$ where $\mathcal{J}$ is the above introduced almost complex structure. The diagonal elements of an Hermitian metric vanish. Indeed :

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = g(\mathcal{J}\partial_\mu, \mathcal{J}\partial_\nu) = -g(\partial_\mu, \partial_\nu) = 0 \quad \text{where} \quad \partial_\mu = \frac{\partial}{\partial z^\mu}. \tag{B.38}$$

It can be shown that a complex manifold always admits such a metric. We now turn our attention to the notion of parallel transport. As the tangent space is the direct sum of T_pM^+ and T_pM^-, one can define the parallel transport in such a way for the vectors of $T_pM^\pm$ to stay in $T_pM^\pm$. In other words, the connection is required to be compatible with the complex structure. In order to satisfy these requirements, the only non-vanishing components of the Christoffel symbols are the totally holomorphic and totally anti-holomorphic ones, *i.e.* $\Gamma^\rho_{\mu\bar{\nu}} = 0$ for example. If in addition we require the connection to be compatible with the metric, it has to satisfy the following equations :

$$\nabla_\rho g_{\mu\bar{\nu}} = \partial_\rho g_{\mu\bar{\nu}} - \Gamma^\sigma_{\rho\mu} g_{\sigma\bar{\nu}} \stackrel{!}{=} 0 \qquad \text{and} \qquad \nabla_{\bar{\rho}} g_{\mu\bar{\nu}} = \partial_{\bar{\rho}} g_{\mu\bar{\nu}} - \Gamma^{\bar{\sigma}}_{\bar{\rho}\bar{\nu}} g_{\mu\bar{\sigma}} \stackrel{!}{=} 0. \tag{B.39}$$

These can be solved for the Christoffel symbols in terms of g and g^{-1} :

$$\Gamma^\rho_{\mu\nu} = g^{\rho\bar{\sigma}} \partial_\mu g_{\nu\bar{\sigma}} \qquad \text{and} \qquad \Gamma^{\bar{\sigma}}_{\bar{\mu}\bar{\nu}} = g^{\rho\bar{\sigma}} \partial_{\bar{\mu}} g_{\rho\bar{\nu}}. \tag{B.40}$$

Note that at this point the Christoffel symbols are not necessarily symmetric in their lower indices. The antisymmetric part is closely related to the torsion defined by $T(A,B) = \nabla_A B - \nabla_B A - [A,B]$ where $A = A^i\partial_i$ and $B = B^i\partial_i$ where i can take both holomorphic and anti-holomorphic values. For example, one finds :

$$T(\partial_\mu, \partial_\nu) = \nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu - [\partial_\mu, \partial_\nu] \qquad i.e. \qquad T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{B.41}$$

which is to say that a complex structure compatible metric is generally not torsionless.

B.2.3 Inner Product and Adjoints

Just as we have defined the Hodge $*$ operation for real forms, we now extend it to complex forms :

Hodge Duality Given a $(r,s)-$form A defined on a $2m-$dimensional manifold M :

$$A = \frac{1}{r!s!} A_{i_1\dots i_r j_1\dots j_s} dz^{i_1} \wedge \dots \wedge dz^{i_r} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_s}, \tag{B.42}$$

one defines its Hodge dual as :

$$\begin{aligned} *A = i^m (-1)^{\frac{m(m+1)}{2}} \frac{\sqrt{g}}{r!(m-r)!s!(m-s)!} A_{i_1\dots i_r j_1\dots j_s} \epsilon^{i_1\dots i_r}{}_{i_{r+1}\dots i_m} \epsilon^{j_1\dots j_s}{}_{j_{s+1}\dots j_m} \\ d\bar{z}^{i_{r+1}} \wedge \dots \wedge d\bar{z}^{i_m} \wedge dz^{j_{s+1}} \wedge \dots \wedge dz^{j_m} \end{aligned} \tag{B.43}$$

where the indices on the $\epsilon-$symbol were raised with the inverse metric. Note that the Hodge dual transforms $(r,s)-$forms into $(m-s, m-r)-$forms.

This convention is chosen in such a way that $\int_M *1 = \int_M \sqrt{\bar{g}} d^{2m}x \equiv V$ where $\bar{g}$ is the metric in real coordinates which satisfies $\bar{g} = 2^{2m} g$ and where $d^{2m}x$ stands for $dx^1 \wedge \dots \wedge dx^{2m}$.

Indeed :

$$
\begin{aligned}
*1 &= i^m(-1)^{\frac{m(m+1)}{2}}\frac{\sqrt{g}}{m!\,m!}\epsilon_{i_1\ldots i_m}\epsilon_{j_1\ldots j_m}\bigwedge_{a=1}^{m} d\bar{z}^{i_a}\bigwedge_{b=1}^{m} dz^{j_b} \\
&= i^m(-1)^{\frac{m(m+1)}{2}}\frac{1}{2^m}\frac{\sqrt{g}}{m!\,m!}\epsilon^2_{i_1\ldots i_m}\epsilon^2_{j_1\ldots j_m}(-1)^{\sum_{i=1}^m i}\bigwedge_{i=1}^{m}\left(dz^i\wedge d\bar{z}^i\right) \\
&= \sqrt{g}d^{2m}x.
\end{aligned}
\tag{B.44}
$$

We can now define a scalar product for complex forms. The following operations can be shown to satisfy all the properties of a scalar product :

$$
\langle A, B\rangle = \int_M A\wedge *\bar{B}. \tag{B.45}
$$

Just as in the context of real forms, the scalar product allows for a definition of the adjoints of the operators ∂ and $\bar{\partial}$ which are respectively denoted by $\partial^\dagger$ and $\bar{\partial}^\dagger$:

$$
\langle\bar{\partial}^\dagger A, B\rangle = \langle A, \bar{\partial}B\rangle \qquad \text{and} \qquad \langle\partial^\dagger A, B\rangle = \langle A, \partial B\rangle. \tag{B.46}
$$

In the context of complex differential forms, one may define several Laplacians :

$$
\begin{aligned}
\Delta &= \left(\mathrm{d}+\mathrm{d}^\dagger\right)^2, \\
\Delta_\partial &= \left(\partial+\partial^\dagger\right)^2, \\
\Delta_{\bar{\partial}} &= \left(\bar{\partial}+\bar{\partial}^\dagger\right)^2
\end{aligned}
\tag{B.47}
$$

which are shown to be closely related for Kähler manifolds :

$$
\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}. \tag{B.48}
$$

See [126] for the proof.

B.2.4 Kähler Manifolds

We now define a class of Hermitian complex manifolds that will be proven to be torsionless. We first introduce the Kähler form J : $J(A,B)\equiv g(\mathcal{J}A,B)$. The elements of the Kähler form are given by :

$$
J_{\mu\nu}=0, \qquad J_{\mu\bar{\nu}}=ig_{\mu\bar{\nu}}, \qquad J_{\bar{\nu}\mu}=-ig_{\mu\bar{\nu}} \qquad \text{and} \qquad J_{\bar{\mu}\bar{\nu}}=0. \tag{B.49}
$$

One can thus write J as a $(1,1)$-form :

$$
J = J_{\mu\bar{\nu}}dz^\mu\otimes d\bar{z}^\nu + J_{\bar{\nu}\mu}d\bar{z}^\nu\otimes dz^\mu = ig_{\mu\bar{\nu}}dz^\mu\wedge d\bar{z}^\nu. \tag{B.50}
$$

The Kähler form may be used to compute the volume of a complex manifold. Indeed the integral of J^m is given by :

$$\begin{aligned}
\frac{1}{m!}\int \underbrace{J\wedge\cdots\wedge J}_{m \text{ times}} &= \frac{i^m}{m!}\int g_{\mu_1\bar{\nu}_1}\cdots g_{\mu_m\bar{\nu}_m}\bigwedge_{i=1}^{m} dz^{\mu_i}\wedge d\bar{z}^{\nu_i} \\
&= \frac{i^m(-2i)^m}{m!}\epsilon_{\mu_1\cdots\mu_m}\epsilon_{\nu_1\cdots\nu_m}\int g_{\mu_1\bar{\nu}_1}\cdots g_{\mu_m\bar{\nu}_m} d^{2m}x \\
&= 2^m\int \det(g_{\mu\bar{\nu}})d^{2m}x \\
&= \int \sqrt{g}d^{2m}x \\
&= V.
\end{aligned} \tag{B.51}$$

The complex manifolds whose Kähler form are closed are called Kähler manifolds. As already mentioned, this condition is equivalent to the torsionless condition. Indeed $\mathrm{d}J = 0$ is equivalent to :

$$\partial_\rho g_{\mu\bar{\nu}}\, dz^\rho\wedge dz^\mu\wedge d\bar{z}^\nu + \partial_{\bar{\rho}} g_{\mu\bar{\nu}}\, d\bar{z}^\rho\wedge dz^\mu\wedge d\bar{z}^\nu = 0 \tag{B.52}$$

which is to say that $\partial_\rho g_{\mu\bar{\nu}}$ is symmetric in its holomorphic indices and that $\partial_{\bar{\rho}} g_{\mu\bar{\nu}}$ is symmetric in its anti-homolorphic ones. If we now look at how the Christoffel symbols are expressed through g and g^{-1}, we immediately see that the Christoffel symbols are symmetric in their lower indices provided the manifold is of the Kähler type and the manifold is thus torsionless. Moreover, due to the properties mentioned above, one can introduce a Kähler potential K from which the metric is derived :

$$\left.\begin{array}{lll}
\partial_\rho g_{\mu\bar{\nu}} = \partial_\mu g_{\rho\bar{\nu}} & \to & g_{\mu\bar{\nu}} = \partial_\mu A_{\bar{\nu}} \\
\partial_{\bar{\rho}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\rho}} & \to & g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} B_\mu
\end{array}\right\} \qquad g_{\mu\bar{\nu}} = \partial_\mu\partial_{\bar{\nu}}K. \tag{B.53}$$

Let us now turn to the Riemann tensor. Let us first assume the manifold to be Hermitian, *i.e.* which do not necessarily satisfy $\mathrm{d}J = 0$. The Riemann tensor is defined as : $R(A,B,C) = \nabla_A\nabla_B C - \nabla_B\nabla_A C - \nabla_{[A,B]}C$ where $[A,B]$ is the Lie bracket and where $X = X^i\partial_i$ for $X \in \{A,B,C\}$ where i can take both holomorphic and anti-holomorphic values. One can easily work out the elements of the Riemann tensor. For example, the totally holomorphic element is obtained as :

$$\begin{aligned}
R(\partial_\mu,\partial_\nu,\partial_\gamma) &= \nabla_\mu\nabla_\nu\partial_\gamma - \nabla_\nu\nabla_\mu\partial_\gamma \\
&= \nabla_\mu\Gamma^\rho_{\nu\gamma}\partial_\rho - \nabla_\nu\Gamma^\rho_{\mu\gamma}\partial_\rho \\
&= \left(\partial_\mu\Gamma^\rho_{\nu\gamma}\right)\partial_\rho + \Gamma^\rho_{\nu\gamma}\Gamma^\sigma_{\mu\rho}\partial_\sigma - \left(\partial_\nu\Gamma^\rho_{\mu\gamma}\right)\partial_\rho - \Gamma^\rho_{\mu\gamma}\Gamma^\sigma_{\nu\rho}\partial_\sigma \\
&= \left(\partial_\mu\Gamma^\rho_{\nu\gamma} - \partial_\nu\Gamma^\rho_{\mu\gamma} + \Gamma^\sigma_{\nu\gamma}\Gamma^\rho_{\mu\sigma} - \Gamma^\sigma_{\mu\gamma}\Gamma^\rho_{\nu\sigma}\right)\partial_\rho \\
&\equiv R^\rho{}_{\gamma\mu\nu}\partial_\rho.
\end{aligned} \tag{B.54}$$

Note that in the torsionless case all the components of R can also be obtained as the quantity appearing in the following type of commutator, which should remind us about the definition of the gauge field strength as the commutator of covariant derivatives ($[D_\mu, D_\nu] \propto F_{\mu\nu}$) :

$$[\nabla_\mu, \nabla_\nu]\, V_\gamma = -R^\rho{}_{\gamma\mu\nu} V_\rho = R_{\mu\nu\gamma}{}^\rho V_\rho \equiv g^{\rho\sigma} g_{\mu\delta} R^\delta{}_{\nu\gamma\sigma} V_\rho \tag{B.55}$$

where we have used that the totally covariant Riemann tensor $R_{\rho\gamma\mu\nu} \equiv g_{\rho\sigma} R^\sigma{}_{\gamma\mu\nu}$ is antisymmetric in both the (ρ,γ) and the (μ,ν) pairs and symmetric under the exchange of those pairs.

If we now apply this last definition of the Riemann tensor to the case of Kähler manifolds we easily find that the only non-vanishing component of the Riemann tensor is $R_{\mu\bar\nu\rho\bar\sigma}$ and its permutations. Since the Christoffel symbols for Kähler manifolds are given by simple expressions, the Riemann tensor itself is quite simple :

$$R_{\mu\bar\nu\rho\bar\sigma} = \partial_\rho \partial_{\bar\sigma} g_{\mu\bar\nu} - g^{\alpha\bar\beta} \partial_\rho g_{\mu\bar\beta} \partial_{\bar\sigma} g_{\alpha\bar\nu}. \tag{B.56}$$

The Ricci tensor is defined as $R_{\mu\bar\nu} \equiv R^\alpha{}_{\alpha\mu\bar\nu} = -\partial_\mu \partial_{\bar\nu} \log\det(g_{\mu\bar\nu})$ and can be used to define the Ricci form $R \equiv i R_{\mu\bar\nu} dz^\mu \wedge d\bar z^\nu = -i\partial\bar\partial \log\det(g_{\mu\bar\nu})$. As $\mathrm{d}\partial\bar\partial = (\partial+\bar\partial)\partial\bar\partial = \bar\partial\partial\bar\partial = -\partial\bar\partial^2 = 0$, the Ricci form is closed.

Let us now examine the restriction imposed by harmonicity on $(1,1)$-form such as $A = A_{\mu\bar\nu} dz^\mu \wedge d\bar z^\nu$. As already shown, harmonicity is equivalent to closeness and co-closeness. The fact that A is closed, $\mathrm{d}A = 0$, implies that both ∂A and $\bar\partial A$ vanish. In components, this is translated by :

$$\mathrm{d}A = \left(\partial + \bar\partial\right) A = \partial_\rho A_{\mu\bar\nu}\, dz^\rho \wedge dz^\mu \wedge d\bar z^\nu + \partial_{\bar\rho} A_{\mu\bar\nu}\, d\bar z^\rho \wedge dz^\mu \wedge d\bar z^\nu = 0 \tag{B.57}$$

which implies that $\partial_\rho A_{\mu\bar\nu} = \partial_\mu A_{\rho\bar\nu}$ and $\partial_{\bar\rho} A_{\mu\bar\nu} = \partial_{\bar\nu} A_{\mu\bar\rho}$. The fact that A is co-closed, $\mathrm{d}^\dagger A \propto *\mathrm{d} * A = 0$, implies that both $*\partial * A$ and $*\bar\partial * A$ vanish. After a little algebra one finds that $\nabla_\mu \sigma^{\bar\nu\mu} = 0$ and $\nabla_{\bar\nu}\sigma^{\bar\nu\mu} = 0$.

From this discussion, we are now in position to show that $g^{\mu\bar\nu}\sigma_{\mu\bar\nu}$ is a constant given that σ is harmonic. Indeed its covariant derivative is given by :

$$\nabla_\rho \left(g^{\mu\bar\nu}\sigma_{\mu\bar\nu}\right) = g^{\mu\bar\nu}\nabla_\rho \sigma_{\mu\bar\nu} = g^{\mu\bar\nu}\nabla_\mu \sigma_{\rho\bar\nu} = \nabla_\mu \left(\sigma^{\bar\sigma\mu} g_{\rho\bar\sigma}\right) = g_{\rho\bar\sigma}\nabla_\mu \sigma^{\bar\sigma\mu} = 0 \tag{B.58}$$

and thus $g^{\mu\bar\nu}\sigma_{\mu\bar\nu}$ is a constant since the covariant and usual derivatives coincide on scalars.

B.2.4.1 The Hodge Dual of Harmonic Forms

The Hodge dual of a harmonic form of bigradation $(1,1)$ on a six-dimensional manifold M can be written as [173] :

$$*\sigma = -\sigma \wedge J + \frac{1}{4V}\left(\int_M \sigma \wedge J \wedge J\right) J \wedge J. \tag{B.59}$$

Proof :

$$\begin{aligned} *\sigma &= i^3 \frac{\sqrt{g}}{4} \sigma_{\mu\bar\nu} \epsilon^\mu{}_{\bar\rho\bar\sigma} \epsilon^{\bar\nu}{}_{\alpha\beta}\, dz^\alpha \wedge dz^\beta \wedge d\bar z^{\bar\rho} \wedge d\bar z^{\bar\sigma} \\ &= \frac{-i}{4} \det\left(g_{\delta\bar\gamma}\right) \sigma^{\bar\nu\mu} \epsilon_{\bar\nu\bar\rho\bar\sigma} \epsilon_{\mu\alpha\beta}\, dz^\alpha \wedge dz^\beta \wedge d\bar z^{\bar\rho} \wedge d\bar z^{\bar\sigma} \\ &= \frac{-i}{4!} \sigma^{\bar\nu\mu} \epsilon_{\mu\alpha\beta} \epsilon_{\delta\gamma\kappa} \epsilon_{\bar\nu\bar\rho\bar\sigma} \epsilon_{\bar\zeta\bar\eta\bar\theta} g_{\delta\bar\zeta} g_{\gamma\bar\eta} g_{\kappa\bar\theta}\, dz^\alpha \wedge dz^\beta \wedge d\bar z^{\bar\rho} \wedge d\bar z^{\bar\sigma}. \end{aligned} \tag{B.60}$$

Using the identity of ϵ-symbols :

$$\epsilon_{\alpha\beta\gamma}\epsilon_{\mu\nu\rho} = \det\begin{pmatrix}\delta_{\alpha\mu} & \delta_{\alpha\nu} & \delta_{\alpha\rho}\\ \delta_{\beta\mu} & \delta_{\beta\nu} & \delta_{\beta\rho}\\ \delta_{\gamma\mu} & \delta_{\gamma\nu} & \delta_{\gamma\rho}\end{pmatrix}, \tag{B.61}$$

one can easily bring the Hodge dual of σ to the following form :

$$*\sigma = -\sigma\wedge J - \frac{1}{2}\left(\sigma^{\bar{\nu}\mu}J_{\mu\bar{\nu}}\right)J\wedge J. \tag{B.62}$$

Since σ is assumed to be a harmonic form, $\sigma^{\bar{\nu}\mu}J_{\mu\bar{\nu}}$ is a constant and can thus trivially be written as :

$$\begin{aligned}\sigma^{\bar{\nu}\mu}J_{\mu\bar{\nu}} &= -\frac{1}{V}\int_M \det\left(g_{\alpha\bar{\beta}}\right)g^{\mu\bar{\nu}}\sigma_{\mu\bar{\nu}}\,d^3z\wedge d^3\bar{z}\\ &= -\frac{1}{2V}\int_M \sigma\wedge J\wedge J\end{aligned} \tag{B.63}$$

where we have used the following relation to perform the second step :

$$\left(M^{-1}\right)_{ij} = \partial_{M_{ji}}\log\det M \quad\rightarrow\quad \det M\left(M^{-1}\right)_{ij} = \frac{1}{2}\epsilon_{iab}\epsilon_{jcd}M_{ac}M_{bd}. \tag{B.64}$$

This finishes the proof.

B.2.4.2 Logarithmic Kähler Potentials

The Kähler potential from which the metric is derived does often take the form $K = -n\log V$ where V is a function of the coordinates and $n\in\mathbb{R}$. We adopt the conventional notation where a lower index stands for the derivation with respect to the corresponding coordinate, *e.g.* $K_\mu \equiv \partial K/\partial z^\mu$ and where indices may be raised using the inverse metric, *e.g.* $K^\mu \equiv K^{\mu\bar{\nu}}K_{\bar{\nu}}$. In the following expressions we will denote by $V^{\mu\bar{\nu}}$ the inverse of $V_{\mu\bar{\nu}}$. The derivatives of K may then be written as :

$$\begin{aligned}K_\mu &= -n\frac{V_\mu}{V},\\ K_{\mu\bar{\nu}} &= -n\frac{V_{\mu\bar{\nu}}}{V} + n\frac{V_\mu V_{\bar{\nu}}}{V^2},\\ K^{\mu\bar{\nu}} &= -\frac{VV^{\mu\bar{\nu}}}{n} + \frac{1}{n}\frac{1}{\theta-1}V^{\mu\bar{\rho}}V_{\bar{\rho}}V^{\sigma\bar{\nu}}V_\sigma,\\ K^\mu &= -\frac{1}{\theta-1}V^{\mu\bar{\nu}}V_{\bar{\nu}},\end{aligned} \qquad \text{where}\quad \theta = \frac{V^{\mu\bar{\nu}}V_\mu V_{\bar{\nu}}}{V}. \tag{B.65}$$

Combining the first and last of these expressions leads to :

$$K_\mu K^\mu = n\frac{\theta}{\theta-1} \tag{B.66}$$

which controls the so-called no-scale property of the manifold. The Riemann tensor is found to be given by :

$$\begin{aligned} R_{\mu\bar\nu\rho\bar\sigma} = & \frac{1}{n}\left(K_{\mu\bar\nu}K_{\rho\bar\sigma} + K_{\mu\bar\sigma}K_{\rho\bar\nu}\right) - \frac{n}{V}V_{\mu\bar\nu\rho\bar\sigma} \\ & - \frac{n}{V^2}\left(nV_{\mu\rho\bar\beta}K^{\alpha\bar\beta}V_{\alpha\bar\nu\bar\sigma} + \frac{1}{\theta-1}V_{\mu\rho}V_{\bar\nu\bar\sigma}\right) \\ & + \frac{n^2}{V^3}\left(V_{\mu\rho}V_{\bar\nu\bar\sigma\alpha}K^{\alpha\bar\beta}V_{\bar\beta} + V_{\bar\nu\bar\sigma}V_{\mu\rho\bar\beta}K^{\alpha\bar\beta}V_\alpha\right). \end{aligned} \tag{B.67}$$

B.3 Calabi-Yau Manifolds and Vector Bundles over them

In this section, we review some notation and results concerning compact Calabi-Yau manifolds X and holomorphic vector bundles V over them. We will focus on those results that concern more directly $(1,1)$-forms on X and 1-forms on X with values in V, since these are the ingredients that we need to work out the results we are interested in.

Consider first a compact Calabi-Yau manifold X. The tangent and cotangent bundles TX and T^*X have structure group $SU(3)$, since this is the holonomy group characterising this kind of manifolds. We can introduce a basis of $h^{1,1}$ independent harmonic $(1,1)$ forms ω_A on X, which provide a basis for the cohomology group $H^{1,1}(X) \simeq H^1(X, T^*X)$. We next consider the dual basis of $(2,2)$ harmonic forms ω^A and the corresponding basis of 4-cycles γ_A, defined in such a way that :

$$\int_X \omega_A \wedge \omega^B = \int_{\gamma_A} \omega^B = \delta_A^B. \tag{B.68}$$

We may then define the intersection numbers d_{ABC}, which are topological invariants of X counting how many times a triplet of 4 cycles γ^A, γ^B and γ^C intersect each other, as :

$$d_{ABC} = \int_X \omega_A \wedge \omega_B \wedge \omega_C. \tag{B.69}$$

Any harmonic $(1,1)$-form σ can be decomposed on the basis ω_A as :

$$\sigma = \sigma^A \omega_A \tag{B.70}$$

with real components σ^A given by :

$$\sigma^A = \int_X \omega^A \wedge \sigma. \tag{B.71}$$

The Hodge dual $*\sigma$ is a $(2,2)$-form which is easily seen to be harmonic and can therefore be decomposed onto the basis of ω^A as :

$$*\sigma = \sigma_A \omega^A \tag{B.72}$$

with real components σ_A given by :

$$\sigma_A = \int_X \omega_A \wedge *\sigma. \tag{B.73}$$

There always exist at least one harmonic $(1,1)$ form defining the Kähler structure. As shown in (B.44), the volume form $*1$ on X can be expressed as the exterior product of three Kähler forms J :

$$*1 = \frac{1}{6} J \wedge J \wedge J. \tag{B.74}$$

Integrating this expression over X one deduces that the volume V of X can be expressed as :

$$V = \frac{1}{6} \int_X J \wedge J \wedge J. \tag{B.75}$$

As a consequence of the existence and the properties of J, the Hodge dual of any harmonic $(1,1)$-form σ on X can be expressed in the following way in terms of J as shown in B.2.4.1 :

$$*\sigma = -J \wedge \sigma + \frac{1}{4V} \left(\int_X \sigma \wedge J \wedge J \right) J \wedge J. \tag{B.76}$$

In particular, one has :

$$*J = \frac{1}{2} J \wedge J. \tag{B.77}$$

Taking the exterior product of (B.76) with any other harmonic $(1,1)$ form ρ and integrating over X, one further deduces that the natural positive-definite scalar product on the space of all the harmonic $(1,1)$-forms can be rewritten as :

$$\int_X \rho \wedge *\sigma = -\int_X \rho \wedge \sigma \wedge J + \frac{1}{4V} \int_X \rho \wedge J \wedge J \int_X \sigma \wedge J \wedge J. \tag{B.78}$$

In particular, one finds :

$$\begin{aligned} &\int_X J \wedge *J = 3V, \\ &\int_X \omega_A \wedge *J = \frac{1}{2} \int_X \omega_A \wedge J \wedge J, \\ &\int_X \omega_A \wedge *\omega_B = -\int_X \omega_A \wedge \omega_B \wedge J + \frac{1}{4V} \int_X \omega_A \wedge J \wedge J \int_X \omega_B \wedge J \wedge J. \end{aligned} \tag{B.79}$$

Dividing by V and using the decomposition $J = J^A \omega_A$, which implies that $\omega_A = \partial J / \partial J^A$, these relations can also be rewritten in the following form :

$$\begin{aligned} &\frac{1}{V} \int_X J \wedge *J = 3, \\ &\frac{1}{V} \int_X \omega_A \wedge *J = \frac{\partial}{\partial J^A} \log V, \\ &\frac{1}{V} \int_X \omega_A \wedge *\omega_B = -\frac{\partial^2}{\partial J^A \partial J^B} \log V. \end{aligned} \tag{B.80}$$

Consider now a holomorphic vector bundle V over X, with structure group S. Out of this we can define a whole family of vector bundles V_r associated to any representation $\mathbf{r}$ of S, by promoting the transition functions of V, which are matrices in the fundamental representation of S, to the corresponding matrices in the representation $\mathbf{r}$ of S. We can then introduce a basis of n_R harmonic 1-forms u_P taking values in the representation $\mathbf{r}$ of the Lie algebra of S, associated to the cohomology group $H^1(X, V_{\rm r})$. By taking the exterior product of such a u_P with a conjugate $\bar{u}_Q$ and tracing over the indices of the representation $\mathbf{r}$, one may construct $(1,1)$-forms on the Calabi-Yau manifold X, which however are generically not harmonic :

$$c_{PQ} = i\mathrm{Tr}\left(u_P \wedge \bar{u}_Q\right). \tag{B.81}$$

One may then define the following quantities, which are a priori not topological invariants and depend in general on the geometry :

$$c^A_{PQ} = \int_X \omega^A \wedge c_{PQ}. \tag{B.82}$$

In the particular cases where the $(1,1)$ forms c_{PQ} are harmonic, the quantities c^A_{PQ} represent their components on the basis defined by the ω_A, and one may then write $c_{PQ} = c^A_{PQ}\omega_A$. More in general, one may write a Hodge decomposition with exact and coexact terms parametrised by generic $(1,0)$ and $(1,2)$-forms α_{PQ} and β_{PQ} :

$$c_{PQ} = c^A_{PQ}\omega_A + \bar{\partial}\alpha_{PQ} + \bar{\partial}^\dagger\beta_{PQ}. \tag{B.83}$$

Notice that by performing general linear transformations one may choose convenient special bases $\{\hat{\omega}_A\}$ and $\{\hat{u}_P\}$ for harmonic $(1,1)$-forms and Lie-algebra-valued 1 forms. For instance, one may define canonical bases by requiring that the $\hat{\omega}_A$ and $\hat{u}_P$ should form orthonormal sets with respect to the positive definite scalar products that can be defined on them. More precisely, we can impose that :

$$\frac{1}{V}\int_X \hat{\omega}_A \wedge *\hat{\omega}_B = \delta_{AB} \qquad \text{and} \qquad \frac{1}{V}\int_X \hat{c}_{PQ} \wedge *J = \delta_{PQ}. \tag{B.84}$$

One may moreover orient these bases with respect to the Kähler form, in such a way that $\hat{\omega}_0 = J/\sqrt{3}$ and thus $*J = \sqrt{3}V\hat{\omega}^0$. By using the equations (B.79), it follows that in such a basis the intersection numbers $\hat{d}_{ABC}$ and the quantities $\hat{c}^A_{PQ}$ have the following structure :

$$\begin{aligned} &\hat{d}_{000} = \frac{2}{\sqrt{3}}\cdot V, \qquad \hat{d}_{00a} = 0\cdot V, \qquad \hat{d}_{0ab} = -\frac{\delta_{ab}}{\sqrt{3}}\cdot V, \qquad \hat{d}_{abc} = \text{generic}\cdot V, \\ &\hat{c}^0_{PQ} = \frac{1}{\sqrt{3}}\delta_{PQ} \qquad \text{and} \qquad \hat{c}^a_{PQ} = \text{generic}. \end{aligned} \tag{B.85}$$

B.4 Symmetric Coset Manifolds

In this section, we summarise some basic facts about the geometry of the symmetric scalar manifolds appearing in the low energy effective theories of orbifold compactifications. These have the form $\mathcal{M} = \mathcal{G}/\mathcal{H}$, where the isometry group $\mathcal{G}$ is a non-compact Lie group and the isotropy group $\mathcal{H}$ is a maximal compact subgroup of it. Rather than studying separately the three kinds of spaces in (7.71), we shall focus on their basic building block, which is the following Grassmannian coset space for $p = 1$, 2 and 3 and arbitrary integer n, which has complex dimension $p(p+n)$:

$$\mathcal{M} = \frac{SU(p,p+n)}{U(1) \times SU(p) \times SU(p+n)}. \tag{B.86}$$

The canonical parametrisation of the above space involves a rectangular $p \times (p+n)$ matrix of complex coordinates Z^{iJ}, with $i = 1, \cdots, p$, $s = 1, \ldots, n$ and $I = i, s$. In this parametrisation, the full stability group $\mathcal{H} = U(1) \times SU(p) \times SU(p+n)$ acts linearly on Z^{iJ}, in the bifundamental representation $(\mathbf{p}, \mathbf{p}+\mathbf{n})_1$. Moreover, at the reference point $Z^{iJ} = 0$ these canonical coordinates correspond to normal coordinates, with trivial metric and vanishing Christoffel symbols. The Kähler potential reads [166] :

$$K = -\log\det\left(1 - ZZ^\dagger\right). \tag{B.87}$$

The parametrisation that naturally emerges in the String Theory context is however a slightly different one. It involves a $p \times p$ matrix of moduli coordinates T^{ij} and a $p \times n$ matrix Φ^{is} of matter coordinates. These are related as follows to the $p \times p$ and $p \times n$ sub-blocks Z^{ij} and Z^{is} of the above canonical coordinates Z^{iJ} :

$$Z^{ij} = \left(\frac{1-2T}{1+2T}\right)^{ij} \qquad \text{and} \qquad Z^{is} = \left(\frac{2\Phi}{1+2T}\right)^{is}. \tag{B.88}$$

In this new parametrisation, the action of $\mathcal{H}$ is more complicated. However, the subgroup $U(1) \times SU(p)_{\rm diag} \times SU(n) \subset \mathcal{H}$ still acts linearly on T^{ij}, Φ^{is}, in the adjoint and bifundamental representations $(\mathbf{1} \oplus \mathbf{p^2} - \mathbf{1}, 1)_0$ and $(\mathbf{p}, \mathbf{n})_1$. In particular, under the universal subgroup $U(p) \simeq U(1) \times SU(p)_{\rm diag}$ that is independent of n, T^{ij} and Φ^{is} transform in the adjoint and the fundamental representations $\mathbf{n^2}$ and $\mathbf{n}$. Moreover, at the reference point $T^{ij} = 1/2\,\delta^{ij}$, $\Phi^{is} = 0$ these new coordinates are only almost normal coordinates, with trivial metric but some non-vanishing Christoffel symbols. The Kähler potential becomes, up to a Kähler transformation, in accordance with [31] :

$$K = -\log\det\left(T + \bar{T} - \Phi\bar{\Phi}\right). \tag{B.89}$$

The manifold under consideration is not only homogeneous but actually symmetric, since the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ is the sum of the Lie algebra $\mathfrak{h}$ of $\mathcal{H}$ and a normal component $\mathfrak{n}$ associated to $\mathcal{G}/\mathcal{H}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$, such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{h}$. This implies that the Riemann curvature tensor is covariantly constant, $\nabla_m R_{i\bar{j}p\bar{q}} = 0$. As a consequence, the metric and the curvature tensors with tangent space indices are both completely fixed in terms of group theoretical properties of $\mathcal{G}$ and $\mathcal{H}$. To be more precise, let us label the generators of $\mathfrak{g}$ with T^X, those of $\mathfrak{h}$ with T^x and finally those of $\mathfrak{n}$ with T^θ. The metric is then given by the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{n}$:

$$g_{\theta\bar{\xi}} = -B_{\theta\xi} = -\mathrm{Tr}\,(\mathrm{ad}_\theta \cdot \mathrm{ad}_\xi)\,. \tag{B.90}$$

The Riemann tensor is instead fixed by the structure constants ruling the part $[\mathfrak{n},\mathfrak{n}] \subseteq \mathfrak{h}$ of the algebra, and reads :

$$R_{\theta\bar{\xi}\sigma\bar{\tau}} = f_{\theta\xi}{}^x f_{\sigma\tau}{}^y B_{xy}. \tag{B.91}$$

Note that although the Killing form B_{XY} on $\mathfrak{g}$ is indefinite, its restriction $B_{\theta\xi}$ to $\mathfrak{n}$ is negative definite, so that the above metric is positive definite, and its restriction B_{xy} to $\mathfrak{h}$ is positive definite, so that the curvature is negative definite.

For the manifold at hand, it is a simple exercise to compute the components of the metric and the Riemann tensor. To do so, it is convenient to switch to the standard two-index labeling of the generators of unitary groups. The generators $T^{\Theta\Gamma}$ of $U(p,p+n)$ satisfy :

$$[T^{\Theta\Gamma},T^{\Sigma\Delta}] = \eta^{\Gamma\Sigma}T^{\Theta\Delta} - \eta^{\Theta\Delta}T^{\Gamma\Sigma}. \tag{B.92}$$

The generators T^{ij} and T^{IJ} of the subgroups $U(p)$ and $U(p+n)$ similarly satisfy $[T^{ij},T^{kl}] = \delta^{jk}T^{il} - \delta^{il}T^{jk}$ and $[T^{IJ},T^{KL}] = -\delta^{JK}T^{IL} + \delta^{IL}T^{JK}$ while $[T^{ij},T^{KL}] = 0$. The remaining generators T^{iJ} and T^{Ij} describing the coset $U(p,p{+}n)/(U(p){\times}U(p{+}n))$, which are associated to the fields Z^{iJ} and their conjugate $\bar{Z}^{Ij}$, satisfy the following commutation relations: $[T^{iJ},T^{kL}] = 0$, $[T^{Ij},T^{Kl}] = 0$, $[T^{iJ},T^{Kl}] = -\delta^{JK}T^{il} - \delta^{il}T^{JK}$, $[T^{Ij},T^{kL}] = \delta^{jk}T^{IL} + \delta^{IL}T^{jk}$. The metric is trivial :

$$g_{iI\bar{j}\bar{J}} = \delta_{ij}\delta_{IJ}. \tag{B.93}$$

The Riemann tensor is instead found to be given by the following simple expression, which can also be verified by a direct computation using canonical coordinates at the reference point as in [166] :

$$R_{iI\bar{j}\bar{J}kK\bar{l}\bar{L}} = \delta_{ij}\delta_{kl}\delta_{IL}\delta_{JK} + \delta_{il}\delta_{jk}\delta_{IJ}\delta_{KL}. \tag{B.94}$$

Finally, one may split the $p(p+n)$ coset generators T^{iJ} into moduli generators T^{im} and matter generators $T^{i\alpha}$. The metric then splits into :

$$g_{im\bar{j}\bar{n}} = \delta_{ij}\delta_{mn}, \qquad g_{i\alpha\bar{j}\bar{\beta}} = \delta_{ij}\delta_{\alpha\beta}, \qquad g_{im\bar{j}\bar{\beta}} = 0 \tag{B.95}$$

and the Riemann tensor decomposes as :

$$\begin{aligned} R_{im\bar{j}\bar{n}kp\bar{l}\bar{q}} &= \delta_{ij}\delta_{kl}\delta_{mq}\delta_{np} + \delta_{il}\delta_{jk}\delta_{mn}\delta_{pq}, \\ R_{i\alpha\bar{j}\bar{\beta}k\gamma\bar{l}\bar{\delta}} &= \delta_{ij}\delta_{kl}\delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{il}\delta_{jk}\delta_{\alpha\beta}\delta_{\gamma\delta}, \\ R_{im\bar{j}\bar{n}k\gamma\bar{l}\bar{\delta}} &= \delta_{il}\delta_{jk}\delta_{mn}\delta_{\gamma\delta}. \end{aligned} \tag{B.96}$$

At this point, one may apply the above results to the coset spaces (7.71) appearing in orbifold models. The resulting expressions can be rewritten more conveniently by relabeling the generators associated to the moduli with a single index. This can be done in parallel for all the three kinds of models by making use of the 3×3 matrices λ^A representing $U(1) \times H$ for the relevant subgroup $H \subset SU(3)$. More precisely, $A = 0, \ldots, 8$ for $H = SU(3)$, $a = 0, \ldots, 3, 8$ for $H = SU(2) \times U(1)$ and $a = 0, 3, 8$ for $H = U(1) \times U(1)$. Using the normalisation condition $\mathrm{Tr}\left(\lambda^A \lambda^B\right) = \delta^{AB}$ and the completeness properties applying to each of the three subsets of matrices, the metric is found to be :

$$g_{A\bar{B}} = \delta_{AB}, \qquad g_{i\alpha\bar{j}\bar{\beta}} = \delta_{ij}\delta_{\alpha\beta}, \qquad g_{A\bar{j}} = 0 \tag{B.97}$$

and the Riemann tensor reads :

$$\begin{aligned}
&R_{A\bar{B}C\bar{D}} = \mathrm{Tr}\left(\lambda^A \lambda^B \lambda^C \lambda^D\right) + \mathrm{Tr}\left(\lambda^A \lambda^D \lambda^C \lambda^B\right), \\
&R_{i\alpha\bar{j}\bar{\beta}k\gamma\bar{l}\bar{\delta}} = \lambda^A_{il}\lambda^A_{kj}\delta_{\alpha\delta}\delta_{\beta\gamma} + \lambda^A_{ij}\lambda^A_{kl}\delta_{\alpha\beta}\delta_{\gamma\delta}, \\
&R_{A\bar{B}k\gamma\bar{l}\bar{\delta}} = (\lambda^B\lambda^A)_{kl}\delta_{\gamma\delta}.
\end{aligned} \tag{B.98}$$

Appendix C

Spinors and Supersymmetry in Various Dimensions

We prove a new theorem on the impossibility of combining space-time and internal symmetries in any but a trivial way.

Coleman and Mandula

C.1 Spinors in Various Dimensions

In this Appendix we will briefly review spinors in various dimensions. Spinors are representations of the Lorentz group $SO(1, d-1)$ whose properties can be extracted for the Clifford algebra satisfied by the Dirac matrices Γ^M :

$$\left\{\Gamma^M, \Gamma^N\right\} = 2\eta^{MN} \tag{C.1}$$

where the indices M and N take value between 0 and the space-time dimensionality d minus one and where η is the Minkowskian metric. The form of the Clifford algebra for an even number of dimensions suggests that it is possible to recast the Dirac matrices in order to obtain raising and lowering operators and that we can find a spinor χ annihilated by all lowering operators. Acting or not on χ with the raising operators will provide us with a $2^{d/2}$-dimensional representation. This representation is called the Dirac representation. It is however reducible. Indeed Γ^d defined as the product of all the Dirac matrices extends the Clifford algebra, where now M and N can take values in the range $(0, d)$. By construction Γ^d has ± 1 eigenvalues. The states with positive eigenvalue form a $2^{d/2-1}$ representation, called the Weyl representation. The states with negative eigenvalue form an inequivalent Weyl representation. Usually spinors are represented by their dimension written in boldface. In four dimensions, the Dirac spinor is thus denoted by $\mathbf{4}$. The previous discussion thus implies that $\mathbf{4} = \mathbf{2} + \bar{\mathbf{2}}$. In a space-time of odd dimensionality the Dirac matrices of the lower even dimensionality have to be supplied with Γ^d to satisfy the Clifford algebra, the representation is $2^{(d-1)/2}$. As a consequence, chirality is not defined in a five-dimensional space-time. The dimension of the irreducible representation

d	Weyl	Majorana	Majorana-Weyl	$2^{\lfloor \frac{d-1}{2} \rfloor}$	#Real Parameters
2	$\circlearrowright$	✓	✓	1	1
3		✓		2	2
4	$\rightleftarrows$	✓		2	4
5				4	8
6	$\circlearrowright$			4	8
7				8	16
8	$\rightleftarrows$	✓		8	16
9		✓		16	16
10	$\circlearrowright$	✓	✓	16	16
11		✓		32	32
12	$\rightleftarrows$	✓		32	64

Table C.1: Spinors in various dimensions

is thus given by $2^{\lfloor \frac{d-1}{2} \rfloor}$ where $\lfloor \cdot \rfloor$ denotes the floor of its argument. In the particular case of $SO(1,1)$ we denote the two irreducible Weyl representations by $\pm 1/2$.

The number of real parameters may be smaller than twice the dimension of the irreducible representation. Indeed one can impose a reality condition on spinors, called the Majorana condition. The Majorana condition can be imposed on a Weyl spinor only if it is self-conjugated (denoted by $\circlearrowright$) and not if the two Weyl representations are each other complex conjugates (denoted by $\rightleftarrows$). Table C.1, taken from [16], summarises our discussion.

C.2 The Supergravity Multiplet in Various Dimensions

As discussed in Chapter 3, the maximal space-time dimensionality compatible with Supersymmetry is eleven. Eleven-dimensional Supergravity has been argued to be the low-energy effective theory of $\mathcal{M}$-theory, yielding the type IIA effective action when compactified on a circle and the $E_8 \otimes E_8$ effective action when compactified on the $\mathbb{S}^1/\mathbb{Z}_2$ segment. Let us derive the pure eleven-dimensional SUGRA spectrum. For sure it contains both the graviton and the gravitino. The D-dimensional graviton is a symmetric traceless representation of the little group $SO(D-2)$, yielding :

$$\text{Graviton} \sim \frac{1}{2}(D-1)(D-2) - 1. \tag{C.2}$$

The D-dimensional gravitino is the product of a $D-2$ vector with its corresponding spinor representation which is found in Table C.1. However not all components of the vector-spinor constructed this way have a $3/2$-spin. One has to project out the spin-$1/2$ part by setting the gravitino trace to zero : $\Gamma^A \Psi_{A\alpha} = 0$. Furthermore, as is always the case with spinors, only half the number of off-shell degrees of freedom survive when going on-shell. The number of propagating degrees of freedom is thus given by :

$$\text{Gravitino} \sim \frac{1}{2}(D-3)\mathcal{S} \tag{C.3}$$

where $\mathcal{S}$ is the number found in the last column of Table C.1.

Whenever the number of degrees of freedom of the graviton do not match the ones of the gravitino, one is to introduce new degrees of freedom in order to enforce SUSY. Let us consider the case of four, five, ten and eleven dimensions which are those appearing in our work.

- ◇ Four dimensions : Graviton ~ 2, Gravitino ~ 2. The spectrum has an equal number of fermionic and bosonic degrees of freedom and thus is SUSY-complete.
- ◇ Five dimensions : Graviton ~ 5, Gravitino ~ 8. One has to introduce 3 bosonic degrees of freedom, which are those of a five-dimensional vector called the *graviphoton*.
- ◇ Ten dimensions : Graviton ~ 35, Gravitino ~ 56. In order to match the fermionic and bosonic degrees of freedom one introduces a 2-form B which has $C_2^8 = 28$ degrees of freedom, a dilaton Φ and a fermion χ called the *dilatino*. The spectrum is then SUSY-complete : $35 + 28 + 1 = 56 + 8$. Note that this spectrum has already been derived by compactifying eleven-dimensional SUGRA on a $\mathbb{S}^1/\mathbb{Z}_2$ segment and may be found in Table 5.2.
- ◇ Eleven dimensions : Graviton ~ 44, Gravitino ~ 128. In order to match the fermionic and bosonic degrees of freedom one introduces a 3-form C which has $C_3^9 = 84$ degrees of freedom which SUSY-completes the spectrum : $128 = 44 + 84$.

C.3 Superfield Representation of SUSY

C.3.1 Lessons from the Poincaré Group

When wanting to find the representation of the Poincaré group in Quantum Field Theory, one introduces fields $\phi(x)$ defined as

$$\phi(x) = R(x)\phi(0)R^{-1}(x) \tag{C.4}$$

where $R(x)$ is the representative of the Poincaré/Lorentz coset defining the Minkowski space. The conventional representative is $R(x) = e^{-ixP}$. Every element of the Poincaré group is uniquely decomposed as $g = R(x) \circ h$ where h belongs to the Lorentz subgroup. The action of the Poincaré group on $\phi(x)$ is then completely fixed once the action of the Lorentz group on $\phi(0)$ is specified.

The field representation of a generator G is then defined by the commutator of the generator with the field itself :

$$[G, \phi(x)] = \text{Rep}(G)\phi(x). \tag{C.5}$$

In order to illustrate the procedure, let us find the field representation of the translation generator P_μ. From (C.4) and using that $[P_\mu, P_\nu] = 0$, one gets :

$$e^{-iyP}\phi(x)e^{iyP} = \phi(x+y) \tag{C.6}$$

which for an infinitesimal displacement y yields :

$$[P_\mu, \phi(x)] = i\partial_\mu\phi(x). \tag{C.7}$$

The representation on fields of the Lorentz generators $M_{\mu\nu}$ is a little bit more involved since these generators do not commute with P_μ. We have :

$$\begin{aligned} e^{\frac{i}{2}\omega M}\phi(x)e^{-\frac{i}{2}\omega M} &= e^{\frac{i}{2}\omega M}e^{-ixP}\phi(0)e^{ixP}e^{-\frac{i}{2}\omega M} \\ &= e^{-i\tilde{x}P}e^{\frac{i}{2}\omega M}\phi(0)e^{-\frac{i}{2}\omega M}e^{i\tilde{x}P} \\ &= e^{-\frac{i}{2}\omega\Sigma}\phi(\tilde{x}) \end{aligned} \tag{C.8}$$

where $\tilde{x}$ is easily determined to be given by $\tilde{x}^\mu = x^\mu - \omega_\rho{}^\mu x^\rho$ at first order in ω by using the Poincaré algebra and where Σ characterises the action of the Lorentz group on $\phi(0)$. Expanding both sides for infinitesimal Lorentz rotations ω yields the representation of the Lorentz generators on fields :

$$[M_{\mu\nu}, \phi(x)] = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\phi(x) - \Sigma_{\mu\nu}\phi(x). \tag{C.9}$$

C.3.2 Super-Poincaré/Lorentz Coset

The same procedure can be carried out to find a representation of Supersymmetry. The coset space in this case is Super-Poincaré/Lorentz and is called Superspace, sometimes denoted by $\mathbb{R}^{4|4}$. The element of Superspace are labelled by x^μ, θ_α and $\bar{\theta}_{\dot{\alpha}}$ where the θ-variables are the anticommuting spinorial parameters of Supersymmetry. An element of the Super-Poincaré group is schematically written as :

$$g = \exp\left[i\left(-xP + \frac{1}{2}\omega M + \theta Q + \bar{\theta}\bar{Q}\right)\right]. \tag{C.10}$$

The analog of the $\phi(x)$ field in this context is thus the Superfield $\Phi(x,\theta,\bar{\theta})$:

$$\Phi(x,\theta,\bar{\theta}) = \exp\left[i(-xP + \theta Q + \bar{\theta}\bar{Q})\right]\Phi(0,0,0)\exp\left[-i(-xP + \theta Q + \bar{\theta}\bar{Q})\right]. \tag{C.11}$$

To find the SUSY generators representation on Superfields, one has to evaluate :

$$e^{i(\xi Q + \bar{\xi}\bar{Q})}\Phi(x,\theta,\bar{\theta})e^{-i(\xi Q + \bar{\xi}\bar{Q})}. \tag{C.12}$$

This is easily done since $[P_\mu, Q_\alpha] = 0$ and gives :

$$e^{i(\xi Q + \bar{\xi}\bar{Q})}\Phi(x,\theta,\bar{\theta})e^{-i(\xi Q + \bar{\xi}\bar{Q})} = \Phi(x + i\theta\sigma\bar{\xi} - i\xi\sigma\bar{\theta}, \theta + \xi, \bar{\theta} + \bar{\xi}). \tag{C.13}$$

Expanding both sides for infinitesimal SUSY transformations reveals the Superfield representation of Q_α and $\bar{Q}_{\dot{\alpha}}$:

$$\begin{aligned}\left[Q_\alpha, \Phi(x,\theta,\bar{\theta})\right] &= -i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right)\Phi(x,\theta,\bar{\theta}),\\ \left[\bar{Q}^{\dot{\alpha}}, \Phi(x,\theta,\bar{\theta})\right] &= -i\left(\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu\right)\Phi(x,\theta,\bar{\theta}).\end{aligned} \tag{C.14}$$

The action of Supersymmetry on Superfields is thus written in the following form where the SUSY generators are understood to be in their Superfield representation defined by the two previous equations :

$$\delta_{\text{SUSY}}\Phi(x,\theta,\bar{\theta}) = i(\xi Q + \bar{\xi}\bar{Q})\Phi(x,\theta,\bar{\theta}). \tag{C.15}$$

Just as in gauge theories, it is possible to define the notion of covariant derivative, *i.e.* a derivative which commutes with the action of a symmetry. In the case of Supersymmetry they are usually denoted by D_α and $\bar{D}_{\dot{\alpha}}$. Asking for $D_\alpha\delta_{\text{SUSY}} = \delta_{\text{SUSY}}D_\alpha$ is equivalent to require :

$$\{D_\alpha, Q_\beta\} = 0, \qquad \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = 0 \qquad \text{and} \qquad [D_\alpha, P_\mu] = 0 \tag{C.16}$$

which are easily solved. The Superfield representations for D_α and $\bar{D}_{\dot{\alpha}}$ are given by :

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu,\\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu.\end{aligned} \tag{C.17}$$

C.3.3 Content of a Superfield

Let us now interpret a Superfield as a collection of Quantum fields. Since the θ-variables anticommute, one can exactly Taylor expand $\Phi(x,\theta,\bar{\theta})$ as :

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = \phi(x) &+ \theta\psi(x) + \bar{\theta}\bar{\psi}(x)\\ &+ \theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta^2 m(x) + \bar{\theta}^2\bar{m}(x)\\ &+ \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\lambda(x) + \theta^2\bar{\theta}^2 d(x)\end{aligned} \tag{C.18}$$

where the different fields appearing in this expansion are all independent. The lowest component of a Superfield is usually named after the Superfield itself. If the Superfield $\Phi(x,\theta,\bar{\theta})$ does not carry any further Lorentz structure then $\phi(x)$, $m(x)$, $\bar{m}(x)$ and $d(x)$ are spinless bosons, $\psi(x)$, $\bar{\psi}(x)$, $\lambda(x)$ and $\bar{\lambda}(x)$ spin-1/2 fermions and $A_\mu(x)$ a spin-1 vector. If, for example, the Superfield $\Phi(x,\theta,\bar{\theta})$ was to carry a Lorentz index, the field $A_\mu(x)$ would then be a spin-2 particle, identified as the graviton, and $\lambda(x)$ a spin-3/2 particle, the gravitino.

Since a generic Superfield contains more states than the irreducible representations discussed in section 3.1, the Superfield representation is either reducible or most of its components are auxiliary and do not propagate. In some sense, both of these possibilities are realised. Let us first count the number of real degrees of freedom. The scalars contribute with 8 real degrees of freedom and the vector also, for a total of 16 real bosonic degrees of freedom. Each of the fermions has 4 real degrees of freedom totalising 16 real fermionic degrees of freedom. However when fermions and vectors propagate, *i.e.* when they are on-shell, their degrees of freedom are reduced.

Indeed, as is known from analytical mechanics, a state φ is characterised by a pair in phase-space (φ, Π_φ) where Π_φ is the derivative of the Lagrangian density with respect to the time-derivative of φ.

- ◇ In the case of a real bosonic field $\phi(x)$ whose kinetic Lagrangian density is $(\partial\phi)^2$ the state is caracterised by the phase-space element $(\phi, \dot{\phi})$ and is thus counted as one state.
- ◇ In the case of a Weyl fermion $\psi_\alpha(x)$ whose kinetic Lagrangian density is $i\bar{\psi}\bar{\sigma}\cdot\partial\psi$ the only two independent phase-space elements are $(\psi_1, i\bar{\psi}_{\dot{1}})$ and $(\psi_2, i\bar{\psi}_{\dot{2}})$. Thus the 4 off-shell real degrees of freedom are translated into 2 on-shell real degrees of freedom, identified with the helicity.
- ◇ In the case of a real vector field $A_\mu(x)$ whose kinetic Lagrangian density is $-1/4 F^2$, the only phase-space elements are (A_i, F^{i0}) with $i = 1, 2, 3$. Indeed A_0 does not propagate since $F^{00} = 0$. The 4 off-shell real degrees of freedom are translated in 3 on-shell degrees of freedom, these are the two transverse and the longitudinal polarisations. In the massless case, the emerging gauge symmetry can be used to set one of the A^i's to zero, effectively reducing the number of real degrees of freedom to 2, identified with the two transversal polarisations.

The number of real on-shell degrees of freedom can now be computed. For bosons, the scalar contribute with 8 units while the gauge field is reduced to 6 units for a total of 14 real degrees of freedom. The fermions degrees of freedom are divided by two for a total of 8 real degrees of freedom. The mismatch is an indication that at least 6 of the bosonic degrees of freedom do not propagate and are thus auxiliary fields.

Let us examine the case of the $(-1/2^1, 0^2, 1/2^1)$ representation. On-shell, the multiplet contains a complex scalar field and a Weyl fermion. Off-shell, the number of fermionic degrees of freedom is increased and thus has to be compensated with a complex scalar field usually denoted by F and characterised by an algebraic equation of motion. We thus have :

$$\left(-1/2^1, 0^2, 1/2^1\right) \quad \leftrightarrow \quad [\phi(x), \psi(x), F(x)] \tag{C.19}$$

and similarly :

$$\left(-1^1, -1/2^1, 1/2^1, 1^1\right) \quad \leftrightarrow \quad [\lambda(x), A_\mu(x), D(x)] \tag{C.20}$$

which means that we have to find constraints which reduce the Superfield content to those of the two previous equations.

C.3.4 Constrained Superfields

In order to reduce the content of a generic Superfield we have to impose constraints which are compatible with Supersymmetry, *i.e.* which are not spoiled by a SUSY transformation. The first possibility is to use covariant derivatives. Indeed D_α and $\bar{D}_{\dot{\alpha}}$ commute with SUSY transformations by construction. A *chiral Superfield* Φ is defined by the constraint :

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{C.21}$$

The second possibility to constrain Superfields is to impose a reality condition. A *vector Superfield* V satisfies :

$$V = V^\dagger. \tag{C.22}$$

It is easily shown that a chiral Superfield field content matches the one of a chiral multiplet and that a vector Superfield content with a gauge symmetry acting as $V \to V + \Phi + \bar{\Phi}$ matches the content of a vector multiplet.

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Colophon

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Citations

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Conclusion	Isidore Ducasse, Comte de Lautréamont *in* Poésies II

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