Tensor Products of Weil Modules

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Tout mathématicien digne de ce nom a connu, parfois seulement à de rares intervalles, ces états d'exaltation lucide où les pensées s'enchaînent comme par miracle, et où l'inconscient (quel que soit le sens qu'on attache à ce mot) paraît aussi avoir sa part. [...]
Qui l'a connu en désire le renouvellement mais est impuissant à le provoquer, sinon tout au plus par un travail opiniâtre dont il apparaît alors comme la récompense; il est vrai que le plaisir qu'on en ressent est sans rapport avec la valeur des découvertes auxquelles il s'associe.
André Weil (1906-1998, France).

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Contents

List of Notation

Throughout this report, we will try to use as much as possible some standard notations. Even if the term "standard" has not really a definition. In our case it will mean that we follow the notations of our references. And thus, for example, our matrices act on the right of our row vectors or in the same manner for x, y two elements of a group G we denote x^y the conjugacy by y which means $x^y := y^{-1}xy$ and $[x, y]$ for the commutator of x and y with $[x, y] = x^{-1}y^{-1}xy$. Following these notations we write x^G for the conjugacy class of x, i.e. $x^G = \{g^{-1}xg \mid g \in G\}$.

In addition, if we define a group by its presentation, for example the quaternion group $Q_8 = \langle x, y \mid x^4 = y^4 = 1, y^{-1}xy = x^{-1}, y^2 = x^2\rangle$, and later in the text we refer to the quaternion group as $Q_8 = \langle x, y\rangle$ or $Q_8 = \langle x_i, y_i\rangle$, the reader must understand that x, y or x_i, y_i are in fact the same as those which appear in the presentation and thus have the same properties.

Without explicit mention, we use the term representation for a complex representation.

Finally, we shall use these notations :

$\Phi(G)$	The Frattini subgroup of a group G.
$G_1 \circ G_2$	The central product of the groups G_1 and G_2.
S_d	The symmetric group of order $d!$.
χ_V	The character associated to the module V.
V^*	The dual of the module V.

Introduction

The aim of this work is to understand the decomposition, in irreducible modules, of tensor products of Weil modules for $\mathrm{Sp}_{2n}(3)$. To do this, we begin by computing the Weil modules for $\mathrm{Sp}_4(3)$, $\mathrm{Sp}_6(3)$ and $\mathrm{Sp}_8(3)$ in order to understand how tensor products decompose for these cases. This leads us to some results and hypotheses for the general case. The understanding of the decomposition of two-fold tensor product of Weil modules for $\mathrm{Sp}_{2n}(3)$ has been treated in [12]. Following it, we try to understand as much as possible the decomposition of three-fold tensor products of Weil modules for $\mathrm{Sp}_{2n}(3)$.

In the first chapter we recall some basic results in group theory and we define the symplectic groups as well as the symmetric and alternating power of a given representation. The goal of this chapter is to give a better understanding for the next chapters.

The second chapter deals with extraspecial p-groups, especially those with exponent p. We will see that we understand this class of groups very well. Indeed, we can describe their structure as a central product of groups of order p^3, which are the smallest cases of extraspecial p-groups. We know their group of automorphisms too and finally we have a good knowledge of their complex representations, which will be the start of our construction of Weil modules for symplectic groups in chapter three. Here, we start to give the general construction for Weil modules of $\mathrm{Sp}_{2n}(p)$ and then we apply the construction to the case $p = 3$ to have an explicit construction. At the end of this chapter, we are able to construct explicitly the Weil modules for $\mathrm{Sp}_{2n}(3)$. This will allow us in chapter five to take some tensor products of Weil modules for $\mathrm{Sp}_4(3)$, $\mathrm{Sp}_6(3)$ and $\mathrm{Sp}_8(3)$ using the algebra programs GAP and Magma. After having shown the decomposition of some tensor products in tabular form for these three groups, we state some general statement for $\mathrm{Sp}_{2n}(3)$. Before that, in the fourth chapter, we focus on the decomposition of $V \otimes \cdots \otimes V$ as $\mathrm{GL}(V)$-module, for a vector space V. This will give us a first idea of how $W \otimes \cdots \otimes W$ will decompose as $\mathrm{Sp}_{2n}(3)$-module, for a Weil module W. For this purpose, we need to introduce the Young diagrams and the Schur functors.

Finally, in the last section, we state all the hypotheses we have generated from consideration of the special cases, but as yet have been unable to prove. If these hypotheses are right, we are able to understand the decomposition of three-fold tensor products of Weil modules for $\mathrm{Sp}_{2n}(3)$. We derive equivalent formulations for some of our open questions, which we think will be easier to prove.

The prerequisites for a good understanding of this work consist of a good knowledge in representation theory, and some elementary facts about classical groups, especially for the symplectic group. A novice reader in one of these topics may refer to the excellent [3] and [14].

For readers interested in how we used GAP and Magma, see the author codes in [13], and the good tutorial [19] which we used to get started.

A few words about the references are sketched before getting started. The second chapter is mainly based on [1], while the third chapter picks up the article [4]. The fourth chapter takes up the ideas of [2].

Chapter 1

Prerequisites

The goal of this chapter is to introduce the prerequisites that we need to understand the construction of Weil modules for the symplectic groups. Thus, after some general reminders, we introduce the definition of the symplectic group and recall some basic properties. We finish this chapter with an overview of the symmetric square and alternating square representations which is essential to understand the decomposition of tensor products of Weil modules. During this chapter p will be a prime number.

1.1 Reminders

We state without demonstration some basic facts from group theory. The demonstrations could be found, for example, in [1] or [3].

Lemma 1.1. *Let G be a group and $x, y, z \in G$. Then has*

$$[xy, z] = y^{-1}[x, z]y[y, z] \quad \textit{and} \quad [x, yz] = [x, z]z^{-1}[x, y]z.$$

Moreover if $[x, y]$ commutes with x and y, then for all $k, n, m \in \mathbb{Z}$ we have

$$[x, y]^k = [x^k, y] = [x, y^k] \quad \textit{and} \quad [x^n, x^m] = [x, y]^{nm}.$$

Lemma 1.2. *Let G be a group and N a central normal subgroup. If G/N is cyclic then G is abelian.*

Lemma 1.3. *Let P be a group of order p^n with $n \geq 1$. If H is a non-trivial proper normal subgroup of P then $H \cap Z(P) \neq \{1\}$. In particular, the center $Z(P)$ is non-trivial.*

1.2 Symplectic Groups

Since our interest is the Weil representations of the symplectic groups we need to present the usual facts concerning these groups.

Definitions 1.4.

(i) Let k be a field and b be a bilinear form on a k-vector space V. The form b is *skew symmetric* if $b(x, y) = -b(y, x)$ for all x, y in V. Moreover, the form is said to be *symplectic* if b is nondegenerate and skew symmetric, and in addition when $\mathrm{char}(k) = 2$ we must have $b(x, x) = 0$ for all x in V.

(ii) Given a symplectic form b on V, we define the *symplectic group* $\mathrm{Sp}(V)$ as the elements of $\mathrm{GL}(V)$ which preserve b, in other words

$$\mathrm{Sp}(V) := \{T \in \mathrm{GL}(V) \mid b(xT, yT) = b(x, y) \text{ for all } x, y \in V\}.$$

Remark 1.5. Let $e_1 \in V$ be non-zero and f_1 such that $b(e_1, f_1) = 1$. Since $b_{|\langle e_1, f_1\rangle}$ is nondegenerate we know that

$$V = \langle e_1, f_1\rangle \oplus \langle e_1, f_1\rangle^{\perp}.$$

Similarly, if we consider the space $\langle e_1, f_1\rangle^{\perp}$ of dimension $\dim(V) - 2$ equipped with the symplectic form $b_{|\langle e_1, f_1\rangle^{\perp}}$ we obtain e_2, f_2 in $\langle e_1, f_1\rangle^{\perp}$ such that

$$V = \langle e_1, f_1\rangle \oplus \langle e_2, f_2\rangle \oplus \langle e_2, f_2\rangle^{\perp}.$$

Continuing in this fashion, we obtain a basis $\{e_1, f_1, e_2, f_2, \ldots, e_m, f_m\}$, called a symplectic basis. The matrix of b in the basis is

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Furthermore, the matrix of b in the basis $\{e_1, e_2, \dots e_m, f_1, f_2, \dots, f_m\}$ is

$$\begin{pmatrix} 0 & \mathrm{id}_m \\ -\mathrm{id}_m & 0 \end{pmatrix}.$$

Finally, we note that if b is a symplectic form on V then V has even dimension.

Proposition 1.6. *The order of the symplectic group over $\mathbb{F}_q$ is*

$$|\operatorname{Sp}_{2m}(q)| = \prod_{j=1}^{m}(q^{2j}-1)q^{2j-1} = q^{m^2}\prod_{j=1}^{m}(q^{2j}-1).$$

Proof. We will just give a sketch of the proof and we refer the interested reader to [14] pages 35 and 36 for the proof. The idea to calculate its order is to count the number of ways of choosing a standard basis. Pick the first vector in $q^{2m}-1$ ways. Of the $q^{2m}-q$ vectors which are linearly independent of the first, $q^{2m-1}-q$ are orthogonal to it, and $q^{2m-1}(q-1)$ have each non-zero inner product with the first. So there are q^{2m-1} choices for the second vector. To finish the proof we can for example do an induction on m. □

Remark 1.7. This is not the purpose of our work, but one can show that $\operatorname{Sp}(V)$ is a subgroup of $\operatorname{SL}(V)$ and that up to conjugation there exists a unique symplectic group, that is to say that if b' is another symplectic form on V then the group of elements in $\operatorname{GL}(V)$ which preserve b' is conjugate to $\operatorname{Sp}(V)$ in $\operatorname{GL}(V)$. One can also show that the groups $\operatorname{Sp}_{2m}(q)$ are perfect except if $m=1$ and $q=2$ or $q=3$ and if $m=2$ and $q=2$. Furthermore, the projective groups $\operatorname{PSp}_{2m}(q)$ are simple except for the cases cited above. In fact, one has that $\operatorname{PSp}_2(2) \cong S_3$, $\operatorname{PSp}_2(3) \cong A_4$ and $\operatorname{PSp}_4(2) \cong S_6$.

1.3 Symmetric square and Alternating square

Let V be a $k[G]$-module with basis $(e_i)_{1\le i\le n}$ where k is a field of characteristic different from 2 and G is a finite group. Then one can define a structure of $k[G]$-module on $V \otimes_k V$ by $(w \otimes v)g = wg \otimes vg$ for $g \in G$ and $v, w \in V$. Let $\vartheta : V \otimes V \to V \otimes V$ be the automorphism given by $(v \otimes w)\vartheta = w \otimes v$. Let λ be an eigenvalue for ϑ. Since $\vartheta^2 = \mathrm{id}$, it follows that $\lambda \in \{-1, 1\}$, because if x is an eigenvector for λ then one has $x = x\vartheta^2 = \lambda^2 x$. We can see that in fact both cases occur, see below.

We define $\mathrm{Sym}^2(V) := V_1$ to be the *symmetric square* and $\mathrm{Alt}^2(V) := V_{-1}$ to be the *alternating square* of the given representation. The elements $(e_i \otimes e_j + e_j \otimes e_i)_{i \leq j}$ form a basis of $\mathrm{Sym}^2(V)$ and the elements $(e_i \otimes e_j - e_j \otimes e_i)_{i<j}$ form a basis of $\mathrm{Alt}^2(V)$. So we have

$$\dim \mathrm{Sym}^2(V) = \frac{n(n+1)}{2} \quad \text{and} \quad \dim \mathrm{Alt}^2(V) = \frac{n(n-1)}{2}.$$

Then ϑ is diagonalizable and $V \otimes V$ decomposes, as a vector space, in a direct sum of the eigenspaces V_1 and V_{-1}.

It's clear that the subspaces $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ are both stable under the action of G and therefore we obtain the following decomposition, as $k[G]$-module,

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Moreover if $k = \mathbb{C}$, we can show that, for g in G,

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}\big(\chi_V(g)^2 + \chi_V(g^2)\big) \quad \text{and} \quad \chi_{\mathrm{Alt}^2(V)}(g) = \frac{1}{2}\big(\chi_V(g)^2 - \chi_V(g^2)\big).$$

Indeed, we can choose the basis $e_1, \ldots, e_n$ of V such that $e_i g = \lambda_i e_i$ for $1 \leq i \leq n$, and some complex numbers λ_i. Then we have

$$(e_i \otimes e_j - e_j \otimes e_i)g = \lambda_i \lambda_j (e_i \otimes e_j - e_j \otimes e_i)$$

and hence $\chi_{\mathrm{Alt}^2(V)}(g) = \sum_{i<j} \lambda_i \lambda_j$. Now computing

$$\begin{aligned} \chi_V(g)^2 &= \sum_{i=1}^n \lambda_i \sum_{j=1}^n \lambda_j = \sum_{i=1}^n \lambda_i^2 + 2\sum_{i<j} \lambda_i \lambda_j \\ &= \chi_V(g^2) + 2\chi_{\mathrm{Alt}^2(V)}(g) \end{aligned}$$

we can see that $\chi_{\mathrm{Alt}^2(V)}(g) = \frac{1}{2}\big(\chi_V(g)^2 - \chi_V(g^2)\big)$. Finally, recall that

$$\chi_V(g)^2 = \chi_{\mathrm{Alt}^2(V)}(g) + \chi_{\mathrm{Sym}^2(V)}(g),$$

and thus we deduce that $\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}\big(\chi_V(g)^2 + \chi_V(g^2)\big)$.

Actually, we can generalize the above definitions. Let W be a finite-dimensional complex vector space. The symmetric group S_d acts on $W^{\otimes d}$ by permuting the factors. We define the *alternating powers* $\mathrm{Alt}^d W$ by the space of anti-invariant vectors of the action of S_d on $W^{\otimes d}$ and the *symmetric powers* $\mathrm{Sym}^d W$ by the space of invariant vectors of the action of S_d on $W^{\otimes d}$. One can check this is coherent with the definition given for $d = 2$.

Chapter 2

Extraspecial Groups

To construct the Weil modules, we start with a representation of an extraspecial group. Thus we need to understand those groups before. So we will, during this chapter, first classify their structures with the help of the notion of central product and then classify their groups of automorphisms. Finally we will parameterize the representations of an extraspecial group, which will be the beginning of our construction of Weil representations of the symplectic groups. Again, during this chapter p will be a prime number.

2.1 Structure of Extraspecial Groups

Definition 2.1. A p-group E is *extraspecial* if $Z(E) = [E, E] = \Phi(E)$ and $Z(E)$ is cyclic.

Remark 2.2. The center of an extraspecial p-group E has order p. Indeed, let $g, h \in E$, then $g^p \in \Phi(E)$ because $E/\Phi(E)$ is elementary abelian. But $Z(E) = \Phi(E)$ so $g^p \in Z(E)$ and $1 = [g^p, h] = [g, h]^p$. Hence $[E, E]$ is elementary abelian. This shows that $Z(E)$ is cyclic and elementary abelian which gives us as the only possibility that $Z(E)$ has order p.

Example 2.3. Denote by Q_8 the quaternion group, then Q_8 is extraspecial. We already know that $Z(Q_8) = \{1, -1\}$ and that $Q_8/Z(Q_8) \cong C_2 \times C_2$. So the commutator and the Frattini subgroups are contained in the center, which has order 2. Thus, because Q_8 is not abelian or elementary abelian, we have $Z(Q_8) = [Q_8, Q_8] = \Phi(Q_8)$ and Q_8 is extraspecial.

Lemma 2.4. *Let E be a non-abelian group of order p^3, then E is extraspecial.*

Proof. By lemma 1.2 , we know that $Z(E)$ has to be cyclic of order p and that $E/Z(E) \cong C_p \times C_p$. Because otherwise this would contradict the fact that E is a non-abelian group. Since $E/Z(E)$ is elementary abelian we have again that the commutator and the Frattini subgroups are non-trivial subgroups of the center, thus we have $Z(E) = [E, E] = \Phi(E)$. □

Lemma 2.5. *There are, up to isomorphism, two non-abelian groups of order p^3. When $p = 2$, they are the dihedral group D_8 and the quaternion group Q_8. When p is odd, there are given by the following presentation*

$$\begin{aligned} X_{p^3} &= \langle x, y \mid x^p = y^p = 1, [x, y]^p = 1, [x, [x, y]] = [y, [x, y]] = 1\rangle, \\ X^-_{p^3} &= \langle x, y \mid x^p = 1, y^p = [x, y], [x, y]^p = 1, [x, [x, y]] = [y, [x, y]] = 1\rangle . \end{aligned}$$

Moreover these two groups have respectively exponent p and p^2.

Proof. We are not going to do this proof because it is a long, not difficult and well-known proof that could be found in any book dealing with finite group theory. The idea is to take $x, y \in E$ such that $\overline{x}, \overline{y}$ are generators of $E/Z(E) \cong C_p \times C_p$ and such that $z := [x, y]$ is a generator of $Z(P)$ and then to distinguish when these two elements have both order p or p^2 and when one has order p and the other one has order p^2. At least you can find a detailed approach in exercise sheet number 4 of [15]. □

Remark 2.6. We can see X_{p^3} as the group of lower triangular matrices of $\mathrm{GL}_3(\mathbb{F}_p)$ with 1 on the diagonal. This can be shown by sending

$$x \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } y \text{ to } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The group $X^-_{p^3}$ can be seen as the semi-direct product of a cyclic group of order p and a cyclic group of order p^2 and is sometimes called the modular p-group, denoted by $\mathrm{Mod}_3(p)$.

Definition 2.7. Let G be a group and $(G_i, 1 \leq i \leq m)$ a family of subgroups of G for some integer m. Then G is said to be a *central product* of the groups G_i, if

(i) $G = \langle G_i \mid 1 \leq i \leq m\rangle$,

(ii) $[G_i, G_j] = 1$ for $i \neq j$.

In this case, we will write $G = G_1 \circ \cdots \circ G_m$.

Remarks 2.8.

- This definition implies in fact that all G_i are normal subgroups of G. Indeed, let $x \in G_k$ and $y \in G$. By the first and the second conditions $y = y_1 \dots y_m$ with $y_i \in G_i$ for all i and by the second one we have $[x, y] = [x, y_1 \dots y_m] = [x, y_k] \in G_k$. Therefore we have shown that $[G_k, G] \leq G_k$ which is equivalent to the statement.

- Moreover, we have $G_i \cap G_j \subseteq Z(G)$ for $i \neq j$. Effectively, let x be an element of $G_i \cap G_j$ and $y = y_1 \dots y_m$ like in the first point. Then $[x, y] = [x, y_1 \dots y_m] = [x, y_i]$ because $x \in G_i$, but x is also in G_j so $[x, y_i] = 1$, which shows that $G_i \cap G_j \subseteq Z(G)$.

- Let G_1 and G_2 be groups and let $Z(G_1)$, respectively $Z(G_2)$, be the center of G_1, respectively G_2. Suppose that the two subgroups $Z(G_1)$ and $Z(G_2)$ are isomorphic. Given an isomorphism $\theta : Z(G_1) \to Z(G_2)$ we construct a central product $G_1 \circ G_2 := (G_1 \times G_2)/N$, where N is the normal subgroup generated by the elements $\{(z, (z^{-1})\theta) \mid z \in Z(G_1)\}$. Now it's an easy exercise to see that this definition is coherent with the above one.

Proposition 2.9. *Let G be a finite group and $(G_i, 1 \leq i \leq m)$ a family of subgroups of G for some integer m such that $G = G_1 \circ \cdots \circ G_m$. Then the map ϕ given by*

$$\begin{array}{rrcl} \phi : & G_1 \times \cdots \times G_m & \to & G \\ & (x_1, \dots, x_m) & \mapsto & x_1 \dots x_m \end{array}$$

is a surjective homomorphism with $D_i\phi = G_i$ and $D_i \cap \ker(\phi) = 1$, where D_i consists of those elements of $G_1 \times \cdots \times G_m$ with 1 in all but the ith component.

Proof. Since G is generated by the G_i as $G = G_1 \dots G_m$, the map ϕ is surjective, and it is a homomorphism since G_i and G_j commute. Moreover we have $D_i\phi \leq G_i$, and since G is a finite group, this map is a bijection. Now suppose $x \in D_i \cap \ker(\phi)$ so $x = (1, \dots, x_i, \dots, 1)$ for some $x_i \in G_i$ and $1 = x\phi = x_i$ and therefore $D_i \cap \ker(\phi) = 1$. $\square$

Proposition 2.10. *Let E be an extraspecial p-group. Regard $Z(E)$ as the field of integers modulo p and $E/Z(E)$ as a vector space over $Z(E)$ and define $\beta : E/Z(E) \times E/Z(E) \to Z(E)$ by $\beta(\bar{x}, \bar{y}) = [x, y]$. Then β is a symplectic form on $E/Z(E)$.*

Proof. Let x, y, w be elements of E then by lemma 1.1, we have

$$[xy, w] = y^{-1}[x, w]y[y, w] = y^{-1}y[x, w][y, w] = [x, w][y, w],$$

since the element $[x, w]$ belongs to $Z(E)$. Which in additive notation is $\beta(\bar{x} + \bar{y}, \bar{w}) = \beta(\bar{x}, \bar{w}) + \beta(\bar{y}, \bar{w})$. Moreover, we have

$$\beta(\lambda\bar{x}, \bar{w}) = [x^\lambda, w] = [x, w]^\lambda = \lambda\beta(\bar{x}, \bar{w}),$$

where $\lambda \in \mathbb{F}_p$. This says β is linear in its first variable and a similar argument gives linearity in the second variable.

If $\beta(\bar{x}, \bar{w}) = 0$ for all $\bar{w} \in E/Z(E)$ then $[x, w] = 1$ for all $w \in E$ and thus x is an element of the center $Z(E)$, which shows that β is nondegenerate.

Finally as $[x, w] = [w, x]^{-1}$ we have that $\beta(\bar{x}, \bar{w}) = -\beta(\bar{w}, \bar{x})$, which means that β is skew-symmetric and therefore a symplectic form. □

Remark 2.11. The difficulty of this proof lies in the passage between additive and multiplicative notation. Another way, but equivalent, to define the symplectic form is the following : let z be a generator of $Z(E)$, now if $x, y \in E$ and the commutator $[x, y]$ is z^k for some $0 \leq k \leq p - 1$, define $\beta : E/Z(E) \times E/Z(E) \to \mathbb{F}_p$ by $\beta(\bar{x}, \bar{y}) = k$.

Theorem 2.12. *Let E be an extraspecial p-group. Then there exists $r \geq 1$ such that $|E| = p^{2r+1}$ and E is the central product of r non-abelian subgroups of order p^3.*

Proof. By proposition 2.10 and the theory of symplectic forms we know that we can write

$$E/Z(E) = \bar{E}_1 \oplus \cdots \oplus \bar{E}_r$$

where $\bar{E}_i = \langle x_i, y_i \rangle$ has dimension 2, and $\beta(x_i, y_i) = 1$ and all E_i and E_j are orthogonal for $i \neq j$. Let $E_1, \ldots, E_r$ be preimages of $\bar{E}_1 \ldots \bar{E}_r$ in E. Then $E_1, \ldots, E_r$ are non-abelian groups of order p^3, generating E, such that $[E_i, E_j] = 1$ if $i \neq j$. Thus E is a central product of $E_1, \ldots, E_r$ as required. □

Remark 2.13. It's important to notice that even if G and H are two non-isomorphic groups, it's possible that $G \circ G$ is isomorphic to $H \circ H$. For example we have $D_8 \circ D_8 \cong Q_8 \circ Q_8$. Indeed, we can take the following presentations for D_8 and Q_8,

$$\begin{aligned} D_8 &= \left\langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1} \right\rangle, \\ Q_8 &= \left\langle x, y \mid x^4 = y^4 = 1, y^{-1}xy = x^{-1}, y^2 = x^2 \right\rangle. \end{aligned}$$

Let $P = Q_8 \circ Q_8$, which by definition is equal to $\langle x_1, y_1, x_2, y_2 \rangle$ with $\langle x_i, y_i \rangle \cong Q_8$ and $\langle x_1, y_1 \rangle$ centralizing $\langle x_2, y_2 \rangle$. Let $H_1 := \langle x_1, x_2y_1 \rangle$, and let $H_2 := \langle x_2, x_1y_2 \rangle$, we want to show that $H_i \cong D_8$. It's clear that $x_1^4 = (x_2y_1)^2 = 1$, furthermore $(x_2y_1)^{-1}x_1(x_2y_1) = y_1^{-1}x_1y_1 = x_1^{-1}$ proving that H_1 is isomorphic to a quotient of D_8, but we can easily see that H_1 has order bigger than four and thus $H_1 \cong D_8$, likewise we can prove that $H_2 \cong D_8$ and so $Q_8 \circ Q_8 \cong D_8 \circ D_8$ because $\langle x_1, x_2y_1, x_2, x_1y_2 \rangle = P$ and $[H_1, H_2] = 1$. We can also prove that $Q_8 \circ D_8 \ncong D_8 \circ D_8$, see for example Chapter 8 in [1] or pages 117 and 118 in [6], which gives us, with theorem 2.12 that if E is an extraspecial 2-group then either $E \cong D_8 \circ \cdots \circ D_8$ or $E = Q_8 \circ D_8 \circ \cdots \circ D_8$.

Our purpose is to have a similar result when p is odd. For that, we need to understand the central products between $X_{p^3}^-$ and X_{p^3}.

Proposition 2.14. *Let E be an extraspecial p-group. If p is odd we have*

$$X_{p^3}^- \circ X_{p^3}^- \cong X_{p^3}^- \circ X_{p^3} \text{ and } X_{p^3}^- \circ X_{p^3}^- \ncong X_{p^3} \circ X_{p^3}.$$

Then either $E \cong X_{p^3}^- \circ \cdots \circ X_{p^3}^-$ or $E \cong X_{p^3} \circ \cdots \circ X_{p^3}$.

Proof. Since X_{p^3} has exponent p and the elements of each subgroup commute, we see that $X_{p^3} \circ X_{p^3}$ has exponent p, whereas $X_{p^3}^- \circ X_{p^3}$ and $X_{p^3}^- \circ X_{p^3}^-$ have exponent p^2, which prove that $X_{p^3}^- \circ X_{p^3}^- \ncong X_{p^3} \circ X_{p^3}$. Now let $P := X_{p^3}^- \circ X_{p^3}^-$, which by definition of the central product means that $P = \langle x_1, y_1, x_2, y_2 \rangle$ with $\langle x_i, y_i \rangle = X_{p^3}^-$ and $\langle x_1, y_1 \rangle$ centralizing $\langle x_2, y_2 \rangle$ and since $\langle y_i^p \rangle = Z(P)$ we can suppose that $y_1^p = y_2^p$. Consider the subgroups $H_1 := \left\langle y_2 y_1^{-1}, x_2 \right\rangle$. Then $H_1 \cong X_{p^3}$, indeed we have $(y_2y_1^{-1})^p = 1$ and x_2 does not centralize $y_2y_1^{-1}$ hence H_1 is a non-abelian group isomorphic to a quotient of X_{p^3}, but since a group of order p^2 or p is abelian one has that H_1 is isomorphic to X_{p^3}. It turns out that if we set $H_2 := \left\langle x_1x_2, y_1y_2^{-p} \right\rangle$, then x_1x_2 has order p, $y_1y_2^{-p}$ has order p^2 and they satisfy the relations of the presentation of $X_{p^3}^-$ and one can check that H_2 is actually isomorphic to $X_{p^3}^-$. Besides H_1 and H_2 generate P and we have $[H_1, H_2] = 1$, so $P \cong X_{p^3} \circ X_{p^3}^-$. □

Proposition 2.15. *Let E be an extraspecial p-group of exponent p and order p^{2r+1} where p is odd. Then E is isomorphic to the subgroup H of $\mathrm{GL}_{r+2}(\mathbb{F}_p)$, where*

$$H := \begin{pmatrix} 1 & & & \\ \star & \ddots & 0 & \\ \vdots & 0 & \ddots & \\ \star & \cdots & \star & 1 \end{pmatrix}.$$

Proof. For $1 < j < r+2$, let H_j be the subgroup of H generated by $\mathrm{id} + e_{j,1}$ and $\mathrm{id} + e_{r+2,j}$. For $1 < j, k < r+2$, an elementary calculation shows that

$$[\mathrm{id} + e_{r+2,k}, \mathrm{id} + e_{j,1}] = \mathrm{id} + \delta_{k,j} e_{r+2,1}$$

and

$$[\mathrm{id} + e_{r+2,k}, \mathrm{id} + e_{r+2,j}] = [\mathrm{id} + e_{k,1}, \mathrm{id} + e_{j,1}] = \mathrm{id}.$$

Thus H_j and H_k commute for $k \neq j$. Furthermore it's clear that

$$\langle H_j \mid 1 < j < r+2 \rangle = H$$

and so H is the central product of $(H_j \mid 1 < j < r+2)$. But since $[\mathrm{id} + e_{r+2,k}, \mathrm{id} + e_{k,1}] = \mathrm{id} + e_{r+2,1}$ we can see that H_k is a non-abelian group of order p^3 and exponent p and thus isomorphic to X_{p^3}. Therefore we have

$$E \cong X_{p^3} \circ \cdots \circ X_{p^3} \cong H_2 \circ \cdots \circ H_{r+1} \cong H.$$

□

2.2 Automorphisms and Extraspecial Groups

Proposition 2.16. *Let E be an extraspecial p-group of order p^{2r+1}, where p is odd. Let α be an automorphism of $Z(E)$, then the automorphism α can be extended to an automorphism $\alpha \nearrow E$ of E.*

Proof. Fix a generator z of $Z(E)$. An automorphism α of $Z(E)$ is given by $z\alpha = z^k$, for some k prime to p. By proposition 2.14 we know that, for $1 \leq i \neq k \leq r$, either

$$\begin{aligned} E &\cong X^-_{p^3} \circ \cdots \circ X^-_{p^3} \\ &= \left\langle x_1, y_1, \ldots, x_r, y_r \mid \langle x_i, y_i \rangle \cong X^-_{p^3} \text{ and } [\langle x_i, y_i \rangle, \langle x_k, y_k \rangle] = 1 \right\rangle \end{aligned}$$

or

$$\begin{aligned} E &\cong X_{p^3} \circ \cdots \circ X_{p^3} \\ &= \Big\langle x_1, y_1, \ldots, x_r, y_r \mid \langle x_i, y_i \rangle \cong X_{p^3} \text{ and } [\langle x_i, y_i \rangle, \langle x_k, y_k \rangle] = 1 \Big\rangle . \end{aligned}$$

In both cases define

$$\begin{aligned} \alpha \nearrow E : E &\to E \\ x_i &\mapsto x_i^k \\ y_i &\mapsto y_i . \end{aligned}$$

An easy calculation shows that all the relations defining E are preserved because x_i has order p and k is prime to p. So that $\alpha \nearrow E$ extends to a endomorphism of E. Now we prove that $\alpha \nearrow E$ is injective. Let $g \in E$ such that $g \in \mathrm{Ker}(\alpha \nearrow E)$, where g can be written as $z^m \prod_i x_i^{s_i} y_i^{r_i}$ for some integers m, s_i, r_i and $z^{km} \prod_i x_i^{ks_i} y_i^{r_i} = 1$. This last condition implies that $ks_i = km = r_i = 0$ modulo p for all i. This is because $\{\overline{x_i}, \overline{y_i}\}_i$ is a basis of the vector space $E/Z(E)$ and thus one has uniqueness in the decomposition. Now, since k is prime to p we deduce that $s_i = m = r_i = 0$ modulo p for all i, thus $g = 1$ and $\alpha \nearrow E$ is injective. Therefore because E is finite we obtain that $\alpha \nearrow E$ is an automorphism of E. □

Remark 2.17. Since, for $1 \leq i \leq r$, one has that x_i and z the have same order, one can see that α and $\alpha \nearrow E$ have the same order as well.

Lemma 2.18. *Let E be an extraspecial p-group and let $\mathrm{Aut}_C(E)$ be the subgroup of $\mathrm{Aut}(E)$ which acts trivially on $Z(E)$. Then $\mathrm{Aut}_C(E)$ is a normal subgroup of $\mathrm{Aut}(E)$.*

Proof. Let $g \in \mathrm{Aut}(E)$, $\alpha \in \mathrm{Aut}_C(E)$ and $z \in Z(E)$; as g is an automorphism of E it sends z to another element of the center and thus

$$(z)(g^{-1}\alpha g) = \big((z)g^{-1}\big)\alpha g \overset{\alpha_{|Z(E)}=\mathrm{id}}{=} (z)g^{-1}g = z.$$

□

Theorem 2.19. *Let E be an extraspecial p-group of exponent p and order p^{2r+1}, where p is odd. Then $\mathrm{Aut}(E) = \mathrm{Aut}_C(E) \langle \theta \rangle$ where θ has order $p-1$, $\mathrm{Aut}_C(E) \cap \langle \theta \rangle = 1$ and $\mathrm{Aut}_C(E)/\mathrm{Inn}(E) \cong \mathrm{Sp}_{2r}(p)$.*

Proof. Let z be a generator of $Z(E)$. We know that $\mathrm{Aut}\big(Z(E)\big) \cong C_{p-1}$ and thus applying proposition 2.16 to a generator of $\mathrm{Aut}\big(Z(E)\big)$ we obtain an automorphism θ of E of order $p-1$ such that $z\theta = z^m$ for some m prime to p. Therefore the first statement is clear since if $\psi \in \mathrm{Aut}(E)$ then $\theta^k\psi \in \mathrm{Aut}_C(E)$ for a suitable power k. Clearly we have $\mathrm{Aut}_C(E) \cap \langle\theta\rangle = 1$ because a non-trivial element of $\langle\theta\rangle$ acts non-trivially on the center.

For the second statement let $\varphi \in \mathrm{Aut}_C(E)$, and denote by $\overline{\varphi}$ the map induced by φ on $E/Z(E)$. Then for $x, y \in E$ we have

$$[x\varphi, y\varphi] = [x,y]\varphi = [x,y].$$

Therefore, $\overline{\varphi}$ preserves the symplectic form β, indeed we have

$$\beta\big((\overline{x})\overline{\varphi}, (\overline{y})\overline{\varphi}\big) = \beta\big(\overline{x\varphi}, \overline{y\varphi}\big) = [x\varphi, y\varphi] = [x,y] = \beta(x,y).$$

This leads us to a homomorphism Φ from $\mathrm{Aut}_C(E)$ to $\mathrm{Sp}\big(E/Z(E)\big)$.

First we prove that the kernel of Φ is the group of inner automorphisms. It's clear that an inner automorphism is sent to the identity by Φ, because $E/Z(E)$ is elementary abelian. Since $\mathrm{Inn}(E) \cong E/Z(E)$ the order of $\mathrm{Inn}(E)$ is p^{2r}. But there are at most p^{2r} elements in $\mathrm{Ker}\,\Phi$ because, by definition, such an element has to send each generator of E to itself modulo the center, so there are p choices for each generator and there are $2r$ generators of E, see proposition 2.16. Therefore we have $\mathrm{Ker}\,\Phi = \mathrm{Inn}(E)$.

Afterward we want to show that Φ is surjective. Recall that

$$E = \Big\langle x_1, y_1, \ldots, x_r, y_r \mid \langle x_i, y_i\rangle \cong X_{p^3} \text{ and } [\langle x_i, y_i\rangle, \langle x_k, y_k\rangle] = 1 \Big\rangle.$$

Let $T \in \mathrm{Sp}\big(E/Z(E)\big)$, $x \in E$ and write x as $z^m \prod_i x_i^{s_i} y_i^{r_i}$ for some integers s_i, r_i. We want to define an automorphism $\psi : E \to E$ acting trivially on $Z(E)$ such that $\overline{\psi} = T$. In order to do this, write

$$(\overline{x})T \text{ as } \prod_{i=1}^{r} \overline{x_i}^{m_i} \overline{y_i}^{n_i} \text{ for some integers } m_i, n_i$$

and now define

$$\begin{aligned} \psi : E &\to E \\ x &\mapsto z^m \prod_{i=1}^{r} x_i^{m_i} y_i^{n_i}. \end{aligned}$$

By construction ψ induces T on $E/Z(E)$ and acts trivially on $Z(E)$ and is injective, moreover since T preserves the symplectic form we have, for all $x, y \in E$,

$$[x, y] = \beta\big(\overline{x}, \overline{y}\big) = \beta\big((\overline{x})T, (\overline{y})T\big) = [x\psi, y\psi].$$

This shows that the elements $\{x_1\psi, y_1\psi, \ldots, x_r\psi, y_r\psi\}$ satisfy the same relation as $\{x_1, y_1, \ldots, x_r, y_r\}$ except the conditions $x_i^p = y_i^p = 1$ and $[x_i, y_i]^p = 1$ for all $1 \leq i \leq r$, but this is always satisfied because E has exponent p. So ψ is an automorphism of E with the desired properties, which shows that Φ is surjective and concludes the proof. $\square$

Remarks 2.20.

- This theorem shows us that $\mathrm{Sp}_{2r}(p)$ acts on E as a group of automorphisms and this action is trivial on the center $Z(E)$.
- The general classification of the automorphism group of an extraspecial p-group is done in [18].

2.3 Representations of Extraspecial Groups

Theorem 2.21. *Let E be an extraspecial p-group of order p^{2r+1} and z a generator for $Z(E)$. Then*

(i) E has exactly $p^{2r} + p - 1$ irreducible representations over $\mathbb{C}$.

(ii) E has p^{2r} linear representations.

(iii) E has $p - 1$ faithful irreducible representations $\phi_1, \ldots, \phi_{p-1}$. Notation can be chosen so that $z\phi_i$ acts via the scalar ω^i on the representation module V_i of ϕ_i, where ω is some fixed primitive pth root of unity in $\mathbb{C}$.

(iv) ϕ_i is of degree p^r for all $1 \leq i \leq p - 1$.

Proof.

(i) We know that the number of irreducible representations is equal to the number of conjugacy classes in E. Let $x \in E$, then if $x \in Z(E)$ we have $x^G = \{x\}$ and otherwise $x^G = xZ(E)$. Indeed, since $Z(E) = [E, E]$ we have

$$x^G = \{g^{-1}xg \mid g \in E\} = \{x[x, g] \mid g \in E\} = xZ(E).$$

But we already know that $E/Z(E)$ has order p^{2r} and thus there are $p^{2r}-1$ different classes of the form $xZ(E)$ with x a non central element. Since $Z(E)$ has order p this gives us $p^{2r}-1+p$ different conjugacy classes in E.

(ii) We already know that the number of linear representations is equal to the order of $E/[E,E]$, but again since $Z(E)=[E,E]$ and $E/Z(E)$ has order p^{2r} we deduce that E has p^{2r} linear representations.

(iii) The first two statements show that there are $p-1$ nonlinear irreducible representations. Let ϕ be such a representation. Suppose $\operatorname{Ker}\phi$ is nontrivial, thus by lemma 1.3 we have $Z(E)\cap\operatorname{Ker}\phi\neq 1$ but since $Z(E)$ has order p this means that $Z(E)\cap\operatorname{Ker}\phi = Z(E)$. Now recall that if ϕ is a nonlinear irreducible representation then $[E,E]\nleq\operatorname{Ker}\phi$ and so $Z(E)\nleq\operatorname{Ker}\phi$ which is a contradiction with $Z(E)\cap\operatorname{Ker}\phi = Z(E)$ and therefore $\operatorname{Ker}\phi$ is trivial and ϕ is faithful. It's well known that $z\phi=\omega\,\mathrm{id}$ for some primitive pth root of unity, see for example [3]. We know that $\operatorname{Aut}\big(Z(E)\big)\cong C_{p-1}$ and thus applying proposition 2.16 to a generator of $\operatorname{Aut}\big(Z(E)\big)$ we obtain an automorphism α of E of order $p-1$ which restricted to $Z(E)$ is still a non-trivial automorphism, given by $z\alpha=z^j$, for some j prime to p. For $1\leq i\leq p-1$ let $\phi_i:=\alpha^{i-1}\phi$ then we have

$$z\phi_i = z\alpha^{i-1}\phi = z^{j^{(i-1)}}\phi = \omega^{j^{(i-1)}}\,\mathrm{id}\,.$$

So, renumbering, we may take $z\phi_i=\omega^i\,\mathrm{id}$ for $1\leq i\leq p-1$.

(iv) Recall that the order of E is equal to the sum of the squares of the dimensions of its irreducible representations. Thus we have

$$p^{2r+1}=|E|=p^{2r}+(\dim V_1)^2+\cdots+(\dim V_{p-1})^2.$$

But by construction, we have $\dim V_1=\dim V_2=\cdots=\dim V_{p-1}$ and so $p^{2r+1}=p^{2r}+(p-1)(\dim V_1)^2$. Therefore we can conclude with the following calculation

$$\dim V_1=\sqrt{\frac{p^{2r+1}-p^{2r}}{p-1}}=\sqrt{\frac{p^{2r}(p-1)}{p-1}}=p^r.$$

□

Corollary 2.22. *Let E be an extraspecial p-group of order p^{2r+1}. Then the characters χ_{V_i} vanish outside $Z(E)$ and satisfy $(\chi_{V_i})_{|Z(E)} = p^r\lambda_i$, where λ_i is a faithful linear character of $Z(E)$.*

Proof. It follows from the last theorem and the fact that the irreducible linear characters of E are the lifts of the irreducible characters of $E/[E,E] \cong C_p \times \cdots \times C_p$, which are well-known. And thus the character table of an extraspecial group is completely described. For more details, a reference may be page 813 of [8]. $\square$

Chapter 3

Construction of Weil Modules

We are now able to give the construction of the Weil representations for the symplectic groups. We first of all present the general and abstract construction, based on paragraph 5 of [4]. This construction is valid for $\mathrm{Sp}_{2n}(q)$, where q is a power of a prime number and n is a positive integer, even if we present it only for $\mathrm{Sp}_{2n}(p)$, where p is an odd prime. After this, we focus on the case $p = 3$ and give an explicit construction, in terms of matrices, of the Weil representations. The second case, which is the interest of this work, is of course almost the first one in the case $p = 3$. But since we wanted an explicit construction we had to make some minor changes in the presentation.

During this chapter ω will be a complex primitive third root of unity.

3.1 The general case

Let E be an extraspecial group of exponent p and order p^{1+2n} for $n \geq 1$. Let M be the unique irreducible $\mathbb{C}[E]$-module of dimension p^n where $Z(E)$ acts via χ, for a fixed character χ of $Z(E)$, see corollary 2.22 for the existence. We have also seen that $\mathrm{Sp}_{2n}(p)$ acts on E as a group of automorphisms, see remark 2.20. Thus we can consider the group $E \ltimes \mathrm{Sp}_{2n}(p)$. It turns out that one can extend M to an irreducible faithful $\mathbb{C}[E \ltimes \mathrm{Sp}_{2n}(p)]$-module, which restricts to the irreducible $\mathbb{C}[E]$-module as given. The proof of this fact is deductive, and since we want something constructive we are only going to discuss it in the case $p = 3$, in the section 3.2, and refer to [4] for more details about the general case. Now we consider M as a $\mathbb{C}[\mathrm{Sp}_{2n}(p)]$-module, which in fact is no longer irreducible. Indeed, denote by z the central involution in $\mathrm{Sp}_{2n}(p)$ and recall the following lemma.

Lemma 3.1. *Let $A, B \in \mathrm{GL}_m(\mathbb{C})$ such that $BA = BA$. Then if $E_1, \ldots, E_k$ are eigenspaces for A we have $E_i B \subset E_i$ for $1 \leq i \leq k$.*

Proof. Let $v \in E_i$ and λ_i the eigenvalues for E_i. Then

$$(vB)A = v(BA) = (vA)B = \lambda_i(vB).$$

Thus $vB \in E_i$ and the result follows. $\square$

So if we show that z has two eigenvalues on M then with the above lemma we can conclude that M is reducible as $\mathbb{C}[\mathrm{Sp}_{2n}(p)]$-module. Suppose by contradiction that z is a scalar, so by Schur's lemma, applied to M as $\mathbb{C}[E \ltimes \mathrm{Sp}_{2n}(p)]$-module, we obtain that $ze = ez$ for all $e \in E$, i.e. $zez^{-1} = e$ for all $e \in E$, which means that z acts trivially on E but since $\mathrm{Sp}_{2n}(p)$ acts on E as a group of automorphisms, it's a contradiction. And thus z has at least two eigenvalues. As z has order 2 this implies that z has in fact exactly two eigenvalues, namely 1 and -1. Finally, we obtain the following decomposition as $\mathbb{C}[\mathrm{Sp}_{2n}(p)]$-module

$$M = V_1 \oplus V_{-1} = C_M(z) \oplus [z, M].$$

We call these $\mathbb{C}[\mathrm{Sp}_{2n}(p)]$-submodules the *Weil modules*.

Proposition 3.2.

(*i*) *The Weil modules are irreducible of dimensions $\frac{(p^n \pm 1)}{2}$.*

(*ii*) *The Weil modules are self-dual if and only if $p \equiv 1 \mod 4$.*

Proof.

(i) In [10], it's proved that the minimal degree of an irreducible nonlinear representation of $\mathrm{Sp}_{2n}(p)$ is $(p^n - 1)/2$. Thus, because the sum of the degrees of our Weil modules is p^n we must be in one of the following cases

- At least one of the Weil modules is a sum of representations of degree 1.

- One Weil module is irreducible of dimension $(p^n-1)/2$ and the other one is a sum of representations of dimension $(p^n-1)/2$ and 1.
- The Weil modules are irreducible of dimensions $\frac{(p^n \pm 1)}{2}$.

Thus if we show that there is not a subrepresentation of degree 1 in a Weil module, the claim will follow. But this is equivalent, except for $\mathrm{Sp}_2(3)$, to saying that there is not a trivial subrepresentation in a Weil module because in these cases $\mathrm{Sp}_{2n}(p)$ is perfect [1].

It's clear that V_{-1} can't contain the trivial representation, since the central involution z acts on V_{-1} as $-\,\mathrm{id}$ and on the trivial representation as id. Now we prove by induction that V_1 doesn't contain the trivial representation. In order to do it, write E_{2n} for an extraspecial group of order p^{2n+1}, M_{2n} for the irreducible module of dimension p^n of $E_{2n} \ltimes \mathrm{Sp}_{2n}(p)$ and $V_1(2n)$ and $V_{-1}(2n)$ for the Weil modules of $\mathrm{Sp}_{2n}(p)$.

The case $n=1$ can be checked with the character table of $\mathrm{Sp}_2(p)$, see [7] page 30, looking at the dimension and the action of the center if $p \equiv 3 \mod 4$ and using the fact that $|\chi_{M_{2n}}(g)|^2 = p^{r(g)}$ where $r(g) = \dim \ker(g - \mathrm{id})$, see proposition 2 of [5], if $p \equiv 1 \mod 4$.

Suppose $V_1(2n-2)$ doesn't contain the trivial representation. We want to prove the same result for $V_1(2n)$ and so we want to look at $\mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_{2n-2}(p)}\big(V_1(2n)\big)$ to use the induction. Now because $E_{2n} = E_2 \circ E_{2n-2}$, one can easily check that $M_{2n} = M_2 \otimes M_{2n-2}$. Moreover if z_{2n} denotes the generator of $Z\big(\mathrm{Sp}_{2n}(p)\big)$, we can write z_{2n} as $z_2 z_{2n-2}$. Therefore, one has

$$\begin{aligned}
\mathrm{Res}^{E_{2n} \ltimes \mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p) \times \mathrm{Sp}_{2n-2}(p)}(M_{2n}) &= \mathrm{Res}^{E_2 \ltimes \mathrm{Sp}_2(p)}_{\mathrm{Sp}_2(p)}(M_2) \otimes \mathrm{Res}^{E_{2n-2} \ltimes \mathrm{Sp}_{2n-2}(p)}_{\mathrm{Sp}_{2n-2}(p)}(M_{2n-2}) \\
&= \big(V_1(2) \oplus V_{-1}(2)\big) \otimes \big(V_1(2n-2) \oplus V_{-1}(2n-2)\big) \\
&= V_1(2) \otimes V_1(2n-2) \oplus V_{-1}(2) \otimes V_{-1}(2n-2) \\
&\oplus\ V_1(2) \otimes V_{-1}(2n-2) \oplus V_{-1}(2) \otimes V_1(2n-2).
\end{aligned}$$

We see that $z_{2n} = z_2 z_{2n-2}$ acts trivially on $V_1(2) \otimes V_1(2n-2)$ since z_2 acts trivially on $V_1(2)$ and z_{2n-2} acts trivially on $V_1(2n-2)$ and since $-\,\mathrm{id} \otimes -\,\mathrm{id} = \mathrm{id} \otimes \mathrm{id}$ we see that z_{2n} acts trivially on $V_{-1}(2) \otimes$

[1] We postpone the case $\mathrm{Sp}_2(3)$ to the next section where we are going to prove (i) by hand.

$V_{-1}(2n-2)$. Using a similar argument we see that z_{2n} acts non-trivially on $V_1(2)\otimes V_{-1}(2n-2)$ and $V_{-1}(2)\otimes V_1(2n-2)$. Therefore, because

$$\begin{aligned}
\mathrm{Res}^{E_{2n}\ltimes \mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p)\times \mathrm{Sp}_{2n-2}(p)}(M_{2n}) &= \mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p)\times \mathrm{Sp}_{2n-2}(p)} \mathrm{Res}^{E_{2n}\ltimes \mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_{2n}(p)}(M_{2n}) \\
&= \mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p)\times \mathrm{Sp}_{2n-2}(p)} \big(V_1(2n)\oplus V_{-1}(2n)\big) \\
&= \mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p)\times \mathrm{Sp}_{2n-2}(p)} \big(V_1(2n)\big) \\
&\quad \oplus \mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p)\times \mathrm{Sp}_{2n-2}(p)} \big(V_{-1}(2n)\big)
\end{aligned}$$

one has

$$\begin{aligned}
\mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p)\times \mathrm{Sp}_{2n-2}(p)} \big(V_1(2n)\big) &= V_1(2)\otimes V_1(2n-2)\oplus V_{-1}(2)\otimes V_{-1}(2n-2) \\
\mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_2(p)\times \mathrm{Sp}_{2n-2}(p)} \big(V_{-1}(2n)\big) &= V_1(2)\otimes V_{-1}(2n-2)\oplus V_{-1}(2)\otimes V_1(2n-2)
\end{aligned}$$

and so finally

$$\begin{aligned}
\mathrm{Res}^{\mathrm{Sp}_{2n}(p)}_{\mathrm{Sp}_{2n-2}(p)} \big(V_1(2n)\big) &= \underbrace{V_1(2n-2)\oplus\cdots\oplus V_1(2n-2)}_{\dim V_1(2)\ \text{times}} \\
&\oplus \underbrace{V_{-1}(2n-2)\oplus\cdots\oplus V_{-1}(2n-2)}_{\dim V_{-1}(2)\ \text{times}}.
\end{aligned}$$

By induction and the above formula, $\mathrm{Sp}_{2n-2}(p)$ centralizes no vector of $V_1(2n)$, thus neither does $\mathrm{Sp}_{2n}(p)$. This implies that there is no trivial subrepresentation in a Weil module and so the Weil modules are irreducible of dimensions $\frac{(p^n\pm 1)}{2}$

(ii) First of all, recall that for a group G a $k[G]$-module is self dual if and only if g and g^{-1} are conjugate for all $g\in G$. It's clear that if g and h are conjugate in G and g is conjugate to g^{-1} then h is conjugate to h^{-1}. Now for $G=\mathrm{Sp}_{2n}(p)$ this is equivalent to say that t is conjugate to t^{-1} for a fixed symplectic transvection t, since the symplectic transvections generate the group and there are all conjugated to a transvection of the form

$$t := \begin{pmatrix} 1 & \lambda & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & 1 \end{pmatrix},$$

for some $\lambda \in \mathbb{F}_p^*$. We can write $t = \mathrm{id} + \lambda e_{12}$ and we have $t^{-1} = \mathrm{id} - \lambda e_{12}$. We see that t is conjugate to t^{-1} in $\mathrm{Sp}_{2n}(p)$ if and only if

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ is conjugate to } \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \text{ in } \mathrm{Sp}_2(p).$$

We want to see when $AtA^{-1} = t$ has a solution, where $A = (a_{ij})$ is an element of $\mathrm{Sp}_2(p)$. So we solve

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix},$$

and we find the following conditions : $a_{21} = 0$, $a_{12} \in \mathbb{F}_p^*$, $a_{11}a_{22} = 1$ and $(a_{11})^2 + 1 = 0$. Therefore, the system has a solution, and thus t is conjugate to t^{-1}, if and only if the last equation has a solution, which occurs if and only if $p \equiv 1 \mod 4$.

□

3.2 The case $p = 3$

Let E be an extraspecial group of exponent 3 and order $3^3 = 27$. By proposition 2.15, we know that E is isomorphic to

$$\left\{ \begin{pmatrix} 1 & & \\ a & 1 & \\ c & b & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{F}_3 \right\}.$$

We want to find a faithful irreducible representation $\rho : E \to \mathrm{GL}_3(\mathbb{C})$. To do it we consider the subgroup $H \leq E$ defined by

$$H := \left\langle \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

Since the first matrix is in the center of E, we can easily see that $H \cong C_3 \times C_3$. So we know that the irreducible characters of H are of the form $\chi_i \times \chi_j$ for $1 \leq i, j \leq 3$ and the χ_i are the irreducible characters of C_3, which we can see in the following tableau, where g is a generator of C_3,

	1	g	g^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

We consider the induced representation $\rho := \mathrm{Ind}_H^E(\chi_2 \times \chi_2)$ acting on the right on the vector space $\mathbb{C}^* \otimes_{\mathbb{C}[H]} \mathbb{C}[E]$ with basis $\{1\otimes 1, 1\otimes x, 1\otimes x^2\}$ where

$$x := \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1 & 1 \end{pmatrix}.$$

Since, by definition of the induced representation, we have $(1\otimes 1)(x\rho) = 1\otimes x$, $(1 \otimes x)(x\rho) = 1 \otimes x^2$ and $(1 \otimes x^2)(x\rho) = 1 \otimes 1$. We see that

$$(x)\rho = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

As $E = \langle H, x\rangle$ we just need to understand the representation of the two generators of H, which we call respectively z and y. Because z is in the center of E and in $\mathbb{C}[H]$ we have $(1 \otimes 1)(z\rho) = 1 \otimes z = \chi_2(z) \otimes 1 = \omega \otimes 1$, $(1 \otimes x)(z\rho) = 1 \otimes xz = 1 \otimes zx = \chi_2(z) \otimes x$ and $(1 \otimes x^2)(z\rho) = \chi_2(z) \otimes x^2$ so we have

$$(z)\rho = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

In the same manner and because $[x, y] = z$, we have $(1 \otimes 1)(y\rho) = 1 \otimes y = \chi_2(y)\otimes 1 = \omega\otimes 1$, $(1\otimes x)(y\rho) = 1\otimes xy = 1\otimes zyx = \chi_2(z)\chi_2(y)\otimes x = \omega^2(1\otimes x)$ and $(1 \otimes x^2)(y\rho) = \chi_2(z^2)\chi_2(y) \otimes x^2 = 1 \otimes x^2$. So we see that

$$(y)\rho = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It's easy to see that $\langle x\rho, y\rho\rangle \cong E$ and thus the representation is faithful. Because the irreducible representations of degree 1 are trivial on the center and there is no irreducible representation of degree 2, see theorem 2.21 and as the character values of elements of the center are imaginary, we have that ρ is irreducible.

Proposition 3.3. *$N_{\mathrm{GL}_3(\mathbb{C})}(E\rho) = Z\big(\mathrm{GL}_3(\mathbb{C})\big)(E\rho)C$, where $C \cong \mathrm{Sp}\,\big(E/Z(E)\big)$.*

Proof. To lighten the notation, we write N for $N_{\mathrm{GL}_3(\mathbb{C})}(E\rho)$. If $A \in N$, then the action by conjugation acts trivially on the center of $E\rho$ which gives us a homomorphism Φ

$$\begin{aligned} \Phi : N &\rightarrow \mathrm{Aut}_C(E) \\ A &\mapsto \text{Action of } A. \end{aligned}$$

which leads us to the injective homomorphism

$$\overline{\Phi} : N/\big(Z\big(\mathrm{GL}_3(\mathbb{C})\big)(E\rho)\big) \rightarrow \mathrm{Aut}_C(E)/\mathrm{Inn}(E)$$

since $(E\rho)\Phi = \mathrm{Inn}(E)$ and $Z\big(\mathrm{GL}_3(\mathbb{C})\big) = \{\lambda\,\mathrm{id} \mid \lambda \in \mathbb{C}^*\}$ is its kernel. Indeed if a matrix A commutes with $y\rho$ then A is diagonal and a matrix which commutes with $x\rho$ has equal values on the diagonal. In order to see that, let $A = (a_{ij})$ be such a matrix. So we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega a_{11} & \omega^2 a_{12} & a_{13} \\ \omega a_{21} & \omega^2 a_{22} & a_{23} \\ \omega a_{31} & \omega^2 a_{32} & a_{33} \end{pmatrix},$$

and

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \omega a_{11} & \omega a_{12} & \omega a_{13} \\ \omega^2 a_{21} & \omega^2 a_{22} & \omega^2 a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Thus we only have equality if A is diagonal. The second condition is the equality between

$$\begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{11} \\ a_{22} & 0 & 0 \\ 0 & a_{33} & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_{33} \\ a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \end{pmatrix}.$$

Therefore, A is a scalar multiple of the identity. Now recall that $\mathrm{Aut}_C(E)/\mathrm{Inn}(E) \cong \mathrm{Sp}\,\big(E/Z(E)\big)$ which shows that

$$N \hookrightarrow Z\big(\mathrm{GL}_3(\mathbb{C})\big)(E\rho)\,\mathrm{Sp}\,\big(E/Z(E)\big),$$

and since clearly $Z\big(\operatorname{GL}_3(\mathbb{C})\big)(E\rho) \leq N$ we just have to identify $\operatorname{Sp}\big(E/Z(E)\big)$ in $\operatorname{GL}_3(\mathbb{C})$ and show that it's contained in N to prove the claim.

Denote by $\overline{x}, \overline{y}$ the class in $E/Z(E)$ of the generators x, y of E. In our case we have

$$\operatorname{Sp}\big(E/Z(E)\big) = \operatorname{SL}_2(3) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle .$$

Denote by t the first generator of $\operatorname{SL}_2(3)$ and s the second one. In additive notation we have $(\overline{x}, \overline{y})t = (\overline{x}+\overline{y}, \overline{y})$, so in multiplicative notation this means that t sends $\overline{x}$ to $\overline{yx}$ and stabilizes $\overline{y}$. In order to preserve the action we are looking for a matrix $T \in \operatorname{GL}_3(\mathbb{C})$ such that $T^{-1}(x\rho)T = (y\rho)(x\rho)$ and $T^{-1}(y\rho)T = (y\rho)$. In the same manner for s we are looking for a matrix $S \in \operatorname{GL}_3(\mathbb{C})$ such that $S^{-1}(x\rho)S = (x\rho)$ and $S^{-1}(y\rho)S = (y\rho)(x\rho)$. From the previous calculation, we know that T has to be diagonal. So we solve the equation $(x\rho)T = T(y\rho)(x\rho)$. If $T = (\lambda_{ij})$ we get the following equality

$$\begin{pmatrix} 0 & 0 & \lambda_{33} \\ \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \omega\lambda_{11} \\ \omega^2\lambda_{22} & 0 & 0 \\ 0 & \lambda_{33} & 0 \end{pmatrix} .$$

So we find, because T has to have order 3, that

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} .$$

The first condition on S implies that

$$S = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \quad \text{for } a, b, c \in \mathbb{C}.$$

So after working out $(y\rho)S = S(y\rho)(x\rho)$ we obtain

$$\begin{pmatrix} \omega a & \omega b & \omega c \\ \omega^2 c & \omega^2 a & \omega^2 b \\ b & c & a \end{pmatrix} = \begin{pmatrix} \omega^2 b & c & \omega a \\ \omega^2 a & b & \omega c \\ \omega^2 c & a & \omega b \end{pmatrix} .$$

So we find that

$$S = a \begin{pmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{pmatrix} ,$$

and in order to have $S^3 = \mathrm{id}$ we see that we need to choose a to be a solution of $a^3 = \frac{\sqrt{3}}{9} i$.

Finally, let $C := \langle S, T \rangle$ then $C \leq N$ by construction and so by sending t to T and s to S we see that $C \cong \mathrm{Sp}\big(E/Z(E)\big)$, which concludes our proof. $\square$

Remark 3.4. So, this proposition shows us what we have not proved in the general case, that is to say that we can extend an irreducible $\mathbb{C}[E]$-module to an irreducible $\mathbb{C}[E \ltimes \mathrm{Sp}_2(3)]$-module. Now we restrict it to $\mathrm{Sp}_2(3)$ to have our first Weil Modules. Indeed, we have obtained a representation $\tau : \mathrm{Sp}_2(3) \to \mathrm{GL}_3(\mathbb{C})$ with

$$\tau(-\mathrm{id}) = \tau([s,t]^2) = [\tau(s), \tau(t)]^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}.$$

According to the general construction we have to look at the eigenspaces of this matrix. We find $V_{-1} = \mathrm{span}\{(1,0,0), (0,\omega,1)\}$ and $V_1 = \mathrm{span}\{(0,-\omega,1)\}$, which are the Weil modules for $\mathrm{Sp}_2(3)$. A careful reader will now remember we have to prove the first part of proposition 3.2 for the case $\mathrm{Sp}_2(3)$, which says that V_1 and V_{-1} are irreducible. Let $U := \mathrm{span}\{\lambda_1(1,0,0) + \lambda_2(0,\omega,1)\}$ be a submodule of V_{-1}, which means that $Ug \subset U$ for all $g \in \mathrm{Sp}_2(3)$, in particular applying T we find that $\lambda_2 = 0$ and applying S we see that $\lambda_1 = 0$ and thus V_{-1} is irreducible as claimed.

Let's go back to the construction of Weil modules for $\mathrm{Sp}_{2n}(3)$. In order to do this, let Q be an extraspecial group of order 3^5 and exponent 3. By, proposition 2.14, we know that $Q = E_1 \circ E_2$ where E_1, E_2 are extraspecial groups of order 27 and exponent 3. For $i = 1, 2$, let $\rho_i := \rho$ be the irreducible representation of E_i defined above. Consider the irreducible representation

$$\begin{aligned} \rho_1 \otimes \rho_2 : Q &\rightarrow \mathrm{GL}_9(\mathbb{C}) \\ q &\mapsto (q_1)\rho_1 \otimes (q_2)\rho_2, \end{aligned}$$

where $q = q_1 q_2$ is the decomposition of an element q in $E_1 \circ E_2$, i.e. $q_1 \in E_1$ and $q_2 \in E_2$. We want to see $\mathrm{Sp}_4(3)$ acting on this nine-dimensional space. To do this let $A := \mathrm{Sp}_2(3) \times \mathrm{Sp}_2(3)$. With the help of proposition 3.3 we see that A can be viewed as a subgroup of $N_{\mathrm{GL}_9(\mathbb{C})}\big(Q(\rho_1 \otimes \rho_2)\big)$. Indeed, using that the tensor product gives us a homomorphism between $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ and $\mathrm{GL}_9(\mathbb{C})$ and the fact that we have seen $\mathrm{Sp}_2(3)$ in $\mathrm{GL}_3(\mathbb{C})$, we just have to take some tensor products of our matrices S, T to see A in $\mathrm{GL}_9(\mathbb{C})$.

On the other hand, since $N_{\mathrm{Sp}_4(3)}(A)$ is maximal in $\mathrm{Sp}_4(3)$, we are looking for an element $g \in \mathrm{GL}_9(\mathbb{C}) \cap \mathrm{Sp}_4(3)$, which is not in $N_{\mathrm{Sp}_4(3)}(A)$, and thus $\langle A, g\rangle = \mathrm{Sp}_4(3)$, because $N_{\mathrm{Sp}_4(3)}(A)$ is the unique maximal subgroup containing A. Since A is the subgroup of $\mathrm{Sp}_4(3)$ which preserves a symplectic decomposition $\langle e_1, f_1, \rangle \oplus \langle e_2, f_2, \rangle$, and $N_{\mathrm{Sp}_4(3)}(A)$ is generated by A and the matrix that sends the first hyperbolic pair to the second one, our element g has to send the first hyperbolic pair to a combination of the first and the second one. For example we can look at this change of basis

$$e_1 \mapsto e_1 + e_2 \qquad f_1 \mapsto \frac{f_1 + f_2}{2} \qquad e_2 \mapsto e_1 - e_2 \qquad f_2 \mapsto \frac{f_1 - f_2}{2}.$$

Which corresponds to the transformation $g \in \mathrm{GL}_9(\mathbb{C})$ such that

$$\begin{aligned} (x\rho_1 \otimes \mathrm{id})^g &= x\rho_1 \otimes x\rho_2, & (y\rho_1 \otimes \mathrm{id})^g &= y^{-1}\rho_1 \otimes y^{-1}\rho_2, \\ (\mathrm{id} \otimes x\rho_2)^g &= x\rho_1 \otimes x^{-1}\rho_2, & (\mathrm{id} \otimes y\rho_2)^g &= y^{-1}\rho_1 \otimes y\rho_2, \end{aligned}$$

where x and y are given at the beginning of this section. So let

$$g := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \text{ where } A_{ij} \in \mathrm{GL}_3(\mathbb{C}).$$

The first condition implies that

$$\begin{aligned} A_{11} &= A_{22}(x\rho_2) = A_{33}(x\rho_2)^2, \\ A_{21} &= A_{32}(x\rho_2) = A_{13}(x\rho_2)^2, \\ A_{31} &= A_{12}(x\rho_2) = A_{23}(x\rho_2)^2. \end{aligned}$$

The second condition implies that

$$\begin{aligned} \omega A_{11} &= A_{11} \begin{pmatrix} \omega & & \\ & 1 & \\ & & \omega^2 \end{pmatrix} \\ \omega A_{12} &= A_{12} \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & \omega \end{pmatrix} \\ \omega A_{13} &= A_{13} \begin{pmatrix} \omega^2 & & \\ & \omega & \\ & & 1 \end{pmatrix}. \end{aligned}$$

So the matrices A_{11}, A_{12}, A_{13} have to be of the following form

$$A_{11} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix}, A_{12} = \begin{pmatrix} 0 & 0 & \alpha_4 \\ 0 & 0 & \alpha_5 \\ 0 & 0 & \alpha_6 \end{pmatrix}, A_{13} = \begin{pmatrix} 0 & \alpha_7 & 0 \\ 0 & \alpha_8 & 0 \\ 0 & \alpha_9 & 0 \end{pmatrix} \text{ for some } \alpha_i.$$

The third condition gives us

$$(x\rho_2)A_{11} = A_{12}(x\rho_2)^2 \quad \text{and} \quad (x\rho_2)A_{12} = A_{13}(x\rho_2)^2,$$

which implies that $\alpha_1 = \alpha_5 = \alpha_9$, $\alpha_2 = \alpha_6 = \alpha_7$ and $\alpha_3 = \alpha_4 = \alpha_8$. Finally the last condition leads us to $(x\rho_2)A_{11} = \omega^2 A_{11}(x\rho_2)$ which implies that $\alpha_1 = \alpha_2 = 0$ and α_3 is arbitrary. So g has the form

$$\alpha_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the transformation has order 4 (it suffices to iterate the action of g on the basis elements) and the matrix $(\alpha_3)^{-1}g$ has already order 4, the only condition on α_3 is that $(\alpha_3)^4 = 1$. So g is not in $N_{\mathrm{GL}_9(\mathbb{C})}(A)$ but not necessarily in $\mathrm{Sp}_4(3)$ viewed as subgroup of $\mathrm{GL}_9(\mathbb{C})$. For example, we see that g doesn't commute with the three central involutions, namely i_1, i_2 and $i_3 = i_1 i_2$, of A, where i_1 and i_2 are the generators of, respectively, the center of the first and second copy of $\mathrm{Sp}_2(3)$ in A. We know that the central involution of $\mathrm{Sp}_4(3)$ is one of them. It turns out that it's the element i_3 because i_1 and i_2 are conjugate, they have same eigenvalues with same multiplicity, so they are both in $Z\big(\mathrm{Sp}_4(3)\big)$ or not, but $Z\big(\mathrm{Sp}_4(3)\big)$ has order two. One can check that g^2 and i_3 are conjugate in $\langle g, i_3 \rangle$ by an element h. Define $\tilde{g} := g^h$ so that $\tilde{g}^2 = i_3$ and thus $\tilde{g} i_3 = i_3 \tilde{g}$. After a few calculations we find that

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0. \end{pmatrix}.$$

So $\tilde{g}$, as g, is not in $N_{\mathrm{GL}_9(\mathbb{C})}(A)$. Actually, using GAP, we can see that this is the element sought.

To summarize, we have obtained a faithful representation of $\mathrm{Sp}_4(3)$ in $\mathrm{GL}_9(\mathbb{C})$ given by the matrices $S \otimes \mathrm{id}, T \otimes \mathrm{id}, \mathrm{id} \otimes T, \mathrm{id} \otimes S$ and $\tilde{g}$.

Proposition 3.5. *For $n \geq 2$ we have $\langle \mathrm{Sp}_4(3), S_n \rangle = \mathrm{Sp}_{2n}(3)$.*

Proof. First of all recall that $\mathrm{Sp}_2(3) \times \mathrm{Sp}_{2n-2}(3)$ is maximal in $\mathrm{Sp}_{2n}(3)$, see [9] page 72. Let $\{e_1, f_1, e_2, f_2, \ldots, e_n, f_n\}$ be a symplectic basis. Now we see $\mathrm{Sp}_4(3)$ in $\mathrm{Sp}_{2n}(3)$ as $\mathrm{Sp}_4(3) \times \mathrm{id}$ and S_n as the group generated by the matrices $A_{(ij)}, 1 \leq i, j \leq n$ where $A_{(ij)}$ is the matrix that send $\langle e_i, f_i \rangle$ to $\langle e_j, f_j \rangle$ and acts trivially on $\langle e_k, f_k \rangle$ if k is different from i or j. So, in $\mathrm{Sp}_6(3)$, we have

$$A_{(13)}^{-1}\big(\mathrm{Sp}_4(3) \times \mathrm{id}\big)A_{(13)} = \mathrm{id} \times \mathrm{Sp}_4(3) := H_1.$$

Besides we can see $H_2 := \mathrm{Sp}_2(3) \times \mathrm{id} \times \mathrm{id}$ in $\mathrm{Sp}_4(3) \times \mathrm{id}$. Thus $\langle \mathrm{Sp}_4(3), S_3 \rangle$ contains $\mathrm{Sp}_4(3) \times \mathrm{id}$ and the maximal subgroup $H_1 H_2 = \mathrm{Sp}_2(3) \times \mathrm{Sp}_4(3)$ of $\mathrm{Sp}_6(3)$ and so we have generated $\mathrm{Sp}_6(3)$. Repeating this argument with $\mathrm{Sp}_6(3)$ instead of $\mathrm{Sp}_4(3)$ and $A_{(14)}$ instead of $A_{(13)}$ we are going to generate $\mathrm{Sp}_8(3)$. Therefore we can conclude with a recursive argument. $\square$

So our next purpose is to find the matrices $A_{(ij)}$. We start with $n = 2$ and we want to find $A_{(12)} \in \mathrm{GL}_9(\mathbb{C})$. Since $A_{(12)}$ permutes the hyperbolic pairs $e_1 \leftrightarrow e_2$ and $f_1 \leftrightarrow f_2$, this correspond to

$$\begin{aligned} (x\rho_1 \otimes \mathrm{id})^{A_{(12)}} &= \mathrm{id} \otimes x\rho_2, & (y\rho_1 \otimes \mathrm{id})^{A_{(12)}} &= \mathrm{id} \otimes y\rho_2, \\ (\mathrm{id} \otimes x\rho_2)^{A_{(12)}} &= x\rho_1 \otimes \mathrm{id}, & (\mathrm{id} \otimes y\rho_2)^{A_{(12)}} &= \mathrm{id} \otimes y\rho_2. \end{aligned}$$

Write P instead of $A_{(12)}$ and $P = (A_{ij})$ with $A_{ij} \in \mathrm{GL}_3(\mathbb{C})$. In the same manner as before, we find that

$$\begin{aligned} A_{11} &= A_{21}(x\rho_2) = A_{31}(x\rho_2)^2, \\ A_{32} &= A_{12}(x\rho_2) = A_{22}(x\rho_2)^2, \\ A_{33} &= A_{13}(x\rho_2) = A_{23}(x\rho_2)^2. \end{aligned}$$

With the second condition we find that

$$A_{11} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix}, A_{31} = \begin{pmatrix} 0 & 0 & \alpha_4 \\ 0 & 0 & \alpha_5 \\ 0 & 0 & \alpha_6 \end{pmatrix}, A_{21} = \begin{pmatrix} 0 & \alpha_7 & 0 \\ 0 & \alpha_8 & 0 \\ 0 & \alpha_9 & 0 \end{pmatrix} \text{ for some } \alpha_i.$$

The fourth condition implies that $\alpha_i = 0$ except if $i = 1, 4, 7$ and the third one shows that $\alpha_1 = \alpha_4 = \alpha_7$ and thus P is of the form

$$\alpha_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As clearly P has to be of order 2 and $(\alpha_1)^{-1}P$ has already order 2 we see that $\alpha_1 = \pm 1$. Now if $n \geq 3$, it's enough to find the matrices $A_{(i\,i+1)}$ to generate S_n. But we have that $A_{(i\,i+1)} = \mathrm{id} \otimes P \otimes \mathrm{id}$ and so it's sufficient to know P.

Thus by proposition 3.5, we can see $\mathrm{Sp}_6(3)$ in $\mathrm{GL}_{27}(\mathbb{C})$ by

$$\begin{aligned} \mathrm{Sp}_6(3) &= \langle \tilde{g} \otimes \mathrm{id}_3, S \otimes \mathrm{id}_3 \otimes \mathrm{id}_3, T \otimes \mathrm{id}_3 \otimes \mathrm{id}_3, \mathrm{id}_3 \otimes T \otimes \mathrm{id}_3, \mathrm{id}_3 \otimes S \otimes \mathrm{id}_3, \mathrm{id}_3 \otimes P \rangle \\ &= \langle \tilde{g} \otimes \mathrm{id}_3, S \otimes \mathrm{id}_9, T \otimes \mathrm{id}_9, \mathrm{id}_3 \otimes T \otimes \mathrm{id}_3, \mathrm{id}_3 \otimes S \otimes \mathrm{id}_3, \mathrm{id}_3 \otimes P \rangle \end{aligned}$$

since $\mathrm{id}_3 \otimes P = A_{(23)}$.

Therefore we can explicitly give a representation of $\mathrm{Sp}_{2n}(3)$ of dimension 3^n, by taking the tensor product of the generators of $\mathrm{Sp}_{2n-2}(3)$ with id_3 and adding $\mathrm{id}_{3^{n-2}} \otimes P$. This representation corresponds to the one discussed in the general case. Thus we know that this representation is reducible and it's irreducible components are the Weil modules of dimension $\frac{3^n \pm 1}{2}$.

Since our construction is now explicit in term of matrices we can use GAP or Magma to decompose our representations and take some tensor products of Weil modules and then decompose them again.

Chapter 4

Young Diagrams and Schur functors

Since one of our purposes is to understand the decomposition of $V\otimes\cdots\otimes V$ where V is a $\mathrm{Sp}_{2n}(3)$-module, it's a good start to know how $V \otimes \cdots \otimes V$ decomposes as $\mathrm{GL}(V)$-module. As this is not the main part of this work, we will omit the proofs. We refer to [2] for a more detailed presentation.

4.1 Young Diagrams

Definitions 4.1.

(i) Let d be a positive integer. A *partition* λ of d is a nonincreasing sequence of positive integers $\lambda_1 \geq \cdots \geq \lambda_k$ such that

$$d = \lambda_1 + \cdots + \lambda_k.$$

We note $\lambda = (\lambda_1, \ldots, \lambda_k)$.

(ii) To a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we associate a *Young diagram* with λ_i boxes in the ith row, the rows of the boxes lined up on the left.

(iii) The *conjugate partition* λ' to the partition λ is defined by interchanging rows and columns in the Young diagram.

Example 4.2. Let $d = 9$ then $\lambda = (3, 3, 2, 1)$ is a partition of d. Its Young diagram is the following

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1} \end{array}$$

Its conjugate is $\lambda' = (4, 3, 2)$.

Definition 4.3. For a given Young diagram, define a *tableau* to be a numbering of the boxes by the integers $1, \dots, d$, where the numbering is done as shown

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} 9 \\ \cline{1-1} \end{array}$$

Definitions 4.4. Given a tableau we define two subgroups of the symmetric group

$$P_\lambda = \{g \in S_d \mid g \text{ preserves each row}\}$$

and

$$Q_\lambda = \{g \in S_d \mid g \text{ preserves each column}\}.$$

In the group algebra $\mathbb{C}[S_d]$ we introduce two elements corresponding to these subgroups,

$$a_\lambda = \sum_{g \in P_\lambda} g \quad \text{and} \quad b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) g.$$

Finally we define the *Young symmetrizer*

$$c_\lambda := a_\lambda b_\lambda.$$

Theorem 4.5. *The image of c_λ by right multiplication on $\mathbb{C}[S_d]$ is an irreducible representation V_λ of S_d. Every irreducible representation of S_d can be obtained in this way for a unique partition.*

Examples 4.6.

- If $\lambda = (d)$ we have $c_{(d)} = a_{(d)} = \sum_{g \in S_d} g$ and so

$$V_{(d)} = \mathbb{C}[S_d] \sum_{g \in S_d} g = \Big\{ \gamma \sum_{g \in S_d} g \mid \gamma \in \mathbb{C} \Big\}.$$

Therefore $V_{(d)}$ is the trivial representation since the action of an element $h \in S_d$ is trivial

$$\Big(\gamma \sum_{g \in S_d} g\Big)h = \gamma \sum_{g \in S_d} (gh) = \gamma \sum_{g \in S_d} g.$$

- If $\lambda = (1, \ldots, 1)$ we have $c_{(1,\ldots,1)} = b_{(1,\ldots,1)} = \sum_{g \in S_d} \mathrm{sgn}(g)g$ and so

$$V_{(1,\ldots,1)} = \mathbb{C}[S_d] \sum_{g \in S_d} \mathrm{sgn}(g)g = \Big\{\gamma \sum_{g \in S_d} \mathrm{sgn}(g)g \mid \gamma \in \mathbb{C}\Big\}.$$

Thus $V_{(1,\ldots,1)}$ is the alternating representation. Indeed, let $h \in S_d$ then one has

$$\Big(\gamma \sum_{g \in S_d} \mathrm{sgn}(g)g\Big)h = \gamma \sum_{g \in S_d} \mathrm{sgn}(g)(gh) \overset{k=gh}{=} \mathrm{sgn}(h)\Big(\gamma \sum_{k \in S_d} \mathrm{sgn}(k)k\Big).$$

- We want to check the theorem for the group S_3. First, recall its character table

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

According to the first two points we know that $\lambda = (3)$ and $\lambda = (1, 1, 1)$ correspond to the trivial and alternating representation. Now we investigate the last case $\lambda = (2, 1)$, which has the following Young diagram

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Therefore we find that $a_{(2,1)} = \mathrm{id} + (1\,2)$ and $b_{(2,1)} = \mathrm{id} - (1\,3)$ and so

$$c_{(2,1)} = \mathrm{id} + (1\,2) - (1\,3) - (1\,3\,2).$$

By definition, we know that

$$V_{(2,1)} = \Big\{ \sum_{g \in S_3} \lambda_g g c_{(2,1)} \Big\}.$$

Computing $gc_{(2,1)}$ for $g \in S_3$ one can see that $V_{(2,1)}$ is spanned by $c_{(2,1)}$ and $(1\,3)c_{(2,1)}$. Indeed, we have $(1\,2)c_{(2,1)} = c_{(2,1)}$ and

$$(2\,3)c_{(2,1)} = (1\,3\,2)c_{(2,1)} = (1\,2\,3)c_{(2,1)} = -\big(c_{(2,1)} + (1\,3)c_{(2,1)}\big).$$

Let ρ be the representation associated to the $\mathbb{C}[S_3]$-module $V_{(12)}$. Using these previous relations we can find $(1\,2)\rho$ and $(1\,2\,3)\rho$. We find

$$(1\,2)\rho = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

because $(1\,2)\big((1\,3)c_{(2,1)}\big) = (1\,3\,2)c_{(2,1)} = -\big(c_{(2,1)} + (1\,3)c_{(2,1)}\big)$ and

$$(1\,2\,3)\rho = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

because $(1\,2\,3)\big((1\,3)c_{(2,1)}\big) = (1\,2)c_{(2,1)} = c_{(2,1)}$. Finally one can see that $\chi_{V_{(2,1)}} = \chi_3$ and thus $V_{(2,1)}$ is, as expected, an irreducible representation of S_3.

Remark 4.7. Actually, the dimensions of the irreducible modules V_λ are well-known, it relies upon the Hook length of a box. The formula can be found in [2] page 50.

4.2 Schur functors

Definition 4.8. Let V be a finite-dimensional complex vector space. The symmetric group S_d acts on $V^{\otimes d}$ by permuting the factors. This action commutes with the action of $\mathrm{GL}(V)$. We denote the image of c_λ on $V^{\otimes d}$ by $\mathbb{S}_\lambda V$

$$\mathbb{S}_\lambda V := \mathrm{Im}(c_{\lambda_{|V^{\otimes d}}}).$$

We call $\mathbb{S}_\lambda V$ the *Schur functor* corresponding to λ. It's again a representation of $\mathrm{GL}(V)$.

Examples 4.9. As before, we want to understand $\mathbb{S}_{(d)}V$, $\mathbb{S}_{(1,\ldots 1)}V$ and $\mathbb{S}_{(2,1)}V$. First, we have to consider the image of the elements a_λ and b_λ, being seen as elements of $\mathrm{End}(V^{\otimes d})$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of d. One can check that

$$\mathrm{Im}(a_\lambda) = \mathrm{Sym}^{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_k} V \subset V^{\otimes d}.$$

Similarly, the image of b_λ on this tensor product is

$$\mathrm{Im}(b_\lambda) = \mathrm{Alt}^{\lambda'_1} V \otimes \cdots \otimes \mathrm{Alt}^{\lambda'_l} V \subset V^{\otimes d},$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_l)$ is the conjugate partition to λ. Therefore, one has

$$\mathbb{S}_{(d)} V = \mathrm{Sym}^d V \quad \text{and} \quad \mathbb{S}_{(1,\ldots 1)} V = \mathrm{Alt}^d(V).$$

Finally, since $c_{(2,1)} = \mathrm{id} + (1\,2) - (1\,3) - (1\,3\,2)$, one can easly see that $\mathbb{S}_{(2,1)} V$ is the subspace of $V^{\otimes 3}$ spanned by all vectors

$$v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2.$$

Theorem 4.10.

(i) *Let $k = \dim V$. Then $\mathbb{S}_\lambda V$ is zero if $\lambda_{k+1} \neq 0$. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ then*

$$\dim \mathbb{S}_\lambda V = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

(ii) *Let m_λ be the dimension of the irreducible representation V_λ of S_d corresponding to λ. Then*

$$V^{\otimes d} \cong \bigoplus_\lambda \mathbb{S}_\lambda (V^{\otimes m_\lambda}).$$

(iii) *Each $\mathbb{S}_\lambda V$ is an irreducible representation of $\mathrm{GL}(V)$.*

Example 4.11. Using theorem 4.10, one can check that

$$V \otimes V \otimes V = \mathrm{Sym}^3 V \oplus \mathrm{Alt}^3 V \oplus 2\mathbb{S}_{(2,1)} V.$$

Using the decomposition $V \otimes V = \mathrm{Sym}^2 V \oplus \mathrm{Alt}^2 V$, one has

$$V \otimes \mathrm{Sym}^2 V = \mathrm{Sym}^3 V \oplus \mathbb{S}_{(2,1)} V \quad \text{and} \quad V \otimes \mathrm{Alt}^2 V = \mathrm{Alt}^3 V \oplus \mathbb{S}_{(2,1)} V.$$

Chapter 5

Tensor Products of Weil Modules

In this chapter we discuss the decomposition of tensor products of Weil modules for $\mathrm{Sp}_{2n}(3)$. Our calculations have been carried out with GAP or Magma, for the groups $\mathrm{Sp}_4(3)$, $\mathrm{Sp}_6(3)$ and $\mathrm{Sp}_8(3)$. We have followed the construction given in chapter 3. First, we show our results for these three cases and then we state some general conclusions about the decomposition of tensor products of Weil modules for $\mathrm{Sp}_{2n}(3)$. We define

$$\mathcal{W}_{2n}^m := \left\{ W_1 \otimes \cdots \otimes W_m \mid W_i \text{ or } W_i^* \text{ is a Weil module of } \mathrm{Sp}_{2n}(3) \right\},$$

to be the set of m-fold tensor products of Weil modules.

5.1 The case $\mathrm{Sp}_4(3)$

Throughout this section, W^- will denote the Weil module of dimension $\frac{(3^2-1)}{2} = 4$ and W^+ the one of dimension $\frac{(3^2+1)}{2} = 5$. To begin with we present a tableau for $\mathcal{W}_4^2$.

	χ_2	χ_5	χ_3	χ_4
χ_2	$\chi_6 + \chi_8$	χ_{11}	$\chi_1 + \chi_9$	χ_{13}
χ_5		$\chi_7 + \chi_{10}$	χ_{14}	$\chi_1 + \chi_{17}$
χ_3			$\chi_6 + \chi_7$	χ_{11}
χ_4				$\chi_8 + \chi_{10}$

We use the same notation as GAP, i.e. $\chi_{W^-} = \chi_2$, $\chi_{W^+} = \chi_5$ and χ_3, χ_4 are the duals of, respectively χ_2 and χ_5. The entries of the tableau are the decompositions of the tensor product of the characters in the corresponding row and column, for example $\chi_2\chi_2 = \chi_6 + \chi_8$, where χ_6, χ_8 are irreducible characters. The characters χ_i are those irreducible characters found with our codes, see [13], for the group $\mathrm{Sp}_4(3)$. Following section 1.3, one can note that $\chi_6 = \chi_{\mathrm{Alt}^2(W^-)}$ and $\chi_8 = \chi_{\mathrm{Sym}^2(W^-)}$. In the same manner, one has $\chi_7 = \chi_{\mathrm{Alt}^2(W^+)}$ and $\chi_{10} = \chi_{\mathrm{Sym}^2(W^+)}$. Now we present the tableau for $\mathcal{W}_4^3$

	χ_2	χ_5	χ_3	χ_4
$\chi_2\chi_2$	$\chi_3 + \chi_{14} + 2\chi_{16}$	$\chi_4 + \chi_{19} + \chi_{25}$	$2\chi_2 + \chi_{15} + \chi_{21}$	$\chi_7 + \chi_{20} + \chi_{24}$
$\chi_2\chi_5$		$\chi_3 + \chi_{22} + \chi_{29}$	$\chi_5 + \chi_{20} + \chi_{26}$	$\chi_2 + \chi_{21} + \chi_{30}$
$\chi_2\chi_3$			$2\chi_3 + \chi_{16} + \chi_{22}$	$\chi_4 + \chi_{19} + \chi_{25}$
$\chi_2\chi_4$				$\chi_{14} + \chi_{16} + \chi_{29}$
$\chi_5\chi_5$		$\chi_4 + \chi_8 + \chi_{18} + 2\chi_{23}$	$\chi_{13} + \chi_{15} + \chi_{30}$	$2\chi_5 + \chi_{18} + \chi_{24} + \chi_{26}$
$\chi_5\chi_3$			$\chi_8 + \chi_{19} + \chi_{23}$	$\chi_3 + \chi_{22} + \chi_{29}$
$\chi_5\chi_4$				$2\chi_4 + \chi_{18} + \chi_{23} + \chi_{25}$
$\chi_3\chi_3$			$\chi_2 + \chi_{13} + 2\chi_{15}$	$\chi_5 + \chi_{20} + \chi_{26}$
$\chi_3\chi_4$				$\chi_2 + \chi_{21} + \chi_{30}$
$\chi_4\chi_4$				$\chi_5 + \chi_7 + \chi_{18} + 2\chi_{24}$

To be able to understand $\mathcal{W}_4^3$ in a better way, we take each irreducible character of $\mathcal{W}_4^2$ and we tensor it with elements of $\mathcal{W}_4^1$. We obtain the following tableau

	χ_2	χ_5	χ_3	χ_4
χ_6	$\chi_3 + \chi_{16}$	χ_{19}	$\chi_2 + \chi_{15}$	χ_{20}
χ_7	$\chi_3 + \chi_{22}$	$\chi_8 + \chi_{28}$	$\chi_{13} + \chi_{15}$	$\chi_5 + \chi_{26}$
χ_8	$\chi_{14} + \chi_{16}$	$\chi_4 + \chi_{25}$	$\chi_2 + \chi_{21}$	$\chi_7 + \chi_{24}$
χ_9	$\chi_2 + \chi_{15} + \chi_{21}$	$\chi_{20} + \chi_{26}$	$\chi_3 + \chi_{16} + \chi_{22}$	$\chi_{19} + \chi_{25}$
χ_{10}	χ_{29}	$\chi_4 + \chi_{18} + \chi_{23}$	χ_{30}	$\chi_5 + \chi_{18} + \chi_{24}$
χ_{11}	$\chi_4 + \chi_{19} + \chi_{25}$	$\chi_3 + \chi_{22} + \chi_{29}$	$\chi_5 + \chi_{20} + \chi_{26}$	$\chi_2 + \chi_{21} + \chi_{30}$
χ_{13}	$\chi_7 + \chi_{20} + \chi_{24}$	$\chi_2 + \chi_{21} + \chi_{30}$	$\chi_4 + \chi_{19} + \chi_{25}$	$\chi_{14} + \chi_{16} + \chi_{29}$
χ_{14}	$\chi_5 + \chi_{20} + \chi_{26}$	$\chi_{13} + \chi_{15} + \chi_{30}$	$\chi_8 + \chi_{19} + \chi_{23}$	$\chi_3 + \chi_{22} + \chi_{29}$
χ_{17}	$\chi_{21} + \chi_{30}$	$\chi_5 + \chi_{18} + \chi_{24} + \chi_{26}$	$\chi_{22} + \chi_{29}$	$\chi_4 + \chi_{18} + \chi_{23} + \chi_{25}$

5.2 The case $\mathrm{Sp}_6(3)$

Throughout this section, W^- will denote the Weil module of dimension $\frac{(3^3-1)}{2} = 13$ and W^+ the one of dimension $\frac{(3^3+1)}{2} = 14$. To begin with, we present a tableau for $\mathcal{W}_6^2$.

	χ_2	χ_4	χ_3	χ_5
χ_2	$\chi_6+\chi_7$	χ_{11}	$\chi_1+\chi_{10}$	χ_{13}
χ_4		$\chi_8+\chi_9$	χ_{12}	$\chi_1+\chi_{14}$
χ_3			$\chi_6+\chi_8$	χ_{11}
χ_5				$\chi_7+\chi_9$

We use the same notation as GAP, i.e. $\chi_{W^-}=\chi_2$, $\chi_{W^+}=\chi_4$ and χ_3, χ_5 are the duals of, respectively χ_2 and χ_4. Again, following section 1.3, one can note that $\chi_6=\chi_{\mathrm{Alt}^2(W^-)}$ and $\chi_7=\chi_{\mathrm{Sym}^2(W^-)}$, $\chi_8=\chi_{\mathrm{Alt}^2(W^+)}$ and $\chi_9=\chi_{\mathrm{Sym}^2(W^+)}$. Now we present the tableau for $\mathcal{W}_6^3$

	χ_2	χ_4	χ_3	χ_5
$\chi_2\chi_2$	$\chi_3+\chi_{15}+\chi_{19}+2\chi_{26}$	$\chi_5+\chi_{34}+\chi_{39}$	$2\chi_2+\chi_{16}+\chi_{27}+\chi_{36}$	$\chi_{18}+\chi_{30}+\chi_{35}$
$\chi_2\chi_4$		$\chi_3+\chi_{37}+\chi_{44}$	$\chi_4+\chi_{35}+\chi_{38}$	$\chi_2+\chi_{36}+\chi_{45}$
$\chi_2\chi_3$			$2\chi_3+\chi_{15}+\chi_{26}+\chi_{37}$	$\chi_5+\chi_{34}+\chi_{39}$
$\chi_2\chi_5$				$\chi_{19}+\chi_{26}+\chi_{44}$
$\chi_4\chi_4$		$\chi_5+\chi_{17}+\chi_{24}+2\chi_{29}$	$\chi_{20}+\chi_{27}+\chi_{45}$	$2\chi_4+\chi_{25}+\chi_{30}+\chi_{38}$
$\chi_4\chi_3$			$\chi_{17}+\chi_{29}+\chi_{34}$	$\chi_3+\chi_{37}+\chi_{44}$
$\chi_4\chi_5$				$2\chi_5+\chi_{24}+\chi_{29}+\chi_{39}$
$\chi_3\chi_3$			$\chi_2+\chi_{16}+\chi_{20}+2\chi_{27}$	$\chi_4+\chi_{35}+\chi_{38}$
$\chi_3\chi_5$				$\chi_2+\chi_{36}+\chi_{45}$
$\chi_5\chi_5$				$\chi_4+\chi_{18}+\chi_{25}+2\chi_{30}$

To be able to understand $\mathcal{W}_6^3$ in a better way, we take each irreducible character of $\mathcal{W}_6^2$ and we tensor it with elements of $\mathcal{W}_6^1$. We obtain the following tableau

	χ_2	χ_4	χ_3	χ_5
χ_6	$\chi_3+\chi_{15}+\chi_{26}$	χ_{34}	$\chi_2+\chi_{16}+\chi_{27}$	χ_{35}
χ_7	$\chi_{19}+\chi_{26}$	$\chi_5+\chi_{39}$	$\chi_2+\chi_{36}$	$\chi_{18}+\chi_{30}$
χ_8	$\chi_3+\chi_{37}$	$\chi_{17}+\chi_{29}$	$\chi_{20}+\chi_{27}$	$\chi_4+\chi_{38}$
χ_9	χ_{44}	$\chi_5+\chi_{24}+\chi_{29}$	χ_{45}	$\chi_4+\chi_{25}+\chi_{30}$
χ_{10}	$\chi_2+\chi_{16}+\chi_{27}+\chi_{36}$	$\chi_{35}+\chi_{38}$	$\chi_3+\chi_{15}+\chi_{26}+\chi_{37}$	$\chi_{34}+\chi_{39}$
χ_{11}	$\chi_5+\chi_{34}+\chi_{39}$	$\chi_3+\chi_{37}+\chi_{44}$	$\chi_4+\chi_{35}+\chi_{38}$	$\chi_2+\chi_{36}+\chi_{45}$
χ_{12}	$\chi_4+\chi_{35}+\chi_{38}$	$\chi_{20}+\chi_{27}+\chi_{45}$	$\chi_{17}+\chi_{29}+\chi_{34}$	$\chi_3+\chi_{37}+\chi_{44}$
χ_{13}	$\chi_{18}+\chi_{30}+\chi_{35}$	$\chi_2+\chi_{36}+\chi_{45}$	$\chi_5+\chi_{34}+\chi_{39}$	$\chi_{19}+\chi_{26}+\chi_{44}$
χ_{14}	$\chi_{36}+\chi_{45}$	$\chi_4+\chi_{25}+\chi_{30}+\chi_{38}$	$\chi_{37}+\chi_{44}$	$\chi_5+\chi_{24}+\chi_{29}+\chi_{39}$

5.3 The case $\mathrm{Sp}_8(3)$

Throughout this section, W^- will denote the Weil module of dimension $\frac{(3^4-1)}{2}=40$ and W^+ the one of dimension $\frac{(3^4+1)}{2}=41$. To begin with, we present a tableau for $\mathcal{W}_8^2$

	χ_2	χ_4	χ_3	χ_5
χ_2	$\chi_6+\chi_7$	χ_{11}	$\chi_1+\chi_{10}$	χ_{12}
χ_4		$\chi_8+\chi_9$	χ_{13}	$\chi_1+\chi_{14}$
χ_3			$\chi_6+\chi_8$	χ_{11}
χ_5				$\chi_7+\chi_9$

We use the same notation as Magma, i.e. $\chi_{W^-}=\chi_2$, $\chi_{W^+}=\chi_4$ and χ_3, χ_5 are the duals of, respectively χ_2 and χ_4. Again, one can note that $\chi_6=\chi_{\mathrm{Alt}^2(W^-)}$ and $\chi_7=\chi_{\mathrm{Sym}^2(W^-)}$, $\chi_8=\chi_{\mathrm{Alt}^2(W^+)}$ and $\chi_9=\chi_{\mathrm{Sym}^2(W^+)}$. Now we present the tableau for $\mathcal{W}_8^3$

	χ_2	χ_4	χ_3	χ_5
$\chi_2\chi_2$	$\chi_3+\chi_{16}+\chi_{19}+2\chi_{24}$	$\chi_5+\chi_{32}+\chi_{36}$	$2\chi_2+\chi_{15}+\chi_{25}+\chi_{34}$	$\chi_{18}+\chi_{26}+\chi_{31}$
$\chi_2\chi_4$		$\chi_3+\chi_{33}+\chi_{38}$	$\chi_4+\chi_{31}+\chi_{35}$	$\chi_2+\chi_{34}+\chi_{37}$
$\chi_2\chi_3$			$2\chi_3+\chi_{16}+\chi_{24}+\chi_{33}$	$\chi_5+\chi_{32}+\chi_{36}$
$\chi_2\chi_5$				$\chi_{19}+\chi_{24}+\chi_{38}$
$\chi_4\chi_4$		$\chi_5+\chi_{17}+\chi_{22}+2\chi_{27}$	$\chi_{20}+\chi_{25}+\chi_{37}$	$2\chi_4+\chi_{21}+\chi_{26}+\chi_{35}$
$\chi_4\chi_3$			$\chi_{17}+\chi_{27}+\chi_{32}$	$\chi_3+\chi_{33}+\chi_{38}$
$\chi_4\chi_5$				$2\chi_5+\chi_{22}+\chi_{27}+\chi_{36}$
$\chi_3\chi_3$			$\chi_2+\chi_{15}+\chi_{20}+2\chi_{25}$	$\chi_4+\chi_{31}+\chi_{35}$
$\chi_3\chi_5$				$\chi_2+\chi_{34}+\chi_{37}$
$\chi_5\chi_5$				$\chi_4+\chi_{18}+\chi_{21}+2\chi_{26}$

To be able to understand $\mathcal{W}_8^3$ in a better way, we take each irreducible character of $\mathcal{W}_8^2$ and we tensor it with elements of $\mathcal{W}_8^1$. We obtain the following tableau

	χ_2	χ_4	χ_3	χ_5
χ_6	$\chi_3+\chi_{16}+\chi_{24}$	χ_{32}	$\chi_2+\chi_{15}+\chi_{25}$	χ_{31}
χ_7	$\chi_{19}+\chi_{24}$	$\chi_5+\chi_{36}$	$\chi_2+\chi_{34}$	$\chi_{18}+\chi_{26}$
χ_8	$\chi_3+\chi_{33}$	$\chi_{17}+\chi_{27}$	$\chi_{20}+\chi_{25}$	$\chi_4+\chi_{35}$
χ_9	χ_{38}	$\chi_5+\chi_{22}+\chi_{27}$	χ_{37}	$\chi_4+\chi_{21}+\chi_{26}$
χ_{10}	$\chi_2+\chi_{15}+\chi_{25}+\chi_{34}$	$\chi_{31}+\chi_{35}$	$\chi_3+\chi_{16}+\chi_{24}+\chi_{33}$	$\chi_{32}+\chi_{36}$
χ_{11}	$\chi_5+\chi_{32}+\chi_{36}$	$\chi_3+\chi_{33}+\chi_{38}$	$\chi_4+\chi_{31}+\chi_{35}$	$\chi_2+\chi_{34}+\chi_{37}$
χ_{12}	$\chi_{18}+\chi_{26}+\chi_{31}$	$\chi_2+\chi_{34}+\chi_{37}$	$\chi_5+\chi_{32}+\chi_{36}$	$\chi_{19}+\chi_{24}+\chi_{38}$
χ_{13}	$\chi_4+\chi_{31}+\chi_{35}$	$\chi_{20}+\chi_{25}+\chi_{37}$	$\chi_{17}+\chi_{27}+\chi_{32}$	$\chi_3+\chi_{33}+\chi_{38}$
χ_{14}	$\chi_{34}+\chi_{37}$	$\chi_4+\chi_{21}+\chi_{26}+\chi_{35}$	$\chi_{33}+\chi_{38}$	$\chi_5+\chi_{22}+\chi_{27}+\chi_{36}$

5.4 The general case

Using the results of the calculations made for the groups $\mathrm{Sp}_4(3)$, $\mathrm{Sp}_6(3)$ and $\mathrm{Sp}_8(3)$ we want to deduce some general results for $\mathrm{Sp}_{2n}(3)$, especially for

$\mathcal{W}_{2n}^3$. The understanding of $\mathcal{W}_{2n}^2$ was treated in [12]. We begin by recalling this in proposition 5.3.

Throughout this section, W_{2n}^- will denote the Weil module of dimension $\frac{(3^n-1)}{2}$ and W_{2n}^+ the one of dimension $\frac{(3^n+1)}{2}$. We start with a general lemma.

Lemma 5.1. *Let χ be an irreducible character for a finite group G. Then $\chi\chi^*$ contains the trivial representation with multiplicity one.*

Proof. Let χ_1 be the trivial representation for G. Then

$$\langle \chi_1, \chi\chi^* \rangle = \langle \chi_1\chi, \chi \rangle = \langle \chi, \chi \rangle = 1.$$

□

Definition 5.2. Let V be an irreducible not self-dual $\mathbb{C}[G]$-module for a finite group G. We note $\mathrm{Adj}(V)$ the submodule of codimension 1 in $V \otimes V^*$.

Proposition 5.3.

(i) *The only irreducible modules in $\mathcal{W}_{2n}^2$ are*

$$W_{2n}^+ \otimes W_{2n}^{-^*}, \quad W_{2n}^{+^*} \otimes W_{2n}^-, \quad W_{2n}^+ \otimes W_{2n}^-, \quad W_{2n}^{+^*} \otimes W_{2n}^{-^*}.$$

Moreover, $W_{2n}^+ \otimes W_{2n}^-$ and $W_{2n}^{+^*} \otimes W_{2n}^{-^*}$ are the only isomorphic ones.

(ii) *Let $W \in \mathcal{W}_{2n}^1$, then the modules $\mathrm{Sym}^2(W)$, $\mathrm{Alt}^2(W)$ and $\mathrm{Adj}(W)$ are irreducible. Moreover, $\mathrm{Sym}^2(W_{2n}^+)$ and $\mathrm{Alt}^2(W_{2n}^-)$ are self-dual.*

Proof. See proposition 5.5 page 17 in [12]. □

Now we start our investigation of the 3-fold tensor products of Weil modules for $\mathrm{Sp}_{2n}(3)$. First, we improve proposition 5.3.

Proposition 5.4. *One has $\chi_{\mathrm{Sym}^2(W_{2n}^-)} = \chi_{\mathrm{Alt}^2(W_{2n}^{+^*})}$ and the same equality holds for the duals.*

Proof. First, note that

$$\left\langle \chi_{\mathrm{Sym}^2(W_{2n}^-)}, \chi_{\mathrm{Alt}^2(W_{2n}^{+^*})} \right\rangle = \left\langle \chi_{\mathrm{Sym}^2(W_{2n}^-)} + \chi_{\mathrm{Alt}^2(W_{2n}^-)}, \chi_{\mathrm{Alt}^2(W_{2n}^{+^*})} + \chi_{\mathrm{Sym}^2(W_{2n}^{+^*})} \right\rangle,$$

because all these characters are irreducible, by proposition 5.3, and an argument of dimension. By definition one has

$$\left\langle \chi_{\mathrm{Sym}^2(W_{2n}^-)} + \chi_{\mathrm{Alt}^2(W_{2n}^-)}, \chi_{\mathrm{Alt}^2(W_{2n}^{+^*})} + \chi_{\mathrm{Sym}^2(W_{2n}^{+^*})} \right\rangle = \left\langle \chi^2_{W_{2n}^-}, \chi^2_{W_{2n}^{+^*}} \right\rangle.$$

So we obtain

$$\left\langle \chi_{\mathrm{Sym}^2(W_{2n}^-)}, \chi_{\mathrm{Alt}^2(W_{2n}^{+*})} \right\rangle = \left\langle \chi^2_{W_{2n}^-}, \chi^2_{W_{2n}^{+*}} \right\rangle = \left\langle \chi_{W_{2n}^-}\chi_{W_{2n}^+}, \chi_{W_{2n}^{+*}}\chi_{W_{2n}^{-*}} \right\rangle = 1,$$

where the last equality holds because of the first part of proposition 5.3. □

In order to prove our main result, theorem 5.7, which describes when Weil modules occurs in the decomposition of a 3-fold tensor products of Weil modules we obtain the following corollary and lemma.

Corollary 5.5. *One has*

$$\left\langle \chi_{W_{2n}^+}\chi_{\mathrm{Sym}^2(W_{2n}^-)}, \chi_{W_{2n}^{+*}} \right\rangle = 1 \quad \textit{and} \quad \left\langle \chi_{W_{2n}^-}\chi_{\mathrm{Alt}^2(W_{2n}^+)}, \chi_{W_{2n}^{-*}} \right\rangle = 1.$$

Proof. It's just a calculation :

$$\left\langle \chi_{W_{2n}^+}\chi_{\mathrm{Sym}^2(W_{2n}^-)}, \chi_{W_{2n}^{+*}} \right\rangle = \left\langle \chi_{W_{2n}^+}\chi_{\mathrm{Alt}^2(W_{2n}^{+*})}, \chi_{W_{2n}^{+*}} \right\rangle = \left\langle \chi_{\mathrm{Alt}^2(W_{2n}^{+*})}, \chi^2_{W_{2n}^{+*}} \right\rangle = 1.$$

In the same manner, we have

$$\left\langle \chi_{W_{2n}^-}\chi_{\mathrm{Alt}^2(W_{2n}^+)}, \chi_{W_{2n}^{-*}} \right\rangle = \left\langle \chi_{W_{2n}^-}\chi_{\mathrm{Sym}^2(W_{2n}^{-*})}, \chi_{W_{2n}^{-*}} \right\rangle = \left\langle \chi_{\mathrm{Sym}^2(W_{2n}^{-*})}, \chi^2_{W_{2n}^{-*}} \right\rangle = 1.$$

□

Lemma 5.6. *One has*

$$\left\langle \chi_{W_{2n}^+}\chi_{\mathrm{Sym}^2(W_{2n}^+)}, \chi_{W_{2n}^{+*}} \right\rangle = 1 \quad \textit{and} \quad \left\langle \chi_{W_{2n}^-}\chi_{\mathrm{Alt}^2(W_{2n}^-)}, \chi_{W_{2n}^{-*}} \right\rangle = 1.$$

Proof. Since $\mathrm{Sym}^2(W_{2n}^+)$ is self-dual, we have

$$\left\langle \chi_{W_{2n}^+}\chi_{\mathrm{Sym}^2(W_{2n}^+)}, \chi_{W_{2n}^{+*}} \right\rangle = \left\langle \chi_{\mathrm{Sym}^2(W_{2n}^+)}, \chi^2_{W_{2n}^{+*}} \right\rangle = \left\langle \chi_{\mathrm{Sym}^2(W_{2n}^{+*})}, \chi^2_{W_{2n}^{+*}} \right\rangle = 1.$$

Since $\mathrm{Alt}^2(W_{2n}^-)$ is self-dual, we have

$$\left\langle \chi_{W_{2n}^-}\chi_{\mathrm{Alt}^2(W_{2n}^-)}, \chi_{W_{2n}^{-*}} \right\rangle = \left\langle \chi_{\mathrm{Alt}^2(W_{2n}^-)}, \chi^2_{W_{2n}^{-*}} \right\rangle = \left\langle \chi_{\mathrm{Alt}^2(W_{2n}^{-*})}, \chi^2_{W_{2n}^{-*}} \right\rangle = 1.$$

□

Theorem 5.7. *Let $U \in \mathcal{W}^3_{2n}$, then there exists $W \in \mathcal{W}^1_{2n}$ such that*

$$\langle \chi_U, \chi_W \rangle \geq 1,$$

except for $U = W^{\pm}_{2n} \otimes W^{\pm}_{2n} \otimes W^{\mp^*}_{2n}$ or $U = W^{\pm^*}_{2n} \otimes W^{\pm^*}_{2n} \otimes W^{\mp}_{2n}$. In other words, when we take the tensor product of three Weil modules, at least one Weil module occurs in the decomposition, except for the cases cited.

Proof. First remark that if W is in the decomposition of U then W^* appears in the decomposition of U^*. Then we can restrict to the following cases

1. $U = V_1 \otimes V_1 \otimes V_1$,
2. $U = V_1 \otimes V_1 \otimes V_2$,
3. $U = V_1 \otimes V_1 \otimes V_1^*$,
4. $U = V_1 \otimes V_1^* \otimes V_2$,
5. $U = V_1 \otimes V_1 \otimes V_2^*$,

where V_1, V_2 are non-isomorphic Weil modules. So we are going to give W for the first four cases and show there is no W in the decomposition of the last one.

1. In this case we claim that $W = V_1^*$. Indeed, we have $V_1 \otimes V_1 \otimes V_1 = \big(\mathrm{Sym}^2(V_1) \oplus \mathrm{Alt}^2(V_1)\big) \otimes V_1$ and so by lemma 5.6 we can conclude that V_1^* occurs, at least, in the decomposition of either $\mathrm{Sym}^2(V_1) \otimes V_1$ or $\mathrm{Alt}^2(V_1) \otimes V_1$.
2. Using a similar method one has $V_1 \otimes V_1 \otimes V_2 = \big(\mathrm{Sym}^2(V_1) \oplus \mathrm{Alt}^2(V_1)\big) \otimes V_2$ and so by corollary 5.5 we conclude that V_2^* appears in the decomposition of $V_1 \otimes V_1 \otimes V_2$.
3. Obviously in this case we can take $W = V_1$ because $V_1 \otimes V_1 \otimes V_1^* = V_1 \otimes \big(1 \oplus \mathrm{Adj}(V_1)\big) = V_1 \oplus V_1 \otimes \mathrm{Adj}(V_1)$. Actually we can be more precise. Using the decomposition $V_1 \otimes V_1 \otimes V_1^* = \big(\mathrm{Sym}^2(V_1) \oplus \mathrm{Alt}^2(V_1)\big) \otimes V_1^*$ we see that V_1 occurs twice. Indeed, we have $\big\langle \chi_{\mathrm{Sym}^2(V_1)} \chi_{V_1^*}, \chi_{V_1} \big\rangle = \big\langle \chi_{\mathrm{Sym}^2(V_1)}, \chi^2_{V_1} \big\rangle = 1$ and $\big\langle \chi_{\mathrm{Alt}^2(V_1)} \chi_{V_1^*}, \chi_{V_1} \big\rangle = \big\langle \chi_{\mathrm{Alt}^2(V_1)}, \chi^2_{V_1} \big\rangle = 1$.
4. For the same reason as the last point we can take $W = V_2$.

5. Like the first cases, write $V_1 \otimes V_1 \otimes V_2^* = \big(\mathrm{Sym}^2(V_1) \oplus \mathrm{Alt}^2(V_1)\big) \otimes V_2^*$ and look at $\big\langle \chi_{\mathrm{Sym}^2(V_1)} \chi_{V_2^*}, \chi_W \big\rangle$ and $\big\langle \chi_{\mathrm{Alt}^2(V_1)} \chi_{V_2^*}, \chi_W \big\rangle$. We treat the different possibilities for W. First we note that W can't be V_1 or V_1^* because in theses cases $\big\langle \chi_{\mathrm{Sym}^2(V_1)}, \chi_{V_2} \chi_W \big\rangle = 0$ since $V_2 \otimes W$ is irreducible as well as $\mathrm{Sym}^2(V_1)$, and the same argument works for the second scalar product. Afterwards, W can't be V_2^* because $V_2 \otimes V_2^* = 1 \oplus \mathrm{Adj}(V_2)$ and looking at the dimension we see that $\mathrm{Adj}(V_2)$ is not isomorphic to $\mathrm{Sym}^2(V_1)$ or $\mathrm{Alt}^2(V_1)$. So the last case to treat is to know if $W = V_2$ is in the decomposition of $V_1 \otimes V_1 \otimes V_2^*$. But since we have that
$$\big\langle \chi_{V_1 \otimes V_1 \otimes V_2^*}, \chi_{V_2} \big\rangle = \big\langle \chi_{V_1 \otimes V_2^*}, \chi_{V_1^* \otimes V_2} \big\rangle = 0,$$
where the last equality follows by proposition 5.3, we can see that V_2 is not in the decomposition of $V_1 \otimes V_1 \otimes V_2^*$ which ends this case.

□

Remark 5.8. We can actually prove without difficulty that in each case, the only Weil module which occurs in the decomposition is the one mentioned with multiplicity mentioned.

Corollary 5.9. *Let W be an element of $\mathcal{W}_{2n}^1$, then W, respectively W^*, occurs in the decomposition of* $\mathrm{Adj}(W) \otimes W$, *respectively* $\mathrm{Adj}(W) \otimes W^*$.

Proof. It follows from the discussion of the point 3 in the previous proof. □

The following propositions partially describe the decomposition of certain terms of 3-fold tensor products of Weil modules.

Proposition 5.10. *The modules* $\mathrm{Sym}^3(W_{2n}^+)$ *and* $\mathrm{Alt}^3(W_{2n}^-)$ *are reducible.*

Proof. We know from the proof of theorem 5.7 that $W_{2n}^{+^*}$ occurs in the decomposition of $W_{2n}^+ \otimes W_{2n}^+ \otimes W_{2n}^+$ which is equal to $(\mathrm{Sym}^2 W_{2n}^+ \oplus \mathrm{Alt}^2 W_{2n}^+) \otimes W_{2n}^+$. But recall that
$$\mathrm{Sym}^2 W_{2n}^+ \otimes W_{2n}^+ = \mathrm{Sym}^3 W_{2n}^+ \oplus \mathbb{S}_{(2,1)} W_{2n}^+$$
and
$$\mathrm{Alt}^2 W_{2n}^+ \otimes W_{2n}^+ = \mathrm{Alt}^3 W_{2n}^+ \oplus \mathbb{S}_{(2,1)} W_{2n}^+.$$
Therefore if $W_{2n}^{+^*}$ occurs in the decomposition of $\mathrm{Sym}^2 W_{2n}^+ \otimes W_{2n}^+$ but not in the one of $\mathrm{Alt}^2 W_{2n}^+ \otimes W_{2n}^+$, we can conclude, looking at the dimension, that

${W_{2n}^{+}}^{*}$ occurs in the decomposition of $\mathrm{Sym}^3(W_{2n}^{+})$. By proposition 5.3 we have that $\mathrm{Sym}^2 W_{2n}^{+}$ is self-dual and $\mathrm{Alt}^2 W_{2n}^{+}$ is not. So one can conclude with the following calculations

$$\left\langle \chi_{\mathrm{Sym}^2 W_{2n}^{+} \otimes W_{2n}^{+}}, \chi_{{W_{2n}^{+}}^{*}} \right\rangle = \left\langle \chi_{\mathrm{Sym}^2 W_{2n}^{+}}, \chi_{\mathrm{Sym}^2 {W_{2n}^{+}}^{*}} \right\rangle = 1$$

and

$$\left\langle \chi_{\mathrm{Alt}^2 W_{2n}^{+} \otimes W_{2n}^{+}}, \chi_{{W_{2n}^{+}}^{*}} \right\rangle = \left\langle \chi_{\mathrm{Alt}^2 W_{2n}^{+}}, \chi_{\mathrm{Alt}^2 {W_{2n}^{+}}^{*}} \right\rangle = 0.$$

The same argument shows that $\mathrm{Alt}^3(W_{2n}^{-})$ is reducible. □

Proposition 5.11.

(*i*) *The module* $\mathbb{S}_{(2,1)}(W_{2n}^{+})$ *occurs in the decomposition of* $W_{2n}^{+} \otimes {W_{2n}^{+}}^{*} \otimes {W_{2n}^{+}}^{*}$ *and of* $W_{2n}^{+} \otimes {W_{2n}^{-}}^{*} \otimes {W_{2n}^{-}}^{*}$.

(*ii*) *The module* $\mathbb{S}_{(2,1)}(W_{2n}^{-})$ *occurs in the decomposition of* $W_{2n}^{-} \otimes {W_{2n}^{-}}^{*} \otimes {W_{2n}^{-}}^{*}$ *and of* $W_{2n}^{-} \otimes {W_{2n}^{+}}^{*} \otimes {W_{2n}^{+}}^{*}$.

(*iii*) *The module* $\mathrm{Alt}^3(W_{2n}^{+})$ *occurs in the decomposition of* $W_{2n}^{+} \otimes {W_{2n}^{-}}^{*} \otimes {W_{2n}^{-}}^{*}$.

(*iv*) *The module* $\mathrm{Sym}^3(W_{2n}^{-})$ *occurs in the decomposition of* $W_{2n}^{-} \otimes {W_{2n}^{+}}^{*} \otimes {W_{2n}^{+}}^{*}$.

Proof.

(i) We trivially have that $\mathbb{S}_{(2,1)}(W_{2n}^{+})$ is in the decomposition of $\mathrm{Sym}^2(W_{2n}^{+}) \otimes (W_{2n}^{+})$, which we note $\mathbb{S}_{(2,1)}(W_{2n}^{+}) \leq \mathrm{Sym}^2(W_{2n}^{+}) \otimes (W_{2n}^{+})$. Therefore, we have $\mathbb{S}_{(2,1)}(W_{2n}^{+})^* \leq \mathrm{Sym}^2(W_{2n}^{+})^* \otimes (W_{2n}^{+})^*$. But $\mathrm{Sym}^2(W_{2n}^{+})$ is self-dual and so

$$\mathbb{S}_{(2,1)}(W_{2n}^{+})^* \leq \mathrm{Sym}^2(W_{2n}^{+}) \otimes (W_{2n}^{+})^* \leq W_{2n}^{+} \otimes W_{2n}^{+} \otimes {W_{2n}^{+}}^{*}.$$

By taking the dual again one has

$$\mathbb{S}_{(2,1)}(W_{2n}^{+}) \leq W_{2n}^{+} \otimes {W_{2n}^{+}}^{*} \otimes {W_{2n}^{+}}^{*}.$$

On the other hand, $\mathbb{S}_{(2,1)}(W_{2n}^{+})$ is in the decomposition of $\mathrm{Alt}^2(W_{2n}^{+}) \otimes (W_{2n}^{+})$. But $\mathrm{Alt}^2(W_{2n}^{+})$ is isomorphic to $\mathrm{Sym}^2({W_{2n}^{-}}^{*})$, by proposition 5.4. Therefore, one has

$$\mathbb{S}_{(2,1)}(W_{2n}^{+}) \leq \mathrm{Sym}^2({W_{2n}^{-}}^{*}) \otimes W_{2n}^{+} \leq {W_{2n}^{-}}^{*} \otimes {W_{2n}^{-}}^{*} \otimes W_{2n}^{+}.$$

(ii) Similarly, replacing W_{2n}^{+} by W_{2n}^{-} and using that $\mathrm{Alt}^2(W_{2n}^{-})$ is self-dual, one has

$$\mathbb{S}_{(2,1)}(W_{2n}^{-}) \leq \mathrm{Alt}^2(W_{2n}^{-})^* \otimes W_{2n}^{-} \leq W_{2n}^{-*} \otimes W_{2n}^{-*} \otimes W_{2n}^{-}$$

and

$$\mathbb{S}_{(2,1)}(W_{2n}^{-}) \leq \mathrm{Sym}^2(W_{2n}^{-}) \otimes W_{2n}^{-} \leq W_{2n}^{+*} \otimes W_{2n}^{+*} \otimes W_{2n}^{-}.$$

(iii) Using that $\mathrm{Alt}^2(W_{2n}^{+})$ is isomorphic to $\mathrm{Sym}^2(W_{2n}^{-*})$ one has

$$\mathrm{Alt}^3(W_{2n}^{+}) \leq \mathrm{Alt}^2(W_{2n}^{+}) \otimes W_{2n}^{+} \leq W_{2n}^{-*} \otimes W_{2n}^{-*} \otimes W_{2n}^{+}.$$

(iv) In the same manner, one has

$$\mathrm{Sym}^3(W_{2n}^{-}) \leq \mathrm{Sym}^2(W_{2n}^{-}) \otimes W_{2n}^{-} \leq W_{2n}^{+*} \otimes W_{2n}^{+*} \otimes W_{2n}^{-}.$$

□

Remark 5.12. If we show that $\mathbb{S}_{(2,1)}(W_{2n}^{+})$, $\mathrm{Alt}^3(W_{2n}^{+})$, $\mathbb{S}_{(2,1)}(W_{2n}^{-})$ and $\mathrm{Sym}^3(W_{2n}^{-})$ are irreducible, then we can actually improve proposition 5.11 saying that they occur only one time in the decomposition of the above modules.

5.5 Open Questions

This section details some results we tried to prove but without success. The following statements are motivated by the fact that they are true for the cases $\mathrm{Sp}_4(3)$, $\mathrm{Sp}_6(3)$ and $\mathrm{Sp}_8(3)$. First, we think that the following modules are irreducible :

$$\begin{array}{lll} \mathrm{Sym}^2(W_{2n}^{+}) \otimes W_{2n}^{-} & \mathrm{Alt}^3(W_{2n}^{+}) & \mathbb{S}_{(2,1)}(W_{2n}^{+}) \\ \mathrm{Alt}^2(W_{2n}^{-}) \otimes W_{2n}^{+} & \mathrm{Sym}^3(W_{2n}^{-}) & \mathbb{S}_{(2,1)}(W_{2n}^{-}). \end{array}$$

Then, we presume that the norm squared of the characters of the following modules is equal to 2 :

$$\begin{array}{ll} \mathrm{Sym}^3(W_{2n}^{+}) & \mathrm{Alt}^2(W_{2n}^{+}) \otimes W_{2n}^{-} \\ \mathrm{Alt}^3(W_{2n}^{-}) & \mathrm{Sym}^2(W_{2n}^{-}) \otimes W_{2n}^{+}. \end{array}$$

We proved in the last section that they are all reducible and we gave an irreducible component of their decomposition. Our hypothesis is that there is just one other irreducible component in each of their decompositions.

Remark 5.13. If these hypotheses are right, then we are able to understand $\mathcal{W}^3_{2n}$.

Now one can notice that $\mathrm{Sym}^2(W^+_{2n}) \otimes W^-_{2n}$ is irreducible if and only if $\mathrm{Adj}(W^-_{2n})$ does not occur in the decomposition of $\mathrm{Sym}^2(W^+_{2n}) \otimes \mathrm{Sym}^2(W^+_{2n})$. Indeed, we have

$$\begin{aligned}
\left\langle \chi_{\mathrm{Sym}^2(W^+_{2n})\otimes W^-_{2n}}, \chi_{\mathrm{Sym}^2(W^+_{2n})\otimes W^-_{2n}} \right\rangle &= \left\langle \chi_{\mathrm{Sym}^2(W^+_{2n})\otimes \mathrm{Sym}^2(W^+_{2n})}, \chi_{{W^-_{2n}}^*\otimes W^-_{2n}} \right\rangle \\
&= \left\langle \chi_{\mathrm{Sym}^2(W^+_{2n})\otimes \mathrm{Sym}^2(W^+_{2n})}, \chi_{\mathrm{Adj}(W^-_{2n})} + \chi_1 \right\rangle \\
&= 1 + \left\langle \chi_{\mathrm{Sym}^2(W^+_{2n})\otimes \mathrm{Sym}^2(W^+_{2n})}, \chi_{\mathrm{Adj}(W^-_{2n})} \right\rangle,
\end{aligned}$$

where χ_1 denotes the trivial representation and we used that $\mathrm{Sym}^2(W^+_{2n})$ is self-dual. Similarly, one can see that $\mathrm{Alt}^2(W^-_{2n})\otimes W^+_{2n}$ is irreducible if and only if $\mathrm{Adj}(W^+_{2n})$ does not occur in the decomposition of $\mathrm{Alt}^2(W^-_{2n}) \otimes \mathrm{Alt}^2(W^-_{2n})$.

Besides, using the same argument, the scalar product of $\mathrm{Sym}^2(W^-_{2n})\otimes W^+_{2n}$ with itself is equal to 2 if and only if one of the following equivalent equalities holds :

- $\left\langle \chi_{\mathrm{Sym}^2(W^-_{2n})\otimes \mathrm{Sym}^2(W^-_{2n})^*}, \chi_{\mathrm{Adj}(W^+_{2n})} \right\rangle = 1,$
- $\left\langle \chi_{\mathrm{Sym}^2(W^-_{2n})\otimes \mathrm{Alt}^2(W^+_{2n})}, \chi_{\mathrm{Adj}(W^+_{2n})} \right\rangle = 1,$
- $\left\langle \chi_{\mathrm{Alt}^2(W^+_{2n})\otimes \mathrm{Alt}^2(W^+_{2n})^*}, \chi_{\mathrm{Adj}(W^+_{2n})} \right\rangle = 1.$

These equations are equivalent because we have $\chi_{\mathrm{Sym}^2(W^-_{2n})} = \chi_{\mathrm{Alt}^2({W^+_{2n}}^*)}$ by proposition 5.4. Likewise, the scalar product of $\mathrm{Alt}^2(W^+_{2n}) \otimes W^-_{2n}$ with itself is equal to 2 if and only if one of the following equivalent equalities holds :

- $\left\langle \chi_{\mathrm{Alt}^2(W^+_{2n})\otimes \mathrm{Alt}^2(W^+_{2n})^*}, \chi_{\mathrm{Adj}(W^-_{2n})} \right\rangle = 1,$
- $\left\langle \chi_{\mathrm{Sym}^2(W^-_{2n})\otimes \mathrm{Alt}^2(W^+_{2n})}, \chi_{\mathrm{Adj}(W^-_{2n})} \right\rangle = 1,$
- $\left\langle \chi_{\mathrm{Sym}^2(W^-_{2n})\otimes \mathrm{Sym}^2(W^-_{2n})^*}, \chi_{\mathrm{Adj}(W^-_{2n})} \right\rangle = 1.$

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