A New Method to Explore Conformal Field Theories in Any Dimension

THÈSE N^{O} 5116 (2011)

PRÉSENTÉE LE 26 AOÛT 2011

À LA FACULTÉ SCIENCES DE BASE
LABORATOIRE DE PHYSIQUE THÉORIQUE DES PARTICULES
PROGRAMME DOCTORAL EN PHYSIQUE

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

PAR

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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Suisse
2011

In loving memory of my father

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Acknowledgements

I thank my supervisor Prof. Riccardo Rattazzi, for introducing me to the study of conformal field theories and for trying to convert me into a phenomenologist (with only partial success!). His constant support and encouragement enabled me to pursue this line of study which finally culminated in this thesis.

Moreover, I am extremely grateful to Vyacheslav Rychkov for guiding me in the research activity and for teaching me the use of numerical algorithms.

Special thanks goes to my supervisors for proofreading this thesis, saving me from making several erroneous and poorly worded statements.

At various times through the course of my PhD I have also enjoyed useful conversations about aspects of this work with many people, including Erik Tonni, Alberto Zaffaroni, David Poland, David Simmons-Duffin, Joao Penedones and Leonardo Rastelli.

In addition I thank all colleagues from EPFL for their friendship, especially Paolo Torrielli, Duccio Pappadopulo, Andrea De Simone, Carlo Maffi and Leonardo Brizi, who have provided me with count-less hours of conversation both on and off topic which have lead me to where I am now.

Finally and more importantly I express my whole hearted gratitude to my parents for their support, for always being there for me and for instilling in me an insatiable thirst for knowledge; and to my wife, Maria, for her love and patience and and for always being next to me. It is with great pleasure that I dedicate this thesis to them.

Abstract

The thesis represents an investigation into Conformal Field Theories (CFT's) in arbitrary dimensions. We propose an innovative method to extract informations about CFT's in a quantitative way. Studying the crossing symmetry of the four point function of scalar operators we derive consistency constraints on the CFT structure, in the form of functional sum rules. The technique we introduce allows to address the feasibility of the sum rule and translate it into restrictions on the CFT spectrum and interactions. Our analysis only assumes unitarity of the CFT, crossing symmetry of the four point function and existence of an Operator Product Expansion (OPE) for scalars. We demonstrate that a CFT satisfying the above hypothesis and containing a scalar operator is not compatible with arbitrary spectra of the operators nor with arbitrary large OPE coefficients. More specifically we prove two main results. First, the spectrum of the CFT must contain a second scalar operator with dimension smaller than a given value. Second, the value of the thee point function of two scalars with equal dimension and a third arbitrary operator is bounded from above. As an application of the fist statement we present the bound on the smallest dimension operator entering the OPE of a real scalar with itself. We perform the analysis in two and four dimensions. The comparison of the two dimensional case with exactly solvable models shows a saturation of the bound. We repeat for CFT's with global symmetries and superconformal field theories in four dimensions. As a demonstration of the second result we provide a lower bound on the central charge for CFT in two and four dimensions without global symmetries and for superconformal field theories. We also discuss in the potentialities of the method and possible future research lines. Finally, we discuss possible implications for model building beyond the Standard Model of particle physics.

Keywords: Conformal Field Theory, crossing symmetry, OPE, operator dimensions, central charge.

Riassunto

Il presente elaborato rappresenta uno studio sulle Teorie di Campo Conformi (CFT) in un numero arbitrario di dimensioni ed introduce un metodo innovativo per estrarne quantitativamente informazioni. Studiando la simmetria di scambio della funzione a quattro punti di uno scalare si ottengono dei vincoli di consistenza sulla teoria. Tali vincoli assumono la forma di regole di somma funzionali. La tecnica introdotta in questo lavoro permette di tradurre la richiesta di esistenza di soluzioni per tale regola di somma in informazioni sullo spettro e sulle interazioni della CFT. Il metodo si basa solamente sulle seguenti assunzioni: unitarietà della teoria, esistenza di una espansione del prodotto di operatori (OPE) scalari e simmetria di scambio della funzione a quattro punti. Data una CFT che soddisfa queste condizioni e contiene un operatore scalare di dimensione assegnata si dimostrano due fondamentali risultati. In primo luogo lo spettro della teoria deve necessariamente contenere un secondo operatore scalare con dimensione minore di un certo valore. Inoltre, per ogni correlatore contenente due scalari di uguale dimensione assegnata e un terzo operatore arbitrario esiste un valore massimo che tale funzione a tre punti può assumere. Come applicazione del primo risultato viene presentato un limite superiore sull'operatore di dimensione minore presente nell' OPE di due campi scalari reali in due e quattro dimensioni. Il confronto con modelli esistenti per il caso due dimensionale mostra come questi saturino completamente il limite. L'analisi è ripetuta per una CFT con simmetrie globali e per una teoria di campo superconforme. Come esemplificazione del secondo risultato viene ricavato un limite inferiore sulla carica centrale in una CFT in due e in quattro dimensioni contenente un campo reale scalare e in una teoria superconforme in quattro dimensioni contente un supercampo chirale. Infine sono discusse possibili implicazioni per la costruzioni di scenari oltre il Modello Standard di fisica delle particelle.

Parole chiave: Teoria di campo conforme, simmetria di scambio, OPE, dimensioni di operatori, carica centrale.

Chapter 1

Introduction

1.1 Conformal Invariance in Modern Physics

In the last century symmetries have undeniably played a fundamental role in theoretical physics, driving the discovery and the understanding of many phenomena. The presence of a symmetry imposes indeed restrictions on the dynamic of a system, thus facilitating the search of solutions. Familiar examples are represented by translations or rotations, however not all the symmetries are equally intuitive; nevertheless their presence is always signaled by a degeneracy of solutions interconnected by a symmetry transformation. This is exemplified by the spectrum of the Hydrogen atom, where an apparently unjustified degeneracy of energy levels signals the existence of a symmetry beyond the rotational one, that is the symmetry responsible for Lenz's vector conservation.

In the real world only few symmetries are exact, such as rotations and Lorentz transformations, however in a great variety of cases the breaking of a symmetry is controlled by small parameters. In those circumstances we expect to be able to organize the description of the system as a perturbative series in the variable parametrizing the symmetry breaking. This naturally leads to the concept of approximate symmetry. Again the Hydrogen atom provides an illustrative example: the Lenz vector conservation is broken by small effects. This results in a perturbative correction of the would-be degeneracy of states in powers of the quantum electromagnetic coupling α, the ratio of the electron mass over the proton mass, the weak coupling, etc. Remarkably this series perfectly matches the observed fine and hyperfine structure of levels.

In this work we concentrate on the consequences of the invariance under scale transformation.

The propagation of light provides a simple example of the relevance of this symmetry. Let us consider a planar wave with a given wavelength λ hitting two slits separated by some distance. Collecting the waves diffracted by the slits we can observe the characteristic interference fringes. If we now consider a second experiment where the light wavelength is $\lambda/2$ and all the distances are rescaled by the same amount we can observe a pattern of interference fringes again rescaled by the same amount. That is, the two systems are connected by a scale transformation. One could repeat this several times producing a family of systems connected by the symmetry. This shows the scale invariance of optics. This invariance is again only approximate. The reason is that the diffraction experiment produces the fringes as long as the length of the wave is much larger than the interspaces of the atoms forming the slits. When this is the case the wave cannot resolve the microscopic structure of matter and we can regard the system as scale invariant with good approximation. On the other hand, when the wavelength is too small the interaction with atoms cannot be neglected.

Albeit very simple, the above example shows a general property of scale symmetry: a necessary requirement for scale invariance is represented by the absence of characteristic scales. More rigorously, dimensionful parameters can exist, however as long as we consider the theory at length scales much lower or higher, their influence is expected to be negligible. Hence, it makes sense to consider asymptotic regimes where the theory is approximately scale invariant.

It is a fact that the presence of scale invariance often leads to conformal invariance: the connection between the two is indeed very strong, the former implying the latter in the majority of the cases (see [1, 2] for recent discussions). We should stress however that a a rigorous proof is only known for unitary two-dimensional field theories ([3]).

Given the above discussion, we expect Conformal Field Theories (CFT's) to have a relevant role in one of the following regimes: i) in the far Ultraviolet (UV), at energies much larger than all the scales of the theory , ii) in the Infrared (IR), at energies much smaller, iii) at intermediate energies between two widely separated scales.

In the last decades we have witnessed an alternate interest toward scale invariant theories. In the '70s a pragmatic stimulus to the investigation of conformal invariant systems came from the study of the renormalization group (RG) flow. In a generic quantum system interactions vary with energy: according to their dependence on the scale at which the process occurs interactions can be denoted irrelevant (relevant), if their strength increases (decreases) while the energy increases, or marginal, if their strength remains constant. Thus at sufficiently small energies we naturally expect only the relevant and marginal interactions to matter[1]. As a consequence

[1] Of course this statement must be quantified taking into account the degree of precision required.

the IR description of a system is broadly specified by its IR degrees of freedom and by a finite number of parameters [4]. Interestingly, choosing properly the RG-flow initial conditions, we can drive the IR description of the theory to a fixed point. This corresponds to setting to zero all relevant operators of the theory.
In condensed matter systems this can be done operatively tuning macroscopic parameters: for instance the IR description of a ferromagnet at the Curie temperature is a *fixed point* of the RG flow. On the contrary, model building in particle physics is usually oriented to eliminate the need of parameters fine tuning. In this case scale invariance in the IR can be enforced with the use of additional symmetries which forbid the existence of relevant operators.

The presence of CFT's is ubiquitous and clearly not limited to the descriptions of infrared (IR) fixed points. As mentioned, CFT's can play an important role even at intermediate scales, provided that all the other scales of the theory are far apart. A simple realization of this scenario (see next section for an application to particle physics) is given by spin systems with a finite correlation length at the critical point[2]. In this case the system has two intrinsic scales: the spin lattice spacing, playing the role of UV scale, and the correlation length, playing the role of IR scale. As long as the two length scales are widely separated the theory at intermediate scales can admit an approximate scale invariant description.

Finally, let us mention that UV fixed points have been proposed also for the description of theories at high scales. The idea that CFT's could play an important role in understanding the high energy dynamics of physical processes goes back to the early $'70$s with the pioneering works by Gross and Wess ([5]) and successively by Migdal ([6]) and others. At that time the most demanding issue was trying to formulate a valid theory of strong interactions. Despite some initial successes it became soon clear that only the free CFT was able to describe the asymptotic behavior of deep inelastic scattering ([7]).
The intriguing aspect of a UV fixed point is the absence of running coupling and divergences of sort. The resulting theory is therefore complete and finite. This ideas , proposed also in the context of gravity ([8]), find their perfect realization in String theory. From the world-sheet point of view, the string is a two dimensional CFT and the embedding fields X^μ represent CFT operators. Although not acting directly on the target space, conformal invariance is responsible for the finiteness of the theory. All the string computations exploit the power of the CFT formalism, not only on flat space but also on more complicated backgrounds.

The remarkable achievements in two dimensional condensed matter systems and in String

[2]In second order phase transitions the correlation length becomes infinite, thus there are no IR characteristic scales. In first order phase transitions this is not the case.

theory, have been obtained due to the special feature of conformal algebra in two dimensions, which can be extended to the so called Virasoro algebra ([9]). This infinite dimensional algebra provides a huge amount of additional information, allowing for instance to completely solve for specific models ([10]).

Exact results for CFT's in dimensions larger then two are much harder to get because of the lack of an higher dimensional version of the Virasoro algebra. Until few years ago only few examples of CFT's were known, either based on perturbative results, such as [11, 12] or on supersymmetry ([13, 14]). Generically, however, it is not known how to extract the spectrum of the theory or how to compute all the correlation functions. Nevertheless it has been conjectured ([10]) that imposing the right consistency conditions on the CFT is sufficient to solve it completely, exactly as in two dimensions. This takes the name of the *bootstrap program*. Unfortunately, despite the significant amount of works on the subject, very few exact results have been achieved in this direction.

A recent boost to CFT's has been given by the discovery ([15]) of a duality between CFT's in D-dimensions and quantum gravity in $D+1$-dimensions. The original conjecture connects the maximally supersymmetric theory in flat space (no gravity) with $SU(N)$ gauge group and quantum gravity in the $AdS_5 \times S_5$ background. In a particular limit both the theories can be exactly solved and the duality can be demonstrated. However it is commonly believed that the duality is more generic and can be applied to a larger class of theories.

A very attractive feature of the AdS/CFT duality is that it can be viewed as a tool to define what a quantum theory of gravity is. Indeed, a CFT in D-dimensions defines holographically a $D+1$-dimensional theory of gravity in a suitable space with boundary. Practically however, the correspondence is used in the opposite direction: although we do know the general properties of a CFT, we don't have many concrete examples to work with. Thus we can start with a weakly-coupled effective field theory of gravity at small curvature and study the CFT side, which, in turn, will be strongly coupled and in some sense effective (see [16] for a recent discussion). Unfortunately, the set of the CFT's admitting a useful holographic description doesn't cover the whole space of CFT's.

Thus we surely need an alternative method to explore systems that cannot be studied via the AdS/CFT correspondence.

1.2 Conformal Invariance and model building in Particle Physics

In this section we would like to explain the phenomenological motivation for the present study. This section is technically disconnected from the rest of the thesis and could be skipped by a reader interested only in the theoretical aspects of the topic. However, we would like to stress that the entire work has been motivated by a concrete question in model building. We therefore find it important to dedicate this section to trace a connection between the formal aspects of CFT's and the needs of particle physics.

CFT's could play a fundamental role for the construction of a high energy alternative to the Standard Model (SM) of particle physics. The reason is related to a tension between the lower bound on the scale of new physics set by flavor experiments and the stability of the electroweak scale with respect to it. We shall show that a possible solution to this issue requires a CFT with some specific spectrum of operators. The needed spectrum cannot be obtained in large-N, quasi-Gaussian CFT's, for which great progress have been made in the last decade using AdS/CFT techniques. The purpose of the present work is to develop new tools to investigate the feasibility of requests on CFT's.

Let us now briefly review the phenomenological context. In the current understanding the Standard Model of particle physics is a low energy description of a more fundamental theory which will show up its nature at ranges of energies higher than the ones so far investigated. In addition to the degrees of freedom so far discovered (quarks, leptons and gauge bosons) those responsible for mass generation and electroweak symmetry breaking are clearly missing.
The presence of a vector mass and the observed structure of trilinear vector coupling imply that, in absence of new degrees of freedom, vector interactions become strong between 1-2 TeV. In that case, we can interpret the loss of perturbativity as the emergence of new physics, given that the theory must be reformulated in terms of new degrees of freedom. On the other hand new states can appear below the mentioned scale and modify the behavior of the scattering amplitude of longitudinal gauge vectors. In this case the theory can admit a weakly-coupled description up to higher scales. In any case we conclude that new degrees of freedom must be present at the electroweak scale. This represents a motivated and non speculative justification for the construction of the Large Hadron Collider (LHC).
A minimal setup, in good agreement with the present Electroweak precision data from LEP, is represented by the addition of a single scalar new degree of freedom, the Higgs particle. The theory describing the SM particles and the Higgs can be extrapolated to very hight energies[3]

[3] In principle up to the Plank scale.

and gives rise to the CKM paradigm, that perfectly describes all the flavor experiments. This because Yukawa interactions are the only non-irrelevant flavor-violating operators that can be constructed out of the SM fields and the Higgs. It is sufficient to assume a large gap between the electroweak scale and all the scales of the theory to have all the other irrelevant interactions highly suppressed.

On the other hand, such a large hierarchy of scales is not naturally stable. Indeed quantum mechanically a scalar particle is particularly sensitive to the UV physics. For instance, if we define the theory with a given cut off, quantum corrections would set the scalar mass to the cut-off, unless a cancellation is enforced by a fine-tuning of the theory parameters.

A more elegant way to make a hierarchy of scales stable is vi the introduction of some symmetry that prevents scalar masses from receiving quantum corrections.

Concerning the SM, there is no such a symmetry protecting the Higgs mass. Thus it's hard to imagine that the natural scale of fundamental interactions is the Plank scale or the Grand Unification scale (respectively 10^{19}, 10^{16} GeV) and an unjustified cancellation of quantum effects gives rise to a Weak scale several order of magnitude smaller. It is fair to stress that no prime principles would be violated by this phenomenon. However we have experienced that whenever a fine tuning is required to cancel a power divergence, new physics intervenes at the proper scale to cut the divergence off and make the theory natural. This is for instance the case for the $K\bar{K}$-mixing mass term: the presence of the charm quark cancels a quadratic dependence on the cut off which would be inconsistent with flavor experiments.

To summarize, in attempting to construct realistic extensions of the SM one is faced with a rigid dichotomy. On one hand, as just discussed, a conceptual problem arises if we push the new physics too high. On the other hand the incredible agreement among SM predictions and experiments indirectly suggests the presence of a large mass gap between the observed spectrum and the degrees of freedom associated to new physics. The tension between these two effects is one way to phrase the so called Hierarchy problem.

Natural Hierarchies

The issue of mass hierarchies in field theory can be conveniently depicted from a CFT viewpoint. Indeed the basic statement that a given field theory contains two widely separated mass scales $\Lambda_{\rm IR} \ll \Lambda_{\rm UV}$ already implies that the energy dependence of physical quantities at $\Lambda_{\rm IR} \ll E \ll \Lambda_{\rm UV}$ is *small*, corresponding to approximate scale (and conformal) invariance. In the case of perturbative field theories the CFT which approximates the behavior in the inter-

mediate mass region is just a free one. For instance, in the case of non-SUSY GUT's, $\Lambda_{\rm IR}$ and $\Lambda_{\rm UV}$ are respectively the Fermi and GUT scale, and the CFT which approximates behavior at intermediate scales is just the free Standard Model. From the CFT viewpoint, the naturalness of the hierarchy $\Lambda_{\rm IR} \ll \Lambda_{\rm UV}$, or equivalently its stability, depends on the dimensionality of the scalar operators describing the perturbations of the CFT Lagrangian around the fixed point. In the language of the RG group, naturalness depends on the relevance of the deformations at the fixed point. If the theory possesses a scalar operator $\mathcal{O}_\Delta$, with dimension $\Delta < 4$, one generically expects UV physics to generate a perturbation

$$\mathcal{L}_{pert} = c\Lambda_{\rm UV}^{4-\Delta}\mathcal{O}_\Delta \,, \tag{1.1}$$

corresponding roughly to an IR scale

$$\Lambda_{\rm IR} = c^{\frac{1}{4-\Delta}}\Lambda_{\rm UV} \,. \tag{1.2}$$

Absence of tuning corresponds to the expectation that c be not much smaller than $O(1)$. If $4-\Delta$ is $O(1)$ (*strongly relevant* operator) a hierarchy between $\Lambda_{\rm IR}$ and $\Lambda_{\rm UV}$ can be maintained only by tuning c to be *hierarchically* smaller than one. This corresponds to an unnatural hierarchy. On the other hand when $4-\Delta$ is close to zero (*weakly-relevant* operator) a mass hierarchy is obtained as soon as both $4-\Delta$ and c are just *algebraically* small[4]. For instance for $4-\Delta = c = 0.1$ the mass hierarchy spans 10 orders of magnitude. Therefore for a weakly-relevant operator a hierarchy is considered natural. The only exception to the above classification of naturalness concerns the case in which the strongly relevant operators transform under some global approximate symmetry. In that case it is natural to assume that the corresponding c's be small. The stability of the hierarchy depends then on the dimension Δ_S of the scalar singlet (under all global symmetries) of lowest dimension. If $4-\Delta_S \ll 1$ the hierarchy is natural.
According to the above discussion, in the SM the hierarchy between the weak scale and any possible UV scale is unnatural because of the presence of a scalar bilinear in the Higgs field $H^\dagger H$ which is a total singlet with dimension ~ 2. Extensions of the SM obviate to this difficulty is different ways: either such bilinear exists but their coefficients can naturally be chosen small because of the presence of additional symmetries, like in supersymmetric models, or gauge invariant relevant operators are simply absent, like in technicolor models. As far as the hierarchy is concerned this second solution is clearly preferable to the SM. However as far as flavor physics is concerned the SM has, over its extensions, an advantage which is also a simple consequence

[4]We stole this definition from ref. [18].

of operator dimensionality. In the SM the flavor violating operators of lowest dimensionality, the Yukawa interactions, have dimension = 4,

$$\mathcal{L}_Y = y^u_{ij} H \bar{q}_L u_R + y^d_{ij} H^\dagger \bar{q}_L d_R + y^e_{ij} H^\dagger \bar{L}_L e_R \quad (\mathrm{SM})\,, \tag{1.3}$$

and provide a very accurate description of flavor violating phenomenology. In particular, the common Yukawa origin of masses and mixing angles leads to a critically important suppression of Flavor Changing Neutral Currents (FCNC) and CP violation. This suppression is often called Natural Flavor Conservation or GIM mechanism [19]. Once the hierarchy $v \ll \Lambda_{\mathrm{UV}}$ is taken as a fact, no matter how unnatural, extra unwanted sources of flavor violation are automatically suppressed. In particular the leading effects are associated to 4-quark interactions, with dimension 6, and are thus suppressed by $v^2/\Lambda^2_{\mathrm{UV}}$. The comparison with technicolor brings us to the heart of the matter. In technicolor the Higgs field is a techni-fermion bilinear $H = \bar{T}T$ with dimension ~ 3. The SM fermions instead remain elementary, i.e. with dimension 3/2. The Yukawa interactions are therefore irrelevant operators of dimension 6,

$$\mathcal{L}_Y = \frac{y_{ij}}{\Lambda^2_{\mathrm{F}}} H \bar{q} q \quad (\mathrm{TC}), \tag{1.4}$$

and are associated to some new dynamics [20], the flavor dynamics, at a scale Λ_{F}, which plays the role of our Λ_{UV}. Very much like in the SM, and as it is found in explicit models [20], we also expect unwanted 4-quark interactions

$$\frac{c_{ijkl}}{\Lambda^2_{\mathrm{F}}} \bar{q}_i q_j \bar{q}_k q_l \tag{1.5}$$

suppressed by the same flavor scale. Unlike in the SM, in technicolor the Yukawa interactions are *not* the single most relevant interaction violating flavor. This leads to a tension. On one hand, in order to obtain the right quark masses, Λ_{F} should be rather low. On the other hand, the bound from FCNC requires Λ_{F} to be generically larger. For instance the top Yukawa implies $\Lambda_{\mathrm{F}} \lesssim 10\,\mathrm{TeV}$. On the other hand the bound from FCNC on operators like Eq. (1.5) is rather strong. Assuming $c_{ijkl} \sim 1$, flavor mixing in the neutral kaon system puts a generic bound ranging from $\Lambda_{\mathrm{F}} > 10^3\,\mathrm{TeV}$, assuming CP conservation and left-left current structure, to $\Lambda_{\mathrm{F}} > 10^5\,\mathrm{TeV}$, with CP violation with left-right current. Of course assuming that c_{ijkl} have a nontrivial structure controlled by flavor breaking selection rules one could in principle obtain a realistic situation. It is however undeniable that the way the SM disposes of extra unwanted sources of flavor violation is more robust and thus preferable. The origin of the problem is the *large* dimension of the Higgs doublet field H. Walking technicolor (WTC) ([21]) was invented

to alleviate this problem. In WTC, above the weak scale the theory is assumed to be near a non-trivial fixed point, where $H = \bar{T}T$ has a sizable negative anomalous dimension. WTC is an extremely clever idea, but progress in its realization has been slowed down by the difficulty in dealing with strongly coupled gauge theories in 4D. Most of the work on WTC relied on gap equations, a truncation of the Schwinger-Dyson equations for the $\bar{T}T$ self-energy. Although gap equations do not represent a fully defendable approximation, they have produced some interesting results. In case of asymptotically free gauge theory they lead to the result that $H = \bar{T}T$ can have dimension 2 at the quasi-fixed point, but not lower [22]. In this case the Yukawa interactions would correspond to dimension 5 operators, which are more relevant than the unwanted dimension 6 operators in Eq. (1.5). However some tension still remains: the top Yukawa still requires a Flavor scale below the bound from the Kaon system, so that the absence of flavor violation, in our definition, is not robust. It is quite possible that the bound $[H] \geq 2$ obtained with the use of gap equation will not be true in general. Of course the closer $[H]$ is to 1, the higher the flavor scale we can tolerate to reproduce fermion masses, and the more suppressed is the effect of Eq. (1.5). However if $[H]$ gets too close to 1 we get back the SM and the hierarchy problem! More formally, a scalar field of dimension exactly 1 in CFT is a free field and the dimension of its composite $H^\dagger H$ is trivially determined to be 2, that is strongly relevant. By continuity we therefore expect that the hierarchy problem strikes back at some point as $[H]$ approaches 1. However the interesting remark made by Luty and Okui [23] is that, after all, we do not really need $[H]$ extremely close to 1. Depending on the assumptions on the flavor structure of the UV theory, $[H] \leq 1.3 - 1.5$ would still be good, in which case the corresponding CFT is not weakly coupled and it could well be that $[H^\dagger H]$ is significantly bigger than $2[H]$ and maybe even close to 4. The motivation of our present work is precisely to find, from prime principles, what is the upper bound on $\Delta_S = [H^\dagger H]$ as $d = [H]$ approaches 1.

Conformal sequestering

The general recipe emerging from the previous discussion is the possibility to use large anomalous dimensions to naturally solve issues that are seemingly impossible to address in usual weakly-coupled theories.

Another scenario exploiting this property is the so called *conformal sequestering* (see, for instance, [24] and references therein). This time the stability of the weak scales with respect to other fundamental scales is addressed with the help of supersymmetry. The decoupling of the (so far) unobserved super-partners is obtained connecting the visible sector with a secluded sector

where supersymmetry is broken. The mechanism responsible for transmitting the breaking to the visible sector is unimportant for the rest of the discussion[5]: generically soft masses scale as

$$m^2_{\rm soft} \simeq \left(\frac{g^2_{SM}}{16\pi^2}\right)^2 \frac{F^2}{M^2_{\rm Planck}}\,, \tag{1.6}$$

where F is the supersymmetry breaking vacuum expectation values of a chiral superfields $\langle X\rangle = \theta^2 F$. Notice that the above mass shows a loop suppression due to the nature of the coupling between visible and secluded sector. In order to produce sufficiently heavy superpartners we must require $\sqrt{F} \sim 10^{11}\,\mathrm{GeV}$. Above this scale the theory is supersymmetric. On the other hand, besides these effects, an additional mediation between the visible and the supersymmetry breaking sector is represented by gravity. Generically one must expect an effective interaction, coming from a direct tree level mediation, of the form

$$\int d^4\theta \frac{c_{ij}}{M^{\Delta-1}_{\rm Planck}} Q^\dagger_i Q_j O_{\rm secl}(X, X^\dagger)\,, \qquad [O_{\rm secl}] = \Delta\,, \tag{1.7}$$

where $O_{\rm secl}$ is an operator belonging to the secluded sector, possibly depending on the chiral field X, while Q_i is a chiral superfield containing the quark q_i. A simple possibility is represented by the composite operator $X^\dagger X$. In absence of further assumptions, the interaction (1.7) produces, at the scale of mediation, a squarks mass term of the form

$$m^2_{ij} \simeq c_{ij} \frac{F^2}{M^2_{\rm Planck}}\,. \tag{1.8}$$

In order to avoid a brutal violation of flavor constraints, at the electroweak scale the above mixing must be suppressed with respect to the soft mass (1.6) by a factor $10^{-3} - 10^{-4}$. A clever way to realize this suppression is to take the secluded sector above the supersymmetry breaking scale conformally invariant with, in addition, large anomalous dimensions for all the operators giving rise to flavor violating interactions. If this is the case, mixing terms like (1.7) undergo a power law running from the mediation scale to the supersymmetry breaking scale (or slightly above) with the final result:

$$m^2_{ij}(\sqrt{F}) \simeq m^2_{ij}(M_{\rm Planck}) \left(\frac{\sqrt{F}}{M_{\rm Planck}}\right)^\gamma\,, \tag{1.9}$$

where γ is the anomalous dimension of the composite operator $X^\dagger X$. We observe therefore the required suppression whenever γ is sufficiently large.

More demanding is instead the solution of the μ-B_μ problem proposed in the context of

[5] In these models high scale mediation mechanism are used, such as anomaly-mediation or gaugino-mediation.

conformal sequestering ([25]): this solution requires not only order one anomalous dimensions but also an order one deviation from factorization, namely

$$\Delta_{X^\dagger X} - 2\Delta_X \sim O(1)\,. \tag{1.10}$$

When this is the case one can naturally generate the supersymmetric μ term and the soft B_μ term of the same size at the electroweak scale[6] simply running the following effective interactions

$$\frac{g_{SM}^2}{16\pi^2}\int d^2\theta \frac{C_\mu}{M_{\text{Planck}}^{\Delta_X-1}} X H_u H_d\,, \qquad \frac{g_{SM}^2}{16\pi^2}\int d^4\theta \frac{C_{B_\mu}}{M_{\text{Planck}}^{\Delta_{X^\dagger X}-1}} X^\dagger X H_u H_d\,, \tag{1.11}$$

from the mediation scale down to the supersymmetry breaking scale:

$$C_\mu(\sqrt{F}) = C_\mu(M_{\text{Planck}})\left(\frac{\sqrt{F}}{M_{\text{Planck}}}\right)^{\Delta_X}, \qquad C_{B_\mu}(\sqrt{F}) = C_{B_\mu}(M_{\text{Planck}})\left(\frac{\sqrt{F}}{M_{\text{Planck}}}\right)^{\Delta_{X^\dagger X}}. \tag{1.12}$$

The ratio B_μ/m^2, naively one loop enhanced, can therefore be of order one if the condition (1.10) is satisfied. We should stress that this requirement is never accomplished in theories with a gravity dual or exhibiting a sort of perturbative expansion. Such a splitting can only be achieved in genuine strongly coupled, small N CFT's.

The natural question that arises out of the above discussion is again, given the absence of explicit examples, whether theories implementing the needs of conformal technicolor or conformal sequestering can be constructed at all or if there are prime principle obstructions to their realization.

1.3 Structure of the Thesis

The goal of the study reported in this thesis is the implementation of a method to explore the structure of CFT's in any dimensions, based only on prime principles such as unitarity and crossing symmetry. This elaborate collects and extends the results published in a series of paper: [26, 27, 28, 29, 30, 31, 32].

So far the analysis of CFT's in dimension bigger than two relied mostly in the use of Ward identities or additional symmetries. In this work we managed to bring the bootstrap program to a higher level, implementing a consistent and general procedure to extract informations about CFT's. One of the reasons why we were successful is because we could put to good use the works

[6] This requirement is needed in order to reproduce the correct electroweak symmetry breaking pattern in the scalar potential.

of Dolan & Osborn [33]. As we will review in Section 4.1, four point functions in conformal field theory can be expressed as sums of building blocks called *conformal blocks*. In their works Dolan & Osborn derived a closed expression for these quantities. In particular the dependence on the spectrum and on the OPE coefficient of the CFT has been made explicit. Using their results we translated the so far somewhat obscure crossing symmetry constraints into precise functional equations, taking the form of functional sum rules. Studying the feasibility of these constraints we obtained informations about the spectrum and the size of OPE coefficients of a general CFT.

Before discussing in more details the results of our studies let us stress the assumption/limitation of the method. Firstly, so far the technique is limited to even dimensions, since the explicit expression for the four point function in odd dimensions has not been derived jet. Moreover the results of [33] are restricted to correlators of four scalar operators, therefore we can explore only CFT's containing at least one scalar operator.
We have no doubt that our method can be extended to odd dimensions and to four point functions involving non-scalar operators[7], but we cannot say how hard it will be at the moment.

Since the majority of CFT reviews present in the literature are focused on two dimensions, we dedicated the first three Chapters of the manuscript to review the CFT formalism in arbitrary dimension, with a specific interest to the four dimensional case. Further details can be found in [34, 35, 36, 37, 38, 39, 40].
Then, in Chapter 4 we introduce the formalism to express the four point functions in terms of conformal blocks, and we derive the constraints imposed by crossing symmetry. The formalism allows to deal with CFT's with global symmetries and with superconformal symmetry, which are reviewed in Chapter 5.
In the sixth Chapter we present our results and we describe the technical procedure to derive them. The informations about general CFT's that we have been able to extract fall into two major categories:

- *Spectrum* of the CFT: we will demonstrate that CFT's are not compatible with arbitrary spectra of the operators. More specifically we prove that given a unitary CFT containing a scalar operator ϕ with dimension $d \geq 1$, the OPE of this field with itself[8] contains necessarily a *scalar* operator with dimension smaller than a certain function depending on d. Similarly, if the theory has some additional symmetry under which the scalar field is non neutral, the operators appearing in the OPE can be classified according to their

[7] Using supersymmetry one con for instance derive the general form of a supersymmetric four point functions involving two scalar and two fermions, or using extended supersymmetry even involving currents or tensors

[8] If the field is complex one can consider two different OPE's: $\phi \times \phi$ and $\phi \times \phi^\dagger$: the arguments applies to both.

representations: independent bounds can be derived for any representation.
The analysis has been performed in two and four dimensions pushing the numerics to the maximal power allowed by computing (with the current algorithm). In two dimension the existence of solvable models allows a comparison between our results and the exact results: the remarkable agreement is shown in Section 6.1.6 . In four dimension we explored the case of real ϕ , the case of $SO(N)$ symmetry[9] and of superconformal field theories. These provide a preliminary answer to the issues raised during the phenomenological discussion of the previous section.
Our method does not allow us to solve CFT's but can provide hints and useful information on where they can be. This is make evident in two dimensions where the presence of the Ising model is hinted by a kink in the everywhere else smooth bound on the dimension of scalar operators. Unfortunately, no similar hints on the existence of a four dimensional CFT are observed.

- *OPE coefficients* of the CFT: we will show that the size of the coefficient of a generic operator entering the OPE of a scalar field ϕ is bounded from above.
  OPE coefficients are per se interesting quantities, however there are two interesting application that we explored in this work . Firstly, the OPE coefficients of conserved currents can be related to the central charges of the CFT. In this way we derived lower bounds on the (energy-momentum) central charge. This study has been carried over for real scalar in two and four dimensions and for supersymmetric theories.
  A second relevant application concerns the possibility to extract information about the spectrum of a CFT. Indeed, when pushing the bounds to their optimum features appear. These *peaks*, emergent out of the background, seems to be intriguing associated to operators that must exist in the CFT. We show that this is precisely the case in the Ising model.

A final comment concerns the potential of the method described in this thesis. These techniques let us deal with many interesting questions. We decided to focus on those we judge relevant, mostly guided by phenomenological needs and the will to compare with known results, but surely many others can be addressed. Furthermore, when performing a given analysis we can decide either not to rely on any restriction on the CFT[10] or easily include precious informations or assumptions. A simple example is given by the lower bound on the central charge. We extracted

[9] Note that $SO(2) \sim U(1)$ allows the treatment complex fields.
[10] As long as the theory is unitary and contains a scalar

a result valid no matter what is the operator content and the values of the OPE coefficients, however, incorporating the restriction that the lowest dimension scalar entering in the OPE is larger than a given hypothetical value, we have been able to derive a stronger bound. Similarly one can implement any additional restriction on the spectrum. A second example is given by two dimensional CFT, where one would like to use the full power of the Virasoro algebra[11]. For instance the presence of a single Verma module of dimension Δ can be incorporated imposing only operators of dimension $\Delta + n$, $n \in \mathbb{N}$, to be present in the OPE[12].
As a final example we mention how one can make contact with perturbative CFT's or with ADS/CFT results. In this class of theories we expect the dimension of the CFT operators to be additive up to small corrections. Enforcing this restrictions allows to derive very severe bounds. For instance we can show that in the limit of exact additivity the central charge grows to infinity, as expected in a generalized free theory, which is believed to be the dual description of a free scalar field in AdS_5 background without gravity. Indeed, the decoupling of gravity interactions translates in the dual picture in the absence of the energy-momentum tensor. We describe this in Section 6.3.3.

[11] We stress that in the present work only the global $SL(2, \mathbb{C})$ is used to derive the results in two dimensions. Nevertheless we have been able to compare with the Minimal models, the discovery and complete solution of which required the Virasoro algebra.

[12] This information is clearly minimal but it is already sufficient to simplify enormously numerical computations.

Chapter 2

Conformal Symmetry

2.1 Conformal Algebra in D-dimensions

Let us consider the metric tensor $g_{\mu\nu}(x)$ of a D-dimensional space-time. The conformal group can be defined as the set of diffeomorphisms that leave the metric unchanged up to a overall scale factor, which in general can be coordinate dependent:

$$g'_{\mu\nu}(x') \doteqdot \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) = \Lambda(x) g_{\mu\nu}(x)\,. \tag{2.1}$$

Writing at infinitesimal level $x'^\mu(x) \doteqdot x^\mu + \varepsilon^\mu(x)$ and $\Lambda = 1 - O(\varepsilon)$ we can obtain the general constraint

$$\partial_\rho \varepsilon_\mu + \partial_\mu \varepsilon_\rho = \frac{2}{D}\partial^\sigma \varepsilon_\sigma g_{\mu\rho}\,. \tag{2.2}$$

Deriving a second time and permuting the indices (we restrict to constant metric tensor) we get

$$\begin{aligned}
\partial_\mu\partial_\nu\varepsilon_\rho + \partial_\rho\partial_\nu\varepsilon_\mu &= \frac{2}{D}\partial_\nu\partial^\sigma\varepsilon_\sigma g_{\mu\rho}\,,\\
\partial_\nu\partial_\mu\varepsilon_\rho + \partial_\rho\partial_\mu\varepsilon_\nu &= \frac{2}{D}\partial_\mu\partial^\sigma\varepsilon_\sigma g_{\nu\rho}\,,\\
\partial_\mu\partial_\rho\varepsilon_\nu + \partial_\nu\partial_\rho\varepsilon_\mu &= \frac{2}{D}\partial_\rho\partial^\sigma\varepsilon_\sigma g_{\mu\nu}\,.
\end{aligned} \tag{2.3}$$

Adding the first and the third equation and subtracting the last one we obtain

$$\partial_\rho\partial_\nu\varepsilon_\mu = \frac{1}{D}\left(\partial_\nu\partial^\sigma\varepsilon_\sigma g_{\mu\rho} - \partial_\mu\partial^\sigma\varepsilon_\sigma g_{\nu\rho} + \partial_\rho\partial^\sigma\varepsilon_\sigma g_{\mu\nu}\right)\,. \tag{2.4}$$

Finally, contracting the indices, we obtain

$$\Box\varepsilon_\mu = \frac{2-D}{D}\partial_\mu(\partial^\sigma\varepsilon_\sigma)\,, \tag{2.5}$$

where $\Box$ is defined with the matrix $g_{\mu\nu}$, whatever signature we have chosen. Finally, applying ∂_ν to the above equation and $\Box$ to eq. (2.2) we get:

$$\begin{aligned} \partial_\nu \Box \varepsilon_\mu &= \frac{2-D}{D} \partial_\mu \partial_\nu (\partial^\sigma \varepsilon_\sigma)\,, \\ \partial_\nu \Box \varepsilon_\mu + \partial_\mu \Box \varepsilon_\nu &= \frac{2}{D} \Box \partial^\sigma \varepsilon_\sigma g_{\mu\nu} \end{aligned} \tag{2.6}$$

Symmetrizing the first equation we can get

$$(2-D)\partial_\mu \partial_\nu (\partial^\sigma \varepsilon_\sigma) = g_{\mu\nu} \Box \partial^\sigma \varepsilon_\sigma\,, \tag{2.7}$$

and taking the trace of the above equation we finally obtain a second order equation for $f(x) \doteqdot (\partial^\sigma \varepsilon_\sigma)$

$$(D-1)\Box f(x) = 0\,. \tag{2.8}$$

Inserting the above constraint in eq.(2.7) we argue that for $D > 2$ the function $f(x)$ must be at least linear in the coordinates. Hence the general solution is $f(x) = a + b_\mu x^\mu$, which translates in the general expression

$$\varepsilon^\mu = c^\mu + a_{\mu\nu} x^\nu + b_{\mu\nu\rho} x^\nu x^\rho \tag{2.9}$$

Plugging the general solution into eq. (2.4) we observe that the coefficient $b_{\mu\nu\rho}$ can be expressed in terms of only one vector $b_\sigma{}^\sigma{}_\mu$:

$$b_{\mu\nu\rho} = \frac{1}{D}\left(b_\sigma{}^\sigma{}_\nu g_{\mu\rho} + b_\sigma{}^\sigma{}_\rho g_{\mu\nu} - b_\sigma{}^\sigma{}_\mu g_{\nu\rho}\right) \tag{2.10}$$

Finally, using eq. (2.2) we show that the symmetric part of $a_{\mu\nu}$ is proportional to the matrix $g_{\mu\nu}$, while the antisymmetric one is completely unconstrained.
Counting the parameters contained in a general infinitesimal transformation we obtain

$$c_\mu : \quad D\,, \qquad a_{\mu\nu} : \quad \frac{D(D-1)}{2} + 1\,,, \qquad b_\mu : \quad D\,. \tag{2.11}$$

for a total of $(D+1)(D+2)/2$ parameters. We can easily recognize the transformations associated to the above parameters:

$$\begin{aligned} x'^\mu &= x^\mu + c^\mu : \quad \text{translations} \\ x'^\mu &= x^\mu + \lambda x^\mu : \quad \text{dilatations} \\ x'^\mu &= x^\mu + w^\mu_{\ \nu} x^\nu : \quad \text{Lorentz rotations} \\ x'^\mu &= x^\mu + 2(b_\rho x^\rho) x^\mu - x^2 b^\mu : \quad \text{Conformal boosts} \end{aligned} \tag{2.12}$$

It is instructive to derive the finite form of the above transformations. Clearly the only non trivial transformation are the conformal boosts, given their non linearity in the coordinates x. For concreteness let us restrict to $D = 4$, but the procedure can be straightforwardly generalized to higher dimensions.

We will make use of the isomorphism between the conformal algebra and the algebra of $O(4,2) \sim SU(2,2)$ (see for instance [35, 38, 36, 41]). We can start observing that the infinitesimal transformations (2.12) can be obtained starting from a vector y^A, defined on 6-dimensional space where we have a metric g_{AB} with signature $(+,-,-,-,-,+)$. Then, defining

$$x^\mu = \frac{y^\mu}{y^5 + y^6} \tag{2.13}$$

we can show that the following generators

$$(J_{AB})^C_D = i\left(\delta^C_A g_{BD} - \delta^C_B g_{AD}\right) \tag{2.14}$$

generate the conformal transformations in the sense if $\vec{y}' = (\mathbb{1} - i\alpha_{AB}J^{AB}/2)\vec{y}$, then $x^\mu(y'^A)$ is related to $x^\mu(y^A)$ by a conformal transformation. A key point is to restrict the 6-dimensional space to the set of zero-norm vectors, $y^A y_A = 0$.

Clearly Lorentz transformations are correctly reproduced by the subset $SO(3,1) \subset O(4,2)$ acting only on y^μ $\mu = 0,1,2,3$. A pure translation corresponds to having only non vanishing parameters $\alpha^{5\mu} = -\alpha^{6\mu}$. A conformal boost instead corresponds to $\alpha^{5\mu} = \alpha^{6\mu}$. Indeed, under an infinitesimal transformation we have:

$$\begin{aligned} y'^C &\simeq y^C - i(\alpha^{\mu 5}J_{\mu 5} + \alpha^{\mu 6}J_{\mu 6})^C_D y^D \\ &= y^C + (\alpha^{\mu 5}(-\delta^C_\mu y^5 - \delta^C_5 y_\mu) + \alpha^{\mu 6}(\delta^C_\mu y^6 - \delta^C_6 y_\mu)) \end{aligned} \tag{2.15}$$

Using the above expression we can compute the infinitesimal change in x^μ:

$$\begin{aligned} x'^\rho = \frac{y'^\rho}{y'^5 + y'^6} &\simeq \frac{y^\rho - (\alpha^{\rho 5}y^5 - \alpha^{\rho 6}y^6)}{y^5 + y^6 - y_\mu(\alpha^{\mu 5} + \alpha^{\mu 6})} \\ &\simeq \frac{y^\rho}{y^5 + y^6}\left(1 + 2\frac{y_\mu}{y^5 + y^6}\frac{(\alpha^{\mu 5} + \alpha^{\mu 6})}{2} - \frac{(\alpha^{\rho 5} - \alpha^{\rho 6})}{2} - \frac{(\alpha^{5\rho} + \alpha^{6\rho})y_\mu y^\mu}{2(y5 + y^6)^2}\right) \end{aligned} \tag{2.16}$$

In the manipulation of the above expression is crucial that $y^A y_A = 0$. We can recognize the infinitesimal translation with parameters $(-\alpha^{5\rho} + \alpha^{6\rho})/2$ and a conformal boost with parameter $(\alpha^{5\rho} + \alpha^{6\rho})/2$. Finally one can integrate the infinitesimal transformation exponentiating the

linear transformation acting on y^A:

$$y'^A = (e^{-ib^\mu(J_{\mu 5}+J_{\mu 6})})^A_{\ B}\, y^B \qquad e^{-ib^\mu(J_{5\mu}-J_{6\mu})} = \left(\begin{array}{c|c|c} \mathbb{1}_4 & -b^\mu & b^\mu \\ \hline -b_\mu & 1+\frac{b^2}{2} & -\frac{b^2}{2} \\ \hline -b_\mu & \frac{b^2}{2} & 1-\frac{b^2}{2} \end{array}\right) \tag{2.17}$$

Using the above expression we achieve the final expression for a conformal boost with parameter b^μ:

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b_\mu x^\mu + b^2 x^2}\,. \tag{2.18}$$

For later convenience we report the transformations under the other element of the group:

$$\begin{aligned} &\text{if} \quad y'^A = (e^{-ia^\mu(-J_{\mu 5}+J_{\mu 6})})^A_{\ B}\, y^B \quad \text{then} \quad x'^\mu = x^\mu + a^\mu \\ &\text{if} \quad y'^A = (e^{-i\lambda J_{65}})^A_{\ B}\, y^B \qquad\quad \text{then} \quad x'^\mu = e^\lambda x^\mu \end{aligned} \tag{2.19}$$

In conclusion we have the following identifications:

$$\begin{aligned} J_{\mu\nu} &\equiv M_{\mu\nu}\,, \qquad J_{\mu 5} = \frac{1}{2}\left(-P_\mu + K_\mu\right)\,, \\ J_{56} &= -D\,, \qquad J_{\mu 6} = \frac{1}{2}\left(P_\mu + K_\mu\right)\,. \end{aligned} \tag{2.20}$$

Starting from the infinitesimal transformations we can define the differential form of the generators acting on functions. Given a coordinate transformation $x \to x' = \xi(x)$ (therefore $x = \xi^{-1}(x')$), we have $f(x) \to f'(x) = f(\xi^{-1}(x))$. The implementation of function can be implemented through differential generators J such that $f'(x) = e^{-iJ} f(x)$. In the case in exam we get:

$$\begin{aligned} &\text{Translations}: && f'(x) = f(x^\mu - c^\mu) = f(x) - c^\mu \partial_\mu f(x) && \Rightarrow P_\mu = -i\partial_\mu \\ &\text{Lorentz}: && f'(x) = f(x^\mu) - w^\mu_\nu x^\nu \partial_\mu f(x) && \Rightarrow M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ &\text{Dilatations}: && f'(x) = f(x^\mu) - \lambda x^\mu \partial_\mu f(x) && \Rightarrow D = -i x^\mu \partial_\mu \\ &\text{Conf. boosts}: && f'(x) = f(x) - (2b^\nu x_\nu x^\mu \partial_\mu - x^2 b^\mu \partial_\mu) f(x) && \Rightarrow K_\mu = -i(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu) \end{aligned} \tag{2.21}$$

At this point we can straightforwardly compute the commutation rules using the explicit differential representation:

$$\begin{aligned}
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu}D - M_{\mu\nu}\right) \\
&[M_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right) \\
&[M_{\mu\nu}, K_\rho] = -i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) \\
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\left(M_{\mu\rho}\eta_{\nu\sigma} - M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\rho}\eta_{\mu\sigma} + M_{\nu\sigma}\eta_{\mu\rho}\right) \\
&[D, M_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = [D, D] = 0
\end{aligned} \tag{2.22}$$

2.2 Unitary representations of conformal symmetry

In this section we review the structures of unitary representation of the conformal symmetry. Given the isomorphism between the conformal group in D-dimension and the isometry-group of the maximally symmetric space with constant negative curvature in $D+1$-dimension, namely AdS_{D+1}, we can exploit the general formalism developed to study representation of Anti-de Sitter space [42], [43].

A simple way to describe AdS_{D+1} is through its embedding in $D+2$-dimensions. Let us consider the hyperboloid defined by the relation

$$\eta_{AB}y^A y^B = (y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 - ... - (y^D)^2 + (y^{D+1})^2 = R^2 \tag{2.23}$$

where R^2 is the radius of the Anti-de Sitter space[1]. We immediately see that the isometry group of this space is $SO(D,2)$. We now study the representation of this group in details. Let us introduce the explicit form for the generators:

$$(J_{AB})^C_F = i\left(\delta^C_A g_{BF} - \delta^C_B g_{AF}\right)\,, \qquad g_{AB} = (+,-,...,-,+)\,, \tag{2.24}$$

which satisfy the $SO(D,2)$ algebra:

$$[J_{AB}, J_{CF}] = i\left(g_{BC}J_{AF} - g_{AC}J_{BF} - g_{BF}J_{AC} + g_{AF}J_{BC}\right)\,. \tag{2.25}$$

[1] As a side comment we notice that the boundary at infinite of the above hyperboloid ($y^i \gg R$) is a cone and has the topology of a projective space and is isomorphic to Minkowski in D-dimension.

An important property of the group $SO(D,2)$ is the non-compactness, which implies that all unitary representation are infinite dimensional. Indeed, from the explicit form (2.24) we verify immediately that

$$\begin{aligned} &J^\dagger_{0(D+1)} = J_{0(D+1)}\,, \qquad J^\dagger_{ab} = J_{ab} \\ &J^\dagger_{0a} = -J_{0a}\,, \qquad J^\dagger_{(D+)b} = -J_{(D+1)b} \end{aligned} \tag{2.26}$$

where we adopted the convention:

$$\begin{aligned} &A, B, C, F... = 0, 1, ..., D+1\,, \\ &\mu, \nu, ... = 0, 1, ..., D-1 \\ &a, b, ... = 1, 2, ..., D\,. \end{aligned} \tag{2.27}$$

From now on we assume to deal with a unitary infinite dimensional representation, such that the generator J_{AB} are hermitian and satisfy the algebra (2.25). The maximal compact subgroup of of the conformal group is represented by $SO(D) \times SO(2)$, which is generated by $J_{0(D+1)}$ and J_{ab}.The former will play the role of the Hamiltonian operator (and we will denote it $\mathcal{H}$ from now on), while the latter identifies the "spin" of the representation. Among the non compact generators we can identify raising and lowering operators

$$\begin{aligned} &J^\pm_a \equiv iJ_{0a} \pm J_{(D+1)a}\,, \qquad (J^-_a)^\dagger = -J^+_a\,, \\ &[\mathcal{H},\, J^\pm_a] = \pm J^\pm_a\,. \end{aligned} \tag{2.28}$$

Notice that J^+_a and J^-_a form two abelian abelian sub-algebra, $\mathcal{A}^+$ and $\mathcal{A}^-$, such that $[\mathcal{A}^+, \mathcal{A}^-] \neq 0$:

$$\begin{aligned} &[J^+_a, J^+_b] = [J^-_a, J^-_b] = 0\,, \\ &[J^+_a, J^-_b] = 2\left(\eta_{ab}\mathcal{H} + iJ_{ab}\right)\,. \end{aligned} \tag{2.29}$$

Depending on the dimension D, the sub-algebra $SO(D)$ generated by J_{ab} has different structures. For instance, if $D = 2$ then there is only one generator, $J_{ab} = \epsilon_{ab}J$, while for $D = 4$ we can decompose J_{ab} as follows:

$$\begin{aligned} &T^L_i = \frac{1}{2}\left(\frac{1}{2}\epsilon_{ijk}J_{jk} + J_{4i}\right)\,, \\ &T^R_i = \frac{1}{2}\left(\frac{1}{2}\epsilon_{ijk}J_{jk} - J_{4i}\right)\,, \\ &[T^{L(R)}_i,\, T^{L(R)}_j] = i\epsilon_{ijk}T^{L(R)}_k\,, \qquad [T^L_i,\, T^R_j] = 0 \end{aligned} \tag{2.30}$$

For $D = 3$ we refer to [42], [43]. Let us proceed in $D = 4$ from now on. The raising and lowering operators $J_a^{\pm}$ transform in the (1/2, 1/2) representation of $SO(4) \simeq SU(2)_L \times SU(2)_R$ as a :

$$\begin{aligned}
&J^{\pm}_{\uparrow\uparrow} = J^{\pm}_1 + iJ^{\pm}_2\,, \qquad J^{\pm}_{\downarrow\downarrow} = J^{\pm}_1 - iJ^{\pm}_2\,,\\
&J^{\pm}_{\uparrow\downarrow} = J^{\pm}_3 - iJ^{\pm}_4\,, \qquad J^{\pm}_{\downarrow\uparrow} = J^{\pm}_3 + iJ^{\pm}_4\,,\\
&[T^L_3, J^{\pm}_{\uparrow *}] = \frac{1}{2}J^{\pm}_{\uparrow *} \qquad [T^L_3, J^{\pm}_{\downarrow *}] = -\frac{1}{2}J^{\pm}_{\downarrow *}\,, \qquad * = \uparrow, \downarrow\\
&[T^R_3, J^{\pm}_{*\uparrow}] = \frac{1}{2}J^{\pm}_{*\uparrow} \qquad [T^R_3, J^{\pm}_{*\downarrow}] = -\frac{1}{2}J^{\pm}_{*\downarrow}
\end{aligned} \tag{2.31}$$

Hence the above operators raise or lower the eigenvalues of $T_3^{L(R)}$ by half unit.
After this long premise let us introduce a highest weight representation. Let us assume we have an infinite dimension representation which has a state with minimal "energy", that is to say the eigenvalues of the Hamiltonian $\mathcal{H}$ are bounded form below. This *ground state* must be annihilated by all the lowering operators J_a^-:

$$J_a^- \left|E_0, s, j_1, j_2, m_1, m_2\right\rangle = 0 \quad \text{for any } a = 1, ..., 4 \tag{2.32}$$

In the above expression we have introduced a basis of the Hilbert space of states which diagonalize $\mathcal{H}$, $T_i^L T_i^L$, $T_i^R T_i^R$, T_3^L and T_3^R and we have switched notation in order to make the formula more compact: $L \equiv 1$, $R \equiv 2$. Thus:

$$\begin{aligned}
&\mathcal{H}\left|E, j_1, j_2, m_1, m_2\right\rangle = E\left|E, j_1, j_2, m_1, m_2\right\rangle\,,\\
&T_i^L T_i^L\left|E, j_1, j_2, m_1, m_2\right\rangle = j_1(j_1+1)\left|E, j_1, j_2, m_1, m_2\right\rangle\,,\\
&T_i^R T_i^R\left|E, j_1, j_2, m_1, m_2\right\rangle = j_2(j_2+1)\left|E, j_1, j_2, m_1, m_2\right\rangle\,,\\
&T_3^L\left|E, j_1, j_2, m_1, m_2\right\rangle = m_1\left|E, j_1, j_2, m_1, m_2\right\rangle\,,\\
&T_3^R\left|E, j_1, j_2, m_1, m_2\right\rangle = m_2\left|E, j_1, j_2, m_1, m_2\right\rangle\,,
\end{aligned} \tag{2.33}$$

$$\tag{2.34}$$

and we denote E_0 the energy of the ground state. Let us consider the result of applying one of the $J^+_{\uparrow\uparrow}$ generator to a the a ground state

$$\left|J^+_{\uparrow\uparrow}\left|E_0, j_1, j_2, m_1, m_2\right\rangle\right|^2 = \left\langle E_0, j_1, j_2, m_1, m_2\right| \left[J^+_{\uparrow\uparrow},\, J^-_{\downarrow\downarrow}\right]\left|E_0, j_1, j_2, m_1, m_2\right\rangle \tag{2.35}$$

The state on the left hand side inside the norm can be decomposed in a set of orthonormal states. Recalling that $J^+_{\uparrow\uparrow}$ transforms under a $(1/2, 1/2)$ representation. Moreover, according to

(2.31) it raises both m_1 and m_2 of half unit while it increases the energy of one unit. Thus

$$\begin{aligned} J^+_{\uparrow\uparrow}\left|E_0, j_1, j_2, m_1, m_2\right\rangle &= R_{++} C^{j_1+\frac{1}{2}, m_1+\frac{1}{2}}_{j_1 m_1, \frac{1}{2}\frac{1}{2}} C^{j_2+\frac{1}{2}, m_2+\frac{1}{2}}_{j_2 m_2, \frac{1}{2}\frac{1}{2}} \left|E_0+1, j_1+\frac{1}{2}, j_2+\frac{1}{2}, m_1+\frac{1}{2}, m_2+\frac{1}{2}\right\rangle \\ &+ R_{+-} C^{j_1+\frac{1}{2}, m_1+\frac{1}{2}}_{j_1 m_1, \frac{1}{2}\frac{1}{2}} C^{j_2-\frac{1}{2}, m_2+\frac{1}{2}}_{j_2 m_2, \frac{1}{2}\frac{1}{2}} \left|E_0+1, j_1+\frac{1}{2}, j_2-\frac{1}{2}, m_1+\frac{1}{2}, m_2+\frac{1}{2}\right\rangle \\ &+ R_{-+} C^{j_1-\frac{1}{2}, m_1+\frac{1}{2}}_{j_1 m_1, \frac{1}{2}\frac{1}{2}} C^{j_2+\frac{1}{2}, m_2+\frac{1}{2}}_{j_2 m_2, \frac{1}{2}\frac{1}{2}} \left|E_0+1, j_1-\frac{1}{2}, j_2+\frac{1}{2}, m_1+\frac{1}{2}, m_2+\frac{1}{2}\right\rangle \\ &+ R_{--} C^{j_1-\frac{1}{2}, m_1+\frac{1}{2}}_{j_1 m_1, \frac{1}{2}\frac{1}{2}} C^{j_2-\frac{1}{2}, m_2+\frac{1}{2}}_{j_2 m_2, \frac{1}{2}\frac{1}{2}} \left|E_0+1, j_1-\frac{1}{2}, j_2-\frac{1}{2}, m_1+\frac{1}{2}, m_2+\frac{1}{2}\right\rangle \end{aligned} \tag{2.36}$$

where we have extracted the Clebsh-Gordan contribution:

$$C^{j+\frac{1}{2}, m+\frac{1}{2}}_{jm, \frac{1}{2}\frac{1}{2}} = \left(\frac{j+m+1}{2j+1}\right)^{1/2}, \tag{2.37}$$

$$C^{j-\frac{1}{2}, m+\frac{1}{2}}_{jm, \frac{1}{2}\frac{1}{2}} = \left(\frac{j-m}{2j+1}\right)^{1/2}, \tag{2.38}$$

Finally, let us express the commutator appearing in the right hand side of (2.35) in terms of the Hamiltonian and the third component of the spins:

$$\left[J^+_{\uparrow\uparrow},\, J^-_{\downarrow\downarrow}\right] = 4\left(\mathcal{H} + T^L_3 + T^R_3\right) \tag{2.39}$$

As final step let us substitute the expansion (2.36) and the above commutator in (2.35). Choosing the four different combinations:

$$(m_1\,, m_2) = \{(j_1, j_2)\,, (j_1-1, j_2)\,, (j_1, j_2-1)\,, (j_1-1, j_2-1)\} \tag{2.40}$$

we get a system of equation that we can solve in terms of the R's coefficients:

$$\begin{aligned} |R_{++}|^2 &= 4(E_0 + j_1 + j_2)\,, \\ |R_{+-}|^2 &= 4(E_0 + j_1 - j_2 - 1)\,, \\ |R_{-+}|^2 &= 4(E_0 - j_1 + j_2 - 1)\,, \\ |R_{--}|^2 &= 4(E_0 - j_1 - j_2 - 2)\,, \end{aligned} \tag{2.41}$$

(2.42)

The unitarity of the representation requires the above coefficients to all positive. Whenever both j_1, j_2 are non vanishing this translates in the *unitarity bound*:

$$E_0 \geq j_1 + j_2 + 2\,. \tag{2.43}$$

Instead if only one of the two indexes, say j_2, is zero, the second and fourth term in (2.36) are not present. Hence the most stringent restriction is imposed by the positivity of $|R_{-+}|^2$, which implies:

$$E_0 \geq j_1 + 1\,, \qquad \text{for } j_2 = 0 \quad j_1 \neq 0\,. \tag{2.44}$$

The case $j_1 = j_2 = 0$ is more subtle. From (2.41) we are induced to think that $E_0 \geq 0$ is sufficient, however with some additional steps one can show that the necessary condition is more restrictive [44]:

$$E_0 \geq 1\,, \qquad \text{for } j_1 = j_2 = 0\,. \tag{2.45}$$

For completeness we report the unitarity bound for other dimensions [44, 45]:

$$D = 2: \quad E_0 \geq j\,, \qquad D = 3: \quad \begin{cases} E_0 \geq j + 1 (j \geq 1) \\ E_0 \geq 1 \quad (j = 1/2) \\ E_0 \geq 0 \quad (j = 0) \end{cases} \tag{2.46}$$

In addition, for any D, there is a state with $\Delta = j_i = 0$.
We conclude this section noting that the usual relation between J_{AB} and the conformal algebra generators would imply

$$\mathcal{H} = \frac{1}{2}(P_0 + K_0)\,. \tag{2.47}$$

hence the we would conclude that the unitarity bounds constrain the eigenvalues of the above combination. On the other hand we can prove that $i\mathcal{H}$, $J_a^{\pm}$ and M_{ab} satisfy the euclidean conformal algebra, and $i\mathcal{H}$ plays the role of the dilatation operators. Note however that $i\mathcal{H}$ is not hermitian thus the representation of this euclidean algebra is not unitary. Nevertheless its dilatation operator is bounded. When Wick rotating to Minkowsky this bound translates in a bound on the states dimension in a unitary representation.

2.3 Coordinates invariants and configurations in space-time

We devote this section to review how we can construct a conformal invariant combination starting from a set of coordinates. Clearly the only combinations invariant under the Poincaré group have the form $|x_i - x_j|$, hence at least two four-vectors are needed. Dilatation requires ratios of differences, hence at least three four-vectors are required in order to have non-trivial ratios. The invariance under conformal boost is more involved. Indeed the simple ratio $|x_i - x_j|/|x_k - x_l|$ is not invariant but instead transforms in the following way:

$$\frac{|x_i - x_j|}{|x_k - x_l|} \longrightarrow \frac{|x_i - x_j|}{|x_k - x_l|} \frac{(1 + 2c \cdot x_k + c^2 x_k^2)^{1/2}(1 + 2c \cdot x_l + c^2 x_l^2)^{1/2}}{(1 + 2c \cdot x_i + c^2 x_i^2)^{1/2}(1 + 2c \cdot x_j + c^2 x_j^2)^{1/2}} \tag{2.48}$$

Hence in order to have an invariant combination we must have i, j, k, l all different and consider

$$u = \frac{|x_i - x_j|^2 |x_k - x_l|^2}{|x_i - x_k|^2 |x_j - x_l|^2} \quad \text{or} \quad v = \frac{|x_i - x_l|^2 |x_k - x_j|^2}{|x_i - x_k|^2 |x_j - x_l|^2} \tag{2.49}$$

In general the number of *conformal invariants* we can construct out of N four-vectors is $N(N-3)/2$, although they are not always independent.

We now show that given three points with space-like separation, they can always be put on a line. Indeed, using the translation we can put one of the points, say P_1 in the origin. Since the points are space-like separated using a Lorentz boost we can set to zero the time component of P_2. Hence, using a rotation we can bring it to $P_2 = (0, 0, 0, p_2)$. Finally, with a boost in the direction orthogonal to the vector joining $P_1 - P_2$, the (x, y) plane in the specific, we can set to zero also the time component of P_3 and bring it to into the plane (y,z) with a rotation around $\hat{z}$. The final configuration (modulo a dilatation) is

$$P_1^\mu = (0, 0, 0, 0)\,, \qquad P_2^\mu = (0, 0, 0, 1)\,, \qquad P_3^\mu = (0, 0, p_3^y, p_3^z) \tag{2.50}$$

The above points lie on a plane with $t = x = 0$.

If we had an additional point we could have used a further boost in the $\hat{x}$ direction to set its time component to zero, but in this case the points could not be put on a 2-plane with the only use of the Lorentz group.

The final step is to use a conformal boost on the (y, z) plane to bring the three points (2.50) on a line. The existence of such a transformation is obvious since we can always find a circle passing through them. Setting the origin in the center of such circle then a conformal boost with parameter equal to one of the points of the circle sends the the circle into a straight line, as depicted in Fig. 2.1. More explicitly, a conformal boost with parameter

$$b^\mu = (0, 0, b^y, 0) \qquad b^y = \frac{p_3^y}{p_3^z(1 - p_3^z) - (p_3^y)^2} \tag{2.51}$$

keeps the origin fixed[2] and sets $\frac{p_2'^y}{p_3'^y} = \frac{p_2'^z}{p_3'^z}$ (hence vectors P_2 and P_3 are parallel in the new coordinates). Upon a rotation and a dilatation, we have

$$P_1 = (0, 0, 0, 0)\,, \qquad P_2 = (0, 0, 0, 1)\,, \qquad P_3 = (0, 0, 0, p_3^z)\,. \tag{2.52}$$

It is also possible to send P_3 to infinity through a conformal boost in the $\hat{z}$ direction with parameter $-p_3^z$.

[2]We recall that conformal boosts belong to the stability group of the origin.

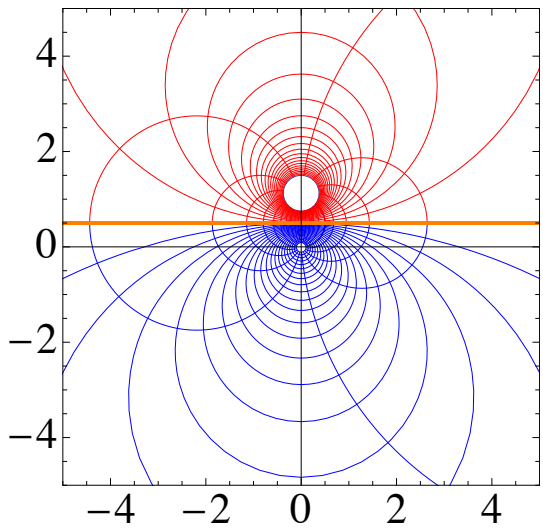


Figure 2.1: *Graphic representation of the effect of a conformal boost with parameter* $b^\mu = (0,0,0,1)$ *on the plane* (y,z)*. The circumference of radius* $r = 1$ *is sent into a straight line (orange thick line) while circles with radius* $r > 1$ $(r < 1)$ *are sent in the upper (lower) half plane and correspond to the red (blue) lines.*

Let us now turn to the more interesting case of four points. As discussed before we can always achieve the configuration where all the point lie on the hyperplane $t = 0$. Following the same procedure as before we can a set three point on the $\hat{z}$ direction (with P_4 at infinity). Upon a rotation we finally have

$$P_1 = (0,0,0,0)\,, \qquad P_2 = (0,0,p_2^y,p_2^z)\,, \qquad P_3 = (0,0,0,1) \qquad P_4 = (0,0,0,\infty) \tag{2.53}$$

This particular choice make manifest the presence of only two degrees of freedom. The above configuration corresponds to a sort of gauge fixing of the conformal symmetry. The harmonic ratios (2.49) are particularly simple in this case:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = x_2^2 + y_2^2\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (x_2 - 1)^2 + y_2^2 \tag{2.54}$$

where we introduced the notation $x_{ij} \doteqdot |x_i - x_j|$. Let us notice that the invariant u corresponds to the norm of the coordinates of the point p_2 in the configuration (2.53).

The above discussion makes manifest the possibility of reducing four mutually space-like points on a two-dimensional plane. The conformal transformations that leave a two dimensional plane

invariant contains two translations, a rotation, the dilatation and two conformal boost. In what follows it will be useful to implement these transformations on complex coordinates: let us map the two-dimensional plane where the points live into a complex plane

$$\text{ex:} \quad P_2 = (0, 0, p_2^y, p_2^z) \quad \longrightarrow \quad z_4 = p_2^y + ip_2^z \tag{2.55}$$

Then is trivial to check that the set of transformations parametrized by

$$z' = \frac{az + b}{cz + d}\,, \qquad a,\, b,\, c,\, d \in \mathbb{C} \tag{2.56}$$

reproduces all the listed transformations. For instance conformal boosts are given by the choice $a = c = 1, b = 0$.

Using the parametrization (2.56) of the conformal transformations we now prove that given four mutually space-like points in four dimensions they can always be brought to a parallelogram configuration. The proof proceeds as follows:

- we reduce the four points to the configuration (2.53) with the procedure explained above;
- we map the problem to the complex plane;
- we then show that given a parallelogram we can bring the vertices to the same configuration (2.53); moreover, spanning all the possible parallelograms, we can obtain any value for $z_2 = p_2^y + ip_2^z$.
- since conformal transformations are invertible we argue that the thesis is true.

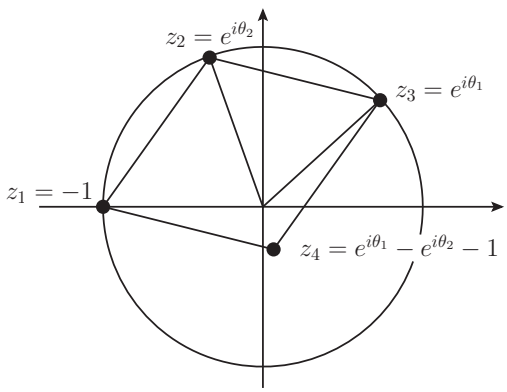


Figure 2.2: *Parallelogram configuration of four point x_i.*

Let us therefore start from four points in a parallelogram configuration. Modulo a rotation, a translation and a dilatation we can always achieve the disposition where three vertices lie on the upper semi-circumference of unitary radius, with one of them sitting on the point $(-1, 0)$, see Fig. 2.2 . This is possible since one of the internal angles of the parallelogram is surely not acute, thus the two sides defining it are inscribable in a semi-circumference. Let us parametrize the vertices with the points:

$$z_1 = -1\,, \qquad z_2 = e^{i\theta_1}\,, \qquad z_3 = e^{i\theta_2}\,, \qquad z_4 = e^{i\theta_1} - e^{i\theta_2} - 1\,. \tag{2.57}$$

As a first step we find a conformal transformation of the form (2.56) such that the circle is sent into the real line $z = \bar{z}$. In addition we require that z_1 in sent in the origin while the interior of the circle is sent in the upper plane. This can be easily obtained through the transformation

$$w(z) = i\frac{1+z}{1-z}\,, \qquad w(-1) = 0\,, \qquad \lim_{z\to 1} w(z) = \infty\,, \qquad w(0) = i\,. \tag{2.58}$$

We can also track the position of the other vertices in the w-plane, for instance:

$$w\left(e^{i\theta_i}\right) = -\cot\left(\frac{\theta_i}{2}\right)\,, \qquad i = 1, 2\,. \tag{2.59}$$

Next we perform a second transformation in order to achieve the configuration (2.53). In particular we send $w(z_2)$ in 1 and $w(z_3)$ to infinity by the transformation

$$w'(w) = \left(\frac{\cot(\theta_2/2) - \cot(\theta_1/2)}{\cot(\theta_2/2)\cot(\theta_1/2)}\right)\frac{w}{1 + \frac{w}{\cot(\theta_1/2)}} \tag{2.60}$$

Finally we can extract $w'(w(z_4))$ and explore the image of the z_4 point when varying θ_1, θ_2:

$$w'(w(z_4)) = -\frac{\sin^2\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos^2\left(\frac{\theta_2}{2}\right)}\,e^{i\theta_1} \tag{2.61}$$

In order not to double count the configuration the range of the parameters must be taken:

$$\theta_2 \in [-\pi,\, \pi]\,, \qquad \theta_1 \in [-\pi, \theta_2]\,; \tag{2.62}$$

One can verifies that for the values of θ_1, θ_2 in (2.62), the point $w'(w(z_4))$ covers almost the entire complex plane. The only region left out is the open interval of the real axis $w' = \bar{w}' \in (1, \infty)$. This because we have chosen to preserve the order, hence the line joining $w'(w(z_2)) = 1$ and $w'(w(z_3)) = \infty$ cannot cross $w'(w(z_4))$. Recall that in a configuration like (2.53) the invariants u and v can be easily computed:

$$u = |w'(w(z_4))|^2 \qquad u = |1 - w'(w(z_4))|^2\,. \tag{2.63}$$

We finally notice that whenever the point lie on a single line the associated parallelogram degenerates to a rectangle. This because four points on a line can be inscribed in a circle. In this sub-case the harmonic ratios have the form:

$$u = \cos^4\theta\,, \qquad v = \sin^4\theta\,, \tag{2.64}$$

where 2θ is the angle at the intersection of the diagonals. Notice that a rectangle is identified by the constraint

$$(1+u-v)^2 - 4u = 0 \tag{2.65}$$

The consequence of the above relation is particularly nice if we introduce the variables

$$u = z\bar{z}\,, \qquad v = (1-z)(1-\bar{z}) \tag{2.66}$$

$$z, \bar{z} = \frac{1}{2}\left(1+u-v \pm \sqrt{(1+u-v)^2 - 4u}\right) \tag{2.67}$$

Notice that z, $\bar{z}$ in the case in exam are complex and one the the conjugate of the other. This because we restricted four points mutually space-like, which imply $u, v > 0$. If we relax this assumption then this statement is not true any more. In particular, in the configuration (2.53) we have:

$$z = x + i|y| \tag{2.68}$$

In these coordinates the rectangle corresponds to $z = \bar{z} = \cos^2(\theta)$, reals. Finally we have that the square coincides with $z = \bar{z} = 1/2$.

Chapter 3

Conformal Quantum Field Theory

3.1 Representations of the Conformal Algebra

3.1.1 Representation on Fields

Let us now consider the representation of the conformal algebra on a set of fields collectively called Φ_α. In order to construct a general representation we need to compute the action of the generator on $\Phi_\alpha(x)$. As a first step we compute the action of the stability group $x = 0$ on $\Phi_\alpha(0)$. Once this is known we can generalize the construction in the following way ([38]):

$$\Phi_\alpha(x) = e^{-iPx}\Phi_\alpha(0)e^{iPx} \quad \Rightarrow \quad [G, \Phi_\alpha(x)] = e^{-iPx}[\widetilde{G}, \Phi_\alpha(0)]e^{iPx} \tag{3.1}$$

where, making use of the Baker-Campbell-Hausdorff expansion, we have

$$\widetilde{G} \doteqdot e^{iPx}Ge^{-iPx} = \sum_n \frac{(i)^n}{n!} x_{\mu_1} \dots x_{\mu_n} \left[P^{\mu_1}, [P^{\mu_2}, \dots [P^{\mu_n}, G] \dots]\right] \tag{3.2}$$

The resummation of the above series is straightforward whenever $[P_\mu, G] \propto P_\nu$. In this case the infinite series can be truncated to the linear order. This happens for the generators D, $M_{\mu\nu}$. For what concerns K_μ, the series must be extended to the second term. Finally,

$$\begin{aligned}
&\widetilde{D} = D + x_\mu P^\mu \,, \\
&\widetilde{M}_{\mu\nu} = M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \,, \\
&\widetilde{K}_\mu = K_\mu + 2x_\mu D + 2x^\rho M_{\rho\mu} + 2x_\mu x^\rho P_\rho - x^2 P_\mu \,,
\end{aligned} \tag{3.3}$$

and we obtain the generator representation on fields:

$$
\begin{aligned}
&[P_\mu, \Phi_\alpha(x)] = i\partial_\mu \Phi_\alpha(x)\,, \\
&[D, \Phi_\alpha(x)] = i\Delta\Phi_\alpha(x) + ix^\mu \partial_\mu \Phi_\alpha(x)\,, \\
&[M_{\mu\nu}, \Phi_\alpha(x)] = i(S_{\mu\nu})_{\alpha\beta}\Phi_\beta(x) - i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi_\alpha(x)\,, \\
&[K_\mu, \Phi_\alpha(x)] = i\mathcal{K}_\mu\Phi_\alpha(x) - 2ix_\mu\Delta\Phi_\alpha(x) + 2x^\rho i(S_{\rho\mu})_{\alpha\beta}\Phi_\beta(x) + i\left(2x_\mu x^\rho\partial_\rho\Phi_\alpha(x) - x^2\partial_\mu\Phi_\alpha(x)\right)\,,
\end{aligned}
\tag{3.4}
$$

where we have introduced the quantities

$$
[D, \Phi_\alpha(0)] = i\Delta\Phi_\alpha(0)\,, \qquad [M_{\mu\nu}, \Phi_\alpha(0)] = i(S_{\mu\nu})_{\alpha\beta}\Phi_\beta(0)\,, \qquad [K_\mu, \Phi_\alpha(0)] = i\mathcal{K}_\mu\Phi_\alpha(0)\,. \tag{3.5}
$$

Let us discuss the representations of the stability group of $x = 0$. In a given irreducible representation of the Lorentz group the generator K_μ vanishes identically. Indeed, D being a Lorentz scalar, by Shur's Lemma, it must to be proportional to the identity, since it commutes with all the generators of the representation. Thus, the commutation relation $[D, K_\mu] = -iK_\mu$ requires $K_\mu = 0$.

Let now start from a reducible *finite dimensional* Lorentz representation. In this case K_μ can be different form zero but it must be is nilpotent. This because K_μ acts as lowering operator for the dilatation: if an operator O has dimension Δ than $[K_\mu, O]$ has dimension $\Delta - 1$. Since the representation is finite dimensional by assumptions, the repeated action of K_μ must give zero after a finite number of steps.

Let us now consider representation of the entire conformal group. The generator P_μ acts as raising operator with respect to the dilatations generator. Hence we conclude that representations on fields cannot be finite dimensional because P_μ cannot be nilpotent. The same conclusion can be achieved requiring the representation to be unitary since the conformal group is non-compact and cannot have unitary finite dimensional representation, as already discussed in Section 2.2.

Following to the above reasoning we can classify representations of the conformal group according to the Lorentz quantum number and the scaling dimension of the lowest dimension operator appearing in the representation. Such an operator is called *primary field* while all the other operators in the representation can be obtained acting with P_μ and are denoted *descendants*.

3.1.2 Transformation properties of Primary fields

As we will show in details in the following sections conformal symmetry constrains the form of correlation functions of two and three operators. Their form is determined once the transfor-

mation properties of the operators are known. In this section we derive them starting from the manifestly covariant formalism introduced in Section 2.1 2.2. Again we restrict to four dimension for concreteness.

We recall that the Minkowski space can be related to the 5-dimensional surface (see for instance [35, 38, 36, 41])

$$g_{AB}\, y^A\, y^B = 0\,, \qquad y^A \in \mathbb{R}^6 \qquad g_{AB} = \mathrm{diag}(+,-,-,-,-,+)\,. \tag{3.6}$$

The above cone is invariant under transformations of $O(4,2)$ and under the rescaling $y^A \to \lambda y^A$, which commutes with the orthogonal group. The projective space formed by the rays of the cone (3.6) is four dimensional and isomorphic to the Minkowski space. The relation between a point in the projective space and a vector in Minkowski is given by the relation (2.13).

In order to construct a field on Minkowski space we need a function defined on the projective space: this can be obtained starting from an homogenous field defined on the cone $\chi(y_A)$:

$$\chi(\lambda y^A) = \lambda^n \chi(y^A)\,, \quad \Rightarrow \quad y^A \frac{\partial}{\partial y^A}\chi(y^A) = -n\chi(y^A)\,. \tag{3.7}$$

Let us take the field $\chi(y^A)$ transforming according to an irreducible representation of the $O(4,2)$ group

$$\chi'(y') = e^{-i\epsilon^{AB} s_{AB}}\chi(y) \simeq \left(1 - i\epsilon^{AB}(s_{AB} + L_{AB}\right)\chi(y')\,, \tag{3.8}$$

where ϵ^{AB} are the infinitesimal parameters of the transformations, s_{AB} are the generators of some finite dimensional irreducible representation of the symmetry group and

$$L_{AB} = -i\left(y_A \frac{\partial}{\partial y^B} - y_B \frac{\partial}{\partial y^A}\right)\,, \tag{3.9}$$

is the differential representation of the generators. Since all irreducible representations of the group $O(4,2) \sim SU(2,2)$ are classified, the form of the generators s_{AB} is known.

We now proceed to the construction of a general field defined on the Minkowski space, such that its infinitesimal transformation properties under the conformal group reproduce eq. (3.4). Let us start expressing the function $\chi(y^A)$ as a function of

$$y^+ = y^5 + y^6\,, \qquad x^\mu = \frac{y^\mu}{y^+}\,, \qquad \mu = 1,..,4\,. \tag{3.10}$$

In principle we should trade the 6 variables y^A for 6 new independent variables; in practice we can choose the sixth to be $y_A y^A$ which vanishes identically on the cone. With the above

definitions we have (we systematically drop $y_A y^A$):

$$\begin{aligned}\frac{\partial}{\partial y^A} &= \left(\frac{\partial x^\mu}{\partial y^A}\right)\frac{\partial}{\partial x^\mu} + \left(\frac{\partial y^+}{\partial y^A}\right)\frac{\partial}{\partial y^+} \\ &= \frac{1}{y^+}\left(\delta^\mu_A - (\delta^5_A + \delta^6_A)x^\mu\right)\frac{\partial}{\partial x^\mu} + (\delta^5_A + \delta^6_A)\frac{\partial}{\partial y^+}\,. \end{aligned} \tag{3.11}$$

Notice that in the new coordinates the homogeneity condition 3.7 reads

$$y_A\frac{\partial}{\partial y_A}\chi(y) = y_+\frac{\partial}{\partial y^+}\chi(x,y^+) = -n\,\chi(x,y^+)\,. \tag{3.12}$$

In terms of the new coordinates the differential generators L_{AB} decompose as:

$$\begin{aligned} L_{\mu\nu} &= -i\left(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\right)\,, \\ L_{\mu 5} - L_{\mu 6} &= i\frac{\partial}{\partial x^\mu}\,, \\ L_{\mu 5} + L_{\mu 6} &= -2ix_\mu x^\rho\frac{\partial}{\partial x^\rho} - 2ix_\mu n + ix^2\frac{\partial}{\partial x^\mu}\,, \\ L_{56} &= i\left(x^\rho\frac{\partial}{\partial x^\rho} + n\right)\,. \end{aligned} \tag{3.13}$$

The above generators have a familiar form: in particular the first two lines represent the generators of Lorentz transformations and translations. Finally we can construct a field depending exclusively on the variables x^μ multiplying $\chi(x,y^+)$ by a pre-factor:

$$\phi(x) = (y^+)^n\chi(x,y^+) \qquad \text{such that} \qquad \frac{\partial}{\partial y^+}\phi(x) = 0\,. \tag{3.14}$$

Finally, let us construct a field that reproduces the correct transformation laws. Under a translation, the function $\phi(x)$ changes according to the following relation (see eq. 3.8):

$$\delta\phi(x) = -i\epsilon^{\mu 5}(L_{\mu 5} - L_{\mu 6} + s_{\mu 5} - s_{\mu 6})\phi(x) = -i\epsilon^{\mu 5}\left(-i\frac{\partial}{\partial x^\mu} + (s_{\mu 5} - s_{\mu 6})\right)\phi(x)\,. \tag{3.15}$$

In order to implement the right transformations we can define

$$\Phi(x) = e^{ix^\mu(s_{\mu 5} - s_{\mu 6})}\phi(x)\,. \tag{3.16}$$

Exploiting the property

$$e^{ix^\mu(s_{\mu 5} - s_{\mu 6})}\frac{\partial}{\partial x^\rho}e^{-ix^\nu(s_{\nu 5} - s_{\nu 6})} = \frac{\partial}{\partial x^\rho} - i(s_{\rho 5} - s_{\rho 6}) \tag{3.17}$$

we can now prove that $\Phi(x)$ has the desired transformation properties under translations:

$$\begin{aligned}\delta\Phi(x) &= i\varepsilon^\mu[J_{\mu 5} - J_{\mu 6}, \Phi(x)] = i\varepsilon^\mu e^{ix^\mu(s_{\mu 5} - s_{\mu 6})}[J_{\mu 5} - J_{\mu 6}, \phi(x)] \\ &= \varepsilon^\mu e^{ix^\mu(s_{\mu 5} - s_{\mu 6})}\left(-i\frac{\partial}{\partial x^\mu} + (s_{\mu 5} - s_{\mu 6})\right)\phi(x) = -i\varepsilon^\mu\frac{\partial}{\partial x^\mu}\Phi(x) \end{aligned} \tag{3.18}$$

One can verify that $\Phi(x)$ reproduces all the conformal transformations at infinitesimal level. We will see in the following that in the case of non scalar representations the above definition must be supplemented by a tranversality condition which ensures that the four dimensional field contains the right number of degrees of freedom.
In order to obtain the finite form of the transformations we must specify the form of the matrices s_{AB}. We present it explicitly for scalars, vectors and we generalize for tensors. A similar procedure can be carried out for any other representation such as fermions [41].

Scalar Fields

As a first simple example let us consider a Lorentz scalar fields. correspondingly the field $\chi(y)$ is scalar under $O(4,2)$, that is to say $s_{AB} = 0$. Hence the quantity

$$\Phi(x) = (y^+)^n \chi(x, y^+) \tag{3.19}$$

transforms under dilatations ($y' = e^{i(\log\lambda)J_{56}} y$) as

$$\Phi'(x) = (y^+)^n \chi'(x, y^+) = (y^+)^n \chi(\lambda^{-1}x, \lambda y^+) = \lambda^{-n}(y^+)^n \chi(\lambda^{-1}x, y^+) = \lambda^{-n}\Phi(\lambda^{-1}x) \tag{3.20}$$

where we used the homogeneity of the field $\chi(x, y^+)$. The degree of homogeneity n is therefore the scaling dimension Δ of the field. Equivalently we can write:

$$\Phi'(x) = \lambda^{\Delta}\Phi(\lambda x) \quad \text{when} \quad x' = \lambda^{-1}x\,. \tag{3.21}$$

Under conformal boosts ($y' = e^{-i\alpha^\mu(J_{\mu 5}+J_{\mu 6})} y$) we have instead:

$$\begin{aligned}\Phi'(x) &= (y^+)^n \chi\left(\frac{x^\mu - \alpha^\mu x^2}{1 - 2x\cdot\alpha + x^2\alpha^2}, y^+(1 - 2x\cdot\alpha + x^2\alpha^2)\right) \\ &= (1 - 2x\cdot\alpha + x^2\alpha^2)^{-n}\Phi\left(\frac{x^\mu - \alpha^\mu x^2}{1 - 2x\cdot\alpha + x^2\alpha^2}\right)\end{aligned} \tag{3.22}$$

Vector Fields

In order to obtain a representation transforming as a four vector under the Lorentz group we start from a six dimensional vector fields $V^A(y)$ transforming under the defining representation of $SO(4,2)$. The generators are given by eq. (2.14). In addition to the homogeneity condition we also require the following transversality condition:

$$y_A V^A(y) = 0 \tag{3.23}$$

The above condition ensures that the differential operator $V^A(y)\frac{\partial}{\partial y^A}$ leaves invariant the cone $y^A y_A = 0$. Moreover this condition guarantees that the number of degrees of freedom contained in $V^A(y)$ matches those of a four dimensional vector field ([41]).
Let us finally consider the four-vector:

$$v^\mu(x^\nu) = (y^+)^n \left(e^{ix^\mu(J_{\mu 5}-J_{\mu 6})}\right)^\mu{}_A V^A(x^\nu, y^+)\,. \tag{3.24}$$

In the present case the exponential of $ix^\mu(J_{\mu 5} - J_{\mu 6})$ is particularly simple:

$$\left(e^{ix^\mu(J_{\mu 5}-J_{\mu 6})}\right) = \left(\begin{array}{c|c|c} \mathbb{1}_4 & -x^\mu & -x^\mu \\ \hline -x_\nu & 1+x^2/2 & x^2/2 \\ \hline x^\nu & -x^2/2 & 1-x^2/2 \end{array}\right)\,. \tag{3.25}$$

In the following we will need the useful relation ([41]):

$$\left(e^{ix^\mu(J_{\mu 5}-J_{\mu 6})}\right)^\mu{}_A = \delta^\mu_A - x^\mu\left(\delta^A_5 + \delta^A_6\right) = y^+\frac{\partial x^\mu}{\partial y^A}\,. \tag{3.26}$$

Substituting the above expression in (3.24) we obtain:

$$v^\mu(x) = (y^+)^{n+1}\frac{\partial x^\mu}{\partial y^A}V^A(x, y^+)\,. \tag{3.27}$$

In order to extract the transformation properties of the above field let us introduce some notation:

$$\begin{aligned} &y'^A = \Lambda^A{}_B\, y^B\,, \qquad y''^A = (\Lambda^{-1})^A{}_B\, y^B = \Lambda_B{}^A\, y^B\,, \qquad \frac{\partial}{\partial y''^A} = \Lambda^B{}_A\frac{\partial}{\partial y^B}\,, \\ &x'^\mu = f^\mu(x), \qquad x''^\mu = (f^{-1})^\mu(x)\,, \qquad \frac{\partial x^\mu}{\partial x''^\rho} = \frac{\partial x'^\mu}{\partial x^\rho}(x'') \end{aligned} \tag{3.28}$$

A final comment is needed:

$$\frac{\partial x^\mu}{\partial y^\rho} = \frac{1}{y^+}\delta^\mu_\rho\,, \qquad \frac{\partial x'^\mu}{\partial y'^\rho} = \frac{1}{y'^+}\delta^\mu_\rho \tag{3.29}$$

$$\frac{\partial x'^\mu}{\partial y'^\rho} = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\frac{\partial x^\nu}{\partial y^\sigma}\Lambda_\rho{}^\sigma = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\frac{1}{y^+}\Lambda_\rho{}^\nu \quad \Rightarrow \quad \left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \frac{y^+}{y'^+}\Lambda_\rho{}^\mu \tag{3.30}$$

Collecting all the results we have:

$$\begin{aligned} v'^\mu(x) &= U^\dagger v^\mu(x) U = (y^+)^{n+1}\frac{\partial x^\mu}{\partial y^A}\Lambda^A{}_B\, V^B(x'', y''^+) = (y^+)^{n+1}\left(\frac{y''^+}{y^+}\right)^{-n}\frac{\partial x^\mu}{\partial y''^A}V^A(x'', y^+) \\ &= (y^+)^{n+1}\frac{\partial x'^\mu}{\partial x^\rho}\left(\frac{y''^+}{y^+}\right)^{-n}\frac{\partial x''^\rho}{\partial y''^A}V^A(x'', y^+) = \left(\frac{y''^+}{y^+}\right)^{-n-1}\frac{\partial x'^\mu}{\partial x^\rho}v^\rho(x'') \\ &= \left|\frac{\partial x'}{\partial x}(x'')\right|^{-\frac{n+1}{4}}\frac{\partial x'^\mu}{\partial x^\rho}(x'')\, v^\rho(x'') \end{aligned} \tag{3.31}$$

Notice the important dependence on x''. The above equation states that the field v^μ transforms as a vector field with density $-(n+1)/4$. Let us check the above relation in the simple case of dilatation: $x' = \lambda^{-1}x$:

$$v'^\mu(x) = \lambda^n v^\mu(\lambda x) \tag{3.32}$$

Hence the conformal dimension of the vector field is $\Delta = n$. Under conformal transformations instead, when $x' = (x + \alpha x^2)/(1 + 2\alpha x + x^2\alpha^2)$:

$$\begin{aligned} v'^\mu(x) &= (1 + 2x'' \cdot \alpha + x''^2\alpha^2)^{n+1} \frac{\partial x'^\mu}{\partial x^\nu} v^\nu \left(\frac{x^\mu - \alpha^\mu x^2}{1 - 2x \cdot \alpha + x^2\alpha^2} \right) \\ &= (1 - 2x \cdot \alpha + x^2\alpha^2)^{-(n+1)} \frac{\partial x'^\mu}{\partial x^\nu} v^\nu \left(\frac{x^\mu - \alpha^\mu x^2}{1 - 2x \cdot \alpha + x^2\alpha^2} \right) \end{aligned} \tag{3.33}$$

Tensor Fields

The analysis of the previous subsection can be extended to tensors of arbitrary rank: given a tensor transforming under $SO(4,2)$ as:

$$T'^{A_1 \dots A_k}(y) = \Lambda^{A_1}_{\ B_1} \dots \Lambda^{A_k}_{\ B_k} T^{B_1 \dots B_k}(\Lambda^{-1} y) \tag{3.34}$$

and satisfying the homogeneity condition and the transverse conditions:

$$T^{A_1 \dots A_k}(\lambda y) = \lambda^{-n} T^{A_1 \dots A_k}(y)\,, \qquad T^{A_1 \dots A_i \dots A_k}(y) y_{A_i} = 0 \quad \text{for any } i \tag{3.35}$$

we can construct a 4-dimensional tensor:

$$t^{\mu_1 \dots \mu_k}(x) = (y^+)^{n+k} \frac{\partial x^{\mu_1}}{\partial y^{A_1}} \dots \frac{\partial x^{\mu_1}}{\partial y^{A_1}} T^{A_1 \dots A_k}(x, y^+) \tag{3.36}$$

and we can show ([41]) that it transforms as a tensor density of degree $(n+k)/4$:

$$t'^{\mu_1 \dots \mu_k}(x) = \left| \frac{\partial x'}{\partial x}(x'') \right|^{-\frac{n+k}{4}} \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \dots \frac{\partial x'^{\mu_k}}{\partial x^{\rho_k}} t^{\mu_1 \dots \mu_k}(x'') \tag{3.37}$$

where again if $x' = f(x)$, then $x'' = f^{-1}(x)$. Moreover it is clear that symmetry properties and tracelessness properties of $T^{A_1 \dots A_k}(y)$ corresponds to the same properties for $t^{\mu_1 \dots \mu_k}(x)$, since for instance ([41])

$$\eta_{\mu_1\mu_2} t^{\mu_1 \dots \mu_k}(x) \propto \eta_{A_1 A_2} T^{A_1 \dots A_k}(y) \tag{3.38}$$

hence irreducible representations of the six-dimensional group are also irreducible representations of the four dimensional Lorentz group.

3.2 Correlation Functions

Conformal symmetry highly restrict the form of correlation functions. We postpone a rigorous definition of correlation functions of a general Conformal invariant theory to another section: for the present discussion we can assume that a the theory is defined in terms of an action $\mathcal{S}[\phi]$. We also denote $\{O_i\}$ the set of primary operators. The euclidean correlation function of n primary operator in this case can be defined as

$$\langle O_{i_1} \dots O_{i_n} \rangle \doteqdot \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, O_{i_1} \dots O_{i_n} \, e^{-S[\Phi]} \tag{3.39}$$

where $\mathcal{Z}$ is the standard partition function. The left hand side (l.h.s.) of the above equation represents therefore the expectation value on a conformally invariant vacuum of a set of primary operators. In this section we demonstrate the following results:

$$\begin{aligned}
&\langle O_i(x_1)\, O_i(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta_i}} \\
&\langle t^{\mu_1 \dots \mu_l}(x_1)\, t^{\nu_1 \dots \nu_l}(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta_t}} \left(\left(\eta^{\mu_1 \nu_1} - 2 \frac{x_{12}^{\mu_1} x_{12}^{\nu_1}}{x_{12}^2} \right) \dots \left(\eta^{\mu_l \nu_l} - 2 \frac{x_{12}^{\mu_l} x_{12}^{\nu_l}}{x_{12}^2} \right) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \text{symmetrized - traces} \right) \\
&\langle O_1(x_1) O_2(x_2) t^{\mu_1 \dots \mu_l}(x_3) \rangle = C_{OOt} \frac{Z_{123}^{\mu_1} \dots . Z_{123}^{\mu_l} - \text{traces}}{x_{12}^{(\Delta_1 + \Delta_2 - \Delta_3 + l)/2} x_{13}^{(\Delta_1 - \Delta_2 + \Delta_t - l)/2} x_{23}^{(\Delta_2 + \Delta_t - \Delta_1 - l)/2}} \\
&Z_{ijk}^{\mu} = \frac{x_{ik}^{\mu}}{x_{ik}^2} - \frac{x_{jk}^{\mu}}{x_{jk}^2}
\end{aligned} \tag{3.40}$$

where $O_i(x)$ are scalar operators of dimension Δ_i and $t^{\mu_1 \dots \mu_l}(x)$ is a symmetric traceless tensor of rank l with dimension Δ_t. The above correlators are the only ones needed for the present work, however similar expression can be derived for generic set of fields [33].

Scalars

In order to keep the notation compact, we denote a generic coordinate transformation by $x'^{\mu} = f^{\mu}(x)$ and its inverse by $x''^{\mu} = (f^{-1})^{\mu}(x)$.

Let us begin considering the correlation functions of two scalar primary fields O_1 and O_2 with scaling dimensions Δ_1 and Δ_2. Inserting in the correlation function the unitary operator implementing the coordinate rescaling $x'^{\mu} = \lambda^{-1} x^{\mu}$ we obtain the relation

$$\langle O_1(x_1)\, O_2(x_2) \rangle = \langle O_1'(x_1)\, O_2'(x_2) \rangle = \lambda^{\Delta_1 + \Delta_2} \langle O_1(\lambda x_1)\, O_2(\lambda x_2) \rangle \,. \tag{3.41}$$

Poincaré invariance of the scalar fields forces the result to depend only on the combination $x_{12} = |x_1 - x_2|$ and the above relation states that it must be an homogenous function of degree $-\Delta_1 - \Delta_2$. The only function with this properties is $x_{12}^{-\Delta_1-\Delta_2}$. Considering in addition conformal boosts and recalling the transformation properties of x_{12} (see for instance eq. (2.48))

$$\begin{aligned}\langle O_1'(x_1)\, O_2'(x_2)\rangle &= \frac{1}{(1-2b\cdot x_1+b^2x_1^2)^{\Delta_1}(1-2b\cdot x_2+b^2x_2^2)^{\Delta_2}}\langle O_1(x_1'')\, O_2(x_2'')\rangle \\ &= \frac{1}{(1-2b\cdot x_1+b^2x_1^2)^{\Delta_1}(1-2b\cdot x_2+b^2x_2^2)^{\Delta_2}}\frac{C_{12}}{(x_{12}'')^{\Delta_1+\Delta_2}} \\ &= \frac{C_{12}}{(x_{12})^{\Delta_1+\Delta_2}}\frac{(1-2b\cdot x_1+b^2x_1^2)^{\frac{\Delta_1+\Delta_2}{2}}(1+2b\cdot x_2+b^2x_2^2)^{\frac{\Delta_1+\Delta_2}{2}}}{(1-2b\cdot x_1+b^2x_1^2)^{\Delta_1}(1-2b\cdot x_2+b^2x_2^2)^{\Delta_2}}\end{aligned} \tag{3.42}$$

The above relation can be verified with a non vanishing constant C_{12} if and only if $\Delta_1 = \Delta_2$. In all the other cases $C_{12} = 0$. Hence:

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta}}\,, & \text{if } \Delta_1 = \Delta_2 = \Delta \\ 0 & \text{otherwise}\end{cases} \tag{3.43}$$

In general there can be non vanishing correlation function between non identical operators: it is sufficient they have the same scaling dimension. On the other hand in unitary theories we can diagonalize the subspaces of operators with equal dimension and rescale the fields such that

$$\langle O_i(x_1)\, O_j(x_2)\rangle = \begin{cases} \frac{1}{|x_1-x_2|^{2\Delta}}\,, & \text{if } i = j \\ 0 & \text{if } i \neq j\end{cases} \tag{3.44}$$

Conformal symmetry also fixes the structure of three point function. Analogously as before coordinate rescaling restricts the generic form of the correlation function to

$$\langle O_1(x_1)\, O_2(x_2) O_3(x_3)\rangle = \frac{C_{123}}{x_{12}^a x_{13}^b x_{23}^c}\,, \qquad \text{and } a+b+c = \Delta_1+\Delta_2+\Delta_3 \tag{3.45}$$

In principle there could be a sum on different terms but we will see in the following that the coefficients a, b, c are completely fixed. Indeed conformal boost covariance forces the following relation:

$$\begin{aligned}&\langle O_1'(x_1)\, O_2'(x_2) O_3'(x_3)\rangle = \\ &\quad = \frac{C_{123}}{x_{12}^a x_{13}^b x_{23}^c}\frac{(1+2b\cdot x_1+b^2x_1^2)^{\frac{a+b}{2}}(1+2b\cdot x_2+b^2x_2^2)^{\frac{a+c}{2}}(1+2b\cdot x_3+b^2x_3^2)^{\frac{b+c}{2}}}{(1+2b\cdot x_1+b^2x_1^2)^{\Delta_1}(1+2b\cdot x_2+b^2x_2^2)^{\Delta_2}(1+2b\cdot x_3+b^2x_3^2)^{\Delta_3}}\,.\end{aligned} \tag{3.46}$$

The only solution satisfying the above constraint for any choice of b_μ is:

$$\langle O_1(x_1)\, O_2(x_2) O_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1-\Delta_2+\Delta_3} x_{23}^{-\Delta_1+\Delta_2+\Delta_3}}\,. \tag{3.47}$$

Finally we explore the correlation function of four scalar primary operators. As shown in Section 2.3 whenever we have four distinct point in the space-time there exist two independent conformal invariant combination, the harmonic ratios. Thus, although we can find a structure that transforms covariantly under conformal boosts and rescaling, it can be always multiplied by an arbitrary function of the two variables u, v. Hence, the general structure can be parametrized:

$$\langle O_1(x_1)\, O_2(x_2) O_3(x_3) O_4(x_4)\rangle = \frac{g(u,v)}{x_{12}^a x_{13}^b x_{14}^c x_{23}^d x_{24}^e x_{34}^f}\,. \tag{3.48}$$

with the constraints imposed by conformal symmetry:

$$a = \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 + f\,, \qquad c = 4\Delta_4 - e - f \tag{3.49}$$

$$b = \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 + e\,, \qquad d = -\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - e - f \tag{3.50}$$

$$\tag{3.51}$$

For instance we can take the following convenient parametrization:

$$\langle O_1(x_1)\, O_2(x_2) O_3(x_3) O_4(x_4)\rangle = \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_1-\Delta_2} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4} \frac{g(u,v)}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}}\,. \tag{3.52}$$

In the next section we will compute the two and three point function of non-scalar operators. A clever way to extract the general form of the these correlators is to start from the associate field defined on the six dimensional space ([41]). In order to appreciate better the power of this formalism let us repeat the computation of the two and three point function of scalar operators. Thus let us write

$$O_i(x_i) = (y_i^+)^{\Delta_i} \Phi_i(x_i, y_i^+) \tag{3.53}$$

where we recall that all the six dimensional fields are homogenous function of degree equal to minus the scaling dimension of the four dimensional fields. Hence the two point function of two scalar fields is proportional to

$$\langle \Phi_1(Y_1) \Phi_2(Y_2)\rangle \tag{3.54}$$

which can only be an $SO(4,2)$ invariant function of the coordinates. The only possibility is $(Y_1 \cdot Y_2)$. This also imposes the two scaling dimensions to be equal, $\Delta_1 = \Delta_2 = \Delta$, otherwise there would be a contradiction because of a different scaling behavior, The correct power $(Y_1 \cdot Y_2)$ of is fixed by the homogeneity condition:

$$\langle O_1(x_1)\, O_2(x_2)\rangle \frac{(y_1^+ y_2^+)^\Delta}{(Y_1 \cdot Y_2)^{2\Delta}} \tag{3.55}$$

Substituting the expression of Y_i in terms of the four dimensional coordinates (see later) we easily recover eq. (3.43). Similarly, the three point function can be reduces to the correlator of three six-dimensional fields:

$$\langle \Phi_1(Y_1)\Phi_2(Y_2)\Phi_3(Y_3)\rangle \,. \tag{3.56}$$

This time there are three possible invariants: $(Y_1 \cdot Y_2)\,(Y_1 \cdot Y_3)\,(Y_2 \cdot Y_2)$. Again the homogeneity condition completely fixes their dependence and the result coincides with eq. (3.47).

Vectors

Two point functions of non scalar operators and three pint functions oft two scalar fields and a third field with spin equal or greater that one are more complicated, since they carry Lorentz indices. Nevertheless they are completely fixed by conformal invariance. Let us start computing the three point function of two scalar operators and a vector.

$$\langle O_1(x_1)O_2(x_2)v^\mu(x_3)\rangle, \qquad \text{with:} \quad \Delta_{O_i} = \Delta_i \,,\ \Delta_v = \Delta_3 \,. \tag{3.57}$$

We directly make use of the six-dimensional formalism. Let us write:

$$O_i(x_i) = (y_i^+)^{\Delta_i}\Phi_i(x_i, y_i^+)\,, \qquad v^\mu(x_3) = (y_3^+)^{\Delta_3+1}\frac{\partial x_3^\mu}{\partial y_3^A}V^A(x, y^+) \tag{3.58}$$

where again all the six dimensional fields are homogenous function of degree equal to minus the scaling dimension of the four dimensional fields and that V^A is transverse: $V^A(Y_3)Y_{3A} = 0$. consider first the three point function. Hence

$$\langle \Phi_1(Y_1)\Phi_2(Y_2)V^A(Y_3)\rangle \tag{3.59}$$

can only be an $SO(4,2)$ invariant function of the coordinates times a transverse vector. We also recall that the coordinates Y_i^A are restricted on the cone $Y_i^A Y_{iA} = 0$, therefore the only transverse vectors are Y_3^A and $(Y_3 \cdot Y_2)Y_1^A - (Y_3 \cdot Y_1)Y_2^A$. Given the expression of v^μ in terms of V^A, only the second vector matters, since the Y_3^A gives no contribution in the contraction, since $y^A \frac{\partial x^\mu}{\partial y^A} = 0$. Using the homogeneity on the fields we can restrict the form of the three point function to the following expression:

$$\begin{aligned}
&\langle \Phi_1(Y_1)\Phi_2(Y_2)V^A(Y_3)\rangle = C_{OOv}\frac{(Y_3 \cdot Y_2)Y_1^A - (Y_3 \cdot Y_1)Y_2^A}{(Y_1 \cdot Y_2)^a (Y_1 \cdot Y_3)^b (Y_2 \cdot Y_3)^c} \\
&a = \frac{1}{2}\left(1 + \Delta_1 + \Delta_2 - \Delta_3\right)\,, \quad b = \frac{1}{2}\left(1 + \Delta_1 + \Delta_3 - \Delta_2\right)\,, \quad c = \frac{1}{2}\left(1 + \Delta_2 + \Delta_3 - \Delta_1\right)\,.
\end{aligned} \tag{3.60}$$

At this point we can easily reconstruct the four-dimensional three point function multiplying for the proper factors and using the relations:

$$(Y_i \cdot Y_j) = -\frac{1}{2} y_i^+ y_j^+ (x_i - x_j)^2 \,,$$
$$y_3^+ \, (Y_3 \cdot Y_i) \frac{\partial x_3^\mu}{\partial Y_3^A} Y_j^A = -\frac{1}{2}(y_i^+ y_j^+ y_3^+)(x_i - x_3)^2 (x_j - x_3)^\mu \tag{3.61}$$

The final results then reads:

$$\langle O_1(x_1) O_2(x_2) v^\mu(x_3) \rangle = C_{OOv} \frac{Z^\mu_{123}}{x_{12}^{(\Delta_1+\Delta_2-\Delta_3+1)/2} x_{13}^{(\Delta_1-\Delta_2+\Delta_3-1)/2} x_{23}^{(-\Delta_1+\Delta_2+\Delta_3-1)/2}}$$
$$Z^\mu_{ijk} = \frac{x^\mu_{ik}}{x^2_{ik}} - \frac{x^\mu_{jk}}{x^2_{jk}} \tag{3.62}$$

Notice the antisymmetry in the x_1, x_2 coordinate that make the correlator vanish in the case of equal bosonic scalar operators.

Similarly we can now derive the structure of the two point function of a vector. The more complicated index structure is somewhat compensated by the presence of we have less coordinates. The homogeneity of the six dimensional fields restrict the correlator to be non zero only if $\Delta_1 = \Delta_2$. In such a case it must have the form

$$\langle V_1^A(Y_1) V_2^B(Y_2) \rangle = \frac{1}{(Y_1 \cdot Y_2)^{2\Delta_1}} I(Y_1, Y_2)_{AB} \tag{3.63}$$

where $I(Y_1, Y_2)_{AB}$ must be an homogeneous tensor of degree zero satisfying the transversality conditions:

$$I(Y_1, Y_2)_{AB} Y_{1A} = I(Y_1, Y_2)_{AB} Y_{2B} = 0 \tag{3.64}$$

Such a tensor exists and has a unique expression(rescaling suitably the vectors V_i^A)[1]:

$$\eta^{AB} - \frac{Y_2^A Y_1^B}{(Y_1 \cdot Y_2)} \tag{3.65}$$

On the contrary if the dimensions are different there is no expression which is proportional to a tensor and has the correct scaling. Again we can use the relations in (3.61) and

$$(y_1^+)(y_2^+) \frac{\partial x_1^\mu}{\partial y_1^A} \frac{\partial x_2^\nu}{\partial y_2^B} \eta^{AB} = \eta^{\mu\nu} \tag{3.66}$$

to finally obtain:

$$\langle v_1^\mu(x_1) v_2^\nu(x_2) \rangle = \frac{1}{(x_{12}^{2\Delta_1}} \left(\eta^{\mu\nu} - 2 \frac{x^\mu_{12} x^\nu_{12}}{x^2_{12}} \right) . \tag{3.67}$$

As in the scalar case we can diagonalize the space of vectors with the same dimension in order to have non vanishing two point functions only among two identical fields.

[1] Again we neglect terms proportional to Y_1^A or Y_2^B since they give no contribution when construction the four dimensional fields.

Tensors

We now find the expression for the three point function of two scalars and one traceless symmetric tensor of rank two: again we start from the three point function of the six-dimensional operators

$$O_i(x_i) = (y_i^+)^{\Delta_i}\Phi_i(x_i, y_i^+)\,, \qquad t^{\mu\nu}(x_3) = (y_3^+)^{\Delta_3+2}\frac{\partial x_3^\mu}{\partial y_3^A}\frac{\partial x_3^\nu}{\partial y_3^B}T^{AB}(x, y^+) \tag{3.68}$$

and imposing the transversality, the tracelessness and the symmetry of T^{AB}, we obtain the general expression:

$$\begin{aligned}\langle\Phi_1(Y_1)\Phi_2(Y_2)T^{AB}(Y_3)\rangle = &\frac{C_{OOT}}{(Y_1\cdot Y_2)^{(\Delta_2+\Delta_3-\Delta_1)/2}(Y_1\cdot Y_3)^{(\Delta_2+\Delta_3-\Delta_1)/2}(Y_2\cdot Y_3)^{(\Delta_2+\Delta_3-\Delta_1)/2}}\times\\ &\left(\frac{\sin^2\theta}{2}\left(\eta^{AB}-\frac{Y_3^AY_2^B+Y_2^AY_3^B}{(Y_3\cdot Y_2)}\right)+\frac{\cos^2\theta}{2}\left(\eta^{AB}-\frac{Y_1^AY_3^B+Y_3^AY_1^B}{(Y_1\cdot Y_3)}\right)\right.\\ &\left.-\left(\frac{Y_1^AY_2^B+Y_2^AY_1^B}{(Y_1\cdot Y_2)}-Y_1^AY_1^B\frac{(Y_3\cdot Y_2)}{(Y_1\cdot Y_3)(Y_1\cdot Y_2)}-Y_2^AY_2^B\frac{(Y_3\cdot Y_1)}{(Y_2\cdot Y_3)(Y_1\cdot Y_2)}\right)\right).\end{aligned} \tag{3.69}$$

In the above θ is a free parameter which will disappear when passing to four dimensions. Indeed all the terms proportional to Y_3^A or Y_3^B vanish and the coefficients of η^{AB} sum to one half. Again using the relations (3.61) and an identity similar to (3.61) we obtain the simple result:

$$\begin{aligned}\langle O_1(x_1)O_2(x_2)t^{\mu\nu}(x_3)\rangle &= C_{OOt}\frac{Z^\mu_{123}Z^\nu_{123}-\eta^{\mu\nu}Z^2_{123}/4}{x_{12}^{(\Delta_1+\Delta_2-\Delta_3+2)/2}x_{13}^{(\Delta_1-\Delta_2+\Delta_3-2)/2}x_{23}^{(-\Delta_1+\Delta_2+\Delta_3-2)/2}}\\ Z^\mu_{ijk} &= \frac{x^\mu_{ik}}{x^2_{ik}}-\frac{x^\mu_{jk}}{x^2_{jk}}\end{aligned} \tag{3.70}$$

Finally, let us derive an expression for the two point function of two traceless symmetric tensors of rank two. As in the case of vectors the correlator will be non vanishing only when the dimensions are equal. In that case we can agin diagonalize the subspace. Hence let us consider directly

$$\langle T^{AB}(Y_1)T^{CD}(Y_2)\rangle = \frac{1}{(Y_1\cdot Y_2)^{2\Delta}}I^{ABCD}(Y_1, Y_2) \tag{3.71}$$

where the tensor I^{ABCD} must be homogeneous of degree zero, symmetric traceless in the the first pair and second pair of indices and transverse. This restrict the form to only one form[2]:

$$\begin{aligned}&\left(\eta^{AC}-\frac{Y_2^AY_1^C}{(Y_1\cdot Y_2)}\right)\left(\eta^{BD}-\frac{Y_2^BY_1^D}{(Y_1\cdot Y_2)}\right)+\left(\eta^{AD}-\frac{Y_2^AY_1^D}{(Y_1\cdot Y_2)}\right)\left(\eta^{BC}-\frac{Y_2^BY_1^C}{(Y_1\cdot Y_2)}\right)\\ &-2\left(\eta^{AB}-\frac{Y_2^AY_1^B+Y_1^AY_2^B}{(Y_1\cdot Y_2)}\right)\left(\eta^{CD}-\frac{Y_2^CY_1^D+Y_1^CY_2^D}{(Y_1\cdot Y_2)}\right)\end{aligned} \tag{3.72}$$

[2]This time we include also term that will give a vanishing contribution, otherwise the expression would not be traceless or symmetric

Although it is not straightforward, the above result can shown to give a result in agreement with the second expression in (3.40). Moreover, we can generalize to traceless symmetric tensors of arbitrary rank.

3.3 Hilbert space and the Operator Product Expansion

By assumption any quantum field theory admits a description in terms of local operator. In the case of a unitary quantum conformal field theory those operators can be classified according to their transformation properties under the conformal symmetry. As discussed in Section 3.1, unitary irreducible representations on fields of $SO(D,2)$ are infinite dimensional lowest weight representations and are completely characterized by the dimension Δ and the $SO(D-1,1)$ quantum numbers $(j_1,...j_n)$ associated to the primary operator. We denote $\Phi_m(x)$, $m = \{\Delta,(j_1,...j_n)\}$ the primary operator of the unitary irreducible representation $\mathcal{R}_m$. We will suppress the dependence on the $SO(D-1,1)$ indexes for compactness.
As we will see in the next chapters, any CFT must contain an infinite number of primary operators.

Let us now define $|0\rangle$ the conformal invariant vacuum state of the theory and let us introduce the set of states obtained acting with an operator $\Phi_m(x)$ on the vacuum state:

$$|m,x\rangle \equiv \Phi_m(x)|0\rangle\,. \tag{3.73}$$

It is not hard to check that the above states transform according to an irreducible representation with parameters m. More generically we can consider states obtained acting with several operators:

$$|\psi\rangle = \int d^Dx_1...d^Dx_n f(x_1,...,x_n)\Phi_{m_1}(x_1)...\Phi_{m_n}(x_n)|0\rangle\,, \tag{3.74}$$

and all their possible linear combination. It can be shown that the entire set of those states form an Hilbert space, which we will denote $\mathcal{H}$. We can decompose $\mathcal{H}$ in a direct sum of irreducible representations of the conformal group. A non trivial result consists in the statement that those irreducible representations are in one to one correspondence with the primary operators of the CFT: for any irreducible representation labeled by m, the states $|m,x\rangle$ represents a complete basis.
Let us investigate the consequence of the above result considering the simplest state built acting repetitively on vacuum with different fields. Clearly the resulting state won't belong to an

irreducible representation but rather can be expanded in a complete set of states. Thus:

$$\begin{aligned}\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)|0\rangle &= \sum_m \int d^4x\, C_{m_1m_2m}(x_1,\, x_2,\, x)|m, x> \\ &= \sum_m \int d^4x C_{m_1m_2m}(x_1,\, x_2,\, x)\Phi_m(x)|0\rangle\,; \end{aligned} \tag{3.75}$$

As we will show in the following, the functions $C_{m_1m_2m}(x_1,\, x_2,\, x)$ are fixed by conformal invariance modulo some multiplicative coefficient.

The above expression naturally leads naturally to conjecture the existence of an *Operator Product Expansion* (OPE), which we will denote

$$\Phi_{m_1}(x_1) \times \Phi_{m_2}(x_2) = \sum_m \int d^4x\, C_{m_1m_2m}(x_1, x_2, x)\Phi_m(x)\,. \tag{3.76}$$

When applied to the vacuum state the above expression reduces to the known decomposition of a general state into a complete basis. On the other hand when we act on a different state the OPE represents a more general statement and we have not proved its validity. In [46] it is shown that the OPE applied to the vacuum state, namely (3.75), converges in the strong sense when smeared with a set of test functions[3]

Due to this fundamental result, it is possible to define iteratively any correlation function [4] in terms of two and three point functions, that are completely fixed by conformal invariance. For instance, in this work we will mainly restrict our attention to four point functions. According to the above discussion, the correlator of four fields admits a convergent expansion of the form

$$\langle 0|\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3)\Phi_{m_4}(x_4)|0\rangle = \sum_m \int\, d^4x\, C_{m_3m_4m}(x_3, x_4, x)\langle 0|\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_m(x)|0\rangle\,. \tag{3.77}$$

In the next chapter we will show how to compute the terms of the sum in the right hand side of the above equation in closed form, in the simple case of four scalar operators Φ_{m_i}.

Any n-point function can be defined via an expansion similar to (3.77).

We conclude this section relating the function $C_{m_1m_2m}(x_1, x_2, x)$ appearing in the OPE (3.76) to the three point function $\langle\Phi_{m_1}\Phi_{m_2}\Phi_m\rangle$. Let us take the OPE of two fields in the following correlator:

$$\langle 0|\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3)|0\rangle = \int\, d^4x\, C_{m_2m_3m_1}(x_2, x_3, x)\langle 0|\Phi_{m_1}(x_1)\Phi_{m_1}(x)|0\rangle \tag{3.78}$$

[3]The proof however is not extended to states different from the vacuum.

[4]Correlation functions are defined as expectation value of a set of fields on the vacuum, hence the argument of [46] applies.

where we have used the fact that the two point function vanishes if the field belongs to different irreducible representations of the conformal group[5] The above relation is an integral equation which can be formally solved passing in Fourier space. Defining the Fourier transforms:

$$\begin{aligned}\langle 0|\Phi_{m_1}(x_1)\Phi_{m_1}(x)|0\rangle &= \int \frac{d^D p}{(2\pi)^D} e^{-ip(x_1-x)}\Delta_{m_1}(p)\\ \langle 0|\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3)|0\rangle &= \int \frac{d^D p}{(2\pi)^D} e^{-ipx_1} W_{m_1 m_2 m_3}(p, x_2, x_3)\end{aligned} \tag{3.79}$$

and plugging those expressions in (3.78) we obtain the relation

$$C_{m_2 m_3 m_1}(x_2, x_3, x_1) = \int \frac{d^D p}{(2\pi)^D} e^{-ipx_1} \Delta^{-1}_{m_1}(p) W_{m_1 m_2 m_3}(p, x_2, x_3)\,. \tag{3.80}$$

Recalling the expression of the two and three point functions (3.40) we conclude that the contribution of a given irreducible representation to the OPE of two scalar[6] fields is totally fixed by conformal invariance, a part from a normalization constant which takes the name of *OPE coefficient*.

The OPE expansion (3.76) still contains an integration over the coordinate of the operator $\Phi_m(x)$. This because all the states in a given irreducible representation contribute to the decomposition of the lhs. On the other hand, according to [47], the product of two local fields should admit, at short distances, an expansion in terms of local fields. One cane verity that this is precisely the case expanding (3.80) in the coordinate difference x^μ_{12}. For instance, in the simplest case of a scalar irreducible representation entering the OPE of two scalar fields we have [36]:

$$\begin{aligned}C_{m_1 m_2 m}(x_1, x_2, x_3) = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}} \Bigg(1 &+ \frac{\Delta_3-\Delta_1+\Delta_2}{2\Delta_3} x^\mu_{12}\partial_\mu \\ &+ \frac{1}{8}\frac{(\Delta_3-\Delta_1+\Delta_2)(\Delta_3-\Delta_1+\Delta_2+2)}{\Delta_3(\Delta_3+1)} x^\mu_{12} x^\nu_{12}\partial_\mu\partial_\nu \\ &- \frac{1}{16}\frac{(\Delta_3-\Delta_1+\Delta_2)(\Delta_3+\Delta_1-\Delta_2)}{\Delta_3(\Delta_3+1-D/2)} x^\mu_{12} x^2_{12}\Box\partial_\mu + \ldots\Bigg)\,\delta^4(x_{23}),\end{aligned} \tag{3.81}$$

where all derivatives are taken with respect to x_2. The integral on x_3 in (3.76) is now trivial: the contribution of a single irreducible representation to the OPE is translated in an infinite sum of local operators. The first term of the series corresponds to the primary operator which identifies the representation, while all the others are the descendant fields, represented as derivatives of

[5] As in Section 3.2 we assume to work in a basis where the two point function has the form (3.43).

[6] Here we discussed only three point functions containing two scalar field and a third operator. Three point functions involving other fields can contain more than one arbitrary constant.

the primary field. The coefficient c_{123} is the OPE coefficient.
Here we considered only the case of a scalar field entering the OPE. In principle all the expansions are calculable. Also, an alternative procedure to compute $C_{m_1 m_2 m_3}(x_1, x_2, x_3)$ is proposed in the first paper of [33], where expression for tensors of rank $l = 1, 2$ are reported.

In conclusion, a CFT is completely determined by the quantum numbers of the primary operators and by their three point functions, or equivalently by their OPE's. Interestingly, as pointed out in [48], not all the OPE's gives rise to correlation functions that satisfy all the Wightman axioms [49]. On the other hand, there is only one condition which is not trivially satisfied by arbitrary OPE's and it turns out to be a sufficient requirement to show that all the other correlation function will obey the those axioms. This constraints take the form of the *crossing symmetry condition*

$$\langle 0|\Phi_{m_1}(x_1)\Phi_{m_2}(x_2)\Phi_{m_3}(x_3)\Phi_{m_4}(x_4)|0\rangle = \pm\langle 0|\Phi_{m_1}(x_1)\Phi_{m_3}(x_3)\Phi_{m_2}(x_2)\Phi_{m_4}(x_4)|0\rangle\,, \tag{3.82}$$

where the sign depends on the statistic of the Φ's. We will see in the next chapter how this condition can be translated in constraints on the OPE's of the theory.

3.4 Euclidean CFT

Although our interest is mainly focused on field theories living in Minkowski space, it is sometimes useful to switch to the euclidean formulation of the theory, given that the structure of singularities of correlation functions is simpler in that case. In this section we review a result that shows the connection among the two formalisms.
First of all let us define what we mean by the *Euclidean formulation* of a field theory. Given a set of fields $\Phi_i(x_i)$ defined on Minkowski space we can consider their correlation functions

$$G_n(x_i^0, \vec{x}_i) = \langle 0|\Phi_1(x_1)...\Phi_n(x_n)|0\rangle \tag{3.83}$$

Then, we define the euclidean correlation functions simply via Wick rotation, $x_0 \to -i\tau$:

$$G_n^E(\tau_i, \vec{x}_i) = G_n(-ix_i^0, \vec{x}_i)\,. \tag{3.84}$$

In the context of CFT's, the equivalence of the two formulations has been proved in [50]. There Mack and Lusher showed that, given a set of correlation functions defined on Minkowski (satisfying the Wightman axioms) such that the associated euclidean correlation functions are invariant under the conformal euclidean group $SO(D+1,1)$, it follows that

- The euclidean correlation functions can be analytically continued to complex coordinates. In particular at imaginary times they coincide with the original ones defined on Minkowski.
- The Minkowski correlation functions can be analytically extended on larger space $\widetilde{M}$ which is invariant under the universal covering group G^* of the Minkowski conformal group $SO(D,2)$. In the coordinates that imbed the Minkowski space into a compact subspace of the Einstein universe [50, 34], the space $\widetilde{M}$ corresponds to the entire Einstein universe.
- The Hilbert space of states generated by the fields Φ_i carries a unitary representation of G^*.
- Given an irreducible representation of G^*, generated by a field Φ_i, the spectrum of the Hamiltonian $\mathcal{H} = \frac{1}{2}(P^0 + K^0)$ is discrete and its eigenvalues are given by

$$E_k = \Delta_i + k, \qquad k \in \mathbb{N}\,, \tag{3.85}$$

  where Δ_i is the dimension of Φ_i. This result shows that the unitary bounds extracted in Section 2.2 on the eigenvalues of $\mathcal{H}$ translate directly on field dimensions.

Given the above results we can always start from a Minkowski CFT, defined in terms of a spectrum of operators and their OPE's, compute the euclidean correlation functions and then rotate back to Minkowski space.

Chapter 4

Constraints from conformal bootstrap

4.1 The Partial Wave decomposition

As we mentioned in Section 3.2, conformal invariance implies that a scalar 4-point function must have the form (3.52), where $g(u, v)$ is an arbitrary function of the cross-ratios. Further information about $g(u, v)$ can be extracted using the OPE. Namely, if we apply the OPE (3.76) to the LHS of (3.52) both in 12 and in 34 channel, we can represent the 4-point function as a sum over primary operators which appear in both OPEs:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \sum_{\mathcal{O}} c_{12\mathcal{O}} c_{34\mathcal{O}} \, \mathtt{CB}_{\mathcal{O}} \, , \tag{4.1}$$

$\mathtt{CB}_{\mathcal{O}} =$  . (4.2)

The non-diagonal terms do not contribute to this equation because the 2-point functions of nonidentical primaries $\mathcal{O} \neq \mathcal{O}'$ vanish, and so do 2-point functions of any two operators belonging to different conformal families. The functions $\mathtt{CB}_{\mathcal{O}}$, which receive contributions from 2-point functions of the operator $\mathcal{O}$ and its descendants, are called *conformal blocks* and represent the terms in the sum (3.77). Conformal invariance of the OPE implies that the conformal blocks transform under the conformal group in the same way as $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$. Thus they can be written in the form of the RHS of (3.52), with an appropriate function $g_{\mathcal{O}}(u, v)$. In terms of these

functions, (4.1) can be rewritten as

$$\begin{aligned}\langle\phi_1\phi_2\phi_3\phi_4\rangle &= \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_1-\Delta_2}\left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4}\frac{g(u,v)}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}\\ g(u,v) &= \sum_{\mathcal{O}} c_{12\mathcal{O}}c_{34\mathcal{O}}\, g_{\mathcal{O}}(u,v).\end{aligned} \tag{4.3}$$

Following [33] [1] we now derive an expression for the conformal block. The main idea is that a conformal block represents the contribution to the four point function of single primary operator and all its descendants, thus it must be eigenvector of all the Casimir operators of the Conformal group. Since different primaries correspond to different eigenvalues, each conformal block satisfies a different differential equation separately. In order to derive the differential equation let us consider the Casimir operator

$$C = \frac{1}{2}M_{\mu\nu}M_{\mu\nu} - D^2 - \frac{1}{2}(P_\mu K_\mu + K_\mu P_\mu) \tag{4.4}$$

A simple way to verify that the above expression corresponds to a Casimir operator is to use the isomorphism between the Conformal group and the $O(4,2)$ group defined in Section 2.1 . Thus recalling the identification of the generators:

$$J_{AB} = \left(\begin{array}{c|c|c} M_{\mu\nu} & \frac{1}{2}(K_\mu - P_\mu) & \frac{1}{2}(K_\mu + P_\mu) \\ \hline -\frac{1}{2}(K_\mu - P^\mu) & 0 & D \\ \hline -\frac{1}{2}(K_\mu + P_\mu) & -D & 0 \end{array}\right), \tag{4.5}$$

the Casimir $\frac{1}{2}J^{AB}J_{AB}$ reduces to equation (4.4). Applying the Casimir to the state $\phi_1(x_1)\phi_2(x_2)|0\rangle$ we obtain

$$C\phi_1(x_1)\phi_2(x_2)|0\rangle = \frac{1}{2}[J_{AB},[J^{AB},\phi_1(x_1)\phi_2(x_2)]]|0\rangle = \mathcal{D}_{x_1,x_2}\phi_1(x_1)\phi_2(x_2)|0\rangle, \tag{4.6}$$

where $\mathcal{D}$ is a second-order partial differential operator acting on the coordinates $x_{1,2}$. On the other hand, using the OPE expansion on the product $\phi_1(x_1)\phi_2(x_2)$[2], we get a sum of terms in which the Casimir operator acts on the primary operators and their descendants:

$$\begin{aligned} C\cdot\phi_1(x_1)\phi_2(x_2)|0\rangle &= \sum_{\Delta_l}\frac{c_{12O}}{x_{12}^{\Delta_1+\Delta_2-\Delta_O}}C\left(O_{\Delta,l}(x_2) + \text{ descendants}\right)\|0\rangle \\ &= -\sum_{\Delta,l}E_{\Delta,l}\,\frac{c_{12O}}{x_{12}^{\Delta_1+\Delta_2-\Delta_O}}\left(O(x_2) + \text{ descendants}\right)\|0\rangle \end{aligned} \tag{4.7}$$

[1] The first work of is a brute force resummation of contributions of all conformal descendants of $\mathcal{O}$ and is not particularly enlightening.

[2] Equivalently we can expand the state $\phi_1(x_1)\phi_2(x_2)|0\rangle$ on a complete basis.

where $E_{\Delta,l}$ in the above expression are nothing but the eigenvalues of the Casimir acting on conformal primaries. For instance in $D = 4$ dimensions we have:

$$C \cdot \mathcal{O}_{(\mu)}(0) = -E_{\Delta,l}\mathcal{O}_{(\mu)}(0)\,, \qquad E_{\Delta,l} = \Delta(\Delta - D) + 2j_1(j_1+1) + 2j_2(j_2+1) \tag{4.8}$$

Where j_1, j_2 label the Lorentz (or $SO(4)$)representation. For different dimensions D, j_1, j_2 are substituted by the $SO(D-1,1)$ quantum numbers. In the case of symmetric traceless tensors of rank l the Casimir eigenvalues look the same in any dimension:

$$c_{\Delta,l} = \Delta(\Delta - D) + l(l+2)\,. \tag{4.9}$$

The explicit form of $\mathcal{D}_{x_1,x_2}$ defined in (4.6) can be found switching to the six-dimensional formalism (or more generically to $D+2$ dimensions). Using the results of Section 3.1.2 we can write the four point function of four scalar operators as:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \Pi_{i=1}^4 (y_i^+)^{\Delta_i} \langle \Phi_1(y_1)\Phi_2(y_2)\Phi_3(y_3)\Phi_4(y_4)\rangle \tag{4.10}$$

$$x_i^\mu = \frac{y_i^\mu}{y_i^+}\,, \qquad y_i^+ = y_i^5 + y_i^6\,. \tag{4.11}$$

and, recalling the relations (3.61), we have:

$$\langle \Phi_1(y_1)\Phi_2(y_2)\Phi_3(y_3)\Phi_4(y_4)\rangle = \left(\frac{y_1\cdot y_4}{y_1\cdot y_3}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{y_2\cdot y_4}{y_1\cdot y_4}\right)^{\frac{\Delta_{34}}{2}} \frac{g(u,v)}{(y_1\cdot y_2)^{(\Delta_1+\Delta_2)/2}(y_3\cdot y_4)^{(\Delta_3+\Delta_4)/2}}\,,$$
$$\Delta_{ij} \equiv \Delta_i - \Delta_j\,. \tag{4.12}$$

The differential representation of the J_{AB} generator in the $D+2$ dimensional space is:

$$J_{AB} = L_{AB} = -i\left(y_A\frac{\partial}{\partial y^B} - y_B\frac{\partial}{\partial y^A}\right) \tag{4.13}$$

Hence we obtain:

$$\frac{1}{2}[L^{AB},\,[L_{AB},\Phi_1(y_1)\Phi_2(y_2)]] = \frac{1}{2}[L^{AB},\,[L_{AB},\Phi_1(y_1)]]\Phi_2(y_2) + \frac{1}{2}\Phi_1(y_1)[L^{AB},\,[L_{AB},\Phi_2(y_2)]]$$
$$+ [L^{AB},\Phi_1(y_1)][L_{AB},\Phi_2(y_2)]\,, \tag{4.14}$$

and using the explicit expression for the generator we can extract a differential equation for the function $g_{\mathcal{O}}(u,v)$. The result ([33]) has the form:

$$L^2 g_{\mathcal{O}}(u,v) - (\Delta_{12}-\Delta_{34})\left((1+u-v)\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right) - (1-u-v)\frac{\partial}{\partial v}\right)g_{\mathcal{O}}(u,v)(u,v)$$
$$-\frac{1}{2}\Delta_{12}\Delta_{34}g_{\mathcal{O}}(u,v) = -E_{\Delta,l}g_{\Delta,l}(u,v)\,, \tag{4.15}$$

where for any $F(u,v)$,

$$\frac{1}{2}L^2F = -\big((1-v)^2 - u(1+v)\big)\frac{\partial}{\partial v}v\frac{\partial}{\partial v}F - (1-u+v)u\frac{\partial}{\partial u}u\frac{\partial}{\partial u}F + 2(1+u-v)uv\frac{\partial^2}{\partial u\partial v}F + D\,u\frac{\partial}{\partial u}F\,. \tag{4.16}$$

The clever change of variables $u,v \to z,\bar{z}$ performed in [33] allows to find explicit solutions. In these variables, related to u,v via

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \tag{4.17}$$

or, equivalently

$$z,\bar{z} = \frac{1}{2}\left(u - v + 1 \pm \sqrt{(u-v+1)^2 - 4u}\right). \tag{4.18}$$

the differential equation takes the form

$$\mathcal{D}_{z,\bar{z}}g_{\mathcal{O}}(z,\bar{z}) = \frac{1}{2}E_{\Delta,l}g_{\mathcal{O}}(z,\bar{z}), \tag{4.19}$$

$$\mathcal{D}_{z.\bar{z}} = z^2(1-z)\frac{\partial^2}{\partial z^2} + \bar{z}^2(1-\bar{z})\frac{\partial^2}{\partial \bar{z}^2} - \left(z^2\frac{\partial}{\partial z} + \bar{z}^2\frac{\partial}{\partial \bar{z}}\right) + (D-2)\frac{z\bar{z}}{\bar{z}-z}\left[(1-z)\frac{\partial}{\partial z} - (1-\bar{z})\frac{\partial}{\partial \bar{z}}\right] + \frac{1}{2}\left(\Delta_{12} - \Delta_{34}\right)\left(z^2\frac{\partial}{\partial z} + \bar{z}^2\frac{\partial}{\partial \bar{z}}\right) + \Delta_{12}\Delta_{34}(z+\bar{z})$$

Notice that the RHS of (4.19) is invariant under $z \leftrightarrow \bar{z}$ and that the second line of the above equation vanishes in the simpler case of operator with the same scalar dimension. The solution for generic values of the dimensions Δ_i can be found in [33]. Here we specialize to the simpler case of equal dimensions: $\Delta_i = d$ for $i = 1,2,3,4$, since in the present work we will focus on the case where the operators entering the four point function are either equal or related by some internal symmetry, hence they belong to the same representation of the Conformal group. Whenever this is the case the solution of the above equation is determined only in terms of Δ and l. From now on we will label conformal blocks with the quantum number of the operator they refer to:

$$g_{\mathcal{O}}(z,\bar{z}) \equiv g_{\Delta,l}(z,\bar{z})\,. \tag{4.20}$$

Rewriting the r.h.s. of eq. (4.19) as:

$$E_{\Delta,l} = \lambda_1(\lambda_1 - 1) + \lambda_2(\lambda_2 - 1 - (D-2))\,, \qquad \lambda_1 = \frac{1}{2}(\Delta + l)\,, \quad \lambda_2 = \frac{1}{2}(\Delta - l)\,,. \tag{4.21}$$

then the OPE fixes the boundary condition of the solution ([33]) to be:

$$g_{\Delta,l}(z,\bar{z}) \sim z^{\lambda_1}\bar{z}^{\lambda_2} \qquad \text{as } z,\bar{z} \longrightarrow 0,. \tag{4.22}$$

Let us re-derive the solution for different values of the space-time dimension D. Unfortunately the differential equation appear to have solution in a closed form only for even values of D.

4.1.1 Two dimensional Conformal Blocks

For $D = 2$ the differential equation splits in two part, without mixed terms between z and $\bar{z}$. Let us introduce the building block for the solution in two and higher dimension:

$$k_\beta(z) = x^{\beta/2} {}_2F_1\left(\beta/2, \beta/2, \beta; z\right) \tag{4.23}$$

which is an eigenfunction of the differential operator $z^2(1-z)\partial_z^2 - z^2\partial_z$:

$$\left(z^2(1-z)\frac{\partial^2}{\partial z^2} - z^2\frac{\partial}{\partial z}\right) k_\beta(z) = \frac{\beta(\beta-2)}{4} k_\beta(z)\,. \tag{4.24}$$

and has the small z behavior: $k_\beta(z) \sim x^{\beta/2}\left(1 + \beta z/4\right)$. Thus we can solve the differential equation for the two dimensional conformal block taking the symmetrized product of two such functions with suitable values of β. The solution is therefore:

$$g_{\Delta,l}^{(D=2)} = k_{\Delta+l}(z) k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})\,, \tag{4.25}$$

which has a small $z, \bar{z}$ expansion[3]

$$g_{\Delta,l}^{(D=2)} \sim z^{(\Delta+l)/2} \bar{z}^{(\Delta-l)/2}\,, \tag{4.26}$$

4.1.2 Four dimensional Conformal Blocks

When $D > 2$ the differential operator contains mixed terms and the solution is not factorized. On the other hand we can make use of an interesting property of the operator (4.19):

$$\mathcal{D}_{z,\bar{z}}^{D=4} \frac{z\bar{z}}{z-\bar{z}} = \frac{z\bar{z}}{z-\bar{z}} \left(D_{z,\bar{z}}^{D=2} - 2\right) \tag{4.27}$$

Thus we can look for a solution of the form

$$g_{\Delta,l}(z,\bar{z}) \sim \frac{z\bar{z}}{z-\bar{z}} \left(k_{\beta_1}(z) k_{\beta_2}(\bar{z}) - (z \leftrightarrow \bar{z})\right) \tag{4.28}$$

The boundary conditions fix either:

$$\beta_1 = \Delta - l - 2\,, \quad \beta_2 = \Delta + l\,, \qquad \text{or} \qquad \beta_1 = \Delta + l\,, \quad \beta_2 = \Delta - l - 2\,. \tag{4.29}$$

Given the symmetry $z \leftrightarrow \bar{z}$ we can take the following convention:

$$g_{\Delta,l}^{(D=4)} = \frac{z\bar{z}}{z-\bar{z}} \left(k_{\Delta+l}(z) k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right)\,. \tag{4.30}$$

which has a small $z, \bar{z}$ expansion

$$g_{\Delta,l}^{(D=4)} \sim \frac{z^{(\Delta+l+2)/2} \bar{z}^{(\Delta-l)/2} - z^{(\Delta-l)/2} \bar{z}^{(\Delta+l+2)/2}}{z-\bar{z}} \sim z^{(\Delta+l)/2} \bar{z}^{(\Delta-l)/2}\,, \tag{4.31}$$

[3]Here and in the expansion of 4D conformal blocks we retain only the leading z term. In other words, we take the $\bar{z} \to 0$ limit first

4.1.3 Recursive relations and conformal blocks in dimension larger than 4

For $D \geq 6$ the search for solutions become more involved and simple ansatz are not sufficient. In [33] a recursive relation is provided which allows to express the solution for general even D in terms of the solution for $D-2$. Although we didn't checked the correctness of the formula we tested its validity for the simple cases $D = 2$. We report the formula for completeness:

$$\begin{aligned}\left(\frac{z-\bar{z}}{z\bar{z}}\right)^2 F^{D+2}_{\Delta,l}(z,\bar{z}) &= F^{D}_{\Delta-2,l+2}(z,\bar{z}) - 4\frac{(D+l+2)(D+l-1)}{(D+2l-2)(D+2l)}F^{D}_{\Delta-2,l}(z,\bar{z}) \\ &- 4\frac{(D-\Delta-1)(D-\Delta)}{(D-2\Delta)(D-2\Delta+2)}\left(\frac{(\Delta+l)^2}{16(\Delta+l-1)(\Delta+l+1)}F^{D}_{\Delta,l+2}(z,\bar{z})\right. \\ &\left.+\frac{(D+l-2)(D+l-1)^2}{4(D+2l-2)(D+2l)(D+l-\Delta-1)(D+l-\Delta+1)}F^{D}_{\Delta,l}(z,\bar{z})\right)\end{aligned} \tag{4.32}$$

4.1.4 Analyticity properties of Conformal blocks

We conclude this section with a comment on the analyticity properties of the conformal blocks, specializing to four dimensions for concreteness. We first recall the definition of the z and $\bar{z}$ variables:

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \tag{4.33}$$

or, equivalently

$$z, \bar{z} = \frac{1}{2}\left(u - v + 1 \pm \sqrt{(u-v+1)^2 - 4u}\right).$$

As shown in 2.3 when the four points lie in a configuration as in Fig. 4.1. than

$$z = \frac{1}{2} + X + iY, \quad \bar{z} = z^*, \tag{4.34}$$

where (X, Y) are the coordinates of x_2 in the plane, chosen so that $X = Y = 0$ corresponds to x_2 halfway between x_1 and x_3. This "self-dual" configuration, for which $u = v$, will play an important role below. We can see that the z variable is a natural extension of the usual complex coordinate of the 2D CFT to the 4D case. According to the above discussion, the OPE is expected to converge for $|z| < 1$. Conformal block decomposition is a partial resummation of the OPE and thus also converges at least in this range. In fact, below we will only use convergence around the self-dual point $z = 1/2$. However, conformal blocks, as given by (4.25), (4.30), are regular (real-analytic) in a larger region, namely in the z-plane with the $(1, +\infty)$ cut along the real axis (see Fig. 4.1). The conformal block decomposition is thus expected to converge in this larger region. One can check that this indeed happens in the free scalar theory.

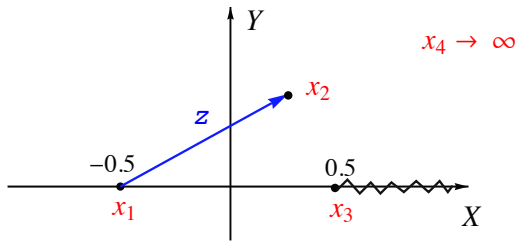


Figure 4.1: *The auxiliary z coordinate. The conformal blocks are regular outside the cut denoted by the zigzag line.*

One can intuitively understand the reason for this extended region of regularity. The condition for the OPE convergence, as stated above, does not treat the points x and 0 symmetrically. On the other hand, the conformal blocks are completely symmetric in $x_1 \leftrightarrow x_2$ and so must be the condition for their regularity. The appropriate condition is as follows: *the conformal block decomposition in the* 12-34 *channel is regular and convergent if there is a sphere separating the points $x_{1,2}$ from the points $x_{3,4}$*. For the configuration of Fig. 4.1, such a sphere exists as long as x_2 is away from the cut.

4.2 The bootstrap equation

We begin with some preliminary comments and notational conventions. We will work in the $D = 4$ Euclidean space. Consider a 4-point function $\langle \phi(x_1)\chi^\dagger(x_2)\chi(x_3)\phi^\dagger(x_4)\rangle$ where ϕ and χ are two primary operators, not necessarily Hermitian, assumed to have equal dimensions $d_\phi = d_\chi = d$. The OPE $\phi \times \chi^\dagger$ will contain a sequence of spin l, dimension Δ primary fields $O_{\Delta,l}$,:

$$\phi \times \chi^\dagger = \sum_{\Delta,l} c_{\Delta,l} O_{\Delta,l}\,. \tag{4.35}$$

Here $c_{\Delta,l}$ are the OPE coefficients, in general complex, meant as the normalization of the three point function $\langle \phi\chi^\dagger O\rangle$. We then normalize the conformal blocks via:

$$\left\langle \begin{array}{cc} \phi(x_1)\bullet & \bullet\phi^\dagger(x_4) \\ \chi^\dagger(x_2)\bullet & \bullet\chi(x_3) \end{array} \right\rangle = \sum_{\Delta,l} \frac{1}{x_{12}^{2d} x_{34}^{2d}}\, p_{\Delta,l}\, g_{\Delta,l}(u,v)\,, \tag{4.36}$$

$$u \equiv x_{12}^2 x_{34}^2/(x_{13}^2 x_{24}^2) = z\bar{z}, \quad v \equiv x_{14}^2 x_{23}^2/(x_{13}^2 x_{24}^2) = (1-z)(1-\bar{z})\,,$$

$$g_{\Delta,l}(u,v) = +\frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right], \tag{4.37}$$

$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1\left(\beta/2, \beta/2, \beta; x\right) . \tag{4.38}$$

The points x_i are assumed to be near the vertices of a square, as the picture suggests. The ordering is important. Eq. (4.36) says that the exchanges of $O_{\Delta,l}$ and of its conformal descendants in the (12)(34) channel (≡s-channel) can be summed up in a 'conformal block' $g_{\Delta,l}(u,v)$. The coefficients $p_{\Delta,l}$ are given by

$$p_{\Delta,l} = \frac{|c_{\Delta,l}|^2}{2^l} > 0. \tag{4.39}$$

Compared to [33], and also to [26, 28, 30], we have dropped the $(-1/2)^l$ pre-factor in the expression for $g_{\Delta,l}$. This normalization is more convenient for the following reason. In the new convention all conformal blocks are positive when operators are inserted at the vertices of a square in the shown order (this corresponds to $z = \bar{z} = 1/2$). This is just as it should be, because this configuration is reflection-positive in the Osterwalder-Shrader sense with respect to the vertical median line (notice that the fields in the two sides of the correlator are complex-conjugate of each other)[4]. Thus *any* s-channel contribution to the correlator, even spin or odd, has to be positive. There is no disagreement with Doland and Osborn [33], because in their notation the extra minus sign would be offset by a change in the sign of the OPE coefficient in the RHS of the correlator.

The (14)(23) channel (≡t-channel) conformal block decomposition can be analyzed similarly. In this case we will need OPEs $\phi \times \phi^\dagger$ and $\chi \times \chi^\dagger$ and only fields appearing in both of these OPEs will give a nonzero contribution, proportional to the product of the two OPE coefficients.

Finally, an important technical remark. Unlike in [26], to extract full information from the 4-point function (4.36), we will have to consider not only the s- and t-channel OPEs, but also the u-channel ones (13)(24). A useful way to keep track of signs is not to consider the u-channel OPE directly, but to instead apply the s- and t-channel decompositions to the 4-point function with the permuted insertion points:

$$\left\langle \begin{array}{cc} \phi(x_1)\bullet & \bullet\chi(x_4) \\ \chi^\dagger(x_2)\bullet & \bullet\phi^\dagger(x_3) \end{array} \right\rangle \tag{4.40}$$

Here, we transposed the fields in the right side of the correlator. Now in the t-channel we have the same OPE as we would have in the u-channel in (4.36). And in the s-channel we have the

[4] Actually, conformal blocks are positive on the whole interval $0 < z = \bar{z} < 1$. Configurations corresponding to such $z, \bar{z}$ can be mapped onto a rectangle, which is reflection-positive.

same OPE as in (4.36), except for the exchange $x_3 \leftrightarrow x_4$. This in turn translates in

$$u \longrightarrow \frac{u}{v}\,, \qquad v \longrightarrow \frac{1}{v} \tag{4.41}$$

and, as shown in the following section, can be taken into account by reversing the sign of the odd-spin contributions in the s-channel (and permuting the flavor indices accordingly when there are flavor symmetries, see below).

4.3 The simplest case: Sum rule without symmetries

We start the analysis focusing on the particular case when ϕ is Hermitean and $\chi = \phi$ ([26, 28]). In this case the s- and t-channels of the four point function:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2d}|x_{34}|^{2d}}, \quad d = [\phi]. \tag{4.42}$$

correspond to the same OPE ($\phi \times \phi$).
The LHS of this equation is invariant under the interchange of any two x_i, and so the RHS should also be invariant, which gives a set of *crossing symmetry* constraints for the function $g(u,v)$. Invariance under $x_1 \leftrightarrow x_2$ and $x_1 \leftrightarrow x_3$ (other permutations do not give additional information) implies:

$$g(u,v) = g(u/v, 1/v) \qquad (x_1 \leftrightarrow x_2), \tag{4.43}$$

$$v^d g(u,v) = u^d g(v,u) \qquad (x_1 \leftrightarrow x_3). \tag{4.44}$$

At the same time, $g(u,v)$ can be expressed via the conformal block decomposition (4.3), which in the case under examination takes the form:

$$g(u,v) = 1 + \sum_{\mathcal{O}_{\Delta,l} \in \phi\times\phi} p_{\Delta,l}\, g_{\Delta,l}(u,v)\,. \tag{4.45}$$

Here in the first term we explicitly separated the contribution of the unit operator, present in the $\phi \times \phi$ OPE. We stress that all conformal blocks appear in (4.45) with positive coefficients.
Let us now see under which conditions Eq. (4.45) is consistent with the crossing symmetry. The $x_1 \leftrightarrow x_2$ invariance turns out to be rather trivial. Transformation properties of any conformal block under this crossing depend only on its spin [33]:

$$g_{\Delta,l}(u,v) = (-)^l g_{\Delta,l}(u/v, 1/v). \tag{4.46}$$

All the operators appearing in the OPE $\phi \times \phi$ have even spin[5]. Thus the first crossing constraint (4.43) will be automatically satisfied for arbitrary coefficients $p_{\Delta,l}$. Let us introduce the notation for the sum of all s-channel contributions:

$$G^+ = \sum_{l \text{ even};\Delta} p_{\Delta,l} g_{\Delta,l}(u,v)\,, \tag{4.47}$$

(+ means that we are summing over even spins only) and a tilde notation for a contribution of the same set of operators in the t-channel:

$$\widetilde{G}^+ = G^+_{u\leftrightarrow v} = \sum p_{\Delta,l} g_{\Delta,l}(v,u)\,. \tag{4.48}$$

Here we used the fact that going from the s- to the t-channel, which means simply rotating the picture by 90°, interchanges u and v. In this notation the crossing symmetry constraint of [26] is written compactly as:

$$G^+ = \left(\frac{u}{v}\right)^d \widetilde{G}^+\,. \tag{4.49}$$

The appearance of the $(u/v)^d$ factor in this relation is due to a nontrivial transformation of the pre-factor $1/(x_{12}^{2d}x_{34}^{2d})$ in (4.36) under crossing.
More explicitly the above constraint can be expressed in the form of the following *sum rule*:

$$\begin{aligned} 1 &= \textstyle\sum_{\Delta,l} p_{\Delta,l} F_{d,\Delta,l}(z,\bar{z}), \quad p_{\Delta,l} > 0\,, \\ F_{d,\Delta,l}(z,\bar{z}) &\equiv \tfrac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}, \end{aligned} \tag{4.50}$$

where the sum is taken over all Δ, l corresponding to the operators $\mathcal{O} \in \phi \times \phi$, $p_{\Delta,l} = \lambda^2_{\mathcal{O}}$, and u, v are expressed via $z, \bar{z}$ via (4.17). As we will see below, this sum rule contains a great deal of information.
Below we will always apply Eq. (4.50) in the space-like diamond $0 < z, \bar{z} < 1$. We also find it convenient to use the coordinates a, b vanishing when the four scalars in (4.36) sit at the vertices of a square:

$$z = \frac{1}{2} + a + b, \quad \bar{z} = \frac{1}{2} + a - b.$$

The sum rule functions $F_{d,\Delta,l}$, in the region $-0.5 \le a, b \le 0.5$:

1. are smooth;

[5] A formal proof of this fact can be given by considering the 3-point function $\langle \phi(x)\phi(-x)\mathcal{O}_{(\mu)}(0)\rangle$. By $x \to -x$ invariance, nonzero value of this correlator is consistent with Eq. (3.40) only if l is even.

2. are even in both a and b, independently:

$$F_{d,\Delta,l}(\pm a, \pm b) = F_{d,\Delta,l}(a, b)\,; \tag{4.51}$$

3. vanish on its boundary:

$$F_{d,\Delta,l}(\pm 1/2, b) = F_{d,\Delta,l}(a, \pm 1/2) = 0. \tag{4.52}$$

Properties 1,2 are shown in Appendix C. Property 3 trivially follows from the definition of $F_{d,\Delta,l}$, since both terms in the numerator contain factors $z\bar{z}(1-z)(1-\bar{z})$.

A consequence of Property 3 is that the sum rule can never be satisfied with finitely many terms in the RHS.

The sum rule in the free scalar theory

To get an idea about what one can expect from the sum rule, we will demonstrate how it is satisfied in the free scalar theory. In this case $d = 1$, and only operators of twist $\Delta - l = 2$ are present in the OPE $\phi \times \phi$ [51],[33]. These are the operators

$$\mathcal{O}_{\Delta,l} \propto \phi\, \partial_{\mu_1} \ldots \partial_{\mu_l} \phi + \ldots \qquad (\Delta = l + 2,\ l = 0, 2, 4, \ldots). \tag{4.53}$$

The first term shown in (4.53) is traceless by ϕ's equation of motion, but it is not conserved. The extra bilinear in ϕ terms denoted by $\ldots$ make the operator conserved for $l > 0$ (in accord with the unitarity bounds of Section 2.2), without disturbing the tracelessness. Their exact form can be found e.g. in [52].

In particular, there is of course the dimension 2 scalar

$$\mathcal{O}_{2,0} = \frac{1}{\sqrt{2}}\phi^2\,,$$

where the constant factor is needed for the proper normalization. At spin 2 we have the energy-momentum tensor:

$$\mathcal{O}_{4,2} \propto \phi\partial_\mu\partial_\nu\phi - 2\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}\delta_{\mu\nu}(\partial\phi)^2\right]\,.$$

The operators with $l > 2$ are the conserved higher spin currents of the free scalar theory.

The OPE coefficients of all these operators (or rather their squares) can be found by decomposing the free scalar 4-point function

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{13}^2 x_{24}^2} + \frac{1}{x_{12}^2 x_{34}^2} + \frac{1}{x_{14}^2 x_{32}^2} = \frac{1}{x_{12}^2 x_{34}^2}\left(1 + u + \frac{u}{v}\right)$$

into the corresponding conformal blocks. Indeed

$$u + \frac{u}{v} = z\bar{z}\left(1 + \frac{1}{(1-z)(1-\bar{z})}\right) = \frac{z\bar{z}}{z-\bar{z}}\left(z - \bar{z} + \frac{1}{1-z} - \frac{1}{1-\bar{z}}\right) \tag{4.54}$$

must be equal to a sum of twist 2 contributions:

$$\sum_{\text{twist 2}} p_{l+2,l} \frac{z\bar{z}(k_{\Delta+l}(z) - k_{\Delta+l}(\bar{z}))}{z-\bar{z}} \tag{4.55}$$

This will be satisfied if and only if

$$\sum p_{l+2,l} k_{\Delta+l}(z) = z + \frac{1}{1-z} + \text{const}, \qquad \text{const} = -1 \tag{4.56}$$

The constant in this equation is fixed by using the fact that

$$k_{\Delta+l}(z/(z-1)) + k_{\Delta+l}(z) = 0$$

The RHS of (4.56) satisfy this relation if and only if const$= -1$. Now Eq. (4.56) can be used to find coefficients c_l order by order, since for $\Delta = l+2$, $k_{\Delta+l}(z) \sim z^{l+1}$. It's not obvious that only even l will enter but one can check that this is indeed true, and

$$p_{l+2,l} = (1 + (-)^l)\frac{(l!)^2}{(2l)!} \tag{4.57}$$

Using these coefficients, we show in Fig. 4.2 how the sum rule (4.50), summed over the first

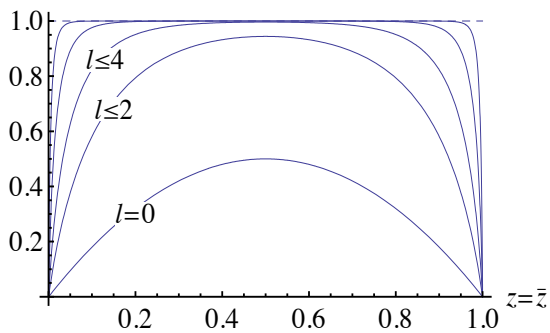


Figure 4.2: The RHS of the sum rule in the free scalar theory, summed over $l \leq 0, 2, 4, 8, 16$ (from below up) and plotted for $0 \leq z = \bar{z} \leq 1$. The asymptotic approach to 1 (dashed line) is evident. Notice the symmetry with respect to $z = 1/2$, a consequence of (4.51).

few terms, converges on the diagonal $z = \bar{z}$ of the space-like diamond. Several facts are worth

noticing. First, notice that the convergence is monotonic, i.e. all $F_{d,\Delta,l}$ entering the infinite series are positive. This feature is not limited to the free scalar case and remains true for a wide range of d, Δ, l; it could be used to limit the maximal size of allowed OPE coefficients. Second, the convergence is uniform on any subinterval $z \in [\varepsilon, 1-\varepsilon]$, $\varepsilon > 0$, but not on the full interval $[0,1]$, because all the sum rule functions vanish at its ends, see Eq. (4.52). Finally, the convergence is fastest near the middle point $z = 1/2$, corresponding to the center $a = b = 0$ of the space-like diamond. Below, when we apply the sum rule to the general case $d > 1$, we will focus our attention on a neighborhood of this point.

4.4 CFT's with Global symmetries

We now will discuss a generalization of the sum rule introduced in the previous section to the case when CFT has a continuous global symmetry G (Abelian or non-Abelian), and the operator ϕ transforms in a nontrivial representation R of G. We will consider the OPE $\phi \times \phi^\dagger$ if R is complex, or $\phi \times \phi$ if R is real. Thus the novelty with respect to the previous section is that we will be able to extract information concerning a given global symmetry sector while the analysis carried on so far is blind to flavor indexes.
It is useful to recall that the original motivation of [26] was to find a bound of precisely this type for the case $G = SO(4)$ and ϕ in the fundamental. This in turn was needed in order to constrain the Conformal Technicolor scenario of electroweak symmetry breaking [23]. We will come back on this in Section 6.1.4.

4.4.1 Fundamental of $SO(N)$

As a first example we will now consider the $SO(N)$ global symmetry case, with a scalar primary operator ϕ_a transforming in the fundamental representation. We normalize the 2-point function of ϕ_a as $\langle \phi_a(x)\phi_b(0)\rangle = \delta_{ab}\left(x^2\right)^{-d}$, $d = d_\phi$. Consider the 4-point function

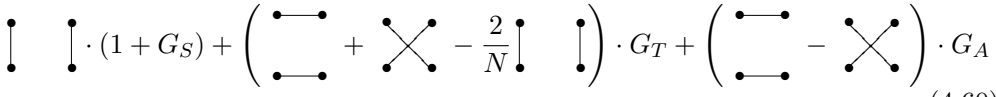
$$\left\langle \begin{array}{cc} \phi_a \bullet & \bullet \phi_d \\ \phi_b \bullet & \bullet \phi_c \end{array} \right\rangle \equiv \frac{1}{x_{12}^{2d} x_{34}^{2d}} \mathcal{G}\left[\begin{array}{c} a\ d \\ b\ c \end{array} \middle| u, v \right]. \tag{4.58}$$

The operator insertion points are assumed numbered in the same order as in (4.36).
Operators appearing in the $\phi_a \times \phi_b$ OPE can transform under the global symmetry as singlets

S, symmetric traceless tensors $T_{(ab)}$, or antisymmetric tensors $A_{[ab]}$:

$$\begin{aligned}\phi_a \times \phi_b = \delta_{ab}\mathbb{1} & \\ &+ \delta_{ab} S^{(\alpha)} \quad \text{(even spins)} \\ &+ T^{(\alpha)}_{(ab)} \quad \text{(even spins)} \\ &+ A^{(\alpha)}_{[ab]} \quad \text{(odd spins)}\end{aligned} \tag{4.59}$$

The index (α) shows that an arbitrary number of operators of each type may in general be present, of various dimensions Δ and spins l. However, permutation symmetry of the $\phi_a\phi_b$ state implies that the spins of the S's and T's will be even, while they will be odd for the A's.
it will be important for us that the unit operator $\mathbb{1}$ is always present in the $\phi_a \times \phi_b$ OPE, with a unit coefficient.
We note in passing that the stress tensor will be an S of $\Delta = 4, l = 2$, while the conserved $SO(N)$ current will be an A of $\Delta = 3$,$l = 1$. The OPE coefficients of these operators are related to the stress tensor and the current central charges by the Ward identities [53], which allow to derive various bounds on these quantities by the method of [29]. The simplest cases of these bounds have already been explored in [30],[54].
We will now see what crossing symmetry says about the relative weights of various contributions in the $\phi \times \phi$ OPE. Applying the conformal block decomposition in the s-channel we get:

$\mathcal{G}\left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right] =$  (4.60)

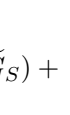
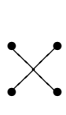
Here $G_{S,T,A}$ are defined as in (4.47), and sum up conformal blocks of all fields of a given symmetry. Remember that $G_{S,T}$ contain only even spins, while G_A only the odd ones. The unit operator contributes together with the singlets, and its conformal block is $\equiv 1$. To keep track of the index structure, we are using the graphical notation for tensors. Every line means that the corresponding indices are contracted with the δ tensor. E.g.:

 $= \delta_{ab}\,\delta_{cd}\,,$ etc. (4.61)

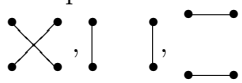
The index structure of the symmetric traceless and the antisymmetric tensor contributions in (4.60) is fixed by the symmetry (and by the tracelessness, in the case of G_T). The signs are fixed from the requirement that for $a = d \neq b = c$ all contributions have to be positive by reflection positivity, see Section4.2. Apart from the sign and the index structure, we do not keep track of the overall, positive, normalization of each term. In other words, we know that

each G contains conformal blocks summed with positive coefficients, but we do not keep track of the normalization of these coefficients. This is sufficient for deriving constraints on the operator spectrum. On the other hand, normalization conventions are important for any study of OPE coefficients.

Next we apply the t-channel conformal block decomposition to the same 4-point function, and we get an alternative representation:

$\mathcal{G}\left[\begin{smallmatrix} a & d \\ b & c \end{smallmatrix}\right] = \left(\frac{u}{v}\right)^d \Big\{$  $\cdot\,(1+\widetilde{G}_S) + \Big($  $+$  $-\frac{2}{N}$  $\Big)\cdot\widetilde{G}_T + \Big($ $-$ $\Big)\cdot\widetilde{G}_A\Big\}$

Note that to get this equation requires only changing the index structure appropriately, permuting $u \leftrightarrow v$ (here we are using the tilde notation introduced in (4.48)), and multiplying by $(u/v)^d$ to take into account how the $1/(x_{12}^{2d}x_{34}^{2d})$ transforms.

Now we equate the s- and t-channel representations and pick up coefficients before each of the 3 inequivalent tensor structures:  . We get 2 independent equations:

$$u^{-d}\left\{G_T - G_A\right\} = v^{-d}\left\{\widetilde{G}_T - \widetilde{G}_A\right\}\,, \tag{4.62a}$$

$$u^{-d}\left\{1 + G_S - \frac{2}{N}G_T\right\} = v^{-d}\left\{\widetilde{G}_T + \widetilde{G}_A\right\}\,, \tag{4.62b}$$

and a third one which can be obtained from the second by $u \leftrightarrow v$:

$$v^{-d}\left\{1 + \widetilde{G}_S - \frac{2}{N}\widetilde{G}_T\right\} = v^{-d}\left\{G_T + G_A\right\}\,, \tag{4.63}$$

Notice that for the $SO(N)$ case using the u-channel OPE would not yield any new equation.

It will be convenient to rewrite the system (4.62a),(4.62b) in the following equivalent form:

$$F_T - F_A = 0\,, \tag{4.64a}$$

$$F_S + \left(1 - \frac{2}{N}\right)F_T + F_A = 1\,, \tag{4.64b}$$

$$H_S - \left(1 + \frac{2}{N}\right)H_T - H_A = -1\,, \tag{4.64c}$$

where we introduced notation for (anti)symmetric linear combinations of G and $\widetilde{G}$:

$$F(u,v) = \frac{u^{-d}G(u,v) - v^{-d}G(v,u)}{v^{-d} - u^{-d}},$$
$$H(u,v) = \frac{u^{-d}G(u,v) + v^{-d}G(v,u)}{u^{-d} + v^{-d}}\,. \tag{4.65}$$

Thus (4.64a) is obtained from (4.62a) just by grouping and dividing by $v^{-d} - u^{-d}$. Eq. (4.64b) is obtained by taking the difference of (4.62b) and (4.63) and moving the contribution of the unit operator to the RHS. Finally, Eq. (4.64c) follows by taking the sum of (4.62b) and (4.63), and again separating the unity contribution.

Note that the functions $F(u,v)$ were already present in Section 4.3, while the appearance of $H(u,v)$ is a new feature of the global symmetry analysis. Writing the equations in terms of these functions is convenient because they are highly symmetric with respect to the $z = \bar{z} = 1/2$ point (they have only even derivatives in $z + \bar{z}$ and $z - \bar{z}$ at this point).

The system (4.64a)-(4.64c) is then the main result of this Section. In an expanded notation, it can be written as a "vectorial sum rule":

$$\sum p^S_{\Delta,l} \begin{pmatrix} 0 \\ F_{\Delta,l} \\ H_{\Delta,l} \end{pmatrix} + \sum p^T_{\Delta,l} \begin{pmatrix} F_{\Delta,l} \\ \left(1 - \frac{2}{N}\right) F_{\Delta,l} \\ -\left(1 + \frac{2}{N}\right) H_{\Delta,l} \end{pmatrix} + \sum p^A_{\Delta,l} \begin{pmatrix} -F_{\Delta,l} \\ F_{\Delta,l} \\ -H_{\Delta,l} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{4.66}$$

Here the functions $F_{\Delta,l}(u,v)$ and $H_{\Delta,l}(u,v)$ are related to the individual conformal blocks $g_{\Delta,l}$ by the same formulas as F and H are related to G. Their dependence on d is left implicit. In each sum we are summing vector-functions corresponding to the dimensions and spins present in this symmetry sector, with positive coefficients. The total must converge to the constant vector in the RHS.

Consequences of this new sum rule for the lowest singlet dimension will be discussed below. Let us do however a quick counting of degrees of freedom. In total we have three G-functions: G_S,G_T,G_A, each of which is restricted only to the odd or even spins. The vectorial sum rule gives three equations for their (anti)symmetric combinations F and H. This coincidence between the number of equations and unknowns is not accidental; see Section 4.5.3. One may hope that the constraining power is similar to the case without global symmetry, when we had one equation for only one function G^+. We will see in Section 6.1.4 how this hope is realized.

The sum rule in the free scalar theory with $O(N)$ symmetry

It is interesting to find an $SO(N)$ decomposition of the 4-point function in the theory of N free real scalars. In doing this, we can check explicitly that we did not make any sign mistakes and all coefficients are positive.

Let us use a particular choice of flavors in the 4-point functions, namely $c = a \neq d = b$.

According to (4.60), we must have

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = G_T - G_A \tag{4.67}$$

On the other hand in free theory the correlator is given by

$$\langle \phi_a(x_1)\phi_b(x_2)\phi_a(x_3)\phi_b(x_4)\rangle = \frac{1}{x_{13}^2 x_{24}^2} = \frac{1}{x_{12}^2 x_{34}^2} u$$

(summation convention suppressed in the last two equations). Thus we have to find an expansion:

$$u = \sum_{\text{even } l\geq 0} p^T_{l+2,l}\, g_{\Delta,l}(u,v) - \sum_{\text{odd } l\geq 1} p^A_{l+2,l}\, g_{\Delta,l}(u,v) \tag{4.68}$$

which just reduces to:

$$z = \sum_{l=0}^{\infty} p_{l+2,l} k_{2l+2}(z)$$

Again matching the taylor expansion of both sides we can deduce

$$p^T_{l+2,l} = (-1)^l \frac{(l!)^2}{(2l)!}$$

which is consistent with the fact that odd l terms have to be negative according to (4.68). Thus

$$p^{T/A}_{l+2,l} = \frac{(l!)^2}{(2l)!} \ (l \text{ even/odd}) \tag{4.69}$$

C_S^2 now can be found from (4.64b) since we know that in the case without global symmetry it was satisfied with coefficients given by (4.57). C_S^2 is the difference between the old coefficients and the C_T^2 contribution:

$$p^S_{l+2,l} = 2\frac{(l!)^2}{(2l)!} - \left(2 - \frac{2}{N}\right) p^T_{l+2,l} = \frac{2}{N}\frac{(l!)^2}{(2l)!} > 0.$$

One can check what with these coefficients the triple sum rule converges rapidly near $z = \bar{z} = 1/2$.

4.5 Special unitary groups

4.5.1 $U(1)$

We next discuss the $U(1)$ global symmetry, as a case intermediate between $SO(N)$ and $SU(N)$. On the one hand, we will be able to check that the $U(1)$ constraints agree with the already

considered $SO(N)$ case for $N = 2$. On the other hand, the derivation will be similar to the $SU(N)$ case which follows. In particular, we will be working with complex fields and will need the u-channel OPE.

We want to derive constraints from crossing in the 4-point function of a charge 1 complex scalar ϕ. Charge normalization is unimportant. The non-vanishing correlators must have zero total charge, thus we are led to consider $\left\langle \phi\phi\phi^\dagger\phi^\dagger \right\rangle$. There are two basic OPEs:

$$\text{Charge 0 sector:} \quad \phi \times \phi^\dagger = \mathbb{1} + \text{spins } 0,1,2\ldots, \tag{4.70}$$

$$\text{Charge 2 sector:} \quad \phi \times \phi = \text{even spins only}\,. \tag{4.71}$$

Let us begin by considering the configuration

$$\left\langle \begin{array}{cc} \phi\bullet & \bullet\phi^\dagger \\ \phi^\dagger\bullet & \bullet\phi \end{array} \right\rangle, \tag{4.72}$$

which is the same as in (4.36) for $\chi = \phi$. By doing the s- and t-channel conformal block decompositions and demanding that the answers agree we get a constraint:

$$u^{-d}\left\{1 + G_0^+ + G_0^-\right\} = v^{-d}\left\{1 + \widetilde{G}_0^+ + \widetilde{G}_0^-\right\}. \tag{4.73a}$$

Here the subscript 0 refers to the charge 0 fields appearing in the relevant $\phi \times \phi^\dagger$ OPE. As indicated in (4.70), this OPE contains both even and odd spin fields, whose contributions we separate in $G_0^\pm$. According to the discussion in Section 4.2, reflection positivity of (4.72) implies that even and odd spins contribute in (4.73a) with the same positive sign.

Next consider the configuration with the transposed right side of the correlator:

$$\left\langle \begin{array}{cc} \phi\bullet & \bullet\phi \\ \phi^\dagger\bullet & \bullet\phi^\dagger \end{array} \right\rangle.$$

Equating the s- and t-channel decompositions we get:

$$u^{-d}\left\{1 + G_0^+ - G_0^-\right\} = v^{-d}\widetilde{G}_2^+\,. \tag{4.73b}$$

The LHS of this equation differs from the LHS of (4.73a) only by the reversed sign of the odd spin contribution (see Section4.2). The t-channel decomposition appearing in the RHS is positive since the configuration is reflection-positive in this direction.

Eqs. (4.73a),(4.73b) solve the problem of expressing crossing constraints in a $U(1)$ symmetric theory. Upon identification

$$G_S = G_0^+, \quad G_A = G_0^-, \quad G_T = \frac{1}{2}G_2^+ \tag{4.74}$$

the $U(1)$ constraints become equivalent to the $N = 2$ case of the $SO(N)$ constraints discussed above. The appearance of a positive factor $1/2$ is consistent with the fact that we are keeping careful track of positivity but not of the normalization.

$U(1)$ free

Again we check our computations finding an $U(1)$ decomposition of the 4-point function in the theory of a free scalar. Let us use a particular choice of flavors in the 4-point functions, namely

$$\left\langle \phi(x_1)\phi(x_2)\phi^\dagger(x_3)\phi^\dagger(x_4)\right\rangle = \frac{1}{x_{14}^2 x_{24}^2} + \frac{1}{x_{13}^2 x_{24}^2} = \frac{1}{x_{12}^2 x_{34}^2}\left(u + \frac{u}{v}\right)$$

On the other hand the above four point function is given by

$$\left\langle \phi(x_1)\phi(x_2)\phi^\dagger(x_3)\phi^\dagger(x_4)\right\rangle = \frac{1}{x_{12}^2 x_{34}^2} \sum_{l=even} p^{(2)}_{l+2,l} g_{l+2,l}(u,v)$$

Thus we have to find expansion:

$$\left(u + \frac{u}{v}\right) = \sum_{\text{even } l\geq 0} p^{(2)}_{l+2,l}\, g_{\Delta,l}(u,v) \tag{4.75}$$

which just reduces to:

$$(z - \bar{z} + \frac{1}{1-z} - \frac{1}{1-\bar{z}}) = \sum_{\text{even } l\geq 0} p^{(2)}_{l+2,l}\, \left(k_{2l+2}(z) - k_{2l+2}(\bar{z})\right) \tag{4.76}$$

We have checked that this equation is satisfied for

$$p^{(2)}_{l+2,l} = \left(1 + (-1)^l\right)\frac{(l!)^2}{(2l)!}$$

Another way to express the four point function is to perform the OPE in the neutral channel $\phi \times \phi^\dagger$

$$\left\langle \phi(x_1)\phi^\dagger(x_2)\phi(x_3)\phi^\dagger(x_4)\right\rangle = \frac{1}{x_{12}^2 x_{34}^2}\left(1 + \sum_{l=even} p^{(0)}_{l+2,l} g_{l+2,l}(u,v) + \sum_{l=odd} p^{(0)}_{l+2,l} g_{l+2,l}(u,v)\right)$$

which is solved by

$$p^{(0)}_{l+2,l} = \frac{(l!)^2}{(2l)!}\ (l \text{ even/odd}) \tag{4.77}$$

4.5.2 Fundamental of $SU(N)$

Our next example is the $SU(N)$ case, with a scalar operator ϕ_i transforming in the fundamental. We have two basic OPEs:

$$\phi_i \times \phi_{\bar{i}}^\dagger = \delta_{i\bar{i}}\mathbb{1} + \delta_{i\bar{i}} \times \text{Singlets(spins 0,1,2...)} + \text{Adjoints(spins 0,1,2...)}\,, \tag{4.78}$$

$$\phi_i \times \phi_j = \square\!\square\text{'s (even spins)} + \begin{smallmatrix}\square\\ \square\end{smallmatrix}\text{'s (odd spins)} \tag{4.79}$$

The representation content of the first OPE is $N \otimes \bar{N} = 1+\text{Adj}$. Notice that, in general, there will be singlets and adjoints of any spin. The adjoint sector will contain the conserved current, but at present we are not using information about its coefficient. The second OPE contains symmetric and antisymmetric tensors, of even and odd spins respectively.

The constraints are now derived by a combination of what we did for $SO(N)$ and $U(1)$. First consider the following 4-point function configuration:

$$\left\langle \begin{matrix} \phi_{i\bullet} & {}_\bullet\phi_{\bar{j}}^\dagger \\ \phi_{\bar{i}}^{\dagger\bullet} & {}^\bullet\phi_j \end{matrix} \right\rangle .$$

The s- and t-channel conformal block decompositions are evaluated using the first OPE. Equating them, we get a constraint:

$$u^{-d}\left\{ \bullet\!\!-\!\!\circ \quad \circ\!\!-\!\!\bullet\,(1+G_S^+ + G_S^-) + \left(\begin{matrix}\bullet\!\!-\!\!\circ\\ \circ\!\!-\!\!\bullet\end{matrix} - \frac{1}{N}\,\bullet\!\!-\!\!\circ \quad \circ\!\!-\!\!\bullet \right)(G_{\text{Adj}}^+ + G_{\text{Adj}}^-) \right\}$$
$$= v^{-d}\left\{ \begin{matrix}\bullet\!\!-\!\!\circ\\ \circ\!\!-\!\!\bullet\end{matrix}\,(1+\widetilde{G}_S^+ + \widetilde{G}_S^-) + \left(\bullet\!\!-\!\!\circ \quad \circ\!\!-\!\!\bullet - \frac{1}{N}\begin{matrix}\bullet\!\!-\!\!\circ\\ \circ\!\!-\!\!\bullet\end{matrix} \right)(\widetilde{G}_{\text{Adj}}^+ + \widetilde{G}_{\text{Adj}}^-) \right\}$$

Here lines denote $SU(N)$-invariant contractions of N (dots) and $\bar{N}$ (circles) indices by $\delta_{i\bar{i}}$. The tensor structure of the Adj contributions is fixed by the tracelessness condition of the $SU(N)$ generators. The sign is fixed by the condition that for $i = \bar{j} \neq j = \bar{i}$ the s-channel contributions must be positive by reflection positivity.

Setting equal the coefficients before $\bullet\!\!-\!\!\circ \quad \circ\!\!-\!\!\bullet$ and $\begin{matrix}\bullet\!\!-\!\!\circ\\ \circ\!\!-\!\!\bullet\end{matrix}$ we get two equations:

$$u^{-d}\left\{1 + G_S^+ + G_S^- - \frac{1}{N}(G_{\text{Adj}}^+ + G_{\text{Adj}}^-)\right\} = v^{-d}\left\{\widetilde{G}_{\text{Adj}}^+ + \widetilde{G}_{\text{Adj}}^-\right\}, \tag{4.80a}$$

and a second one which is just the $u \leftrightarrow v$ version of the first.

Next we consider the transposed 4-point configuration:

$$\left\langle \begin{matrix} \phi_{i\bullet} & {}_\bullet\phi_j \\ \phi_{\bar{i}}^{\dagger\bullet} & {}^\bullet\phi_{\bar{j}}^\dagger \end{matrix} \right\rangle .$$

Equating the s- and t-channel decompositions, we get:

The s-channel decomposition is obtained from the previous case by transposing the index structure *and* flipping the sign of the odd-spin contributions. The t-channel decomposition is obtained by using the second OPE (4.79). The index structure is fixed by (anti)symmetry of the exchanged fields, while the signs are determined by demanding positive contributions for $i = \bar{\imath} \neq j = \bar{\jmath}$ (which makes the configuration reflection-positive in the t-channel).
Collecting coefficients before two inequivalent tensor structures, we get two more equations, which this time are independent:

$$u^{-d}\left\{1 + G_S^+ - G_S^- - \frac{1}{N}G_{\text{Adj}}^+ + \frac{1}{N}G_{\text{Adj}}^-\right\} = v^{-d}\left\{\widetilde{G}_{\square\square} + \widetilde{G}_{\substack{\square\\\square}}\right\}, \tag{4.80b}$$

$$u^{-d}\left\{G_{\text{Adj}}^+ - G_{\text{Adj}}^-\right\} = v^{-d}\left\{\widetilde{G}_{\square\square} - \widetilde{G}_{\substack{\square\\\square}}\right\}. \tag{4.80c}$$

The system (4.80a)-(4.80c) solves the problem of expressing the crossing symmetry constraints. Like in the $SO(N)$ case, we will find it convenient to rewrite it by separating the unit operator contributions and (anti)symmetrizing with respect to u and v. We end up with the following equivalent "vectorial sum rule":

$$\begin{array}{rrrrrrcr}
F_S^+ & +F_S^- & +\left(1-\frac{1}{N}\right)F_{\text{Adj}}^+ & +\left(1-\frac{1}{N}\right)F_{\text{Adj}}^- & & & = & 1\\
H_S^+ & +H_S^- & -\left(1+\frac{1}{N}\right)H_{\text{Adj}}^+ & -\left(1+\frac{1}{N}\right)H_{\text{Adj}}^- & & & = & -1\\
F_S^+ & -F_S^- & -\frac{1}{N}F_{\text{Adj}}^+ & +\frac{1}{N}F_{\text{Adj}}^- & +F_{\square\square} & +F_{\substack{\square\\\square}} & = & 1\\
H_S^+ & -H_S^- & -\frac{1}{N}H_{\text{Adj}}^+ & +\frac{1}{N}H_{\text{Adj}}^- & -H_{\square\square} & -H_{\substack{\square\\\square}} & = & -1\\
 & & F_{\text{Adj}}^+ & -F_{\text{Adj}}^- & +F_{\square\square} & -F_{\substack{\square\\\square}} & = & 0\\
 & & H_{\text{Adj}}^+ & -H_{\text{Adj}}^- & -H_{\square\square} & +H_{\substack{\square\\\square}} & = & 0
\end{array} \tag{4.81}$$

Just like for $SO(N)$, the number of components, six, is again equal to the number of G-functions restricted to spins of definite parity: $G_S^{\pm}$,$G_{\text{Adj}}^{\pm}$,$G_{\square\square}$, $G_{\substack{\square\\\square}}$.

$SU(N)$ **Free case**

Let us start from the s-channel considering the case $i = \bar{j} \neq \bar{\imath} = j$. Then the four point function reads

$$\left\langle \begin{array}{cc} \phi_i(x_1) & \bar{\phi}_i(x_4) \\ \bar{\phi}_j(x_2) & \phi_j(x_3) \end{array} \right\rangle = \frac{1}{(x_{14}x_{23})^2} = \frac{1}{(x_{13}x_{24})^2}\frac{1}{v} = \frac{G^{Adj}_{tot}}{(x_{13}x_{24})^2} \tag{4.82}$$

The above relation can be restated as follows:

$$\frac{z-\bar{z}}{(1-z)(1-\bar{z})} \equiv \frac{1}{1-z} - \frac{1}{1-\bar{z}} = \sum_l p^{Adj}_{l+2,l}\left(k_{2l+2}(z) - k_{2l+2}(\bar{z})\right) \tag{4.83}$$

where we used $k_0(x) = 1$. We can easily check that the above equation is solved by the choice:

$$p^{Adj}_{l+2,l} = \frac{(l!)^2}{(2l)!} \tag{4.84}$$

Indeed we have the following two important relations:

$$\sum_{l=0}^{\infty} \frac{(l!)^2}{(2l)!} k_{2l+2}(z) = \frac{1}{1-z} - 1 \tag{4.85}$$

$$\sum_{l=0}^{\infty} (-1)^l \frac{(l!)^2}{(2l)!} k_{2l+2}(z) = z \tag{4.86}$$

Let us now consider again the s-channel with a different choice of indices:

$$\left\langle \begin{array}{cc} \phi_i(x_1) & \bar{\phi}_j(x_4) \\ \bar{\phi}_i(x_2) & \phi_j(x_3) \end{array} \right\rangle = \frac{1}{(x_{12}x_{34})^2} = \frac{1}{(x_{12}x_{34})^2}\left(1 + \sum p^S_{l+2,l}\ldots - \frac{1}{N}\sum p^{Adj}_{l+2,l}\ldots\right) \tag{4.87}$$

Hence:

$$p^S_{l+2,l} = \frac{1}{N}\frac{(l!)^2}{(2l)!} \tag{4.88}$$

Similarly we can start from equation (4.80c) and get:

$$\tilde{G}_{\square\square} + \tilde{G}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{u} \quad \Rightarrow \quad G_{\square\square} + G_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{v} \tag{4.89}$$

which again is solved by

$$p^{\square\square}_{l+2,l} = \frac{(l!)^2}{(2l)!} \qquad p^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{l+2,l} = \frac{(l!)^2}{(2l)!} \tag{4.90}$$

4.5.3 General case

In this Section we will consider the case of an arbitrary global symmetry group G, with ϕ_α transforming in an irreducible representation R. We aim at a general analysis of crossing symmetry constraints. In particular, we would like to understand why the number of constraints came out equal to the number of unknown functions in the explicit $SO(N)$ and $SU(N)$ examples above. We will assume that R is complex. The case of R real is analogous but simpler; necessary changes will be indicated below.

To understand the group theory aspect of the problem, we begin by counting the number of scalar invariants which can be made out of two ϕ's and two $\phi^\dagger$'s. These invariants can be constructed by decomposing the products $\phi_\alpha \times \phi^\dagger_{\bar\alpha}$ and $\phi_\beta \times \phi^\dagger_{\bar\beta}$ into irreducible representations and contracting those. The tensor product representation decomposes as:

$$R \otimes \bar{R} = \bigoplus_{i=1}^{n} r_i(+\bar{r}_i)\,, \tag{4.91}$$

where $(+\bar{r}_i)$ indicates that the representations in the RHS must be either real or come in complex conjugate pairs. To simplify the discussion, assume for now that all r_i are real and different. In accord with the above decomposition, we have

$$\phi_\alpha \times \phi^\dagger_{\bar\alpha} = \sum_i \sum_{A_i} C^i_{\alpha\bar\alpha A_i} \Psi^i_{A_i}\,, \tag{4.92}$$

where the objects Ψ_{A_i} transform in the r_i, and $C^i_{\alpha\bar\alpha A_i}$ are the Clebsch-Gordan coefficients (A_i is the index in the r_i). Then we can construct exactly n invariant tensors by contracting two Clebsch-Gordan coefficients:

$$T^i_{\alpha\bar\alpha\beta\bar\beta} = \sum_{A_i} C^i_{\alpha\bar\alpha A_i} C^i_{\beta\bar\beta A_i}\,, \tag{4.93}$$

so that the product of two ϕ's and two $\phi^\dagger$'s can be decomposed into a sum of T's:

$$\begin{aligned} \phi_\alpha \phi^\dagger_{\bar\alpha} \phi_\beta \phi^\dagger_{\bar\beta} &= \sum_i \xi_i T^i_{\alpha\bar\alpha\beta\bar\beta} \\ &= \sum_i \tilde{\xi}_i T^i_{\alpha\bar\beta\beta\bar\alpha}\,, \end{aligned} \tag{4.94}$$

where in the second line we indicated that we can do the same construction in a crossed fashion, by starting with the $\phi_\alpha \times \phi^\dagger_{\bar\beta}$ product. The fact that both decompositions exist means that the invariant tensors satisfy a linear relation ('Fierz identity')

$$T^i_{\alpha\bar\alpha\beta\bar\beta} = \mathcal{F}^i_{\ i'} T^{i'}_{\alpha\bar\beta\beta\bar\alpha}\,. \tag{4.95}$$

The matrix $\mathcal{F}$ is invertible and must satisfy $\mathcal{F}^2 = \mathbb{1}$, since crossing is a $\mathbb{Z}_2$ operation.
It is also possible to construct invariants by starting from $\phi_\alpha \times \phi_\beta$, which requires the tensor product

$$R \otimes R = \bigoplus_{j=1}^{n} \tilde{r}_j. \tag{4.96}$$

Assume for now that all $\tilde{r}$'s appearing in this product are also distinct (excluding as well the possibility for the same representation to occur both in the symmetric and antisymmetric part of the tensor product). Under this simplifying assumption, the number of $\tilde{r}$'s is the same as the number of r's. Indeed, we can construct invariant tensors

$$\widetilde{T}^j_{\alpha\beta\bar{\alpha}\bar{\beta}} = \sum_{A_j} C^j_{\alpha\beta A_j} C^j_{\bar{\alpha}\bar{\beta}A_j}\,, \tag{4.97}$$

where $C^j_{\alpha\beta A_j}$ (resp. $C^j_{\bar{\alpha}\bar{\beta}A_j}$) are the Clebsch-Gordan coefficients for $\tilde{r}_j$ in $R \times R$ (resp. $\bar{\tilde{r}}_j$ in $\bar{R} \times \bar{R}$). These must be related to T's by another Fierz identity

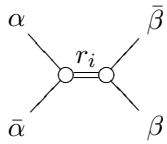
$$T^i_{\alpha\bar{\alpha}\beta\bar{\beta}} = \widetilde{\mathcal{F}}^i{}_j \widetilde{T}^j_{\alpha\beta\bar{\alpha}\bar{\beta}}\,, \tag{4.98}$$

where $\widetilde{\mathcal{F}}$ is again an invertible matrix. Notice however that $T \neq \widetilde{T}$ and thus $\widetilde{\mathcal{F}}^2 \neq \mathbb{1}$.

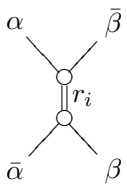
After this prelude, we come back to our problem of analyzing the crossing symmetry constraints of the CFT 4-point function.

Step 1. Let us compare the s- and t-channel conformal block decompositions:

$$\left\langle \begin{matrix} \phi_\alpha \bullet & \bullet \phi^\dagger_{\bar{\beta}} \\ \phi^\dagger_{\bar{\alpha}} \bullet & \bullet \phi_\beta \end{matrix} \right\rangle = \sum_i$$



$$= \sum_i$$



$$. \tag{4.99}$$

Introduce functions G_i which sum up conformal blocks of operators in the representation r_i (which will in general occur in both even and odd spins). The tensor structure of these contributions will be given precisely by the invariant tensors T introduced above. The crossing symmetry constraint then takes the form:

$$\sum_i T^i_{\alpha\bar{\alpha}\beta\bar{\beta}} G_i(u,v) = \sum_i T^i_{\alpha\bar{\beta}\beta\bar{\alpha}} G_i(v,u)\,. \tag{4.100}$$

Here we assume that the signs of T's have been chosen in agreement with reflection positivity. To simplify the notation we included the u^{-d}, v^{-d} prefactors in the definition of G_i. We also do

not separate the unit operator explicitly.
Eq. (4.100) will be consistent with the first Fierz identity (4.95) if and only if

$$G_i(u,v) = \mathcal{F}_i^{\ i'} G_{i'}(v,u)\,. \tag{4.101}$$

Let us now define even and odd combinations:

$$^{(\pm)}G_i = G_i(u,v) \pm G_i(v,u)\,. \tag{4.102}$$

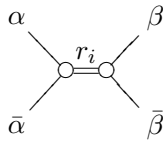
These are the analogues of the F and H functions from Eq. (4.65). We put the index $(\pm)$ on the left to stress that it has nothing to do with the spin parity index used in the previous Sections; these functions receive contributions from both even and odd spins. We have

$$(\mathbb{P}_\pm)_i^{\ i'} G_{i'}^{(\pm)} = 0\,, \tag{4.103}$$

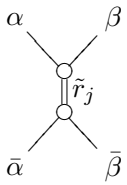
where $\mathbb{P}_\pm = (1 \mp \mathcal{F})/2$ are projectors, $(\mathbb{P}_\pm)^2 = \mathbb{P}_\pm$ by using $\mathcal{F}^2 = \mathbb{1}$. Going to the diagonal basis for $\mathbb{P}_\pm$, it is clear that Eq. (4.103) represents a total of n constraints.

Step 2. We next compare the s- and t-channel conformal block decompositions of the transposed 4-point function:

$$\left\langle \begin{matrix} \phi_\alpha \bullet & \bullet \phi_\beta \\ \phi^\dagger_{\bar\alpha} \bullet & \bullet \phi^\dagger_{\bar\beta} \end{matrix} \right\rangle = \sum_i$$



$$= \sum_j$$



. (4.104)

The crossing symmetry constraint can be written in terms of the invariant tensors introduced above as:

$$\sum_i T^i_{\alpha\bar\alpha\beta\bar\beta}[G_i^+(u,v) - G_i^-(u,v)] = \sum_j \widetilde{T}^j_{\alpha\beta\bar\alpha\bar\beta} G_j(v,u)\,. \tag{4.105}$$

Here we have shown explicitly that the odd spin parts G_i^- of the G_i flip signs compared to the above configuration (4.99). Note as well that each of the functions G_j will include even or odd spins only, depending if $\tilde r_j$ occurs in the symmetric or antisymmetric part of $R \times R$.
For Eq. (4.105) to be consistent with the second Fierz identity (4.98), we must have

$$G_i^+(u,v) - G_i^-(u,v) = \widetilde{\mathcal{F}}_i^{\ j} G_j(v,u)\,. \tag{4.106}$$

Since the functions in the RHS and LHS now refer to completely different OPE channels (r_i in $\phi \times \phi^\dagger$ vs $\tilde r_j$ in $\phi \times \phi$), this equation gives exactly $2n$ constraints when (anti)symmetrizing in

u, v.

To summarize, we expect $3n$ constraints for $3n$ channels $r_i^{\pm}$,$\tilde{r}_j$. In particular, $n = 3$ for the fundamental of $SU(N)$.

In case when R is a real representation, we only have one set of invariant tensors, whose Fierz dictionary matrix satisfies $\mathcal{F}^2 = \mathbb{1}$. In this case each of n representations in the $R \times R$ product will contribute with only even or odd spins. Only the first step of the above analysis is needed in this case. We will get n constraints for n channels. The fundamental of $SO(N)$ corresponds to $n = 3$.

Generalizations. Let us now discuss how one can relax the assumptions on the content of $R \otimes \bar{R}$ and $R \otimes R$ taken in the above argument. In general, $R \otimes \bar{R}$ may contain repetitions of the same representation as well as conjugate pairs, while $R \otimes R$ may contain the same representation in both symmetric (s) and antisymmetric (a) part. As it will become clear below, these two things must happen simultaneously. A sufficiently representative example is $R = \overline{15}$ of $G = SU(3)$ [55]:

$$15 \otimes \overline{15} = 1 + 64 + (8_1 + 8_2) + (27_1 + 27_2) + (10 + \overline{10}) + (35 + \overline{35}) \,, \tag{4.107}$$

$$\overline{15} \otimes \overline{15} = 3_a + \bar{6}_s + 15'_s + 24_a + 42_a + 60_s + (15_s + 15_a) + (24_a + 24_s) \,. \tag{4.108}$$

In $15 \otimes \overline{15}$ we have 8 and 27 appearing twice each, and also two conjugate pairs ($10 + \overline{10}$ and $35 + \overline{35}$), while in $\overline{15} \otimes \overline{15}$, 15 and 24 appear both as s and a.

In cases like this, it is slightly more involved to count the quartic invariants. When counting in the $R \otimes \bar{R}$ channel, every conjugate pair $r + \bar{r}$ gives two invariants which for future purposes we (anti)symmetrize with respect to $(\alpha\bar{\alpha}) \leftrightarrow (\beta\bar{\beta})$:

$$\sum_A C^{r}_{\alpha\bar{\alpha}A} C^{\bar{r}}_{\beta\bar{\beta}A} \pm C^{\bar{r}}_{\alpha\bar{\alpha}A} C^{r}_{\beta\bar{\beta}A} \,. \tag{4.109}$$

In the same channel, a k-fold repetition of a real representation r gives rise to k^2 invariants:

$$\sum_A C^{r_i}_{\alpha\bar{\alpha}A} C^{r_j}_{\beta\bar{\beta}A} \qquad (i, j = 1 \ldots k) \,, \tag{4.110}$$

which can be (anti)symmetrized with respect to $(\alpha\bar{\alpha}) \leftrightarrow (\beta\bar{\beta})$, producing $k(k+1)/2$ symmetrics and $k(k-1)/2$ antisymmetrics.

When counting in the $R \otimes R$ channel, every representation r occurring both as s and a gives rise to 4 invariants

$$\sum_A C^{r_{s/a}}_{\alpha\beta A} C^{\bar{r}_{s/a}}_{\bar{\alpha}\bar{\beta}A} \,, \tag{4.111}$$

out of which two are symmetric and two antisymmetric in $(\alpha\bar{\alpha}) \leftrightarrow (\beta\bar{\beta})$.

The total number of invariants must of course be the same counted in $R \otimes \bar{R}$ and in $R \otimes R$

channel. This is indeed true in the above example, when both $15 \otimes \overline{15}$ and $\overline{15} \otimes \overline{15}$ give 14. The number of symmetric in $(\alpha\bar{\alpha}) \leftrightarrow (\beta\bar{\beta})$ invariants also agrees (10 in both channels). This is also true in general. An intuitive argument is as follows. The total number of invariants equals the number of independent coupling constants in the scalar potential $V(\phi_1, \phi_2^\dagger, \phi_3, \phi_4^\dagger)$ where ϕ_i are four non-identical scalars transforming in R. This number should be the same whether you begin by contracting ϕ_1 with $\phi_2^\dagger$ or ϕ_3. Analogously, the number of *symmetric* invariants is the number of quartic couplings if we identify $\phi_3 \equiv \phi_1$, $\phi_4 \equiv \phi_2$.

Each of the two Fierz identities (4.95) and (4.98) will now split into two, one for symmetric and one for antisymmetric invariants.

Let us now proceed to the crossing symmetry analysis of the 4-point function $\left\langle \phi_\alpha \phi_{\bar{\alpha}}^\dagger \phi_\beta \phi_{\bar{\beta}}^\dagger \right\rangle$. To begin with, out of all the invariant tensors discussed above, only the symmetric ones will appear as the coefficients in the conformal block expansions of this correlator[6]. The $(\alpha\bar{\alpha}) \leftrightarrow (\beta\bar{\beta})$ symmetry is made manifest by applying a conformal transformation which maps a generic 4-point configuration in (4.99) onto a parallelogram (see Section 2.3). The 180° rotation symmetry of the parallelogram then acts on the indices as $(\alpha\bar{\alpha}) \leftrightarrow (\beta\bar{\beta})$, see Fig. 4.3.

To see how this symmetry arises in the conformal block decomposition, consider the OPE

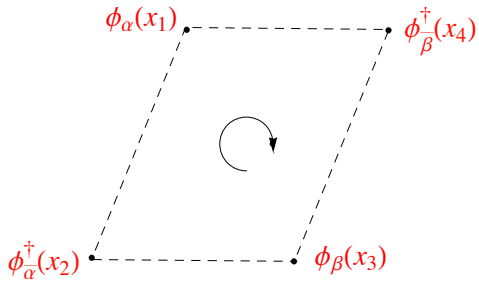


Figure 4.3: *For any 4-point configuration, there exists a conformal transformation which maps it onto a parallelogram.*

$$\phi_\alpha \times \phi_{\bar{\alpha}}^\dagger = \sum_{r \text{ real}} \left(\sum_{i=1}^{k_r} \lambda_O^i C^{r_i}_{\alpha\bar{\alpha}A} \right) O_A \tag{4.112}$$

$$+ \sum_{r+\bar{r} \text{ pairs}} \lambda_O C^r_{\alpha\bar{\alpha}A} O_A + (-1)^l \lambda_O^* C^{\bar{r}}_{\alpha\bar{\alpha}A} O_A^\dagger \,. \tag{4.113}$$

[6]In a general Lorentz-invariant theory, the flavor structure of this correlator will involve both symmetric and antisymmetric tensors.

Here in the first line we include all operators belonging to the real representations. If the representation is repeated k times in the $R\otimes\bar{R}$ product, there are k independent Clebsch-Gordan coefficients, and k independent real OPE coefficients λ_O^i. In the second line we have operators from the complex-conjugate pairs, whose OPE coefficients are always complex-conjugate, up to a spin-dependent minus sign.

By using this OPE in the s-channel conformal block decomposition of (4.99), we see that indeed only symmetric invariant tensors arise. Notice that the off-diagonal invariant tensors ($i \neq j$) in the case of repeated representations will appear with coefficients $\lambda_O^i\lambda_O^j$ ($\times$conformal block), which are not positive definite. We will discuss below what this means for the subsequent application of the derived constraints.

We then consider the t-channel conformal block decomposition of (4.99), and repeat the analysis of Step 1. The resulting number of constraints is equal to the number n_{sym} of symmetric invariants, while the number of representation$\times$(spin parity) channels is $2n_{\text{sym}}$.

To generalize Step 2, we have to consider the t-channel decomposition of (4.104). In this channel, the OPE parity selection rules imply immediately that only symmetric tensor structures appear, in agreement with the above general result. If the s and a representations are not repeated, as in the $\overline{15}\otimes\overline{15}$ example, then only diagonal terms are present, and all conformal blocks enter with positive coefficients. Compared to Step 1, we will have n_{sym} new representation$\times$(spin parity) channels and $2n_{\text{sym}}$ new constraints.

We are done: we have a total $3n_{\text{sym}}$ constraints for $3n_{\text{sym}}$ channels. Moreover, these constraints distinguish not only different representations appearing in the OPE, but also different copies of the same representation, and how they 'interfere' among each other.

Let us now come back to the fact that if repeated representations are present in $R\otimes\bar{R}$, the off-diagonal 'interference' channels have coefficients $\lambda_O^i\lambda_O^j$. To appreciate the difficulty that this creates, readers unfamiliar with our method of linear functionals are encouraged to read the rest of this Section after having read Chapter 6.

Consider then our abstract way of representing the vectorial sum rule as an equation in a linear space V of functions from two variables u, v into $\mathbb{R}^3$ (vector space of vector-functions):

$$\sum p_\alpha \mathbf{x}_\alpha = \mathbf{y}\,. \tag{4.114}$$

Here vectors $\mathbf{x}_\alpha$ represent all vector-functions appearing in the LHS of vector sum rules such as (4.66), while the $\mathbf{y}$ is the vector corresponding to the RHS.

It is crucial for us that when all coefficients p_α are allowed to vary subject to the positivity constraints $p_\alpha \geq 0$, linear combinations in the LHS fill a convex cone. In particular, this allows

us to use the dual formulation of the problem in terms of linear functionals satisfying positivity properties[7]

$$\begin{aligned} &\Lambda[\mathbf{y}] < 0, \\ &\Lambda[\mathbf{x}_\alpha] \geq 0 \qquad \forall \text{ scalar singlets with } \Delta \geq \Delta_S \text{ and} \\ &\qquad\qquad\qquad \forall \text{ other fields (subject to the unitarity bounds)}\,. \end{aligned} \tag{4.115}$$

Since the off-diagonal coefficients may be negative, the geometric interpretation in this case is not as obvious. Notice however that the off-diagonal coefficients cannot become arbitrarily negative since they are not independent of the diagonal ones. For a sharp formulation, consider a symmetric real matrix

$$P_{ij} = \sum_O \lambda_O^i \lambda_O^j\,, \tag{4.116}$$

where we allow for presence of more than one operator O with a given dimension, spin, and representation. The characterizing property of P is positive-definiteness:

$$P_{ij} s_i s_j \geq 0 \qquad \forall s_i \in \mathbb{R}\,. \tag{4.117}$$

Now, as can be seen from this equation, the set of positive-definite matrices forms by itself a convex cone. It follows that the set of vectors in the LHS of the vectorial sum rule will remain a convex cone even if repeated representations are present. Constraints (4.117) replace the simple inequality $p_\alpha \geq 0$. In practical applications these constraints may have to be discretized by choosing a finite set of vectors s_i.

The dual formulation (4.115) is extended to the present case as follows. For the vectors $\mathbf{x}_{ij}$ in the LHS of the sum rule corresponding to diagonal ($i = j$) and off-diagonal ($i \neq j$) channels of the repeated representation, the simple condition $\Lambda[\mathbf{x}_\alpha] \geq 0$ must be replaced by the following condition on the matrix $\Lambda[\mathbf{x}_{ij}]$:

$$P_{ij}\Lambda[\mathbf{x}_{ij}] \geq 0 \quad \forall P \text{ positive-definite.} \tag{4.118}$$

In other words, $\Lambda[\mathbf{x}_{ij}]$ must belong to the cone *dual* to the cone of positive-definite matrices. However, the latter cone is in fact self-dual, as can be easily inferred from the representation (4.116). Thus, $\Lambda[\mathbf{x}_{ij}]$ must be itself positive-definite.

In the above discussion, only real representations were allowed to repeat in $R \otimes \bar{R}$. However, repetitions of complex pairs could be treated similarly; the only difference is that the corresponding P matrices will be positive-definite Hermitian rather than real.

[7] The positivity property 4.115 is used in bounding dimensions of singlet operators (see Section 6.1.2). Different constraints can be imposed for bounds on OPE coefficients see Section 6.2).

Chapter 5

Superconformal field theories

The interest in superconformal field theories is twofold. First of all we would like to be able to compare the results obtained with our analysis with exact results derived for superconformal field theories. In the context of supersymmetric conformal field theories not only there exist exactly soluble models but also there are quantities that can be exactly derived even without solving completely the theory. This includes dimensions of chiral operators, since they are connected with the R-charge, and central charges. In addition, as we will discuss in the next chapter, the analysis of conformal field theories with global symmetries becomes quickly numerically challenging; the presence of supersymmetry provides a relation between the coefficients of otherwise independent conformal blocks. This in turns makes the numerical procedure more powerful and precise, allowing to derive strong bounds for the case of $U(1)$ symmetry (corresponding in this case to the R-charge).
In this chapter we present the investigation of the four point functions of a complex scalar which is the lowest component field of a chiral superfied. All the formalism derived in Section 4.5.1 can be applied in the present context: we will review it briefly in the following. In the analysis we will exploit the following properties of superconformal invariance:

- The operators of the theory are arranged in irreducible representations of the superconformal algebra. These, decomposed with respect to the usual conformal sub-algebra contain a finite number of conformal primary operators, but only one of them is a *superconformal primary*. The others can be obtained acting with the supercharges, which play the role of raising operators in the superspace. Hence the contribution to the four point function of a superconformal representation will be the sum, with fixed coefficients, of a finite number

of conformal blocks.

- Unitarity bounds in presence of the superconformal algebra are more restrictive.
- The OPE of two chiral (or a chiral and an anti-chiral) operators is constrained by superconformal invariance, thus the operators contributing to the four point function must obey to restrictions more stringent then those imposed only by unitarity.

Superconformal theories essentially represent for us an application of the methods discussed in the previous chapter. A complete treatment of this topic would go beyond the goal of this thesis. We will therefore review the basic ingredients and results necessary for the discussion and we will refer to the original works for a complete and more detailed discussions.

5.1 Superconformal algebra

The superconformal algebra represents an extension to superspace of the ordinary conformal algebra. One of the possible way to define it is as the set of transformation acting on the superspace $(x^\mu,\, \theta_\alpha,\, \bar{\theta}_{\dot{\beta}})$ that preserve the super-line element

$$ds^2 = (dx_\mu + i\theta\sigma^\mu\, d\bar{\theta} + i\bar{\theta}\bar{\sigma}^\mu\, d\theta)^2 \tag{5.1}$$

up to a conformal factor. The above condition provides the following differential representation for the superconformal generators:

$$\begin{aligned}
&P_\mu = -i\partial_\mu \qquad Q_\alpha = \frac{\partial}{\partial\,\theta^\alpha} - i\left(\bar{\theta}\bar{\sigma}^\mu\right)_\alpha \partial_\mu\,, \qquad \bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial\,\bar{\theta}^{\dot{\alpha}}} - i\left(\theta\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu \\
&M_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) + i\theta\sigma_{\mu\nu}\frac{\partial}{\partial\,\theta} + i\bar{\theta}\bar{\sigma}_{\mu\nu}\frac{\partial}{\partial\,\bar{\theta}} \\
&D = -i\left(x^\mu\partial_\mu + \frac{1}{2}\theta\frac{\partial}{\partial\,\theta} + \frac{1}{2}\bar{\theta}\frac{\partial}{\partial\,\bar{\theta}}\right) \qquad (5.2)\\
&A = -i\theta\frac{\partial}{\partial\,\theta} + i\bar{\theta}\frac{\partial}{\partial\,\bar{\theta}} \\
&S_\alpha = -2\theta^2\frac{\partial}{\partial\,\theta^\alpha} + i\left(x_\mu - i\xi_\mu\right)\left(\sigma^\mu\frac{\partial}{\partial\,\bar{\theta}}\right)_\alpha - \left(x_\mu + i\xi_\mu\right)\left(\theta\sigma^\mu\bar{\sigma}^\nu\epsilon\right)_\alpha \partial_\nu \qquad (5.3)\\
&\bar{S}_{\dot{\alpha}} = -2\bar{\theta}^2\frac{\partial}{\partial\,\bar{\theta}^{\dot{\alpha}}} + \left(x_\mu + i\xi_\mu\right)\left(\bar{\sigma}^\mu\frac{\partial}{\partial\,\theta}\right)_{\dot{\alpha}} - \left(x_\mu - i\xi_\mu\right)\left(\bar{\theta}\bar{\sigma}^\nu\bar{\sigma}^\mu\epsilon\right)_{\dot{\alpha}} \partial_\nu \\
&K_\mu = ix^2\partial_\mu - i2x_\mu(x^\nu\partial_\nu) - i\xi^2\partial_\mu + 2i\xi_\mu(\xi^\nu\partial_\nu) + i\left(x_\nu + i\xi_\nu\right)\left(\theta\sigma_\mu\bar{\sigma}^\nu\frac{\partial}{\partial\,\theta}\right) \\
&\qquad\qquad + i\left(x_\nu - i\xi_\nu\right)\left(\bar{\theta}\bar{\sigma}_\mu\sigma^\nu\frac{\partial}{\partial\,\bar{\theta}}\right)
\end{aligned}$$

where $\xi^\mu = \theta\sigma^\mu\bar{\theta}$. Exploiting the above representation we can finally extract the $N = 1$ superconformal algebra:

$$
\begin{aligned}
&\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu\,, \qquad [M_{\mu\nu},\, Q_\alpha] = i(\sigma_{\mu\nu}Q)_\alpha\,, \qquad [M_{\mu\nu},\, S_\alpha] = i(\sigma_{\mu\nu}S)_\alpha\,,\\
&[D,\, Q_\alpha] = \frac{i}{2}Q_\alpha \qquad [D,\, \bar{Q}_{\dot\alpha}] = \frac{i}{2}\bar{Q}_{\dot\alpha} \qquad [D,\, S_\alpha] = -\frac{i}{2}S_\alpha \qquad [D,\, \bar{S}_{\dot\alpha}] = -\frac{i}{2}\bar{S}_{\dot\alpha}\\
&[A,\, Q_\alpha] = iQ_\alpha \qquad [A,\, \bar{Q}_{\dot\alpha}] = -i\bar{Q}_{\dot\alpha} \qquad [A,\, S_\alpha] = -iS_\alpha \qquad [A,\, \bar{S}_{\dot\alpha}] = i\bar{S}_{\dot\alpha}\\
&[K_a, Q_\alpha] = -\sigma_{a\alpha\dot\beta}\bar{S}^{\dot\beta}\,, \qquad [K_a, \bar{Q}^{\dot\alpha}] = -\bar{\sigma}_a^{\dot\alpha\beta}S_\beta\,,\\
&[S_\alpha, P_a] = -\sigma_{a\alpha\dot\beta}\bar{Q}^{\dot\beta}\,, \qquad [\bar{S}^{\dot\alpha}, P_a] = -\bar{\sigma}_a^{\dot\alpha\beta}Q_\beta\,,\\
&\{S_\alpha, Q_\beta\} = i\left(2D\epsilon_{\alpha\beta} - 2M_{\alpha\beta} + 3iA\epsilon_{\alpha\beta}\right) \qquad \{\bar{S}^{\dot\alpha}, \bar{Q}^{\dot\beta}\} = i\left(2D\epsilon^{\dot\alpha\dot\beta} - 2M^{\dot\alpha\dot\beta} - 3A\epsilon^{\dot\alpha\dot\beta}\right)\,,\\
&\{S_\alpha,\, \bar{S}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}K_\mu\,.
\end{aligned}
\tag{5.4}
$$

The above algebra coincides with the results of [56] once the generators are redefined in order to match (5.2).

5.2 Representation of the Superconformal Algebra and unitarity bounds

Let us start describing qualitatively the structure of highest weight representations of the superconformal algebra and how they decompose with respect to the conformal sub-algebra. The lowest dimension state of the representation is called *superconformal primary* and satisfies the condition

$$[S_\alpha,\, O] = [\bar{S}_{\dot\alpha},\, O] = [K_\mu,\, O] = 0\,, \tag{5.5}$$

The higher states of the representation can be obtained actioning with the raising operators P_μ ,Q_α, $\bar{Q}_{\dot\alpha}$. The effect of P_μ has already been discussed and reproduces the ordinary descendants operators. The action of the supercharges instead can produces operators that are still conformal primaries but not superconformal primaries any more. For instance, given a field satisfying the condition (5.5) we have

$$[K_\mu,\, [Q_\alpha\,, O]] = i(\sigma_\mu)_{\alpha\dot\beta}[\bar{S}^{\dot\beta}, O] = 0 \tag{5.6}$$

implying that $[Q_\alpha\,, O]$ is again a primary operator. This means that a super field contains several primary operators, corresponding to different powers of θ, $\bar{\theta}$. Notice that all the non-superconformal primaries appearing in a given representation have dimension higher than the

dimension of the superconformal primary. This is by construction a consequence of the fact that the lowest state of the representation is defined as the state annihilated by all the operators that lower the dimension. On the other hand, the lowest state has not necessarily the minimal spin among the primaries. Indeed there can be primary operators with lower spin. Finally, the R-charge of the non-superconformal primary can be at maximum one unit larger or smaller than the one of the superconformal primary.[1]

An irreducible representation of the super conformal algebra with $\mathcal{N} = 1$ is labelled by 4 numbers:

$$(q\,,\bar{q}\,,j_1\,,j_2) \tag{5.7}$$

where $q\,,\bar{q}$ are related to the scaling dimension and the $R-$charge of the superconformal primary according to:

$$\Delta = q + \bar{q}\,, \qquad \frac{3}{2}R = q - \bar{q} \tag{5.8}$$

The unitarity bounds read ([57]):

$$\begin{aligned} & q \geq j_1 + 1\,, \qquad \bar{q} \geq j_2 + 1\,, \\ \Rightarrow \quad & \Delta \geq \left|\frac{3}{2}R - j_1 + j_2\right| + j_1 + j_2 + 2\,. \end{aligned} \tag{5.9}$$

The equality in the first line of (5.9) is realized by the so called semi-conserved currents, for which either q or $\bar{q}$ saturates the bound. For instance if $\bar{q} = j_2 + 2$ then we have

$$\frac{3}{2}R = j_1 - j_2 \qquad \Delta = \frac{3}{2}R + 2j_2 + 2 = -\frac{3}{2}R + 2j_1 + 2 \tag{5.10}$$

In the language of superfields the above operators corresponds to currents satisfying the constraint

$$D^{\alpha_1} J_{\alpha_1 \ldots \alpha_{j_1}, \dot{\beta}_1 \ldots \dot{\beta}_{j_2}} = 0\,. \tag{5.11}$$

If both q and $\bar{q}$ saturate the unitarity bounds then:

$$\Delta = j_1 + j_2 + 2 \qquad \frac{3}{2}R = j_1 - j_2\,. \tag{5.12}$$

In this case the operator is a conserved current and satisfies:

$$D^{\alpha_1} J_{\alpha_1 \ldots \alpha_{j_1}, \dot{\beta}_1 \ldots \dot{\beta}_{j_2}} = \bar{D}^{\dot{\beta}_1} J_{\alpha_1 \ldots \alpha_{j_1}, \dot{\beta}_1 \ldots \dot{\beta}_{j_2}} = 0\,. \tag{5.13}$$

[1] This because the expansion in Grassman variables ends at quadratic order.

Finally, if the superfield is real then its R-charge vanishes and $j_1 = j_2$. These operators corresponds to traceless symmetric tensors of rank $l = 2j_1 = 2j_2$.
As in usual supersymmetry we can have multiplet shortening, with therefore a different structure of the unitarity bounds. In the case in which one of the parameters, say $\bar{q}$, vanishes while the other is non-zero, $q \geq j + 1j$ we have:

$$\Delta = \frac{3}{2}R \geq j + 1 \tag{5.14}$$

The above relation defines the so called *chiral* superfields, for which it holds

$$\bar{D}^{\dot{\beta}} J_{\alpha_1 \ldots \alpha_j} = 0. \tag{5.15}$$

Chiral superfields saturating the unitarity bound satisfy also the condition

$$D^{\alpha_1} J_{\alpha_1 \ldots \alpha_j} = 0. \tag{5.16}$$

We could also define antichiral operators, for which we have

$$q = 0\, \bar{q} > j + 1\, \Delta = -\frac{3}{2}R \geq j + 1 \tag{5.17}$$

Finally the unit operator corresponds to $q = \bar{q} = j_1 = j_2 = 0$.

5.3 $\mathcal{N} = 1$ OPE of Chiral Superfields

Before presenting the explicit form of the superconformal blocks let us review the OPE expansion for chiral fields. This will be crucial to understand what are the operators allowed to contribute to the four point function an will let us restrict the number of constraints to impose.
Consider now the OPE of a chiral superfields $\Phi(x, \theta, \bar{\theta})$ with dimension $d = \frac{3}{2}R_\Phi$ with itself:

$$\Phi \times \Phi \tag{5.18}$$

According to the usual relation between the OPE coefficient and the three point function the above expression will contain the superconformal primary operator O_I (here I represents the space-time indices) if and only if the correlator

$$\langle \Phi \Phi O_I^\dagger \rangle \tag{5.19}$$

is non vanishing. A complete characterization of the general form of three point function of generic superconformal primaries can be found in [58]. For the case of two scalar superfields and

a third spin-l superfield, it can be written as

$$\begin{aligned}
&\langle \Phi(x_1,\theta_1,\bar\theta_1)\Phi(x_2,\theta_2,\bar\theta_2)O_I^\dagger(x_3,\theta_3,\bar\theta_3)\rangle = \frac{t_I(X_3,\Theta_3,\bar\Theta_3)}{x_{\bar 31}^{2d}x_{\bar 32}^{2d}}\,,\\
&x_{\bar i j} = x_{i-} + x_{j+} + 2i\theta_i\sigma\bar\theta_j\,, \qquad x_{i\pm} = x \pm i\theta_i\sigma\bar\theta_i\,,\\
&X_3^\mu = -\frac{1}{2}\frac{x_{\bar 31}^\nu x_{\bar 12}^\rho x_{\bar 23}^\gamma}{x_{\bar 31}^2 x_{\bar 32}^2}\mathrm{Tr}[\bar\sigma^\mu\sigma_\nu\bar\sigma_\rho\sigma_\gamma]\\
&\Theta_3 = i\frac{x_{\bar 31}^\mu}{x_{\bar 13}^2}\sigma_\mu\bar\theta_{31} - i\frac{x_{\bar 32}^\mu}{x_{\bar 23}^2}\sigma_\mu\bar\theta_{32}\,, \qquad \bar\Theta_3 = \Theta_3^\dagger\,.
\end{aligned} \tag{5.20}$$

where $t_I(X_3,\Theta_3,\bar\Theta_3)$ has to be determined. In addition, further restrictions must be imposed. First, the correct transformation properties under the superconformal group are realized if t_I satisfies the homogeneity condition:

$$t_I(\lambda\bar\lambda X_3,\lambda\Theta_3,\bar\lambda\bar\Theta_3) = \lambda^{2a}\bar\lambda^{2\bar a}t_I(X_3,\Theta_3,\bar\Theta_3)\,, \tag{5.21}$$

$$a = \frac{1}{3}\left(2q_O + \bar q_O - 4d\right)\,, \qquad \bar a = \frac{1}{3}\left(2\bar q_O + q_O - 2d\right)\,. \tag{5.22}$$

Moreover, the chirality of Φ translates in the condition:

$$\begin{aligned}
\bar D_{\dot\alpha}\frac{t_I(X_3,\Theta_3,\bar\Theta_3)}{x_{\bar 31}^{2d}x_{\bar 32}^{2d}} &= \frac{\bar D_{\dot\alpha}t_I(X_3,\Theta_3,\bar\Theta_3)}{x_{\bar 31}^{2d}x_{\bar 32}^{2d}}\\
&= -i\frac{1}{x_{\bar 31}^{2d}x_{\bar 32}^{2d}}\frac{(x_{\bar 13})^{\dot\alpha\alpha}}{x_{\bar 31}}\left(\frac{\partial}{\partial\Theta_3^\alpha} - 2i(\sigma^\mu\bar\Theta_3)_\alpha\frac{\partial}{\partial X_3^\mu}\right)t_I(X_3,\Theta_3,\bar\Theta_3)
\end{aligned} \tag{5.23}$$

which has the general solution:

$$\begin{aligned}
&t_I(X_3,\Theta_3,\bar\Theta_3) = t_I(\bar X_3,\bar\Theta_3)\,,\\
&\bar X_3 = X_3 + 2i\Theta\sigma\bar\Theta\,.
\end{aligned} \tag{5.24}$$

At this point we can look for a generic function of the above form that satisfies the homogeneity condition (5.21):

$$\begin{aligned}
t_I(\bar X_3,\bar\Theta_3) \sim \bar X_3^m\bar\Theta^n\,, \qquad n = 0,1,2\,,\\
2a = m\,, \qquad 2\bar a = n+m
\end{aligned} \tag{5.25}$$

A final constraint come from the invariance under exchange $1 \leftrightarrow 2$, since the chiral operators are the same. This requirement translate in the invariance under $\bar X_3 \to -X_3\,, \bar\Theta_3 \to -\Theta_3\,,$. We found three possible solutions consistent with all the constraints:

$$\begin{aligned}
&\text{constant}, && R_O = \frac{4}{3}d\,, \quad \Delta_O = 2d\,, \quad l = 0\\
&\bar\Theta_3\bar X_3^{\mu_1}...\bar X_3^{\mu_l}\,, && R_O = \frac{4}{3}d - 1\,, \quad \Delta_O = 2d + l + \frac{1}{2}\,, \quad \text{l odd}\\
&\bar\Theta_3^2\bar X_3^{\Delta_O - 2d - 1 - l}\bar X_3^{\mu_1}...\bar X_3^{\mu_l}\,, && R_O = \frac{4}{3}d - 2\,, \quad \Delta_O \geq |2d-3| + l + 2 \quad \text{l even.}
\end{aligned} \tag{5.26}$$

We finally conclude that only the following superconformal primary can be present in the OPE (5.18):

- Chiral operator with $\Delta = \frac{3}{2}R = 2d$ and $l = 0$. We will denote it Φ^2.
- Non-Chiral operators with $j_1 - j_2 = 1/2$, $l = 2j_2$ odd, $R = \frac{4}{3}d - 1$ and dimension
$$\Delta_O = 2d + l + \frac{1}{2}\,. \tag{5.27}$$
Notice that this operators saturate the unitarity bound (5.10) and corresponds to a semi-conserved current in the sense of (5.11).
- Non-Chiral operators with $l = 2j_1 = 2j_2$ even, $R = \frac{4}{3}d - 2$ and dimension
$$\Delta_O \geq |2d - 3| + l + 2\,. \tag{5.28}$$

We should also stress that none of the expressions in (5.26) can be further expanded in $\Theta_3\sigma\bar{\Theta}_3$ (which is hidden inside $\bar{X}_3$)[2]. This fact has a very crucial consequence: if we take $\theta_1 = \theta_2 = 0$ in (5.19) and we expand in $\theta_3, \bar{\theta}_3$ we obtain a series in the Grassman variables whose coefficients are three point function of two scalars with the lowest component of O_I or one of its super-descendants. On the other hand, the expressions in (5.26) contain only one term each. This means that there is only one operator in the superconformal representation, with the correct R-charge, that have a non vanishing three point function with $\phi\phi$.
From the above results we can infer what are the operators appearing in the OPE of the lowest component of the chiral superfield Φ with itself. Call ϕ this scalar field with dimension d than $\phi \times \phi$ receive contributions from the lowest component of

- Chiral superfield Φ^2, call it ϕ^2.
- super-descendant $(\sigma^\mu)^{\beta\dot{\alpha}}\bar{Q}_{\dot{\alpha}}O_\beta^{\mu_1\cdots\mu_l}$. In this case the operator contributing to the OPE $\phi \times \phi$ is a $(l+1)$-rank tensor, $(l+1)$ even and:
$$R_{\bar{Q}O} = \frac{4}{3}d\,, \qquad \Delta_{\bar{Q}O} = 2d + (l+1)\,. \tag{5.29}$$
Notice that in the free case, $d = 1$, the above operators are precisely twist-2 operators with even spin, as expected from the expansion of the four point function $\langle\phi\phi\phi^\dagger\phi^\dagger\rangle$ in the s-channel in the non supersymmetric case.

[2] While in the third line of (5.26) the expansion is trivial since $\bar{\Theta}_3^2$ already saturates the antisymmetric properties of Grassman variables, the second line of requires a more delicate analysis.

- super-descendant $\bar{Q}^2 O$. In this case the operator contributing to the OPE $\phi \times \phi$ is a l-rank tensor, l even and:

$$R_{\bar{Q}^2 O} = \frac{4}{3} d \,, \qquad \Delta_{\bar{Q}^2 O} \geq |2d-3| + l + 3 \,. \tag{5.30}$$

We will also need the general structure of the OPE

$$\Phi \times \Phi^\dagger \tag{5.31}$$

As before we need to study the form of the correlator

$$\langle \Phi \Phi^\dagger O_I^\dagger \rangle \tag{5.32}$$

allowed by superconformal symmetry. Starting from the general form ([58])

$$\begin{aligned}
&\langle \Phi(x_1, \theta_1, \bar{\theta}_1) \Phi^\dagger(x_2, \theta_2, \bar{\theta}_2) O_I^\dagger(x_3, \theta_3, \bar{\theta}_3) \rangle = \frac{t_I(X_3, \Theta_3, \bar{\Theta}_3)}{x_{\bar{3}1}^{2d} x_{\bar{3}2}^{2d}} \,, \\
&t_I(\lambda \bar{\lambda} X_3, \lambda \Theta_3, \bar{\lambda} \bar{\Theta}_3) = \lambda^{2a} \bar{\lambda}^{2\bar{a}} t_I(X_3, \Theta_3, \bar{\Theta}_3) \,, && (5.33) \\
&a = \frac{1}{3}\left(2 q_O + \bar{q}_O - 3d\right) \,, \qquad \bar{a} = \frac{1}{3}\left(2 \bar{q}_O + q_O - 3d\right) \,, && (5.34)
\end{aligned}$$

we can supplement the chirality condition of the Φ, imposing $t_I(X_3, \Theta_3, \bar{\Theta}_3) = t_I(\bar{X}_3, \bar{\Theta}_3)$, with the anti-chirality condition of $\Phi^\dagger$:

$$0 = D_\alpha \frac{t_I(\bar{X}_3, \bar{\Theta}_3)}{x_{\bar{3}1}^{2d} x_{\bar{3}2}^{2d}} = -i \frac{1}{x_{\bar{3}1}^{2d} x_{\bar{3}2}^{2d}} \frac{(x_{2\bar{3}})^{\dot{\alpha}\alpha}}{x_{\bar{2}3}^2} \frac{\partial}{\partial \bar{\Theta}_3^{\dot{\alpha}}} t_I(\bar{X}_3, \bar{\Theta}_3) \tag{5.35}$$

The only possibility is to have t^I depending only on $\bar{X}_3^\mu$. In the end the only structure admitted for the three point function is

$$\begin{aligned}
&\langle \Phi(x_1, \theta_1, \bar{\theta}_1) \Phi^\dagger(x_2, \theta_2, \bar{\theta}_2) O_I^\dagger x_3, \theta_3, \bar{\theta}_3 \rangle = \frac{C_{\Phi \Phi^\dagger O}}{x_{\bar{3}1}^{2d} x_{\bar{3}2}^{2d}} \left(\bar{X}_3^{\Delta_O - 2d - l} \bar{X}_3^{\mu_1} ... \bar{X}_3^{\mu_l} - \text{traces} \right) \,, \\
&R_O = 0 \,, \quad \text{l integer} \,, \quad \Delta_O \geq l + 2. && (5.36)
\end{aligned}$$

where we used the unitarity bound (5.9) for traceless rank$-l$ operator with vanishing R-charge. Setting $\theta_{1,2} = \bar{\theta}_{1,2} = 0$ in the above expression one can therefore reduce to the three point function of the lowest component $\phi\, \phi^\dagger$, with the a third superfield O. Expanding in $\theta_3, \bar{\theta}_3$ and matching the corresponding terms one can finally extract the contribution of the superconformal primary (the zeroth order term) and those of the other conformal primaries contained in the superfield O. Clearly only operators with vanishing R-charge will contribute to the three point function, since only those operators can appear in the $\phi \times \phi^\dagger$ OPE. This operators corresponds

to the term linear and quadratic in the combination $\theta\sigma^\mu\bar{\theta}$ and therefore corresponds to spin and dimension $(\Delta+1, l+1)$, $(\Delta+1, l-1)$, and $(\Delta+2, l)$. Moreover the relative coefficients are totally fixed by supersymmetry and the only unknown quantity is the overall constant $C_{\Phi\Phi^\dagger O}$ appearing in the three point function (5.36).
This analysis has been performed in in [54] and we refer to the original paper for all the details.

5.4 $\mathcal{N}=1$ Superconformal blocks

According to the discussion of the previous section we can now infer the most general form of the four point function

$$\langle\phi(x_1)\phi^\dagger(x_2)\phi(x_3)\phi^\dagger(x_4)\rangle\,, \tag{5.37}$$

allowed by superconformal invariance. Here ϕ is the lowest component of a chiral superfield with dimension d. Generically the above correlator can be expressed in a sum of conformal blocks. Recalling also the discussion of Section 4.5.1 we notice that there are two alternative and equivalent ways to express the above four point function. The s-channel corresponds to take the OPE $\phi(x_1)\times\phi^\dagger(x_2)$ and $\phi(x_3)\times\phi^\dagger(x_4)$. These OPE's contain operators with vanishing R-charge[3] and integer spin, even and odd. Moreover the contribution of conformal primaries belonging to the same superconformal representation are related by known coefficients and can be grouped in *superconformal blocks*:

$$\langle\phi(\underbrace{x_1)\phi^\dagger(}x_2)\phi(\underbrace{x_3)\phi^\dagger(}x_4)\rangle = \frac{1}{x_{12}^{2d}x_{34}^{2d}}\left(1+\sum_{\Delta\geq l+2}p_{\Delta,l}\mathcal{G}_{\Delta,l}(u,v)\right)\,, \tag{5.38}$$

where we defined

$$\begin{aligned}\mathcal{G}_{\Delta,l}(u,v) = g_{\Delta,l}(u,v) &+ \frac{(\Delta+l)}{4(\Delta+l+1)}g_{\Delta+1,l+1}(u,v) + \frac{(\Delta-l-2)}{4(\Delta-l-1)}g_{\Delta+1,l-1}(u,v)\\ &+ \frac{(\Delta+l)(\Delta-l-2)}{16(\Delta+l+1)(\Delta-l-1)}g_{\Delta+2,l}(u,v)\,.\end{aligned} \tag{5.39}$$

As discussed at the end of the previous section the superconformal blocks encodes the contribution of four operators with dimension and spin $(\Delta+1, l+1)$, $(\Delta+1, l-1)$, and $(\Delta+2, l)$. The difference from [54] is only due to a different normalization of the ordinary conformal block: we have removed the pre-factor $(-1/2)^l$ (see eq. 4.30).

[3] In supersymmetry the role of the $U(1)$ symmetry is played by the R-charge. Although this symmetry is not properly a global symmetry in superspace, it commutes with the conformal subalgebra.

Performing the OPE expansion in the t-channel produces again an expansion in terms of superconformal blocks, with a different dependence on coordinates:

$$\langle \phi(x_1)\phi^\dagger(x_2)\phi(x_3)\phi^\dagger(x_4)\rangle = \frac{1}{x_{14}^{2d}x_{23}^{2d}}\left(1+\sum_{\Delta\geq l+2} p_{\Delta,l}\mathcal{G}_{\Delta,l}\left(v,u\right)\right) \tag{5.40}$$

Equating the two expansions we get one of the crossing symmetry constraint for $U(1)$-invariant theories, except that now only superconformal blocks enter in the sum rule:

$$\begin{aligned}\sum_{\Delta\geq l+2} p_{\Delta,l}\mathcal{F}_{\Delta,l} &= 1\,, \\ \mathcal{F}_{\Delta,l} &= \frac{u^{-d}\mathcal{G}_{\Delta,l}\left(u,v\right)-v^{-d}\mathcal{G}_{\Delta,l}\left(v,u\right)}{v^{-d}-u^{-d}}\end{aligned} \tag{5.41}$$

Additional constraints can be derived as usual considering the $s-$ and $t-$ channel expansions of a different four point function. Let us start from the following decomposition

$$\begin{aligned}\langle \phi(x_1)\phi^\dagger(x_2)\phi^\dagger(x_3)\phi(x_4)\rangle &= \frac{1}{x_{12}^{2d}x_{34}^{2d}}\left(1+\sum_{\Delta\geq l+2} p_{\Delta,l}\mathcal{G}_{\Delta,l}\left(\frac{u}{v},\frac{1}{v}\right)\right) \\ &= \frac{1}{x_{12}^{2d}x_{34}^{2d}}\left(1+\sum_{\Delta\geq l+2} (-1)^l p_{\Delta,l}\widetilde{\mathcal{G}}_{\Delta,l}\left(u,v\right)\right)\,,\end{aligned} \tag{5.42}$$

where we have used the known property of the conformal blocks (4.46) and we have defined

$$\begin{aligned}\widetilde{\mathcal{G}}_{\Delta,l}(u,v) = g_{\Delta,l}(u,v) &- \frac{(\Delta+l)}{4(\Delta+l+1)}g_{\Delta+1,l+1}(u,v) - \frac{(\Delta-l-2)}{4(\Delta-l-1)}g_{\Delta+1,l-1}(u,v) \\ &+ \frac{(\Delta+l)(\Delta-l-2)}{16(\Delta+l+1)(\Delta-l-1)}g_{\Delta+2,l}(u,v)\,.\end{aligned} \tag{5.43}$$

The $t-$channel decomposition corresponds to take OPE's $\phi\times\phi$ and its conjugate. These OPE's contains operators with $R-$charge twice the R-charge of ϕ. As discussed in the previous section only one primary operator per representation contributes, that is to say superconformal blocks in this channel reduce to ordinary conformal blocks. Moreover the dimensions of the primary operators are subject to the constraint (5.29) or the bound (5.30). Hence:

$$\langle \phi(x_1)\phi^\dagger(x_2)\phi^\dagger(x_3)\phi(x_4)\rangle = \frac{1}{x_{14}^{2d}x_{23}^{2d}}\left(\sum_{l\text{ even}} p^{R=2}_{2d+l,l}\, g_{2d+l,l}\left(v,u\right) + \sum_{\substack{\Delta\geq|2d-3|+l+3\\ l\text{even}}} p^{R=2}_{2d+l,l}\, g_{\Delta,l}\left(v,u\right)\right) \tag{5.44}$$

Notice that the first term in parenthesis, when $l=0$, contains the contribution of the chiral operator Φ^2. Equating the two expansion we finally get two more sum rules:

$$\sum_{\Delta\geq l+2} p_{\Delta,l}(-1)^l\widetilde{\mathcal{F}}_{\Delta,l} + \sum_{\substack{\Delta=2d+l\\ \Delta\geq|2d-3|+l+3\\ l \text{ even}}} p^{R=2}_{\Delta,l}F_{\Delta,l}=1\,, \tag{5.45}$$
$$\sum_{\Delta\geq l+2} p_{\Delta,l}(-1)^l\widetilde{\mathcal{H}}_{\Delta,l} - \sum_{\substack{\Delta=2d+l\\ \Delta\geq|2d-3|+l+3\\ l \text{ even}}} p^{R=2}_{\Delta,l}H_{\Delta,l}=-1\,,$$

where F,H are defined in (4.65) and

$$\widetilde{\mathcal{F}}_{\Delta,l}=\frac{u^{-d}\widetilde{\mathcal{G}}_{\Delta,l}\left(u,v\right)-v^{-d}\widetilde{\mathcal{G}}_{\Delta,l}\left(v,u\right)}{v^{-d}-u^{-d}}$$
$$\widetilde{\mathcal{H}}_{\Delta,l}=\frac{u^{-d}\widetilde{\mathcal{G}}_{\Delta,l}\left(u,v\right)+v^{-d}\widetilde{\mathcal{G}}_{\Delta,l}\left(v,u\right)}{v^{-d}+u^{-d}} \tag{5.46}$$

The sum rule (5.41) has been used in [54] to derive bounds on the dimension of the lowest dimension scalar operator entering in in the OPE $\phi\times\phi^\dagger$. In the next chapter we will reproduce their results and show that the additional use of (5.45) allows the extraction of stronger results with less numerical effort.

Free Theory

In order to check that we didn't miss any sign we can compute the expansion in superconformal blocks and verity that the obtained spectrum and coefficient solve the vectorial sum rule. A theory of a complex free scalar can be trivially made supersymmetric adding a free Weyl fermion. Since both the fields are free there is no modification of the OPE's nor of the correlation functions. Hence we know that the OPE of a complex field contains only twist-2 operators (see Section 4.5.1). From the analysis of $U(1)$ theories we also know the decomposition of the four point function in terms of conformal blocks. On the other hand here we are interested in the superconformal block decomposition:

$$\left\langle\phi(x_1)\phi^\dagger(x_2)\phi(x_3)\phi^\dagger(x_4)\right\rangle=\frac{1}{x_{12}^2x_{34}^2}\left(1+\sum_l p_{l+2,l}\left(g_{l+2,l}(u,v)+\frac{2l+2}{2l+3}g_{l+3,l+1}(u,v)\right)\right)$$

In the above expansion we have collected together the contribution to the four point function of an entire supermultiplet. It easy to see that the coefficients satisfy

$$p_{l+2,l}=\frac{(l!)^2}{(2l)!}\frac{l+1}{2l+1} \tag{5.47}$$

On the other hand, the coefficient $p^{R=2}_{\Delta,l}$ are exactly the same as in the free complex scalar:

$$p^{R=2}_{l+2,l} = (1 + (-1)^l)\frac{(l!)^2}{(2l)!} \tag{5.48}$$

One can construct plots similar to Fig. 4.2 and verify the convergence of the three sum rules (5.41), (5.45).

Chapter 6

Bounds and numerical results

We now show how we can extract non trivial information from the sum rules derived in the previous chapters. We assume that we are given a unitary CFT with a set of primary scalar operators ϕ_i of dimension $d \geq 1$. We consider 4-point functions involving only the above fields and the constraints imposed by crossing symmetry. We will use only the most general information about the operators appearing in the OPE $\phi_i \times \phi_j$, such as:

1. only the operators satisfying the unitarity bounds)2.2) may appear;
2. Bose symmetry can forbid even or odd spins to appear;
3. all the coefficients $p_{\Delta,l}$ entering the sum rule are non-negative.

We will focus on two particular classes of information: the spectrum of the operators entering the OPE's and the size of their OPE coefficients. More specifically we will be able to put an upper bound on the lower-dimension scalar operator which appears in the OPE of two scalar fields; in other words, we will show that if only scalar operators of dimension $\Delta > f(d)$ are allowed to appear in the OPE, the sum rule cannot be satisfied no matter what are the dimensions, spins, and OPE coefficients of all the other operators (as long as they satisfy the above assumptions 1,2,3). Thus such a CFT cannot exist! In the process of proving this, we will also derive the value of $f(d)$.
A second kind of information concerns, as mentioned, the value of OPE coefficients. Again we will show that the sum rules admit solution only if the OPE coefficient $c_{\Delta,l}$ of a chosen operator $\mathcal{O}$ is smaller than a given function of d.

We would like to stress that, although for phenomenological reasons we always concentrate on the dimension of the lowest scalar operator entering an OPE, the method we present in this chapter has a wider range of application and can be use to explore further the structure of CFT's.

6.1 Bounding scalar operator dimension

Let us begin with a very simple example which should convince the reader that some sort of bound should be possible, at least for d sufficiently close to 1. For our preliminary discussion we restrict to the case without symmetries: in this case we have to deal with the single sum:

$$\sum_{\Delta,l} p_{\Delta,l} F_{\Delta,l}(u,v) = 1\,, \tag{6.1}$$

All the quantities appearing in the above expression have been defined in Section. 4.3.
The argument involves some numerical exploration of functions $F_{d,\Delta,l}$ entering the sum rule (6.1), easily done e.g. with MATHEMATICA. These functions depend on two variables $z, \bar{z}$, but for now it will be enough to explore the case $0 < z = \bar{z} < 1$. We begin by making a series of plots of $F_{d,\Delta,l}$ for $l = 2, 4$ and for Δ satisfying the unitarity bound $\Delta \geq l + 2$ appropriate for these spins (Fig. 6.1). The scalar case $l = 0$ will be considered below. We take $d = 1$ in these plots.

We see is that all these functions have a rather similar shape: they start off growing mono-

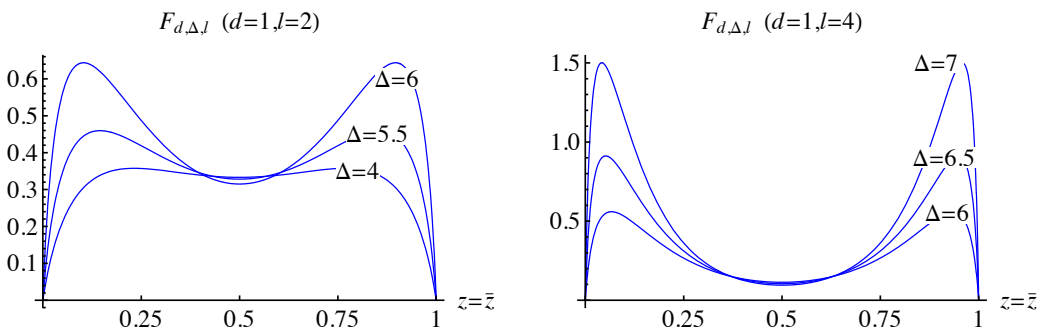


Figure 6.1: The shape of $F_{d,\Delta,l}$ for $d = 1$, $l = 2, 4$ and several values of Δ satisfying the unitarity bound.

tonically as z deviates from the symmetric point $z = 1/2$, and after a while decrease sharply as

$z \to 0, 1$. These characteristics become more pronounced as we increase l and/or Δ. We invite the reader to check that, for $d = 1$, these properties are in fact true for all $F_{d,\Delta,l}$ for even $l \geq 2$ and $\Delta \geq l + 2$. By continuity, they are also true for $d = 1 + \varepsilon$ as long as $\varepsilon > 0$ is sufficiently small.[1] Mathematically, we can express the fact that $F_{d,\Delta,l}$ is downward convex near $z = 1/2$ as:

$$\begin{aligned} &F''_{d,\Delta,l} > 0 \quad \text{at } z = \bar{z} = 1/2, \\ &l = 2, 4, 6 \ldots, \quad \Delta \geq l + 2, \\ &1 \leq d \leq 1 + \varepsilon\,. \end{aligned} \tag{6.2}$$

Even before addressing the existence of the bound, let us now ask a very basic question, namely whether a CFT without any scalars in the OPE $\phi \times \phi$ could exist. Eq. (6.2) immediately implies that the answer is negative, at least if d is sufficiently close to 1.

In fact, in such a CFT, the sum rule (6.1) would have to be satisfied with only $l \geq 2$ terms present in the RHS. Applying the second derivative to the both sides of (6.1) and evaluating at $z = \bar{z} = 1/2$, the LHS is identically zero, while in the RHS, by (6.2), we have a sum of positive terms with positive coefficients. This is a clear contradiction, and thus such a CFT does not exist.

To rephrase what we have just seen, the sum rule must contain some terms with negative $F''(z = 1/2)$ to have a chance to be satisfied, and by (6.2) such terms can come only from $l = 0$. Thus, the next natural step is to check the shape of $F_{d,\Delta,l}$ for $l = 0$, which we plot for several $\Delta \geq 2$ in Fig. 6.2. We see that the second derivative in question is negative at $\Delta = 2$ (it better be since this corresponds to the free scalar theory which surely exists!). By continuity, it is also negative for Δ near 2. However, and this is crucial, it turns *positive* for Δ above certain critical dimension Δ_c between 3 and 4. It is not difficult to check that in fact $\Delta_c \simeq 3.61$ for d near 1.

We arrive at our main conclusion: not only do some scalars have to be present in the OPE, but at least one of them should have $\Delta \leq \Delta_c$, otherwise such a CFT will be ruled out by the same argument as a CFT without any scalars in the OPE. In other words, we have just established the bound $\Delta_{\min} \leq \Delta_c$ for d near 1.

Admittedly, this first result is extremely crude: for instance, the obtained bound does not approach 2 as $d \to 1$. However, what is important is that it already contains the main idea of the method which will be developed and used with increasing refinement below. This idea is that we have to look for a differential operator which gives zero acting on the unit function in the

[1] One can check that they are true up to $d \simeq 1.12$. For larger d, $F''_{d,4,2}(z = 1/2)$ becomes negative.

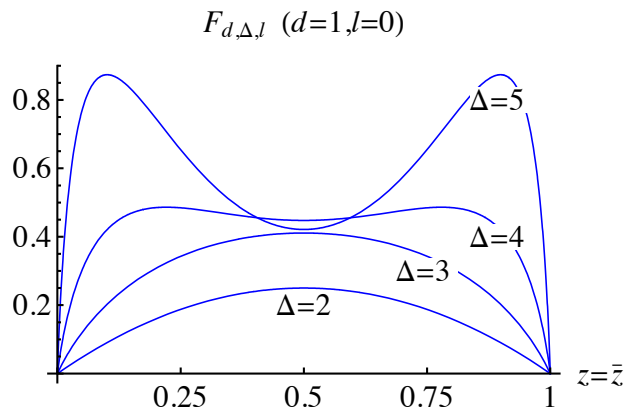


Figure 6.2: Same as Fig. 6.1, for $l = 0$.

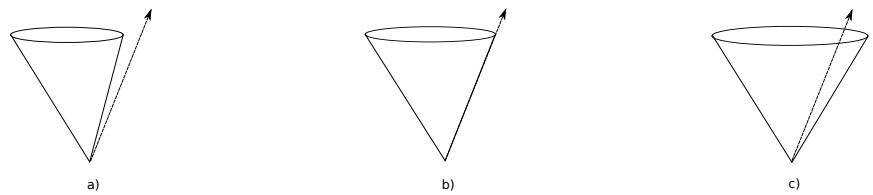
LHS of the sum rule, but stays positive when applied to the functions $F_{d,\Delta,l}$ in the RHS.

6.1.1 Geometry of the sum rule

To proceed, it is helpful to develop a geometric understanding of the sum rule. Given d and a spectrum $\{\Delta, l\}$ of $\mathcal{O} \in \phi \times \phi$, and allowing for arbitrary positive coefficients $p_{\Delta,l}$, the linear combinations in the RHS of (6.1) form, in the language of functional analysis, a *convex cone* $\mathcal{C}$ in the function space $\{F(a, b)\}$. For a fixed spectrum, the sum rule can be satisfied for *some* choice of the coefficients if and only if the unit function $F(a, b) \equiv 1$ belongs to this cone.
Obviously, when we expand the spectrum by allowing more operators to appear in the OPE, the cone gets wider. Let us consider a one-parameter family of spectra:

$$\Sigma(\Delta_{\min}) = \{\Delta, l \ \mid \ \Delta \geq \Delta_{\min}\,(l = 0), \quad \Delta \geq l + 2\ (l = 2, 4, 6 \ldots)\}\ . \tag{6.3}$$

Thus we include all scalars of dimension $\Delta \geq \Delta_{\min}$, and all higher even spin primaries allowed by the unitarity bounds.
The crucial fact which makes a bound possible is that in the limit $\Delta_{\min} \to \infty$ the convex cone generated by the above spectrum does *not* contain the function $F \equiv 1$. In other words, CFTs without any scalars in the OPE $\phi \times \phi$ cannot exist, as we already demonstrated in Section6.1 for d sufficiently close to 1.
As we lower $\Delta_{\min}$, the spectrum expands, and the cone gets wider. There exists a critical value Δ_c such that for $\Delta_{\min} > \Delta_c$ the cone is not yet wide enough and the function $F \equiv 1$ is still



$\Delta_{\min} > \Delta_c$	$\Delta_{\min} = \Delta_c$	$\Delta_{\min} < \Delta_c$
sum rule violated	critical case	sum rule satisfied

Figure 6.3: The three geometric situations described in the text. The dashed line denotes the vector corresponding to the function $F \equiv 1$.

outside, while for $\Delta_{\min} < \Delta_c$ the $F \equiv 1$ function is inside the cone. For $\Delta_{\min} = \Delta_c$ the function belongs to the cone boundary. This geometric picture is illustrated in Fig. 6.3.

For $\Delta_{\min} > \Delta_c$, the sum rule cannot be satisfied, and a CFT corresponding to the spectrum $\Sigma(\Delta_{\min})$ (or any smaller spectrum) cannot exist. By contradiction, the bound must be true in any CFT. The problem thus reduces to determining Δ_c.

Any concrete calculation must introduce a coordinate parametrization of the above function space. We will parametrize the functions by an infinite vector $\left\{F^{(2m,2n)}\right\}$ of even-order mixed derivatives at $a = b = 0$:

$$F^{(2m,2n)} \equiv \partial_a^{2m} \partial_b^{2n} F(a,b)\Big|_{a=b=0} . \tag{6.4}$$

Notice that all the odd-order derivatives of the functions entering the sum rule vanish at this point due to the symmetry expressed by Eq. (4.51):

$$F^{(2m+1,2n)} = F^{(2m,2n+1)} = F^{(2m+1,2n+1)} = 0 .$$

The choice of the $a = b = 0$ point is suggested by this symmetry, and by the fact that it is near this point that the sum rule seems to converge the fastest, at least in the free scalar case, see Fig. 4.2.

The derivatives (6.4) are relatively fast to evaluate numerically. Presumably, there is also no loss of generality in choosing these coordinates on the function space, since the functions entering the sum rule are analytic in the space-like diamond.

In terms of the introduced coordinates, the sum rule becomes a sequence of linear equations for the coefficients $p_{\Delta,l} \geq 0$. We have one inhomogeneous equation:

$$1 = \sum p_{\Delta,l} F^{(0,0)}_{d,\Delta,l} , \tag{6.5}$$

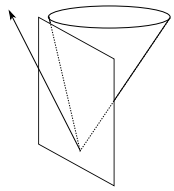


Figure 6.4: The existence of a hyper-plane separating the cone and the vector 1 corresponds to the existence of a functional positively defined on the cone and strictly negative on the function $F \equiv 1$.

and an infinite number of homogeneous ones:

$$\begin{aligned} 0 &= \sum p_{\Delta,l} F^{(2,0)}_{d,\Delta,l}\,, \\ 0 &= \sum p_{\Delta,l} F^{(0,2)}_{d,\Delta,l}\,, \\ &\dots \end{aligned} \tag{6.6}$$

We have to determine if, for a given $\Delta_{\min}$, the above system has a solution or not[2] Each of the 3 cases shown in Fig. 6.3 can be characterized more technically in terms of linear functionals:

- $\Delta_{\min} > \Delta_c$: In this case the function $F \equiv 1$ is outside the cone defined by all the other vectors included in the sum rule. No matter what is the values of the coefficients $p_{\Delta,l}$ in (6.1) , no combination of vectors belonging to the cone can reproduce $F \equiv 1$. This situation is signaled by the existence of a plane separating the vector $F \equiv 1$ and the cone (see Fig.). In the language of linear functionals, it must exist a functional Λ strictly positive on the vectors defining the cone and strictly negative on $F \equiv 1$.

- $\Delta_{\min} = \Delta_c$: In this case the boundary of the cone must contains the vector $F \equiv 1$.

- $\Delta_{\min} < \Delta_c$: This case is realized if the function $F \equiv 1$ is strictly contained in the cone. Hence there couldn't exist a plane separating the cone from the unit vector. In terms of linear functional this means that there isn't any functional Λ which strictly positive on the vectors defining the cone and strictly negative on $F \equiv 1$.

[2]It turns out that in the range $d \geq 1$ and $\Delta_{\min} \geq 2$ which is of interest for us, all $F^{(0,0)}_{d,\Delta,l} > 0$ in the RHS of the inhomogeneous equation (6.5). In such a situation, if a nontrivial solution of the homogeneous system (6.6) is found, a solution of the full system (6.5), (6.6) can be obtained by a simple rescaling. Let us however keep all the equation for later purposes.

For practical reasons we will have to work with finitely many derivatives, i.e. with a finite-dimensional subspace of the function space or, equivalently, with a finite subset of the homogeneous system (6.6). The above geometric picture applies also within such a subspace. Satisfaction of the sum rule on a subspace gives (in general) weaker but necessary condition, so that we still get a valid bound. As we expand the subspace by including more and more derivatives, the critical scalar dimension Δ_c will go down, monotonically converging to the optimal value corresponding to the full system.

Warmup example: $d = 1$

Let us use this philosophy to examine what the sum rule says about the spectrum of operators appearing in the $\phi \times \phi$ OPE when ϕ has dimension $d = 1$. Of course we know that $d = 1$ corresponds to the free scalar, see, and thus we know everything about this theory. In particular, we know that only twist 2 operators appear in the OPE, see Section 4.3. Our interest here is to derive this result directly from the sum rule. We expect the sum rule based approach to be robust: if we make it work for $d = 1$, chances are it will also give us a nontrivial result for $d > 1$. In this section we truncate the system (6.6) to the first *two* equations . As we will see now, this truncation already contains enough information to recover the free theory operator dimensions from $d = 1$. Following the discussion in Section 6.1.1 , we consider the projected cone—the cone generated by the vectors $F = F_{1,\Delta,l}$ projected into the two-dimensional plane $(F^{(2,0)}, F^{(0,2)})$. For each $l = 0, 2, 4, \ldots$ we get a curve in this plane, starting at the point corresponding to the lowest value of Δ allowed by the unitarity bound (see Section 2.2), see Fig. 6.5.
Notice that in the plane $(F^{(2,0)}, F^{(0,2)})$ the function $F \equiv 1$ is projected into the origin and the case $\Delta_{min} < \Delta_c$ depicted in Fig. 6.4 corresponds to the case when all the vectors of the sum rule are strictly contained in a half-plane passing through the origin.
It can be seen from this figure that the vectors corresponding to the twist 2 operators $\Delta = l + 2$ lie on the line $F^{(2,0)} = F^{(0,2)}$,[3] while all the other vectors lie to the right of this line. Moreover, the $l = 0$, $\Delta = 2$ vector points in the direction opposite to the higher-spin twist 2 operators. The boundary of the projected cone is thus given by the line $F^{(2,0)} = F^{(0,2)}$ *if* the spectrum includes the $\Delta = 2$ scalar and at least one higher-spin twist 2 operator (e.g. the energy-momentum tensor). Otherwise the boundary will be formed by two rays forming an angle less than π.[4]. It is only in the former case that the sum rule can have a solution. This case corresponds to

[3]This fact is easy to check analytically using the definition of $F_{d,\Delta,l}$ at $d = 1$.
[4]We ignore such subtleties as the possibility of a continuous scalar spectrum ending at $\Delta = 2$.

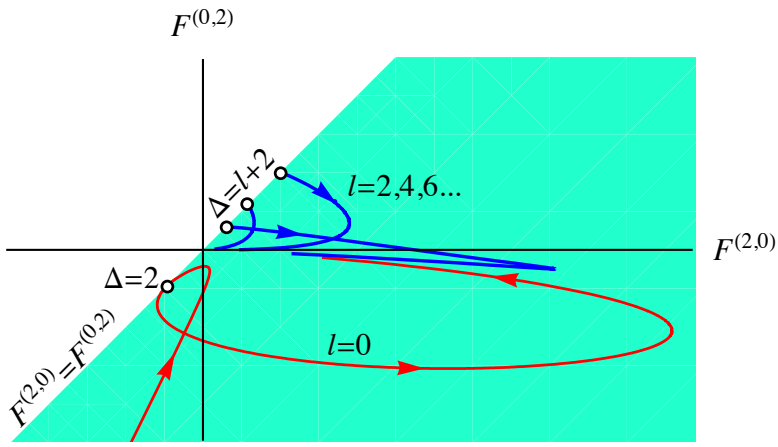


Figure 6.5: The sum rule terms $F_{1,\Delta,l}$ in the $(F^{(2,0)}, F^{(0,2)})$ plane. The shown curves correspond to $l = 0, 2, 4, 6$. The arrows are in the direction of increasing Δ. The $l = 0$ curve starts at $\Delta = 1$ ($\Delta \lesssim 1.01$ part is outside the plotted range); the $l = 2, 4, 6$ curves—at $\Delta = l + 2$. For large Δ the curves asymptote to the positive $F^{(2,0)}$ axis, see Appendix C. The shaded half-plane is the projected cone for a spectrum which includes the $\Delta = 2$ scalar.

$\Delta_{\min} = \Delta_c$: the boundary of the projected cone contains a linear subspace passing through the origin. Thus we also have additional information: only the vectors from the boundary, i.e. those of the twist 2 operators, may be present in the sum rule with nonzero coefficients.

The above argument appealed to the geometric intuition. For illustrative purposes we will also give a more formal proof. Taking the difference of the two first equations in (6.6), we get:

$$0 = \sum p_{\Delta,l} \left(F^{(2,0)}_{d,\Delta,l} - F^{(0,2)}_{d,\Delta,l} \right), \qquad p_{\Delta,l} \geq 0.$$

As stated above, for $d = 1$ all the terms in the RHS of this equation are strictly positive unless $\Delta = l + 2$. Thus, only twist 2 operators may appear with nonzero coefficients.

It is interesting to note that in Fig. 6.5 the $l = 0$ curve is tangent to the line $F^{(2,0)} = F^{(0,2)}$ at $\Delta = 2$.[3] Were it not so, we would not be able to exclude the existence of solutions to the sum rule involving scalar operators of $\Delta < 2$.

To conclude, we have shown that the spectrum of operators appearing in the sum rule, and hence in the OPE, of a $d = 1$ scalar consists solely of twist 2 fields and that, moreover, a $\Delta = 2$ scalar must be necessarily present in this spectrum.

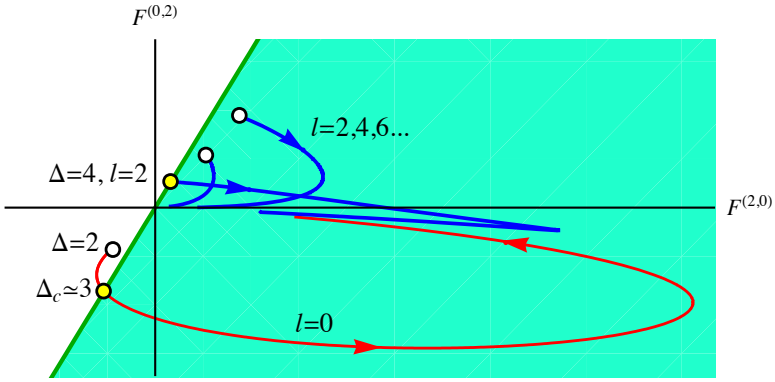


Figure 6.6: The analogue of Fig. 6.5 for $d = 1.05$. In this plot we started the $l = 0$ curve at $\Delta = 2$. The green line is the boundary of the projected cone for $\Delta_{\min} = \Delta_c \simeq 3$, see Fig. 6.7. The slope of this line is determined by the energy-momentum tensor vector.

Simplest bound satisfying $f(1) = 2$

We will now present the simplest bound of the form $\Delta_{min} < f(d)$ which, unlike the bound discussed in the previous section, approaches 2 as $d \to 1$. The argument uses the projection on the $(F^{(2,0)}, F^{(0,2)})$ plane similarly to the $d = 1$ case from the previous section. Since that method gave $\Delta_{\min} = 2$ for $d = 1$, by continuity we expect that it should give $\Delta_{\min} \simeq 2$ for d sufficiently close to 1.

To demonstrate how the procedure works, we pick a d close to 1, say $d = 1.05$, and produce the analogue of the plot in Fig. 6.5, see Fig. 6.6. We see several changes with respect to Fig. 6.5. The energy-momentum tensor determines one part of the projected cone boundary (the green line), while the spins $l = 4, 6, \ldots$ lie in the bulk of the cone. Continuation of the green line to the other side of the origin intersects the $l = 0$ curve at the point corresponding to $\Delta = \Delta_c \simeq 3$. This gives the critical value of $\Delta_{\min}$. Namely, if $\Delta_{\min} > \Delta_c$ in the spectrum (6.3), the projected cone will have an angle less than π and the sum rule will have no solutions. On the other hand, for $\Delta_{\min} < \Delta_c$ the projected cone covers the full plane, see Fig. 6.7, and a nontrivial solution to the first two equations of the system (6.6) will exist. For $\Delta_{\min} = \Delta_c$ the projected cone covers the half-plane shaded in Fig. 6.6. One can check that the same situation is realized for any $d > 1$. In particular, the slope of the critical cone boundary, described by the linear equation

$$F^{(2,0)} - \lambda(d) F^{(0,2)} = 0\,, \tag{6.7}$$

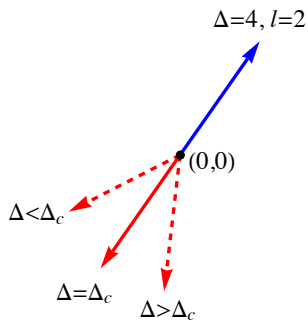


Figure 6.7: The relative position of the $l = 0$ vectors (red) with respect to the energy-momentum tensor vector (blue, pointing to upper right) determines the shape of the projected cone, see the text. If the cone contains the blue vector and both dashed red vectors, it covers the whole plane by their convex linear combinations.

is always determined by the energy-momentum tensor:

$$\lambda(d) = \frac{F^{(2,0)}}{F^{(0,2)}}\,, \quad F = F_{d,4,2}. \tag{6.8}$$

Once $\lambda(d)$ is fixed, the critical value of $\Delta_{\min}$ is determined from the intersection of the line (6.7) with the $l = 0$ curve:

$$F^{(2,0)} - \lambda(d)F^{(0,2)} = 0, \quad F = F_{d,\Delta_c,0}\,. \tag{6.9}$$

The $l = 0$, $\Delta > \Delta_c$ points then lie strictly inside the half-plane $F^{(2,0)} - \lambda(d)F^{(0,2)} \geq 0$. For $\Delta_{\min} > \Delta_c$ the cone angle is less than π, and the sum rule has no solution.

In Fig. 6.8 we plot the corresponding value of $f(d)$ found numerically from Eq. (6.8), (6.9), denoted $f_2(d)$ to reflect the order of derivatives used to derive this bound. As promised, the free field theory value $\Delta = 2$ is approached continuously as $d \to 1$.

The asymptotic behavior of $f_2(d)$ for $d \to 1$ can be determined by expanding the equations defining Δ_c in power series in $d - 1$ and $\Delta_c - 2$. We find:

$$\begin{aligned} f_2(d) &= 2 + \gamma\sqrt{d-1} + O(d-1)\,, \\ \gamma &\equiv [2(K+1)/3]^{1/2} \simeq 2.929, \\ K &\equiv (192\ln 2 - 133)^{-1}. \end{aligned} \tag{6.10}$$

This asymptotic provides a good approximation for $d - 1 \lesssim 10^{-3}$, see Fig. 6.8. The square root dependence in (6.10) can be traced to the fact that for $d = 1$ the $l = 0$ curve was *tangent*

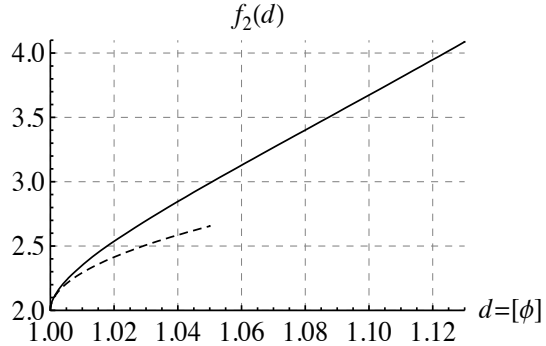


Figure 6.8: $f_2(d) = \Delta_c$ as determined by solving Eq. (6.9). We plot it only for d rather close to 1 because in any case this bound will be significantly improved below. The dashed line shows the asymptotic behavior (6.10), which becomes a good approximation for $d \lesssim 1.001$.

to the projected cone boundary at $\Delta = 2$. The bound of Fig. 6.8 will be improved below by taking more derivatives into account, however the square root behavior will persist (albeit with a different coefficient).

6.1.2 Improved bounds: general method

As we already mentioned in Section6.1.1, the bound will improve monotonically as we include more and more equations from the infinite system (6.6) in the analysis, i.e. increase the number of derivatives $F^{(2m,2n)}$ that we are controlling. We thus consider a finite basis $\mathcal{B}$, adding several higher-order derivatives to the $F^{(2,0)}$ and $F^{(0,2)}$ included in the previous section:

$$\mathcal{B} = \{F^{(2m,2m)}\} = \{F^{(2,0)}, F^{(0,2)}, \ldots\}. \tag{6.11}$$

According to the discussion in previous sections , we have to study the boundary of the projected cone in the finite-dimensional space with coordinates (6.11). The logic in principle is the same: we will have a family of curves corresponding to $l = 0, 2, 4 \ldots$ generating the projected cone. As we lower $\Delta_{\min}$ in the spectrum (6.3), the projected cone grows. For $\Delta_{\min} < \Delta_c$ it will cover the whole space. However, in this many-dimensional situation it is not feasible to look for Δ_c by making plots similar to Fig. 6.6. We need a more formal approach.
Such an approach uses the language of linear functionals, already encountered in Section 6.1.1.

A linear functional Λ on the finite-dimensional subspace with basis $\mathcal{B}$ is given by

$$\Lambda = \sum_{m,n} \lambda_{2m,2n} \partial_a^{2m} \partial_b^{2m}\,, \qquad \Lambda(F) = \sum_{\mathcal{B}} \lambda_{2m,2n} F^{(2m,2n)}, \tag{6.12}$$

where $\lambda_{2m,2n}$ are some fixed numbers characterizing the functional and all derivatives are evaluated at $a = b = 0$. They generalize the single parameter $\lambda(d)$ from Section 6.1.1 to the present situation.

Assume thus that for certain fixed d and $\Delta_{\min}$, we manage to find a linear functional of this form such that (*"positivity property"*)

$$\begin{aligned} \Lambda[F_{d,\Delta,l}] \geq 0 \qquad & \text{for all } \Delta \geq \Delta_{\min}\,(l=0) \\ \text{and} \qquad & \text{for all } \Delta \geq l+2\,(l=2,4,6\ldots)\,. \end{aligned} \tag{6.13}$$

Moreover, assume that all but a finite number of these inequalities are actually strict: $\Lambda[F] > 0$. Then the sum rule cannot be satisfied, and such a spectrum, corresponding to a putative OPE $\phi \times \phi$, is ruled out.

The proof uses the above "positivity argument". Since $\Lambda[1] = 0$, the positivity property implies that only those primaries for which $\Lambda[F] = 0$ would be allowed to appear in the RHS of the sum rule with nonzero coefficients. By assumption, there are at most a finite number of such primaries. However, as noted in Section4.3, finitely many terms can never satisfy the sum rule globally, because of the behavior near $z = 0, 1$.

Using linear functionals, the two non-critical cases of Fig. 6.4 can be distinguished as follows:

- $\Delta_{\min} > \Delta_c \Longleftrightarrow$ there *IS* a functional Λ such that the positivity property (6.13) is satisfied
- $\Delta_{\min} < \Delta_c \Longleftrightarrow$ there is *NO* functional Λ such that the positivity property (6.13) is satisfied

A numerical procedure which for any given $\Delta_{\min}$ finds such a positive Λ or shows that a non-negative Λ does not exists will be explained below. Assuming that we know how to do this, determination of Δ_c becomes an easy task. First, we bracket Δ_c from above and below by trying out a few values of $\Delta_{\min}$ and checking to which of the two above sets, $\Delta_{\min} > \Delta_c$ or $\Delta_{\min} < \Delta_c$, they belong. Second, we apply the division-in-two algorithm, *i.e.* reduce the length of the bracketing interval by checking its middle point, etc. This achieves exponential precision after a finite number of steps.

We will now explain the numerical procedure. Let us begin with the non-negative functional

defined by (6.13). The positivity property can be viewed as a system of infinitely many linear inequalities for the coefficients $\lambda_{2m,2n}$. The infinitude is due to three reasons:

- there are infinitely many spins l;
- for each spin l the dimension Δ can be arbitrary large;
- the dimension Δ varies continuously.

To be numerically tractable, this system needs to be truncated to a finite system, removing each of the three infinities. We do it by imposing inequalities in (6.13) not for all Δ, l but only for a 'trial set' such that

- only finite number of spins $l \leq l_{\max}$ are included;
- only dimensions up to a finite $\Delta = \Delta_{\max}$ are included;
- Δ is discretized.

To ensure that we are not losing important information by truncating at $l_{\max}$ and $\Delta_{\max}$, we include into the trial set the vectors corresponding to the large l and large Δ asymptotics of the derivatives. The relevant asymptotics have the form (see Eq. (C.4) in Appendix C):

$$F^{(2m,2n)}_{d,\Delta,l} \sim \frac{const}{(2m+1)(2n+1)}(2\sqrt{2}l\,(1+x))^{2m+1}(2\sqrt{2}l)^{2n+1}, \quad x \equiv \frac{\Delta - l - 2}{l} \geq 0\,, \tag{6.14}$$

where a constant $const > 0$ is independent of m and n. This asymptotics is valid for $l \to \infty$, $x \ll l$ fixed.

Upon truncation to the trial set, Eq. (6.13) becomes a finite system of linear inequalities, a particular case of the *linear programming problem*[5]. It is thus possible to determine if a solution exists (and find it if it does) using one of several existing efficient numerical algorithms (see [59]). In our work we used the classic Simplex Method as realized by the `LinearProgramming` function of Mathematica.

A detailed review of the numerical method is presented in appendix D

[5] A general linear programming problem consists in minimizing a linear function of several variables subject to a set of linear constraints (equalities and inequalities). Our problem is a particular case when all constraints are inequalities and the function to be minimized is absent (or, equivalently, it is constant).

6.1.3 Results in 4D

In this section we present the numerical results for the upper bound for the dimension $\Delta_{\min}$ of the leading scalar in the OPE $\phi_d \times \phi_d$, universal for all unitary 4D CFTs. The bound has been first computed in [26], using the sum rule of Section 4.3 truncated to the $N = 6$ derivative order and then improved in [28].

We report the results obtained for largest numerically accessible values of N. These results[6] are plotted in Fig 6.9 as a collection of curves $f_N(d)$, $N = 6 \dots 18$, where the index N denotes the number of derivatives used to obtain the bound. The bound naturally gets stronger as N increases (see below), and thus the lowest curve $f_{18}(d)$ is the strongest bound to date. In the considered interval $1 \leq d \leq 1.7$ this bound is well approximated (within 0.5%) by

$$f_{18}(d) \simeq 2 + 0.7\gamma^{1/2} + 2.1\gamma + 0.43\gamma^{3/2}, \quad \gamma = d - 1. \tag{6.15}$$

For points lying on the curves $\Delta_{\min} = f_N(d)$ we are able to find a linear functional of the form (6.12) satisfying the positivity property (6.13).[7]

Several comments are in order here. We have actually computed the bound only for a discrete number of d values, shown as points in Fig. 6.9. The behavior for $d \to 1$ can be better appreciated from the logarithmic-scale plot in Fig. 6.10.

We do not see any significant indication which could suggest that the curves $f_N(d)$ do not interpolate smoothly in between the computed points. Small irregularities in the slope are however visible at several points in Figs. 6.9,6.10. These irregularities are understood; they originate from the necessity to *discretize* the infinite system of inequalities (6.13), see appendix D for a discussion. In our computations the discretization step was chosen so that these irregularities are typically much smaller than the improvement of the bound that one gets for $N \to N + 2$.

For each N the bound $f_N(d)$ is near-optimal, in the sense that no positive functional involving derivatives up to order N exists for

$$\Delta_{\min} - 2 < (1 - \varepsilon)[f_N(d) - 2].$$

We estimate $\varepsilon \simeq 1\%$ from the analysis of residuals in the fit of $f_N(d)$ by a smooth curve like in (6.15).

On the other hand, by increasing N we are allowing more general functionals, and thus the bound $f_N(d)$ can and does get stronger. This is intuitively clear since for larger N the Taylor-expanded sum rule includes more and more constraints.

[6] See Appendix B of [28] for the same results in tabular form.

[7] Thus actually the bound is strict: $\Delta_{\min} < f_N(d)$, except at $d = 1$.

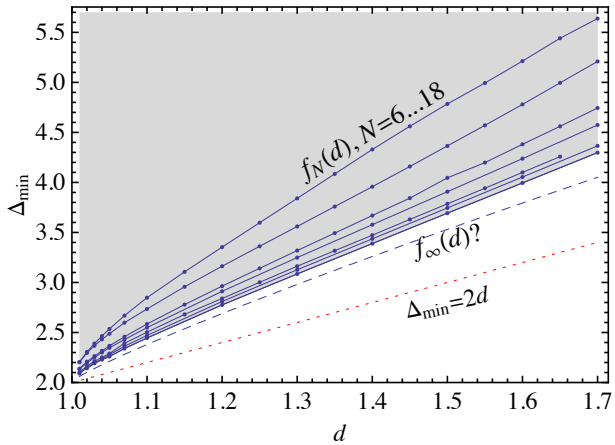


Figure 6.9: *The solid curves are the bounds* $f_N(d)$, $N = 6 \ldots 18$. *The bounds get stronger as* N *increases, thus* $N = 6$ *is the weakest bound (highest curve), and* $N = 18$ *is the current best bound (lowest curve). The shaded region is thus excluded. The dashed curve* $f_\infty(d)$ *is an approximation to the best possible bound, obtained by extrapolating* $N \to \infty$. *The dotted line* $\Delta_{\min} = 2d$ *is realized in a family of "generalized free scalar" CFTs, and is compatible with our bounds.*

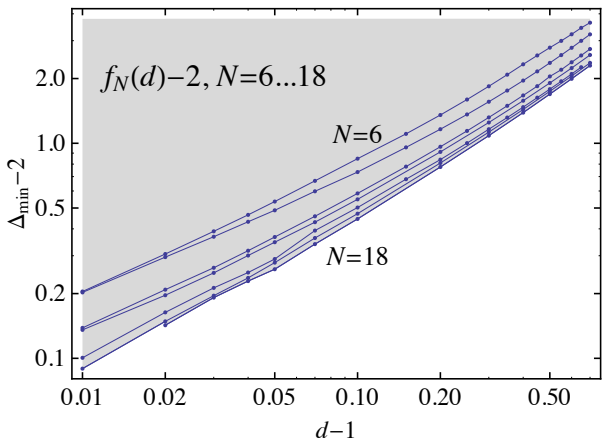


Figure 6.10: *Same as Fig. 6.9 but with anomalous dimensions* $d-1$, $\Delta-2$ *in logarithmic scale. The shaded region is excluded.*

Comparing the results for $N = 6$ and $N = 18$ derivatives, the bound on the anomalous dimension $\Delta_{\min} - 2$ is improved by $\sim 30 \div 50\%$ in the range $1 \leq d \leq 1.7$ that we explored.

We have pushed our analysis to such large values of N in the hope of seeing that the bound saturates as $N \to \infty$. Indeed, we do observe signs of convergence in Figs. 6.9,6.10, especially at $d \gtrsim 1.1$. In fact, we have observed that the bounds $f_N(d)$ starting from $N = 8$ follow rather closely the asymptotic behavior

$$f_N(d) \simeq f_\infty(d) + \frac{c(d)}{N^2}, \quad (1 \leq d \leq 1.7).$$

An approximation to the optimal bound $f_\infty(d)$ can thus be found by performing for each d a fit to this formula. This approximation is shown by a dashed line in Fig. 6.9. From this rough analysis we conclude that the optimal bound on the anomalous dimension $\Delta_{\min} - 2$ is probably within $\sim 10\%$ from our current bound.

We have $f_N(d) \to 2$ continuously as $d \to 1$. The point $d = 1$, $\Delta_{\min} = 2$ corresponds to the free scalar theory.
We don't know of any unitary CFTs that saturate our bound at $d > 1$, see the discussion in Section 6 of [26]. We know however a family of unitary 4D CFTs in which $\Delta_{\min} = 2d$ and which are consistent with our bound (the red dotted line in Fig. 6.9). This "generalized free scalar" theory is defined for a fixed d by specifying the 2-point function

$$\langle \phi(x)\phi(0)\rangle = |x|^{-2d} \ ,$$

and defining all other correlators of ϕ via Wick's theorem. This simple procedure gives a well-defined CFT, unitary as long as $d \geq 1$. In these class of models the leading scalar in this OPE has dimension $2d$ .We will come back to this topic in Sec. 6.3.3.

6.1.4 Results for CFT's with global symmetries

As discussed in Sec.4.4, 4.5, the presence of global symmetries in the CFT implies that the operators appearing in the OPE of two scalar field can be classified according to their representation. Moreover, in Sec. 4.5.3 we showed that generically the number of sum rules arising from crossing symmetry constraints is equal in number to the number of structures. We recall that by *structure* we denote the contribution to the four point function of all the operators with the same spin parity belonging to the same representation of the global symmetry. For instance, in the case of a complex scalar field transforming under a global $U(1)$ symmetry with unitary charge we have three structures: charge zero even or odd spin, charge two with even

spin. Correspondingly we have three sum rules, as shown in Sec. 4.5.1.

In order to extract information from the system of sum rules we adopt again the method of linear functionals. Here we denote Λ a functional defined on the space of vector functions. Let us review the procedure for the simple case of $SO(N)$. Recalling eq. 4.66 :

$$\sum p^S_{\Delta,l} \underbrace{\begin{pmatrix} 0 \\ F_{\Delta,l} \\ H_{\Delta,l} \end{pmatrix}}_{\vec{V}^S_{\Delta,l}} + \sum p^T_{\Delta,l} \underbrace{\begin{pmatrix} F_{\Delta,l} \\ \left(1-\frac{2}{N}\right) F_{\Delta,l} \\ -\left(1+\frac{2}{N}\right) H_{\Delta,l} \end{pmatrix}}_{\vec{V}^T_{\Delta,l}} + \sum p^A_{\Delta,l} \underbrace{\begin{pmatrix} -F_{\Delta,l} \\ F_{\Delta,l} \\ -H_{\Delta,l} \end{pmatrix}}_{\vec{V}^A_{\Delta,l}} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{\vec{V}^{RHS}} \tag{6.16}$$

we need a functional defined on the vectors function $\vec{V}_i$. We can again project the functions appearing in the sum rule into a finite dimensional subspace spanned by their Taylor coefficients up to a given order[8] N_{der} . Such a linear functional can be parametrized as follows. Given a vector of functions of two variables $\vec{v} = (v_1(a,b),\, v_2(a,b)\, v_3(a,b))$ we define

$$\Lambda[\vec{v}] = \sum_{j=1}^{3} \sum_{n,m=0}^{N_{der}} c_j^{2n,2m} v_j^{(2m,2n)}(0,0)\,. \tag{6.17}$$

We immediately notice that, at the same order of truncation, the number of coefficients needed to parametrize a linear functional in the $SO(N)$ case are the triple of the non-symmetric case. More in general, calling $N_{structures}$ the number of structure and sum rules, the number of coefficients will be

$$\frac{(N_{der}+2)(N_{der}+4)}{8} \times N_{structures} \tag{6.18}$$

The degree of numerical complexity is therefore enhanced. However this not the only complication: even the number of constraints that must be satisfied increases proportionally to $N_{structures}$.

The functional Λ must satisfy the suitable generalized positivity property. Suppose for instance that we are interested in extracting a bound on the smallest dimension scalar singlet operator contributing to the sum rule; then we have to look for a functional subjected to

$$\begin{aligned} &\Lambda[\vec{V}^S_{d,\Delta,l}] \geq 0\,, && \text{for all } \Delta \geq \Delta_{\min}\,(l=0) \qquad (6.19)\\ &\Lambda[\vec{V}^T_{d,\Delta,l}] \geq 0\,, && \text{for all } \Delta \geq 1\,(l=0)\\ &\Lambda[\vec{V}^i_{d,\Delta,l}] \geq 0\ ,\ i=S,T, && \text{for all } \Delta \geq l+2\,(l=2,4,6\ldots)\,.\\ &\Lambda[\vec{V}^A_{d,\Delta,l}] \geq 0\,, && \text{for all } \Delta \geq l+2\,(l=1,3,5\ldots)\,.\\ &\Lambda[\vec{V}^{RHS}] \leq 0\,. && \qquad (6.20) \end{aligned}$$

[8] Given the presence of several N symbols, from now on N will refer to $SO(N)$ while N_{der} the order of truncation of the sum rule.

As mentioned, generically the number of constraints is multiplied by $N_{structures}$ with respect to the non-symmetric case. Nevertheless the above is again a standard LinearProgramming problem and can be solve with the same method explained in Section 6.1.2.
Notice that in the scalar symmetric traceless structure ($\vec{V}^T$, $l = 0$) we impose the positivity on all the $\Delta \geq 1$ since this is the constraint imposed by unitarity (see Section 2.2).

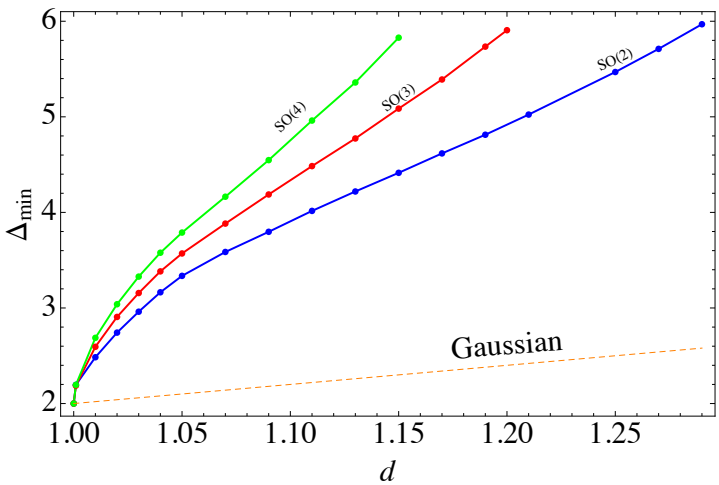


Figure 6.11: *Bound for the smallest dimension of a scalar operators singlet under a global $SO(N)$ symmetries. The bounds corresponds, from the strongest to the weaker, to $SO(N)$, $N = 2, 3, 4$ and have been computed with 4 derivatives. The line is an interpolation between the points where the bound has been computed exactly. We assume as usual a smooth interpolation.*

In [31] the existence of a bound was proved and few numerical results are provided. Here we push further the numerical calculations. The results for $N_{der} = 4$ are shown in Fig. for several theories. We observe that the bound gets weaker as N increases.

$SO(4)$ and Conformal Technicolor

When in Section 1.2 we discussed the phenomenological motivations for the present work we raised the question of what is the allowed separation between the dimension of a scalar field, identified with the Higgs H, and the dimension of the first scalar operator appearing in its OPE, identified with the Higgs mass term $H^\dagger H$. A realistic scenario of BSM, where the Higgs

is an operator of a strongly interacting CFT , must be able to accommodate a custodial $SO(4)$ symmetry in the strong sector, under which the Higgs field transforms as a fundamental representation while its mass term is a singlet[9]. As consequence we cannot directly apply stringent results of Sec. 6.1.3 to Conformal Technicolor, since the bounds shown in 6.9 refer to the smallest operator appearing in the OPE without informations concerning its transformation properties under the global symmetry.

Using the formalism developed in this section we are now able to distinguish between different structures.

Recalling the OPE of a field transforming in the fundamental of $SO(4)$, call it h_a[10],

$$h_a(x)h_b(0) \sim \frac{1}{|x|^{2d}} \left(\delta_{ab}\mathbb{1} + C_S|x|^{\Delta_S}\delta_{ab}\,(H^\dagger H)(0) + C_T|x|^{\Delta_T}\mathcal{T}_{(ab)}(0) + C_J|x|^2 x^\mu J_\mu^{[ab]}(0) + \dots\right) . \tag{6.21}$$

we can extract a bound for the dimension Δ_S of the singlet operator and an independent bound on the dimension Δ_T on the symmetric traceless operator $T_{(ab)}$. The former makes use of the of the positivity properties as in (6.19), while the latter uses similar constraints, reversing the role of S and T, namely:

$$\begin{aligned}
&\Lambda[\vec{V}^T_{d,\Delta,l}] \geq 0\,, && \text{for all } \Delta \geq \Delta_{\min}\,(l=0) && (6.22)\\
&\Lambda[\vec{V}^S_{d,\Delta,l}] \geq 0\,, && \text{for all } \Delta \geq 1\,(l=0) \\
&\Lambda[\vec{V}^i_{d,\Delta,l}] \geq 0\ ,\ i=S,T, && \text{for all } \Delta \geq l+2\,(l=2,4,6\dots)\,. \\
&\Lambda[\vec{V}^A_{d,\Delta,l}] \geq 0\,, && \text{for all } \Delta \geq l+2\,(l=1,3,5\dots)\,. \\
&\Lambda[\vec{V}^{RHS}] \leq 0\,. && && (6.23)
\end{aligned}$$

The extracted bounds are shown in Fig. , for $N_{der} = 6$, and compared with the correspondent bound for non-symmetric theories computed with the same number of derivatives. The bound on Δ_T is the strongest one. This results points towards the generic expectation that the singlet operators should have higher dimension, confirming the explicit calculation for $O(N)$ models in $4-\epsilon$ dimensions ([26]). However we can't make a general statement about whether $\Delta_S - \Delta_T \gtrless 0$.

Let us now come back on the requirements of Conformal Technicolor ([23]) and compare them with the obtained results.

To be more quantitative about the needed pattern of field dimensions, assumptions on the physics of flavor must be made. Making the very favorable, but strong, assumption that flavor

[9]Notice that part of the custodial symmetry is gauged by the SM gauge group, hence if we want to include a CFT operator in the SM Lagrangian this has to be a singlet under custodial symmetry.

[10]In standard notation h_a is related to H by $H = \frac{1}{\sqrt{2}}\binom{h_1+ih_2}{h_3+ih_4}$

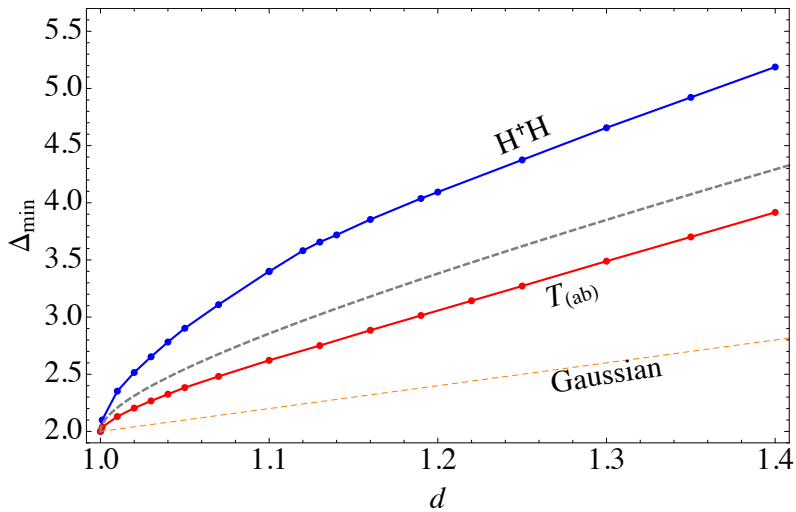


Figure 6.12: *Bounds for the smallest dimension operators appearing in the OPE of two scalar fields transforming under the fundamental representation of a global $SO(4)$. The weaker bound (blue line) corresponds to scalar operators neutral under $SO(4)$. The strongest bound (red) refers to scalar operators transforming as a symmetric traceless tensor. Again we assumed a smooth interpolation between the points where the bound has been computed exactly.*

violation in the light families is either suppressed by their mixing to the third family of by their Yukawa couplings [60], the range $d \lesssim 1.9$, $\Delta_S \gtrsim 4$ is sufficient. In that situation the scale where the top Yukawa becomes strong can be as low as $\sim 100TeV$, so that the window where CT is active is not very big. On the other hand, by the making the more conservative, but robust, assumption that all flavor violating operators are equally important at the Flavor scale requires the more constrained pattern $d \lesssim 1.2, \Delta_S 4$. That second situation corresponds to a flavor scale around $10^5 TeV$, with the CFT describing physics in a sizable window of scales ([60]).
At the present stage both the scenarios are still allowed, although the more conservative one is restricted to a small corner close to $d \sim 1.2$. On the other hand theories containing clever assumptions on the flavor structure are allowed to live in sizable region of the parameter space (d, Δ_S). We should stress that increasing the numerical power in the non-symmetric case produced an improvement of 30-50% in passing form $N_{der} = 6$ to $N_{der} = 18$. If a similar amelioration is repeated only theories exhibiting a clever flavor structure would remain unruled out, while the others could not not be realized in a unitary CFT.

6.1.5 Results for supersymmetric theories

Superconformal field theories theories represent another interesting class of theories where we can apply our formalism. Suppose we are given a superconformal field theory containing a chiral scalar field Φ, the lowest component of which is a scalar complex field of dimension d. The crossing symmetry constraints arising from the four point function of $\langle\phi\phi^\dagger\phi\phi^\dagger\rangle$ have been revised in Sec. 5.4 and have the a similar structure of the vectorial sum rule (6.16). This time

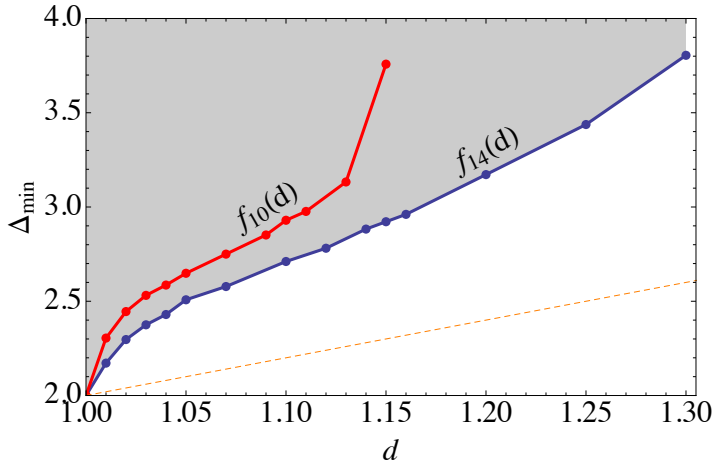


Figure 6.13: *Bound for the smallest dimension of a vector superfield appearing in the OPE of a chiral field with its conjugate obtained using only the first sum rule of (6.24). The bound $f_4(d)$ reproduces the results of [54].*

we denote the three structures as $V^{0\pm}$, corresponding to operators with vanishing R-charge and even/odd spin, which appear in the $\phi\times\phi^\dagger$ OPE, and V^2, corresponding to operators with R-charge twice the ϕ R-charge appearing in the $\phi\times\phi$. The fundamental difference among superconformal case and a pure $U(1)$ symmetric case is that supersymmetry relates some of the coefficient in the sum rule, grouping the sum in superconformal blocks. Moreover supersymmetry and the chirality of Φ restrict the allowed values of Δ in the sum. Recalling the definition of Sec. 5.4 we can write the vectorial sum rule in the schematic form:

$$\sum p^{0+}_{\Delta,l}\underbrace{\begin{pmatrix}\mathcal{F}_{\Delta,l}\\ \widetilde{\mathcal{F}}_{\Delta,l}\\ \widetilde{\mathcal{H}}_{\Delta,l}\end{pmatrix}}_{\vec{V}^{0+}_{\Delta,l}}+\sum p^{0-}_{\Delta,l}\underbrace{\begin{pmatrix}\mathcal{F}_{\Delta,l}\\ \widetilde{\mathcal{F}}_{\Delta,l}\\ -\widetilde{\mathcal{H}}_{\Delta,l}\end{pmatrix}}_{\vec{V}^{0-}_{\Delta,l}}+\sum p^{2}_{\Delta,l}\underbrace{\begin{pmatrix}0\\ F_{\Delta,l}\\ -H_{\Delta,l}\end{pmatrix}}_{\vec{V}^{2}_{\Delta,l}}=\underbrace{\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}}_{\vec{V}^{RHS}} \tag{6.24}$$

In [54] a bound on the smallest dimension non-chiral scalar field with vanishing R-charge is derived making use of only the first sum rule of eq. (6.24). This method requires the use of an high number of derivatives, since no functional satisfying the suitable positivity properties exists for $N_{der} < 10$. This is a consequence of the constraint incompleteness. Before exploiting the full power of the triple sum rule we review and improve the results of [54]. Hence we look for a linear functional defined on the functions $\mathcal{F}$ satisfying

$$\begin{aligned} \Lambda[\mathcal{F}_{d,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta \geq \Delta_{\min}\,(l=0) \\ \text{and} \qquad & \text{for all } \Delta \geq l+2\,(l=1,2,3,4...) \end{aligned} \tag{6.25}$$

The results are shown in Fig. 6.13 for different values of N_{der}. We observe that the bounds rapidly become weak as d increases, ending in an absence of constraints on $\Delta_m in$ for $d \gtrsim 1.2-1.3$. This behavior is understood as follows: as d increases the convex hull spanned by the vectors of derivatives become wider and wider; at a certain value the projected cone fills the entire sub-space even without the need of $l = 0$ vectors. In turns, the bounds on the scalar sector is absent. As we increase N_{der} the value of d where this degeneracy is reached grows. Instead using the entire set of constraints we do not find such a peculiarity.
Exploiting the triple sum rule requires again a linear functional defined on vector functions $\vec{V}^i_{\Delta,l}$. Unitarity bounds, superconformal symmetry and chirality restrict the positivity condition on Λ (see Section. 5.2, 5.3, 5.4):

$$\begin{aligned} \Lambda[\vec{V}^{0+}_{d,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta \geq \Delta_{\min}\,(l=0) \\ \Lambda[\vec{V}^{0\pm}_{d,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta \geq l+2\,(l=1,2,3,4,\ldots)\,. \\ \Lambda[\vec{V}^{2}_{d,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta = 2d+l\,(l=0,2,4\ldots)\,. \\ \Lambda[\vec{V}^{2}_{d,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta \geq |2d-3|+l+3\,(l=0,2,4\ldots)\,. \end{aligned} \tag{6.26}$$

The results obtained for $N_{der} = 6$ are shown in Fig. 6.14 . Compared to the results of Fig. 6.13 the bound has become more stringent, even exploiting a smaller number of derivatives. Moreover a fundamental difference is represented by the behavior of the bound close to the free theory. Compared to all the other cases so far investigated the bound approaches the free theory value linearly. This crucial difference could allow a direct comparison with perturbative calculations. Notice that the dimension of a chiral field is fixed by its R-charge, which usually takes rational values. We didn't find any example in the literature where the dimension of Φ is sufficiently close to 1 such that a perturbative correction to the dimension of the non-chiral operator $\Phi^\dagger\Phi$ has a chance to saturate the bound. In addition, most of the known $\mathcal{N} = 1$

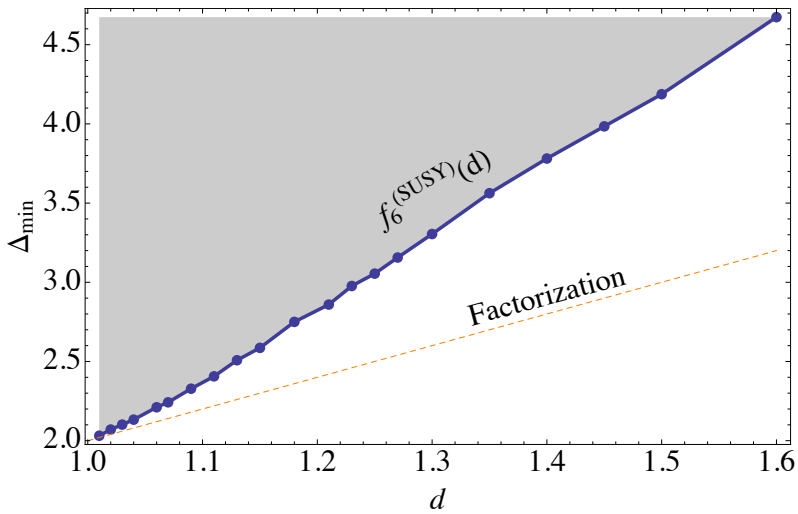


Figure 6.14: *Bound for the smallest dimension of a vector superfield appearing in the OPE of a chiral field with its conjugate obtained with 6 derivatives. The line is an interpolation between the blue points where the bound has been computed exactly. We assume as usual a smooth interpolation. Irregularities are due to the gap in dimension of the operators allowed by superconformal symmetry to appear in the $\Phi \times \Phi$ OPE. The shaded region is excluded.*

superconformal field theories have additional global symmetries (for instance the IR fixed points of [13]) under which the chiral superfield Φ is charged. Whenever this is the case the OPE $\phi \times \phi^\dagger$ contains scalar field sitting in the same multiplet of a conserved global symmetry current. Since the current is conserved the dimension of the scalar field is constrained to be exactly 2.

6.1.6 Results in 2D

Although our main interest is in the 4D CFTs, our methods allow a parallel treatment of the 2D case. We recall that we only exploit the finite-dimensional $SL(2,\mathbb{C})$ symmetry and we do not take advantage of the full Virasoro algebra.

The sum rules for the non-symmetric case (or the symmetric ones) have the same structures as in four dimensions, provided that the two dimensional conformal blocks (4.25) are used. Moreover, in two dimensions the unitarity bounds for $SL(2,\mathbb{C})$ primaries[11] have the form

$$\Delta \geq l, \quad l = 0, 1, 2 \ldots,$$

[11] Known as quasi-primaries in 2D CFT literature.

where l is the Lorentz spin. Thus, when looking for a linear functional, the positivity properties like (6.13) must be modified appropriately.
With same formalism used so far, we can try to answer the same question as in 4D. Namely, for a $SL(2,\mathbb{C})$ scalar primary ϕ of dimension d, we can look for an upper bound on the dimension $\Delta_{\min}$ of the first scalar operator appearing in the OPE $\phi \times \phi$. Since the free scalar is dimensionless in 2D, the region of interest is $d > 0$.

Fig. 6.15 summarizes our current knowledge of this bound. The dotted line is the $N_{der} = 2$ bound presented in [26]. The solid line is the $N_{der} = 12$ improved bound obtained in [28]. A numerical fit to this bound is given by:

$$f_{12}^{(2D)}(d) \simeq \begin{cases} 4.3d + 8d^2 - 87d^3 + 2300d^4, & d \lesssim 0.122\,, \\ 0.64 + 2.87d, & d \gtrsim 0.122\,. \end{cases} \tag{6.27}$$

The dashed line and scattered crosses correspond to various OPEs realized in explicit examples of exactly solvable unitary 2D CFTs. In addition to generalized free CFT's one can compare with the 2D unitary minimal models (see appendix B for a concise review) $\mathcal{M}(m, m+1)$, $m = 3, 4, \ldots$, where we have the OPEs

$$\sigma \times \sigma = 1 + \epsilon + \ldots, \quad \Delta_\sigma = \frac{1}{2} - \frac{3}{2(m+1)}, \quad \Delta_\epsilon = 2 - \frac{4}{m+1}, \tag{6.28}$$

Not only these models respect the bound but come very close to saturate it. More precisely, our 2D bound starts at $(0,0)$ tangentially to the line $\Delta = 4d$ realized in the free scalar theory, then grows monotonically and passes remarkably closely above the Ising model point $(\Delta_\sigma, \Delta_\varepsilon) = (1/8, 1)$. After a "knee" at the Ising point, the bound continues to grow linearly, passing in the vicinity of the higher minimal model points (6.28). We have checked that increasing the value of N_{der} up to 16 there is no violation of the bound in correspondence of the minimal models: the only effect is slight modification of the slope of the bound after $d = 1/8$.

It is curious to note that if we did not know beforehand about the Ising model, we could have conjectured its field dimensions and the basic OPE $\sigma \times \sigma = 1 + \varepsilon$ based on the singular behavior of the 2D bound at $d = 1/8$.
On the other hand, nothing special happens with the 2D bound at the higher minimal model points, it just interpolates linearly in between[12]. Most likely, this does not mean that there exist other unitary CFTs with intermediate operator dimensions. Rather, this behavior suggests that

[12]The straight line fitting the bound would cross the dashed Free theory line just above $d = 0.5$, which is the accumulation point of the minimal models $\mathcal{M}(m, m+1)$. For larger values of d we expect that the bound modifies its slope and eventually asymptotes to the Free line.

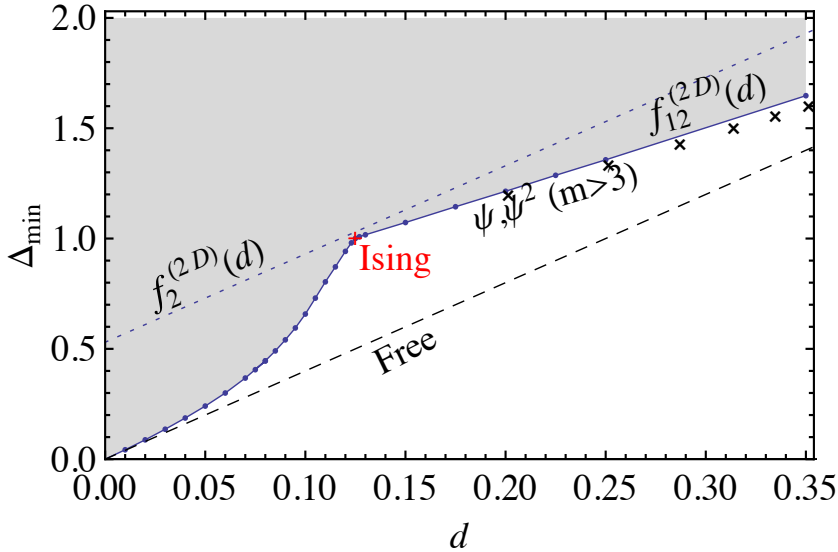


Figure 6.15: *See the text for an explanation. The red cross denotes the position of the Ising model, the black crosses marked ψ, ψ^2 correspond to the OPEs realized in the higher minimal models. The shaded region is excluded.*

the single conformal bootstrap equation used to derive the bound is not powerful enough to fully constrain a CFT.

In comparison, it is a bit unfortunate that the 4D bound does not exhibit any similar singular points which would immediately stand out as CFT candidates. Nevertheless, if we assume that the shape of the 4D bound is a result of an interpolation between existing CFTs (as it is the case in 2D), we may conjecture that the downward convex behavior of the functions $f_N(d)$ in Fig. 6.9 is due to the presence of a family of points satisfying the sum rule that can correspond to exact CFTs. This observation, though speculative, shows how the presented method can provide a guideline in the study of 4D CFTs.

6.2 Bounding OPE coefficients

So far we have used the sum rules to constrain the maximal allowed gap in the scalar sector. In order to constrain the size of the OPE coefficients $p_{\Delta,l}$, we proceed as follows [29]. For definiteness we consider a theory with a single sum rule and later on we will generalize. Let us

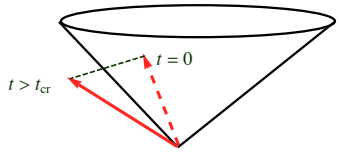


Figure 6.16: *Geometric interpretation of Eq. (6.29). As t increases, the vector $1 - t\,F_{d,4,2}$ eventually leaves the cone.*

rewrite the sum rule extracting part the contribution of one particular operator and transferring into the LHS:

$$1 - t\,F_{d,\Delta_*,l_*} = (p_{\Delta_*,l_*} - t)F_{d,\Delta_*,l_*} + \sum_{(\Delta,l)\neq(\Delta_*,l_*)} p_{\Delta,l}F_{d,\Delta,l} \tag{6.29}$$

The geometric interpretation of this equation is that the t-dependent vector $1 - t\,F_{d,\Delta_*,l_*}(u,v)$ belongs to the same cone as before as long as $t \leq p_{\Delta_*,l_*}$. In other words, the maximal allowed value of p_{Δ_*,l_*} can be determined as the value $t = t_{cr}$ for which the curve $1 - t\,F_{d,\Delta_*,l_*}(u,v)$ crosses the cone boundary, Fig. 6.16. Analytically, we can detect that the crossing happened if there exists a linear functional such that

$$\Lambda[F_{d,\Delta,l}] \geq 0 \tag{6.30}$$

for all functions generating the cone, and

$$\Lambda[1 - t\,F_{d,\Delta_*,l_*}] = 0. \tag{6.31}$$

Note that in the present situation the function $f \equiv 1$ must of course belong to the cone, otherwise the CFT simply does not exist and there is no point of discussing an upper bound on the OPE coefficients. Thus we are assuming from the beginning $\Lambda[1] \geq 0$, unlike in (6.13). Since the functional is linear, Eq. (6.31) is satisfied for

$$t = \Lambda[1]/\Lambda[F_{d,\Delta_*,l_*}], \tag{6.32}$$

and for larger t the functional will become negative as long as $\Lambda[F_{d,\Delta_*,l_*}] > 0$. Thus we obtain the following result: each functional Λ satisfying (6.30) gives a bound on the maximal allowed value of p_{Δ_*,l_*}:

$$\max p_{\Delta_*,l_*} \leq \Lambda[1]/\Lambda[F_{d,\Delta_*,l_*}]\,. \tag{6.33}$$

This bound can be optimized by choosing the functional judiciously. Here comes the technical difference between the case of a single sum rule and theories with additional symmetries where we can derive vectorial sum rules. In the former case the functional optimizing the bound on p_{Δ_*,l_*} corresponds to the functional maximizing the quantity $\Lambda[F_{d,\Delta_*,l_*}]$ with the constraint $\Lambda[1] = 1$, which can always be achieved with a rescaling since the constraint that $f = 1$ is inside the cone translates into the condition that the the zeroth order derivative term in Λ is positive. When there more than a single sum rule the condition 6.33 is replaced by

$$\max p_{\Delta_*,l_*} \leq \frac{\Lambda[\vec{V}^{RHS}]}{\Lambda[\vec{V}_{d,\Delta_*,l_*}]} \,. \tag{6.34}$$

Finding the optimal functional requires this time the minimization of the right hand size of the above equation over a set of constraints like (6.19). This problem is no more a standard Linear Programming problem but is called a *Linear-fractional Programming* problem.

6.2.1 Results in 2D

The method just described was first applied in [29] to constrain the size of the OPE coefficients of scalar operators. Here we exploit the same method, combined with a higher numerical effort and assumptions on the spectrum of the theory, to explore the potentials of our method. Suppose for instance that the Ising model was not known. As already mentioned in Section 6.1.6, form the inspection of Fig. 6.15 one can guess the existence of a CFT containing the the following scalar primaries:

$$\sigma(x) \times \sigma(0) \sim \frac{1}{|x|^{1/4}} \left(1 + c_\epsilon \epsilon + ...\right) \qquad [\sigma] = d = \frac{1}{8} \,, \qquad [\epsilon] = \Delta_\epsilon = 1 \tag{6.35}$$

All the other operators appearing in the above OPE have unknown quantum numbers; the only information is that ϵ has the smallest dimension among the scalar operators. Imposing this restriction on the theory we can study what is the bound on the OPE coefficient of a generic primary operator with dimension Δ and even spin l. The results of this study are depicted the left column of Fig. 6.17, for different values of the spin. A remarkable result consists in the evident emergence of peaks out of a smooth background. Although these structures are already trackable[13] using $N_{der} = 6$ derivatives, their presence is made more evident by the use of $N_{der} = 12$. Indeed using more derivatives we observe a global improvement of the bound

[13] The shown plots are in logarithmic scale. Some peak appear more evident using the linear scale.

except for determined values, which we identify with

$$\begin{aligned} l = 0, \qquad & \Delta = 1, 4, (8?)\,, \\ l = 2, \qquad & \Delta = 2, 6\,, \\ l = 4, \qquad & \Delta = 4, 5, 8\,. \end{aligned} \tag{6.36}$$

If we interpret the presence of a peak in the bound on OPE coefficients as a signal that the corresponding operator gives a dominant contribution in satisfying the sum rule we conclude that this operator must be present in the CFT we are trying to study. Thus, the peaks position (6.36) suggests that in addition to the operator ϵ, other operators with integer dimension should appear in the OPE (6.35). Of course we know that this is exactly the case, since the σ OPE contains only two Virasoro-primary operators ($\mathbb{1}$ and ϵ) and all the other primaries are obtained acting on them with the Virasoro generators which increases the dimension of one unit each (see later).

However, let us proceed for a while with our agnostic argument. The restriction that only operators with integer dimension enter in the OPE can be easily implemented enforcing the general positivity property with

$$\begin{aligned} \Lambda[F_{1/8,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta \geq 1\,, \quad \Delta \in \mathbb{N}\,(l = 0) \\ \text{and} \qquad & \text{for all } \Delta \in \mathbb{N}\,(l = 2, 4, 6, ..) \\ \Lambda[1] \geq 0\,. \end{aligned} \tag{6.37}$$

Notice that replacing the last inequality of (6.37) with $\Lambda[1] \leq 0$ would correspond to test whether the sum rule can be satisfied by the chosen spectrum. In that case one would discover, as it should, that such a functional doesn't exist. Instead, using (6.37) we can recompute the bound on the OPE coefficients, restricting only to integer values of the dimension. The results are shown in the right column of Fig. 6.17. Comparing with the bound for the corresponding spin we observe a remarkable agreement, which suggests that all the information derived for $d = 1/8$ are driven by operators with integer dimensions. In the geometric interpretation of the sum rule this is understood as follows: the vectors corresponding to integer dimensions form the edges of the convex hull (the cone). The inclusion of all the other operators in the positivity property 6.37 doesn't bring any additional restriction nor information[14].

[14] Their inclusion however complicates the numerical analysis.

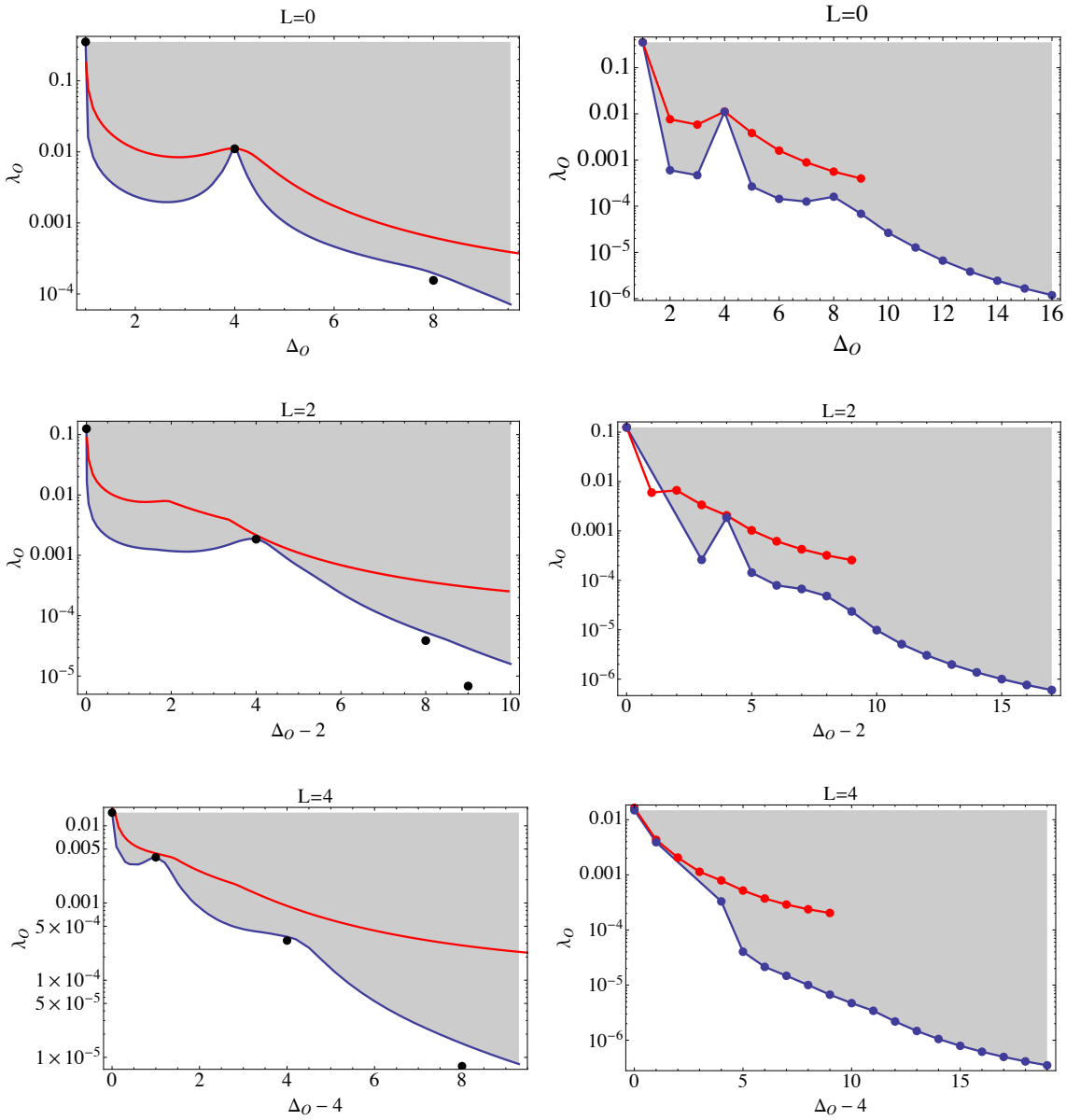


Figure 6.17: *Upper bound on the coefficients λ_O of operators O appearing in the OPE of a scalar field with dimension $d = 1/8$ as function of the operator dimension Δ_O. The plots on the left has been computed with the unique assumption that $\Delta_O \geq 1$ for $l = 0$. The curves are continuos due to the fine discretization on Δ_O.*

The plots on the right has been computed including only operators with integer dimension and $\Delta_O \geq 1$ for $l = 0$. See the text for explanation. The plots shows in pairs the same peaks. Red curves (points) have been computed with 6 derivative, blue curves (points) with 12 derivatives.

We now would like to show quantitatively how close to reality can be the bounds on OPE coefficients. As a first check let us consider the central charge of the theory. Postponing a rigorous discussion to the next section, we anticipate that the coefficient squared of the OPE energy momentum tensor can be related to inverse of the central charge. Choosing $d = 1/8$ and imposing the positivity condition in the form 6.37 we extracted, with $N_{der} = 12$:

$$C_T \geq 0.499977 \tag{6.38}$$

which must be compared with the exact value for the Ising model: $C_T = 1/2$. For the case under examination we can even go further and compute exactly the OPE coefficients of the theory. The four point function of the two dimensional Ising model is indeed known and takes the following form [61]:

$$\begin{aligned}\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle &= \left(\frac{1}{x_{12}x_{34}}\right)^{1/4}(f_+(z)f_+(\bar{z}) + f_-(z)f_-(\bar{z})) \\ f_\pm(z)\frac{1}{\sqrt{2}}&\left(\frac{1}{\sqrt[4]{1-z}} \pm \sqrt[4]{1-z}\right)^{1/2}\end{aligned} \tag{6.39}$$

The above expression can be expanded in conformal blocks using techniques similar to those used in Section 4.3 to extract the OPE coefficients of the free theory. Although we have not been able to derive a closed expression we report in the following table the first terms of the expansion:

$$\langle\sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle = \left(\frac{1}{x_{12}x_{34}}\right)^{1/4}\left(1 + \sum_{\substack{\Delta \text{ integer} \\ l \text{ even}}} p_{\Delta,l}\, g^{D=2}_{\Delta,l}(u,v)\right) \tag{6.40}$$

	$\Delta - l$										
	0	1	2	3	4	5	6	7	8	9	10
$l = 0$		$\frac{1}{8}$	0	0	$\frac{1}{8192}$	0	0	0	$\frac{81}{3355443200}$	$\frac{1}{2147483648}$	0
$l = 2$	$\frac{1}{64}$	0	0	0	$\frac{9}{2621440}$	0	0	0	$\frac{45}{30064771072}$	$\frac{1}{21474836480}$	0
$l = 4$	$\frac{9}{40960}$	$\frac{1}{65536}$	0	0	$\frac{25}{234881024}$	0	0	0	$\frac{46581}{786150813859840}$	$\frac{1125}{492581209243648}$	0

The values corresponding to the above table are depicted in the left column of Fig. 6.17 as black points. Remarkably our bound is saturated by those OPE coefficients[15]. With the use

[15] We recall that Fig. 6.17 shows bound on the OPE coefficients, while values in 6.40 refer $p_{\Delta,l}$. The two are related by eq. (6.52).

of $N_{der} = 12$ we are sensitive only to $\Delta - l \geq 8$, but we are confident that a higher number of derivatives can make manifest also other peaks.

We conclude the section with some final comment. First, we decided to start with a gap in the scalar spectrum, motivated by the behavior in Fig. 6.15, however there is a second, more technical, motivation. If we replace the first condition of (6.37) with

$$\Lambda[F_{1/8,\Delta,l}] \geq 0\,, \qquad \text{for all } \Delta \geq \frac{1}{8}\,, \tag{6.41}$$

we wouldn't find any functional. Namely, we discovered a peculiar property of the two dimensional sum rule: at least using $N_{der} = 12$, the simple assumption that σ is the smallest dimension scalar in the theory is not sufficient to extract a bond on OPE coefficients. In other words, a bound on the OPE coefficients exists only if there is a gap between d and Δ_ϵ in the scalar sector. Geometrically this means that the the convex hull described by all the vectors in the sum rule and $f = 1$ fills all the space, unless we restrict the spectrum via some assumption. In this case we imposed the restriction that for $l = 0, \Delta \geq 1$.
Finally, we carried out the analysis for the Ising model, however a similar study can be done for the other minimal models as well. Again the bounds show the presence of peaks. Notice that in other minimal models the dimension Δ_ϵ of the first scalar appearing in the OPE $\sigma \times \sigma$ is not integer, hence we must include in the positivity property additional conditions like $\Lambda[F_{d,\Delta,l}] \geq 0$, with $\Delta = l + n + \Delta_\epsilon$, $n \in \mathbb{N}$.Moreover, as discussed in Appendix B, OPE's are more involved, since they contain additional Virasoro primaries.

6.3 Central charge

We now concentrate on a particular OPE coefficient, the one associated to the energy momentum tensor. Before proceeding further it is useful to recall the relation between the OPE coefficient of a given operator $O_{\Delta,l}$ and the contribution of the corresponding conformal block to the four scalar correlator $g_{\Delta,l}$. Let us start from a basis where all the operators are unit normalized:

$$\langle \phi(x)\phi(0)\rangle = (x^2)^{-d}, \tag{6.42}$$

$$\langle O_{\mu_1\ldots\mu_l}(x) O_{\lambda_1\ldots\lambda_l}(0)\rangle = \frac{1}{(x^2)^\Delta}\left[\frac{1}{l!}(I_{\mu_1\lambda_1}\ldots I_{\mu_l\lambda_l} + \text{perms}) - \text{traces}\right], \tag{6.43}$$

and the OPE coefficient $c_{\Delta,l}$ is defined in terms of the 3-point function

$$\langle\phi(x_1)\phi(x_2)O_{\mu_1\ldots\mu_l}(0)\rangle = \frac{c_{\Delta,l}}{(x_{12}^2)^{d-\frac{\Delta-l}{2}}(x_1^2)^{\frac{\Delta-l}{2}}(x_2^2)^{\frac{\Delta-l}{2}}}\left(Z_{\mu_1}\ldots Z_{\mu_l} - \text{traces}\right), \tag{6.44}$$

$$Z_\mu = x_{1\mu}/x_1^2 - x_{2\mu}/x_2^2. \tag{6.45}$$

At the same time, the *central charge* is defined as the normalization of the energy momentum tensor

$$\langle T_{\mu\nu}(x)T_{\lambda\sigma}(0)\rangle = \frac{C_T}{S_D^2}\frac{1}{(x^2)^D}\left[\frac{1}{2}(I_{\mu\lambda}I_{\nu\sigma} + I_{\mu\sigma}I_{\nu\lambda}) - \frac{1}{D}\delta_{\mu\nu}\delta_{\lambda\sigma}\right],$$
$$I_{\mu\nu} = \delta_{\mu\nu} - 2x_\mu x_\nu/x^2\,, \quad S_D = 2\pi^{D/2}/\Gamma(D/2) \tag{6.46}$$

when the tensor is consistent with the Ward identity:

$$\partial_\mu\langle T_{\mu\nu}(x)\phi(x_1)\ldots\phi(x_n)\rangle = -\sum_i \delta(x-x_i)\langle\phi(x_1)\ldots\partial_\nu\phi(x_i)\ldots\phi(x_n)\rangle\,. \tag{6.47}$$

The central charge C_T is an interesting quantity because it provides a certain measure of the number of degrees of freedom in the theory. For example, for a free conformal theory of N_ϕ real scalars, N_ψ Weyl fermions, and N_A vectors in 4D, we have

$$C_T = \frac{4}{3}N_\phi + 4N_\psi + 16N_A. \tag{6.48}$$

In the normalization 6.47 is satisfied, the three point function of scalar operators and the energy momentum tensor is fixed [53]:

$$\langle\phi(x_1)\phi(x_2)T_{\mu\nu}(0)\rangle = -\frac{Dd}{(D-1)S_D}\frac{1}{(x_{12}^2)^{d-1}x_1^2x_2^2}\left(Z_\mu Z_\nu - \frac{1}{D}\delta_{\mu\nu}Z^2\right). \tag{6.49}$$

Being a conserved tensor of rank two, the energy-momentum tensor has dimension exactly D in D dimensions. Rescaling properly the energy momentum tensor to bring (6.46) in the form of (6.42) we can express the OPE coefficient $c_{D,2}$ in terms of the central charge C_T and the dimension of ϕ [33]:

$$c_{D,2} = -\frac{Dd}{D-1}\frac{1}{\sqrt{C_T}}\,. \tag{6.50}$$

The above equation is valid in arbitrary dimension D. In 2D the total central charge is defined as the sum of the central charges of holomorphic and antiholomorphic modes, $C_T = c + \bar{c}$ (see [37, 39]). Finally we should recall that, when expressing the four point function as a sum of conformal blocks

$$\langle\phi\phi\phi\phi\rangle = \frac{1}{x_{12}^{2d}x_{34}^{2d}}\left(1 + \sum p_{\Delta,l}g_{\Delta,l}(u,v)\right) \tag{6.51}$$

in the present conformal block convention (see (4.30)) we have the relation:

$$p_{\Delta,l} = \frac{(c_{\Delta,l})^2}{2^l}\,, \quad \Rightarrow \quad p_{4,2} = \frac{D^2 d^2}{(D-1)^2}\frac{1}{4C_T} \tag{6.52}$$

The above relation implies that for large C_T, the contribution of the stress tensor to the 4-point function of ϕ decreases as $1/C_T$.

6.3.1 Results in 4D

We now present our numerical results. First of all, let us consider the most general case when we are not making any assumption concerning the gap in the scalar sector of the OPE. This means that the scalar operators appearing in the OPE are allowed to have any dimension $\Delta \geq d$. Operators with lower dimensions are a priori excluded if ϕ is the lowest dimension scalar. Under this assumption, we use the method of linear functionals to bound $p_{4,2}$ from above. Operationally we impose a positivity property of the form (6.30), which in our case means

$$\begin{aligned} &\Lambda[F_{d,\Delta,0}] \geq 0 \quad \text{for all } \Delta \geq d\,, \\ &\Lambda[F_{d,\Delta,l}] \geq 0 \quad \text{for all } \Delta \geq l+2, \quad l = 2, 4, \ldots \end{aligned} \tag{6.53}$$

We will choose $\lambda_{0,0} = 1$ to have $\Lambda[1] = 1$. Then to optimize the bound (6.34), the coefficients of the functional must be chosen so that

$$\Lambda[F_{d,4,2}] \to \max\,, \tag{6.54}$$

We will consider the functionals with the maximal derivative order up to $N = 16$.. Pushing to higher N values is likely to somewhat improve the bound. In principle N as large as 18 were demonstrated feasible, see for instance Fig.6.9.

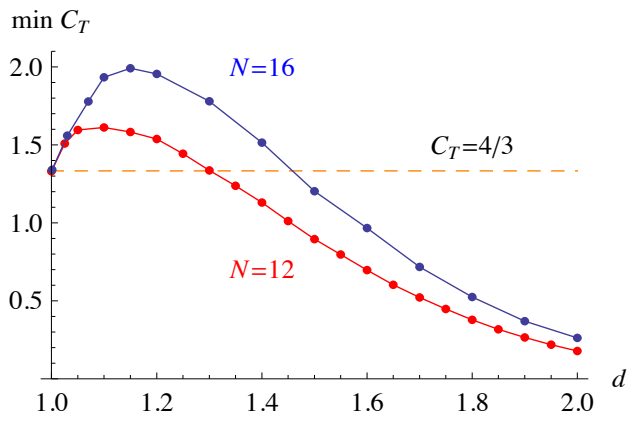


Figure 6.18: *The lower bound on the central charge C_T in terms of the dimension d of the lowest-dimension scalar primary. The stronger bound (upper blue curve) is obtained with $N = 16$. For comparison we give a weaker bound obtained with $N = 12$ (lower red curve), which corresponds to the horizontal axis $\Delta_* = d$ in the following Fig. 6.19. The horizontal dashed line $C_T = 4/3$ shows where our bound stays above the free scalar central charge.*

Using this procedure, we computed a bound on $p_{4,2}$ from above, which via (6.52) translates into a bound on C_T from below. The latter bound is plotted in Fig. 6.18 as a function of the dimension of ϕ in the range $1 \leq d \leq 2$. We plot our best bound for $N = 16$ and, for comparison, a weaker bound obtained with a smaller value $N = 12$.

The first interesting point about this bound is that in the limit $d \to 1$ it approaches the free scalar central charge value $C_T^{\text{free}} = 4/3$, see Eq. (6.48). In other words, our method shows that free theory limit is approached continuously. Next, we see that for $1 < d \lesssim 1.4$ our bound stays *above* C_T^{free}, thus showing that an interacting theory necessarily has larger central charge than the free one. Unfortunately, for larger d our bound drops below C_T^{free}. We do not know if this means that there are CFTs with $C_T < C_T^{\text{free}}$. More likely, this indicates that our bound is not best-possible in this range. One could speculate that the best-possible bound should stay above C_T^{free} in the whole range $1 < d < 2$. The fact that it should necessarily come down to C_T^{free} (or lower) for $d = 2$ can be inferred by considering the dimension 2 operator φ^2 in the free scalar theory and its OPE with itself.

Let us now consider what happens with the bound when additional information about the CFT are exploited. For instance we can assume the in presence of a gap in the scalar spectrum.

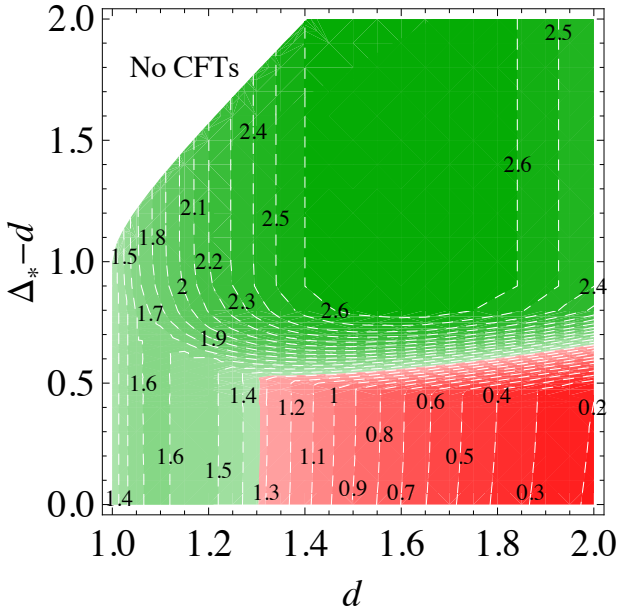


Figure 6.19: *Contour plot of the C_T lower bound as a function of d and of the gap $\Delta_* - d$, where Δ_* is the dimension of the first scalar in the $\phi \times \phi$ OPE. The gap is nonnegative, since we assume that ϕ is the lowest dimension scalar. On the horizontal axis the bound reduces to the $N = 12$ curve in Fig. 6.18. The lighter green color marks the region where the bound is above $C_T^{\text{free}} = 4/3$, while in the darker red region the bound is below this value. As the gap increases, the bound gets stronger, so that a rather weak assumption about the gap is already enough to have $C_T > C_T^{free}$*

In other words, we now impose that the first scalar operator in the $\phi \times \phi$ OPE has dimension Δ_* *strictly bigger* than d. Technically, this problem is analyzed exactly as the previous one, except that the first set of constraints (6.53) is replaced by a shorter list:

$$\Lambda[F_{d,\Delta,0}] \geq 0 \quad \text{for all } \Delta \geq \Delta_* \,. \tag{6.55}$$

Because of considerable computer time involved, we solved this problem by using linear functionals with $N = 12$ only. The bound is given in Fig. 6.19 as a contour plot in the $(d, \Delta_* - d)$ plane. On the horizontal axis $\Delta_* = d$ the bound reduces to the $N = 12$ bound from Fig. 6.18. Naturally, when Δ_* increases, the bound on C_T gets stronger. This is clear since the number of constraints to which the functional Λ is subjected is reduced. We see that Δ_* somewhat bigger than d is already sufficient to raise the bound above C_T^{free} for all d (the lighter green region in the

plot). The points with $\Delta_* \sim 2d$ (i.e. with an approximate factorization of operator dimensions) belong to the green region by a big margin. The white region in upper left corresponds to

$$\Delta_* > 2 + 0.7(d-1)^{1/2} + 2.1(d-1) + 0.43(d-1)^{3/2} \tag{6.56}$$

and is excluded, since such a large gap cannot be realized in any CFT according to the results of Section 6.1.3.

In summary, we have shown in this work that if a unitary 4D CFT is non-trivial (in that it contains at least one primary scalar operator), then its central charge C_T cannot be arbitrarily low.

6.3.2 Results in 2D

We now presents our results for 2 dimensions. Fig. 6.20 represents the analog of Fig. 6.19. In the vertical axis we put directly Δ_*, the value of the smallest scalar operator dimension contributing to the four point function. A crucial difference with the four dimensional case id the presence of a region in the plane (d, Δ_*) where it was not possible to find a functional satisfying the positivity property

$$\begin{aligned} \Lambda[F_{d,\Delta,0}] &\geq 0 \quad \text{for all } \Delta \geq \Delta_* \,, \\ \Lambda[F_{d,\Delta,l}] &\geq 0 \quad \text{for all } \Delta \geq l, \quad l = 2, 4, \ldots \\ \Lambda[1] &= 1 \,. \end{aligned} \tag{6.57}$$

This happens because the cone projected into the subspace of $N_{der} = 12$ fills all the space when to many scalar operators are allowed to contribute in delineating the cone. The interpretation can be twofold: either the truncation to a finite N_{der} makes a large quantity of information to get lost, or there exist CFT's in two dimensions with arbitrary large OPE coefficients.
We observe that the bound on the central charge is always smaller than 1. Models with $C_T < 1$ are completely classified in 2 dimension and consists in the Minimal models (see appendix B), hence we only expect to make contact with their central charges for the suitable values of (d, Δ_*). In Table 6.1 we report the theoretical value of the central charges, the computed bounds subjected to the positivity property (6.57). Again we verify that the two dimensional sum rule captures important features of CFT's.

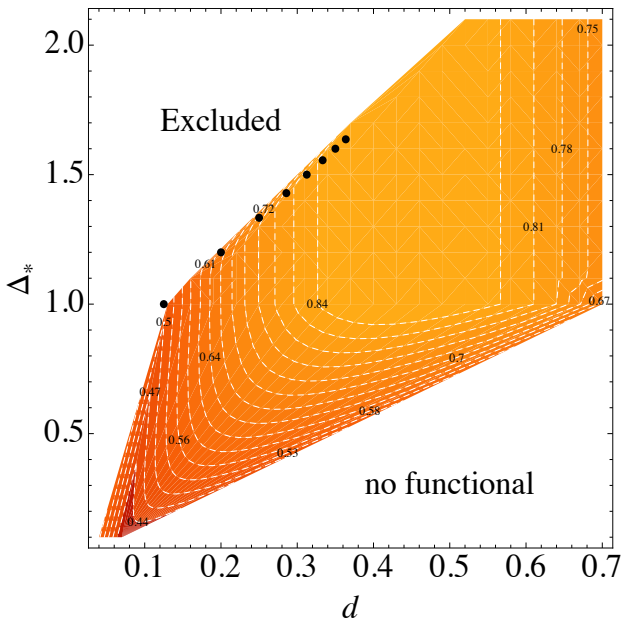


Figure 6.20: *Contour plot of the C_T lower bound as a function of d and of the gap Δ_*, where Δ_* is the dimension of the first scalar in the $\phi \times \phi$ OPE. The gap is nonnegative, since we assume that ϕ is the lowest dimension scalar. As the gap increases, the bound gets stronger. For values of (d, Δ_*) lying in the lower-right white zone the method can't provide a bound. See the text for explanation. The upper-left white zone corresponds to the excluded region. See Fig. 6.15. The framed values report the central charges of Minimal models.*

6.3.3 Generalized Free Theories and AdS/CFT

Generalized Free CFT's, also called Gaussian CFT's, are probably the simplest non trivial example of CFT's. They are defined specifying the two point functions of a given operator (scalar for the present applications) of dimensions $d \geq 1$:

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{x_{12}^2 d} \tag{6.58}$$

and computing all the other quantities via Wick theorem. Because of this, the OPE $\phi \times \phi$ contains only operators of the form

$$\phi \times \phi \sim \frac{1}{x_{12}^2 d}\mathbb{1} + \sum_{n,l} \frac{c_{n,2l}}{x_{12}^{n+2l}} \left(\phi \Box^n \partial_{\mu_1} ... \partial_{\mu_{2l}} \phi - \text{traces}\right) \tag{6.59}$$

Table 6.1: Central charge for the Minimal models. The second column from right is the theoretical value. The first column from right is the lower bounds in theories where only scalar operators with dimension larger than Δ_{ϕ^2} enter the OPE $\phi \times \phi$.

d_ϕ	Δ^2_ϕ	C_T	min C_T
0.125	1	0.5	0.5
0.2	1.2	0.7	0.6918
0.25	1.3333	0.8	0.7821
0.2857	1.4286	0.8571	0.8295
0.3125	1.5	0.8928	0.8597

The coefficients $c_{n,l}$ can be computed exactly either from the above OPE or more straightforwardly expanding in conformal blocks the four point function:

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \frac{1}{x_{12}^2 d x_{34}^2 d}\left(1+u^d+\left(\frac{u}{v}\right)^d\right)\\ &= \frac{1}{x_{12}^2 d x_{34}^2 d}\left(1+\sum_{n,l} p_{n,l} g_{2d+2n+2l,2l}(u,v)\right)\end{aligned} \tag{6.60}$$

from which we derive [62]:

$$p_{n,l} = (1+(-1)^l)\frac{(2l+1)(2d+2n+l-2)}{(d-1)^2}A_n A_{n+l+2}\,, \qquad A_n = \frac{\Gamma^2(d-1+n)\Gamma(2d-3+n)}{n!\Gamma^2(d-1)\Gamma(2d-3+2n)} \tag{6.61}$$

while $c_{n,l}$ are determined consequently by the relation in (6.52). A possible way to describe a gaussian scalar CFT it through AdS/CFT correspondence, viewing the model as the dual description of a scalar field propagating in AdS_5. If we call R the radius of the five dimensional space, than the AdS/CFT dictionary determines the mass m^2 of the scalar field to be

$$m^2 R^2 = d(d-4)\,. \tag{6.62}$$

The exact dual description of the gaussian CFT is therefore a free scalar propagating in the bulk of AdS_5, in the limit in which gravity is decoupled. According to the AdS/CFT prescription this is equivalent to the absence of the energy momentum tensor in the CFT. This is correctly realized by our OPE analysis and partial wave decomposition (6.59)(6.60). On the other hand, the five dimensional theory makes sense only as an effective field theory, which is valid only

whenever the gravitational expansion parameter is small. This expansion, usually referred as *large-$\mathcal{N}$ limit* corresponds to the limit $R \ll M_{Planck}$.

As mentioned in the introduction, the AdS/CFT conjecture suggests that the dual description

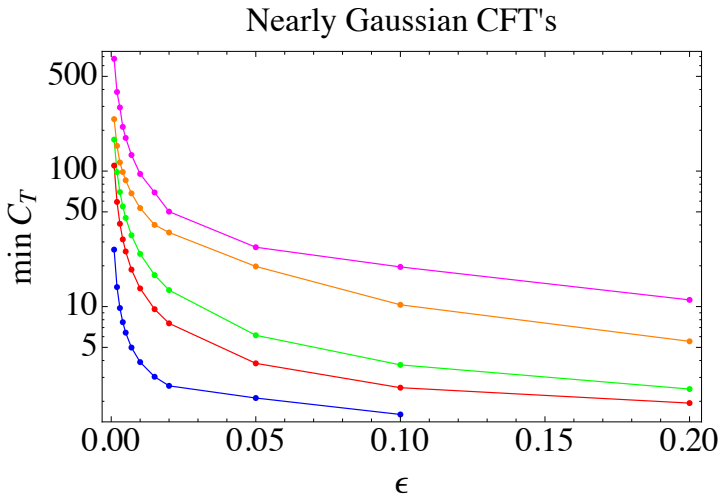


Figure 6.21: *Lower bound on the central charge in a nearly gaussian CFT as a function on the maximal allowed correction to dimension operators ϵ. The points corresponds to CFT's with a scalar operator of dimension (from the above): $d = 1.5, 1.2, 1.1, 1.05, 1.01$. The bound has been obtained with 12 derivatives. Notice the logarithmic scale.*

of the five dimensional theory is a four-dimensional CFT, at all orders in $1/\mathcal{N}$. If this is the case we expect this CFT to contains a sort of expansion that parametrizes the difference with respect to pure gaussian scalar CFT. With an abuse of notation we will refer to the this expansion parameter again as $1/\mathcal{N}$. We expect therefore that the OPE of ϕ will still resemble (6.59), however the dimension of of the operators will be modified according to

$$\Delta_{n,l} = 2d + 2n + l + \frac{1}{\mathcal{N}^2}\gamma_{n,l} + O\left(\frac{1}{\mathcal{N}^4}\right) \tag{6.63}$$

Similarly the OPE coefficients will be corrected as well. Remarkably, imposing the crossing symmetry constraints to the four point function order by order in $1/\mathcal{N}^2$ allows to solve the theory perturbatively. For instance, an exact expression for the anomalous dimensions $\gamma_{n,l}$ and for the OPE coefficients can be derived ([62]).

In what follows we show how our method can be used to extract information about perturbed gaussian CFT's. We will therefore consider a CFT with a real scalar field of dimension d with

an OPE containing only operators as in (6.59) and eventually the energy-momentum tensor. Since we are interested in perturbative modification of gaussian CFT their dimension is allowed to differ from the standard one by a small quantity. We will call ϵ the maximum deviation from the standard dimension. As $\epsilon \to 0$ we expect the theory to reduce to a gaussian CFT.
As a first check we present here the investigation of how the bound on the central charge depend on the corrections to the anomalous dimensions. We repeated our analysis for the theory in exam: technically this reduces to impose the positivity property:

$$\begin{aligned}
&\Lambda[F_{d,\Delta,0}] \geq 0 \quad \text{for all } \Delta = 2d + 2n + 2l + \gamma \quad , n \in \mathbb{N}\,, \quad \gamma \in [-\epsilon,\, \epsilon], \quad l = 0, 2, 4... \\
&\Lambda[F_{d,4,2}] \geq 0 \\
&\Lambda[1] = 1.
\end{aligned} \tag{6.64}$$

Fig. 6.21 shows the results of this study. We observe the expected behavior: as ϵ decreases the bound on the central charge becomes stronger, eventually growing to infinity when reaching the limit of gaussian CFT.
The bound gets weaker as d decreases. This is because as $d \to 1$ the theory becomes exactly free and we know in that case the central charge has a finite value, $C_T = 4/3$. On the other hand, for different values of d the central charge is exactly infinite[16]. It is therefore evident that the limit $\epsilon \to 0$ and $d \to 1$ don't commute. When ϵ is small but non-zero, the bound on the central charge approaches the free theory value continuously (see Fig. 6.22), while for gaussian theories the the bound has a discontinuity at $d = 1$.

We conclude this section comparing our results with the expectations from AdS/CFT. In the large $\mathcal{N}$ limit the central charge is expected to scale with $\mathcal{N}$ and therefore with an inverse power of ϵ. The bounds in Fig. 6.21 show however an exponential-like dependence. The disagreement is due to our assumption of existence of a maximal anomalous dimension $\gamma_{n,l}/\mathcal{N}^2 \leq \epsilon$. In fact, the expression found in [62] for $\gamma_{n,l}$ grows with n. We believe this represents a very interesting line of research that requires a dedicated analysis. For instance, it would be interesting to input into the analysis that only OPE coefficients of the form 6.61, with eventually $1/\mathcal{N}^2$ corrections, can be used to solve the sum rule.

[16] More formally, the contribution to the four point function of the energy momentum tensor vanishes for gaussian theories with $d > 1$.

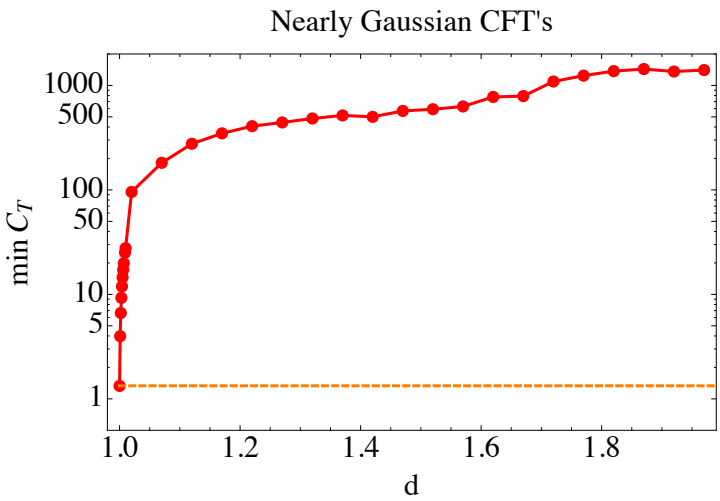


Figure 6.22: *Lower bound on the central charge in a nearly gaussian CFT as a function of the d. Here* $\epsilon = 0.05$*. The bound approaches the free value for* $d \longrightarrow 1$ *and rapidly increases as* $d \neq 1$*. The bound has been obtained with 12 derivatives. Notice the logarithmic scale. When* $\epsilon \longrightarrow 0$ *the steep slope near* $d \sim 1$ *becomes a true discontinuity.*

6.3.4 Results for supersymmetric theories

We conclude this chapter presenting the bound on the central charge in superconformal field theories.

In superconformal field theories the energy momentum tensor is contained in the same supermultiplet of the R-current, which represents the lowest component. Hence the contribution of the energy-momentum tensor is encoded in the superconformal block $\mathcal{G}_{3,1}(u,v)$. Using eq. (5.38), (5.39) and (6.52) we can derive the relation between the conformal block coefficient and the central charge:

$$p_{3,1}^{(susy)} = \frac{D^2 d^2}{(D-1)^2} \frac{5}{4C_T} \,. \tag{6.65}$$

Notice that using the decomposition of the four point function for a free complex scalar in therm of superconformal blocks (5.47) we can verify that

$$C_T = \frac{20}{3} \,, \tag{6.66}$$

which is the correct value for a free theory containing one complex scalar and one Weyl fermion [33]. As for the bound on operator dimensions we can compute the lower bound on C_T using only

the information encoded in the first sum rule in (6.24) or, more correctly, using the whole set of equations. In the former case no bound can be extracted for $N_{der} < 10$. The plot in Fig. 6.23 reports the result obtained with only one sum rule. The weaker bound reproduces the results of [54], while the strongest one, derived with $N_{der} = 14$, represents a numerical improvement. We see that using more computing effort we can constraint the central charge to be larger than the free theory value in a non negligible interval. This method however is not able to capture the correct behavior for $d \to 1$, limit in which the central charge is expected to reach the free value.

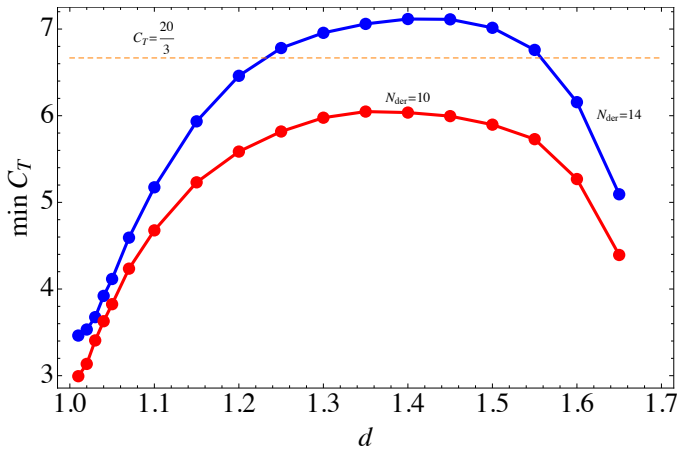


Figure 6.23: *Lower bound on the central charge computed with $N_{der} = 10$ (red) and $N_{der} = 14$ (blue) using only one sum rule. The dashed line corresponds to the central charge of a supersymmetric theory with one chiral superfield.*

The use of the entire set of crossing symmetry constraints allows instead to extract a bound even for small values of N_{der}. Here we report the results obtained with $Nder = 6$. Although the methods can be pushed to higher values of N_{der} the bound becomes irregular and we decided not to present it. The plot in Fig. 6.24 shows the bound obtained imposing a positivity property of the form

$$\begin{aligned} \Lambda[\vec{V}^{0\pm}_{d,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta \geq l+2\,(l=0,1,2,3,4,\ldots)\,. \\ \Lambda[\vec{V}^{2}_{d,\Delta,l}] \geq 0\,, \qquad & \text{for all } \Delta = 2d+l\,(l=0,2,4\ldots)\,. \\ \text{and} \qquad & \text{for all } \Delta \geq |2d-3|+l+3\,(l=0,2,4\ldots)\,. \\ \Lambda[\vec{V}^{RHS}] = 1\,, & \end{aligned} \tag{6.67}$$

where the the vectors $\vec{V}$ are defined in (6.24). As a first crucial difference with the plot Fig.

6.23 we notice that the bound at $d = 1$ is very close to the free values 20/3 . In addition it remains above the free value for $d \leq 1.4$. For larger values of d the bound decreases as usual [28, 54]. We expect however the optimal bound to stay above the free value in all the open interval $1 < d < 2$.

A comparison with explicit models ([54]) shows that the bound is never saturated: for instance IR fixed points of SQCD theories in the conformal windows have a central charge at least a factor of 20 larger. This discrepancy is somewhat expected since all these models contain additional global symmetries which increases the number of degrees of freedom and consequently the central charge. We expect that a combined use of vector sum rule for the proper global symmetry group and superconformal blocks could allow a non trivial comparison with these models.

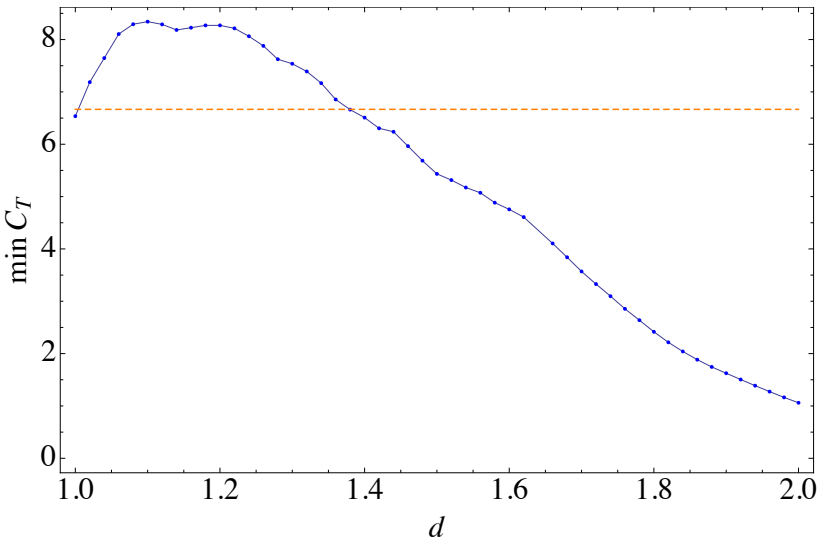


Figure 6.24: *Lower bound on the central charge computed with $N_{der} = 6$ using the vectorial sum rule. The dashed line corresponds to the central charge of a supersymmetric theory with one chiral superfield.*

Chapter 7

Conclusions

7.1 Summary and outlook

In this thesis we have shown that prime principles of Conformal Field Theory, such as unitarity, OPE, and conformal block decomposition, can be translated in restrictions on the spectrum and the size of OPE coefficients of the CFT.

We developed a method which allows numerical determination of those constraints with arbitrary desired accuracy. The method is based on the investigation of *sum rules*, function-space identities satisfied by the conformal block decomposition of the 4-point function $\langle \phi_i \phi_j \phi_k \phi_l \rangle$, which follow from the crossing symmetry constraints. When the the CFT is invariant under global symmetries or supersymmetry we have been able to disentangle different contributions arising from operators transforming in different representations. We also show that the number of sum rules equals the number of such contributions in all the examples we have been able to construct.

In practical application of the method the sum rules are Taylor-expanded and replaced by finitely many equations for the derivatives. The constraints we derived improve monotonically as more and more derivatives are included.

As a first applications of the technique we derived bounds on dimension $\Delta_{\min}$ of the first scalar ϕ^2 entering the OPE $\phi_i \times \phi_j$. These bounds takes the form $\Delta_{\min} < f(d)$, where d is the dimension of ϕ_i.

In 4D we first presented a bound for the case of CFT's without global symmetries. We show a convergence of the bound to an optimal one as the number of derivatives is increased. Next we

consider theories with $SO(N)$ symmetry and superconformal theories. In both cases crossing symmetry gives rise to three sum rules. In the former case we compute independent bounds on the first singlet and the first $SO(N)$ tensor appearing in the OPE of two scalars in the fundamental representation. In the latter case we computed the bound on the dimension of the first non-chiral operator entering the OPE of a chiral with an anti-chiral superfield.

As a further demonstration of the method we computed upper bounds on OPE coefficients. Given the inverse relation between the central charge and the energy momentum OPE coefficient squared, a limit on the latter translates in a lower bound on the former. Again we presented results for several theories.

Our bounds are satisfied, by a large margin, in all weakly coupled 4D CFTs that we are able to construct. For comparison we considered 2D CFT's without global symmetries and checked it against exact 2D CFT results. Again, the bound is satisfied, and in a less trivial way than in 4D, since the Minimal models saturate it.

All the mentioned analysis can be carried out without need of any assumptions on the CFT: bounds independent of what is the spectrum or the OPE coefficients of the CFT can be extracted. On the other hand, stronger limits arise when we incorporate informations about the CFT. For instance, we showed that more stringent limits on the central charge can be derived when we restrict the operators spectrum.
Moreover, we showed how to impose the presence of Virasoro algebra in 2D or the existence of a perturbative expansion around a Gaussian CFT in 4D.

We conclude our discussion listing some possible developments and applications of our method. A first obvious option is a numerical improvement of the results presented in this manuscript. Although we expect bounds for CFT's without symmetries to be rather close to the optimal ones, bounds for $SO(N)$ theories and supersymmetric theories have probably a large margin of improvement. In particular, it would be very interesting to push the numerical analysis for $SO(4)$, given the connection with theories of Conformal Technicolor.
Similarly, the 6 sum rules derived for $SU(N)$ global symmetry can be used to extract bounds on operator dimensions and central charges.
As we did for $SO(N)$, independent bounds on different symmetry representations entering the scalar OPE's can in principle be extracted. Moreover, bounds on non scalar operators can be computed too.
Finally, various assumptions on the theory can be input in the method. In the thesis we largely showed how to implement restrictions on the CFT spectrum; moreover if an OPE coefficient is exactly known we could exploit this information extracting the corresponding term in the sum

rule and bringing it on the other side of the equation. Using this procedure we can for instance input the value of the central charge.

The goal of these investigations, despite the intrinsic theoretical interest of the existence of a bound, is to make contact with known examples of CFT, as in two dimensions, and eventually discover new ones. So far however, we haven't observed any irregularities in the 4D bounds that could suggest the presence of a CFT.
It is probable that 4D CFT's always contains global symmetries and their structure can only be revealed studying vector sum rules. This would be the case of theories in the non supersymmetric conformal window under investigation using lattice techniques [63].

A different research line would consist in generalizing the analysis for $\mathcal{N} = 1$ supersymmetry to extended supersymmetry. In [64] the expression for $\mathcal{N} = 2, 4$ superconformal blocks are derived for the four point function of specific multiplets. As in the simpler $\mathcal{N} = 1$ case, they are expressed as combinations of conformal blocks. Combining this results with the proper vector sum rules[1] it should be possible to extract bounds. This would enable a comparison with a large class of exact results and possibly the inference of new ones.

Finally, all the analysis presented and suggested so far can be repeated in any even dimensions, simply using the suitable expression for the conformal block reported in Sec. 4.1.3.

In conclusion, we believe the method discussed in this work represents a powerful tool to shed light on the still obscure structure of CFT's. The numerical results obtained with our technique can confirm analytical results, disprove conjectures and guide further investigations in this field.

[1] In these theories the R-symmetry is respectively $SU(2)$ and $SU(4)$.

Appendix A

Representation on functions and fields

In this section we review how symmetries are implemented in a unitary representation. Let us consider a group of transformations acting on the space-time coordinates as

$$f: \quad x^\mu \longrightarrow x'^\mu = f(x^\mu) \simeq x^\mu + \delta x^\mu \tag{A.1}$$

Given a function $\phi(x)$, in terms the transformed coordinates we have

$$\phi'(x') = \mathcal{F}[\phi(x)] = \mathcal{F}[\phi(f^{-1}(x'))] \tag{A.2}$$

In particular if we say that the function is *invariant* under the symmetry if $\mathcal{F} = \mathbb{1}$. In this case the value of the function at a given point must be the same, no matter if the point is described in terms of coordinates x or x'. On the other hand one can define the variation of a function at fixed coordinates:

$$\delta\phi = \phi'(x) - \phi(x) \tag{A.3}$$

The above variation is non-vanishing also for invariant functions but in that case is entirely due to the fact that coordinates do transform under the symmetry.
Let us now consider the implementation of the above set of transformations on the Hilbert space of physical states. The wave-function of a state $\psi(x)$ behaves exactly as a function under transformations (let us restrict to the invariant case):

$$\psi'(x') = \psi(x), \qquad \text{where, for instance:} \quad \psi(x) = \int \frac{d^3k}{2k_0(2\pi)^3} c(k) e^{ikx} a^\dagger(k)|0\rangle \tag{A.4}$$

The transformation at fixed coordinate can be obtained through the use of a unitary operator $U(g)$ acting on the states of the Hilbert space; notice that this realizes the variation at fixed coordinate:

$$\psi'(x) = U(f)\psi(x) \tag{A.5}$$

Starting from the above expression and expanding for infinitesimal value of the parameters of the symmetry transformation both sides of the equation we get:

$$\begin{aligned} & \psi'(x^\mu) \simeq \psi'(x'^\mu - \delta x^\mu) \simeq \psi(x^\mu) - \delta x^\mu \partial_\mu \psi(x) \\ & U(f) \doteqdot e^{-i\alpha^a T^a} = \mathbb{1} - i\alpha^a T^a \\ \Rightarrow \quad & \alpha^a T^a \psi = -i\delta x^\mu \partial_\mu \psi(x) \end{aligned} \tag{A.6}$$

The above expression specify the action of the generator on states. They are represented as differential operators acting on the wave function associated to the state. Starting from the found differential expressions one can compute the algebra of the group simply taking the commutators of the generators.

As usual we can take an alternative approach and consider the states of the Hilbert space untouched by symmetry transformation and transform instead the operators. In order to find the right transformation properties we impose the equivalence of the two methods:

$$\langle \psi_1' | O(x) | \psi_2' \rangle = \langle \psi_1 | O'(x) | \psi_2 \rangle \,, \qquad \Rightarrow O' = U^\dagger(f)\, O\, U(f) \tag{A.7}$$

Expanding again for infinitesimal transformations we get

$$\alpha^a \left[T^a, O(x) \right] = i\delta^\rho \partial_\rho O(x) \,, \tag{A.8}$$

where again we stick to invariant operators. Notice an important minus factor between the above result and eq. (A.6). This is crucial in order to correctly reproduce the algebra of the generators. Let us see how this works for the simple case of translation and dilatations in one dimension:

$$\begin{aligned} & \text{trasl.:} \quad x' = x + a, \qquad\qquad \text{dilat.:} \quad x' = x + \lambda x \\ & T\phi = -i\partial\psi \qquad\qquad D\psi = -ix\,\partial\phi \quad \Rightarrow \quad [D, T] = iT \end{aligned} \tag{A.9}$$

In the Heisenberg picture instead we have:

$$\begin{aligned} & [T, O(x)] = i\partial O \qquad\qquad [D, O(x)] = ix\partial O \\ & [[D,T], O(x)] = [D, [T, O(x)]] - [T, [D, O(x)]] = i\partial[D, O(x)] - ix\partial[T, O(x)] \\ & \qquad = -\partial O = [iT, O(x)] \end{aligned} \tag{A.10}$$

Notice that differential operators exit the commutation relations and act on the result of the second generator. This compensate the extra minus factor in the definitions of the generators.

Appendix B

Minimal Models

In this appendix we review a class a class of exactly solvable models in two dimensions.
In 2D CFTs, we must make distinction between the global conformal group $SL(2,C)$, and the infinite-dimensional Virasoro algebra of local conformal trasnformations, of which $SL(2,C)$ is a finite-dimensional subgroup Virasoro algebra plays crucial role in solving these theories exactly, but it has no analogue in higher dimensions.
When we speak about primaries, descendants, conformal blocks in 2D theories, we must specify with respect to which group we define these concepts, Virasoro or $SL(2,C)$. The former is standard in the 2D CFT literature, while it is the latter that is directly analogous to 4D situation.[1]
Every Virasoro primary is a $SL(2,C)$ primary, but the converse is not true. E.g. the stress tensor in any 2D CFT is a Virasoro descendant of the unit operator. To find $SL(2,C)$ primaries, we need to decompose the sequence of all Virasoro descendants of each Virasoro primary (the so called Verma module) into irreducible $SL(2,C)$ representations. While this is possible in principle, it may not be straightforward in practice. Nevertheless we know that $SL(2,C)$ primaries have dimensions of the form

$$\Delta_{SL(2,C)} = \Delta_{\text{Vir}} + n, \quad n = 0 \text{ or } n \geq 2,$$

where Δ_{Vir} is a Virasoro primary dimension, and n is an integer. This is because the Virasoro operators which are not in $SL(2,C)$ raise the dimension by at least 2 units.
The unitarity bound for bosonic fields in 2D is

$$\Delta \geq l,$$

[1] $SL(2,C)$ primaries are sometimes called *quasi-primaries* in the 2D CFT literature.

where $l = 0, 1, 2, \ldots$ is the Lorentz spin. A very interesting example involves the *minimal model* family of exactly solvable 2D CFT. The unitary minimal models (see [37],[40]) are numbered by an integer $m = 3, 4, \ldots$, and describe the universality class of the multicritical Ginzburg-Landau model:

$$\mathcal{L} \sim (\partial\phi)^2 + \lambda\phi^{2m-2}\,. \tag{B.1}$$

For $m = 3$, the Ising model is in the same universality class. The central charge of the model,

$$c = 1 - \frac{6}{m(m-1)},$$

monotonically approaches the free scalar value $c_{\text{free}} = 1$ as $m \to \infty$. Intuitively, as m increases, the potential becomes more and more flat, allows more states near the origin (c grows), and disappears as $m \to \infty$ (free theory).

Minimal models are called so because they have finitely many Virasoro primary fields (the number of $SL(2, C)$ primaries is infinite). All Virasoro primaries are scalar fields $O_{r,s}$ numbered by two integers $1 \leq s \leq r \leq m-1$, whose dimension is

$$\Delta_{r,s} = \frac{(r + m(r-s))^2 - 1}{2m(m+1)}\,. \tag{B.2}$$

The $O_{1,1}$ is the unit operator ($\Delta_{1,1} = 0$), while the field $\phi \equiv O_{2,2}$ has the smallest dimension among all nontrivial operators:

$$d_\phi = \Delta_{2,2} = \frac{3}{2m(m+1)}. \tag{B.3}$$

This field is identified with the Ginzburg-Landau field in (B.1). For $m = 3$ we have $\Delta_{2,2} = 1/8$, which is the spin field dimension in the Ising model.

It is convenient to extend the Virasoro primary fields to a larger range $1 \leq r \leq m-1$, $1 \leq s \leq m$, subject to the identification

$$(r, s) \leftrightarrow (m - r, m + 1 - s)\,. \tag{B.4}$$

The *fusion rules*, which say which operators appear in the OPE $O_{r_1 s_1} \times O_{r_2 s_2}$ (but do not specify the coefficients) can now be written in a relatively compact form:

$$\begin{aligned} &O_{r_1 s_1} \times O_{r_2 s_2} \sim \sum O_{r,s} \\ &r = |r_1 - r_2| + 1, |r_1 - r_2| + 3, \ldots \min(r_1 + r_2 - 1, 2m - 1 - r_1 - r_2) \\ &s = |s_1 - s_2| + 1, |s_1 - s_2| + 3, \ldots \min(s_1 + s_2 - 1, 2m + 1 - s_1 - s_2) \end{aligned} \tag{B.5}$$

For any m, the fusion rules respect a discrete Z_2 symmetry

$$O_{r,s} \to \pm O_{r,s}, \tag{B.6}$$

where $\pm = (-1)^{s-1}$ for m odd, $(-1)^{r-1}$ for m even (this choice is dictated by consistency with (B.4)). This symmetry corresponds to the $\phi \to -\phi$ symmetry of the Ginzburg-Landau model; in particular $\phi = O_{2,2}$ is odd under (B.6).

We are interested in OPEs of the form $O \times O \sim 1 + \tilde{O} + \ldots$ where both O and $\tilde{O}$ have small dimensions. Two such interesting OPEs are

$$\phi \times \phi \sim 1 + \phi^2 + \ldots \tag{B.7}$$

$$\psi \times \psi \sim 1 + \psi^2 + \ldots, \quad \psi \equiv O_{1,2}, \quad d_\psi = \frac{1}{2} - \frac{3}{2(m+1)}. \tag{B.8}$$

Here ϕ^2 and ψ^2 are just notation for the lowest dimension operators appearing in the RHS. Note that for $m = 3$ we have $\psi \equiv \phi$ via (B.4). Using the fusion rule (B.5) and the operator dimensions (B.2) it is not difficult to make identification:

$$m = 3: \quad \phi^2 \equiv O_{1,3}, \ \Delta_{\phi^2} = 1, \quad \text{(Ising)} \tag{B.9}$$

$$m > 3: \quad \phi^2 \equiv O_{3,3}, \ \Delta_{\phi^2} = \frac{4}{m(m+1)},$$

$$\psi^2 \equiv O_{1,3}, \ \Delta_{\psi^2} = 2 - \frac{4}{m+1}. \tag{B.10}$$

In particular, we see that the $\psi \times \psi$ OPE does not contain ϕ^2, which is precisely the reason why we are considering it[2].

[2]In general, ϕ^2 does not appear in the OPE $O_{r,s} \times O_{r,s}$ for $r = 1$ or $s = 1$. The operators ψ has the lowest dimension among all these fields.

Appendix C

Asymptotic behavior

In this Appendix we find large l and Δ asymptotics of derivatives of $F_{d,\Delta,l}$ at $a = b = 0$. It is useful to rewrite the definition of $F_{d,\Delta,l}$ as follows:

$$F_{d,\Delta,l}(a,b) = h_d(a,b)\frac{\tilde{g}_{d,\Delta,l}(a,b) - \tilde{g}_{d,\Delta,l}(-a,b)}{a}\,, \tag{C.1}$$

where we introduced the functions

$$\begin{aligned} \tilde{g}_{d,\Delta,l} &\equiv [(1-z)(1-\bar{z})]^d g_{\Delta,l}\,, \\ h_d(a,b) &\equiv \frac{a}{(z\bar{z})^d - [(1-z)(1-\bar{z})]^d}\,. \end{aligned}$$

These functions are smooth in the spacelike diamond. Moreover, it is not difficult to see that

$$\tilde{g}(a,-b) = \tilde{g}(a,b), \quad h_d(\pm a, \pm b) = h_d(a,b)\,.$$

In particular, from (C.1) we see the property (4.51).
Let us introduce the parameter

$$\delta \equiv \Delta - l - 2.$$

As we will see below, there are three relevant asymptotic limits to consider:

- l large, $\delta = O(1)$;
- l large, δ large, $\delta \ll l^2$;
- δ large, $\delta \gg l^2$.

In all these cases the large asymptotic behavior of derivatives will come from differentiating $g_{\Delta,l}$, which we write in the form

$$g_{\Delta,l} = const(-)^l \frac{z\bar{z}}{b} \left[k_{2l+\delta+2}(z) k_\delta(\bar{z}) - (b \to -b) \right] . \tag{C.2}$$

In this Appendix by *const* we denote various *positive* constants which may depend on d, δ or l but are independent of the derivative order $\partial_a^{2m}\partial_b^{2n}$. These constant factors are irrelevant for controlling the positivity of the linear functionals defined on the cones.
Starting from the following integral representation for the hypergeometric function (see [65])

$${}_2F_1\,(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty e^{-bt}(1-e^{-t})^{c-b-1}(1-x\,e^{-t})^{-a}\,dt \quad (\mathrm{Re}\ c > \mathrm{Re}\ b > 0)$$

and using the steepest descent method, we derive the large β asymptotics:

$$\begin{aligned} k_\beta(x) &= e^{\,(\beta/2)h(x)}\,[q(x) + O(1/\beta)]\,, \\ h(x) &= \ln\left(\frac{4(1-\sqrt{1-x}\,)^2}{x}\right), \quad q(x) = \frac{x}{2\left(1-\sqrt{1-x}\right)\sqrt[4]{1-x}}\,. \end{aligned}$$

The leading asymptotic behavior appears when all the derivatives fall on the exponential factors in f_β containing large exponents. Various prefactors appearing in (C.1) and (C.2) are not differentiated in the leading asymptotics. However, the a^{-1} and b^{-1} factors are responsible for changing the order of the needed derivative, as follows:

$$\begin{aligned} F_{d,\Delta,l}^{(2m,2n)} &\sim \frac{const}{2m+1}(g_{\Delta,l})^{(2m+1,2n)} \\ &\sim \frac{const(-)^l}{(2m+1)(2n+1)}(\exp A)^{(2m+1,2n+1)}\,, \\ A &= (l+\delta/2)h(1/2+a+b) + (\delta/2)h(1/2+a-b)\,. \end{aligned}$$

To find the leading asymptotics, we expand A near $a = b = 0$:

$$\begin{aligned} A &= (l+\delta)[h(1/2) + a\,h'(1/2)] + lb\,h'(1/2) + (\delta/2)b^2 h''(1/2) + \ldots, \\ h'(1/2) &= 2\sqrt{2}, \quad h''(1/2) = -2\sqrt{2}. \end{aligned} \tag{C.3}$$

In the case $\delta \ll l^2$ the last term in (C.3) plays no role, and we get:

$$F_{d,\Delta,l}^{(2m,2n)} \sim \frac{const(-)^l}{(2m+1)(2n+1)}[h'(1/2)(l+\delta)]^{2m+1}[h'(1/2)l]^{2n+1}, \quad \delta \ll l^2\,. \tag{C.4}$$

This asymptotic is applicable for l large, while δ can be small or large, as long as the condition $\delta \ll l^2$ is satisfied; i.e. it covers the first two cases mentioned above. If on the other hand $\delta \gg l^2$, it is the last term in (C.3) which determined the asymptotics of b-derivatives, and we get

$$F_{d,\Delta,l}^{(2m,2n)} \sim const(-)^l \frac{(2n-1)!!}{2m+1}[h'(1/2)\delta]^{2m+1}[h''(1/2)\delta]^n, \quad \delta \gg l^2 \,.$$

Because $h''(1/2) < 0$, the last asymptotics changes sign depending on the parity of n.

Appendix D

Details about numerical algorithms

We now discuss more in detail the issues introduced in Section 6.1.2, namely how we can find in practice a linear functional $\Lambda[F]$ of the form (6.12) satisfying the positivity property (6.13). We will first describe the general procedure and how it can implemented in a computer code, and then mention possible algorithmic improvements and shortcuts that we found useful in our analysis.

Given the complexity of the functions $F_{d,\Delta,l}$, the search for a positive functional is too hard a task to be attacked analytically. As already mentioned, we reduce the complexity of the problem by looking for a functional which is a linear combination of derivatives up to a given order N. The derivative are taken w.r.t. the selfdual point $X = Y = 0$, since the sum rule is expected to converge fastest around this point and, in addition, the functions $F_{d,\Delta,l}(X, Y)$ are even in both arguments. The choice of the functional (6.12) simplifyes our task enormously since we can now work in a finite dimensional space, and the only information concerning $F_{d,\Delta,l}$ that we need are their derivatives up to a certain order. Put another way, the F-functions are now considered as elements not of a function space but of a finite-dimensional vector space $\mathbb{R}^s$, $s = N(N + 6)/8$.

In this appendix we will restrict to the case of single sum rule, however the discussion can be straightforwardly generalized to more complex cases. This interpretation is discussed in Section 6.1.1.

The sum rule (4.50) in this picture represents a constraints on these vectors that, in any CFT, must sum to zero. Here we adopt an equivalent point of view in terms of the dual space of linear functionals defined on $\mathbb{R}^s$ since we find this prospective closer to the method used to obtain numerically $\Lambda[F]$.

Let us fix the notation. We define the s-dimensional vector of Taylor coefficients:

$$\mathcal{F}_0[d,\delta,l] \equiv \left\{ \frac{1}{m!n!} F^{(m,n)}_{d,\Delta,l} \mid m,n \text{ even}, \, 2 \leq m+n \leq N \right\} \ , \tag{D.1}$$
$$F^{(m,n)}_{d,\Delta,l} \equiv \partial^m_X \partial^n_Y F_{d,\Delta,l}|_{X=Y=0}, \qquad \delta \equiv \Delta - l - 2,$$

and the same vector normalized to the unit length:

$$\mathcal{F}[d,\delta,l] \equiv \frac{\mathcal{F}_0}{\|\mathcal{F}_0\|} \ , \tag{D.2}$$

where the norm $\|\mathcal{F}_0\|$ is the usual Euclidean length of the vector $\mathcal{F}_0$.

We form the vector $\mathcal{F}_0$ out the Taylor coefficients of the function $F_{d,\Delta,,l}$ rather then of its derivatives, because this way all elements turn out to have approximately the same order of magnitude, which is preferable in the subsequent numerical computation. Definition of the normalized vector $\mathcal{F}$ serves the same purpose. Indeed, as explained in the following, our numerical analysis consist in finding a solution of a system of linear inequalities where the coefficient are given by the elements of $\mathcal{F}[d,\delta,l]$. The solution is more accurate and easier to extract if all the coefficient are of the same order of magnitude. Since the existence of the functional Λ is not affected by these rescalings, we opted for the definition (D.1), (D.2).

According to the positivity property we look for a functional which is strictly positive on all but finitely many vectors $\mathcal{F}[d,\delta,l]$. Let us fix the dimension d of the scalar ϕ_d. Then each pair $\Delta\,,l$ identifies the semi-space of $(\mathbb{R}^s)^*$ of the functionals positive-definite on the vectors $\mathcal{F}[d,\delta,l]$; let us call this open sets $U_{d,\Delta,l}$. With this notation the positivity property (6.13) can be restated in the following way: *If for fixed d and $\Delta_{\min}$*

$$\bigcap_{\substack{\Delta \geq \Delta_{\min},\, l=0 \\ \Delta \geq l+2,\, l=2,4,\ldots}} U_{d,\Delta,l} \neq \emptyset \ , \tag{D.3}$$

then the sum rule cannot be satisfied. The issue is thus to be able to check whether the intersection (D.3) is non-empty, and to compute the smallest $\Delta_{\min}$ for which this is the case.

Clearly it is not possible nor needed to check all the values of Δ as required by the condition (D.3). We can indeed consider only a finite number of them and check if they admit the existence of a functional or not. This can be achieved with a double simplification. First, we consider values of Δ and l only up to a given maximum value ("truncation"), and secondly, we discretize the kept range of Δ ("discretization"). The truncation does not produce a loss of information since we take into account of the large Δ and l contribution using the asymptotic expressions computed in Appendix C. The discretization step requires special care, see below.

We used MATHEMATICA 7 to perform the computations. The algorithm to extract the smallest value of $\Delta_{\min}$ proceeds in several steps:

1. Setting up an efficient procedure to compute vectors $\mathcal{F}[d, \delta, l]$.

2. Selection of the l's and δ's to be used in checking the positivity property (D.3). For each l the range of δ was discretized, and a discrete set of points was chosen, called Γ_l below. The derivatives of the F-functions approach zero as $\delta \to \infty$ and reach the asymptotic behavior for sufficiently large values. We take a finer discretization where the function are significantly varying while we can allow to increase the step in the asymptotic region. More details are given below.

3. Reduction to a Linear Programming problem. With only a finite number of equations to check, the determination of the intersection of the $U_{d,\Delta,l}$ becomes a standard problem of *Linear Programming* which can be solved in finite amount of time. Hence we look for a solution of the linear system of inequalities

$$\Lambda[\mathcal{F}[d, \delta, l]] \equiv \sum \tilde{\lambda}_{m,n} \mathcal{F}_{m,n} \geq 0 \ , \qquad \text{(D.4)}$$
$$\delta \in \Gamma_l, \quad l = 0 \ldots l_{\max} \ .$$

   Clearly, the coefficients $\tilde{\lambda}_{m,n}$ are related to those appearing in (6.12) by a trivial rescaling depending on m, n:

$$\tilde{\lambda}_{m,n} = m! n! \lambda_{m,n}$$

   Further, the asymptotic behavior of the F-functions (see below) tells us that for large δ the inequality is dominated by the $(N, 0)$ derivative

$$\Lambda[F_{d,\Delta,l}] \longrightarrow \tilde{\lambda}_{N,0} F^{(N,0)}_{d,\Delta,l} \qquad (\delta \gg l^2), \qquad \text{(D.5)}$$

   hence $\tilde{\lambda}_{N,0}$ needs to be positive. By an overall rescaling of Λ we can always achieve

$$\tilde{\lambda}_{N,0} = 1 \, ,$$

   which we choose as a normalization condition. When dealing with multiple sum rules we must pay attention to this condition.

4. Extraction of the smallest $\Delta_{\min}$ for which a positive functional exists. We begin by selecting two points $\delta_{\min} = \delta_1$ and $\delta_{\min} = \delta_2 > \delta_1$ such that we know a priori that in the first case

a positive functional does not exists, while in the second case it does[1]. Starting from these values we apply the bisection method to determine the critical $\delta_{\min}$ up to the desired precision: we test if a functional exists for $\delta_{\min} = (\delta_2 + \delta_1)/2$ and we increase or decrease the extremes of the interval $[\delta_1, \delta_2]$ depending on the outcome. The procedure we follow is such that in the end the critical $\delta_{\min}$ is contained in an interval of relative width 10^{-3}, i.e. we terminate if $\delta_2 - \delta_1 \leq 10^{-3}\delta_1$. The plots presented in this work correspond to the upper end δ_2 of the final interval, for which we have found a functional, while we know for sure that the positivity property is not satisfied for the lower end δ_1.

Let us now come back to the point 1. Although computation of the derivatives can be carried on by brute force Taylor-expanding the F-functions, we can save time decomposing the computation in various blocks. From thre expression of the F functions (the same is true for H-functions) we see the rather simple dependence on the parameter d, which translates in a polynomial dependence once the function is Taylor-expanded in X and Y. We therefore separately computed the dependence on d once and for all as a matrix $M(d)_{mn|ij}$. To compute Taylor coefficients of F-functions, this matrix is contracted with two vectors containing one-dimensional Taylor coefficients of the function $k_\beta(x)$, see (4.23). The latter derivatives are pre-computed for several values of β with a fine step and stored. For definiteness we report the interval we used:

$$0 \leq \beta \leq 10^2 \qquad \text{step: } 10^{-3}\,. \tag{D.6}$$

For larger β we made use of the analytic expression of the asymptotics instead of computing the derivatives numerically (see below).

Finally let us discuss the choice of the discretization and the truncation in Δ and l. This step is of fundamental importance in order to reduce the time needed to perform the computation. In Appendix C it is shown that for large values of δ and l the functions $F_{d,\Delta,l}$ approach an asymptotic behavior. With respect to [28] we changed approach. Instead of deciding a priori which was the maximal value of δ, l to include, we choose to include values of δ and l below which the difference between a vector and its asymptotic expression was larger than a given value[2]. A measure of this is given by the norm of the difference between a vector and its asymptotic expression. This new method turned out very useful to extract the bounds in the case of triple

[1] We can choose these points blindly as $\delta_1 = 0$, $\delta_2 \gg 1$, however prior experience can suggest a choice closer to the final $\delta_{\min}$

[2] We estimated that 0.1 is a safe value. For values larger than 0.2 we truncate in δ and l too early, while values smaller than 0.05 make the Linear programming too complicated.

sum rules.

When a vector enters in the asymptotic regime we can safely use the approximate expression

$$F^{(m,n)}_{d,\Delta,l} \sim const.(2\sqrt{2})^{m+n+2}\frac{(l+\delta)^{m+1}l^{n+1}}{(m+1)(n+1)}\,,$$

For large l,δ the vector $\mathcal{F}_{d,\Delta,l}$ is dominated by the components where $m+n$ assumes the highest allowed value N. Hence we can take into account this large l,δ behavior imposing additional constraints:

$$\Lambda[\mathcal{F}_\theta] \geq 0\ , \qquad \mathcal{F}_\theta = \begin{cases} \frac{(\cos\theta+\sin\theta)^{m+1}\cos\theta^{n+1}}{(m+1)(n+1)} & \text{if} \quad m+n=N \\ 0 & \text{otherwise} \end{cases}\ , \qquad \tan\theta \equiv \frac{\delta}{l}\,,$$

where we have dropped irrelevant positive constants not depending on $m\,,n$.

Now comes the discretization: in the range of values of Δ, as well as in the interval $\theta \in [0,\pi/2]$, we can allow to take only a discrete, finite number of points. For θ we take a fixed small step. However, for δ we try to concentrate the points in the region where the unit vector $\mathcal{F}_{d,\Delta,l}$ is significantly varying. A measure of this is given by the norm of its derivative w.r.t. δ:[3]

$$\mathcal{N} = \left\| \frac{\partial}{\partial\delta}\mathcal{F}[d,\delta,l] \right\|\,.$$

We discretize by taking the spacing between two consecutive values of δ equal $c/\mathcal{N}$, where c is a small fixed number ($c = 0.02 \div 0.05$ was typically taken in our work). Clearly when the unit vector is slowly varying the discretization step is large, while it is refined where they are significantly changing, and where presumably more information is encoded. Typically we get about a hundred δ values for each l, but only a few dozen of those above $\delta > 50$.Moreover as l increases the asymptotic regimes is reached earlier in δ. Eventually only $\delta = 0$ is included for large spins.

The sets Γ_l, one for each l, of values of δ obtained in this way are the ones referred to at point 2 above. In constructing the linear system that we use at point 3 we consider additional intermediate points between two subsequent δ's. In order to understand why we do this, let us assume that we have found a functional Λ which is positive for all the values of δ contained in Γ_l. Since we considered a discrete set of values, it can and actually does happen that for intermediate values of δ (which were not included in Γ_l) the functional becomes slightly negative. In [26] this issue was solved looking for solution of the form $\Lambda[F_{d,\Delta,l}] > \varepsilon$, so that for intermediate values this condition could be violated but the positivity was safe. In the current work we found it more convenient to build the linear system in the following way:

[3]In practice the derivative $\partial/\partial\delta$ is evaluated by using the finite-difference approximation.

- for each $\delta \in \Gamma_l = \{\delta_1, ..., \delta_i, \delta_{i+1}, ...\}$ we evaluate the vector $\mathcal{F}[d, \delta, l]$.
- for any two consecutive points δ_i, δ_{i+1}, we consider the first-order Taylor expansion of the vector $\mathcal{F}[d, \delta, l]$ around $\delta = \delta_i$ and evaluate it at half-spacing between δ_i and δ_{i+1}:[3]

$$\mathcal{F}_{1/2}[d, \delta_i, l] \equiv \mathcal{F}[d, \delta_i, l] + \left(\frac{\delta_{i+1} - \delta_i}{2}\right) \frac{\partial}{\partial \delta} \mathcal{F}[d, \delta_i, l] \tag{D.7}$$

  and we add the constraints $\Lambda[\mathcal{F}_{1/2}] \geq 0$ to the linear system (D.4).

These additional constraints become important to keep the functional positive near the δ's for which the inequalities $\Lambda[\mathcal{F}] \geq 0$ are close to saturation, while they are redundant away from those points. Indeed, assume that for some δ_i and δ_{i+1} the functional is exactly vanishing. Then at the intermediate point the functional would be strictly negative, which is not allowed. However, in the presence of the additional constraint $\Lambda[\mathcal{F}_{1/2}] \geq 0$ this cannot happen, since $\Lambda[\mathcal{F}]$ is generically a convex function of δ near the minimum. See Figure D.1 for an illustration. Thus we can be certain that the found functional will be positive also for those δ which were not included into Γ_l. This certainty has a price. Namely, the opposite side of the coin is that the

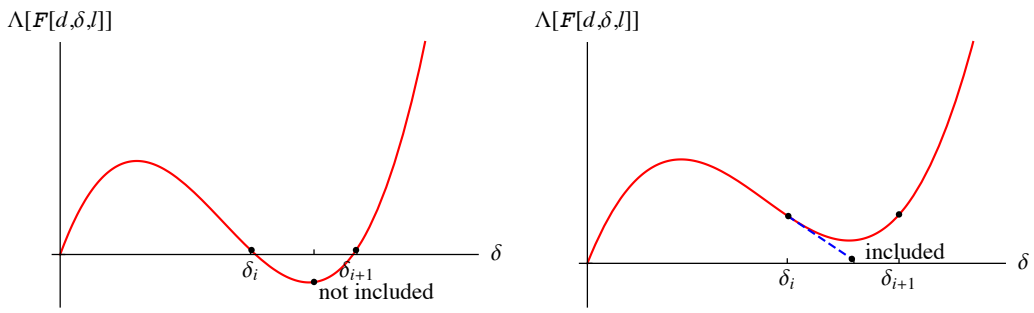


Figure D.1: *Imposing the positivity of the functional on a discrete set of points, it could happens that the intermediate points don't satisfy* $\Lambda[\mathcal{F}[d, \delta, l]] \geq 0$ *(on the left). However adding the constraint* $\Lambda[\mathcal{F}_{1/2}[d, \delta_i, l]] \geq 0$*, see (D.7), we can be sure that the functional is positive on all the neglected points (on the right).*

added $\mathcal{F}_{1/2}$ constraints are somewhat *stronger* than needed, and the bigger the discretization parameter c, the bigger the difference. As a result, the found critical value of $\Delta_{\min}$ will be somewhat *above* the optimal critical value, corresponding to $c \to 0$. This observation explains why the curves in Figs. 6.9,6.10 have small irregularities in the slope. These irregularities could

be decreased by decreasing the value of c.

Several comments concerning the numerical accuracy are in order. The components of the vector $\mathcal{F}[d, \delta, l]$ have been computed using standard double-precision arithmetic (16 digits). As a consequence all the numerical results must be rounded to this precision. In particular, quantities smaller that 10^{-16} are considered zero.

In addition, the built-in MATHEMATICA 7 function `LinearProgramming`, which we used, has an undocumented `Tolerance` parameter. Most of the computations were done with `Tolerance` equal 10^{-6} (default value). However for $N = 16$ and $N = 18$, and for $d < 1.1$, we found that `LinearProgramming` terminates prematurely, concluding that no positive linear functional exists, even for some values of $\delta_{\min}$ for which a positive functional for smaller N was in fact found. The problem disappeared once we set `Tolerance` to a lower value (10^{-12}). In our opinion, `Tolerance` is probably the so-called *pivot tolerance*, the minimal absolute value of a number in the pivot column of the Simplex Method to be considered nonzero. Recall that a nonzero (actually negative) pivot element is necessary in each step of the Simplex Method [59]. This interpretation explains why the above problem could occur, and why it could be overcome by lowering `Tolerance`. Notice that when increasing the number of sum rules, and therefore the dimensionality of the vectors, this issues manifests for smaller values of N.

As described above, our numerical procedure has been designed to be robust with respect to the effects of truncation and discretization. In addition, for each d, we have tested the last found functional (i.e. for $\delta_{\min}$ at the upper end δ_2 of the final interval $[\delta_1, \delta_2]$) on the much bigger set of δ, l:

$$\begin{aligned} &2 \leq l \leq 500\ , \quad 0 \leq \delta \leq 500\ , \text{step} = 0.1\ , \\ &l = 0\ , \quad \delta_{\min} \leq \delta \leq 500\ , \text{step} = 0.1\ , \end{aligned} \tag{D.8}$$

and found that indeed $\Lambda[\mathcal{F}] \geq 0$, within the declared 10^{-16} accuracy.

Finally, we have checked that in all cases the found functionals Λ are such that the inequality $\Lambda[\mathcal{F}] \geq 0$ is in fact strict: $\Lambda[\mathcal{F}] > 0$, for all but finitely many values of δ and l. Thus they satisfy the requirements stated in Section 6.1.

List of Figures

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	T-parity: its problems and their solution
June 7 2010	PLANCK 2010, Geneve, Switzerland.
June 17 2010	CERN, Geneve, Switzerland.
	Theoretical constraints on the Higgs effective couplings
August 7 2009	CERN, Geneve, Switzerland.
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Publications

Bounds in 4D conformal field theories with global symmetry, R. Rattazzi, S. Rychkov, A. Vichi, Journal of Physics A44 (2011) 035402

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