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Electromagnetic fields and beam coupling impedances in a multilayer flat chamber

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Summary

This paper aims at giving a derivation of the six electromagnetic field components created by an offset point charge travelling at any speed in a flat chamber, i.e. a two dimensional structure consisting of two infinite parallel plates surrounding a vacuum region. The plates are infinite both horizontally and along the direction of the point charge movement. We study the general case where the two plates are made of several layers of linear materials up to infinity in the vertical direction, with no a priori top-bottom symmetry. We find new results for the electromagnetic fields and impedances, generalizing in particular the so-called "Yokoya factors" that are valid under certain conditions only.

Contents

1 Introduction

Beam coupling impedances, and more generally the electromagnetic fields created by a travelling beam (also called self-generated fields), are a subject of interest for synchrotrons since many years. Ideally the best approach to compute the impedance in a given synchrotron would be to solve the tridimensional electromagnetic problem for all the elements around the beam, i.e. the beam pipe, collimators, pumping ports, beam position monitors, magnets, and so on. This task requires the extensive and time-consuming use of electromagnetic codes; on the other hand, a first step to evaluate the impedance of a machine that proved to be quite efficient, is to assume first two-dimensional geometries for the main impedance contributors (basically, those contributors that are the nearest to the beam and/or have the highest resistivities), i.e. to assume that they have an infinite length and to compute the fields created by a beam near their center. This way one can actually analytically compute beam-coupling impedances for simple geometries, which has the asset to be much quicker than the use of a tridimensional code, and can also have less limitations, for instance with respect to the frequency range, the beam velocity or the material properties.

Such two-dimensional analytical computations of the beam-coupling impedances have been developped for more than forty years [1]. In particular the axisymmetric case is now fully solved in frequency domain, for a beam pipe made of any multilayer structure with any linear materials, for any frequency and any beam velocity, thanks to e.g. Zotter's formalism [2–6]. For other simple two-dimensional geometries, the usual approach is to deduce the impedance from the axisymmetric case multiplied by some constant form factors [7] depending on the geometry, often called Yokoya [8] or Laslett [9] factors.

However, recently it has been shown that this approach to compute the beam coupling impedances of a flat chamber fails in the case of non metallic materials such as ferrite [10]. Indeed, the hypotheses on which the form factors theories rely break down for general non conductive materials and/or over certain frequency range: in Ref. [9] one is concerned only about perfectly conductive materials in the static case (i.e. at zero frequency), whereas in Refs. [7, 8] one assumes that the beam is ultrarelativistic and that the chamber material is conductive with a skin depth [11, p. 220] much smaller than both the chamber thickness and its half gap. Since the skin depth is a monotically decreasing function of frequency, the latter assumption implies a lower bound in frequency. This approximation is known to break down in the case of many high impedance contributors of the LHC, namely the collimators made of poorly conductive graphite [12], in particular around 8 kHz which is the frequency of the first unstable betatron line in the LHC [5, 13].

There also exists a double-layer formula for the impedances (and wakes fields) of a flat chamber, for a longitudinally gaussian beam, in the specific case of a thin dielectric layer coating (typically an oxide) on metallic plates [14] under again some assumptions on the skin depth in the metallic layer. The general multilayer case with any linear materials has been studied analytically in Ref. [15] with certain approximations (in particular the high frequencies are not considered) and restricted to transverse impedances. In Ref. [16] one can find a detailed analysis for a longitudinally gaussian beam in a chamber made of one or two single-layer conductive plates, giving nonlinearities of both the time domain wake fields and the frequency domain impedances, which were rewritten and used in Refs. [17, 18]. In time domain, some analytic formulae for the wake fields are also given in Ref. [19], restricted to single-layer conductive flat chambers and to the ultrarelativistic case.

Given the limitations of the current analytic formulae and the fact that the multilayer case is certainly of significance when for instance a metallic coating is present on poorly conductive collimator jaws of finite thickness, in particular if the skin depth exceeds the coating thickness, it is highly desirable to provide a general theory on the impedance of a multilayer flat chamber, valid for any linear materials, any beam velocity and any frequency, and giving all nonlinear terms. We will try here to present a general formalism with similar ideas as the original Zotter's formalism for an axisymmetric structure [4, 6], which will enable us to compute analytically the electromagnetic fields and impedance created by a beam in a flat chamber, infinitely long along the beam's direction of movement, infinitely large horizontally, and with an infinite thickness vertically. Both longitudinal and transverse impedances are studied, for a chamber made of any resistive, dielectric or magnetic materials, assuming only their linearity, isotropy, homogeneity (in each layer) and the validity of Ohm's law. We derive here the most general and exact (within the assumptions made, see next section) formulae in frequency domain, in the multilayer case and with all the possible nonlinearities of the electromagnetic fields.

The paper is structured as follows. We start by giving the source creating the electromagnetic fields in the flat chamber in Section 2. Then we introduce Maxwell equations and the various material constants used in Section 3. After that we derive the general expressions for the longitudinal components of the fields for a given horizontal wave number in Section 4, from which the transverse components are then computed in Section 5. The various constants are computed in Section 6, where closed-form formulae are found in the multilayer case. Expressions for the electromagnetic forces on a test particle are then obtained as a function of the longitudinal component of the electric field in Section 7, which is followed by the derivation of the total result for this component (upon integration over the horizontal wave number, i.e. giving the total response from the point-like source) in Section 8. The impedance calculation appears after that in Section 9. Some specific configurations are studied in Section 10. Our concluding remarks follow finally in Section 11.

Note that the whole paper is expressed in SI (or MKSA) units.

2 Source charges and currents

We consider as the source of the electromagnetic fields a point-like macroparticle of charge Q travelling at a speed v along the synchrotron beam line (Os), using the cartesian coordinates (O, x, y, s). Note that s is also assumed to be the azimuthal coordinate along the beam reference orbit in the accelerator – we therefore neglect all curvature effects which is a good approximation for accelerators of long radius of curvature like the LHC; we refer the reader to Refs. [20–25] for details about such effects.

The beam line along the s coordinate is parallel to a flat chamber, i.e. a structure made of two infinitely thick (in the y direction) and large (in the x direction) plates with vertical separation of $2b$ between them, located at $y = \pm b$, where b is called the half-gap. The flat chamber is supposed to be infinitely long along the ring beam line, so that the structure considered is in a sense two-dimensional (no side effects). In time domain, the source macroparticle charge is supposed to be slightly offset from the plane (Oxs) equidistant to each plate, by an amount y_1 along the vertical direction, so that its coordinates are $x = 0$ (we take the origin since there is a invariance of the problem upon any translation along the horizontal direction), $y = y_1$ and $s = vt$ [26, p. 5]. In time domain, the corresponding charge density is then[1]

$$\rho(x, y, s; t) = Q\delta(x)\delta(y - y_1)\delta(s - vt), \tag{2.1}$$

which has the same transverse part as the source charge density of Ref. [14]. Here δ is the Dirac distribution, i.e. such that for any function f, $\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0)$. As expected we get $\iiint_\Omega \rho(x, y, s; t)\mathrm{d}x\mathrm{d}y\mathrm{d}s = Q$ for any volume Ω around $x = 0$, $y = y_1$ and $s = vt$.

It is convenient to solve Maxwell equations in frequency domain (neglecting then any transient effects) and this will enable us to get the beam coupling impedances which are the quantities used in many synchrotron instability theories [27, 28]. To do so we write the factor $\delta(s - vt)$ in terms of its Fourier

[1] This charge density (and the corresponding current density) is valid for a single passage through the flat chamber. In circular rings we should in principle take into account multiturn effects, in that case a bunch is passing several times at a position s, all separated by an integer number of T_{rev}, the revolution period along the orbit. This means that we have to replace $\delta(s - vt)$ by $\sum_{l=-\infty}^{\infty} \delta(s - v(t - lT_{rev}))$ if at $t = 0$ the bunch is at $s = 0$. For this study only single-turn effects are considered. Maxwell equations being linear we could anyway perform such a multiturn sum on the resulting fields we compute in this paper.

spectrum [29, 30]

$$\begin{aligned}\delta(s-\upsilon t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jk(s-\upsilon t)}\mathrm{d}k \\ &= \frac{1}{2\pi\upsilon}\int_{-\infty}^{\infty} e^{j\omega\left(t-\frac{s}{\upsilon}\right)}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega e^{j\omega t}\frac{e^{-jks}}{\upsilon},\end{aligned} \tag{2.2}$$

where j is the imaginary constant, and

$$k \equiv \frac{\omega}{\upsilon}, \tag{2.3}$$

is the wave number. We drop the factor $\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega e^{j\omega t}$ to proceed to the frequency domain, to get

$$\rho(x,y,s;\omega) = \frac{Q}{\upsilon}\delta(x)\delta(y-y_1)e^{-jks}. \tag{2.4}$$

Since the macroparticle is supposed to travel at the velocity υ along the s axis, the current density is obtained in general by [31]

$$\vec{J} = \rho\upsilon\vec{e_s},$$

$\vec{e_s}$ being the unit vector along the s axis. Therefore we get for our source, in frequency domain

$$\vec{J}(x,y,s;\omega) = Q\delta(x)\delta(y-y_1)e^{-jks}\vec{e_s}. \tag{2.5}$$

We can rewrite Eqs. (2.4) and (2.5) thanks to the horizontal cosine Fourier transform of the $\delta(x)$ factor [14]: similarly as above we have

$$\begin{aligned}\delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jk_x x}\mathrm{d}k_x \\ &= \frac{1}{2\pi}\int_{0}^{\infty}\left(e^{-jk_x x}+e^{jk_x x}\right)\mathrm{d}k_x \\ &= \frac{1}{\pi}\int_{0}^{\infty}\cos(k_x x)\mathrm{d}k_x\end{aligned} \tag{2.6}$$

We then obtain for the charge density:

$$\rho(x,y,s;\omega) = \frac{Q}{\pi\upsilon}\int_{0}^{+\infty}\mathrm{d}k_x\cos(k_x x)\delta(y-y_1)e^{-jks}, \tag{2.7}$$

We drop now the $\int_0^{+\infty}\mathrm{d}k_x$ factor and study first the electromagnetic response to the following source:

$$\tilde{\rho}(k_x,y,s;\omega) = \frac{Q}{\pi\upsilon}\cos(k_x x)\delta(y-y_1)e^{-jks}, \tag{2.8}$$

which corresponds to a surface charge density on the plane $y=y_1$. The associated current density along the s axis is

$$\tilde{J}(k_x,y,s;\omega) = \frac{Q}{\pi}\cos(k_x x)\delta(y-y_1)e^{-jks}. \tag{2.9}$$

For a given horizontal wave number k_x, Eqs. (2.8) and (2.9) give the charge and current densities in frequency domain that we will use to solve the electromagnetic fields. They are those of a wave of frequency ω propagating along the s axis with a longitudinal wave number k. Since Maxwell equations are linear in ρ and $\vec{J}$ (see next section), to obtain the fields for the initial sources in $\delta(x)$ from Eqs. (2.4) and (2.5) we will need to put back the $\int_0^{\infty}\mathrm{d}k_x$ factor and integrate the responses from all the horizontal wave numbers k_x, as in Eq. (2.7). Then to get back to time domain we should put back the $\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega e^{j\omega t}$ factor and integrate our frequency domain solutions.

3 Maxwell equations

The macroscopic Maxwell equations in frequency domain for the electric and magnetic fields $\vec{E}$ and $\vec{H}$ in a general linear, homogeneous and isotropic medium are [4]

$$\begin{aligned}
\mathrm{div}\vec{D} &= \tilde{\rho}, && (3.1)\\
\vec{\mathrm{curl}}\vec{H} - j\omega\vec{D} &= \tilde{\vec{J}}, && (3.2)\\
\vec{\mathrm{curl}}\vec{E} + j\omega\vec{B} &= 0, && (3.3)\\
\mathrm{div}\vec{B} &= 0, && (3.4)
\end{aligned}$$

where $\tilde{\rho}$ and $\tilde{\vec{J}} = \tilde{J}\vec{e_s}$ are given in the whole space by Eqs. (2.8) and (2.9). The electric displacement $\vec{D}$ and the magnetic induction $\vec{B}$ are defined using complex permittivities and permeabilities ε_c and μ

$$\begin{aligned}
\vec{D} &= \varepsilon_c(\omega)\vec{E} = \varepsilon_0\varepsilon_1(\omega)\vec{E}, && (3.5)\\
\vec{B} &= \mu(\omega)\vec{H} = \mu_0\mu_1(\omega)\vec{H}, && (3.6)
\end{aligned}$$

where ε_c and μ are general frequency dependent complex permittivity and permeability. We will also often use the quantities ε_1 and μ_1 which are the relative complex permittivity and permeability of the medium. ε_0 and μ_0 are the permittivity and permeability of vacuum. As in Ref. [6] (although it is not stated clearly there) we don't have to assume any particular frequency dependence of these properties, but the following expressions [12] can be considered as a relevant example, since they have a fairly general range of validity:

$$\begin{aligned}
\varepsilon_c(\omega) &= \varepsilon_0\varepsilon_1(\omega) = \varepsilon_0\varepsilon_b\left[1 - j\ \mathrm{sign}(\omega)\tan\vartheta_E\right] + \frac{\sigma_{DC}}{j\omega\left(1 + j\omega\tau\right)}, && (3.7)\\
\mu(\omega) &= \mu_0\mu_1(\omega) = \mu_0\mu_r\left[1 - j\ \mathrm{sign}(\omega)\tan\vartheta_M\right]. && (3.8)
\end{aligned}$$

In these expressions, μ_r is the real part of the relative complex permeability, $\tan\vartheta_M$ is the magnetic loss tangent, ε_b is the dielectric constant and $\tan\vartheta_E$ is the dielectric loss tangent. We also consider in this model a simple AC conductivity following the Drude model (see Refs. [11, p. 312] and [32, p. 16], with an opposite sign convention for ω in both references) where σ_{DC} is the DC conductivity of the material and τ its relaxation time. It is here important to note that we assume that Ohm's law (in its local sense, i.e. the proportionality between the induced conductive current density and the electric field, at any point) holds for the media involved. Doing so we neglect magnetoresistance effects (see Refs. [32, pp. 11-15 and 234-239] and [33]) and the so-called "anomalous skin effect" [33–40]. Both might appear at low temperature, and very high magnetic fields for the former (several Teslas), or very high frequencies for the latter (see Ref. [41] for some examples of relevant limits).

Note that Eqs. (3.5) and (3.6) can be derived from the general microsopic Maxwell equations, as shown in Refs. [11, p. 248] and [6, App. A]. Sometimes in the literature the conductivity part of the complex permittivity is not included into the expression of $\vec{D}$, which was found to lead to an inconsistency when writing the boundary conditions for the electric displacement component perpendicular to a surface between different media, unless some surface charges or currents are taken into account. Therefore here we prefer to include the conductivity part in the complex permittivity to avoid this problem. We refer the reader to Ref. [6] for more details on this subject.

We have written above Maxwell equations for one single homogeneous material; now to describe the whole electromagnetic problem we introduce the superscript (p) to all the quantities related to the properties of the medium when considering a region of space made of one homogeneous material, i.e. with uniform values of ε_c and μ. The space is thus divided into $N + M$ flat parallel layers, the outer boundary of each of them being located at $y = b^{(p)}$, as shown in Fig. 1. N layers are located in the upper part of the chamber (including the vacuum layer between $y = y_1$ and $y = b = b^{(1)}$), for which the superscript of the quantities ε_c, μ and b is with a plus sign, and M layers are located in the lower part of the chamber (including the vacuum layer between $y = -b = b^{(-1)}$ and $y = y_1$), for which the superscript of the quantities ε_c, μ and b is with a minus sign. In these p is between $-M$ and N and is different from zero,

and either M or N can be equal to one (case of a structure with no top or bottom plate).
Finally, when needed we will assume a positive angular frequency ω. The fields in frequency domain for $\omega < 0$ can be obtained by noticing that all the time domain field components should be real, which means that for any field component φ the quantity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{j\omega t}\varphi(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{d}\omega \left[e^{j\omega t}\varphi(\omega) + e^{-j\omega t}\varphi(-\omega)\right] = \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{d}\omega \left[e^{j\omega t}\varphi(\omega) + (e^{j\omega t}\varphi(-\omega)^*)^*\right], \tag{3.9}$$

is real ($*$ denotes the complex conjugate). This is true if

$$\varphi(-\omega) = \varphi(\omega)^*. \tag{3.10}$$

Therefore we can use the above equation to compute the field components for negative frequencies.

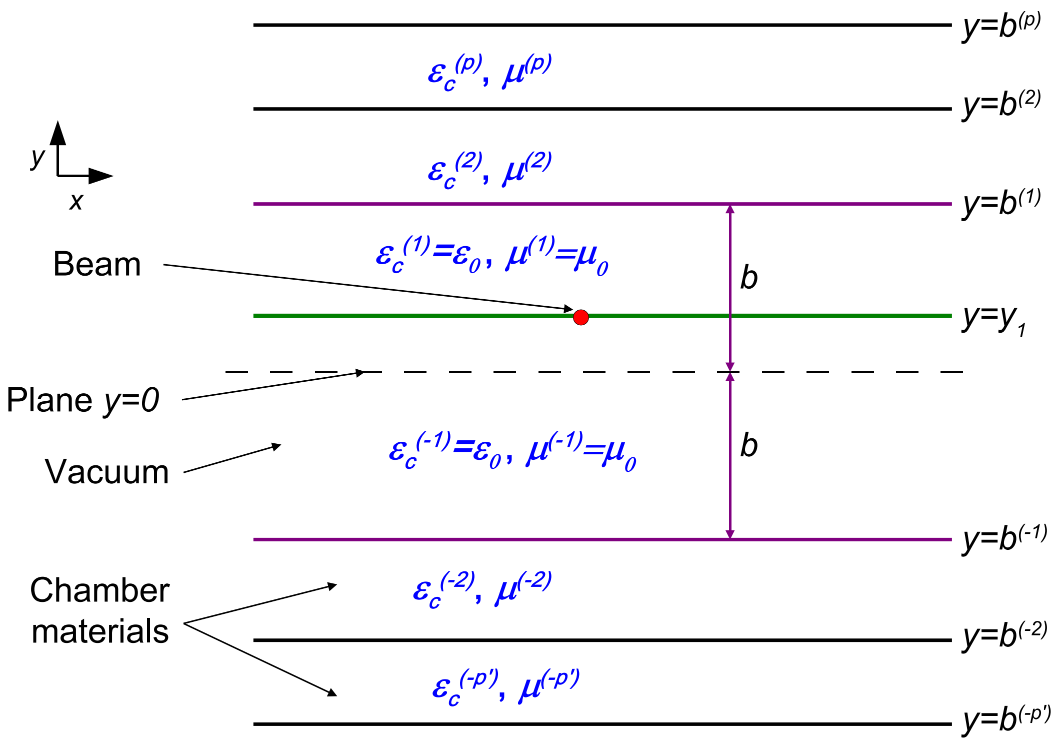

Figure 1: Cross section of the flat chamber. The regions denoted by the superscript (± 1) are the vacuum regions inside the structure, with the surface density on the plane $y = y_1$ as a separation between them. Subsequent layers can be made of any medium. The last layers, denoted at the top by the superscript (N) and at the bottom by the superscript $(-M)$, go to infinity ($b^{(N)} = -b^{(-M)} = \infty$). We have also sketched in red the initial beam source, before its horizontal Fourier decomposition using Eq. (2.6), which is a point-like charge at $x = 0$ and $y = y_1$.

4 Longitudinal components of the electromagnetic fields

We apply Maxwell equations in a region where ε_c and μ are constant (boundary conditions will be considered in Section 6), so that we will omit the superscript $(\pm p)$.
Applying the $\vec{\text{curl}}$ operator to Maxwell equation (3.3), we obtain

$$\vec{\text{curl}}\left(\vec{\text{curl}}\vec{E}\right) + j\omega\mu\vec{\text{curl}}\vec{H} = 0.$$

Using the "$\vec{\text{curl}}\,\vec{\text{curl}}$" relation (Eq. (B.1) of Appendix B), injecting Maxwell equations (3.1) and (3.2), and knowing that $\vec{\tilde{J}} = \tilde{\rho} v \vec{e_s}$, we then get

$$\nabla^2 \vec{E} + \omega^2 \varepsilon_c \mu \vec{E} = \frac{1}{\varepsilon_c} \vec{\text{grad}} \tilde{\rho} + j\omega\mu\tilde{\rho} v \vec{e_s}. \tag{4.1}$$

Similarly, we can apply the $\vec{\text{curl}}$ operator to Maxwell equation (3.2) to obtain

$$\vec{\text{curl}} \left(\vec{\text{curl}} \vec{H} \right) - j\omega\varepsilon_c \vec{\text{curl}} \vec{E} = \vec{\text{curl}} \left(\tilde{\rho} v \vec{e_s} \right),$$

which gives, with Eqs. (3.3) and (3.4), using also the expression of the $\vec{\text{curl}}$ operator in cartesian coordinates from Eq. (A.3) of Appendix A for the right-hand side

$$\nabla^2 \vec{H} + \omega^2 \varepsilon_c \mu \vec{H} = v \frac{\partial \tilde{\rho}}{\partial x} \vec{e_y} - v \frac{\partial \tilde{\rho}}{\partial y} \vec{e_x}. \tag{4.2}$$

Using the expressions of the gradient and the laplacian in cartesian coordinates (see Eqs. (A.1), (A.4) and (A.5) of Appendix A), for the longitudinal field components we can turn the wave equations (4.1) and (4.2) into the following scalar Helmholtz equations

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2} + \omega^2 \varepsilon_c \mu \right] E_s = \frac{1}{\varepsilon_c} \frac{\partial \tilde{\rho}}{\partial s} + j\omega\mu\tilde{\rho} v, \tag{4.3}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2} + \omega^2 \varepsilon_c \mu \right] H_s = 0. \tag{4.4}$$

In any region where $y \neq y_1$, those equations are homogeneous and we can seek solutions by separation of variables, in the general form $X(x)Y(y)S(s)$. For both E_s and H_s we have

$$Y(y)S(s)X''(x) + X(x)S(s)Y''(y) + X(x)Y(y)S''(s) + \omega^2 \varepsilon_c \mu X(x)Y(y)S(s) = 0, \tag{4.5}$$

where the prime $'$ denotes the derivative with respect to the argument of the function (e.g. $X'(x) = \frac{dX}{dx}$). This gives, when dividing by $X(x)Y(y)S(s)$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{S''(s)}{S(s)} + \omega^2 \varepsilon_c \mu = 0,$$

or equivalently

$$\begin{cases} \frac{X''(x)}{X(x)} = -\omega^2 \varepsilon_c \mu - \frac{Y''(y)}{Y(y)} - \frac{S''(s)}{S(s)} = \text{constant} = A, \\ \frac{S''(s)}{S(s)} = -\omega^2 \varepsilon_c \mu - \frac{X''(x)}{X(x)} - \frac{Y''(u)}{Y(y)} = \text{constant} = B. \\ \frac{Y''(y)}{Y(y)} = -\omega^2 \varepsilon_c \mu - \frac{X''(x)}{X(x)} - \frac{S''(s)}{S(s)} = \text{constant} = -A - B - \omega^2 \varepsilon_c \mu, \end{cases} \tag{4.6}$$

Therefore X, Y and S are solutions of the harmonic differential equation. The constants A and B above can be a priori complex. Defining the square root of a complex number by

$$\sqrt{\alpha e^{j\varphi}} = \sqrt{\alpha} e^{j\frac{\varphi}{2}} \text{ with } -\pi < \varphi \leq \pi, \tag{4.7}$$

we can then write for X

$$X(x) = \kappa_1 e^{\sqrt{A}x} + \kappa_2 e^{-\sqrt{A}x}.$$

Now we know that $X(\pm\infty)$ should stay finite so that the fields stay finite. If $\Re\left(\sqrt{A}\right) \neq 0$, this can be the case only for the trivial solution $\kappa_1 = \kappa_2 = 0$, therefore we can state that $\Re\left(\sqrt{A}\right) = 0$. We will write respectively for the electric field longitudinal component E_s and the magnetic field one H_s:

$$\begin{aligned} \sqrt{A_{E_s}} &= jk_{x_e} \Rightarrow A_{E_s} = -\left(k_{x_e}\right)^2, \\ \sqrt{A_{H_s}} &= jk_{x_h} \Rightarrow A_{H_s} = -\left(k_{x_h}\right)^2, \end{aligned} \tag{4.8}$$

where k_{x_e} and k_{x_h} are real numbers. $X(x)$ is therefore a combination of cosine and sine functions with real arguments. Now we notice that the $x = 0$ plane is a symmetry plane of the whole problem – including boundary conditions and source charges and currents from Eqs. (2.8) and (2.9), corresponding to its invariance with the sign of x. Therefore the electric field should also be symmetric with respect to that plane, and in particular the longitudinal component (which is parallel to the symmetry plane). So $E_s(-x) = E_s(x)$ and there is no sine term in X_{E_s}. This is the opposite for the longitudinal component of H_s: we should imagine current loops creating the magnetic field, perpendicular to it; for the longitudinal part of the field, such loops are perpendicular to the $x = 0$ symmetry plane, and their reflected images are current loops where the current flows in the opposite direction, in such a way that the magnetic field created by them is opposite. Therefore $H_s(-x) = -H_s(x)$, which means that there is no cosine term in X_{H_s}.

Therefore we finally get[2]

$$\begin{aligned} X_{E_s}(x) &\propto \cos(k_{x_e} x), \\ X_{H_s}(x) &\propto \sin(k_{x_h} x). \end{aligned}$$

In addition, the whole problem exhibits a translational invariance along the s axis, the translation vector being $\frac{2\pi}{k}\vec{e_s}$. Therefore the resulting fields should exhibit the same invariance, which means that like A, B must be real and negative, to prevent any exponentially growing or decaying term in the longitudinal dependence $S(s)$. $S(s)$ can be then taken of the form:

$$S(s) = \kappa_3 e^{-j\sqrt{-B}s} + \kappa_4 e^{j\sqrt{-B}s}.$$

From the translational invariance, $\sqrt{-B}$ should then be an integer multiple of k, that we will write lk. If we (temporary) get back to the time domain, applying the inverse Fourier transform $\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{j\omega t}$ we see that we get two terms, one whose integrand is proportional to

$$\kappa_3 e^{j\omega\left(t-\frac{ls}{v}\right)},$$

and the other whose integrand is proportional to

$$\kappa_4 e^{j\omega\left(t+\frac{ls}{v}\right)}.$$

Since this must be invariant with respect to the transformation $(t_0, s_0) \to (t_0 + t, s_0 + vt)$ for any (t_0, s_0, t) (which is an invariance property of the time domain initial problem), and recalling that $k \equiv \frac{\omega}{v}$, this clearly means that the second term is zero ($\kappa_4 = 0$) and that $l = 1$. We finally get[3]

$$\begin{aligned} B &= -k^2, \\ S_{E_s}(s) &\propto e^{-jks}, \\ S_{H_s}(s) &\propto e^{-jks}. \end{aligned} \tag{4.9}$$

For the vertical dependence of the electric field longitudinal component $Y_{E_s}(y)$, we get using Eqs. (4.6), (4.8) and (4.9)

$$Y''_{E_s}(y) - \left(k_{x_e}^2 + k^2 - \omega^2\varepsilon_c\mu\right) Y_{E_s}(y) = 0,$$

and a very similar equation for Y_{H_s}. Now we define the propagation constant as in the cylindrical case [5] (using the definitions of Eqs. (3.5) and (3.6), and the identity $\varepsilon_0\mu_0 c^2 = 1$ where c is the speed of light in vacuum):

$$\nu^2 = k^2 - \omega^2\varepsilon_c\mu = k^2\left(1 - \beta^2\varepsilon_1\mu_1\right), \tag{4.10}$$

[2]Note that A can be zero, in which case the solution of the harmonic differential equation for $X(x)$ is in the form $X(x) = \kappa_1 x + \kappa_2$, but κ_1 has to be zero otherwise $X(\pm\infty)$ goes to infinity, and for X_{H_s} κ_2 has also to be zero since $H_s(-x) = -H_s(x)$. Therefore this kind of solution (constant for X_{E_s}, zero for X_{H_s}) is recovered in the general form $X_{E_s} \propto \cos(k_{x_e}x)$ and $X_{H_s} \propto \sin(k_{x_h}x)$ when $k_{x_e} = k_{x_h} = 0$.

[3]The same as for the horizontal dependence $X(x)$ applies in the case when $B = k = 0$ (see footnote 2), except that in this case both S_{E_s} and S_{H_s} are non zero constants.

so that

$$\nu = |k|\sqrt{1-\beta^2\varepsilon_1\mu_1}, \tag{4.11}$$

where $\beta \equiv \frac{v}{c}$ is the relativistic velocity factor. Defining then for E_s and H_s the following vertical complex wave numbers

$$\begin{aligned} k_{y_e} &= \sqrt{k_{x_e}^2+\nu^2}, \\ k_{y_h} &= \sqrt{k_{x_h}^2+\nu^2}, \end{aligned} \tag{4.12}$$

we get for the vertical dependence of e.g. E_s (the same is applicable to H_s):

$$Y_{E_s}(y) = \kappa_5 e^{k_{y_e} y} + \kappa_6 e^{-k_{y_e} y}.$$

Putting all the integration constants together into $Y(y)$ and reinserting the superscript (p) (p positive or negative) for each layer (note that at that point, $k_{x_e}^{(p)}$ and $k_{x_h}^{(p)}$ could be in principle dependent on the layer), the longitudinal components of the electromagnetic fields in the layer (p) can be finally written

$$E_s^{(p)} = \cos\left(k_{x_e}^{(p)} x\right) e^{-jks}\left[C_{e+}^{(p)} e^{k_{ye}^{(p)} y} + C_{e-}^{(p)} e^{-k_{ye}^{(p)} y}\right], \tag{4.13}$$

$$H_s^{(p)} = \sin\left(k_{x_h}^{(p)} x\right) e^{-jks}\left[C_{h+}^{(p)} e^{k_{y_h}^{(p)} y} + C_{h-}^{(p)} e^{-k_{y_h}^{(p)} y}\right], \tag{4.14}$$

where the subscripts in the integration constants $C_{e+}^{(p)}$, $C_{e-}^{(p)}$, $C_{h+}^{(p)}$ and $C_{h-}^{(p)}$ have the following meaning: the letter (e or h) stands respectively for the electric or magnetic field, and the + or − sign stands for the sign in front of $k_y^{(p)} y$ in the exponential corresponding to the constant.[4]

5 Transverse components of the electromagnetic fields

Again, we apply Maxwell equations in a region where ε_c and μ are constant and we omit the superscript (p). Writing the transverse components of Eqs. (3.2) and (3.3) in cartesian coordinates (see Appendix A) and assuming $y \neq y_1$, we have the relations

$$\begin{aligned} \frac{\partial H_s}{\partial y} - \frac{\partial H_y}{\partial s} &= j\omega\varepsilon_c E_x, && (5.1) \\ \frac{\partial H_x}{\partial s} - \frac{\partial H_s}{\partial x} &= j\omega\varepsilon_c E_y, && (5.2) \\ \frac{\partial E_s}{\partial y} - \frac{\partial E_y}{\partial s} &= -j\omega\mu H_x, && (5.3) \\ \frac{\partial E_x}{\partial s} - \frac{\partial E_s}{\partial x} &= -j\omega\mu H_y. && (5.4) \end{aligned}$$

Differentiating with respect to s Eq. (5.4) and combining it to Eq. (5.1), we get, knowing the longitudinal dependence of E_s

$$\frac{\partial^2 E_x}{\partial s^2} + \omega^2\varepsilon_c\mu E_x = -jk\frac{\partial E_s}{\partial x} - j\omega\mu\frac{\partial H_s}{\partial y}. \tag{5.5}$$

[4] Note that with our definition of ν, we have $\nu(-\omega) = \nu(\omega)^*$. This comes from the fact that $\nu^2(-\omega) = \left(\nu(\omega)^2\right)^*$ since $\varepsilon_1(-\omega) = \varepsilon_1(\omega)^*$ and a similar relation for μ_1, which can be seen in Eqs. (3.7) and (3.8) or more generally in Ref. [11, p. 332]. Then if $\nu^2 = k^2\alpha e^{j\varphi}$, we have $\alpha(-\omega) = \alpha(\omega)$ and $\varphi(-\omega) = -\varphi(\omega)$ which gives $\nu(-\omega) = \nu(\omega)^*$ from $\nu = |k|\sqrt{\alpha}e^{j\frac{\varphi}{2}}$ (note that to insure this is still true in the particular case when $\varphi(\omega) = \pi$ we need to change the square root definition for negative frequencies, taking $-\pi \leq \varphi(-\omega) < \pi$ instead of $-\pi < \varphi(-\omega) \leq \pi$ in Eq. (4.7)). Since k_{x_e} and k_{x_h} are real, with a similar argument we have $k_{y_e}(-\omega) = k_{y_e}(\omega)^*$ and $k_{y_h}(-\omega) = k_{y_h}(\omega)^*$ from Eqs. (4.12). The same kind of relations apply to the integration constants as they are obtained from linear equations whose coefficients depend on $\varepsilon_1^{(p)}$, $\mu_1^{(p)}$, $\nu^{(p)}$ and exponential functions whose arguments exhibit the same property as above (see Section 6). Therefore from Eqs. (4.13) and (4.14) we have $E_s^{(p)}(-\omega) = E_s^{(p)}(\omega)^*$ and $H_s^{(p)}(-\omega) = H_s^{(p)}(\omega)^*$, and the same relation for the transverse components (from the equations of the next section), which means that Eq. (3.10) is true for the field components without having to apply it "by hand".

In the same way, we can differentiate with respect to s Eqs. (5.3), (5.2) and (5.1), then combine them respectively to Eqs. (5.2), (5.3) and (5.4), to get

$$\frac{\partial^2 E_y}{\partial s^2} + \omega^2 \varepsilon_c \mu E_y = -jk\frac{\partial E_s}{\partial y} + j\omega\mu\frac{\partial H_s}{\partial x}, \tag{5.6}$$

$$\frac{\partial^2 H_x}{\partial s^2} + \omega^2 \varepsilon_c \mu H_x = j\omega\varepsilon_c\frac{\partial E_s}{\partial y} - jk\frac{\partial H_s}{\partial x}, \tag{5.7}$$

$$\frac{\partial^2 H_y}{\partial s^2} + \omega^2 \varepsilon_c \mu H_y = -j\omega\varepsilon_c\frac{\partial E_s}{\partial x} - jk\frac{\partial H_s}{\partial y}. \tag{5.8}$$

At given x and y, these four equations can all be written in the form

$$\frac{d^2\psi}{ds^2} + \omega^2\varepsilon_c\mu\psi = C_\psi e^{-jks},$$

ψ being the field component considered, and C_ψ a constant with respect to s. The general solution of this equation is

$$\psi(s) = \eta_1 e^{-j\omega\sqrt{\varepsilon_c\mu}s} + \eta_2 e^{j\omega\sqrt{\varepsilon_c\mu}s} + \eta_3 e^{-jks},$$

where η_3 is related to C_ψ. The first two terms are obviously not invariant with respect to the translation of vector $\frac{2\pi}{k}\vec{e_s}$ (see Section 4) which means that we have to drop them: $\eta_1 = \eta_2 = 0$. Therefore ψ is proportional to e^{-jks}, the proportionallity constant depending only on x and y. So the transverse components have the same longitudinal dependence as the longitudinal components (i.e. in e^{-jks}), and we can rewrite Eqs. (5.5) to (5.8) in the form (reinserting the superscript (p) to avoid any confusion)

$$E_x^{(p)} = \frac{jk}{{\nu^{(p)}}^2}\left(\frac{\partial E_s^{(p)}}{\partial x} + \upsilon\mu^{(p)}\frac{\partial H_s^{(p)}}{\partial y}\right), \tag{5.9}$$

$$E_y^{(p)} = \frac{jk}{{\nu^{(p)}}^2}\left(\frac{\partial E_s^{(p)}}{\partial y} - \upsilon\mu^{(p)}\frac{\partial H_s^{(p)}}{\partial x}\right), \tag{5.10}$$

$$H_x^{(p)} = \frac{jk}{{\nu^{(p)}}^2}\left(-\upsilon\varepsilon_c^{(p)}\frac{\partial E_s^{(p)}}{\partial y} + \frac{\partial H_s^{(p)}}{\partial x}\right), \tag{5.11}$$

$$H_y^{(p)} = \frac{jk}{{\nu^{(p)}}^2}\left(\upsilon\varepsilon_c^{(p)}\frac{\partial E_s^{(p)}}{\partial x} + \frac{\partial H_s^{(p)}}{\partial y}\right). \tag{5.12}$$

From their linearity, the above relations are also true for the transverse components of the total fields, i.e. those resulting from the initial point-like source in Eqs. (2.4) and (2.5), after integration over k_x. Note that we have implicitely assumed that $\nu^{(p)} \neq 0$ and will continue to make this assumption in all the following sections. Actually, $\nu^{(p)} = 0$ is possible at the onset of Cherenkov radiation in the layer (p) (see e.g. Refs. [11, p. 637] or [42, p. 406]) but it requires very particular conditions: from the definition of the propagation constant in Eq. (4.11) together with Eqs. (3.7) and (3.8) this would require no losses in the layer, zero conductivity and $\beta = \frac{1}{\sqrt{\varepsilon_b\mu_r}}$.

6 Field matching

To specify the field components we need to express the boundary conditions between all the layers. For conciseness of the notations, we will assume from this section onward that the angular frequency ω is positive[5].

[5]To recover the results at any frequency we would simply need to replace $\frac{k}{\gamma}$ by $\frac{|k|}{\gamma}$ in the expression of the propagation constant of vacuum. See also the end of Section 3 and footnote 4.

6.1 Boundary conditions at $y = y_1$

We know (from e.g. Ref. [11, p. 18]) that the electric field components tangential to a boundary between media is always continuous, giving in particular at $y = y_1$, from Eq. (4.13)

$$\cos(k_{x_e}^{(-1)}x)\left[C_{e+}^{(-1)}e^{k_{ye}^{(-1)}y_1}+C_{e-}^{(-1)}e^{-k_{ye}^{(-1)}y_1}\right]=\cos(k_{x_e}^{(1)}x)\left[C_{e+}^{(1)}e^{k_{ye}^{(1)}y_1}+C_{e-}^{(1)}e^{-k_{ye}^{(1)}y_1}\right]. \tag{6.1}$$

Since this is valid for any x, $k_{x_e}^{(-1)}$ and $k_{x_e}^{(1)}$ are necessarily equal[6].
Equation (4.3) is valid across $y = y_1$, and following what is done in Ref. [30], we can integrate it over y between $y_1 - \delta y_1$ and $y_1 + \delta y_1$. Using also Eqs. (2.8) and (4.13) we obtain

$$\begin{aligned}\left.\frac{\partial E_s}{\partial y}\right|_{y_1+\delta y_1}-\left.\frac{\partial E_s}{\partial y}\right|_{y_1-\delta y_1}+\int_{y_1-\delta y_1}^{y_1+\delta y_1}\mathrm{d}y\left(-k_{x_e}^{(1)2}-k^2+\omega^2\varepsilon_0\mu_0\right)E_s \\ =\frac{jQ}{\pi}\cos(k_x x)e^{-jks}\left(\frac{-k}{\varepsilon_0 v}+\omega\mu_0\right)\int_{y_1-\delta y_1}^{y_1+\delta y_1}\mathrm{d}y\delta(y-y_1) \\ =\frac{-jQ\omega\mu_0}{\pi\beta^2\gamma^2}\cos(k_x x)e^{-jks},\end{aligned}$$

where we have replaced ε_c and μ by their values in vacuum ε_0 and μ_0, and where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the relativistic mass factor of the source macroparticle. When δy_1 goes to zero, the integral term in the left-hand side vanishes since E_s is not infinite at $y = y_1$. Replacing E_s by its expression on each side of the boundary and dropping the e^{-jks} factor, we can rewrite the equation as

$$\begin{aligned}\cos(k_{x_e}^{(1)}x)\left[k_{y_e}^{(1)}C_{e+}^{(1)}e^{k_{ye}^{(1)}y_1}-k_{y_e}^{(1)}C_{e-}^{(1)}e^{-k_{ye}^{(1)}y_1}\right] \\ -\cos(k_{x_e}^{(-1)}x)\left[k_{y_e}^{(-1)}C_{e+}^{(-1)}e^{k_{ye}^{(-1)}y_1}-k_{y_e}^{(-1)}C_{e-}^{(-1)}e^{-k_{ye}^{(-1)}y_1}\right]=\frac{-jQ\omega\mu_0}{\pi\beta^2\gamma^2}\cos(k_x x).\end{aligned} \tag{6.2}$$

This relation, valid at any x, means that we necessarily have

$$k_{x_e}^{(-1)} = k_{x_e}^{(1)} = k_x, \tag{6.3}$$

and consequently, recalling Eq. (4.12) and $\nu^{(-1)} = \nu^{(1)} = \frac{k}{\gamma}$ in vacuum

$$k_{y_e}^{(-1)} = k_{y_e}^{(1)} \equiv k_y^{(1)} = \sqrt{k_x^2 + \frac{k^2}{\gamma^2}}. \tag{6.4}$$

We can now divide by $\cos(k_x x)$ both sides of Eq. (6.2). Injecting then Eq. (6.1) taken in the form

$$\begin{aligned}C_{e+}^{(-1)}e^{k_y^{(1)}y_1}+C_{e-}^{(-1)}e^{-k_y^{(1)}y_1} &= C_{e+}^{(1)}e^{k_y^{(1)}y_1}+C_{e-}^{(1)}e^{-k_y^{(1)}y_1} \\ \Longleftrightarrow \qquad C_{e-}^{(-1)}-C_{e-}^{(1)} &= \left(C_{e+}^{(1)}-C_{e+}^{(-1)}\right)e^{2k_y^{(1)}y_1},\end{aligned}$$

into Eq. (6.2), we get

$$\begin{aligned}C_{e+}^{(1)}-C_{e+}^{(-1)} &= -\mathcal{C}\frac{e^{-k_y^{(1)}y_1}}{k_y^{(1)}}, \\ \text{and} \quad C_{e-}^{(1)}-C_{e-}^{(-1)} &= \mathcal{C}\frac{e^{k_y^{(1)}y_1}}{k_y^{(1)}},\end{aligned} \tag{6.5}$$

[6] If the multiplicating factors in front of both $\cos(k_{x_e}^{(-1)}x)$ and $\cos(k_{x_e}^{(1)}x)$ are zero, we can use the relation (6.2) below (whose demonstration is still valid if $k_{x_e}^{(-1)} \neq k_{x_e}^{(1)}$) to prove that $k_{x_e}^{(-1)} = k_{x_e}^{(1)}$, unless again in the latter relation one of the multiplicating factors in front of either $\cos(k_{x_e}^{(-1)}x)$ or $\cos(k_{x_e}^{(1)}x)$ is zero. But the latter case would mean that for one of the vacuum region (-1) or (1) both E_s and $\frac{\partial E_s}{\partial y}$ are zero at $y = y_1$ which would mean that $E_s = 0$ in the whole region since it is determined by two constants only. Then we could still write $k_{x_e}^{(-1)} = k_{x_e}^{(1)}$ provided the appropriate integration constants are set to zero.

with the definition

$$\mathcal{C} \equiv \frac{jQ\omega\mu_0}{2\pi\beta^2\gamma^2}. \tag{6.6}$$

At $y = y_1$ there is a surface current density flowing along the s axis. From Ref. [11, p. 18] we know that only the tangential component of the magnetic field H_x is discontinuous at that point. So H_s is continuous, which gives a relation analogous to Eq. (6.1). Therefore we have[7]

$$k_{x_h}^{(-1)} = k_{x_h}^{(1)}, \tag{6.7}$$

$$k_{y_h}^{(-1)} = k_{y_h}^{(1)}, \tag{6.8}$$

and

$$C_{h-}^{(-1)} - C_{h-}^{(1)} = \left(C_{h+}^{(1)} - C_{h+}^{(-1)}\right) e^{2k_{y_h}^{(1)} y_1},$$

which we can plug into a similar integration of Eq. (4.4) as what was done above on E_s, but since this time the right-hand side is zero, we get[8]

$$\begin{aligned} C_{h+}^{(1)} &= C_{h+}^{(-1)}, \\ C_{h-}^{(1)} &= C_{h-}^{(-1)}. \end{aligned} \tag{6.9}$$

No further information can be obtained from the boundary condition at $y = y_1$. Indeed, the discontinuity of H_x coming from the surface charge density [11, p. 18] is proportional to the discontinuity of $\frac{\partial E_s}{\partial y}$ as can been seen from Eq. (5.11), since both H_s and its derivative with respect to x are continuous at $y = y_1$. It will then give the same relation as above (Eq. (6.2)).

6.2 Boundary conditions at the flat chamber inner surfaces and between each of its layers

We will now consider the boundary conditions for the subsequent layers, i.e. at each $y = b^{(p)}$ for $1 \leq p \leq N-1$ and $-M+1 \leq p \leq -1$. There are no externally imposed surface charge or currents between each layer and all boundaries are perpendicular to $\vec{e_y}$, so according to Ref. [11, p. 18] and Eqs. (3.5) and (3.6) the following relations hold across those boundaries (for any x, s and ω)

$$E_x^{(p)}\left(x, b^{(p)}, s; \omega\right) = E_x^{(p\pm1)}\left(x, b^{(p)}, s; \omega\right), \tag{6.10}$$

$$\varepsilon_c^{(p)} E_y^{(p)}\left(x, b^{(p)}, s; \omega\right) = \varepsilon_c^{(p\pm1)} E_y^{(p\pm1)}\left(x, b^{(p)}, s; \omega\right), \tag{6.11}$$

$$E_s^{(p)}\left(x, b^{(p)}, s; \omega\right) = E_s^{((p\pm1))}\left(x, b^{(p)}, s; \omega\right), \tag{6.12}$$

$$H_x^{(p)}\left(x, b^{(p)}, s; \omega\right) = H_x^{(p\pm1)}\left(x, b^{(p)}, s; \omega\right), \tag{6.13}$$

$$\mu^{(p)} H_y^{(p)}\left(x, b^{(p)}, s; \omega\right) = \mu^{(p\pm1)} H_y^{(p\pm1)}\left(x, b^{(p)}, s; \omega\right), \tag{6.14}$$

$$H_s^{(p)}\left(x, b^{(p)}, s; \omega\right) = H_s^{(p\pm1)}\left(x, b^{(p)}, s; \omega\right), \tag{6.15}$$

where in the superscripts the plus sign is selected for the layers in the upper part ($y > 0$), and the minus sign for the layers in the lower part ($y < 0$).

We focus first on the upper layers and assume therefore $1 \leq p \leq N-1$. Using Eqs. (4.13) and (4.14), Eqs. (6.12) and (6.15) read respectively

$$\cos\left(k_{x_e}^{(p)} x\right)\left[C_{e+}^{(p)} e^{k_{y_e}^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_{y_e}^{(p)} b^{(p)}}\right] = \cos\left(k_{x_e}^{(p+1)} x\right)\left[C_{e+}^{(p+1)} e^{k_{y_e}^{(p+1)} b^{(p)}} + C_{e-}^{(p+1)} e^{-k_{y_e}^{(p+1)} b^{(p)}}\right], \tag{6.16}$$

$$\sin\left(k_{x_h}^{(p)} x\right)\left[C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} + C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}}\right] = \sin\left(k_{x_h}^{(p+1)} x\right)\left[C_{h+}^{(p+1)} e^{k_{y_h}^{(p+1)} b^{(p)}} + C_{h-}^{(p+1)} e^{-k_{y_h}^{(p+1)} b^{(p)}}\right], \tag{6.17}$$

[7] When the multiplicating factors in front of the sines are zero, the same as for E_s applies (see footnote 6).

[8] This is actually obvious since in Eq. (4.4) there is no source of discontinuity at $y = y_1$, so no reason for H_s to have a different expression from one side to the other of the plane $y = y_1$.

while Eqs. (6.13) and (6.10), using Eqs. (5.11) and (5.9), can be written

$$\frac{1}{\nu^{(p)2}}\left(-\upsilon\varepsilon_c^{(p)}\left.\frac{\partial E_s^{(p)}}{\partial y}\right|_{b^{(p)}}+\frac{\partial H_s^{(p)}}{\partial x}\left(b^{(p)}\right)\right) = \frac{1}{\nu^{(p+1)2}}\left(-\upsilon\varepsilon_c^{(p+1)}\left.\frac{\partial E_s^{(p+1)}}{\partial y}\right|_{b^{(p)}}+\frac{\partial H_s^{(p+1)}}{\partial x}\left(b^{(p)}\right)\right),$$

$$\frac{1}{\nu^{(p)2}}\left(\frac{\partial E_s^{(p)}}{\partial x}\left(b^{(p)}\right)+\upsilon\mu^{(p)}\left.\frac{\partial H_s^{(p)}}{\partial y}\right|_{b^{(p)}}\right) = \frac{1}{\nu^{(p+1)2}}\left(\frac{\partial E_s^{(p+1)}}{\partial x}\left(b^{(p)}\right)+\upsilon\mu^{(p+1)}\left.\frac{\partial H_s^{(p+1)}}{\partial y}\right|_{b^{(p)}}\right),$$

which, using again Eqs. (4.13) and (4.14), become

$$\frac{1}{\nu^{(p)2}}\left[-\upsilon\varepsilon_c^{(p)}\cos\left(k_{x_e}^{(p)}x\right)k_{y_e}^{(p)}\left\{C_{e+}^{(p)}e^{k_{y_e}^{(p)}b^{(p)}}-C_{e-}^{(p)}e^{-k_{y_e}^{(p)}b^{(p)}}\right\}\right.$$
$$\left.+k_{x_h}^{(p)}\cos\left(k_{x_h}^{(p)}x\right)\left\{C_{h+}^{(p)}e^{k_{y_h}^{(p)}b^{(p)}}+C_{h-}^{(p)}e^{-k_{y_h}^{(p)}b^{(p)}}\right\}\right]$$

$$=\frac{1}{\nu^{(p+1)2}}\left[-\upsilon\varepsilon_c^{(p+1)}\cos\left(k_{x_e}^{(p+1)}x\right)k_{y_e}^{(p+1)}\left\{C_{e+}^{(p+1)}e^{k_{y_e}^{(p+1)}b^{(p)}}-C_{e-}^{(p+1)}e^{-k_{y_e}^{(p+1)}b^{(p)}}\right\}\right.$$
$$\left.+k_{x_h}^{(p+1)}\cos\left(k_{x_h}^{(p+1)}x\right)\left\{C_{h+}^{(p+1)}e^{k_{y_h}^{(p+1)}b^{(p)}}+C_{h-}^{(p+1)}e^{-k_{y_h}^{(p+1)}b^{(p)}}\right\}\right], \quad (6.18)$$

and

$$\frac{1}{\nu^{(p)2}}\left[-k_{x_e}^{(p)}\sin\left(k_{x_e}^{(p)}x\right)\left\{C_{e+}^{(p)}e^{k_{y_e}^{(p)}b^{(p)}}+C_{e-}^{(p)}e^{-k_{y_e}^{(p)}b^{(p)}}\right\}\right.$$
$$\left.+\upsilon\mu^{(p)}\sin\left(k_{x_h}^{(p)}x\right)k_{y_h}^{(p)}\left\{C_{h+}^{(p)}e^{k_{y_h}^{(p)}b^{(p)}}-C_{h-}^{(p)}e^{-k_{y_h}^{(p)}b^{(p)}}\right\}\right]$$

$$=\frac{1}{\nu^{(p+1)2}}\left[-k_{x_e}^{(p+1)}\sin\left(k_{x_e}^{(p+1)}x\right)\left\{C_{e+}^{(p+1)}e^{k_{y_e}^{(p+1)}b^{(p)}}+C_{e-}^{(p+1)}e^{-k_{y_e}^{(p+1)}b^{(p)}}\right\}\right.$$
$$\left.+\upsilon\mu^{(p+1)}\sin\left(k_{x_h}^{(p+1)}x\right)k_{y_h}^{(p+1)}\left\{C_{h+}^{(p+1)}e^{k_{y_h}^{(p+1)}b^{(p)}}-C_{h-}^{(p+1)}e^{-k_{y_h}^{(p+1)}b^{(p)}}\right\}\right]. \quad (6.19)$$

Equation (6.16), valid for any x, tells us that $k_{x_e}^{(p)}=k_{x_e}^{(p+1)}$ except in the case when

$$C_{e+}^{(p)}e^{k_{y_e}^{(p)}b^{(p)}}+C_{e-}^{(p)}e^{-k_{y_e}^{(p)}b^{(p)}}=C_{e+}^{(p+1)}e^{k_{y_e}^{(p+1)}b^{(p)}}+C_{e-}^{(p+1)}e^{-k_{y_e}^{(p+1)}b^{(p)}}=0.$$

If this happens, we can give another expression of Eq. (6.18) using the derivative with respect to x of Eq. (6.17) (that is, the continuitity at $y=b^{(p)}$ of $\frac{\partial H_s}{\partial x}$):

$$-\frac{1}{\nu^{(p)2}}\upsilon\varepsilon_c^{(p)}\cos\left(k_{x_e}^{(p)}x\right)k_{y_e}^{(p)}\left[C_{e+}^{(p)}e^{k_{y_e}^{(p)}b^{(p)}}-C_{e-}^{(p)}e^{-k_{y_e}^{(p)}b^{(p)}}\right]+$$
$$\left(\frac{1}{\nu^{(p)2}}-\frac{1}{\nu^{(p+1)2}}\right)k_{x_h}^{(p)}\cos\left(k_{x_h}^{(p)}x\right)\left[C_{h+}^{(p)}e^{k_{y_h}^{(p)}b^{(p)}}+C_{h-}^{(p)}e^{-k_{y_h}^{(p)}b^{(p)}}\right]$$

$$=-\frac{1}{\nu^{(p+1)2}}\upsilon\varepsilon_c^{(p+1)}\cos\left(k_{x_e}^{(p+1)}x\right)k_{y_e}^{(p+1)}\left[C_{e+}^{(p+1)}e^{k_{y_e}^{(p+1)}b^{(p)}}-C_{e-}^{(p+1)}e^{-k_{y_e}^{(p+1)}b^{(p)}}\right]. \quad (6.20)$$

Then the case $k_{x_e}^{(p)}\neq k_{x_e}^{(p+1)}$ is possible only if one of the two terms

$$k_{y_e}^{(p)}\left[C_{e+}^{(p)}e^{k_{y_e}^{(p)}b^{(p)}}-C_{e-}^{(p)}e^{-k_{y_e}^{(p)}b^{(p)}}\right]$$

or

$$k_{y_e}^{(p+1)}\left[C_{e+}^{(p+1)}e^{k_{y_e}^{(p+1)}b^{(p)}}-C_{e-}^{(p+1)}e^{-k_{y_e}^{(p+1)}b^{(p)}}\right]$$

is zero, since $k_{x_h}^{(p)}$ can be equal to only one of $k_{x_e}^{(p)}$ or $k_{x_e}^{(p+1)}$ so that the $\cos\left(k_{x_h}^{(p)}x\right)$ term will combine to at most only one of the two other cosine terms. In such a case in either layer p or layer $p+1$ both

E_s and $\frac{\partial E_s}{\partial y}$ are zero at $y = b^{(p)}$ which means that the longitudinal component of the electric field is zero in the whole layer (it is determined by two constants only). Should this happen we can still impose $k_{x_e}^{(p)} = k_{x_e}^{(p+1)}$ by taking the value of the layer that has a non zero E_s, since in the other layer the constants are zeros so the value of the horizontal wave number does not play any role.
We can repeat this argument at each boundary, and also for $-M+1 \leq p \leq -1$, and therefore drop the superscript for the quantity $k_{x_e}^{(p)}$. Using also the definition of k_{y_e} in Eq. (4.12) together with Eqs. (6.3) and (6.4) we get

$$\text{For any } p \neq 0 \text{ between } -M \text{ and } N, \quad k_{x_e}^{(p)} = k_x \text{ and } k_{y_e}^{(p)} = k_y^{(p)} \equiv \sqrt{k_x^2 + {\nu^{(p)}}^2}. \tag{6.21}$$

Very similar arguments can be applied to H_s and $k_{x_h}^{(p)}$: using first Eq. (6.17), and then, if the vertical part of H_s is zero at $y = b^{(p)}$, Eq. (6.19) together with the continuity of $\frac{\partial E_s}{\partial x}$, we prove that $k_{x_h}^{(p)} = k_{x_h}^{(p+1)}$. Applying this at each layer boundary and using Eq. (6.7) enable us to write

$$\text{For any } p \neq 0 \text{ between } -M \text{ and } N, \quad k_{x_h}^{(p)} = k_{x_h}^{(1)} \text{ and } k_{y_h}^{(p)} = \sqrt{{k_{x_h}^{(1)}}^2 + {\nu^{(p)}}^2}. \tag{6.22}$$

We now want to prove that $k_{x_h}^{(1)} = k_x$. To do so we rewrite Eqs. (6.19) and (6.20) using Eqs. (6.16), (6.21) and (6.22), obtaining

$$\begin{aligned}
&\left(\frac{1}{{\nu^{(p+1)}}^2} - \frac{1}{{\nu^{(p)}}^2}\right) k_x \sin(k_x x) \left[C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}}\right] = \\
&\qquad \upsilon \sin\left(k_{x_h}^{(1)} x\right) \left[\frac{k_{y_h}^{(p+1)} \mu^{(p+1)}}{{\nu^{(p+1)}}^2} \left\{C_{h+}^{(p+1)} e^{k_{y_h}^{(p+1)} b^{(p)}} - C_{h-}^{(p+1)} e^{-k_{y_h}^{(p+1)} b^{(p)}}\right\}\right. \\
&\qquad\qquad \left. - \frac{k_{y_h}^{(p)} \mu^{(p)}}{{\nu^{(p)}}^2} \left\{C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} - C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}}\right\}\right],
\end{aligned} \tag{6.23}$$

and

$$\begin{aligned}
&\left(\frac{1}{{\nu^{(p)}}^2} - \frac{1}{{\nu^{(p+1)}}^2}\right) k_{x_h}^{(1)} \cos\left(k_{x_h}^{(1)} x\right) \left[C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} + C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}}\right] = \\
&\qquad \upsilon \cos(k_x x) \left[\frac{k_y^{(p)} \varepsilon_c^{(p)}}{{\nu^{(p)}}^2} \left\{C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}}\right\}\right. \\
&\qquad\qquad \left. - \frac{k_y^{(p+1)} \varepsilon_c^{(p+1)}}{{\nu^{(p+1)}}^2} \left\{C_{e+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} - C_{e-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}}\right\}\right].
\end{aligned} \tag{6.24}$$

The two above equations, valid for any x, show that $k_x = k_{x_h}^{(1)}$ unless (with $\upsilon \neq 0$)

$$\begin{aligned}
&\left(\frac{1}{{\nu^{(p+1)}}^2} - \frac{1}{{\nu^{(p)}}^2}\right) k_x \left[C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}}\right] \\
&= \frac{k_{y_h}^{(p+1)} \mu^{(p+1)}}{{\nu^{(p+1)}}^2} \left[C_{h+}^{(p+1)} e^{k_{y_h}^{(p+1)} b^{(p)}} - C_{h-}^{(p+1)} e^{-k_{y_h}^{(p+1)} b^{(p)}}\right] - \frac{k_{y_h}^{(p)} \mu^{(p)}}{{\nu^{(p)}}^2} \left[C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} - C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}}\right] \\
&= \frac{k_y^{(p+1)} \varepsilon_c^{(p+1)}}{{\nu^{(p+1)}}^2} \left[C_{e+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} - C_{e-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}}\right] - \frac{k_y^{(p)} \varepsilon_c^{(p)}}{{\nu^{(p)}}^2} \left[C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}}\right] \\
&= \left(\frac{1}{{\nu^{(p+1)}}^2} - \frac{1}{{\nu^{(p)}}^2}\right) k_{x_h}^{(1)} \left[C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} + C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}}\right] \\
&= 0.
\end{aligned} \tag{6.25}$$

Should this happen we could express the same boundary conditions on the subsequent layers, leading to the same conclusion ($k_x = k_{x_h}^{(1)}$). In the end the only case that is problematic is when Eq. (6.25) is true at every boundary, i.e. for $p \neq 0$ between $-M+1$ and $N-1$. We can actually show (see Appendix D) that in this case the layers must have all the same propagation constant $\nu^{(p)}$, that is, that of vacuum, such that $\varepsilon_1^{(p)} \mu_1^{(p)} = 1$ for any p, and that $H_s = 0$ everywhere in space, meaning that we can still write $k_{x_h}^{(1)} = k_x$ provided we set the constants $C_{h+}^{(p)}$ and $C_{h-}^{(p)}$ to zero for all the layers. The case where $\varepsilon_1 \mu_1 = 1$ is in principle possible for a medium different from vacuum, for instance if it's diamagnetic (i.e. $\mu_r < 1$) and slightly dielectric (i.e. $\varepsilon_b > 1$ with $\varepsilon_b - 1$ small compared to 1), with no loss and an infinite resistivity. As a consequence, we can state that

$$k_x = k_{x_h}^{(1)},$$

and consequently

$$\text{For any } p \neq 0 \text{ between } -M \text{ and } N, \quad k_{x_h}^{(p)} = k_x \text{ and } k_{y_h}^{(p)} = k_y^{(p)} \equiv \sqrt{k_x^2 + {\nu^{(p)}}^2}. \tag{6.26}$$

The only unknown coefficients remain the constants in front of the exponential functions in the expression of E_s and H_s of Eqs. (4.13) and (4.14). We have four such constants per layer, so $4(N+M)$ of them in total. To determine them we will use the continuity at the boundaries between the different materials of $E_x^{(p)}$, $E_s^{(p)}$, $H_x^{(p)}$ and $H_s^{(p)}$. Continuity of $\varepsilon_c^{(p)} E_y^{(p)}$ and $\mu^{(p)} H_y^{(p)}$ give redundant equations, which can be readily seen from Eqs. (5.2) and (5.4) (see also the discussion in Ref. [6] on that subject).
To solve for all the constants of the problem we will first introduce as in Ref. [4] the free space impedance Z_0 and the field $\vec{G}$ which has the same dimension as the electric field $\vec{E}$

$$Z_0 = \frac{1}{\varepsilon_0 c} = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \tag{6.27}$$

$$\vec{G} = Z_0 \vec{H}, \tag{6.28}$$

and the corresponding constant coefficients for $\vec{G}$

$$\begin{aligned} C_{g+}^{(p)} &= Z_0 C_{h+}^{(p)}, \\ C_{g-}^{(p)} &= Z_0 C_{h-}^{(p)}. \end{aligned} \tag{6.29}$$

The continuity of E_s and H_s is given by Eqs. (6.16) and (6.17) which we can now simplify by virtue of Eqs. (6.21) and (6.26):

$$C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} = C_{e+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} + C_{e-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}}, \tag{6.30}$$

$$C_{g+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{g-}^{(p)} e^{-k_y^{(p)} b^{(p)}} = C_{g+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} + C_{g-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}}. \tag{6.31}$$

The continuity of E_x and H_x can be written, from Eqs. (6.23) and (6.24) where the cosine and sine factors have been dropped:

$$\begin{aligned} &\left(\frac{1}{{\nu^{(p+1)}}^2} - \frac{1}{{\nu^{(p)}}^2} \right) k_x \left[C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right] = \\ &\beta \left[\frac{k_y^{(p+1)} \mu_1^{(p+1)}}{{\nu^{(p+1)}}^2} \left\{ C_{g+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} - C_{g-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}} \right\} - \frac{k_y^{(p)} \mu_1^{(p)}}{{\nu^{(p)}}^2} \left\{ C_{g+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{g-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right\} \right], \end{aligned} \tag{6.32}$$

and

$$\begin{aligned} &\left(\frac{1}{{\nu^{(p+1)}}^2} - \frac{1}{{\nu^{(p)}}^2} \right) k_x \left[C_{g+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{g-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right] = \\ &\beta \left[\frac{k_y^{(p+1)} \varepsilon_1^{(p+1)}}{{\nu^{(p+1)}}^2} \left\{ C_{e+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} - C_{e-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}} \right\} - \frac{k_y^{(p)} \varepsilon_1^{(p)}}{{\nu^{(p)}}^2} \left\{ C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right\} \right]. \end{aligned} \tag{6.33}$$

We can write Eqs. (6.30) and (6.33) in matrix form:

$$\begin{bmatrix} e^{k_y^{(p+1)} b^{(p)}} & e^{-k_y^{(p+1)} b^{(p)}} \\ \frac{\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{\nu^{(p+1)^2}} e^{k_y^{(p+1)} b^{(p)}} & \frac{-\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{\nu^{(p+1)^2}} e^{-k_y^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \end{bmatrix} =$$

$$\begin{bmatrix} C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \\ \frac{\beta k_y^{(p)} \varepsilon_1^{(p)}}{\nu^{(p)^2}} \left\{ C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right\} + \\ \left(\frac{1}{\nu^{(p+1)^2}} - \frac{1}{\nu^{(p)^2}} \right) k_x \left\{ C_{g+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{g-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right\} \end{bmatrix}.$$

This can be readily solved for $\begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \end{bmatrix}$, knowing that the determinant of the left hand side matrix is equal to $-\frac{2\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{\nu^{(p+1)^2}}$. We get, assuming that[9] $k_y^{(p+1)} \neq 0$ and using the inversion formula of a 2×2 matrix (see Appendix E)

$$\begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \end{bmatrix} = -\frac{\nu^{(p+1)^2}}{2\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}} \begin{bmatrix} \frac{-\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{\nu^{(p+1)^2}} e^{-k_y^{(p+1)} b^{(p)}} & -e^{-k_y^{(p+1)} b^{(p)}} \\ \frac{-\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{\nu^{(p+1)^2}} e^{k_y^{(p+1)} b^{(p)}} & e^{k_y^{(p+1)} b^{(p)}} \end{bmatrix} .$$

$$\left(\begin{bmatrix} e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \\ \frac{\beta k_y^{(p)} \varepsilon_1^{(p)}}{\nu^{(p)^2}} e^{k_y^{(p)} b^{(p)}} & \frac{-\beta k_y^{(p)} \varepsilon_1^{(p)}}{\nu^{(p)^2}} e^{-k_y^{(p)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ C_{e-}^{(p)} \end{bmatrix} + \right.$$

$$\left. \left\{ \frac{1}{\nu^{(p+1)^2}} - \frac{1}{\nu^{(p)^2}} \right\} k_x \begin{bmatrix} 0 & 0 \\ e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{g+}^{(p)} \\ C_{g-}^{(p)} \end{bmatrix} \right). \quad (6.34)$$

[9] $k_y^{(p+1)} = 0$ can happen if $\nu^{(p+1)}$ is purely imaginary (Cherenkov radiation in the layer considered, see e.g. Refs. [11, p. 637], [42, p. 406] or [4, p. 11]) and if $k_x^2 = -\nu^{(p+1)^2}$ from Eq. (6.21). This can be the case for only a finite number of k_x values (one for each layer where $\nu^{(p+1)}$ is purely imaginary), so is not a concern since we will in the end perform a continuous integration over k_x.

Very similarly we can write for $\begin{bmatrix} C_{g+}^{(p+1)} \\ C_{g-}^{(p+1)} \end{bmatrix}$, from Eqs. (6.31) and (6.32)

$$\begin{bmatrix} C_{g+}^{(p+1)} \\ \\ C_{g-}^{(p+1)} \end{bmatrix} = -\frac{{\nu^{(p+1)}}^2}{2\beta k_y^{(p+1)} \mu_1^{(p+1)}} \begin{bmatrix} \frac{-\beta k_y^{(p+1)} \mu_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{-k_y^{(p+1)} b^{(p)}} & -e^{-k_y^{(p+1)} b^{(p)}} \\ \\ \frac{-\beta k_y^{(p+1)} \mu_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{k_y^{(p+1)} b^{(p)}} & e^{k_y^{(p+1)} b^{(p)}} \end{bmatrix} .$$

$$\left(\begin{bmatrix} e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \\ \\ \frac{\beta k_y^{(p)} \mu_1^{(p)}}{{\nu^{(p)}}^2} e^{k_y^{(p)} b^{(p)}} & \frac{-\beta k_y^{(p)} \mu_1^{(p)}}{{\nu^{(p)}}^2} e^{-k_y^{(p)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{g+}^{(p)} \\ \\ C_{g-}^{(p)} \end{bmatrix} + \right.$$

$$\left. \left\{ \frac{1}{{\nu^{(p+1)}}^2} - \frac{1}{{\nu^{(p)}}^2} \right\} k_x \begin{bmatrix} 0 & 0 \\ \\ e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ \\ C_{e-}^{(p)} \end{bmatrix} \right) . \quad (6.35)$$

Let us now define the four following 2×2 matrices, enabling the computation of the values of the constants for the $p+1$ region knowing those of the p region:

$$P^{p+1,p} = \frac{-{\nu^{(p+1)}}^2}{2\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}} \begin{bmatrix} \frac{-\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{-k_y^{(p+1)} b^{(p)}} & -e^{-k_y^{(p+1)} b^{(p)}} \\ \\ \frac{-\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{k_y^{(p+1)} b^{(p)}} & e^{k_y^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \\ \\ \frac{\beta k_y^{(p)} \varepsilon_1^{(p)}}{{\nu^{(p)}}^2} e^{k_y^{(p)} b^{(p)}} & \frac{-\beta k_y^{(p)} \varepsilon_1^{(p)}}{{\nu^{(p)}}^2} e^{-k_y^{(p)} b^{(p)}} \end{bmatrix},$$

$$Q^{p+1,p} = \frac{k_x \left(\frac{{\nu^{(p+1)}}^2}{{\nu^{(p)}}^2} - 1 \right)}{2\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}} \begin{bmatrix} \frac{-\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{-k_y^{(p+1)} b^{(p)}} & -e^{-k_y^{(p+1)} b^{(p)}} \\ \\ \frac{-\beta k_y^{(p+1)} \varepsilon_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{k_y^{(p+1)} b^{(p)}} & e^{k_y^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ \\ e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \end{bmatrix},$$

$$R^{p+1,p} = \frac{-{\nu^{(p+1)}}^2}{2\beta k_y^{(p+1)} \mu_1^{(p+1)}} \begin{bmatrix} \frac{-\beta k_y^{(p+1)} \mu_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{-k_y^{(p+1)} b^{(p)}} & -e^{-k_y^{(p+1)} b^{(p)}} \\ \\ \frac{-\beta k_y^{(p+1)} \mu_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{k_y^{(p+1)} b^{(p)}} & e^{k_y^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \\ \\ \frac{\beta k_y^{(p)} \mu_1^{(p)}}{{\nu^{(p)}}^2} e^{k_y^{(p)} b^{(p)}} & \frac{-\beta k_y^{(p)} \mu_1^{(p)}}{{\nu^{(p)}}^2} e^{-k_y^{(p)} b^{(p)}} \end{bmatrix},$$

$$S^{p+1,p} = \frac{k_x \left(\frac{{\nu^{(p+1)}}^2}{{\nu^{(p)}}^2} - 1 \right)}{2\beta k_y^{(p+1)} \mu_1^{(p+1)}} \begin{bmatrix} \frac{-\beta k_y^{(p+1)} \mu_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{-k_y^{(p+1)} b^{(p)}} & -e^{-k_y^{(p+1)} b^{(p)}} \\ \\ \frac{-\beta k_y^{(p+1)} \mu_1^{(p+1)}}{{\nu^{(p+1)}}^2} e^{k_y^{(p+1)} b^{(p)}} & e^{k_y^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ \\ e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \end{bmatrix},$$

such that Eqs. (6.34) and (6.35) become

$$\begin{bmatrix} C_{e+}^{(p+1)} \\ \\ C_{e-}^{(p+1)} \end{bmatrix} = P^{p+1,p} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ \\ C_{e-}^{(p)} \end{bmatrix} + Q^{p+1,p} \cdot \begin{bmatrix} C_{g+}^{(p)} \\ \\ C_{g-}^{(p)} \end{bmatrix}, \quad (6.36)$$

$$\begin{bmatrix} C_{g+}^{(p+1)} \\ \\ C_{g-}^{(p+1)} \end{bmatrix} = R^{p+1,p} \cdot \begin{bmatrix} C_{g+}^{(p)} \\ \\ C_{g-}^{(p)} \end{bmatrix} + S^{p+1,p} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ \\ C_{e-}^{(p)} \end{bmatrix}. \quad (6.37)$$

We can rewrite these four matrices in the following way

$$P^{p+1,p} = \frac{1}{2}\begin{bmatrix} \left(1+\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p+1)}}\right)e^{\left(k_y^{(p)}-k_y^{(p+1)}\right)b^{(p)}} & \left(1-\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p+1)}}\right)e^{\left(-k_y^{(p)}-k_y^{(p+1)}\right)b^{(p)}} \\ \left(1-\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p+1)}}\right)e^{\left(k_y^{(p)}+k_y^{(p+1)}\right)b^{(p)}} & \left(1+\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p+1)}}\right)e^{\left(k_y^{(p+1)}-k_y^{(p)}\right)b^{(p)}} \end{bmatrix}, \tag{6.38}$$

$$Q^{p+1,p} = \frac{k_x\left(\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}-1\right)}{2\beta k_y^{(p+1)}\varepsilon_1^{(p+1)}}\begin{bmatrix} -e^{\left(k_y^{(p)}-k_y^{(p+1)}\right)b^{(p)}} & -e^{\left(-k_y^{(p)}-k_y^{(p+1)}\right)b^{(p)}} \\ e^{\left(k_y^{(p)}+k_y^{(p+1)}\right)b^{(p)}} & e^{\left(k_y^{(p+1)}-k_y^{(p)}\right)b^{(p)}} \end{bmatrix}, \tag{6.39}$$

$$R^{p+1,p} = \frac{1}{2}\begin{bmatrix} \left(1+\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p+1)}}\right)e^{\left(k_y^{(p)}-k_y^{(p+1)}\right)b^{(p)}} & \left(1-\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p+1)}}\right)e^{\left(-k_y^{(p)}-k_y^{(p+1)}\right)b^{(p)}} \\ \left(1-\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p+1)}}\right)e^{\left(k_y^{(p)}+k_y^{(p+1)}\right)b^{(p)}} & \left(1+\frac{\nu^{(p+1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p+1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p+1)}}\right)e^{\left(k_y^{(p+1)}-k_y^{(p)}\right)b^{(p)}} \end{bmatrix}, \tag{6.40}$$

$$S^{p+1,p} = \frac{\varepsilon_1^{(p+1)}}{\mu_1^{(p+1)}}Q^{p+1,p}. \tag{6.41}$$

Then we define the 4×4 matrix $M^{p+1,p}$ by

$$M^{p+1,p} = \begin{bmatrix} P^{p+1,p} & Q^{p+1,p} \\ S^{p+1,p} & R^{p+1,p} \end{bmatrix}, \tag{6.42}$$

such that

$$\begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \\ C_{g+}^{(p+1)} \\ C_{g-}^{(p+1)} \end{bmatrix} = M^{p+1,p}\cdot\begin{bmatrix} C_{e+}^{(p)} \\ C_{e-}^{(p)} \\ C_{g+}^{(p)} \\ C_{g-}^{(p)} \end{bmatrix}. \tag{6.43}$$

When successively applying this relation for subsequent layers in the upper part, we get

$$\begin{bmatrix} C_{e+}^{(N)} \\ C_{e-}^{(N)} \\ C_{g+}^{(N)} \\ C_{g-}^{(N)} \end{bmatrix} = M^{N,N-1}\cdot M^{N-1,N-2}\cdots M^{2,1}\cdot\begin{bmatrix} C_{e+}^{(1)} \\ C_{e+}^{(1)} \\ C_{g-}^{(1)} \\ C_{g-}^{(1)} \end{bmatrix} = \mathcal{M}\cdot\begin{bmatrix} C_{e+}^{(1)} \\ C_{e-}^{(1)} \\ C_{g+}^{(1)} \\ C_{g-}^{(1)} \end{bmatrix}, \tag{6.44}$$

with the definition

$$\mathcal{M} = M^{N,N-1}\cdot M^{N-1,N-2}\cdots M^{2,1}. \tag{6.45}$$

All the above was performed for the upper part of the chamber, i.e. the layers 1 to N. For the lower part of the chamber we can essentially obtain the same results: for this we simply need to consider p

between $-M$ and -1 and replace the $(p+1)$ superscript by $(p-1)$. This gives the following definitions:

$$P^{p-1,p} = \frac{1}{2}\begin{bmatrix} \left(1+\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p-1)}}\right)e^{\left(k_y^{(p)}-k_y^{(p-1)}\right)b^{(p)}} & \left(1-\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p-1)}}\right)e^{\left(-k_y^{(p)}-k_y^{(p-1)}\right)b^{(p)}} \\ \left(1-\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p-1)}}\right)e^{\left(k_y^{(p)}+k_y^{(p-1)}\right)b^{(p)}} & \left(1+\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\varepsilon_1^{(p)}}{\varepsilon_1^{(p-1)}}\right)e^{\left(k_y^{(p-1)}-k_y^{(p)}\right)b^{(p)}} \end{bmatrix}, \tag{6.46}$$

$$Q^{p-1,p} = \frac{k_x\left(\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}-1\right)}{2\beta k_y^{(p-1)}\varepsilon_1^{(p-1)}}\begin{bmatrix} -e^{\left(k_y^{(p)}-k_y^{(p-1)}\right)b^{(p)}} & -e^{\left(-k_y^{(p)}-k_y^{(p-1)}\right)b^{(p)}} \\ e^{\left(k_y^{(p)}+k_y^{(p-1)}\right)b^{(p)}} & e^{\left(k_y^{(p-1)}-k_y^{(p)}\right)b^{(p)}} \end{bmatrix}, \tag{6.47}$$

$$R^{p-1,p} = \frac{1}{2}\begin{bmatrix} \left(1+\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p-1)}}\right)e^{\left(k_y^{(p)}-k_y^{(p-1)}\right)b^{(p)}} & \left(1-\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p-1)}}\right)e^{\left(-k_y^{(p)}-k_y^{(p-1)}\right)b^{(p)}} \\ \left(1-\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p-1)}}\right)e^{\left(k_y^{(p)}+k_y^{(p-1)}\right)b^{(p)}} & \left(1+\frac{\nu^{(p-1)^2}}{\nu^{(p)^2}}\frac{k_y^{(p)}}{k_y^{(p-1)}}\frac{\mu_1^{(p)}}{\mu_1^{(p-1)}}\right)e^{\left(k_y^{(p-1)}-k_y^{(p)}\right)b^{(p)}} \end{bmatrix}, \tag{6.48}$$

$$S^{p-1,p} = \frac{\varepsilon_1^{(p-1)}}{\mu_1^{(p-1)}}Q^{p-1,p}. \tag{6.49}$$

We can then define the 4×4 matrix $M^{p-1,p}$ by

$$M^{p-1,p} = \begin{bmatrix} P^{p-1,p} & Q^{p-1,p} \\ S^{p-1,p} & R^{p-1,p} \end{bmatrix}, \tag{6.50}$$

and the relations between the integration constants for $-M+1 \leq p \leq -1$ become

$$\begin{bmatrix} C_{e+}^{(p-1)} \\ C_{e-}^{(p-1)} \\ C_{g+}^{(p-1)} \\ C_{g-}^{(p-1)} \end{bmatrix} = M^{p-1,p} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ C_{e-}^{(p)} \\ C_{g+}^{(p)} \\ C_{g-}^{(p)} \end{bmatrix}. \tag{6.51}$$

When successively applying this relation for subsequent layers in the lower part, we get

$$\begin{bmatrix} C_{e+}^{(-M)} \\ C_{e-}^{(-M)} \\ C_{g+}^{(-M)} \\ C_{g-}^{(-M)} \end{bmatrix} = \mathcal{M}' \cdot \begin{bmatrix} C_{e+}^{(-1)} \\ C_{e-}^{(-1)} \\ C_{g+}^{(-1)} \\ C_{g-}^{(-1)} \end{bmatrix}, \tag{6.52}$$

with

$$\mathcal{M}' = M^{-M,-M+1} \cdot M^{-M+1,-M+2} \cdots M^{-2,-1}. \tag{6.53}$$

Finally, for reasons that will appear later in Section 8, it is better to rewrite Eqs. (6.44) and (6.52) in a form involving three constants of layer 1 ($C_{e+}^{(1)}$, $C_{g+}^{(1)}$ and $C_{g-}^{(1)}$) and one of layer -1 ($C_{e-}^{(-1)}$). We can do so using Eqs. (6.5) and (6.9):

$$\begin{bmatrix} C_{e+}^{(N)} \\ C_{e-}^{(N)} \\ C_{g+}^{(N)} \\ C_{g-}^{(N)} \end{bmatrix} = \mathcal{M} \cdot \begin{bmatrix} C_{e+}^{(1)} \\ C_{e-}^{(-1)} + \mathcal{C} \frac{e^{k_y^{(1)} y_1}}{k_y^{(1)}} \\ C_{g+}^{(1)} \\ C_{g-}^{(1)} \end{bmatrix}, \tag{6.54}$$

$$\begin{bmatrix} C_{e+}^{(-M)} \\ C_{e-}^{(-M)} \\ C_{g+}^{(-M)} \\ C_{g-}^{(-M)} \end{bmatrix} = \mathcal{M}' \cdot \begin{bmatrix} C_{e+}^{(1)} + \mathcal{C} \frac{e^{-k_y^{(1)} y_1}}{k_y^{(1)}} \\ C_{e-}^{(-1)} \\ C_{g+}^{(1)} \\ C_{g-}^{(1)} \end{bmatrix}. \tag{6.55}$$

6.3 Field matching for the outer layers and final solution for the integration constants

The outer layers go to infinity in the y direction, which can have two different implications in terms of the electromagnetic fields at infinity. If in an outer layer $k_y^{(p)}$ has a non zero real part, this must be strictly positive according to the definition of $k_y^{(p)}$ in Eq. (6.21) and that of the square root in Eq. (4.7). Then in the layer considered the only possible exponential solution in y is the one with a sign opposite to that of y in the layer, since the other exponential would grow to infinity. In other words for the layer N we necessarily have $C_{e+}^{(N)} = C_{g+}^{(N)} = 0$ while for the layer $-M$ we get $C_{e-}^{(-M)} = C_{g-}^{(-M)} = 0$.
The condition will be the same (but for a different reason) if k_y is purely imaginary in the layer considered, which can happen for sufficiently low k_x if ν^2 is real and strictly negative in that layer. In that situation Cherenkov radiation [11, p. 637] occurs in the outer layer, and since there is (in our geometrical model) no other material beyond the outer layer, we cannot have any incoming wave: there should be only outcoming radiation whose wave vector is directed toward the outside of the chamber. Due to our choice of convention for the Fourier transform – see e.g. Eq. (2.2) – we must have an exponential term of the form $e^{-jk_y^{rad}y}$ in the field components, with $k_y^{rad} \geq 0$ in the top layer N and $k_y^{rad} \leq 0$ in the bottom layer $-M$, because in time domain the factor $e^{j\left(\omega t - k_y^{rad} y\right)}$ represents outcoming propagation in these conditions only (this can also be seen in Eq. (2.4) for the propagation along s). Then, since the imaginary part of the square root of a negative number is always positive according to Eq. (4.7), we have in any case

$$\begin{aligned} C_{e+}^{(N)} = C_{g+}^{(N)} = 0 \qquad &\text{for the top layer,} \\ C_{e-}^{(-M)} = C_{g-}^{(-M)} = 0 \qquad &\text{for the bottom one.} \end{aligned} \tag{6.56}$$

Note that Cherenkov radiation can occur in the air due to its small dielectric susceptibility, for γ sufficiently high. The effect on the beam-coupling impedance was discussed for instance in Ref. [43].

From Eqs. (6.54), (6.55) and (6.56), we get the following system

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \mathcal{M}_{14} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \mathcal{M}_{34} \\ \mathcal{M}'_{21} & \mathcal{M}'_{22} & \mathcal{M}'_{23} & \mathcal{M}'_{24} \\ \mathcal{M}'_{41} & \mathcal{M}'_{42} & \mathcal{M}'_{43} & \mathcal{M}'_{44} \end{bmatrix} \cdot \begin{bmatrix} C_{e+}^{(1)} \\ C_{e-}^{(-1)} \\ C_{g+}^{(1)} \\ C_{g-}^{(1)} \end{bmatrix} = -\frac{\mathcal{C}}{k_y^{(1)}} \begin{bmatrix} \mathcal{M}_{12} e^{k_y^{(1)} y_1} \\ \mathcal{M}_{32} e^{k_y^{(1)} y_1} \\ \mathcal{M}'_{21} e^{-k_y^{(1)} y_1} \\ \mathcal{M}'_{41} e^{-k_y^{(1)} y_1} \end{bmatrix}, \tag{6.57}$$

where e.g. $\mathcal{M}_{rs}$ is the component in row r and column s of the matrix $\mathcal{M}$. If we now call $\mathcal{P}$ the 4×4 matrix on the left hand side, we get the constants we look for as

$$\begin{bmatrix} C_{e+}^{(1)} \\ C_{e-}^{(-1)} \\ C_{g+}^{(1)} \\ C_{g-}^{(1)} \end{bmatrix} = -\frac{\mathcal{C}}{k_y^{(1)}} \mathcal{P}^{-1} \cdot \begin{bmatrix} \mathcal{M}_{12} e^{k_y^{(1)} y_1} \\ \mathcal{M}_{32} e^{k_y^{(1)} y_1} \\ \mathcal{M}'_{21} e^{-k_y^{(1)} y_1} \\ \mathcal{M}'_{41} e^{-k_y^{(1)} y_1} \end{bmatrix}, \tag{6.58}$$

or more explicitely:

$$\begin{aligned}
C_{e+}^{(1)} = -\frac{\mathcal{C}}{k_y^{(1)}} \Big[&\left\{ \left(\mathcal{P}^{-1}\right)_{11} \mathcal{M}_{12} + \left(\mathcal{P}^{-1}\right)_{12} \mathcal{M}_{32} \right\} e^{k_y^{(1)} y_1} \\
&+ \left\{ \left(\mathcal{P}^{-1}\right)_{13} \mathcal{M}'_{21} + \left(\mathcal{P}^{-1}\right)_{14} \mathcal{M}'_{41} \right\} e^{-k_y^{(1)} y_1} \Big], \\
C_{e-}^{(-1)} = -\frac{\mathcal{C}}{k_y^{(1)}} \Big[&\left\{ \left(\mathcal{P}^{-1}\right)_{21} \mathcal{M}_{12} + \left(\mathcal{P}^{-1}\right)_{22} \mathcal{M}_{32} \right\} e^{k_y^{(1)} y_1} \\
&+ \left\{ \left(\mathcal{P}^{-1}\right)_{23} \mathcal{M}'_{21} + \left(\mathcal{P}^{-1}\right)_{24} \mathcal{M}'_{41} \right\} e^{-k_y^{(1)} y_1} \Big], \\
C_{g+}^{(1)} = -\frac{\mathcal{C}}{k_y^{(1)}} \Big[&\left\{ \left(\mathcal{P}^{-1}\right)_{31} \mathcal{M}_{12} + \left(\mathcal{P}^{-1}\right)_{32} \mathcal{M}_{32} \right\} e^{k_y^{(1)} y_1} \\
&+ \left\{ \left(\mathcal{P}^{-1}\right)_{33} \mathcal{M}'_{21} + \left(\mathcal{P}^{-1}\right)_{34} \mathcal{M}'_{41} \right\} e^{-k_y^{(1)} y_1} \Big], \\
C_{g-}^{(1)} = -\frac{\mathcal{C}}{k_y^{(1)}} \Big[&\left\{ \left(\mathcal{P}^{-1}\right)_{41} \mathcal{M}_{12} + \left(\mathcal{P}^{-1}\right)_{42} \mathcal{M}_{32} \right\} e^{k_y^{(1)} y_1} \\
&+ \left\{ \left(\mathcal{P}^{-1}\right)_{43} \mathcal{M}'_{21} + \left(\mathcal{P}^{-1}\right)_{44} \mathcal{M}'_{41} \right\} e^{-k_y^{(1)} y_1} \Big].
\end{aligned} \tag{6.59}$$

From this all the constants for all the layers p can be computed thanks to Eqs. (6.5), (6.9), (6.43) and (6.51).

Note that the matrices $\mathcal{P}$, $\mathcal{M}$ and $\mathcal{M}'$ do not depend on y_1, the offset of the source, since the matrices $M^{p+1,p}$ and $M^{p-1,p}$ do not depend on y_1. We can therefore define the following functions of k_x (that are also functions of ω, β and the materials properties, but independent of y_1):

$$\begin{aligned}
\chi_1(k_x) &= \left(\mathcal{P}^{-1}\right)_{11} \mathcal{M}_{12} + \left(\mathcal{P}^{-1}\right)_{12} \mathcal{M}_{32}, \\
\chi_2(k_x) &= \left(\mathcal{P}^{-1}\right)_{21} \mathcal{M}_{12} + \left(\mathcal{P}^{-1}\right)_{22} \mathcal{M}_{32}, \\
\eta_1(k_x) &= \left(\mathcal{P}^{-1}\right)_{13} \mathcal{M}'_{21} + \left(\mathcal{P}^{-1}\right)_{14} \mathcal{M}'_{41}, \\
\eta_2(k_x) &= \left(\mathcal{P}^{-1}\right)_{23} \mathcal{M}'_{21} + \left(\mathcal{P}^{-1}\right)_{24} \mathcal{M}'_{41},
\end{aligned} \tag{6.60}$$

such that we can write the constants for the electric fields in the following compact way

$$C_{e+}^{(1)} = -\frac{\mathcal{C}}{k_y^{(1)}}\left[\chi_1(k_x)e^{k_y^{(1)}y_1} + \eta_1(k_x)e^{-k_y^{(1)}y_1}\right],$$
$$C_{e-}^{(-1)} = -\frac{\mathcal{C}}{k_y^{(1)}}\left[\chi_2(k_x)e^{k_y^{(1)}y_1} + \eta_2(k_x)e^{-k_y^{(1)}y_1}\right]. \quad (6.61)$$

Therefore, to compute the constants of the electric field longitudinal component (which is the one needed to calculate the electromagnetic force and impedances as we will see below), we only need to perform multiplications of 4×4 matrices and one inversion of a 4×4 matrix. The final inversion can even be limited to the computation of only 8 coefficients of the inverted matrix, which can be done using e.g. the cofactor method.

7 Electromagnetic force inside the chamber

To study the dynamics of a passing beam inside the chamber, we need to calculate the Lorentz electromagnetic force $\vec{F}$ on a given test particle. We assume such a particle has a charge of q and the same velocity as the source, namely $\vec{v} = v\vec{e_s}$. The longitudinal component of the force acting on this particle in the vacuum region is written (dropping the superscript (-1) or (1) for conciseness)

$$F_s = qE_s, \quad (7.1)$$

while the transverse components are (using Eqs. (5.9) to (5.12) and recalling that $\nu = \frac{k}{\gamma}$ in vacuum)

$$F_x = q\left(E_x - v\mu_0 H_y\right) = \frac{jq\gamma^2}{k}\left(1-\beta^2\right)\frac{\partial E_s}{\partial x},$$
$$F_y = q\left(E_y + v\mu_0 H_x\right) = \frac{jq\gamma^2}{k}\left(1-\beta^2\right)\frac{\partial E_s}{\partial y}.$$

After simplification we obtain

$$F_x = \frac{jq}{k}\frac{\partial E_s}{\partial x}, \quad (7.2)$$
$$F_y = \frac{jq}{k}\frac{\partial E_s}{\partial y}. \quad (7.3)$$

The three relations (7.1), (7.2) and (7.3) are linear so valid also for the total fields obtained after integration over k_x from 0 to ∞, i.e. the fields coming from our initial source in Eqs. (2.4) and (2.5). Therefore the force components can be computed with the knowledge of the longitudinal component of the electric field only.

8 Electric field longitudinal component after integration over k_x

We now want to go back to our initial problem with the source charge and current densities in frequency domain given by Eqs. (2.4) and (2.5). To do so we have to perform the integration over k_x, i.e (see Section 2) to plug back the factor $\int_0^\infty \mathrm{d}k_x$ in front of the field responses we have computed in the previous sections, and perform the integration. We will do this only for E_s since this is the component needed to compute the electromagnetic force (see Section 7). The following analysis could be done in a very similar way for H_s as well (for the wall term only since there is no direct space-charge term in H_s). From Eqs. (4.13), (6.4) and (6.5) we first write E_s in both vacuum layers, for a given k_x:

$$E_s^{(1)} = \cos(k_x x)\, e^{-jks}\left[C_{e+}^{(1)}e^{k_y^{(1)}y} + C_{e-}^{(-1)}e^{-k_y^{(1)}y} + \frac{\mathcal{C}}{k_y^{(1)}}e^{k_y^{(1)}(y_1-y)}\right],$$
$$E_s^{(-1)} = \cos(k_x x)\, e^{-jks}\left[C_{e+}^{(1)}e^{k_y^{(1)}y} + C_{e-}^{(-1)}e^{-k_y^{(1)}y} + \frac{\mathcal{C}}{k_y^{(1)}}e^{k_y^{(1)}(y-y_1)}\right]. \quad (8.1)$$

We will essentially identify two terms in E_s: one is the direct space charge term, that would be the only one present in the absence of any chamber (i.e. if the whole space were considered to be vacuum) and the other one will be called the wall term, entirely due to the flat chamber's presence.

8.1 Direct space charge term of the longitudinal electric field

If there were no chamber around the beam, the finiteness of the fields for $y \to \pm\infty$ prevents any growing exponential in y in the expression of E_s. Therefore, since $k_y^{(1)} = \sqrt{k_x^2 + \frac{k^2}{\gamma^2}}$ is real and strictly positive, we must have $C_{e+}^{(1)} = C_{e-}^{(-1)} = 0$. From the above Eqs. (8.1) we clearly have

$$E_s^{(1),SC} = \frac{\mathcal{C}}{k_y^{(1)}} \cos(k_x x)\, e^{-jks} e^{k_y^{(1)}(y_1 - y)} \quad \text{and} \quad E_s^{(-1),SC} = \frac{\mathcal{C}}{k_y^{(1)}} \cos(k_x x)\, e^{-jks} e^{k_y^{(1)}(y - y_1)}, \tag{8.2}$$

where the superscript "SC" stands for "direct space-charge". Recalling that $k_y^{(1)} = \sqrt{k_x^2 + \frac{k^2}{\gamma^2}}$ from Eq. (6.4) and noticing that in layer (1) we have $y_1 - y < 0$ while in layer (-1), $y - y_1 < 0$, we can integrate over k_x those formulae using Eq. (C.17). This gives the same result for both regions:

$$E_{s,tot}^{vac,SC} = \mathcal{C} e^{-jks} K_0 \left(\frac{k}{\gamma} \sqrt{x^2 + (y - y_1)^2} \right), \tag{8.3}$$

where K_0 is the modified Bessel function of the second kind of order 0, the subscript "*tot*" stands for the total fields created by the source in Eqs. (2.4) and (2.5) and the superscript "*vac*" stands for the fields in the vacuum region.
We can generalize this formula for a source at $x = x_1$ instead of $x = 0$, which is straightforward from the continuous translation invariance of the electromagnetic configuration along the x axis: we simply need to replace x by $x - x_1$:

$$E_{s,tot}^{vac,SC} = \mathcal{C} e^{-jks} K_0 \left(\frac{k}{\gamma} \sqrt{(x - x_1)^2 + (y - y_1)^2} \right). \tag{8.4}$$

8.2 Wall term of the longitudinal electric field

The wall term of the fields is the part of the fields due to the chamber, or in other words the total fields minus the direct space-charge term we have considered in the previous section. From Eqs. (8.1) and (8.2) the wall part of the field has the same expression in both regions 1 and -1, and we can write:

$$E_s^{vac,W} = \cos(k_x x)\, e^{-jks} \left[C_{e+}^{(1)} e^{k_y^{(1)} y} + C_{e-}^{(-1)} e^{-k_y^{(1)} y} \right],$$

where the superscript "W" stands for the wall part of the field component. Upon integration over k_x, the wall part of the total longitudinal electric field from our initial source in Eqs. (2.4) and (2.5) is

$$E_{s,tot}^{vac,W} = \int_0^{\infty} \mathrm{d}k_x \cos(k_x x)\, e^{-jks} \left[C_{e+}^{(1)} e^{k_y^{(1)} y} + C_{e-}^{(-1)} e^{-k_y^{(1)} y} \right]. \tag{8.5}$$

A direct analytical integration of this equation in the general case (given the very complicated expression of $C_{e+}^{(1)}$ and $C_{e-}^{(-1)}$, see Section 6) looks like an impossible task, or at least very difficult. But we will show below how we can put this into another form in order to identify the dependencies in the test particle coordinates as well as in the source macroparticle offset y_1.
To do so the idea is to try to get back to a formula in the form of a series involving modified Bessel functions of the first kind I_m of argument proportional to the coordinates of the source and test particles, in a similar way to what was found in the case of an axisymmetric geometry [6]. The advantage of this would be that it is easy to approximate these Bessel functions in the case of small displacements from the origin, in such a way that one can restric ourselves to the first terms of the series.
We begin by introducing the cylindrical coordinates (r, θ, s) associated to the cartesian ones we have

considered up to now. We have then $x = r\cos\theta$ and $y = r\sin\theta$. For convenience we make an additional change of angle, defining ϕ as

$$\phi = \theta - \frac{\pi}{2},$$

such that

$$x = -r\sin\phi \quad \text{and} \quad y = r\cos\phi.$$

Then from the invariance of $E_{s,tot}^{vac,W}$ with the sign of x we have the symmetry relation

$$\begin{aligned}
E_{s,tot}^{vac,W}(r,-\phi,s) &= E_{s,tot}^{vac,W}(x = r\sin\phi, y = r\cos\phi, s) \\
&= E_{s,tot}^{vac,W}(x = -r\sin\phi, y = r\cos\phi, s) \\
&= E_{s,tot}^{vac,W}(r,\phi,s).
\end{aligned}$$

The main idea is then to decompose $E_{s,tot}^{vac,W}$ into a Fourier series with respect to the angle ϕ. Since this function is even, there is no sine term in the series and the coefficients are given by [44]:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{d}\phi \cos(n\phi)\, E_{s,tot}^{vac,W},$$

such that

$$E_{s,tot}^{vac,W} = \sum_{n=0}^{\infty} \frac{a_n}{1+\delta_{n0}} \cos(n\phi), \tag{8.6}$$

where $\delta_{n0} = 1$ if $n = 0$, 0 otherwise. Using first the parity of $E_{s,tot}^{vac,W}$ with ϕ, recalling that $k_y^{(1)} = \sqrt{k_x^2 + \frac{k^2}{\gamma^2}}$ and inverting then the order in the integrals in k_x and ϕ, we can obtain for a_n:

$$\begin{aligned}
a_n &= \frac{2}{\pi} e^{-jks} \int_0^{\pi} \mathrm{d}\phi \cos(n\phi) \int_0^{\infty} \mathrm{d}k_x \cos(k_x r\sin\phi) \left[C_{e+}^{(1)} e^{\sqrt{k_x^2+\frac{k^2}{\gamma^2}}\, r\cos\phi} + C_{e-}^{(-1)} e^{-\sqrt{k_x^2+\frac{k^2}{\gamma^2}}\, r\cos\phi} \right] \\
&= \frac{2}{\pi} e^{-jks} \int_0^{\infty} \mathrm{d}k_x \left[C_{e+}^{(1)} \int_0^{\pi} \mathrm{d}\phi \cos(n\phi) \cos(k_x r\sin\phi)\, e^{r\cos\phi\sqrt{k_x^2+\frac{k^2}{\gamma^2}}} \right. \\
&\qquad \left. + C_{e-}^{(-1)} \int_0^{\pi} \mathrm{d}\phi \cos(n\phi) \cos(k_x r\sin\phi)\, e^{-r\cos\phi\sqrt{k_x^2+\frac{k^2}{\gamma^2}}} \right] \\
&= \frac{2}{\pi} e^{-jks} \int_0^{\infty} \mathrm{d}k_x \left[C_{e+}^{(1)} + (-1)^n C_{e-}^{(-1)} \right] \int_0^{\pi} \mathrm{d}\phi \cos(n\phi) \cos(k_x r\sin\phi)\, e^{r\cos\phi\sqrt{k_x^2+\frac{k^2}{\gamma^2}}},
\end{aligned}$$

thanks to the change of variable $\phi \to \pi - \phi$ for the second integral in ϕ. It turns out that the final integral in ϕ can be computed analytically as shown in Appendix C.6. By virtue of Eq. (C.27) we get

$$\begin{aligned}
a_n &= e^{-jks} I_n\left(\frac{kr}{\gamma}\right) \int_0^{\infty} \mathrm{d}k_x \left[C_{e+}^{(1)} + (-1)^n C_{e-}^{(-1)} \right] \left[\left(k_x + \sqrt{k_x^2 + \frac{k^2}{\gamma^2}} \right)^{-n} \frac{k^n}{\gamma^n} + \left(k_x + \sqrt{k_x^2 + \frac{k^2}{\gamma^2}} \right)^{n} \frac{\gamma^n}{k^n} \right] \\
&= e^{-jks} \frac{k}{\gamma} I_n\left(\frac{kr}{\gamma}\right) \int_0^{\infty} \mathrm{d}u \left[C_{e+}^{(1)} + (-1)^n C_{e-}^{(-1)} \right] \cosh u \left[(\cosh u + \sinh u)^{-n} + (\cosh u + \sinh u)^n \right] \\
&\qquad\qquad \text{with the change of variable } k_x = \frac{k}{\gamma}\sinh u \\
&= e^{-jks} \frac{2k}{\gamma} I_n\left(\frac{kr}{\gamma}\right) \int_0^{\infty} \mathrm{d}u \left[C_{e+}^{(1)} + (-1)^n C_{e-}^{(-1)} \right] \cosh u \cosh(nu).
\end{aligned} \tag{8.7}$$

This, when combined with (8.6), gives the dependence of E_s in the test particle transverse position (r,ϕ). To get the dependence in the source particle position we need some additional steps. First, in a_n

we replace $C_{e+}^{(1)}$ and $C_{e-}^{(-1)}$ by their expressions from Eqs. (6.61), using the fact that $k_x = \frac{k}{\gamma}\sinh u$ and $k_y^{(1)} = \frac{k}{\gamma}\cosh u$ in our change of variable:

$$a_n = -2\mathcal{C}e^{-jks} I_n\left(\frac{kr}{\gamma}\right)\int_0^\infty \mathrm{d}u \left[\left\{\chi_1\left(\frac{k}{\gamma}\sinh u\right)e^{\frac{ky_1}{\gamma}\cosh u} + \eta_1\left(\frac{k}{\gamma}\sinh u\right)e^{-\frac{ky_1}{\gamma}\cosh u}\right\}\right.$$
$$\left. +(-1)^n\left\{\chi_2\left(\frac{k}{\gamma}\sinh u\right)e^{\frac{ky_1}{\gamma}\cosh u} + \eta_2\left(\frac{k}{\gamma}\sinh u\right)e^{-\frac{ky_1}{\gamma}\cosh u}\right\}\right]\cosh(nu).$$

From Eqs. (C.12) and (C.13) applied with $z = \frac{ky_1}{\gamma}$, this can be decomposed into

$$a_n = -4\mathcal{C}e^{-jks} I_n\left(\frac{kr}{\gamma}\right)\int_0^\infty \mathrm{d}u\cosh(nu)\sum_{m=0}^{\infty}\frac{\cosh(mu)}{1+\delta_{m0}} I_m\left(\frac{ky_1}{\gamma}\right)\left[\chi_1\left(\frac{k}{\gamma}\sinh u\right)\right.$$
$$\left. +(-1)^m\eta_1\left(\frac{k}{\gamma}\sinh u\right) + (-1)^n\chi_2\left(\frac{k}{\gamma}\sinh u\right) + (-1)^{m+n}\eta_2\left(\frac{k}{\gamma}\sinh u\right)\right].$$

Now we can invert the order of the integral and the sum, before plugging the result into Eq. (8.6), to obtain finally (using also $\phi = \theta - \frac{\pi}{2}$):

$$E_{s,tot}^{vac,W} = -4\mathcal{C}e^{-jks}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}\cos\left[n\left(\theta-\frac{\pi}{2}\right)\right]}{(1+\delta_{m0})(1+\delta_{n0})} I_m\left(\frac{ky_1}{\gamma}\right) I_n\left(\frac{kr}{\gamma}\right), \tag{8.8}$$

with α_{mn} defined by the integral

$$\alpha_{mn} = \int_0^\infty \mathrm{d}u\cosh(mu)\cosh(nu)\left[\chi_1\left(\frac{k}{\gamma}\sinh u\right) + (-1)^m\eta_1\left(\frac{k}{\gamma}\sinh u\right)\right.$$
$$\left. +(-1)^n\chi_2\left(\frac{k}{\gamma}\sinh u\right) + (-1)^{m+n}\eta_2\left(\frac{k}{\gamma}\sinh u\right)\right]. \tag{8.9}$$

The coefficients α_{mn} depend only on the functions η_1, η_2, χ_1 and χ_2 so only on the chamber properties and on ω and β (see Section 6.3). The infinite integrals involved are fastly converging, so even though it does not seem to be possible to compute them analytically in the general case, they can be calculated numerically.
The decomposition into azimuthal modes of $E_{s,tot}^{vac,W}$ in Eq. (8.8) has a similar form as the one that arises in the case of an axisymmetric structure [6]. The clear advantage of this formula is that $I_m\left(\frac{ky_1}{\gamma}\right)$ and $I_n\left(\frac{kr}{\gamma}\right)$ are fastly decaying with m or n when the argument is small (see Eq. (C.10)). Therefore only the first few terms of the series will be sufficient in most applications.
We can also write $E_{s,tot}^{vac,W}$ in cartesian coordinates: using $r = \sqrt{x^2+y^2}$ and

$$\cos\left[n\left(\theta-\frac{\pi}{2}\right)\right] = \cos(n\phi) = \Re\left(e^{jn\phi}\right) = \Re\left[(\cos\phi + j\sin\phi)^n\right]$$
$$= \frac{1}{r^n}\Re\left[(y-jx)^n\right] = \frac{1}{2r^n}\left[(y-jx)^n + (y+jx)^n\right], \tag{8.10}$$

we obtain

$$E_{s,tot}^{vac,W} = -2\mathcal{C}e^{-jks}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}\left[(y-jx)^n+(y+jx)^n\right]}{(x^2+y^2)^{\frac{n}{2}}(1+\delta_{m0})(1+\delta_{n0})} I_m\left(\frac{ky_1}{\gamma}\right) I_n\left(\frac{k\sqrt{x^2+y^2}}{\gamma}\right). \tag{8.11}$$

As we did for the direct space charge in the previous section, we can generalize this formula for a source at $x = x_1$ instead of $x = 0$, by replacing x with $x - x_1$:

$$E_{s,tot}^{vac,W} = -2\mathcal{C}e^{-jks}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}\left[\{y-j(x-x_1)\}^n+\{y+j(x-x_1)\}^n\right]}{\{(x-x_1)^2+y^2\}^{\frac{n}{2}}(1+\delta_{m0})(1+\delta_{n0})} I_m\left(\frac{ky_1}{\gamma}\right) I_n\left\{\frac{k\sqrt{(x-x_1)^2+y^2}}{\gamma}\right\}. \tag{8.12}$$

The sum of Eqs. (8.3) and (8.8) give the total general longitudinal electric field in the vacuum region due to the source in Eqs. (2.4) and (2.5), from which the total force and impedances can be derived.

9 Beam-coupling impedances

9.1 Definitions

We consider a test particle of charge q located at position (x_2, y_2) in the transverse plane, while the source is at $(0, y_1)$ as defined in Section 2. Several definitions of the impedance exist in e.g. Refs. [30, 45, 46] and [47, p. 74]. We will here write the total longitudinal impedance in a general way, inspired by Ref. [46][10]

$$Z_{\parallel} = -\frac{1}{Qq} \int \mathrm{d}V E_s J_t^*(x_2, y_2), \tag{9.1}$$

where the integration is performed over the volume of the structure considered (usually on a finite length L[11]), and where E_s is given by the sum of Eqs. (8.3) and (8.8). The $*$ stands for the complex conjugate and $\vec{J_t} = J_t \vec{e_s}$ is the density of the current flowing at the test particle position, whose expression is therefore (see Eq. (2.5))

$$J_t(x_2, y_2) = q e^{-jks} \delta(x - x_2) \delta(y - y_2). \tag{9.2}$$

With the same notations we define the total transverse impedances as [46]

$$Z_x = \frac{j}{Qq} \int \mathrm{d}V \left[\vec{E} + \beta \vec{e_s} \times \vec{G} \right] \cdot \vec{e_x} J_t^*(x_2, y_2) = \frac{j}{Qq} \int \mathrm{d}V \frac{F_x}{q} J_t^*(x_2, y_2), \tag{9.3}$$

$$Z_y = \frac{j}{Qq} \int \mathrm{d}V \left[\vec{E} + \beta \vec{e_s} \times \vec{G} \right] \cdot \vec{e_y} J_t^*(x_2, y_2) = \frac{j}{Qq} \int \mathrm{d}V \frac{F_y}{q} J_t^*(x_2, y_2). \tag{9.4}$$

These three definitions can be cast into a form involving only an integral over the length L, by plugging in Eqs. (7.2), (7.3) and (9.2):

$$Z_{\parallel} = -\frac{1}{Q} \int^L \mathrm{d}s E_s(x_2, y_2, s; \omega) e^{jks}, \tag{9.5}$$

$$Z_x = -\frac{1}{kQ} \int^L \mathrm{d}s \frac{\partial E_s}{\partial x}(x_2, y_2, s; \omega) e^{jks}, \tag{9.6}$$

$$Z_y = -\frac{1}{kQ} \int^L \mathrm{d}s \frac{\partial E_s}{\partial y}(x_2, y_2, s; \omega) e^{jks}. \tag{9.7}$$

9.2 Direct space-charge impedances

When plugging in Eq. (9.5) the longitudinal electric field due to the direct space charge from Eq. (8.3) together with the definition of $\mathcal{C}$ in Eq. (6.6), one gets the following longitudinal space-charge impedance:

$$Z_{\parallel}^{SC,direct} = -\frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^2} K_0 \left(\frac{k}{\gamma} \sqrt{x_2^2 + (y_2 - y_1)^2} \right). \tag{9.8}$$

[10]Note that in Ref. [6] there is a mistake in the definition of the impedances (without impact on the subsequent derivations): a_2 should not appear in the factor in front of the integral, in Eqs. (8.13) and (8.24).

[11]This seems somehow in contradiction with our initial assumption on the infinite length of the chamber considered, but if we were to integrate over an infinite length we would obtain an infinite result, as nothing depend on the position in the integral. In practice one wants to compute the effect of the self-fields for a beam passing in a structure of finite length, which is the length L used here. Our initial assumption then simply state that we neglect all side effects due to the fact that the actual structure has some edges instead of being infinitely prolonged.

Using Eqs. (9.6) and (9.7) with Eq. (C.5) we find for the transverse space-charge impedances:

$$Z_x^{SC,direct} = \frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^3} K_1\left(\frac{k}{\gamma}\sqrt{x_2^2+(y_2-y_1)^2}\right)\frac{x_2}{\sqrt{x_2^2+(y_2-y_1)^2}}, \tag{9.9}$$

$$Z_y^{SC,direct} = \frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^3} K_1\left(\frac{k}{\gamma}\sqrt{x_2^2+(y_2-y_1)^2}\right)\frac{y_2-y_1}{\sqrt{x_2^2+(y_2-y_1)^2}}. \tag{9.10}$$

9.3 Wall impedances

The term of the impedance coming from the part of the fields due to the flat chamber presence is the so-called wall impedance, which is not exactly the same as the resistive-wall impedance and has been introduced in Ref. [12]. It contains both the impedance that we would have with a flat chamber made of a perfect conductor (this part is usually called the indirect space-charge impedance) and the part of the impedance coming from the resistivity (or more generally the electromagnetic properties) of the layer(s). We will discuss more the indirect space-charge impedance taken alone in Section 10.
The total longitudinal wall impedance is therefore obtained when plugging Eq. (8.8) into the definition (9.5):

$$Z_\parallel^{Wall} = 4\frac{\mathcal{C}L}{Q}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}\cos(n\phi_2)}{(1+\delta_{m0})(1+\delta_{n0})} I_m\left(\frac{ky_1}{\gamma}\right) I_n\left(\frac{kr_2}{\gamma}\right), \tag{9.11}$$

where $r_2 = \sqrt{x_2^2+y_2^2}$ and ϕ_2 is such that $x_2 = -r_2\sin\phi_2$ and $y_2 = r_2\cos\phi_2$ (see Section 8.2). This expression gives the general nonlinear longitudinal wall impedance, but usually one is interested only in the first order terms in the source and test positions, so for small $\frac{ky_1}{\gamma}$ and $\frac{kr_2}{\gamma}$. From the development in Taylor series of I_m and I_n in Eq. (C.9) we see that to go up to the second order in both the source and test positions, we need to go up to $m = 2$ and $n = 2$ in the above summation. Approximating then up to second order the Bessel functions thanks to Eq. (C.9) and using $\cos\phi_2 = \frac{y_2}{r_2}$ and $\cos 2\phi_2 = 2\cos^2\phi_2 - 1 = \frac{2y_2^2}{r_2^2} - 1$ we get

$$\begin{aligned} Z_\parallel^{Wall} &\approx 4\frac{\mathcal{C}L}{Q}\left[\frac{\alpha_{00}}{4}\left\{1+\left(\frac{ky_1}{2\gamma}\right)^2+\left(\frac{kr_2}{2\gamma}\right)^2\right\}+\frac{\alpha_{01}}{2}\frac{y_2}{r_2}\frac{kr_2}{2\gamma}+\frac{\alpha_{02}}{2}\left(\frac{2y_2^2}{r_2^2}-1\right)\frac{1}{2}\left(\frac{kr_2}{2\gamma}\right)^2\right. \\ &\qquad\left.+\frac{\alpha_{10}}{2}\frac{ky_1}{2\gamma}+\alpha_{11}\frac{y_2}{r_2}\frac{ky_1}{2\gamma}\frac{kr_2}{2\gamma}+\frac{\alpha_{20}}{2}\frac{1}{2}\left(\frac{ky_1}{2\gamma}\right)^2\right] \\ &\approx \frac{\mathcal{C}L}{Q}\left[\alpha_{00}+\frac{k\alpha_{10}}{\gamma}y_1+\frac{k\alpha_{01}}{\gamma}y_2+k^2\left(\frac{\alpha_{00}+\alpha_{20}}{4\gamma^2}\right)y_1^2+k^2\left(\frac{\alpha_{00}-\alpha_{02}}{4\gamma^2}\right)x_2^2\right. \\ &\qquad\left.+k^2\left(\frac{\alpha_{00}+\alpha_{02}}{4\gamma^2}\right)y_2^2+\frac{k^2\alpha_{11}}{\gamma^2}y_1y_2\right]. \end{aligned} \tag{9.12}$$

In a similar way, the total horizontal wall impedance is obtained when plugging Eq. (8.8) into the definition (9.6):

$$\begin{aligned} Z_x^{Wall} = 4\frac{\mathcal{C}L}{kQ}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}}{(1+\delta_{m0})(1+\delta_{n0})} I_m\left(\frac{ky_1}{\gamma}\right)&\left[I_n\left(\frac{kr_2}{\gamma}\right)\left.\frac{\partial\cos(n\phi)}{\partial x}\right|_{x_2,y_2}\right. \\ &\left.+\frac{k\cos(n\phi_2)}{\gamma}I_n'\left(\frac{kr_2}{\gamma}\right)\left.\frac{\partial r}{\partial x}\right|_{x_2,y_2}\right]. \end{aligned}$$

Now we have, using in particular Eq. (8.10)

$$\begin{aligned}
\left.\frac{\partial r}{\partial x}\right|_{x_2,y_2} &= \frac{x_2}{r_2}, \\
\left.\frac{\partial \cos(n\phi)}{\partial x}\right|_{x_2,y_2} &= \frac{nj}{2r_2^n}\left[-\left(y_2 - jx_2\right)^{n-1} + \left(y_2 + jx_2\right)^{n-1}\right] - \frac{nx_2}{2r_2^{n+2}}\left[\left(y_2 - jx_2\right)^n + \left(y_2 + jx_2\right)^n\right] \\
&= \frac{nj}{2r_2}\left[-\left(\cos\phi_2 + j\sin\phi_2\right)^{n-1} + \left(\cos\phi_2 - j\sin\phi_2\right)^{n-1}\right] + \frac{n\sin\phi_2}{r_2}\cos(n\phi_2) \\
&= \frac{n}{r_2}\left[\sin\left\{(n-1)\phi_2\right\} + \sin\phi_2\cos(n\phi_2)\right] \\
&= \frac{ny_2}{r_2^2}\sin(n\phi_2).
\end{aligned}$$

We then obtain

$$Z_x^{Wall} = 4\frac{\mathcal{C}L}{kQ}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}}{(1+\delta_{m0})(1+\delta_{n0})} I_m\left(\frac{ky_1}{\gamma}\right)\frac{1}{r_2}\left[\frac{ny_2\sin(n\phi_2)}{r_2} I_n\left(\frac{kr_2}{\gamma}\right) + \frac{kx_2\cos(n\phi_2)}{\gamma} I_n'\left(\frac{kr_2}{\gamma}\right)\right]. \tag{9.13}$$

Upon differentiation of Eq. (C.9) we have for small z

$$I_n'(z) \approx \frac{1}{2(n-1)!}\left(\frac{z}{2}\right)^{n-1} + \frac{n+2}{2(n+1)!}\left(\frac{z}{2}\right)^{n+1},$$

where $l!$ is the factorial of the integer l, with the convention $(-1)! = \infty$. From this and Eq. (C.9) we see that we need to go up to $m = 2$ and $n = 3$ to obtain Z_x^{Wall} up to the second order in the source and test coordinates. Approximating the Bessel functions up to that order we obtain

$$\begin{aligned}
Z_x^{Wall} \approx 4\frac{\mathcal{C}L}{kQ}\Bigg[& \frac{\alpha_{00}}{4}\frac{k^2x_2}{2\gamma^2} + \frac{\alpha_{01}}{2}\left\{1 + \frac{k^2y_1^2}{4\gamma^2}\right\}\left\{\frac{k\sin(\phi_2)\,y_2}{2\gamma r_2} + \frac{k^3\sin(\phi_2)\,y_2r_2}{16\gamma^3}\right. \\
& \left. + \frac{kx_2\cos\phi_2}{2\gamma r_2} + \frac{3k^3x_2\cos\phi_2 r_2}{16\gamma^2}\right\} + \frac{\alpha_{02}}{2}\left\{\frac{2y_2\sin(2\phi_2)k^2}{8\gamma^2} + \frac{k^2x_2\cos(2\phi_2)}{4\gamma^2}\right\} \\
& + \frac{\alpha_{03}}{2}\left\{\frac{3y_2\sin(3\phi_2)\,k^3r_2}{48\gamma^3} + \frac{k^3x_2\cos(3\phi_2)\,r_2}{16\gamma^3}\right\} + \frac{\alpha_{10}}{2}\frac{ky_1}{2\gamma}\frac{k^2x_2}{2\gamma^2} \\
& + \alpha_{11}\frac{ky_1}{2\gamma}\left\{\frac{y_2\sin(\phi_2)\,k}{2\gamma r_2} + \frac{kx_2\cos(\phi_2)}{2\gamma r_2}\right\} + \alpha_{12}\frac{ky_1}{2\gamma}\left\{\frac{y_2\sin(2\phi_2)\,k^2}{4\gamma^2} + \frac{k^2x_2\cos(2\phi_2)}{4\gamma^2}\right\} \\
& + \alpha_{21}\frac{k^2y_1^2}{8\gamma^2}\left\{\frac{y_2\sin(\phi_2)\,k}{2\gamma r_2} + \frac{kx_2\cos(\phi_2)}{2\gamma r_2}\right\}\Bigg], \\
\approx 4\frac{\mathcal{C}L}{kQ}\Bigg[& \frac{\alpha_{00}k^2}{8\gamma^2}x_2 + \frac{\alpha_{01}k^3}{16\gamma^3}x_2y_2 - \frac{\alpha_{02}k^2}{8\gamma^2}x_2 - \frac{\alpha_{03}k^3}{16\gamma^3}x_2y_2 + \frac{\alpha_{10}k^3}{8\gamma^3}y_1x_2 - \frac{\alpha_{12}k^3}{8\gamma^3}y_1x_2\Bigg],
\end{aligned}$$

where we have used various trigonometric identities and

$$\cos\phi_2 = \frac{y_2}{r_2}, \quad \sin\phi_2 = -\frac{x_2}{r_2}, \quad \sin(2\phi_2) = -2\frac{x_2y_2}{r_2^2}.$$

The total horizontal impedance up to second order can be rewritten

$$Z_x^{Wall} \approx \frac{\mathcal{C}Lk}{2\gamma^2 Q}\left[(\alpha_{00} - \alpha_{02})\,x_2 + k\frac{\alpha_{10} - \alpha_{12}}{\gamma}y_1x_2 + k\frac{\alpha_{01} - \alpha_{03}}{2\gamma}x_2y_2\right]. \tag{9.14}$$

Similarly, the total vertical wall impedance is obtained when plugging Eq. (8.8) into the definition (9.7):

$$\begin{aligned}
Z_y^{Wall} = 4\frac{\mathcal{C}L}{kQ}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}}{(1+\delta_{m0})(1+\delta_{n0})} I_m\left(\frac{ky_1}{\gamma}\right)\Bigg[& I_n\left(\frac{kr_2}{\gamma}\right)\left.\frac{\partial\cos(n\phi)}{\partial y}\right|_{x_2,y_2} \\
& + \frac{k\cos(n\phi_2)}{\gamma} I_n'\left(\frac{kr_2}{\gamma}\right)\left.\frac{\partial r}{\partial y}\right|_{x_2,y_2}\Bigg].
\end{aligned}$$

As above we can write

$$\left.\frac{\partial r}{\partial y}\right|_{x_2,y_2} = \frac{y_2}{r_2},$$

$$\begin{aligned}\left.\frac{\partial\cos(n\phi)}{\partial y}\right|_{x_2,y_2} &= \frac{n}{2r_2^n}\left[(y_2-jx_2)^{n-1}+(y_2+jx_2)^{n-1}\right]-\frac{ny_2}{2r_2^{n+2}}\left[(y_2-jx_2)^n+(y_2+jx_2)^n\right]\\ &= \frac{n}{r_2}\left[\cos\{(n-1)\phi_2\}-\cos\phi_2\cos(n\phi_2)\right]\\ &= -\frac{nx_2}{r_2^2}\sin(n\phi_2)\,.\end{aligned}$$

We then get

$$Z_y^{Wall} = 4\frac{\mathcal{C}L}{kQ}\sum_{m,n=0}^{\infty}\frac{\alpha_{mn}}{(1+\delta_{m0})(1+\delta_{n0})}I_m\left(\frac{ky_1}{\gamma}\right)\frac{1}{r_2}\left[-\frac{nx_2\sin(n\phi_2)}{r_2}I_n\left(\frac{kr_2}{\gamma}\right)\right.$$
$$\left.+\frac{ky_2\cos(n\phi_2)}{\gamma}I_n'\left(\frac{kr_2}{\gamma}\right)\right]. \quad (9.15)$$

With the same approximation as above for I_m, I_n and I_n', and going up to second order we can approximate the vertical impedance by

$$\begin{aligned}Z_y^{Wall} \approx\ & 4\frac{\mathcal{C}L}{kQ}\left[\frac{\alpha_{00}}{4}\frac{k^2y_2}{2\gamma^2}+\frac{\alpha_{01}}{2}\left\{1+\frac{k^2y_1^2}{4\gamma^2}\right\}\left\{-\frac{x_2\sin\phi_2 k}{2\gamma r_2}-\frac{x_2\sin\phi_2 k^3 r_2}{16\gamma^3}+\frac{ky_2\cos\phi_2}{2\gamma r_2}+\frac{3k^3y_2\cos\phi_2 r_2}{16\gamma^3}\right\}\right.\\ &+\frac{\alpha_{02}}{2}\left\{-\frac{2x_2\sin(2\phi_2)k^2}{8\gamma^2}+\frac{k^2y_2\cos(2\phi_2)}{4\gamma^2}\right\}+\frac{\alpha_{03}}{2}\left\{-\frac{3x_2\sin(3\phi_2)k^3r_2}{48\gamma^3}+\frac{k^3y_2\cos(3\phi_2)r_2}{16\gamma^3}\right\}\\ &+\frac{\alpha_{10}}{2}\frac{ky_1}{2\gamma}\frac{k^2y_2}{2\gamma^2}+\alpha_{11}\frac{ky_1}{2\gamma}\left\{-\frac{x_2\sin\phi_2 k}{2\gamma r_2}+\frac{ky_2\cos\phi_2}{2\gamma r_2}\right\}\\ &\left.+\alpha_{12}\frac{ky_1}{2\gamma}\left\{-\frac{2x_2\sin(2\phi_2)k^2}{8\gamma^2}+\frac{k^2y_2\cos(2\phi_2)}{4\gamma^2}\right\}+\alpha_{21}\frac{k^2y_1^2}{8\gamma^2}\left\{-\frac{x_2\sin\phi_2 k}{2\gamma r_2}+\frac{ky_2\cos\phi_2}{2\gamma r_2}\right\}\right]\\ \approx\ & 4\frac{\mathcal{C}L}{kQ}\left[\frac{\alpha_{00}k^2}{8\gamma^2}y_2+\frac{\alpha_{01}k}{4\gamma}+\frac{\alpha_{01}k^3}{16\gamma^3}y_1^2+\frac{\alpha_{01}k^3}{32\gamma^3}x_2^2+\frac{3\alpha_{01}k^3}{32\gamma^3}y_2^2+\frac{\alpha_{02}k^2}{8\gamma^2}y_2+\frac{\alpha_{03}k^3}{32\gamma^3}y_2^2\right.\\ &\left.-\frac{\alpha_{03}k^3}{32\gamma^3}x_2^2+\frac{\alpha_{10}k^3}{8\gamma^3}y_1y_2+\frac{\alpha_{11}k^2}{4\gamma^2}y_1+\frac{\alpha_{12}k^3}{8\gamma^3}y_1y_2+\frac{\alpha_{21}k^3}{16\gamma^3}y_1^2\right],\end{aligned}$$

using again some trigonometric identities and

$$\cos\phi_2 = \frac{y_2}{r_2},\quad \sin\phi_2 = -\frac{x_2}{r_2},\quad \cos(2\phi_2) = 2\frac{y_2^2}{r_2^2}-1 = \frac{y_2^2-x_2^2}{r_2^2}.$$

Finally our approximation of the total vertical impedance up to second order can be rewritten

$$Z_y^{Wall} \approx \frac{\mathcal{C}L}{\gamma Q}\left[\alpha_{01}+\frac{\alpha_{11}k}{\gamma}y_1+k\frac{\alpha_{00}+\alpha_{02}}{2\gamma}y_2+k^2\frac{\alpha_{01}+\alpha_{21}}{4\gamma^2}y_1^2+k^2\frac{\alpha_{01}-\alpha_{03}}{8\gamma^2}x_2^2\right.$$
$$\left.+k^2\frac{3\alpha_{01}+\alpha_{03}}{8\gamma^2}y_2^2+k^2\frac{\alpha_{10}+\alpha_{12}}{2\gamma^2}y_1y_2\right]. \quad (9.16)$$

9.4 Generalization to a source at $x = x_1$ and $y = y_1$

We can generalize the impedances computed above in the case of a source at $x = x_1$ instead of $x = 0$. From the continuous translation invariance of the problem along the x axis, or in other words the fact that the place of the source along the horizontal axis does not change the problem such that only the difference between the test and the source x coordinates matters, the results can be obtained from those in the above analysis by replacing x_2 with the difference between the two horizontal coordinates $x_2 - x_1$. First of all, for the exact direct space-charge impedances we obtain from Eqs. (9.8), (9.9) and (9.10)

$$Z_{\parallel}^{SC,direct} = -\frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^2} K_0\left(\frac{k}{\gamma}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right), \tag{9.17}$$

$$Z_{x}^{SC,direct} = \frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^3} K_1\left(\frac{k}{\gamma}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right)\frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}, \tag{9.18}$$

$$Z_{y}^{SC,direct} = \frac{j\omega\mu_0 L}{2\pi\beta^2\gamma^3} K_1\left(\frac{k}{\gamma}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right)\frac{y_2-y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}. \tag{9.19}$$

Those expressions are exactly the same as those obtained for an axisynmmetric geometry [6], which was expected since the direct space-charge is the part of the impedance due to the direct interaction between the source and test particles without the mediation of the surrounding structure, so is independent of it. For the second order approximations of the wall impedances we obtain, from Eqs. (9.12), (9.14) and (9.16) and plugging the value of the constant $\mathcal{C}$ from Eq. (6.6):

$$\begin{aligned}Z_{\parallel}^{Wall} \approx \frac{jkZ_0L}{2\pi\beta\gamma^2}\Bigg[&\alpha_{00} + \frac{k\alpha_{10}}{\gamma}y_1 + \frac{k\alpha_{01}}{\gamma}y_2 + k^2\left(\frac{\alpha_{00}-\alpha_{02}}{4\gamma^2}\right)x_1^2 + k^2\left(\frac{\alpha_{00}+\alpha_{20}}{4\gamma^2}\right)y_1^2 \\ &+k^2\left(\frac{\alpha_{00}-\alpha_{02}}{4\gamma^2}\right)x_2^2 + k^2\left(\frac{\alpha_{00}+\alpha_{02}}{4\gamma^2}\right)y_2^2 - k^2\left(\frac{\alpha_{00}-\alpha_{02}}{2\gamma^2}\right)x_1x_2 + \frac{k^2\alpha_{11}}{\gamma^2}y_1y_2\Bigg],\end{aligned} \tag{9.20}$$

$$\begin{aligned}Z_{x}^{Wall} \approx \frac{jk^2Z_0L}{4\pi\beta\gamma^4}\Bigg[&-(\alpha_{00}-\alpha_{02})\,x_1 + (\alpha_{00}-\alpha_{02})\,x_2 - k\frac{\alpha_{10}-\alpha_{12}}{\gamma}x_1y_1 - k\frac{\alpha_{01}-\alpha_{03}}{2\gamma}x_1y_2 \\ &+k\frac{\alpha_{10}-\alpha_{12}}{\gamma}y_1x_2 + k\frac{\alpha_{01}-\alpha_{03}}{2\gamma}x_2y_2\Bigg],\end{aligned} \tag{9.21}$$

$$\begin{aligned}Z_{y}^{Wall} \approx \frac{jkZ_0L}{2\pi\beta\gamma^3}\Bigg[&\alpha_{01} + \frac{\alpha_{11}k}{\gamma}y_1 + k\frac{\alpha_{00}+\alpha_{02}}{2\gamma}y_2 + k^2\frac{\alpha_{01}-\alpha_{03}}{8\gamma^2}x_1^2 + k^2\frac{\alpha_{01}+\alpha_{21}}{4\gamma^2}y_1^2 + k^2\frac{\alpha_{01}-\alpha_{03}}{8\gamma^2}x_2^2 \\ &+k^2\frac{3\alpha_{01}+\alpha_{03}}{8\gamma^2}y_2^2 - k^2\frac{\alpha_{01}-\alpha_{03}}{4\gamma^2}x_1x_2 + k^2\frac{\alpha_{10}+\alpha_{12}}{2\gamma^2}y_1y_2\Bigg].\end{aligned} \tag{9.22}$$

The longitudinal wall impedance is often reduced to its zeroth order (constant) term, which is given by

$$Z_{\parallel}^{Wall,0} = \frac{jkZ_0L}{2\pi\beta\gamma^2}\alpha_{00}. \tag{9.23}$$

Z_x^{Wall} has no constant term due to the left-right symmetry (see Fig. 1) but we can notice that Z_y^{Wall} has a non-zero constant term due to the absence of top-bottom symmetry: the layers at the bottom can be different from those at the top so the electromagnetic vertical force has no reason to be zero for $y = 0$. This term is given by

$$Z_{y}^{Wall,0} = \frac{jkZ_0L}{2\pi\beta\gamma^3}\alpha_{01}. \tag{9.24}$$

Then, the terms usually considered for the transverse impedances are the linear ones, i.e. proportional to x_1, y_1, x_2 or y_2, the coefficient of proportionality being called dipolar impedance (when considering a term proportional to the source particle coordinates) or quadrupolar impedance (when considering a term proportional to the test particle coordinates). We write therefore:

$$Z_x^{Wall,dip} = -\frac{jk^2 Z_0 L}{4\pi\beta\gamma^4}\left(\alpha_{00} - \alpha_{02}\right), \tag{9.25}$$

$$Z_y^{Wall,dip} = \frac{jk^2 Z_0 L}{2\pi\beta\gamma^4}\alpha_{11}, \tag{9.26}$$

$$Z_x^{Wall,quad} = \frac{jk^2 Z_0 L}{4\pi\beta\gamma^4}\left(\alpha_{00} - \alpha_{02}\right), \tag{9.27}$$

$$Z_y^{Wall,quad} = \frac{jk^2 Z_0 L}{4\pi\beta\gamma^4}\left(\alpha_{00} + \alpha_{02}\right). \tag{9.28}$$

We notice here that $Z_x^{Wall,dip} = -Z_x^{Wall,quad}$ which is a direct consequence of the continuous translation invariance along the x axis of the configuration. On the other hand, contrary to usual ultrarelativistic results (see e.g. Ref. [48]), we see that $Z_x^{Wall,quad} \neq -Z_y^{Wall,quad}$, which is due to the term proportional to α_{00} (it is actually similar to the quadrupolar term found for the impedance in an axisymmetric structure [6]).
Finally, one can check without much difficulty that Eq. (9.20) and the linear terms of Z_x^{Wall} and Z_y^{Wall} in Eq. (9.21) and (9.22) are conform with Panofsky-Wenzel theorem as stated in Ref. [47, p. 90], namely

$$kZ_x^{Wall} = \frac{\partial Z_\parallel^{Wall}}{\partial x_2},$$
$$kZ_y^{Wall} = \frac{\partial Z_\parallel^{Wall}}{\partial y_2}.$$

Note that for the direct space-charge part the conformity with Panofsky-Wenzel theorem was already checked in Ref. [6].

10 Some particular cases

We apply here our formalism to several particular configurations.

10.1 General simplifications in case of top-bottom symmetry

Typically flat chamber have a symmetry between the top and bottom parts, which simplifies the analysis. The top-bottom symmetry means in particular that if we replace y_1 by $-y_1$, we should obtain the same results provided we switch the roles of $C_{e+}^{(1)}$ and $C_{e-}^{(-1)}$. Therefore in Eqs. (6.61) we should have

$$\chi_2 = \eta_1 \qquad \text{and} \qquad \eta_2 = \chi_1. \tag{10.1}$$

So we only need to compute the functions η_1 and χ_1. Note that the matrices $\mathcal{M}$ and $\mathcal{M}'$ defined in Eqs. (6.45) and (6.53) are not identical, since $M^{p+1,p} \neq M^{-p-1,-p}$ because $b^{(p)}$ changes sign in Eqs. (6.38) to (6.42) with respect to Eqs. (6.46) to (6.50). So one still needs to compute both $\mathcal{M}$ and $\mathcal{M}'$ to compute χ_1 and η_1.
We can write now a simpler formula for the α_{mn} from Eq. (8.9):

$$\alpha_{mn} = \left[(-1)^{m+n} + 1\right] \int_0^\infty \mathrm{d}u \cosh\left(mu\right) \cosh\left(nu\right) \left[\chi_1\left(\frac{k}{\gamma}\sinh u\right) + (-1)^m \eta_1\left(\frac{k}{\gamma}\sinh u\right)\right]. \tag{10.2}$$

This means in particular that $\alpha_{mn} = 0$ whenever $m+n$ is odd, so in particular $\alpha_{01} = \alpha_{03} = \alpha_{10} = \alpha_{12} = \alpha_{21} = 0$. For the wall linear terms in the beam-coupling impedances, this has for consequence to cancel out the constant term in Z_y^{Wall}, as expected:

$$Z_y^{Wall,0} = 0. \tag{10.3}$$

On the other hand the impedances used most often, namely $Z_{\parallel}^{Wall,0}$ from Eq. (9.23) and the dipolar and quadrupolar transverse impedances from Eqs. (9.25) to (9.28) remain with the same expressions.
More generally the second order approximation of the longitudinal and transverse total wall impedance become:

$$Z_{\parallel}^{Wall} \approx \frac{jkZ_0L}{2\pi\beta\gamma^2}\left[\alpha_{00} + k^2\left(\frac{\alpha_{00}-\alpha_{02}}{4\gamma^2}\right)x_1^2 + k^2\left(\frac{\alpha_{00}+\alpha_{20}}{4\gamma^2}\right)y_1^2 + k^2\left(\frac{\alpha_{00}-\alpha_{02}}{4\gamma^2}\right)x_2^2 \right. \\ \left. + k^2\left(\frac{\alpha_{00}+\alpha_{02}}{4\gamma^2}\right)y_2^2 - k^2\left(\frac{\alpha_{00}-\alpha_{02}}{2\gamma^2}\right)x_1x_2 + \frac{k^2\alpha_{11}}{\gamma^2}y_1y_2\right], \tag{10.4}$$

$$Z_x^{Wall} \approx \frac{jk^2Z_0L}{4\pi\beta\gamma^4}\left[-\left(\alpha_{00}-\alpha_{02}\right)x_1 + \left(\alpha_{00}-\alpha_{02}\right)x_2\right], \tag{10.5}$$

$$Z_y^{Wall} \approx \frac{jk^2Z_0L}{4\pi\beta\gamma^4}\left[2\alpha_{11}y_1 + \left(\alpha_{00}+\alpha_{02}\right)y_2\right], \tag{10.6}$$

where we see that the nonlinear terms in the transverse impedances are at least of third order.

10.2 Case of two perfectly conducting plates

Now we consider that at $y = \pm b$ we have plates of infinite conductivity. In this case (where the results above on top-bottom symmetry apply), considering first a given k_x, we must have $\vec{E} = \vec{0}$ all along the plates so in particular, from Eqs. (8.1):

$$C_{e+}^{(1)}e^{k_y^{(1)}b} + C_{e-}^{(-1)}e^{-k_y^{(1)}b} + \frac{\mathcal{C}}{k_y^{(1)}}e^{k_y^{(1)}(y_1-b)} = 0,$$
$$C_{e+}^{(1)}e^{-k_y^{(1)}b} + C_{e-}^{(-1)}e^{k_y^{(1)}b} + \frac{\mathcal{C}}{k_y^{(1)}}e^{k_y^{(1)}(-b-y_1)} = 0,$$

or, in matrix form:

$$\begin{bmatrix} e^{k_y^{(1)}b} & e^{-k_y^{(1)}b} \\ e^{-k_y^{(1)}b} & e^{k_y^{(1)}b}\end{bmatrix}\cdot\begin{bmatrix} C_{e+}^{(1)} \\ C_{e-}^{(-1)}\end{bmatrix} = -\frac{\mathcal{C}}{k_y^{(1)}}\begin{bmatrix} e^{k_y^{(1)}(y_1-b)} \\ e^{-k_y^{(1)}(y_1+b)}\end{bmatrix}.$$

Using the inversion formula of a 2×2 matrix (see Appendix E) we obtain for $C_{e+}^{(1)}$:

$$C_{e+}^{(1)} = -\frac{\mathcal{C}}{2\sinh\left(2k_y^{(1)}b\right)k_y^{(1)}}\left[e^{k_y^{(1)}y_1} - e^{-k_y^{(1)}(y_1+2b)}\right],$$

which gives for χ_1 and η_1, from Eq. (6.61) and Eq. (6.4)

$$\chi_1 = \frac{1}{2\sinh\left(2b\sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)} \qquad \text{and} \qquad \eta_1 = \frac{-e^{-2b\sqrt{k_x^2+\frac{k^2}{\gamma^2}}}}{2\sinh\left(2b\sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)}, \tag{10.7}$$

which can be plugged into Eq. (10.2):

$$\alpha_{mn}^{PC} = \left[(-1)^{m+n}+1\right]\int_0^\infty \mathrm{d}u\frac{\cosh(mu)\cosh(nu)}{2\sinh\left(2\frac{kb}{\gamma}\cosh u\right)}\left[1-(-1)^m e^{-2\frac{kb}{\gamma}\cosh u}\right], \tag{10.8}$$

where the superscript PC stands for "perfect conductor". Then, the linear part of the wall impedances for such perfectly conducting plates (also called indirect space-charge) will be given by Eq. (9.23) and Eqs. (9.25) to (9.28).
We can check that the longitudinal electric field (and therefore the electromagnetic force and impedances)

given by our approach would have been identical if we had used instead the method of images [49, 50] to impose the boundary conditions. This method consists in placing several sources (i.e. point-like charge and current density as in Eqs. (2.4) and (2.5)) such as to enforce automatically the boundary condition $E_s = 0$ on the plates at $y = \pm b$, which is the only condition needed to get the longitudinal component of the electric field. The basic idea is then that if one such source is at $x = 0$ and $y = y_1$ and another one with an opposite charge $-Q$ is at $x = 0$ and $y = 2b - y_1$ (i.e. symmetrically placed with respect to the plate at $y = b$), all of this being in vacuum, then we get automatically $E_s = 0$ on the plate at $y = b$. This can be seen using Eq. (8.3) which represents E_s created by the first of those source in vacuum: from the superposition principle the longitudinal component of the electric field created by those two sources would be

$$E_s = \mathcal{C}e^{-jks}\left[K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-y_1)^2}\right) - K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y+y_1-2b)^2}\right)\right],$$

which is obviously zero for $y = b$[12]. Then, to get $E_s = 0$ at $y = -b$ one needs as well to symmetrize in the same way the two sources previously mentioned, with respect to the plane $y = -b$, obtaining two new sources, one of charge $-Q$ at $y = -2b - y_1$ and one of charge Q at $y = -4b + y_1$. Those again should have their symmetric counterparts with respect to the first plate, thus we need two more images on the upper part, and we repeat the process to get an infinite number of sources farther and farther away from the plates, such that in the end the overall sum converges to the solution with the desired boundary condition $E_s = 0$ for $y = \pm b$. The images arrangement is illustrated in Fig. 2: we get in the end sources of charge Q at positions $4lb + y_1$ and sources of charge $-Q$ at positions $2(2l+1)b - y_1$ (l being a positive of negative integer).

Finally the total longitudinal component of the electric field will be given by the sum of E_s from all the sources, each being obtained thanks to Eq. (8.3):

$$\begin{aligned}
E_{s,tot}^{PC,images} &= \mathcal{C}e^{-jks}\left[\sum_{l=-\infty}^{\infty} K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-4lb-y_1)^2}\right) - \sum_{l=-\infty}^{\infty} K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-2b(2l+1)+y_1)^2}\right)\right] \\
&= \mathcal{C}e^{-jks}\left[\sum_{l=-\infty}^{\infty}(-1)^{2l}K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-2b(2l)-(-1)^{2l}y_1)^2}\right)\right. \\
&\qquad \left. + \sum_{l=-\infty}^{\infty}(-1)^{2l+1}K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-2b(2l+1)-(-1)^{2l+1}y_1)^2}\right)\right] \\
&= \mathcal{C}e^{-jks}\sum_{l=-\infty}^{\infty}(-1)^l K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-2lb-(-1)^l y_1)^2}\right). \qquad (10.9)
\end{aligned}$$

[12] More general arguments exist [49, 50] to prove that E_s (and even the tangential component) is zero at the plane equidistant to two such sources. For instance it can be seen in time domain (and therefore in frequency domain from the linearity of the Fourier transform) by changing the reference frame to the rest frame of the two point-like sources (which go at the same velocity v). Upon application of Coulomb's law in the rest frame for each source, one can clearly see that the tangential component on the plate is zero after summation of the two Coulomb's fields. Then, using the Lorentz transform of the fields one would directly obtain the result.

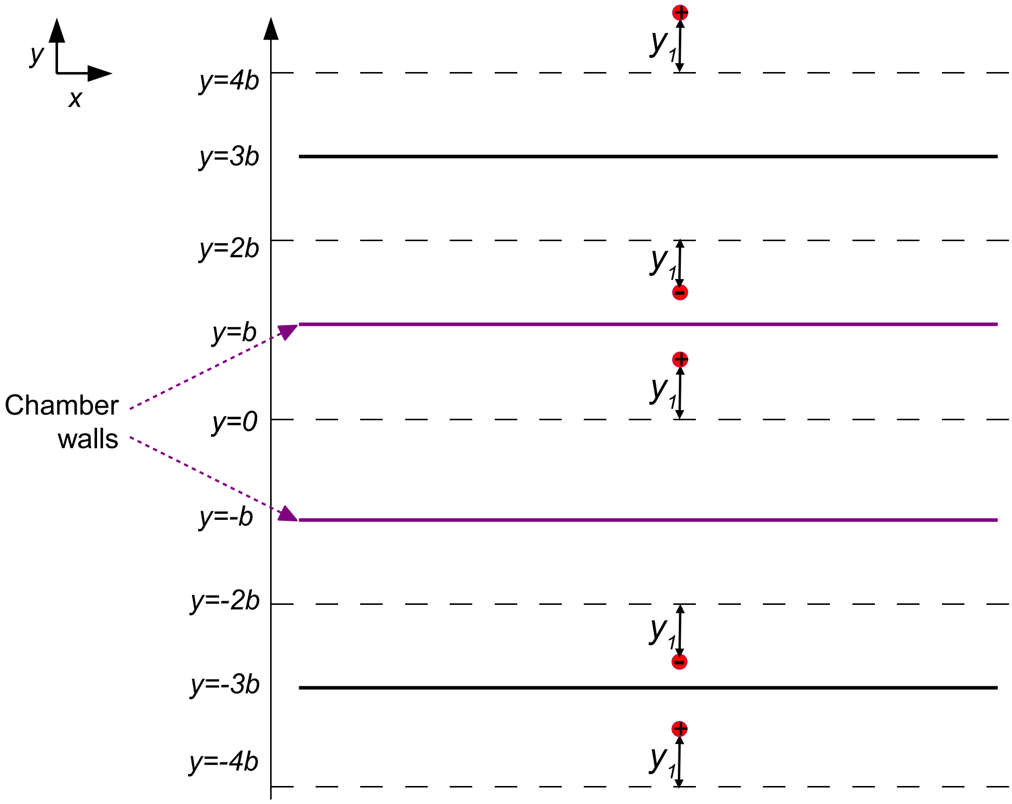

Figure 2: Several of the electric images used to impose the boundary condition $E_s = 0$ on the plates at $y = \pm b$. Images with a plus sign denotes a charge of Q while those with a minus sign have a charge of $-Q$.

Now, we would like to compare this expression with the one obtained from the approach developped throughout the paper. To do so we first rewrite the α_{mn} constants from Eq. (10.8) as

$$\begin{aligned}
\alpha_{mn}^{PC} &= \left[(-1)^{m+n}+1\right]\int_0^\infty \mathrm{d}u \frac{\cosh(mu)\cosh(nu)}{e^{2\frac{kb}{\gamma}\cosh u}\left(1-e^{-4\frac{kb}{\gamma}\cosh u}\right)}\left[1-(-1)^m e^{-2\frac{kb}{\gamma}\cosh u}\right]\\
&= \frac{(-1)^{m+n}+1}{2}\int_0^\infty \mathrm{d}u \Bigg\{\left[\cosh\{(m+n)u\}+\cosh\{(m-n)u\}\right]e^{-2\frac{kb}{\gamma}\cosh u}\sum_{l=0}^{\infty}e^{-4l\frac{kb}{\gamma}\cosh u}\\
&\qquad \left[1-(-1)^m e^{-2\frac{kb}{\gamma}\cosh u}\right]\Bigg\} \qquad \text{using a geometric series decomposition of } \frac{1}{1-e^{-4\frac{kb}{\gamma}\cosh u}}\\
&= \frac{(-1)^{m+n}+1}{2}\int_0^\infty \mathrm{d}u \Bigg\{\left[\cosh\{(m+n)u\}+\cosh\{(m-n)u\}\right]\Bigg[\sum_{l=0}^{\infty}(-1)^{(m+1)(2l+2)}e^{-2(2l+1)\frac{kb}{\gamma}\cosh u}\\
&\qquad +\sum_{l=0}^{\infty}(-1)^{(m+1)(2l+3)}e^{-2(2l+2)\frac{kb}{\gamma}\cosh u}\Bigg]\Bigg\}\\
&= \frac{(-1)^{m+n}+1}{2}\int_0^\infty \mathrm{d}u\left[\cosh\{(m+n)u\}+\cosh\{(m-n)u\}\right]\left[\sum_{l=1}^{\infty}(-1)^{(m+1)(l+1)}e^{-2l\frac{kb}{\gamma}\cosh u}\right]\\
&= \frac{(-1)^{m+n}+1}{2}\sum_{l=1}^{\infty}(-1)^{(m+1)(l+1)}\int_0^\infty \mathrm{d}u\left[\cosh\{(m+n)u\}+\cosh\{(m-n)u\}\right]e^{-2l\frac{kb}{\gamma}\cosh u}\\
&= \frac{(-1)^{m+n}+1}{2}\sum_{l=1}^{\infty}(-1)^{(m+1)(l+1)}\left[K_{m+n}\left(2l\frac{kb}{\gamma}\right)+K_{m-n}\left(2l\frac{kb}{\gamma}\right)\right],
\end{aligned}$$

where we used Eq. (C.18) for the last step. Reinserting this into Eq. (8.8), adding also the direct space-charge part from Eq. (8.3), we get, with $\phi = \theta - \frac{\pi}{2}$:

$$
\begin{aligned}
E_{s,tot}^{PC} &= \mathcal{C}e^{-jks}\left\{K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-y_1)^2}\right) - 2\sum_{l=1}^{\infty}\sum_{m,n=0}^{\infty}(-1)^{l+1}\frac{\left((-1)^{m(l+1)}+(-1)^{ml+n}\right)\cos(n\phi)}{(1+\delta_{m0})(1+\delta_{n0})}\right.\\
&\qquad \left. I_m\left(\frac{ky_1}{\gamma}\right)I_n\left(\frac{kr}{\gamma}\right)\left[K_{m+n}\left(2l\frac{kb}{\gamma}\right)+K_{m-n}\left(2l\frac{kb}{\gamma}\right)\right]\right\}\\
&= \mathcal{C}e^{-jks}\left\{K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-y_1)^2}\right) + 2\sum_{l=1}^{\infty}(-1)^l\sum_{n=0}^{\infty}\frac{\cos(n\phi)}{(1+\delta_{n0})}I_n\left(\frac{kr}{\gamma}\right)\right.\\
&\qquad \left.\sum_{m=0}^{\infty}\frac{(-1)^{m(l+1)}+(-1)^{ml+n}}{1+\delta_{m0}}I_m\left(\frac{ky_1}{\gamma}\right)\left[K_{m+n}\left(2l\frac{kb}{\gamma}\right)+K_{-m+n}\left(2l\frac{kb}{\gamma}\right)\right]\right\} \quad \text{using Eq. (C.4)}\\
&= \mathcal{C}e^{-jks}\left\{K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-y_1)^2}\right) + 2\sum_{l=1}^{\infty}(-1)^l\sum_{n=0}^{\infty}\frac{\cos(n\phi)}{(1+\delta_{n0})}I_n\left(\frac{kr}{\gamma}\right)\right.\\
&\qquad \left.\sum_{m=-\infty}^{\infty}\left[(-1)^{m(l+1)}+(-1)^{ml+n}\right]I_m\left(\frac{ky_1}{\gamma}\right)K_{m+n}\left(2l\frac{kb}{\gamma}\right)\right\} \qquad \text{using Eq. (C.3)}\\
&= \mathcal{C}e^{-jks}\left\{K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-y_1)^2}\right) + 2\sum_{l=1}^{\infty}(-1)^l\sum_{n=0}^{\infty}\frac{\cos(n\phi)}{(1+\delta_{n0})}I_n\left(\frac{kr}{\gamma}\right)\left[K_n\left(\frac{k}{\gamma}\left(2lb+(-1)^l y_1\right)\right)\right.\right.\\
&\qquad \left.\left.+(-1)^n K_n\left(\frac{k}{\gamma}\left(2lb+(-1)^{l+1}y_1\right)\right)\right]\right\} \qquad \text{from Eq. (C.16), since } |y_1|<2lb \text{ for } l\geq 1\\
&= \mathcal{C}e^{-jks}\left\{K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-y_1)^2}\right)\right.\\
&\qquad +\sum_{l=1}^{\infty}(-1)^l\left[K_0\left(\frac{k}{\gamma}\sqrt{r^2+(2lb+(-1)^l y_1)^2-2r(2lb+(-1)^l y_1)\cos\phi}\right)\right.\\
&\qquad\qquad \left.\left.+K_0\left(\frac{k}{\gamma}\sqrt{r^2+(2lb+(-1)^{l+1}y_1)^2+2r(2lb+(-1)^{l+1}y_1)\cos\phi}\right)\right]\right\}\\
&\qquad \text{from Eq. (C.15), since } 0<r<b<2lb\pm y_1 \text{ for } l\geq 1\\
&= \mathcal{C}e^{-jks}\left\{K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-y_1)^2}\right)+\sum_{l=1}^{\infty}(-1)^l\left[K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-2lb-(-1)^l y_1)^2}\right)\right.\right.\\
&\qquad \left.\left.+K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y+2lb-(-1)^l y_1)^2}\right)\right]\right\} \qquad \text{from } y=r\cos\phi \text{ and } r^2=x^2+y^2\\
&= \mathcal{C}e^{-jks}\sum_{l=-\infty}^{\infty}(-1)^l K_0\left(\frac{k}{\gamma}\sqrt{x^2+(y-2lb-(-1)^l y_1)^2}\right).
\end{aligned} \tag{10.10}
$$

This is exactly the result in Eq. (10.9) obtained with the method of images.

10.3 A note on the case of a single plate

The case of a single plate is included into our formalism and can be treated with the same equations as those for a flat chamber with two plates. If for instance there is a plate only on the top part, we simply notice that $M = 1$ (number of layers in the lower part of the chamber) and that the matrix $\mathcal{M}'$ in Eq. (6.53) is the identity matrix, as can be readily seen in Eq. (6.52).

11 Conclusion

This paper used a general approach to give an as complete and detailed as possible derivation of the electromagnetic fields and impedances created by an offset point charge travelling at any speed in a

multilayer flat chamber, infinite along the charge direction of movement and along the horizontal direction. The basic assumptions of the derivation are the geometry of the chamber and the linearity of the wall materials together with the validity of local Ohm's law (thus neglecting magnetoresitance and the anomalous skin effect).
New results were obtained within our very general assumptions: the beam-coupling impedances up to any order can be computed thanks to closed-form formulae, involving a general matrix formalism for the field matching between adjacent layers and final integrals that can be computed numerically. In particular, one can compute generalized form factors between the impedances in an axisymmetric structure and the impedances computed here for a flat chamber, and compare them with the limiting case of the Yokoya factors [8], or with other aproaches. Such comparisons are shown in Refs. [51, 52].
In this study some elementary verifications were done, in particular the direct space-charge impedances are the same as the ones for an axisymmetric structure, the longitudinal electric field for two perfectly conducting plates is the same as the one obtained througth the method of images, and Panofsky-Wenzel theorem was checked on our resulting impedances.
In short, to apply this formalism for the geometry shown in Fig. 1 the way to proceed is to first use the formulae in Section 6.2, namely Eqs. (6.38) to (6.42), Eq. (6.45), Eqs. (6.46) to (6.50) and Eq. (6.53), with the definitions from Eqs. (3.5), (3.6), (4.11) and (6.21). This gives two 4×4 matrices $\mathcal{M}$ and $\mathcal{M}'$, each depending on the horizontal wave number k_x, from which we can construct the matrix $\mathcal{P}$ on the left hand side of Eq. (6.57), that we finally use in Eqs. (6.60) to compute the $\eta_{1,2}$ and $\chi_{1,2}$ functions. The latter are then plugged into Eq. (8.9) to get the α_{mn} constants, before computing the beam-coupling impedances from Eqs. (9.20) to (9.22), or (9.23) to (9.28) for the first linear terms. In the case of a structure with top-bottom symmetry one can use the results of Section 10.1 to reduce the number of computations to be performed.

Acknowledgements

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A Appendix A: Vector operations in cartesian coordinates

The following formulas can be found in many textbooks of mechanics or electrodynamics, and in particular in Refs. [42, p. 452] and [11, cover page] as well as in Ref. [53].

A.1 Gradient

For any scalar field f, the gradient in cartesian coordinates (x, y, s) (the basis unit vectors being $\vec{e_x}$, $\vec{e_y}$ and $\vec{e_s}$) is given by

$$\vec{\mathrm{grad}} f = \frac{\partial f}{\partial x}\vec{e_x} + \frac{\partial f}{\partial y}\vec{e_y} + \frac{\partial f}{\partial s}\vec{e_s}. \tag{A.1}$$

A.2 Divergence

For any vector field $\vec{A}$, the divergence in cartesian coordinates (x, y, s) is given by

$$\mathrm{div}\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_s}{\partial s}. \tag{A.2}$$

A.3 Curl

For any vector field $\vec{A}$, the curl in cartesian coordinates (x, y, s) is given by

$$\begin{aligned} \left(\vec{\mathrm{curl}}\vec{A}\right)_x &= \frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}, \\ \left(\vec{\mathrm{curl}}\vec{A}\right)_y &= \frac{\partial A_x}{\partial s} - \frac{\partial A_s}{\partial x}, \\ \left(\vec{\mathrm{curl}}\vec{A}\right)_s &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \end{aligned} \tag{A.3}$$

A.4 Scalar laplacian

For any scalar field f, the laplacian in cartesian coordinates (x, y, s) is given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial s^2}. \tag{A.4}$$

A.5 Vector laplacian

For any vector field $\vec{A}$, the vector laplacian in cartesian coordinates (x, y, s) is given by

$$\begin{aligned} \left(\nabla^2\vec{A}\right)_x &= \nabla^2 A_x, \\ \left(\nabla^2\vec{A}\right)_y &= \nabla^2 A_y, \\ \left(\nabla^2\vec{A}\right)_s &= \nabla^2 A_s. \end{aligned} \tag{A.5}$$

B Appendix B: Various relations between vector operations

From for instance Ref. [11, cover page], we know that

$$\vec{\mathrm{curl}}\left(\vec{\mathrm{curl}}\right) = \vec{\mathrm{grad}}\,(\mathrm{div}) - \nabla^2, \tag{B.1}$$

$$\mathrm{div}\left(\vec{\mathrm{curl}}\right) = 0. \tag{B.2}$$

C Appendix C: Various properties of the modified Bessel functions

Unless stated otherwise, in all the following ν and z are complex numbers while m is an integer.

C.1 Definitions

Modified Bessel functions $I_\nu(z)$ (first kind) and $K_\nu(z)$ (second kind) are independent solutions of the differential equation [54, p. 374]

$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} - \left(z^2 + \nu^2\right) y = 0. \tag{C.1}$$

The Bessel function of the first kind $J_\nu(z)$ is a solution of [54, p. 358]

$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \nu^2\right) y = 0. \tag{C.2}$$

C.2 General properties

From Ref. [54, p. 375-376], we have the following relations between the modified Bessel functions (z^* stands for the complex conjugate of z)

$$I_{-m}(z) = I_m(z), \tag{C.3}$$

$$K_{-m}(z) = K_m(z), \tag{C.4}$$

$$K_0'(z) = -K_1(z), \tag{C.5}$$

$$J_m(jz) = j^m I_m(z) \quad \text{for} \quad -\pi < \arg(z) \leq \frac{\pi}{2}, \tag{C.6}$$

$$I_\nu\left(ze^{jm\pi}\right) = e^{j\nu m\pi} I_\nu(z). \tag{C.7}$$

From Ref. [54, p. 361] we have the additional relation

$$J_\nu\left(ze^{jm\pi}\right) = e^{j\nu m\pi} J_\nu(z). \tag{C.8}$$

C.3 Expansion for small arguments

Reference [54, p. 375] gives expansions for small arguments. Assuming m integer with $m \geq 0$, we have

$$I_m(z) = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{k!(m+k)!}, \tag{C.9}$$

$$I_m(z) \sim_{|z|\to 0} \frac{\left(\frac{1}{2}z\right)^m}{m!}, \tag{C.10}$$

where $m!$ is the factorial of the integer m.

C.4 Summation formulae

From Ref. [54, p. 376] we have

$$e^{\frac{z}{2}\left(t+\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} t^m I_m(z) \quad \text{with } t \text{ non zero}. \tag{C.11}$$

We can use it to derive two other summation formulae: using Eq. (C.3)

$$e^{z\cosh u} = e^{\frac{z}{2}\left(e^u + \frac{1}{e^u}\right)} = \sum_{m=-\infty}^{\infty} e^{mu} I_m(z) = I_0(z) + 2\sum_{m=1}^{\infty} \cosh(mu) I_m(z), \tag{C.12}$$

and, thanks to (C.7)

$$e^{-z\cosh u} = I_0(z) + 2\sum_{m=1}^{\infty}(-1)^m \cosh(mu) I_m(z). \tag{C.13}$$

Another useful relation is obtained from Ref. [55, p. 102]: for any complex numbers ϕ, ν, z_1 and z_2 such that $|z_1 e^{\pm j\phi}| < |z_2|$ we have

$$\sum_{m=-\infty}^{\infty} I_m(z_1) K_{\nu+m}(z_2) e^{jm\phi} = K_\nu\left(\sqrt{z_1^2 + z_2^2 - 2z_1z_2\cos\phi}\right)\left(\frac{z_2 - z_1 e^{-j\phi}}{\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}}\right)^{\nu}. \tag{C.14}$$

This in particular gives for $\nu = 0$, $0 < z_1 < z_2$ and ϕ real numbers, taking only the real part of the formula, and recalling Eqs. (C.3) and (C.4)

$$\sum_{m=0}^{\infty} \frac{1}{1+\delta_{m0}} I_m(z_1) K_m(z_2) \cos(m\phi) = \frac{1}{2} K_0\left(\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi}\right), \tag{C.15}$$

where $\delta_{m0} = 1$ if $m = 0$, 0 otherwise. We can also write Eq. (C.14) in the case when $\nu = n$, $|z_1| < z_2$ and $\phi = p\pi$ where n and p are integers, obtaining

$$\sum_{m=-\infty}^{\infty} I_m(z_1) K_{m+n}(z_2)(-1)^{mp} = K_n\left(z_2 + (-1)^{p+1} z_1\right). \tag{C.16}$$

C.5 Integral formulae

The following integral formula can be found in Ref. [56, p. 17]

$$\int_0^{\infty} \mathrm{d}k_x \, \cos(k_x x) \frac{e^{-b\sqrt{k_x^2+a^2}}}{\sqrt{k_x^2+a^2}} = K_0\left(a\sqrt{b^2+x^2}\right) \qquad \text{with } \Re(a) > 0 \text{ and } \Re(b) > 0. \tag{C.17}$$

Another useful integral representation is [54, p. 376]

$$K_\nu(z) = \int_0^{\infty} \mathrm{d}t \, e^{-z\cosh t}\cosh(\nu t) \qquad \text{with } |\arg(z)| < \frac{\pi}{2}. \tag{C.18}$$

C.6 A generalization of Schläfli's integrals

References [55, p. 82] and [57, p. 903] give an analytical expression for the following sum of integrals given for x, y complex numbers and ν real[13]:

$$\int_0^{\pi} e^{y\cos t}\cos(x\sin t - \nu t)\,\mathrm{d}t - \sin\nu\pi \int_0^{\infty} e^{-\nu t - y\cosh t - x\sinh t}\mathrm{d}t \qquad \text{with } \Re(x+y) > 0. \tag{C.19}$$

The expressions given in both references are different: according to Ref. [55, p. 82] the result should be

$$\pi\left(\frac{x+y}{x-y}\right)^{\frac{\nu}{2}} J_\nu\left[\left(x^2-y^2\right)^{\frac{1}{2}}\right].$$

In Ref. [57, p. 903] we find on the contrary that this sum of integrals should be

$$\pi\,(x+y)^{\nu}\,\frac{J_\nu\left[\left(x^2+y^2\right)^{\frac{1}{2}}\right]}{\left(x^2-y^2\right)^{\frac{\nu}{2}}}.$$

When comparing those formulae with numerical computations of the integrals (where the one on an infinite range converges very fast so is not computationally intensive), it turns out that both the above

[13] In Ref. [57] there is already an error in the upper bound of the first integral, which is taken as ∞ instead of π.

formulae seem incorrect. In particular, the first analytical expression above is not even continuous with x when going from $x < y$ to $x > y$ for certain values of ν. Therefore we rederive here the formula, following step by step the derivation given in Ref. [58].
The starting point is the following generalization of Bessel's integral due to Schläfli, which can be found in equivalent forms in Refs. [54, p. 360], [57, p. 903], [58] and [59, p. 176]:

$$\text{For } \Re(a) > 0\ ,\ J_\nu(a) = \frac{1}{\pi}\int_0^\pi \cos\left(\nu t - a\sin t\right)\mathrm{d}t - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu t - a\sinh t}\mathrm{d}t. \tag{C.20}$$

Then, for any y complex and t real, we have the general formula

$$\sum_{n=0}^\infty \frac{y^n}{n!}e^{jnt} = \sum_{n=0}^\infty \frac{\left(ye^{jt}\right)^n}{n!} = e^{ye^{jt}} = e^{y\cos t}\left[\cos\left(y\sin t\right) + j\sin\left(y\sin t\right)\right].$$

Using this relation for both t and $-t$ and adding (respectively subtracting) the results, we get

$$\sum_{n=0}^\infty \frac{y^n}{n!}\cos(nt) = e^{y\cos t}\cos\left(y\sin t\right),$$
$$\sum_{n=0}^\infty \frac{y^n}{n!}\sin(nt) = e^{y\cos t}\sin\left(y\sin t\right).$$

From these and the relation

$$\cos\left[\left(\nu+n\right)t - a\sin t\right] = \cos\left(\nu t - a\sin t\right)\cos\left(nt\right) - \sin\left(\nu t - a\sin t\right)\sin\left(nt\right),$$

we obtain

$$\begin{aligned}\sum_{n=0}^\infty \frac{y^n}{n!}\cos\left[(\nu+n)t - a\sin t\right] &= e^{y\cos t}\left[\cos\left(y\sin t\right)\cos\left(\nu t - a\sin t\right) - \sin\left(y\sin t\right)\sin\left(\nu t - a\sin t\right)\right]\\ &= e^{y\cos t}\cos\left[\nu t - (a-y)\sin t\right].\end{aligned} \tag{C.21}$$

We also have

$$\begin{aligned}\sum_{n=0}^\infty \frac{y^n}{n!}\sin\left[\left(\nu+n\right)\pi\right]e^{-nt} &= \sum_{n=0}^\infty \frac{\left(-ye^{-t}\right)^n}{n!}\sin\left(\nu\pi\right)\\ &= e^{-ye^{-t}}\sin\left(\nu\pi\right),\end{aligned}$$

such that

$$\begin{aligned}\sum_{n=0}^\infty \frac{y^n}{n!}\sin\left[\left(\nu+n\right)\pi\right]e^{-nt-\nu t-a\sinh t} &= \sin\left(\nu\pi\right)e^{-y(\cosh t - \sinh t) - \nu t - a\sinh t}\\ &= \sin\left(\nu\pi\right)e^{-\nu t - y\cosh t - (a-y)\sinh t}.\end{aligned} \tag{C.22}$$

Thanks to Eqs. (C.20), (C.21) and (C.22) we have upon integration

$$\begin{aligned}&\int_0^\pi e^{y\cos t}\cos\left[\nu t - (a-y)\sin t\right]\mathrm{d}t - \sin\left(\nu\pi\right)\int_0^\infty e^{-\nu t - y\cosh t - (a-y)\sinh t}\mathrm{d}t\\ &= \sum_{n=0}^\infty \frac{y^n}{n!}\int_0^\pi \cos\left[(\nu+n)t - a\sin t\right]\mathrm{d}t - \sum_{n=0}^\infty \frac{y^n}{n!}\sin\left[\left(\nu+n\right)\pi\right]\int_0^\infty e^{-nt-\nu t - a\sinh t}\mathrm{d}t\\ &= \pi\sum_{n=0}^\infty \frac{y^n}{n!}J_{\nu+n}(a) \qquad \text{for } \Re(a) > 0.\end{aligned} \tag{C.23}$$

The latter sum can be computed thanks to Ref. [59, p. 141], giving

$$\sum_{n=0}^\infty \frac{y^n}{n!}J_{\nu+n}(a) = \left(1 - \frac{2y}{a}\right)^{-\frac{\nu}{2}} J_\nu\left(a\sqrt{1 - \frac{2y}{a}}\right). \tag{C.24}$$

Note that Eq. (C.24) is not exactly the same as the correspondant formula in Ref. [58]: in the latter the argument of the Bessel function has been replaced by $\sqrt{a^2 - 2ay}$ which turns out not to be same when $\arg a + \arg\left(1 - \frac{2y}{a}\right) > \pi$ or $\arg a + \arg\left(1 - \frac{2y}{a}\right) \leq -\pi$ as we know from e.g. Ref. [54, p. 70]. Here only a is restricted to have an argument between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, but not $1 - \frac{2y}{a}$.
Combining Eqs. (C.23) and (C.24) and letting $x = a - y$ we get

$$\int_0^\pi e^{y\cos t}\cos\left(x\sin t - \nu t\right)\mathrm{d}t - \sin\nu\pi\int_0^\infty e^{-\nu t - y\cosh t - x\sinh t}\mathrm{d}t = \pi\left(\frac{x-y}{x+y}\right)^{-\frac{\nu}{2}} J_\nu\left[(x+y)\left(\frac{x-y}{x+y}\right)^{\frac{1}{2}}\right], \qquad \text{valid for } \Re(x+y) > 0. \quad \text{(C.25)}$$

There are two differences between Eq. (C.25) and the equation given in Ref. [55, p. 82]. The first one concerns the argument of the Bessel function, as we already pointed out. The second one is in the way the factor $\left(\frac{x-y}{x+y}\right)^{-\frac{\nu}{2}}$ is written. It was replaced by $\left(\frac{x+y}{x-y}\right)^{\frac{\nu}{2}}$ which does not give an identical value in some cases, for the same reason as already mentioned above: from Ref. [54, p. 70] $a^\nu\left(\frac{1}{a}\right)^\nu \neq 1$ if $\arg a = \pi$ (because then $\arg\frac{1}{a} = \pi$ so $\arg a + \arg\frac{1}{a} > \pi$).
One can get a slightly more compact form after some additional algebra. Letting $x - y = ae^{j\theta}$ and $x + y = be^{j\phi}$ with $a > 0$, $b > 0$, $-\pi < \theta \leq \pi$ and $-\pi < \phi \leq \pi$, we write

$$(x+y)\sqrt{\frac{x-y}{x+y}} = (x+y)e^{\frac{1}{2}\ln\left(\frac{x-y}{x+y}\right)} = be^{j\phi}e^{\frac{1}{2}(\ln a - \ln b + j\theta - j\phi + 2jm\pi)} = \sqrt{ab}e^{\frac{j}{2}(\theta+\phi)}e^{jm\pi},$$

with m an integer such that $m = 0$ if $-\pi < \theta - \phi \leq \pi$, $m = 1$ if $\theta - \phi \leq -\pi$ and $m = -1$ if $\theta - \phi > \pi$. Similarly, we have

$$\sqrt{x^2 - y^2} = e^{\frac{1}{2}\ln[(x-y)(x+y)]} = \sqrt{ab}e^{\frac{j}{2}(\theta+\phi)}e^{jn\pi} = (x+y)\sqrt{\frac{x-y}{x+y}}e^{j(n-m)\pi},$$

with n an integer such that $n = 0$ if $-\pi < \theta + \phi \leq \pi$, $n = 1$ if $\theta + \phi \leq -\pi$ and $n = -1$ if $\theta + \phi > \pi$. Therefore, using Eq. (C.8) we can write

$$J_\nu\left(\sqrt{x^2 - y^2}\right) = J_\nu\left[(x+y)\left(\frac{x-y}{x+y}\right)^{\frac{1}{2}}\right]e^{j\nu(n-m)\pi}.$$

Now, we also have, with the same m and n defined above:

$$\left(\frac{x-y}{x+y}\right)^{-\frac{\nu}{2}} = e^{-\frac{\nu}{2}\ln\left(\frac{x-y}{x+y}\right)} = \left(\frac{b}{a}\right)^{\frac{\nu}{2}} e^{j\frac{\nu}{2}(\phi-\theta)}e^{-j\nu m\pi}$$

and

$$\frac{(x+y)^\nu}{(x^2-y^2)^{\frac{\nu}{2}}} = \frac{b^\nu e^{j\nu\phi}}{(ab)^{\frac{\nu}{2}}e^{j\frac{\nu}{2}(\theta+\phi)}e^{j\nu n\pi}} = \left(\frac{b}{a}\right)^{\frac{\nu}{2}} e^{j\frac{\nu}{2}(\phi-\theta)}e^{-j\nu n\pi} = \left(\frac{x-y}{x+y}\right)^{-\frac{\nu}{2}} e^{j\nu(m-n)\pi}.$$

We finally obtain:

$$J_\nu\left[(x+y)\left(\frac{x-y}{x+y}\right)^{\frac{1}{2}}\right]\left(\frac{x-y}{x+y}\right)^{-\frac{\nu}{2}} = J_\nu\left(\sqrt{x^2-y^2}\right)\frac{(x+y)^\nu}{(x^2-y^2)^{\frac{\nu}{2}}},$$

such that we can write an equivalent form of Eq. (C.25)

$$\int_0^\pi e^{y\cos t}\cos\left(x\sin t - \nu t\right)\mathrm{d}t - \sin\nu\pi\int_0^\infty e^{-\nu t - y\cosh t - x\sinh t}\mathrm{d}t = \pi J_\nu\left(\sqrt{x^2-y^2}\right)\frac{(x+y)^\nu}{(x^2-y^2)^{\frac{\nu}{2}}}, \qquad \text{valid for } \Re(x+y) > 0. \quad \text{(C.26)}$$

This relation, which is a generalization of Schläfli's integrals, is quite similar to Eq. 8.413 in Ref. [57, p. 903] that we have quoted above, but according to our derivation there seems to be a sign error in the argument of the Bessel function, in addition to the already mentioned incorrect upper limit of the first integral.
From this equation we would like to compute the following integral, used in the Section 8.2:

$$\int_0^\pi \mathrm{d}\phi \cos(n\phi) \cos(k_x r \sin\phi)\, e^{r\cos\phi\sqrt{k_x^2+\frac{k^2}{\gamma^2}}}.$$

To do so we first write

$$\cos(n\phi)\cos(k_x r\sin\phi) = \frac{1}{2}\left[\cos(k_x r\sin\phi + n\phi) + \cos(k_x r\sin\phi - n\phi)\right].$$

For the first term we can apply Eq. (C.26) with $x = k_x r$, $y = r\sqrt{k_x^2 + \frac{k^2}{\gamma^2}}$ and $\nu = -n$, giving (since $\sin\nu\pi = 0$ here):

$$\begin{aligned}\frac{1}{2}\int_0^\pi \mathrm{d}\phi \cos(k_x r\sin\phi + n\phi)\, e^{r\cos\phi\sqrt{k_x^2+\frac{k^2}{\gamma^2}}} &= \frac{\pi}{2}\frac{r^{-n}\left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{-n}}{\left(-\frac{k^2r^2}{\gamma^2}\right)^{-\frac{n}{2}}} J_{-n}\left(j\frac{kr}{\gamma}\right)\\
&= \frac{\pi}{2} j^n \left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{-n} \frac{k^n}{\gamma^n} j^{-n} I_n\left(\frac{kr}{\gamma}\right)\\
&\qquad \text{using Eqs. (C.3) and (C.6)}\\
&= \frac{\pi}{2}\left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{-n} \frac{k^n}{\gamma^n} I_n\left(\frac{kr}{\gamma}\right).\end{aligned}$$

Similarly we get for the other term, this time with $\nu = n$ in Eq. (C.26)

$$\begin{aligned}\frac{1}{2}\int_0^\pi \mathrm{d}\phi \cos(k_x r\sin\phi - n\phi)\, e^{r\cos\phi\sqrt{k_x^2+\frac{k^2}{\gamma^2}}} &= \frac{\pi}{2}\frac{r^{n}\left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{n}}{\left(-\frac{k^2r^2}{\gamma^2}\right)^{\frac{n}{2}}} J_{n}\left(j\frac{kr}{\gamma}\right)\\
&= \frac{\pi}{2} j^{-n} \left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{n} \frac{\gamma^n}{k^n} j^{n} I_n\left(\frac{kr}{\gamma}\right)\\
&= \frac{\pi}{2}\left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{n} \frac{\gamma^n}{k^n} I_n\left(\frac{kr}{\gamma}\right).\end{aligned}$$

Adding the two we finally get

$$\int_0^\pi \mathrm{d}\phi \cos(n\phi)\cos(k_x r\sin\phi)\, e^{r\cos\phi\sqrt{k_x^2+\frac{k^2}{\gamma^2}}} =$$

$$\frac{\pi}{2} I_n\left(\frac{kr}{\gamma}\right)\left[\left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{-n}\frac{k^n}{\gamma^n} + \left(k_x + \sqrt{k_x^2+\frac{k^2}{\gamma^2}}\right)^{n}\frac{\gamma^n}{k^n}\right]. \quad \text{(C.27)}$$

D Appendix D: Consequences of Eq. (6.25) for all the layers boundaries

In Section 6.2, we could not prove that the horizontal wave numbers k_x and $k_{x_h}^{(1)}$ are equal in the case when for all the boundaries Eq. (6.25) is true. We will show here what are the consequences for the fields of such a situation.

From this equation together with the continuity of E_s and H_s as stated in Eqs. (6.16) and (6.17) (dropping the sine and cosine factors thanks to Eqs. (6.21) and (6.22)), we have at each boundary $y = b^{(p)}$ with $1 \leq p \leq N-1$:

$$C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} = C_{e+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} + C_{e-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}}, \tag{D.1}$$

$$\frac{k_y^{(p)} \varepsilon_c^{(p)}}{\nu^{(p)2}} \left[C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right] = \frac{k_y^{(p+1)} \varepsilon_c^{(p+1)}}{\nu^{(p+1)2}} \left[C_{e+}^{(p+1)} e^{k_y^{(p+1)} b^{(p)}} - C_{e-}^{(p+1)} e^{-k_y^{(p+1)} b^{(p)}} \right], \tag{D.2}$$

$$C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} + C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}} = C_{h+}^{(p+1)} e^{k_{y_h}^{(p+1)} b^{(p)}} + C_{h-}^{(p+1)} e^{-k_{y_h}^{(p+1)} b^{(p)}}, \tag{D.3}$$

$$\frac{k_{y_h}^{(p)} \mu^{(p)}}{\nu^{(p)2}} \left[C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} - C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}} \right] = \frac{k_{y_h}^{(p+1)} \mu^{(p+1)}}{\nu^{(p+1)2}} \left[C_{h+}^{(p+1)} e^{k_{y_h}^{(p+1)} b^{(p)}} - C_{h-}^{(p+1)} e^{-k_{y_h}^{(p+1)} b^{(p)}} \right]. \tag{D.4}$$

Putting together first Eqs. (D.1) and (D.2), then Eqs. (D.3) and (D.4), and writing them in matrix form, we have

$$\begin{bmatrix} e^{k_y^{(p+1)} b^{(p)}} & e^{-k_y^{(p+1)} b^{(p)}} \\ \frac{k_y^{(p+1)} \varepsilon_c^{(p+1)}}{\nu^{(p+1)2}} e^{k_y^{(p+1)} b^{(p)}} & -\frac{k_y^{(p+1)} \varepsilon_c^{(p+1)}}{\nu^{(p+1)2}} e^{-k_y^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \end{bmatrix} = \begin{bmatrix} C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \\ \frac{k_y^{(p)} \varepsilon_c^{(p)}}{\nu^{(p)2}} \left\{ C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} \right\} \end{bmatrix},$$

and

$$\begin{bmatrix} e^{k_{y_h}^{(p+1)} b^{(p)}} & e^{-k_{y_h}^{(p+1)} b^{(p)}} \\ \frac{k_{y_h}^{(p+1)} \mu^{(p+1)}}{\nu^{(p+1)2}} e^{k_{y_h}^{(p+1)} b^{(p)}} & -\frac{k_{y_h}^{(p+1)} \mu^{(p+1)}}{\nu^{(p+1)2}} e^{-k_{y_h}^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{h+}^{(p+1)} \\ C_{h-}^{(p+1)} \end{bmatrix} = \begin{bmatrix} C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} + C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}} \\ \frac{k_{y_h}^{(p)} \mu^{(p)}}{\nu^{(p)2}} \left\{ C_{h+}^{(p)} e^{k_{y_h}^{(p)} b^{(p)}} - C_{h-}^{(p)} e^{-k_{y_h}^{(p)} b^{(p)}} \right\} \end{bmatrix}.$$

These can be solved readily using the inversion formula of a 2×2 matrix (see Appendix E), noticing that the determinant of the left hand side matrix is equal to $-\frac{2k_y^{(p+1)} \varepsilon_c^{(p+1)}}{\nu^{(p+1)2}}$ for the first equation and $-\frac{2k_{y_h}^{(p+1)} \mu^{(p+1)}}{\nu^{(p+1)2}}$ for the second one. We get

$$\begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \end{bmatrix} = \frac{-\nu^{(p+1)2}}{2k_y^{(p+1)} \varepsilon_c^{(p+1)}} \begin{bmatrix} -\frac{k_y^{(p+1)} \varepsilon_c^{(p+1)}}{\nu^{(p+1)2}} e^{-k_y^{(p+1)} b^{(p)}} & -e^{-k_y^{(p+1)} b^{(p)}} \\ -\frac{k_y^{(p+1)} \varepsilon_c^{(p+1)}}{\nu^{(p+1)2}} e^{k_y^{(p+1)} b^{(p)}} & e^{k_y^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} e^{k_y^{(p)} b^{(p)}} & e^{-k_y^{(p)} b^{(p)}} \\ \frac{k_y^{(p)} \varepsilon_c^{(p)}}{\nu^{(p)2}} e^{k_y^{(p)} b^{(p)}} & -\frac{k_y^{(p)} \varepsilon_c^{(p)}}{\nu^{(p)2}} e^{-k_y^{(p)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ C_{e-}^{(p)} \end{bmatrix}, \tag{D.5}$$

and

$$\begin{bmatrix} C_{h+}^{(p+1)} \\ C_{h-}^{(p+1)} \end{bmatrix} = \frac{-\nu^{(p+1)2}}{2k_{y_h}^{(p+1)} \mu^{(p+1)}} \begin{bmatrix} -\frac{k_{y_h}^{(p+1)} \mu^{(p+1)}}{\nu^{(p+1)2}} e^{-k_{y_h}^{(p+1)} b^{(p)}} & -e^{-k_{y_h}^{(p+1)} b^{(p)}} \\ -\frac{k_{y_h}^{(p+1)} \mu^{(p+1)}}{\nu^{(p+1)2}} e^{k_{y_h}^{(p+1)} b^{(p)}} & e^{k_{y_h}^{(p+1)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} e^{k_{y_h}^{(p)} b^{(p)}} & e^{-k_{y_h}^{(p)} b^{(p)}} \\ \frac{k_{y_h}^{(p)} \mu^{(p)}}{\nu^{(p)2}} e^{k_{y_h}^{(p)} b^{(p)}} & -\frac{k_{y_h}^{(p)} \mu^{(p)}}{\nu^{(p)2}} e^{-k_{y_h}^{(p)} b^{(p)}} \end{bmatrix} \cdot \begin{bmatrix} C_{h+}^{(p)} \\ C_{h-}^{(p)} \end{bmatrix}. \tag{D.6}$$

We call $M_e^{p+1,p}$ and $M_h^{p+1,p}$ the matrices relating respectively $\begin{bmatrix} C_{e+}^{(p+1)} \\ \\ C_{e-}^{(p+1)} \end{bmatrix}$ to $\begin{bmatrix} C_{e+}^{(p)} \\ \\ C_{e-}^{(p)} \end{bmatrix}$ and $\begin{bmatrix} C_{h+}^{(p+1)} \\ \\ C_{h-}^{(p+1)} \end{bmatrix}$ to $\begin{bmatrix} C_{h+}^{(p)} \\ \\ C_{h-}^{(p)} \end{bmatrix}$, i.e. the product of the two 2×2 matrices in Eqs. (D.5) and (D.6), multiplied respectively by $\frac{-\nu^{(p+1)^2}}{2k_y^{(p+1)}\varepsilon_c^{(p+1)}}$ and $\frac{-\nu^{(p+1)^2}}{2k_{y_h}^{(p+1)}\mu^{(p+1)}}$. When successively applying the relations (D.5) and (D.6) for each boundary, we get[14]

$$\begin{bmatrix} C_{e+}^{(N)} \\ \\ C_{e-}^{(N)} \end{bmatrix} = M_e^{N,N-1} \cdot M_e^{N-1,N-2} \cdots M_e^{2,1} \cdot \begin{bmatrix} C_{e+}^{(1)} \\ \\ C_{e-}^{(1)} \end{bmatrix}, \tag{D.7}$$

$$\begin{bmatrix} C_{h+}^{(N)} \\ \\ C_{h-}^{(N)} \end{bmatrix} = M_h^{N,N-1} \cdot M_h^{N-1,N-2} \cdots M_h^{2,1} \cdot \begin{bmatrix} C_{h+}^{(1)} \\ \\ C_{h-}^{(1)} \end{bmatrix}. \tag{D.8}$$

We focus now on the electric field longitudinal component E_s. There are two possible cases, depending if $k_y^{(N)}$ is purely imaginary or not. In the first case, as seen in Section 6.3 we have Cherenkov radiation in the outer infinite layer, and only the outgoing wave can be present, therefore $C_{e+}^{(N)} = 0$. In the second case, the real part of $k_y^{(N)}$ being strictly positive from Eq. (6.21) and the definition of the square root in Eq. (4.7), we have also necessarily $C_{e+}^{(N)} = 0$ since the exponentially growing solution in y in not physical. We can therefore rewrite Eq. (D.7), with the definition $\mathcal{M}(e) = M_e^{N,N-1} \cdot M_e^{N-1,N-2} \cdots M_e^{2,1}$:

$$\begin{bmatrix} 0 \\ C_{e-}^{(N)} \end{bmatrix} = \mathcal{M}(e) \cdot \begin{bmatrix} C_{e+}^{(1)} \\ C_{e-}^{(1)} \end{bmatrix}.$$

Then

$$\mathcal{M}_{11}(e) C_{e+}^{(1)} + \mathcal{M}_{12}(e) C_{e-}^{(1)} = 0. \tag{D.9}$$

Very similar calculations for the lower layers $(-M)$ to (-1) lead to (with $\mathcal{M}'(e)$ a 2×2 matrix defined in a similar way as $\mathcal{M}(e)$)

$$\begin{bmatrix} C_{e+}^{(-M)} \\ 0 \end{bmatrix} = \mathcal{M}'(e) \cdot \begin{bmatrix} C_{e+}^{(-1)} \\ C_{e-}^{(-1)} \end{bmatrix}, \tag{D.10}$$

which gives

$$\mathcal{M}'_{21}(e) C_{e+}^{(-1)} + \mathcal{M}'_{22}(e) C_{e-}^{(-1)} = 0. \tag{D.11}$$

---

[14] Note that when writing Eq. (D.5) we have implicitely assumed that $k_y^{(p+1)} \neq 0$. Actually $k_y^{(p+1)} = 0$ can happen if $\nu^{(p+1)}$ is purely imaginary and if $k_x^2 = -\nu^{(p+1)^2}$ from Eq. (6.21). If this happens, in layer $(p+1)$ E_s is independent of y and what matters is only the constant $C_{e+}^{(p+1)} + C_{e-}^{(p+1)}$, so we can arbitrarily choose that $C_{e+}^{(p+1)} = C_{e-}^{(p+1)} = \frac{C_{e+}^{(p+1)} + C_{e-}^{(p+1)}}{2}$. Then from Eq. (D.2) we have two possible cases: either $k_y^{(p)} = 0$ or $k_y^{(p)} \neq 0$. In the former case, in layer (p) E_s is also independent of y, and then, arbitrarily choosing $C_{e+}^{(p)} = C_{e-}^{(p)} = \frac{C_{e+}^{(p)} + C_{e-}^{(p)}}{2}$ we can still write from Eq. (D.1)

$$\begin{bmatrix} C_{e+}^{(p+1)} \\ C_{e-}^{(p+1)} \end{bmatrix} = M_e^{p+1,p} \cdot \begin{bmatrix} C_{e+}^{(p)} \\ C_{e-}^{(p)} \end{bmatrix},$$

with $M_e^{p+1,p}$ the 2×2 identity matrix. If $k_y^{(p)} \neq 0$ then Eq. (D.1) with $C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} - C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} = 0$ from Eq. (D.2) will give also the same kind of matrix relation. Therefore the general conclusion is conserved, namely Eq. (D.7).
This argument also applies to H_s when $k_{y_h}^{(p+1)} = 0$.

Writing then the equations (6.5), (D.9) and (D.11) in matrix form we get

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ \mathcal{M}_{11}(e) & \mathcal{M}_{12}(e) & 0 & 0 \\ 0 & 0 & \mathcal{M}'_{21}(e) & \mathcal{M}'_{22}(e) \end{bmatrix} \cdot \begin{bmatrix} C_{e+}^{(1)} \\ C_{e-}^{(1)} \\ C_{e+}^{(-1)} \\ C_{e-}^{(-1)} \end{bmatrix} = \begin{bmatrix} -\mathcal{C}\frac{e^{-k_y^{(1)}y_1}}{k_y^{(1)}} \\ \mathcal{C}\frac{e^{k_y^{(1)}y_1}}{k_y^{(1)}} \\ 0 \\ 0 \end{bmatrix}. \tag{D.12}$$

This means that those four integration constants of E_s (and consequently all the others) are fully determined, except in the rather exceptional case when the determinant of the 4×4 matrix above – namely $\mathcal{M}_{11}(e)\mathcal{M}'_{22}(e) - \mathcal{M}_{12}(e)\mathcal{M}'_{21}(e)$ – is zero (this would require a very particular relation to hold between k_x and all the material constants, so is probably possible only for discrete values of k_x which is then of no significance for the total fields obtained after continuous integration over k_x). Then we necessarily have $\nu^{(p+1)} = \nu^{(p)}$ for all the layers: if this was not the case for one layer then Eq. (6.25) gives for that layer the additional relation, assuming $k_x \neq 0$ (the case $k_x = 0$ is again of no significance, for the same reason as above)

$$C_{e+}^{(p)} e^{k_y^{(p)} b^{(p)}} + C_{e-}^{(p)} e^{-k_y^{(p)} b^{(p)}} = 0,$$

for e.g. the upper part of the chamber (the same applying also for the lower part). This relation will propagate to all the other layers through the continuity of E_s (see Eq. (D.1)), and will give in particular

$$C_{e+}^{(1)} e^{k_y^{(1)} b^{(1)}} + C_{e-}^{(1)} e^{-k_y^{(1)} b^{(1)}} = 0.$$

This gives another equation incompatible with the system (D.12) above, unless $\mathcal{M}_{11}(e)e^{-k_y^{(1)}b^{(1)}} - \mathcal{M}_{12}(e)e^{k_y^{(1)}b^{(1)}} = 0$, which again should concern at worst some discrete values of k_x, so is of no significance.

Therefore one of the consequences of Eq. (6.25) is that all the layers have the same propagation constant, namely that of vacuum $\nu = \frac{k}{\gamma}$ (γ being the relativistic mass factor), which also means that $\varepsilon_1\mu_1 = 1$ for all the flat chamber materials.

Very similar considerations apply to H_s. Defining $\mathcal{M}(h) = M_h^{N,N-1} \cdot M_h^{N-1,N-2} \cdots M_h^{2,1}$ we have indeed from Eq. (D.8)

$$\begin{bmatrix} 0 \\ C_{h-}^{(N)} \end{bmatrix} = \mathcal{M}(h) \cdot \begin{bmatrix} C_{h+}^{(1)} \\ C_{h-}^{(1)} \end{bmatrix}.$$

For the layers in the lower part a similar relation is obtained:

$$\begin{bmatrix} C_{h+}^{(-M)} \\ 0 \end{bmatrix} = \mathcal{M}'(h) \cdot \begin{bmatrix} C_{h+}^{(-1)} \\ C_{h-}^{(-1)} \end{bmatrix}, \tag{D.13}$$

with a different 2×2 matrix $\mathcal{M}'(h)$. In the end, as for E_s we obtain the two following relations:

$$\mathcal{M}_{11}(h)C_{h+}^{(1)} + \mathcal{M}_{12}(h)C_{h-}^{(1)} = 0, \tag{D.14}$$

and

$$\mathcal{M}'_{21}(h)C_{h+}^{(-1)} + \mathcal{M}'_{22}(h)C_{h-}^{(-1)} = 0. \tag{D.15}$$

These two equations together with Eqs. (6.9) can be put in matrix form:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ \mathcal{M}_{11}(h) & \mathcal{M}_{12}(h) & 0 & 0 \\ 0 & 0 & \mathcal{M}'_{21}(h) & \mathcal{M}'_{22}(h) \end{bmatrix} \cdot \begin{bmatrix} C_{h+}^{(1)} \\ C_{h-}^{(1)} \\ C_{h+}^{(-1)} \\ C_{h-}^{(-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{D.16}$$

Contrary to the case of E_s, the right hand side of this system is zero, therefore the only solution is when all the constants of H_s for the layers (-1) and (1) are zero (except again in the rather exceptional case when the determinant of the 4×4 matrix above, namely $\mathcal{M}_{11}(h)\mathcal{M}'_{22}(h)-\mathcal{M}_{12}(h)\mathcal{M}'_{21}(h)$, is zero). From the matrix relations between the constants of subsequent layers this will then have as a consequence that all the constants of H_s in all the layers are zero, so $H_s = 0$ everywhere.
In conclusion, if Eq. (6.25) is true for all the layers, it has for consequences that $\varepsilon_1\mu_1 = 1$ for all the layers and $H_s = 0$ everywhere.

E Appendix E: Inversion of a 2×2 matrix

Given a 2×2 matrix of the form

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{E.1}$$

whose determinant $ad - bc$ is non zero, its inverse is given by

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{E.2}$$

This can be checked simply by multiplying the two matrices.

List of notations

$(\vec{e_x},\, \vec{e_y},\, \vec{e_s})$	Basis vectors in the cartesian system of coordinates	5
(p)	Superscript added to all the quantities related to the layer $\pm p$ (with p strictly positive integer)	6
$(r,\, \theta,\, s)$	Cylindrical coordinates	24
$(x,\, y,\, s)$	Cartesian coordinates	4
$*$	Complex conjugate	7
b	Half gap of the flat chamber	4
$b^{(p)}$	Vertical coordinate of the outer boundary of the layer (p)	6
$\vec{B}$	$\mu\vec{H}$ i.e. magnetic induction (components: B_x, B_y, B_s)	6
c	Speed of light in vacuum (299792458 m/s)	9
$\mathcal{C}$	$\frac{jQ\omega\mu_0}{2\pi\beta^2\gamma^2}$	13
$C_{e+}^{(p)}$	Integration constant in front of the exponential having a plus sign in its argument, in $E_s^{(p)}$	10
$C_{e-}^{(p)}$	Integration constant in front of the exponential having a minus sign in its argument, in $E_s^{(p)}$	10
$C_{g+}^{(p)}$	Integration constant in front of the exponential having a plus sign in its argument, in $G_s^{(p)}$	16
$C_{g-}^{(p)}$	Integration constant in front of the exponential having a minus sign in its argument, in $G_s^{(p)}$	16
$C_{h+}^{(p)}$	Integration constant in front of the exponential having a plus sign in its argument, in $H_s^{(p)}$	10
$C_{h-}^{(p)}$	Integration constant in front of the exponential having a minus sign in its argument, in $H_s^{(p)}$	10
$\vec{D}$	Electric displacement (components: D_x, D_y, D_s)	6
$\vec{E}$	Electric field (components: E_x, E_y, E_s)	6
$E_{s,tot}^{vac,SC}$	Longitudinal component of the total electric field in the vacuum region, direct space charge part	24
$E_{s,tot}^{vac,W}$	Longitudinal component of the total electric field in the vacuum region, wall part	24
$\vec{F}$	Electromagnetic (or Lorentz) force (components: F_x, F_y, F_s)	23
$\vec{G}$	$Z_0\vec{H}$ (components: G_x, G_y, G_s)	16
$\vec{H}$	Magnetic field (components: H_x, H_y, H_s)	6
I_ν	Modified Bessel function of the first kind of order ν	24
j	Imaginary constant ($\sqrt{-1}$)	5
J	Current density of the source macroparticle (along the s axis)	5
$\tilde{J}$	Horizontal cosine Fourier transform of J, keeping the cosine factor	5
k	Longitudinal wave number	5
k_x	Horizontal wave number of the source surface charge density and of the electromagnetic fields	5
k_{x_e}	Horizontal wave number of the longitudinal component of the electric field	9
k_{x_h}	Horizontal wave number of the longitudinal component of the magnetic field	9
k_y	Vertical wave number of the electromagnetic fields	12
k_{y_e}	Vertical wave number of the longitudinal component of the electric field	10
k_{y_h}	Vertical wave number of the longitudinal component of the magnetic field	10
K_ν	Modified Bessel function of the second kind of order ν	24
L	Length of the flat chamber (along the s axis)	27
M	Number of layers under the plane $y = y_1$ (including one layer of vacuum for $-b < y < y_1$)	6
N	Number of layers above the plane $y = y_1$ (including one layer of vacuum for $y_1 < y < b$)	6
q	Test particle charge	23
Q	Source macroparticle charge	4
t	Time	4
x_1	Horizontal coordinate of the source macroparticle	24
x_2	Horizontal coordinate of the test particle	27
y_1	Vertical coordinate of the source macroparticle	4
y_2	Vertical coordinate of the test particle	27
Z_0	Free space impedance	16
$Z_\parallel$	Total longitudinal impedance	27
$Z_\parallel^{SC,direct}$	Longitudinal direct space-charge impedance	28
$Z_\parallel^{Wall}$	Longitudinal wall impedance	28
$Z_\parallel^{Wall,0}$	Zeroth order term in the longitudinal wall impedance	31
Z_x	Total horizontal transverse impedance	27
$Z_x^{SC,direct}$	Horizontal transverse direct space-charge impedance	28
Z_x^{Wall}	Horizontal transverse wall impedance	29
$Z_x^{Wall,dip}$	Horizontal dipolar wall impedance	32
$Z_x^{Wall,quad}$	Horizontal quadrupolar wall impedance	32

Z_y	Total vertical transverse impedance	27
$Z_y^{SC,direct}$	Vertical transverse direct space-charge impedance	28
Z_y^{Wall}	Vertical transverse wall impedance	30
$Z_y^{Wall,0}$	Zeroth order term in the vertical wall impedance	31
$Z_y^{Wall,dip}$	Vertical dipolar wall impedance	32
$Z_y^{Wall,quad}$	Vertical quadrupolar wall impedance	32
α_{mn}	Coef. of the decomposition of $E_{s,tot}^{vac,W}$ into azimuthal modes (m^{th} of the source and n^{th} of the test)	26
β	Relativistic velocity factor of the source and test particles	10
γ	Relativistic mass factor of the source and test particles	12
δ	Dirac distribution (or delta function)	4
δ_{n0}	1 if $n = 0$, 0 otherwise	25
ε_0	Permittivity of vacuum	6
ε_1	Relative complex permittivity of the medium (including conductivity)	6
ε_b	Dielectric constant of the medium (real)	6
ε_c	Complex permittivity of the medium (including conductivity)	6
η_1	Coefficient relating $-C_{e+}^{(1)}\frac{k_y^{(1)}}{\mathcal{C}}$ to $e^{-k_y^{(1)}y_1}$	23
η_2	Coefficient relating $-C_{e-}^{(-1)}\frac{k_y^{(1)}}{\mathcal{C}}$ to $e^{-k_y^{(1)}y_1}$	23
$\tan\vartheta_E$	Dielectric loss tangent	6
$\tan\vartheta_M$	Magnetic loss tangent	6
μ	Complex permeability of the medium	6
μ_0	Permeability of vacuum ($4\pi 10^{-7}$ H/m)	6
μ_1	Relative complex permeability of the medium	6
μ_r	Real part of the relative complex permeability of the medium	6
ν	Propagation constant in the medium	10
ρ	Charge density of the source macroparticle	4
$\tilde{\rho}$	Horizontal cosine Fourier transform of ρ, keeping the cosine factor	5
σ_{DC}	DC (i.e. at zero frequency) conductivity (real)	6
τ	Relaxation time of the complex AC conductivity	6
υ	Velocity of the source and test particles	4
ϕ	$\theta - \frac{\pi}{2}$	25
χ_1	Coefficient relating $-C_{e+}^{(1)}\frac{k_y^{(1)}}{\mathcal{C}}$ to $e^{k_y^{(1)}y_1}$	23
χ_2	Coefficient relating $-C_{e-}^{(-1)}\frac{k_y^{(1)}}{\mathcal{C}}$ to $e^{k_y^{(1)}y_1}$	23
ω	Angular frequency	5

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