Dirac Group(oid)s and Their Homogeneous Spaces

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Il arrive quelquefois qu'on ne peut rien répondre, et qu'on n'est pas persuadé. On est atterré sans pouvoir être convaincu. On sent dans le fond de son âme un scrupule, une répugnance qui nous empêche de croire ce qu'on nous a prouvé. Un géomètre vous démontre qu'entre un cercle et une tangente vous pouvez faire passer une infinité de lignes courbes, et que vous n'en pouvez faire passer une droite: vos yeux, votre raison vous disent le contraire. Le géomètre vous répond gravement que c'est là un infini du second ordre. Vous vous taisez, et vous vous en retournez tout stupéfait, sans avoir aucune idée nette, sans rien comprendre et sans rien répliquer.
Vous consultez un géomètre de meilleure foi, qui vous explique le mystère. ≪Nous supposons, dit-il, ce qui ne peut être dans la nature, des lignes qui ont de la longueur sans largeur: il est impossible, physiquement parlant, qu'une ligne réelle en pénètre une autre. Nulle courbe ni nulle droite réelle ne peut passer entre deux lignes réelles qui se touchent: ce ne sont là que des jeux de l'entendement, des chimères idéales; et la véritable géométrie est l'art de mesurer les choses existantes.≫
Je fus très content de l'aveu de ce sage mathématicien, et je me mis à rire, dans mon malheur, d'apprendre qu'il y avait de la charlatanerie jusque dans la science qu'on appelle la haute science.

Voltaire, ≪*L'homme aux quarante écus*≫, Entretien avec un géomètre, 1768.

Résumé

Un théorème de Drinfel′d (Drinfel′d (1993)) classifie les espaces *Poisson-homogènes* d'un *groupe de Poisson-Lie* à l'aide d'une certaine classe de sous-algèbres Lagrangiennes de la *bialgèbre de Lie* associée au groupe de Poisson-Lie. Ce théorème est généralisé par Liu et al. (1998) en une classification des espaces Poisson-homogènes de *groupoïdes de Poisson* par certaines structures de Dirac dans l'algébroïde de Courant défini par le *bialgébroïde de Lie* associé au groupoïde de Poisson.
Il est donc naturel de se demander ce qui correspond dans cette classification, ou une extension de cette classification, à des structures de Dirac plus générales dans la bialgèbre ou le bialgébroïde de Lie associés au group(oïd)e de Poisson. Nous montrons dans cette thèse qu'une plus grande famille de structures de Dirac peut être considérée. Ces structures de Dirac définissent de manière naturelle des *espaces Dirac-homogènes du group(oïd)e de Poisson.*

Les groupes de Lie et groupoïdes Dirac ont été introduits par Ortiz (2009) et généralisent à la fois les groupes de Poisson Lie, les groupoïdes de Poisson et les groupoïdes présymplectiques. Nous prouvons d'une nouvelle manière l'existence d'une bialgèbre de Lie associée à un groupe de Lie Dirac, et nous trouvons des objets semblant jouer le rôle des objets infinitésimaux associés à un groupoïde Dirac. Nous trouvons un carré de morphismes d'algébroïdes de Lie associés à une structure de Dirac multiplicative et intégrable, qui généralisent à la fois les bialgébroïdes de Lie des groupoïdes de Poisson (Mackenzie and Xu (1994)), et les *deux formes infinitésimales* associées aux groupoïdes présymplectiques (Bursztyn et al. (2004)). Nous prouvons aussi l'existence d'un algébroïde de Courant associé à ce diagramme. Cette structure d'algébroïde de Courant est induite de manière naturelle par la structure standard d'algébroïde de Courant sur $TG \times_G T^*G$ et est isomorphe à l'algébroïde de Courant $AG \times_P A^*G$ dans le cas d'un groupoïde de Poisson.
Nous montrons ensuite que les *espaces homogènes Dirac* des *groupes de Lie Dirac* (respectivement des *groupoïdes Dirac*) correspondent à une certaine famille de structures de Dirac dans cette bialgèbre de Lie (respectivement dans cet algébroïde de Courant). Ces résultats généralisent les théorèmes de classification de Drinfel′d (1993) et Liu et al. (1998).
Nous montrons aussi quand et comment une structure de groupoïde est induite sur l'espace des feuilles d'un sous-fibré multiplicatif et involutif du tangent d'un groupoïde. Ceci est toujours vrai pour une sous-distribution multiplicative du tangent d'un groupe de Lie, puisque c'est automatiquement l'image invariante à gauche et à droite d'un idéal dans l'algèbre de Lie du groupe de Lie. La distribution est donc dans ce cas automatiquement

de rang constant et involutive et l'espace des feuilles est l'ensemble des classes à gauche et à droite d'un sous-groupe normal du groupe de Lie.

Mots clés: Groupes de Poisson-Lie, espaces homogènes, variétés Dirac, groupoïdes de Lie, algébroïdes de Lie, algébroïdes de Courant.

Zusammenfassung

Ein Satz in Drinfel′d (1993) klassifiziert die *Poisson-homogenen Räume* einer *Poisson-Lie-Gruppe* mittels einer speziellen Klasse Lagrangescher Unteralgebren der *Lie-Bialgebra* der Poisson-Gruppe. Diese Klassifikation wird in Liu et al. (1998) zu einer Klassifikation der Poisson-homogenen Räumen eines *Poisson-Gruppoids* erweitert. Die klassifizierenden Objekte sind diesmal spezielle Dirac-Strukturen in dem Courant-Algebroid, der durch das *Lie-Bialgebroid* des Poisson-Gruppoids definiert wird.

Es erscheint also folgende natürliche Frage: was würde allgemeineren Dirac-Strukturen in der Bialgebra oder dem Bialgebroid in dieser Klassifikation entsprechen? Wir zeigen dass es tatsächlich eine größere Familie von Dirac-Strukturen gibt, die in dieser Klassifikation, oder eher einer Erweiterung davon, Sinn machen. Sie entsprechen in einer natürlichen Weise *Dirac-homogenen Räumen* der Poisson-Lie-Gruppe oder des Poisson-Gruppoids.

Die Begriffe der Dirac-Lie-Gruppe und des Dirac-Gruppoids sind von Ortiz (2009) als eine gleichzeitige Verallgemeinerung von Poisson-Lie-Gruppen, Poisson-Gruppoiden und präsymplektischen Gruppoiden eingeführt worden. Wir zeigen mit einer neuen Methode, dass es eine natürliche Lie-Bialgebra für jede Dirac-Lie-Gruppe gibt, und wir finden Objekte, die die infinitesimale Information eines Dirac-Gruppoids (zumindest teilweise) wiedergeben: ein quadratisches Diagramm von Lie-Algebroid-Morphismen, die in natürlicher Weise einem (integrablen) Dirac-Gruppoid zugeordnet werden. In den speziellen Fällen der Poisson-Gruppoide und der präsymplektischen Gruppoide finden wir die dazugehörigen Lie-Bialgebroide (Mackenzie and Xu (1994)) und "IM-2-Formen" (Bursztyn et al. (2004)). Wir zeigen auch die Existenz eines zur integrablen multiplikativen Dirac-Struktur assoziierten Courant-Algebroids. Diese Struktur wird von dem standard Courant-Algebroid $TG \times_G T^*G$ induziert, und entspricht genau dem Courant-Algebroid $AG \times_P A^*G$ im Poisson Fall.

Wir zeigen dass *Dirac-homogene Räume* von *Dirac-Lie-Gruppen* (von *Dirac-Gruppoiden*) in eins-zu-eins Korrespondenz zu spezielle Klassen von Dirac-Strukturen in der Lie-Bialgebra stehen (in dem Courant-Algebroid). Diese Klassifikationen verallgemeinern die Sätze in Drinfel′d (1993) und Liu et al. (1998).

Wir untersuchen zudem involutive, multiplikative Blätterungen auf Lie Gruppoide und zeigen wann und wie eine natürliche Gruppoidstruktur auf den Quotienten der Objekte und der Pfeile durch die Blätterung induziert wird. Dies ist immer der Fall, wenn das Lie-Gruppoid eine Lie-Gruppe ist. In diesem Fall ist eine multiplikative Distribution automatisch das links- und rechtsinvariante Bild eines Ideals in der Lie-Algebra, das heißt insbesondere dass sie konstanten Rang hat und involutiv ist. Demnach ist sie integrabel, und ihre Blätter sind die Nebenräume einer normalen Untergruppe der Lie-Gruppe.

Stichwörter: Poisson-Lie-Gruppen, homogene Räume, Dirac-Mannigfaltigkeiten, Lie-Gruppoide, Lie-Algebroide, Courant-Algebroide.

Abstract

A theorem of Drinfel′d (Drinfel′d (1993)) classifies the *Poisson homogeneous spaces* of a *Poisson Lie group* (G, π_G) via a special class of Lagrangian subalgebras of the Drinfel′d double of its *Lie bialgebra*. This result is extended in Liu et al. (1998) to a classification of the Poisson homogeneous spaces of a *Poisson groupoid* $(G \rightrightarrows P, \pi_G)$ via a special class of Dirac structures in the Courant algebroid defined by the corresponding *Lie bialgebroid*. It is hence natural to ask what corresponds via these classifications, or extensions of them, to arbitrary Dirac structures in the Drinfel′d double or the Courant algebroid associated to a Poisson group(oid). We show in this thesis that there is a bigger class of Dirac structures that fits in this correspondence. They correspond naturally to *Dirac homogeneous spaces* of the Poisson group(oid).

Dirac Lie groups and Dirac groupoids have been defined by Ortiz (2009) as a generalization of Poisson Lie groups, Poisson groupoids and presymplectic groupoids. We prove in an alternative manner the existence of a natural Lie bialgebra associated to a Dirac Lie group, and we find good candidates for the infinitesimal data of a Dirac groupoid; a square of morphisms of Lie algebroids associated to the multiplicative Dirac structure. These objects generalize simultaneously the Lie bialgebroid of Poisson groupoids and the IM-2-forms associated to presymplectic groupoids (Bursztyn et al. (2004)). We show also the existence of a Courant algebroid associated to these algebroids. This Courant algebroid structure is induced in a natural way by the ambient standard Courant algebroid $TG \times_G T^*G$ and is isomorphic to the Courant algebroid $AG \times_P A^*G$ in the Poisson case. We show that *Dirac homogeneous spaces* of *Dirac Lie groups* (respectively *Dirac groupoids*) correspond to certain classes of Dirac structures in the Drinfel′d double of the Lie bialgebra (respectively in the Courant algebroid). These results generalize the classification theorems in Drinfel′d (1993) and Liu et al. (1998).
Along the way, we study involutive multiplicative foliations on Lie groupoids and show when and how there is a natural groupoid structure defined on the leaf spaces in the set of objects and the set of units of the Lie groupoid. This is a fact that is always true in the Lie group case, where a multiplicative foliation is automatically the left and right invariant image of an ideal in the Lie algebra, i.e., the leaves are the cosets of a normal subgroup of the Lie group.

Keywords: Poisson Lie groups, homogeneous spaces, Dirac manifolds, Lie groupoids,

Lie algebroids, Courant algebroids.

Contents

Introduction

A *Poisson groupoid* is a Lie groupoid endowed with a Poisson bracket that is compatible with the Lie groupoid structure. *Poisson Lie groups* were introduced by Drinfel′d (1983) and studied by Semenov-Tian-Shansky (1985). Their aim was to understand the Hamiltonian structure of the group of dressing transformations of a completely integrable system. The systematic study of the geometry of Poisson Lie groups was initiated in the works of Lu and Weinstein (see Lu and Weinstein (1989), Lu (1990), Lu and Weinstein (1990), among others). The notion of Poisson Lie group was generalized to the one of Poisson groupoid by Weinstein (1988) and studied in Mackenzie and Xu (1994), Xu (1995), Mackenzie and Xu (1998), among others.
A *Poisson homogeneous space* of a Poisson groupoid is a homogeneous space of the Lie groupoid that is endowed with a Poisson bracket such that the left action of the Lie groupoid on the homogeneous space is compatible with the Poisson structures. Poisson homogeneous spaces of a Poisson Lie group are in correspondence with suitable Lagrangian subspaces of the double of the Lie bialgebra (see Drinfel′d (1993)). This result is extended in Liu et al. (1998) to a classification of the Poisson homogeneous spaces of Poisson groupoids in terms of Dirac structures in their Lie bialgebroids. We show in this thesis that these correspondence results fit in a more general and maybe more natural context: the one of Dirac manifolds, which are objects that generalize Poisson manifolds.

Multiplicative and homogeneous Poisson structures. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A Poisson bivector field $\pi_G \in \Gamma(\bigwedge^2 TG)$ is *multiplicative*, and (G, π_G) is a *Poisson Lie group*, if the group multiplication is compatible with the Poisson structure in the sense that the multiplication map

$$\mathsf{m} : G \times G \to G$$

is a Poisson map, where $G \times G$ is endowed with the product Poisson structure defined by π_G. Equivalently, the graph $\operatorname{Graph}(\pi_G^\sharp) \subseteq TG \times_G T^*G$ of the vector bundle homomorphism $\pi_G^\sharp : T^*G \to TG$ associated to π_G is a subgroupoid of the *Pontryagin groupoid* $TG \times_G T^*G \rightrightarrows \mathfrak{g}^*$ of G.
The multiplicative Poisson bivector π_G on G induces then a *Lie bialgebra* structure on $(\mathfrak{g}, \mathfrak{g}^*)$ and hence a Lie algebra structure on the direct sum of $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. By a theorem in Drinfel′d (1983), the Lie bialgebra structures on $(\mathfrak{g}, \mathfrak{g}^*)$ over the Lie algebra $\mathfrak{g}$ of a connected and simply connected Lie group G are in one-to-one correspondence with multiplicative Poisson structures on G (see also Lu (1990) among others).
More generally, let $G \rightrightarrows P$ be a Lie groupoid endowed with a bivector field $\pi_G \in \Gamma(\bigwedge^2 TG)$. The bivector field π_G is *multiplicative* if the vector bundle map $\pi_G^\sharp : T^*G \to TG$ is a Lie

groupoid morphism over some map $A^*G \to TP$, where A^*G is the dual of the Lie algebroid of $G \rightrightarrows P$, and $TG \rightrightarrows TP$ and $T^*G \rightrightarrows A^*G$ are endowed with the induced Lie groupoid structures (see Coste et al. (1987), Pradines (1988), Mackenzie (2005)). Equivalently, the graph $\operatorname{Graph}(\pi_G^\sharp) \subseteq TG \times_G T^*G$ of the vector bundle homomorphism $\pi_G^\sharp : T^*G \to TG$, associated to π_G is a subgroupoid of the Pontryagin groupoid $(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$ of G. If the bivector field is a Poisson bivector field, then $(G \rightrightarrows P, \pi_G)$ is a *Poisson groupoid.* A Poisson groupoid $(G \rightrightarrows P, \pi_G)$ induces a Lie algebroid structure on the dual A^*G of the Lie algebroid AG of $G \rightrightarrows P$ and a Courant algebroid structure on the direct sum of AG with A^*G. This was shown by Weinstein (1988), Mackenzie and Xu (1994) and Liu et al. (1997). The pair (AG, A^*G) is the *Lie bialgebroid* associated to $(G \rightrightarrows P, \pi_G)$. The one-to-one correspondence between multiplicative Poisson structures on a target simply connected Lie groupoid, and Lie bialgebroid structures on its Lie algebroid, was established in Mackenzie and Xu (2000).

A *Poisson homogeneous space* (X, π_X) of a Poisson Lie group (G, π_G) is a homogeneous space X of G endowed with a Poisson structure π_X such that the transitive left action of G on X

$$\sigma : G \times X \to X$$

is a Poisson map, where $G \times X$ is endowed with the product Poisson structure (see for example Lu (2008)).

Consider the pairing on $\mathfrak{g} \times \mathfrak{g}^*$ defined by $\langle (x, \xi), (y, \eta) \rangle_{\mathfrak{g}} = \xi(y) + \eta(x)$ for all $(x, \xi), (y, \eta) \in \mathfrak{g} \times \mathfrak{g}^*$. Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. A theorem of Drinfel′d (see Drinfel′d (1993)) states that there is a one-to-one correspondence between π_G-homogeneous Poisson structures $\pi_{G/H}$ on G/H and Lagrangian subalgebras $\mathfrak{D}$ of the double $\mathfrak{g} \times \mathfrak{g}^*$ satisfying $\mathfrak{D} \cap (\mathfrak{g} \times \{0\}) = \mathfrak{h} \times \{0\}$ and that are invariant under the restriction to H of an action of G on $\mathfrak{g} \times \mathfrak{g}^*$ that is induced by π_G. This classification and the variety of Lagrangian subalgebras of the Lie bialgebra of a Poisson Lie group are studied in detail in Evens and Lu (2001) and Evens and Lu (2006).

The theorem of Drinfel′d has been extended in Liu et al. (1998) to a correspondence between a certain class of Dirac subspaces of the Courant algebroid $AG \times_P A^*G$ defined by a Poisson groupoid and its *Poisson homogeneous spaces.* A Poisson homogeneous space (X, π_X) of a Poisson groupoid $(G \rightrightarrows P, \pi_G)$ is a homogeneous space X of $G \rightrightarrows P$ endowed with a Poisson structure π_X that is compatible with the action of $G \rightrightarrows P$ on $J : X \to P$.

Main results. Because of Drinfel′d's theorem and its generalization by Liu-Weinstein-Xu, it appears natural to ask what would correspond in these classifications, or extensions of them, to more arbitrary *Dirac subspaces* of $\mathfrak{g} \times \mathfrak{g}^*$ and $AG \times_P A^*G$. A Dirac structure in a Courant algebroid $(\mathsf{E}, \rho, \langle \cdot\,,\cdot \rangle, [\cdot\,,\cdot])$ is a subbundle of E that is Lagrangian relative to the fiberwise pairing $\langle \cdot\,,\cdot \rangle$. It is integrable if its set of sections is closed under the Courant bracket $[\cdot\,,\cdot]$. Dirac structures in $TM \times_M T^*M$ generalize Poisson brackets in the sense that the graph of the homomorphism of vector bundles $\pi^\sharp : T^*M \to TM$ associated to a Poisson bivector field π on M defines an integrable Dirac structure on the manifold M.

In this thesis, we study *Dirac homogeneous spaces* of *Dirac Lie groups* and more generally of *Dirac groupoids.* We show how Drinfel′d's result generalizes to a classification of Dirac homogeneous spaces of Dirac Lie groups, and extend then this result to the more general situation of Dirac groupoids and their homogeneous spaces. Our main theorem (Theorem 6.3.4) generalizes the theorems in Drinfel′d (1993), Liu et al. (1997) and the classification of Dirac homogeneous spaces of Dirac Lie groups that is given in Chapter 4.

Strategy. As a preparation for the more complicated general groupoid case, we choose to study first the group case in a separate chapter (Chapter 4). Dirac Lie groups, which have been defined independently by Ortiz (2008), have the important feature that the characteristic distribution $\mathsf{G}_0 = \mathsf{D}_G \cap TG$ of a multiplicative Dirac structure D_G and its characteristic codistribution $\mathsf{P}_1 = \mathrm{Proj}_{T^*M}(\mathsf{D}_G)$ are always left and right invariant and have thus constant dimensional fibers on the Lie group G. Hence, integrable multiplicative Dirac structures on Lie groups are, in a sense, only a slight generalization of the graphs of multiplicative Poisson bivector fields. The approach in Ortiz (2008) uses this fact to define the Lie bialgebra of an integrable Dirac Lie group. Here, we formulate everything in the Dirac setting and obtain the known results, such as the definition of the Lie bialgebra of a Poisson Lie group, as corollaries in the class of examples given by the Poisson Lie groups.

We choose this approach because the situation is quite different in the case of a Dirac groupoid, i.e., a groupoid endowed with a Dirac structure that is a subgroupoid of the Pontryagin groupoid $(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$ (see Ortiz (2009)). The key point in the construction (of Ortiz (2008)) of the Lie bialgebra of a Dirac Lie group is the fact that the Lie group is foliated by the characteristic distribution of the Dirac structure, which is multiplicative. If regular, the leaf space inherits the structure of a Poisson Lie group that naturally defines the Lie bialgebra. In the general groupoid case, there is no way of controlling the characteristic distribution. Even in the special case of regular Dirac groupoids, i.e., integrable Dirac groupoids whose characteristic distributions are subbundles of the tangent space, there are topological conditions on the leaf space of the foliation for it to become a groupoid. Hence, for an arbitrary Dirac groupoid, there is no chance of finding a Poisson groupoid that is naturally associated to it as in the group case. Chapter 3 is dedicated to the study of multiplicative foliations on Lie groupoids and illustrates in this way the difference between the (easier) theory of Dirac Lie groups and the theory of Dirac groupoids.

Hence, we need to construct the object that will play the role of the Lie bialgebroid in this more general setting. We recover the Lie bialgebroid in the particular case of a Poisson groupoid, and this new approach shows how to see the Courant algebroid $AG \times_P A^*G$ defined by the Lie bialgebroid (AG, A^*G) of a Poisson groupoid as induced by the ambient Courant algebroid structure on $TG \times_G T^*G$. The results known for Poisson groupoids are guidelines, but it is not possible to use them as it is done in Ortiz (2008) in the particular case of Dirac Lie groups. Instead of that, the methods used for the study of the "infinitesimal objects" associated to Dirac groupoids are oriented on our constructions in the group case.

Along the way, we show indeed that if a Dirac groupoid is integrable, there are several Lie algebroids over the units that are induced by the Dirac structure. In particular, there is a Lie algebroid structure on the set of units $\mathfrak{A}(\mathsf{D}_G)$ of the multiplicative Dirac structure. This was predicted by Ortiz (2009) and generalizes the fact that the dual bundle $A^*G \to P$ of the Lie algebroid associated to a Lie groupoid endowed with a multiplicative Poisson structure inherits the structure of a Lie algebroid over P. We observe that every Dirac groupoid gives rise to these infinitesimal objects, that encode completely the integrability of the Dirac structure (Sections 5.2, 5.3 and 5.4). These objects generalize the Lie bialgebroid in the Poisson case, and the IM-2-form in the presymplectic case (Bursztyn et al. (2004)). The results in these sections are of independent interest since they give insights on the infinitesimal geometry associated with a Dirac groupoid, something that is not fully understood yet.

Outline of the thesis

The background on Lie groupoids and Lie algebroids as well as on Dirac manifolds is summarized in Chapter 1. Facts about Poisson Lie groups, Poisson groupoids, their classifications and the classifications of their homogeneous spaces are presented in Section 2.1. The definitions of the Dirac counterpart are given in Section 2.2, before the homogeneous spaces of Dirac groupoids are defined in Section 2.3. The results in this last part are new. In Chapter 3, we study the leaf space of an involutive, multiplicative subbundle of the tangent space of a Lie groupoid. We show that, under some regularity conditions, there is a Lie groupoid structure on it, such that the quotient map is a fibration of Lie groupoids. In Chapter 4, we study Dirac Lie *groups* and their homogeneous spaces. The geometry of Dirac *groupoids* is studied in Chapter 5, and, finally, their Dirac homogeneous spaces are classified in Chapter 6.

Notation

Throughout this thesis the manifolds that are considered are all *paracompact* manifolds, that is, they are Hausdorff and every open covering admits a locally finite refinement.

Let M and N be smooth manifolds. We write TM for the tangent space of M and T^*M for its cotangent space. The push forward of a map $f : M \to N$ will be written $Tf : TM \to TN$.
The *Pontryagin bundle* of M is the direct sum $TM \oplus T^*M \to M$, that will be written $TM \times_M T^*M$. In general, we will write $\mathsf{E} \times_M \mathsf{F}$ for the direct sum of a subbundle $\mathsf{E} \to N$ of $TM \to M$ and a subbundle $\mathsf{F} \to N$ of $T^*M \to M$, and $\mathsf{E} \oplus \mathsf{F}$ if E and F are subbundles of the same vector bundle. The zero section in TM will be considered as a trivial vector bundle over M and written 0_M, and the zero section in T^*M will be written 0_M^*. The pullback or restriction of a vector bundle $\mathsf{E} \to M$ to an embedded submanifold N of M will be written $\mathsf{E}|_N$. In the special case of the tangent and cotangent spaces of M, we will write T_NM and T_N^*M. The annihilator in T^*M of a smooth subbundle $F \subseteq TM$ will be written $F^\circ \subseteq T^*M$.
We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of (local) smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $\mathsf{E} \to M$, the space

of (local) sections of E will be written $\Gamma(\mathsf{E})$ and we call $\mathrm{Dom}(\sigma)$ the open subset of the smooth manifold M where the local section $\sigma \in \Gamma(\mathsf{E})$ is defined.
A *distribution* Δ on M is a subset Δ of TM such that for each $m \in M$, the set $\Delta(m) := \Delta \cap T_m M$ is a vector subspace of $T_m M$. The number $\dim \Delta(m)$ is called the *rank* of Δ at $m \in M$.
A local *differentiable section* of Δ is a smooth section $\sigma \in \mathfrak{X}(M)$ defined on some open subset $U \subset M$ such that $\sigma(u) \in \Delta(u)$ for each $u \in U$. We denote by $\Gamma(\Delta)$ the space of local sections of Δ. A subdistribution is said to be *differentiable* or *smooth* if for every point $m \in M$ and every vector $v \in \Delta(m)$, there is a differentiable section $\sigma \in \Gamma(\Delta)$ defined on an open neighborhood U of m such that $\sigma(m) = v$.

1 Preliminary definitions and facts

In this chapter, we introduce necessary background definitions and facts about Lie groupoids and Lie algebroids, and Dirac manifolds. Along the way, references to books and articles where to find more details are given.

1.1 Lie groupoids and Lie algebroids

In order to set conventions and notations, definitions and facts about Lie groupoids and their Lie algebroids are recalled here. Most of the material and comments in this section are taken from Mackenzie (1987) and Mackenzie (2005) (see also Moerdijk and Mrčun (2003)).

1.1.1 Lie groupoids

Definition 1.1.1 *A* groupoid *consists of two sets G and P, called respectively the groupoid and the base, together with two maps $\mathsf{s},\mathsf{t}: G \to P$ called respectively the* source *and* target projections*, a map $\epsilon : P \to G$, $p \mapsto 1_p$ called the* object inclusion map*, and a partial multiplication $\mathsf{m} : (g,h) \mapsto g \star h =: gh$ in G defined on the set*

$$G \times_P G = \{(g,h) \in G \times G \mid \mathsf{s}(g) = \mathsf{t}(h)\},$$

all subject to the following conditions:

(i) $\mathsf{s}(gh) = \mathsf{s}(h)$ *and* $\mathsf{t}(gh) = \mathsf{t}(g)$ *for all* $(g,h) \in G \times_P G$*;*

(ii) $(g \star h) \star l = g \star (h \star l)$ *for all* $g,h,l \in G$ *such that* $\mathsf{s}(g) = \mathsf{t}(h)$ *and* $\mathsf{s}(h) = \mathsf{t}(l)$*;*

(iii) $\mathsf{s}(1_p) = \mathsf{t}(1_p) = p$ *for all* $p \in P$*;*

(iv) $g \star 1_{\mathsf{s}(g)} = 1_{\mathsf{t}(g)} \star g = g$ *for all* $g \in G$*;*

(v) each $g \in G$ *has a (two-sided) inverse* g^{-1} *such that* $\mathsf{s}(g^{-1}) = \mathsf{t}(g)$, $\mathsf{t}(g^{-1}) = \mathsf{s}(g)$ *and* $g \star g^{-1} = 1_{\mathsf{t}(g)}$, $g^{-1} \star g = 1_{\mathsf{s}(g)}$.

A groupoid G with base P will be written $G \rightrightarrows P$. Elements of P may be called *objects* of the groupoid G and elements of G may be called *arrows*. The arrow 1_p corresponding to $p \in P$ may also be called the *unity* or *identity* corresponding to p. The set P is often considered as a subset of G, that is, 1_p is identified with p for all $p \in P$.
Note that the inverse in (v) is unique. To see this, one can use the following implications (see Mackenzie (2005)), which are easy to verify.

Proposition 1.1.2 *Let $G \rightrightarrows P$ be a groupoid and consider $g \in G$.*

1. *If $h \in G$ satisfies $\mathsf{s}(h) = \mathsf{t}(g)$ and $hg = g$, then $h = 1_{\mathsf{t}(g)}$.*
   If $l \in G$ satisfies $\mathsf{t}(l) = \mathsf{s}(g)$ and $gl = g$, then $l = 1_{\mathsf{s}(g)}$.
2. *If $h \in G$ satisfies $\mathsf{s}(h) = \mathsf{t}(g)$ and $hg = 1_{\mathsf{s}(g)}$, then $h = g^{-1}$.*
   If $l \in G$ satisfies $\mathsf{t}(l) = \mathsf{s}(g)$ and $gl = 1_{\mathsf{t}(g)}$, then $l = g^{-1}$.

Definition 1.1.3 *A* Lie groupoid *is a groupoid G on base P together with smooth structures on G and P such that the maps $\mathsf{s}, \mathsf{t} : G \to P$ are* surjective submersions, *and such that the object inclusion map $\epsilon : P \to G$, $p \mapsto 1_p$ and the partial multiplication $G \times_P G \to G$ are smooth.*

If $G \rightrightarrows P$ is a Lie groupoid, then $G \times_P G \subset G \times G$ is a closed embedded submanifold, since s and t are submersions. The set P of units is then also a closed embedded submanifold of G via $\epsilon : P \hookrightarrow G$. It is also shown in Mackenzie (2005) that the inversion map of a Lie groupoid is then a smooth diffeomorphism.
Since t and s are smooth surjective submersions, the kernels $\ker(T\mathsf{t})$ and $\ker(T\mathsf{s})$ are smooth subbundles of TG. These two vector bundles over G will be written $T^{\mathsf{s}}G := \ker(T\mathsf{s})$ and $T^{\mathsf{t}}G := \ker(T\mathsf{t})$.

Next, we can define the notions of groupoid morphisms and subgroupoids.

Definition 1.1.4

1. *Let $(G \rightrightarrows P, \mathsf{s}, \mathsf{t}, \mathsf{m})$ and $(G' \rightrightarrows P', \mathsf{s}', \mathsf{t}', \mathsf{m}')$ be groupoids. A groupoid morphism $G \to G'$ is a pair of maps $F : G \to G'$, $f : P \to P'$ such that $\mathsf{s}' \circ F = f \circ \mathsf{s}$, $\mathsf{t}' \circ F = f \circ \mathsf{t}$ and $F(gh) = F(g)F(h)$ for any $(g, h) \in G \times_P G$. We also say that F is a morphism over f. If $P = P'$ and $f = \mathrm{Id}_P$, then F is called a* morphism over P *or a* base-preserving morphism. *If $G \rightrightarrows P$ and $G' \rightrightarrows P'$ are Lie groupoids, then (F, f) is a* morphism of Lie groupoids *if both F and f are smooth.*
2. *Let $G \rightrightarrows P$ be a groupoid. A* subgroupoid *of G is a groupoid $G' \rightrightarrows P'$ together with injective maps $I : G' \hookrightarrow G$ and $i : P' \hookrightarrow P$ such that the pair (I, i) is a morphism of groupoids. If $G \rightrightarrows P$ is a Lie groupoid, a* Lie subgroupoid *of G is a subgroupoid $G' \rightrightarrows P'$ such that $I : G' \hookrightarrow G$ and $i : P' \hookrightarrow P$ are injective immersions.*

The closed subset $\mathcal{I}G := \{g \in G \mid \mathsf{s}(g) = \mathsf{t}(g)\}$ is a subgroupoid of G, but it does not necessarily inherit a smooth structure from G. It is called the *inner subgroupoid* of $G \rightrightarrows P$. A Lie subgroupoid $G' \rightrightarrows P'$ of $G \rightrightarrows P$ is *embedded* if G' and P' are embedded submanifolds of G and P, and it is *wide* if $P' = P$ and $i = \mathrm{Id}_P$.
Let for instance $G \rightrightarrows P$ be a Lie groupoid and let C_p be the connectedness component of p in $\mathsf{t}^{-1}(p)$. Then the union

$$C(G) := \bigcup_{p \in P} C_p$$

is a wide Lie subgroupoid of $G \rightrightarrows P$ that is open in G (see Mackenzie (2005)). It is called the *identity-component subgroupoid* of $G \rightrightarrows P$. If $G \rightrightarrows P$ is t-*connected, that is, if all the* t-*fibers of G are connected*, then $C(G) = G$.

Example 1.1.5 Any manifold M may be regarded as a Lie groupoid on itself with $\mathsf{s} = \mathsf{t} = \mathrm{Id}_M$ and every element a unity. A groupoid in which every element is a unity is called a *base groupoid.* $\Diamond$

Example 1.1.6 Consider a smooth manifold M and a Lie group G. Then the product $M \times G \times M$ is a Lie groupoid over M with the source map $\mathsf{s} = \mathrm{pr}_3$, the target map $\mathsf{t} = \mathrm{pr}_1$ and the multiplication

$$(m, g, n) \star (n, h, p) = (m, gh, p)$$

for all $m, n, p \in M$ and $g, h \in G$. The unity corresponding to $m \in M$ is then (m, e, m), where e is the neutral element of G and the inverse of an element $(m, g, n) \in M \times G \times M$ is (n, g^{-1}, m). This Lie groupoid is the *trivial groupoid on M with group G.*
In particular, if $M = \{p\}$ is just one point, we get the Lie group G, and if $G = \{e\}$ is the trivial Lie group, we get the *pair Lie groupoid $M \times M \rightrightarrows M$ associated to M.* Since this example will appear often later on, we give the structure maps explicitly. The source and target maps are defined by $\mathsf{t}, \mathsf{s} : M \times M \to M$, $\mathsf{s}(x, y) = y$ and $\mathsf{t}(x, y) = x$. The product $(x, y) \star (y, z)$ of (x, y) with (y, z) is the pair (x, z). The unit 1_x associated to $x \in M$ is the pair (x, x) and the set of units $\{1_x \mid x \in M\}$ is equal to Δ_M, which is an embedded submanifold of $M \times M$, via the smooth map $\epsilon : M \to \Delta_M$, $m \mapsto (m, m)$. The subset of composable pairs $(M \times M) \times_M (M \times M) = M \times \Delta_M \times M$ is an embedded submanifold of $(M \times M) \times (M \times M)$. Finally, the inverse of $(x, y) \in M \times M$ is (y, x). $\Diamond$

Example 1.1.7 Let $\mathsf{p} : E \to M$ be a vector bundle over a manifold M. Then E is a Lie groupoid with base M, the source and target maps are both equal to the projection p and the partial multiplication is just the addition in every fiber. $\Diamond$

The last example will not be used in this thesis, but it is also interesting and worth to be mentioned here.

Example 1.1.8 Let M be a manifold. Then the set $\Pi(M)$ of homotopy classes $\langle\gamma\rangle$ relative endpoints of smooth paths $\gamma : [0, 1] \to M$ is a groupoid on M with respect to the following structure: the source and target projections are $\mathsf{s}(\langle\gamma\rangle) = \gamma(0)$, $\mathsf{t}(\langle\gamma\rangle) = \gamma(1)$, the object inclusion map is $m \mapsto 1_m = \langle\kappa_m\rangle$, where κ_m is the constant path at m, and the partial multiplication is $\langle\delta\rangle \star \langle\gamma\rangle = \langle\delta * \gamma\rangle$, where $\delta * \gamma$ is the standard concatenation of γ followed by δ, namely $(\delta * \gamma)(t) = \gamma(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $(\delta * \gamma)(t) = \delta(2t - 1)$ for $\frac{1}{2} \leq t \leq 1$. The inverse of $\langle\gamma\rangle$ is $\langle\bar{\gamma}\rangle$, where $\bar{\gamma}$ is the reverse of the path γ, i.e., $\bar{\gamma}(t) = \gamma(1 - t)$ for all $t \in [0, 1]$.
With this structure, $\Pi(M)$ is the *fundamental groupoid* of M. The *vertex* groups $\mathsf{s}^{-1}(p) \cap \mathsf{t}^{-1}(p)$ for $p \in M$ are the fundamental groups of M, and the s-fibers are the universal covering spaces of M. $\Diamond$

Left and right translations, bisections

Definition 1.1.9 *Let $G \rightrightarrows P$ be a Lie groupoid and choose $g \in G$. The* left translation *by g is defined by:*

$$L_g : \mathsf{t}^{-1}(\mathsf{s}(g)) \to \mathsf{t}^{-1}(\mathsf{t}(g)), \qquad h \mapsto L_g(h) = g \star h.$$

In the same manner, the right translation *by g is*

$$R_g : \mathsf{s}^{-1}(\mathsf{t}(g)) \to \mathsf{s}^{-1}(\mathsf{s}(g)), \qquad h \mapsto R_g(h) = h \star g.$$

A right translation on G *is a pair of diffeomorphisms $\Phi : G \to G$, $\phi : P \to P$ such that $\mathsf{s} \circ \Phi = \phi \circ \mathsf{s}$, $\mathsf{t} \circ \Phi = \mathsf{t}$ and, for all $p \in P$, the map $\Phi|_{\mathsf{s}^{-1}(p)} : \mathsf{s}^{-1}(p) \to \mathsf{s}^{-1}(\phi(p))$ is R_g for some $g \in G$. A* bisection *of $G \rightrightarrows P$ is a smooth map $K : P \to G$ which is right-inverse to $\mathsf{t} : G \to P$ and such that $\mathsf{s} \circ K$ is a diffeomorphism. The set of bisections of G will be denoted by $\mathcal{B}(G)$.*

If $K : P \to G$ is a bisection of $G \rightrightarrows P$, then the *right translation by K* is a right translation:

$$R_K : G \to G, \qquad g \mapsto R_{K(\mathsf{s}(g))}(g) = g \star K(\mathsf{s}(g)).$$

We will also use the *left translation by K*,

$$L_K : G \to G, \qquad g \mapsto L_{K((s \circ K)^{-1}(\mathsf{t}(g)))}(g).$$

The set $\mathcal{B}(G)$ of bisections of G has the structure of a group. For $K, L \in \mathcal{B}(G)$, the product $L \star K$ is given by

$$L \star K : P \to G, \quad (L \star K)(p) = L(p) \star K((\mathsf{s} \circ L)(p)) \quad \forall p \in P.$$

The composition $\mathsf{t} \circ (L \star K)$ is equal to Id_P since $(\mathsf{t} \circ (L \star K))(p) = \mathsf{t}(L(p)) = p$ for all $p \in P$, and the composition $\mathsf{s} \circ (L \star K)$ is equal to $(\mathsf{s} \circ K) \circ (\mathsf{s} \circ L)$, which is a diffeomorphism of P.
The identity element in $\mathcal{B}(G)$ is the identity section $\epsilon : P \hookrightarrow G$. The inverse $K^{-1} : P \to G$ of $K \in \mathcal{B}(G)$ is given by

$$K^{-1}(p) = \left(K\left((\mathsf{s} \circ K)^{-1}(p)\right)\right)^{-1}$$

for all $p \in P$. Indeed, it is easy to verify that $\mathsf{t} \circ K^{-1} = \mathrm{Id}_P$, $\mathsf{s} \circ K^{-1} = (\mathsf{s} \circ K)^{-1}$ and we compute for all $p \in P$:

$$\begin{aligned}
\left(\left(K^{-1}\right) \star K\right)(p) &= K^{-1}(p) \star K\left(\left(\mathsf{s} \circ K^{-1}\right)(p)\right) \\
&= \left(K\left((\mathsf{s} \circ K)^{-1}(p)\right)\right)^{-1} \star K\left((\mathsf{s} \circ K)^{-1}(p)\right) \\
&= \mathsf{s}\left(K\left((\mathsf{s} \circ K)^{-1}(p)\right)\right) = p
\end{aligned}$$

and

$$\left(K\star\left(K^{-1}\right)\right)(p)=K(p)\star K^{-1}((\mathsf{s}\circ K)(p))=K(p)\star\left(K\left((\mathsf{s}\circ K)^{-1}((\mathsf{s}\circ K)(p))\right)\right)^{-1}$$
$$=K(p)\star(K(p))^{-1}=(\mathsf{t}\circ K)(p)=p.$$

We check finally that $R_{L\star K}=R_K\circ R_L$ for all $K,L\in\mathcal{B}(G)$: for an arbitrary $g\in G$, we have

$$R_{L\star K}(g)=g\star(L\star K)(\mathsf{s}(g))=g\star L(\mathsf{s}(g))\star K((\mathsf{s}\circ L)(\mathsf{s}(g)))$$
$$=(R_L(g))\star K(\mathsf{s}(R_L(g)))=R_K(R_L(g)).$$

Since $R_\epsilon=\mathrm{Id}_G$, we find then simultaneously the equality $R_{K^{-1}}=R_K^{-1}$ for all $K\in\mathcal{B}(G)$.
In the following, we will also consider *local bisections* of G without saying it always explicitly. A local bisection of $G\rightrightarrows P$ is a map $K:U\to G$ defined on an open set $U\subseteq P$ such that $\mathsf{t}\circ K=\mathrm{Id}_U$ and $\mathsf{s}\circ K$ is a diffeomorphism on its image. The set of local bisections of $G\rightrightarrows P$ with the domain of definition $U\subseteq P$ is written $\mathcal{B}_U(G)$. The local right translation induced by the local bisection $K:U\to G$ is the map $R_K:\mathsf{s}^{-1}(U)\to\mathsf{s}^{-1}((\mathsf{s}\circ K)(U))$, $g\mapsto g\star K(\mathsf{s}(g))$.

Example 1.1.10 A bisection of a Lie group G is just an element $g\in G$. The right translation associated to it is the right translation by g. $\Diamond$

Example 1.1.11 Let M be a smooth manifold and consider the pair Lie groupoid $M\times M\rightrightarrows M$ associated to it.
A map $K:M\simeq\Delta_M\to M\times M$ is a bisection of $M\times M\rightrightarrows M$ if and only if $\mathrm{pr}_1\circ K=\mathrm{Id}_M$ and $\mathrm{pr}_2\circ K$ is a diffeomorphism of M. The group of bisections of $M\times M\rightrightarrows M$ is hence exactly the group $\mathrm{Diff}(M)$ of diffeomorphisms of M; a bisection $K\in\mathcal{B}(M\times M)$ is given by $K(m)=(m,\phi_K(m))$ with some diffeomorphism $\phi_K:M\to M$. The map $R_K:M\times M\to M\times M$ is then given by $R_K=\mathrm{Id}_M\times\phi_K$ and its inverse equals $R_K^{-1}=\mathrm{Id}_M\times\phi_K^{-1}$. $\Diamond$

Groupoid actions Let $G\rightrightarrows P$ be a groupoid and M a set with a map $\mathsf{J}:M\to P$. Consider the set $G\times_P M=\{(g,m)\in G\times M\mid \mathsf{s}(g)=\mathsf{J}(m)\}$.
A groupoid action of $G\rightrightarrows P$ on $\mathsf{J}:M\to P$ is a map $\Phi:G\times_P M\to M$, $\Phi(g,m)=g\cdot m=gm$ such that

- $\mathsf{J}(g\cdot m)=\mathsf{t}(g)$ for all $(g,m)\in G\times_P M$,
- $g\cdot(h\cdot m)=(g\star h)\cdot m$ for all $(h,m)\in G\times_P M$, and $g\in G$ such that $\mathsf{s}(g)=\mathsf{t}(h)$,
- $1_{\mathsf{J}(m)}\cdot m=m$ for all $m\in M$.

The map J is sometimes called the *moment map*.

Example 1.1.12 Let $G\rightrightarrows P$ be a groupoid.

1. $G\rightrightarrows P$ acts obviously on $\mathsf{t}:G\to P$ via the multiplication.
2. $G\rightrightarrows P$ acts on $\mathrm{Id}_P:P\to P$ via $\Phi:G\times_P P\to P$, $(g,p)\mapsto\mathsf{t}(g\star p)=\mathsf{t}(g)$. $\Diamond$

1.1.2 Quotients by normal subgroupoids or normal subgroupoid systems.

We follow the definitions and conventions of Mackenzie (1987) for normal subgroupoids and the quotient defined by a normal subgroupoid.

Definition 1.1.13 *Let $G\rightrightarrows P$ be a groupoid. A normal subgroupoid N of $G\rightrightarrows P$ is a wide subgroupoid of G such that for all $n \in N \cap \mathcal{I}(G)$ and $g \in G$ such that $\mathsf{s}(g) = \mathsf{s}(n) = \mathsf{t}(n)$, the product gng^{-1} is again an element of N.*

If $N \subseteq G$ is a normal subgroupoid of $G\rightrightarrows P$, one can define the two equivalence relations $\sim_\circ$ on P and $\sim$ on G as follows:

$$p \sim_\circ q \qquad \Longleftrightarrow \qquad \exists\, n \in N \text{ such that } \mathsf{t}(n) = p, \mathsf{s}(n) = q$$

and

$$g \sim h \qquad \Longleftrightarrow \qquad \exists\, n_1, n_2 \in N \text{ such that } g = n_1 h n_2.$$

It is easy to check that the quotient $G/\sim$ has the structure of a groupoid over $P/\sim_\circ$.

If $(F, f) : (G\rightrightarrows P) \to (G'\rightrightarrows P')$ is a Lie groupoid morphism, then the *kernel* of (F, f) is the set $K = \{g \in G \mid F(g) \in P'\}$, which is a normal subgroupoid of $G\rightrightarrows P$. The induced groupoid $G/\sim_K \rightrightarrows P/\sim_{K,\circ}$ as above is in general not equal to $G'\rightrightarrows P'$, even if the Lie groupoid morphism is surjective. Hence, this concept of kernel does not adequately measure injectivity, in the sense that surjective morphisms are not determined by their kernel as in the group case (Examples 3.1.1 and 3.1.2 will illustrate this). This is why there is also a more general notion of quotient of a Lie groupoid by a normal object in Mackenzie (2005).
The Lie groupoid morphism (F, f) is a *fibration* if both $f : P \to P'$ and $F^! : G \to f^! G'$ are surjective submersions, where $f^! G'$ is the pullback and $F^! : G \to f^! G'$ the induced base preserving morphism. The map f defines a wide subgroupoid $\mathcal{R}(f)$ of $P \times P \rightrightarrows P$ defined by $\{(p, q) \in P \times P \mid f(p) = f(q)\}$. Set $G/K := \{gK \mid g \in G\}$, where $gK = \{g \star k \mid k \in K, \mathsf{s}(g) = \mathsf{t}(k)\}$. Then there is an induced action θ of $\mathcal{R}(f)$ on $\mathsf{J} : G/K \to P$, $\mathsf{J}(gK) = \mathsf{t}(g)$ given by $(p, q) \cdot gK = hK$, where $h \in G$ is such that $F(h) = F(g)$ and $\mathsf{t}(h) = p$. The triple $(K, \mathcal{R}(f), \theta)$ is called the *kernel system of* (F, f) (see Mackenzie (2005)). It is a normal groupoid system in the following sense, and the Lie groupoid $G'\rightrightarrows P'$ is the quotient of $G\rightrightarrows P$ by $(K, \mathcal{R}(f), \theta)$.

Definition 1.1.14 (Mackenzie (2005)) *Let $G\rightrightarrows P$ be a Lie groupoid. A* normal groupoid system *in $G\rightrightarrows P$ is a triple $\mathcal{N} = (N, \mathcal{R}, \theta)$ where N is a closed, embedded, wide Lie subgroupoid of G, $\mathcal{R}$ is a closed, embedded, wide Lie subgroupoid of the pair groupoid $P \times P \rightrightarrows P$ and θ is an action of $\mathcal{R}$ on the map $\mathsf{J} : G/N \to P$, $\mathsf{J}(gN) = \mathsf{t}(g)$ such that the following conditions hold*

1. *For $(p, q) \in \mathcal{R}$ and $gN \in G/N$ such that $\mathsf{t}(g) = q$, if $\theta((p, q), gN) = hN$, then $(\mathsf{s}(h), \mathsf{s}(g)) \in \mathcal{R}$.*

2. *For* $(p,q) \in \mathcal{R}$*, we have* $\theta((p,q), qN) = pN$*.*

3. *Consider* $(p,q) \in \mathcal{R}$ *and* $gN \in G/N$ *with* $\mathsf{J}(gN) = q$*, and* $h \in G$ *with* $\mathsf{t}(h) = \mathsf{s}(g)$*. Then if* $\theta((p,q), gN) = g'N$ *and* $\theta((\mathsf{s}(g'), \mathsf{s}(g)), hN) = h'N$*, then* $\theta((p,q), ghN) = g'h'N$*.*

Set
$$\mathcal{S} = \{(h,g) \in G \times G \mid (\mathsf{t}(h), \mathsf{t}(g)) \in \mathcal{R} \text{ and } \theta((\mathsf{t}(h), \mathsf{t}(g)), gN) = hN\}.$$
Then $\mathcal{S}$ is a closed embedded wide Lie subgroupoid of $G \times G \rightrightarrows G$ and $\mathcal{S} \rightrightarrows \mathcal{R}$ is a Lie subgroupoid of the Cartesian product Lie groupoid $G \times G \rightrightarrows P \times P$. If $\mathrm{pr}_{\mathcal{S}} : \mathcal{S} \to \mathrm{G}$, $(g,h) \mapsto h$ and $\mathrm{pr}_{\mathcal{R}} : \mathcal{R} \to \mathrm{P}$, $(p,q) \mapsto q$ are the projections, the square

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\mathrm{pr}_{\mathcal{S}}"] \arrow[d, "\mathsf{t}\times\mathsf{t}"'] & G \arrow[d, "\mathsf{t}"] \\ \mathcal{R} \arrow[r, "\mathrm{pr}_{\mathcal{R}}"'] & P \end{tikzcd}$$

is *versal*, i.e., the pullback $\mathcal{R} \times_P G$ exists as a submanifold of $\mathcal{R} \times G$ and the induced map $\mathcal{S} \to \mathcal{R} \times_P G$ is a surjective submersion.
If $\mathcal{N} = (N, \mathcal{R}, \theta)$ is a normal subgroupoid system, then $\mathcal{R} = \mathcal{R}(f)$ with a surjective submersion $f : P \to P'$. If G' is the set of orbits of G/N under θ and $< gN >$ is the orbit of $gN \in G/N$, then we can write $F : G \to G'$, $F(g) =< gN >$ and we can define groupoid projections on G' with base P' by $\mathsf{t}' \circ F = f \circ \mathsf{t}$ and $\mathsf{s}' \circ F = f \circ \mathsf{s}$. The multiplication of $< gN >, < hN >\in G'$ with $\mathsf{s}' < gN >= \mathsf{t}' < hN >$ (which is equivalent to $(\mathsf{s}(g), \mathsf{t}(h)) \in \mathcal{R}$) is given by
$$< gN > \star < hN >=< gh'N >,$$
where $h'N = \theta((\mathsf{s}(g), \mathsf{t}(h)), hN)$. With these structure maps, the pair $G' \rightrightarrows P'$ has the structure of a Lie groupoid such that (F, f) is a fibration. It is called the *quotient Lie groupoid of* $G \rightrightarrows P$ *by the normal subgroupoid system* $\mathcal{N}$. It corresponds to the unique Lie groupoid structure on the quotient sets $G' = G/\mathcal{S}$, $P' = P/\mathcal{R}$ such that the natural projections $\mathrm{pr}_{\mathcal{S}} : G \to G'$, $\mathrm{pr}_{\mathcal{R}} : P \to P'$ form a morphism of Lie groupoids (see Mackenzie (2005)).

In the following, we will use the name *regular normal subgroupoid system* for the smooth object defined above, and say *normal subgroupoid system* for a triple $\mathcal{N} = (N, \mathcal{R}, \theta)$ without the condition that N and $\mathcal{R}$ are embedded in G and $P \times P$, respectively. We get a groupoid structure on $G/\mathcal{S} \rightrightarrows P/\mathcal{R}$, but without necessarily a smooth structure.

1.1.3 Lie algebroids

The concept of a Lie algebroid is due to Pradines (1967) and generalizes at the same time the concept of Lie algebra and of the tangent bundle of a manifold. We will see below that the Lie algebroid of a Lie groupoid is obtained in a process that generalizes the construction of the Lie algebra of a Lie group.

Definition 1.1.15 *Let M be a manifold. A* Lie algebroid *on M is a vector bundle (A, q, M) together with a vector bundle map $\mathsf{a} : A \to TM$ over M, called* the anchor *of A, and a bracket $[\cdot\,,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ which is $\mathbb{R}$-bilinear and alternating, satisfies the Jacobi identity, and is such that*

$$[X, f \cdot Y] = f \cdot [X, Y] + \mathsf{a}(X)(f) \cdot Y. \tag{1.1}$$

for all $X, Y \in \Gamma(A)$.

The identity

$$\mathsf{a}([X, Y]) = [\mathsf{a}(X), \mathsf{a}(Y)] \tag{1.2}$$

follows from (1.1) for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$ by expending out $[X, [Y, fZ]]$ in two ways (see Herz (1953), Kosmann-Schwarzbach and Magri (1990)).

The manifold M is then *the base of A*. If $A' \to M$ is a second Lie algebroid on the same base M, then a *morphism of Lie algebroids* $\varphi : A \to A'$ over M, or a *base-preserving morphism of Lie algebroids* is a vector bundle morphism such that $\mathsf{a}' \circ \varphi = \mathsf{a}$ and $\varphi[X, Y] = [\varphi(X), \varphi(Y)]$ for all $X, Y \in \Gamma(A)$.

Example 1.1.16 Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is a Lie algebroid over a point. $\Diamond$

Example 1.1.17 Let M be a smooth manifold. Then the tangent space TM of M is a Lie algebroid on M with anchor the identity and bracket the Lie bracket on vector fields. $\Diamond$

The Lie algebroid of a Lie groupoid Let $G \rightrightarrows P$ be a Lie groupoid. A smooth vector field $X \in \mathfrak{X}(G)$ is *left invariant* if it is tangent to the t-fibers, i.e., $X \in \Gamma(T^{\mathsf{t}}G)$ and

$$X(gh) = T_h L_g X(h)$$

for any composable pair $(g, h) \in G \times_P G$. It is easy to see that X is then completely determined by its restriction to P, since it satisfies $X(g) = T_{\mathsf{s}(g)} L_g X(\mathsf{s}(g))$ for all $g \in G$. Given $X \in \Gamma(T^{\mathsf{t}}G|_P)$, we write X^l for the left invariant vector field defined by X, i.e., $X^l(g) = T_{\mathsf{s}(g)} L_g X(\mathsf{s}(g))$ for all $g \in G$. If X, Y are sections of $T^{\mathsf{t}}G|_P =: AG$, then the Lie bracket $\left[X^l, Y^l\right]$ is again left invariant. The bracket $[X, Y]_{AG} \in \Gamma(AG)$ of $X, Y \in \Gamma(AG)$ is defined as the smooth section of AG such that

$$\left[X^l, Y^l\right] = \left([X, Y]_{AG}\right)^l.$$

The vector bundle $AG \to P$ inherits then a Lie algebroid structure, with the bracket defined above and the anchor map $\mathsf{a} : AG \to TP$ defined by $\mathsf{a}(u_p) = T_p\mathsf{s}(u_p) \in T_pP$ for all $u_p \in A_pG$, $p \in P$. The triple $(AG, \mathsf{a}, [\cdot\,,\cdot]_{AG})$ is the *Lie algebroid of the Lie groupoid G*. For simplicity, we will write $[\cdot\,,\cdot]$ for the Lie algebroid bracket.

Note that if $X \in \Gamma(AG)$, the vector field X^l satisfies $X^l \sim_{\mathsf{s}} \mathsf{a}(X) \in \mathfrak{X}(P)$ since we have $T_g\mathsf{s}X^l(g) = T_g\mathsf{s}(T_{\mathsf{s}(g)} L_g X(\mathsf{s}(g))) = T_{\mathsf{s}(g)}\mathsf{s}X(\mathsf{s}(g))$ for all $g \in G$.

In Mackenzie (2000), the Lie algebroid AG of $G \rightrightarrows P$ is defined as $AG = T^{\mathsf{s}}G|_P$ with $T\mathsf{t}$ as anchor map and the bracket defined with the right invariant vector fields. To avoid confusions, we will write $\left(\widetilde{AG}, [\cdot\,,\cdot]_{\widetilde{AG}}, \tilde{\mathsf{a}}\right)$ for this Lie algebroid over P.

The exponential map We recall here also the definition of the *exponential map for a Lie groupoid*, see Mackenzie (2005).
Let $G \rightrightarrows P$ be a Lie groupoid and choose $X \in \Gamma(AG)$. Let $\{\phi_t^X : U \to U_t\}$ be a local flow for $X^l \in \mathfrak{X}(G)$. Since $T_g \mathsf{t} X^l(g) = 0$ for all $g \in G$, we have $(\mathsf{t} \circ \phi_t^X)(g) = \mathsf{t}(g)$ for all $t \in \mathbb{R}$ and $g \in G$ where this makes sense. For each $t \in \mathbb{R}$ where this is defined and $p \in P$, the map ϕ_t^X restricts to $\phi_t^X : \mathsf{t}^{-1}(p) \to \mathsf{t}^{-1}(p)$. Choose $h \in G$ such that $\mathsf{s}(h) = p$. We have then $L_h : \mathsf{t}^{-1}(p) \to \mathsf{t}^{-1}(\mathsf{t}(h))$ and $L_h \circ \phi_t^X = \phi_t^X \circ L_h$ since the vector field X^l satisfies $X^l(h \star g) = T_g L_h X^l(g)$ for all $g \in \mathsf{t}^{-1}(p)$. Recall that $\bar{X} := \mathsf{a}(X) \in \Gamma(P)$ is defined on $\mathsf{s}(U) := V \subseteq P$ and is such that $X^l \sim_{\mathsf{s}} \bar{X}$. Let $\bar{\phi}^X$ be the flow of $\bar{X}$. Then we have $\{\bar{\phi}_t^X : V \to V_t\}$, where $V_t = \mathsf{s}(U_t)$, and $\bar{\phi}_t^X \circ \mathsf{s} = \mathsf{s} \circ \phi_t^X$ for all t where this makes sense.
Set $\operatorname{Exp}(tX)(p) := g^{-1} \star \phi_t^X(g)$ for any $g \in U \cap \mathsf{s}^{-1}(p)$. We have then $\mathsf{t} \circ \operatorname{Exp}(tX) = \mathrm{Id}_V$ and $\mathsf{s} \circ \operatorname{Exp}(tX) = \bar{\phi}_t^X$ is a local diffeomorphism on its image V_t. The map $\operatorname{Exp}(tX) : P \to G$ is thus a local bisection of G and, for any $g \in U$, we have by definition

$$\phi_t^X(g) = g \star \operatorname{Exp}(tX)(\mathsf{s}(g)) = R_{\operatorname{Exp}(tX)}(g).$$

Each ϕ_t^X is hence the restriction to U of a unique local right translation $R_{\operatorname{Exp}(tX)}$ with $\operatorname{Exp}(tX) \in \mathcal{B}_V(G)$. This is summarized in the following proposition.

Proposition 1.1.18 (Mackenzie (2005)) *Let $G \rightrightarrows P$ be a Lie groupoid, choose $X \in \Gamma(AG)$ and set $W = \operatorname{Dom}(X) \subseteq P$. For all $p \in W$ there exists an open neighborhood U of p in W, a flow neighborhood for X, an $\varepsilon > 0$ and a unique smooth family of local bisections $\operatorname{Exp}(tX) \in \mathcal{B}_U(G)$, $|t| < \varepsilon$, such that:*

1. $\left.\frac{d}{dt}\right|_{t=0} \operatorname{Exp}(tX) = X$,
2. $\operatorname{Exp}(0X) = \mathrm{Id}_U$,
3. $\operatorname{Exp}((t+s)X) = \operatorname{Exp}(tX) \star \operatorname{Exp}(sX)$, *if* $|t|, |s|, |s+t| < \varepsilon$,
4. $\operatorname{Exp}(-tX) = (\operatorname{Exp}(tX))^{-1}$,
5. $\{\mathsf{s} \circ \operatorname{Exp}(tX) : U \to U_t\}$ *is a local* 1*-parameter group of transformations for* $\mathsf{a}(X) \in \mathfrak{X}(P)$.

Note that we can show in the same manner that the flow of a right invariant vector field Y^r is the left translation by a family of bisections $\{L_t\}$ of G satisfying $\mathsf{s} \circ L_t = \mathrm{Id}$ on their domains of definition and such that $\mathsf{t} \circ L_t$ are diffeomorphisms on their images. Hence, the flow of Y^r commutes with the flow of X^l for any left invariant vector field X^l and we get the fact that

$$\left[Y^r, X^l\right] = 0 \tag{1.3}$$

for all $Y \in \Gamma\left(T_P^{\mathsf{s}} G\right)$ and $X \in \Gamma(AG)$.
Note also that if $G \rightrightarrows P$ is a Lie groupoid, then the set of values $\operatorname{Exp}(tX)(p)$, for all $X \in \Gamma(AG)$, $p \in P$ and $t \in \mathbb{R}$ where this makes sense, is the identity-component subgroupoid $C(G)$ of $G \rightrightarrows P$ (see Mackenzie and Xu (2000), Mackenzie (2005)).

1.1.4 Lie bialgebroids, associated Courant algebroids

Let M be a smooth manifold and $(A \to M, \mathsf{a}, [\cdot\,,\cdot])$ a Lie algebroid on M. For $k \geq 0$, let $\bigwedge^k A^*$ denote the k-th exterior power bundle on M. The *exterior derivative* $\mathbf{d}$: $\Gamma\left(\bigwedge^k A^*\right) \to \Gamma\left(\bigwedge^{k+1} A^*\right)$ is defined by

$$\begin{aligned}\mathbf{d}\phi(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \mathsf{a}(X_i)\left(\phi\left(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}\right)\right)\\ + \sum_{i<j} (-1)^{i+j} \phi\left([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j \ldots, X_{k+1}\right)\end{aligned}$$

for $\phi \in \Gamma\left(\bigwedge^k A^*\right)$, $X_i \in \Gamma(A)$, $1 \leq i \leq k+1$ and $(\mathbf{d}f)(X) = \mathsf{a}(X)(f)$ for all $f \in C^\infty(M) = \Gamma\left(\bigwedge^0 A^*\right)$ and $X \in \Gamma(A)$.
For $X \in \Gamma(A)$ and $k \geq 0$, the Lie derivative $\pounds_X : \Gamma\left(\bigwedge^k A^*\right) \to \Gamma\left(\bigwedge^k A^*\right)$ is defined by

$$(\pounds_X \phi)(Y_1, \ldots, Y_k) = \mathsf{a}(X)\left(\phi\left(Y_1, \ldots, Y_k\right)\right) - \sum_{i=1}^{k} \phi\left(Y_1, \ldots, [X, Y_i], \ldots, Y_k\right)$$

for $\phi \in \Gamma\left(\bigwedge^k A^*\right)$, $Y_i \in \Gamma(A)$, $1 \leq i \leq k$ (see for instance Mackenzie and Xu (1994) for a quick review of these objects). In a similar way, the Schouten bracket and Lie derivative of multivector fields extend to A. The *generalized Schouten bracket*

$$[\cdot\,,\cdot] : \Gamma\left(\bigwedge^k A\right) \times \Gamma\left(\bigwedge^m A\right) \to \Gamma\left(\bigwedge^{k+m-1} A\right) \tag{1.4}$$

is characterized by the conditions that $[\cdot\,,\cdot] : \Gamma\left(\bigwedge^1 A\right) \times \Gamma\left(\bigwedge^1 A\right) \to \Gamma\left(\bigwedge^1 A\right)$ coincide with the Lie algebroid bracket, that $[X, f] = \mathsf{a}(X)(f)$ for $X \in \Gamma(A)$ and $f \in C^\infty(M)$ and that the properties

$$\begin{aligned}[D_1, D_2] &= -(-1)^{(k-1)(m-1)}[D_2, D_1],\\ (-1)^{(k-1)(n-1)}[[D_1, D_2], D_3] &+ (-1)^{(m-1)(k-1)}[[D_2, D_3], D_1]\\ &+ (-1)^{(n-1)(m-1)}[[D_3, D_1], D_2] = 0,\\ [D_1, D_2 \wedge D_3] &= [D_1, D_2] \wedge D_3 + (-1)^{k(n-1)} D_2 \wedge [D_1, D_3]\end{aligned}$$

hold for all $D_1 \in \Gamma\left(\bigwedge^k A\right)$, $D_2 \in \Gamma\left(\bigwedge^m A\right)$ and $D_3 \in \Gamma\left(\bigwedge^n A\right)$ (see Mackenzie and Xu (1994), Kosmann-Schwarzbach and Magri (1990)).
Assume that the dual $A^* \to M$ of A is endowed with a Lie algebroid structure $(A^* \to M, \mathsf{a}_*, [\cdot\,,\cdot]_*)$ such that the following (equivalent) identities holds for the induced maps $\mathbf{d}$: $\Gamma\left(\bigwedge^\bullet A^*\right) \to \Gamma\left(\bigwedge^\bullet A^*\right)$ and $\mathbf{d}_* : \Gamma\left(\bigwedge^\bullet A\right) \to \Gamma\left(\bigwedge^\bullet A\right)$ and the brackets $[\cdot\,,\cdot]$ (respectively $[\cdot\,,\cdot]_*$) induced on $\Gamma\left(\bigwedge^\bullet A\right)$ (respectively $\Gamma\left(\bigwedge^\bullet A^*\right)$):

$$\mathbf{d}[\cdot\,,\cdot]_* = [\mathbf{d}\cdot\,,\cdot]_* + [\cdot\,,\mathbf{d}\cdot]_* \quad \text{and} \quad \mathbf{d}_*[\cdot\,,\cdot] = [\mathbf{d}_*\cdot\,,\cdot] + [\cdot\,,\mathbf{d}_*\cdot]$$

(see for instance Kosmann-Schwarzbach (1995)). Then the pair (A, A^*) is a *Lie bialgebroid* and the direct sum vector bundle $A \oplus A^* \to M$ endowed with the map $\rho = \mathsf{a} \oplus \mathsf{a}_*$, the symmetric non degenerate bilinear form $\langle \cdot\,, \cdot \rangle$ given by $\langle (x_m, \alpha_m), (y_m, \beta_m) \rangle = \alpha_m(y_m) + \beta_m(x_m)$ for all $(x_m, \alpha_m), (y_m, \beta_m) \in (A \oplus A^*)(m)$, $m \in M$, and the bracket on its sections given by

$$[(X, \alpha), (Y, \beta)] = \Big([X, Y] + \pounds_\alpha Y - \pounds_\beta X - \frac{1}{2}\mathbf{d}_*(\alpha(Y) - \beta(X)), \qquad (1.5)$$
$$[\alpha, \beta]_* + \pounds_X \beta - \pounds_Y \alpha + \frac{1}{2}\mathbf{d}(\alpha(Y) - \beta(X)) \Big)$$

for all $(X, \alpha), (Y, \beta) \in \Gamma(A \oplus A^*)$ is a Courant algebroid in the sense of the definition below.

Definition 1.1.19 (Liu et al. (1997)) *A Courant algebroid over a manifold M is a vector bundle $\mathsf{E} \to M$ equipped with a fiberwise non degenerate symmetric bilinear form $\langle \cdot\,, \cdot \rangle$, a skew-symmetric bracket $[\cdot\,, \cdot]$ on the smooth sections $\Gamma(\mathsf{E})$, and a vector bundle map $\rho : \mathsf{E} \to TM$ called the anchor, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(\mathsf{E})$ and $f \in C^\infty(M)$:*

1. $[[e_1, e_2], e_3]] + \text{c.p.} = \frac{1}{3}\mathcal{D}\big(\langle [e_1, e_2], e_3 \rangle + \text{c.p.}\big)$,
2. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$,
3. $[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - \langle e_1, e_2 \rangle \mathcal{D} f$,
4. $\rho \circ \mathcal{D} = 0$, *i.e., for any* $f, g \in C^\infty(M)$, $\langle \mathcal{D} f, \mathcal{D} g \rangle = 0$,
5. $\rho(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2] + \mathcal{D}\langle e_1, e_2 \rangle, e_3 \rangle + \langle e_2, [e_1, e_3] + \mathcal{D}\langle e_1, e_3 \rangle \rangle$,

where $\mathcal{D} : C^\infty(M) \to \Gamma(\mathsf{E})$ is defined by

$$\langle \mathcal{D} f, e \rangle = \frac{1}{2}\rho(e)(f)$$

for all $f \in C^\infty(M)$ and $e \in \Gamma(\mathsf{E})$, that is, $\mathcal{D} = \frac{1}{2}\beta^{-1} \circ \rho^* \circ \mathbf{d} : C^\infty(M) \to \Gamma(\mathsf{E})$. Here, $\beta : \mathsf{E} \to \mathsf{E}^*$ is the isomorphism defined by the non degenerate bilinear form $\langle \cdot\,, \cdot \rangle$.

Example 1.1.20 Consider a smooth manifold M, the Lie algebroid $(TM, [\cdot\,, \cdot], \mathsf{a} = \mathrm{Id}_{TM})$ and its dual, the cotangent space T^*M endowed with the trivial bracket $[\cdot\,, \cdot]_* = 0$ and the trivial anchor map $\mathsf{a}_* = 0$. The map $\mathbf{d}$ induced by TM on the sections of $\bigwedge^\bullet T^*M$ is here simply the usual de Rham derivative. The map $\mathbf{d}_*$ induced by $(T^*M, 0, 0)$ on the sections of $\bigwedge^\bullet TM$ is trivial since $\mathbf{d}_*(f)(\alpha) = \mathsf{a}_*(\alpha)(f) = 0$ for all $\alpha \in \Omega^1(M)$.
Hence, the pair (TM, T^*M) is a Lie bialgebroid since the equation $\mathbf{d}[\cdot\,, \cdot]_* = [\mathbf{d}\cdot\,, \cdot]_* + [\cdot\,, \mathbf{d}\cdot]_*$ is trivially satisfied.

The direct sum $\mathsf{P}_M = TM \times_M T^*M$ endowed with the projection on TM as anchor map, $\rho = \mathrm{pr}_{TM}$, the symmetric bracket $\langle \cdot\,, \cdot\rangle$ given by

$$\langle (v_m, \alpha_m), (w_m, \beta_m)\rangle = \alpha_m(w_m) + \beta_m(v_m) \tag{1.6}$$

for all $m \in M$, $v_m, w_m \in T_mM$ and $\alpha_m, \beta_m \in T_m^*M$ and the *Courant bracket* given by

$$\begin{aligned} [(X,\alpha),(Y,\beta)] &= \left([X,Y], \pounds_X\beta - \pounds_Y\alpha + \frac{1}{2}\mathbf{d}(\alpha(Y) - \beta(X))\right) \\ &= \left([X,Y], \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha - \frac{1}{2}\mathbf{d}\langle (X,\alpha),(Y,\beta)\rangle\right) \end{aligned} \tag{1.7}$$

for all $(X,\alpha),(Y,\beta) \in \Gamma(\mathsf{P}_M)$, is then a Courant algebroid. The map $\mathcal{D} : C^\infty(M) \to \Gamma(\mathsf{P}_M)$ is given by $\mathcal{D}f = \frac{1}{2}(0, \mathbf{d}f)$. $\Diamond$

1.1.5 "Higher" Lie groupoids

The tangent prolongation of a Lie groupoid Let $G\rightrightarrows P$ be a Lie groupoid. Applying the tangent functor to each of the maps defining G yields a Lie groupoid structure on TG with base TP, source $T\mathsf{s}$, target $T\mathsf{t}$ and multiplication $T\mathsf{m} : T(G \times_P G) \to TG$. The identity at $v_p \in T_pP$ is $1_{v_p} = T_p\epsilon v_p$. This defines the *tangent prolongation $TG\rightrightarrows TP$ of $G\rightrightarrows P$* or the *tangent groupoid associated to $G\rightrightarrows P$*.

The cotangent Lie groupoid defined by a Lie groupoid If $G\rightrightarrows P$ is a Lie groupoid, then there is also an induced Lie groupoid structure on $T^*G\rightrightarrows A^*G = (TP)^\circ$. The source map $\hat{\mathsf{s}} : T^*G \to A^*G$ is given by

$$\hat{\mathsf{s}}(\alpha_g) \in A^*_{\mathsf{s}(g)}G \text{ for } \alpha_g \in T_g^*G, \qquad \hat{\mathsf{s}}(\alpha_g)(u_{\mathsf{s}(g)}) = \alpha_g(T_{\mathsf{s}(g)}L_g u_{\mathsf{s}(g)})$$

for all $u_{\mathsf{s}(g)} \in A_{\mathsf{s}(g)}G$, and the target map $\hat{\mathsf{t}} : T^*G \to A^*G$ is given by

$$\hat{\mathsf{t}}(\alpha_g) \in A^*_{\mathsf{t}(g)}G, \qquad \hat{\mathsf{t}}(\alpha_g)(u_{\mathsf{t}(g)}) = \alpha_g\big(T_{\mathsf{t}(g)}R_g(u_{\mathsf{t}(g)} - T_{\mathsf{t}(g)}\mathsf{s}u_{\mathsf{t}(g)})\big)$$

for all $u_{\mathsf{t}(g)} \in A_{\mathsf{t}(g)}G$. If $\hat{\mathsf{s}}(\alpha_g) = \hat{\mathsf{t}}(\alpha_h)$, then the product $\alpha_g \star \alpha_h$ is defined by

$$(\alpha_g \star \alpha_h)(v_g \star v_h) = \alpha_g(v_g) + \alpha_h(v_h)$$

for all composable pairs $(v_g, v_h) \in T_{(g,h)}(G \times_P G)$. This Lie groupoid structure was introduced in Coste et al. (1987) and is explained for instance in Coste et al. (1987), Pradines (1988) and Mackenzie (2005). Note that the original definition was the following: let Λ_G be the graph of the partial multiplication m in G, i.e.,

$$\Lambda_G = \{(g, h, g \star h) \mid g, h \in G, \mathsf{s}(g) = \mathsf{t}(h)\}.$$

The isomorphism $\psi : (T^*G)^3 \to (T^*G)^3$, $\psi(\alpha, \beta, \gamma) = (\alpha, \beta, -\gamma)$ sends the conormal space $T\Lambda_G{}^\circ \subseteq (T^*G)^3|_{\Lambda_G}$ to a submanifold Λ_* of $(T^*G)^3$. It is shown in Coste et al. (1987) that Λ_* is the graph of a groupoid multiplication on T^*G, which is exactly the multiplication above.

The "Pontryagin groupoid" of a Lie groupoid If $G \rightrightarrows P$ is a Lie groupoid, there is hence an induced Lie groupoid structure on $\mathsf{P}_G = TG \times_G T^*G$ over $TP \times_P A^*G$. We will write $\mathbb{T}\mathsf{t}$ for the target map

$$\begin{array}{rcl} \mathbb{T}\mathsf{t} : TG \times_G T^*G & \to & TP \times_P A^*G \\ (v_g, \alpha_g) & \mapsto & \big(T\mathsf{t}(v_g), \hat{\mathsf{t}}(\alpha_g)\big) \end{array},$$

$\mathbb{T}\mathsf{s}$ for the source map

$$\mathbb{T}\mathsf{s} : TG \times_G T^*G \to TP \times_P A^*G$$

and $\mathbb{T}\epsilon$, $\mathbb{T}\iota$, $\mathbb{T}\mathsf{m}$ for the embedding of the units, the inversion map and the multiplication of this Lie groupoid.
The canonical projection $TG \times_G T^*G \to G$ is a Lie groupoid morphism.

Example 1.1.21 (The Pontryagin groupoid of a Lie group) Let G be a Lie group. Then the Lie groupoid structure of the Pontryagin bundle P_G over $\mathfrak{g}^*$ simplifies as follows. The target and source maps t and s are defined by

$$\begin{array}{rccc} \mathbb{T}\mathsf{t} : & TG \times_G TG^* & \to & \mathfrak{g}^* \\ & (v_g, \alpha_g) \in T_gG \times T_gG^* & \mapsto & (T_eR_g)^*\alpha_g \end{array}$$

and

$$\begin{array}{rccc} \mathbb{T}\mathsf{s} : & TG \times_G TG^* & \to & \mathfrak{g}^* \\ & (v_g, \alpha_g) \in T_gG \times T_gG^* & \mapsto & (T_eL_g)^*\alpha_g \end{array}.$$

If $\mathbb{T}\mathsf{s}(v_g, \alpha_g) = \mathbb{T}\mathsf{t}(w_h, \beta_h)$, then the product $(v_g, \alpha_g) \star (w_h, \beta_h)$ makes sense and is equal to

$$\begin{aligned} (v_g, \alpha_g) \star (w_h, \beta_h) &= \big(T_gR_hv_g + T_hL_gw_h, (T_{gh}R_{h^{-1}})^*\alpha_g\big) \\ &= \big(T_gR_hv_g + T_hL_gw_h, (T_{gh}L_{g^{-1}})^*\beta_h\big). \end{aligned}$$

The identity map $\mathbb{T}\epsilon : \mathfrak{g}^* \to \mathsf{P}_G$ is given by $\mathbb{T}\epsilon(\xi) = (0, \xi) \in \mathfrak{g} \times \mathfrak{g}^*$ and the inverse map $\mathbb{T}\mathsf{i} : \mathsf{P}_G \to \mathsf{P}_G$ is defined by

$$\mathbb{T}\mathsf{i} : (v_g, \alpha_g) \mapsto \big(-T_g(L_{g^{-1}}R_{g^{-1}})v_g, T_{g^{-1}}(L_gR_g)^*\alpha_g\big)$$

for all $(v_g, \alpha_g) \in TG \times_G T^*G$. $\Diamond$

Example 1.1.22 Consider a smooth manifold M and the pair Lie groupoid $M \times M \rightrightarrows M$. The tangent groupoid $T(M \times M) \rightrightarrows TM$ of $M \times M \rightrightarrows M$ is easily seen to be $TM \times TM \rightrightarrows TM$, the pair groupoid associated to TM.
The Lie algebroid $A(M \times M)$ of $M \times M \rightrightarrows M$ is the set $T^{\mathsf{t}}_{\Delta_M}(M \times M)$. A vector $(v_m, w_m) \in T_{(x,x)}(M \times M)$ lies in $T^{\mathsf{t}}_{(m,m)}(M \times M)$ if $0 = T_{(m,m)}\mathsf{t}(v_m, w_m) = v_m$. Hence, we have $A(M \times M) = (0_{TM} \times_M TM)|_{\Delta_M}$ and its dual $A^*(M \times M) \simeq (T\Delta_M)^\circ \subseteq T^*(M \times M)|_{\Delta_M}$ is given by $A^*_{(m,m)}(M \times M) = \{(-\alpha_m, \alpha_m) \mid \alpha_m \in T^*_mM\}$ for all $m \in M$.

Next, we give the structure of the cotangent groupoid $T^*(M\times M) \rightrightarrows A^*(M\times M)$. If $(\alpha_m,\alpha_n)\in T^*_{(m,n)}(M\times M)$, then $\hat{\mathsf{t}}(\alpha_m,\alpha_n)\in(\{0_m\}\times T_mM)^* = A^*_{(m,m)}(M\times M)$,

$$\begin{aligned}\hat{\mathsf{t}}(\alpha_m,\alpha_n)(0_m,v_m) &= (\alpha_m,\alpha_n)(T_{(m,m)}R_{(m,n)}((0_m,v_m)-T\mathsf{s}(0_m,v_m)))\\ &= (\alpha_m,\alpha_n)(T_{(m,m)}R_{(m,n)}((0_m,v_m)-(v_m,v_m)))\\ &= (\alpha_m,\alpha_n)(-T_{(m,m)}R_{(m,n)}(v_m,0_m))) = -(\alpha_m,\alpha_n)(v_m,0_n) = -\alpha_m(v_m)\end{aligned}$$

for all $v_m\in T_mM$, and hence $\hat{\mathsf{t}}(\alpha_m,\alpha_n)=(\alpha_m,-\alpha_m)$. In the same manner, we show that $\hat{\mathsf{s}}(\alpha_m,\alpha_n)=(-\alpha_n,\alpha_n)$. The product of (α_m,α_n) and $(-\alpha_n,\alpha_p)$ is then given by

$$\begin{aligned}((\alpha_m,\alpha_n)\star(-\alpha_n,\alpha_p))(v_m,v_p) &= ((\alpha_m,\alpha_n)\star(-\alpha_n,\alpha_p))((v_m,v_n)\star(v_n,v_p))\\ &= (\alpha_m,\alpha_n)(v_m,v_n)+(-\alpha_n,\alpha_p)(v_n,v_p) = (\alpha_m,\alpha_p)(v_m,v_p)\end{aligned}$$

for any $(v_m,v_p)\in T_{(m,p)}(M\times M)$ (and any choice of $v_n\in T_nM$), and hence

$$(\alpha_m,\alpha_n)\star(-\alpha_n,\alpha_p)=(\alpha_m,\alpha_p).$$

$\Diamond$

1.1.6 Homogeneous spaces

The notion of homogeneous space for a groupoid action is more subtle than for groups (Liu et al. (1998), Mackenzie (1987), Brown et al. (1976)). One natural candidate for such a space is G acting on itself by left translations, but this action is not transitive in the usual sense, since $\mathsf{s}(gx)=\mathsf{s}(x)$, so that the action is transitive only on each s-fiber. The following intrinsic definition is given in Liu et al. (1998).

Definition 1.1.23 *A G-space X over P is* homogeneous *if there is a section σ of the moment map $\mathsf{J}: X\to P$ which is* saturating *for the action in the sense that $G\star\sigma(P)=X$. The* isotropy subgroupoid *of the section σ consists of those $g\in G$ for which $g\star\sigma(P)\subseteq\sigma(P)$.*

Let $G\rightrightarrows P$ be a Lie groupoid and $H\rightrightarrows P$ a wide subgroupoid of G. Define the equivalence relation

$$g\sim_H g' \iff \exists\ h\in H \text{ such that } g\star h=g'$$

on G and

$$G/H := G/\sim_H = \{gH \mid g\in G\},$$

where

$$gH=\{g\star h \mid \mathsf{s}(g)=\mathsf{t}(h) \text{ and } h\in H\}.$$

Since $\mathsf{t}(g\star h)=\mathsf{t}(g)$ for all $g\star h\in gH$, the map t factors to a map

$$\mathsf{J}: G/H\to P, \qquad \mathsf{J}(gH)=\mathsf{t}(g)$$

for all $gH \in G/H$. The multiplication $\mathsf{m} : G \times_P G \to G$ factors to a groupoid action Φ of $G \rightrightarrows P$ on $\mathsf{J} : G/H \to P$,

$$\Phi(g, g'H) = (g \star g')H$$

for all $(g, g'H) \in G \times_P (G/H) = \{(g, g'H) \mid \mathsf{s}(g) = \mathsf{J}(g'H) = \mathsf{t}(g')\}$.
It is shown in Liu et al. (1998) that a G-space is homogeneous if and only if it is isomorphic to G/H for some wide subgroupoid $H \subseteq G$. This is the definition that will be used here.

Example 1.1.24 Let $G \rightrightarrows P$ be a groupoid. The two extreme examples of homogeneous spaces of G are the following.

1. In the case where the wide subgroupoid is P, the equivalence classes are $gP = \{g \star p \mid p \in P, p = \mathsf{s}(g)\} = \{g\}$ and the quotient is just $G/P = G$ with the first action of Example 1.1.12.

2. If the wide subgroupoid is G itself, then the equivalence classes are $gG = \{g \star h \mid h \in G, \mathsf{t}(h) = \mathsf{s}(g)\} = \mathsf{t}^{-1}(\mathsf{t}(g))$ and the quotient is $G/G = P$, with projection equal to the target map $\mathsf{t} : G \to G/G \simeq P$ and with the second action in Example 1.1.12. $\Diamond$

Assume that H *is a* t*-connected wide Lie subgroupoid of* G and that G/H is a smooth manifold such that the projection $q : G \to G/H$ is a smooth surjective submersion. We will say for simplicity that G/H is a *smooth homogeneous space of* $G \rightrightarrows P$.
Consider the Lie algebroid $AH = T_P^{\mathsf{t}} H \subseteq T_P H \subseteq T_P G$ seen as a subbundle of AG over P and the subbundle $\mathcal{H} \subseteq TG$ defined as the left invariant image of AH, i.e., $\mathcal{H}(g) = T_{\mathsf{s}(g)} L_g \left(A_{\mathsf{s}(g)} H \right)$ for all $g \in G$. We show that $\mathcal{H} = \ker Tq$ and G/H is the leaf space of the foliation on G defined by the *involutive subbundle* $\mathcal{H} \subseteq TG$.
The vector bundle $\mathcal{H}$ is spanned by the left invariant vector fields X^l, for $X \in \Gamma(AH) \subseteq \Gamma(AG)$. Since H is an immersed submanifold of G, AH is a subalgebra of AG, and $\Gamma(\mathcal{H})$ is hence closed under the Lie bracket.
Choose $g \in G$ and $v_g \in \mathcal{H}(g)$. Then $v_g = T_{\mathsf{s}(g)} L_g u_{\mathsf{s}(g)}$ for some $u_{\mathsf{s}(g)} \in A_{\mathsf{s}(g)} H$ and there exists a curve $c : (-\varepsilon, \varepsilon) \to \mathsf{t}^{-1}(\mathsf{s}(g)) \cap H$ such that $c(0) = \mathsf{s}(g)$ and $\dot{c}(0) = u_{\mathsf{s}(g)}$. We have then $v_g = \dot{\gamma}(0)$, if $\gamma : (-\varepsilon, \varepsilon) \to \mathsf{t}^{-1}(\mathsf{t}(g))$ is defined by $\gamma(t) = g \star c(t)$ for all $t \in (-\varepsilon, \varepsilon)$. We can hence compute $T_g q v_g = \frac{d}{dt}\big|_{t=0} (q \circ \gamma)(t) = \frac{d}{dt}\big|_{t=0} (g \star c(t))H = 0$ since $c(t) \in H$ for all t and thus

$$(g \star c(t))H = \{g \star c(t) \star h \mid \mathsf{s}(c(t)) = \mathsf{t}(h), h \in H\} = \{g \star h \mid \mathsf{s}(g) = \mathsf{t}(h), h \in H\} = gH.$$

Conversely, if $p \in P$ and $v_p \in \ker(T_p q)$, then we have $T_p \mathsf{t}(v_p) = T_{pH} \mathsf{J}(T_p q v_p) = 0$ and hence $v_p \in A_p G$. Let $c : (-\varepsilon, \varepsilon) \to G$ be such that $c(0) = p$ and $\dot{c}(0) = v_p$. Since $p \in H$ and $\frac{d}{dt}\big|_{t=0} c(t)H = T_p q v_p = 0$, we find that v_p is tangent to H, and hence $v_p \in A_p H$. If $g \in G$ and $v_g \in \ker(T_g q)$, then $T_g \mathsf{t} v_g = 0$ and hence $u_{\mathsf{s}(g)} := T_g L_{g^{-1}} v_g \in A_{\mathsf{s}(g)} G$. We find then $T_{\mathsf{s}(g)} q(T_g L_{g^{-1}} v_g) = T_{gH} \Phi_{g^{-1}}(T_g q v_g) = 0$ and hence $T_g L_{g^{-1}} v_g \in A_{\mathsf{s}(g)} H$, which yields $v_g \in \mathcal{H}(g)$.
If g and g' are in the same leaf of $\mathcal{H}$, we find without loss of generality one invariant vector field $X^l \in \Gamma(\mathcal{H})$, $X \in \Gamma(AH)$ and $t \in \mathbb{R}$ such that $g' = \phi_t^X(g)$, where ϕ^X is the flow

of X^l (in general, g and g' can be joined by finitely many of such paths). We have then $T_{\phi_s^X(g)}q(X^l(\phi_s^X(g))) = 0$ for all $s \in [0,t]$, and hence $(q \circ \phi_s^X)(g) = q(g)$ for all $s \in [0,t]$. This leads to $g'H = gH$. Conversely, if $g'H = gH$, it is easy to show, using the fact that H is t-connected, and hence $H = C(H) = \{\mathrm{Exp}(tX) \mid t \in \mathbb{R}, X \in \Gamma(AH)\}$ that g and g' are in the same leaf of $\mathcal{H}$.
Consider the set $\mathcal{B}(H)$ of (local) bisections $K : U \subseteq P \to H$ of H such that $\mathsf{t} \circ K = \mathrm{Id}_U$ and $\mathsf{s} \circ K$ is a diffeomorphism. We have $gH = \{R_K(g) \mid K \in \mathcal{B}(H)\}$ and G/H is the quotient of G by the right action of $\mathcal{B}(H)$ on G. A function $f \in C^\infty(G)$ pushes forward to the quotient G/H if and only if it is invariant under R_K for all bisections $K \in \mathcal{B}(H)$.

1.2 Generalities on Dirac structures

Dirac structures (Courant (1990), Courant and Weinstein (1988)) provide a unified framework for the study of (closed) 2-forms, (Poisson) bivectors, (regular) foliations and also a convenient geometric setting for the theory of nonholonomic systems and circuit theory. They also have a wide range of applications in geometry and theoretical physics.
We give in this section the definition and important properties of Dirac structures. In the last subsection, we show how infinitesimal symmetries of a subbundle of the Pontryagin bundle integrate to symmetries under flows. This result is standard and widely used, but its proof is difficult to find in the literature.

1.2.1 Dirac manifolds

As we have seen in Example 1.1.20, the *Pontryagin bundle* $\mathsf{P}_M := TM \times_M T^*M$ of a smooth manifold M is endowed with the non-degenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$ given by (1.6). The orthogonal space relative to this pairing of a subbundle $\mathsf{E} \subseteq \mathsf{P}_M$ will be written $\mathsf{E}^\perp$ in the following. A *Dirac structure* (see Courant (1990)) on M is a Lagrangian vector subbundle $\mathsf{D} \subset \mathsf{P}_M$. That is, D coincides with its orthogonal relative to (1.6), $\mathsf{D} = \mathsf{D}^\perp$, and so its fibers are necessarily $\dim M$-dimensional.

Let (M, D) be a Dirac manifold. For each $m \in M$, the Dirac structure D defines two subspaces $\mathsf{G}_0(m), \mathsf{G}_1(m) \subset T_mM$ by

$$\mathsf{G}_0(m) := \{v_m \in T_mM \mid (v_m, 0) \in \mathsf{D}(m)\}$$

and

$$\mathsf{G}_1(m) := \{v_m \in T_mM \mid \exists\, \alpha_m \in T_m^*M \ : (v_m, \alpha_m) \in \mathsf{D}(m)\}\,,$$

and two subspaces $\mathsf{P}_0(m), \mathsf{P}_1(m) \subset T_m^*M$ defined in an analogous manner. The distributions $\mathsf{G}_0 = \cup_{m\in M}\mathsf{G}_0(m)$ and $\mathsf{P}_0 = \cup_{m\in M}\mathsf{P}_0(m)$ are not necessarily smooth. The distributions $\mathsf{G}_1 = \cup_{m\in M}\mathsf{G}_1(m)$ (respectively $\mathsf{P}_1 = \cup_{m\in M}\mathsf{P}_1(m)$) are smooth since they are the projections on TM (respectively T^*M) of D. The distribution G_0 is called the *characteristic distribution* of the Dirac structure.

Integrable Dirac manifolds A Dirac structure D on a manifold M is *integrable* if $[\Gamma(\mathsf{D}), \Gamma(\mathsf{D})] \subset \Gamma(\mathsf{D})$, where $[\cdot\,,\cdot]$ is the bracket defined in (1.7). Since $\langle (X,\alpha),(Y,\beta)\rangle = 0$ if $(X,\alpha),(Y,\beta) \in \Gamma(\mathsf{D})$, integrability of the Dirac structure is expressed relative to a non-skew-symmetric bracket that differs from (1.7) by eliminating in the second line the third term of the second component. This truncated expression is called the *Courant-Dorfman bracket* in the literature:

$$[(X,\alpha),(Y,\beta)] = ([X,Y], \boldsymbol{\pounds}_X\beta - \mathbf{i}_Y\mathbf{d}\alpha) \tag{1.8}$$

for all $(X,\alpha),(Y,\beta) \in \Gamma(\mathsf{D})$. The restriction of the Courant-Dorfman bracket to the sections of an integrable Dirac bundle is skew-symmetric and satisfies the Jacobi identity. It satisfies also the Leibniz rule:

$$[(X,\alpha), f(Y,\beta)] = f[(X,\alpha),(Y,\beta)] + X(f)\cdot(Y,\beta) \tag{1.9}$$

for all $(X,\alpha),(Y,\beta) \in \Gamma(\mathsf{D})$ and $f \in C^\infty(M)$. Note that this equality is true for all sections $(X,\alpha),(Y,\beta) \in \Gamma(\mathsf{P}_M)$ (see for instance Bursztyn et al. (2007)), but the Courant-Dorfman bracket is not skew-symmetric on arbitrary sections of P_M.

The Dirac manifold (M,D) is integrable if and only if the *tensor* $\mathsf{T}_\mathsf{D} \in \Gamma\left(\bigwedge^3 \mathsf{D}^*\right)$ defined on sections $(X,\alpha),(Y,\beta),(Z,\gamma)$ of D by

$$\mathsf{T}_\mathsf{D}\Big((X,\alpha),(Y,\beta),(Z,\gamma)\Big) = \Big\langle [(X,\alpha),(Y,\beta)],(Z,\gamma)\Big\rangle \tag{1.10}$$

vanishes identically on M (see Courant (1990)). Note that we have

$$\mathsf{T}_\mathsf{D}((X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3)) = \alpha_1([X_2,X_3]) + X_1(\alpha_2(X_3)) + \text{c.p.} \tag{1.11}$$

for all $(X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3) \in \Gamma(\mathsf{D})$. It is also easy to check that the cotangent part of

$$[(X_1,\alpha_1),[(X_2,\alpha_2),(X_3,\alpha_3)]] + \text{c.p.}$$

is equal to

$$\mathbf{d}\left(\mathsf{T}_\mathsf{D}((X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3))\right)$$

for all $(X_1,\alpha_1),(X_2,\alpha_2),(X_3,\alpha_3) \in \Gamma(\mathsf{D})$.

More generally, if $(\mathsf{E} \to M, \rho, \langle\cdot\,,\cdot\rangle, [\cdot\,,\cdot])$ is a Courant algebroid, we can define in the same manner as above a *Dirac structure* D *in* E as a Lagrangian subbundle. It is integrable if its set of sections is closed under the bracket $[\cdot\,,\cdot]$.

Poisson manifolds as integrable Dirac manifolds The following example shows how Dirac manifolds generalize Poisson manifolds. This example will be very important in the following, since we will generalize results known for Poisson manifolds to results about Dirac manifolds.

Example 1.2.1 Recall that a Poisson manifold is a manifold M which set of smooth functions is endowed with a skew-symmetric bracket

$$\{\cdot\,,\cdot\}:C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$

that satisfies the Jacobi identity and such that $\{f,\cdot\}:C^\infty(M)\to C^\infty(M)$ is a derivation for any $f\in C^\infty(M)$. A Poisson bracket on a manifold M is equivalent to a skew-symmetric bivector field $\pi\in\mathfrak{X}^2(M)$ given by $\pi(\mathbf{d}f,\mathbf{d}g)=\{f,g\}$ for all $f,g\in C^\infty(M)$. The Jacobi identity is equivalent to $[\pi,\pi]=0$, where the bracket is the Schouten bracket on multivector fields (see (1.4)).
Let M be a smooth manifold endowed with a globally defined bivector field $\pi\in\Gamma\left(\bigwedge^2 TM\right)$. Then the subdistribution $\mathsf{D}_\pi\subseteq\mathsf{P}_M$ defined by

$$\mathsf{D}_\pi(m)=\left\{(\pi^\sharp(\alpha_m),\alpha_m)\mid\alpha_m\in T_m^*M\right\}\qquad\text{for all }\ m\in M,$$

where $\pi^\sharp:T^*M\to TM$ is defined by $\pi^\sharp(\alpha)=\pi(\alpha,\cdot)\in\mathfrak{X}(M)$ for all $\alpha\in\Omega^1(M)$, is a Dirac structure on M. It is integrable if and only if the bivector field satisfies $[\pi,\pi]=0$, that is, if and only if (M,π) is a Poisson manifold. $\Diamond$

Note that in the following, we will also be interested by non integrable Dirac structures (and hence also by graphs of skew-symmetric bivector fields that are not Poisson), since the objects that we will study also have interesting properties in the non integrable case.

The product of two Dirac manifolds. Let $(M,\mathsf{D}_\mathsf{M})$ and (N,D_N) be Dirac manifolds. Consider the product $M\times N$. We identify in the following always (without mentioning it) the tangent space $T(M\times N)$ with $TM\times TN$, and write (v_p,w_q) for the elements of $T_{(p,q)}(M\times N)=T_pM\times T_qN$. That is, an element of $\mathfrak{X}(M\times N)$ is written (X,Y) with $X\in\mathfrak{X}(M)$ and $Y\in\mathfrak{X}(N)$. We identify in the same manner $T^*(M\times N)$ with $T^*M\times T^*N$. The *product Dirac structure* $\mathsf{D}_M\oplus\mathsf{D}_N$ on $M\times N$ is the product of D_M and D_N: the pair $((X,Y),(\alpha,\beta))$ is a section of $\mathsf{D}_M\oplus\mathsf{D}_N$ if and only if $(X,\alpha)\in\Gamma(\mathsf{D}_M)$ and $(Y,\beta)\in\Gamma(\mathsf{D}_N)$. The Dirac manifold $(M\times N,\mathsf{D}_M\oplus\mathsf{D}_N)$ is integrable if and only if (M,D_M) and (N,D_N) are integrable.

Symmetries of Dirac manifolds Let (M,D) be a Dirac manifold and G a Lie group with $\Phi:G\times M\to M$ a smooth left action. Then G is called a *symmetry Lie group of* D if for every $g\in G$ the condition $(X,\alpha)\in\Gamma(\mathsf{D})$ implies that $\left(\Phi_g^*X,\Phi_g^*\alpha\right)\in\Gamma(\mathsf{D})$. We say then that the Lie group G acts *canonically* or *by Dirac actions* on M.
Let $\mathfrak{g}$ be a Lie algebra and $\xi\in\mathfrak{g}\mapsto\xi_M\in\mathfrak{X}(M)$ be a smooth left Lie algebra action. The Lie algebra $\mathfrak{g}$ is said to be a *symmetry Lie algebra of* D if for every $\xi\in\mathfrak{g}$ the condition $(X,\alpha)\in\Gamma(\mathsf{D})$ implies that $(\boldsymbol{\pounds}_{\xi_M}X,\boldsymbol{\pounds}_{\xi_M}\alpha)\in\Gamma(\mathsf{D})$.

1.2.2 Dirac maps and Dirac reduction

Let (M,D_M) and (N,D_N) be two Dirac manifolds and $F:M\to N$ a smooth map. Then F is a *forward Dirac map* if for all $n\in N$, $m\in F^{-1}(n)$ and $(v_n,\alpha_n)\in\mathsf{D}_N(n)$ there exists $(v_m, \alpha_m) \in \mathsf{D}_M(m)$ such that $T_mFv_m = v_n$ and $\alpha_m = (T_mF)^*\alpha_n$. The map F is a *backward Dirac map* if for all $m \in M$, $n = F(m)$ and $(v_m, \alpha_m) \in \mathsf{D}_M(m)$ there exists $(v_n, \alpha_n) \in \mathsf{D}_N(n)$ such that $T_mFv_m = v_n$ and $\alpha_m = (T_mF)^*\alpha_n$. If F is a diffeomorphism, then it is easy to check that it is a backward Dirac map if and only if it is a forward Dirac map.

Let M and N be smooth manifolds and $\phi : M \to N$ a smooth map. Assume that N is endowed with a Dirac structure D_N. The *pullback* $\phi^*\mathsf{D}_N$ of D_N is the subdistribution of P_M defined by

$$(\phi^*\mathsf{D}_N)(m) = \left\{ (v_m, \alpha_m) \in \mathsf{P}_M(m) \;\middle|\; \begin{array}{c} \exists (w_{\phi(m)}, \beta_{\phi(m)}) \in \mathsf{D}_N(\phi(m)) \\ \text{such that } T_m\phi(v_m) = w_{\phi(m)} \\ \text{and } \alpha_m = (T_m\phi)^*\beta_{\phi(m)} \end{array} \right\} \tag{1.12}$$

for all $m \in M$. Each fiber of $\phi^*\mathsf{D}_N$ is Lagrangian in $TM \times_M T^*M$ (see for instance Jotz and Ratiu (2011)). Hence, if $\phi^*\mathsf{D}_N$ is smooth, it is a Dirac structure on M such that ϕ is a backward Dirac map. The Dirac structure $\phi^*\mathsf{D}_N$ is then the *backward Dirac image* of D_N under ϕ. The *forward Dirac image* of a Dirac structure under a map is not always Lagrangian. We will see below a situation where it is a Dirac structure.

Let (M, D) be a smooth Dirac manifold with a smooth proper Dirac action of a Lie group G on it, such that all isotropy subgroups of the action are conjugated. Then the space $\bar{M} := M/G$ of orbits of the action is a smooth manifold and the quotient map $q : M \to \bar{M}$ is a smooth surjective submersion. Set $\mathcal{K} = \mathcal{V} \times_M 0_{T^*M}$, where $\mathcal{V}$ is the vertical space of the action. We have the following theorem (see Jotz et al. (2011a)).

Theorem 1.2.2 *Let G be a connected Lie group acting in a proper Dirac manner on the Dirac manifold (M, D), such that all isotropy subgroups are conjugated. Assume that $\mathsf{D} \cap \mathcal{K}^\perp$ has constant rank on M, where $\mathcal{K}^\perp = TM \times_G \mathcal{V}^\circ$. Then the Dirac structure D on M induces a Dirac structure $\bar{\mathsf{D}}$ on the quotient $\bar{M} = M/G$ given by*

$$\bar{\mathsf{D}}(\bar{m}) = \left\{ \left(\bar{X}(\bar{m}), \bar{\alpha}(\bar{m})\right) \in T_{\bar{m}}\bar{M} \times T^*_{\bar{m}}\bar{M} \;\middle|\; \begin{array}{c} \exists X \in \mathfrak{X}(M) \text{ such that } X \sim_q \bar{X} \\ \text{and } (X, q^*\bar{\alpha}) \in \Gamma(\mathsf{D}) \end{array} \right\}$$

for all $\bar{m} \in \bar{M}$. If (M, D) is integrable, then $(\bar{M}, \bar{\mathsf{D}})$ is also integrable.

The Dirac structure $\bar{\mathsf{D}}$ is then the *forward Dirac image $q(\mathsf{D})$ of D under q.*

Let $G \rightrightarrows P$ be a Lie groupoid and H a t-connected wide Lie subgroupoid of $G \rightrightarrows P$ such that $G/H = G/\mathcal{H}$ has a smooth manifold structure and the projection $q : G \to G/H$ is a smooth surjective submersion (see the background notions about homogeneous spaces of Lie groupoids in Subsection 1.1.6). We have the following pull-back diagram of vector bundles

$$\begin{array}{ccc} TG \times_G \mathcal{H}^\circ/(\mathcal{H} \times_G 0_{T^*G}) & \longrightarrow & T(G/H) \times_{G/H} T^*(G/H)\,. \\ \downarrow & & \downarrow \\ G & \longrightarrow & G/H \end{array}$$

Set $\mathcal{K}_H := \mathcal{H} \times_G 0_{T^*G}$ and hence $\mathcal{K}_H^\perp = TG \times_G \mathcal{H}^\circ$. Let D be a Dirac structure on G and assume that

$$(R_K^* X, R_K^* \alpha) \in \Gamma(\mathsf{D}) \text{ for all } (X, \alpha) \in \Gamma(\mathsf{D}) \text{ and } K \in \mathcal{B}(H), \tag{1.13}$$

i.e., the Dirac structure is invariant under the right action of H on G. Then we can show that

$$\left[\Gamma\left(\mathcal{K}_H\right), \Gamma\left(\mathsf{D} \cap \mathcal{K}_H^\perp\right)\right] \subset \Gamma(\mathsf{D} + \mathcal{K}_H). \tag{1.14}$$

If D is integrable, then we have in a trivial manner

$$\left[\Gamma\left(\mathsf{D} \cap \mathcal{K}_H^\perp\right)^H, \Gamma\left(\mathsf{D} \cap \mathcal{K}_H^\perp\right)^H\right] \subset \Gamma(\mathsf{D} + \mathcal{K}_H), \tag{1.15}$$

where $\Gamma\left(\mathsf{D} \cap \mathcal{K}_H^\perp\right)^H$ is the set of sections of $\mathsf{D} \cap \mathcal{K}_H^\perp$ that satisfies

$$(R_K^* X, R_K^* \alpha) = (X, \alpha)$$

for all $K \in \mathcal{B}(H)$. Hence, all the hypotheses for the Dirac reduction theorem in Zambon (2008) (see also Jotz et al. (2011a)) are satisfied, and we get the following result in our particular situation.

Theorem 1.2.3 *Let $G \rightrightarrows P$ be a Lie groupoid and H a t-connected wide Lie subgroupoid of $G \rightrightarrows P$ such that $G/H = G/\mathcal{H}$ has a smooth manifold structure such that the projection $q : G \to G/H$ is a smooth surjective submersion. Then we have in particular the reduced standard Courant algebroid $T(G/H) \times_{G/H} T^*(G/H)$ on G/H.*
Let D be a Dirac structure on G such that $\mathsf{D} \cap \mathcal{K}_H^\perp$ has constant rank and (1.13) *is satisfied. Then D descends to a Dirac structure $q(\mathsf{D})$ on G/H. If furthermore D is integrable, then $q(\mathsf{D})$ is integrable.*

The Dirac structure $q(\mathsf{D})$ on the quotient G/H is given by

$$\Gamma(q(\mathsf{D})) = \left\{ (\bar{X}, \bar{\alpha}) \in \Gamma(\mathsf{P}_{G/H}) \;\middle|\; \begin{array}{c} \exists X \in \mathfrak{X}(G) \text{ such that} \\ X \sim_q \bar{X} \text{ and } (X, q^*\bar{\alpha}) \in \Gamma(\mathsf{D}) \end{array} \right\}. \tag{1.16}$$

In other words, $q(\mathsf{D})$ is the forward Dirac image of D under $q : G \to G/H$. If $\mathcal{K}_H \subseteq \mathsf{D}$, then $\mathsf{D} = q^*(q(\mathsf{D}))$.

1.2.3 Invariant Dirac structures on a Lie group

Definition 1.2.4 *A Dirac structure $\mathsf{D} \subseteq TG \times_G TG^*$ on a Lie group G is called* left invariant *(respectively* right invariant*) if it is invariant under the action of G on $TG \times_G T^*G$ induced from the left (right) action of G on itself.*

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{D}$ be a Dirac subspace of $\mathfrak{g}\times\mathfrak{g}^*$, that is, $\mathfrak{D}$ is a vector subspace of $\mathfrak{g}\times\mathfrak{g}^*$ that is orthogonal to itself relative to the pairing $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ defined on $\mathfrak{g}\times\mathfrak{g}^*$ by $\langle(x,\xi),(y,\eta)\rangle_{\mathfrak{g}}=\eta(x)+\xi(y)$ for all $x,y\in\mathfrak{g}$ and $\xi,\eta\in\mathfrak{g}^*$. We set

$$\mathfrak{g}_0:=\{x\in\mathfrak{g}\mid(x,0)\in\mathfrak{D}\},\quad \mathfrak{g}_1:=\{x\in\mathfrak{g}\mid\exists\xi\in\mathfrak{g}^*:(x,\xi)\in\mathfrak{D}\},$$

$$\mathfrak{p}_0:=\{\xi\in\mathfrak{g}^*\mid(0,\xi)\in\mathfrak{D}\}\quad\text{and}\quad\mathfrak{p}_1:=\{\xi\in\mathfrak{g}^*\mid\exists x\in\mathfrak{g}:(x,\xi)\in\mathfrak{D}\}.$$

Then we have $\mathfrak{g}_0^\circ=\mathfrak{p}_1,\mathfrak{p}_1^\circ=\mathfrak{g}_0,\mathfrak{g}_1^\circ=\mathfrak{p}_0$, and $\mathfrak{p}_0^\circ=\mathfrak{g}_1$.

Let $\mathfrak{D}$ be a Dirac subspace of $\mathfrak{g}\times\mathfrak{g}^*$, and define $\mathfrak{D}^l$ on G by

$$\mathfrak{D}^l(g)=\{(T_eL_gx,(T_gL_{g^{-1}})^*\xi)\mid(x,\xi)\in\mathfrak{D}\}$$

for all $g\in G$. Then $\mathfrak{D}^l$ is a left invariant Dirac structure on G. Conversely, if D is a left invariant Dirac structure on a Lie group G, then $\mathsf{D}=\mathfrak{D}^l$, where $\mathfrak{D}:=\mathsf{D}(e)\subseteq\mathfrak{g}\times\mathfrak{g}^*$.
The next proposition shows that the integrability of $\mathfrak{D}^l$ depends only on $\mathfrak{D}$ (see also Milburn (2007)).

Proposition 1.2.5 *The Dirac structure $\mathfrak{D}^l$ is integrable if and only if $\zeta([x,y])+\xi([y,z])+\eta([z,x])=0$ for all pairs $(x,\xi),(y,\eta)$ and $(z,\zeta)\in\mathfrak{D}$.*

Proof: Recall that $\mathfrak{D}^l$ is integrable if for all sections $(X,\alpha),(Y,\beta)\in\Gamma(\mathfrak{D}^l)$, we have $[(X,\alpha),(Y,\beta)]=([X,Y],\pounds_X\beta-\mathbf{i}_Y\mathbf{d}\alpha)\in\Gamma(\mathfrak{D}^l)$.
By (1.9), it suffices to show this for a set of spanning sections of $\mathfrak{D}^l$. For $(x,\xi)\in\mathfrak{D}$, the left invariant pair (x^l,ξ^l), defined by $(x^l(g),\xi^l(g))=(T_eL_gx,(T_gL_{g^{-1}})^*\xi)$ for all $g\in G$, is a section of $\mathfrak{D}^l$. Choose (x,ξ), (y,η) and $(z,\zeta)\in\mathfrak{D}$. Then we have $\left[x^l,y^l\right]=[x,y]^l$ by definition of the Lie bracket on $\mathfrak{g}$, $\pounds_{x^l}\eta^l=(\mathrm{ad}_x^*\eta)^l$, $\mathbf{i}_{y^l}\mathbf{d}\xi^l=(\mathrm{ad}_y^*\xi)^l$, where for $\xi\in\mathfrak{g}^*$ and $x\in\mathfrak{g}$, the element $\mathrm{ad}_x^*\xi\in\mathfrak{g}^*$ is defined by $\mathrm{ad}_x^*\xi(y)=\xi([y,x])$ for all $y\in\mathfrak{g}$. We get

$$\left\langle\left([x,y]^l,\pounds_{x^l}\eta^l-\mathbf{i}_{y^l}\mathbf{d}\xi^l\right),\left(z^l,\zeta^l\right)\right\rangle=\zeta([x,y])+\eta([z,x])+\xi([y,z]).$$

Hence, since the sections $\left(z^l,\zeta^l\right)$, for all $(z,\zeta)\in\mathfrak{D}$, are spanning sections for $\mathfrak{D}^l$, we conclude that $\left[\left(x^l,\xi^l\right),\left(y^l,\eta^l\right)\right]=([x,y]^l,\mathbf{i}_{x^l}\mathbf{d}\eta^l-\mathbf{i}_{y^l}\mathbf{d}\xi^l)$ is a section of $\mathfrak{D}^l$ if and only if $\zeta([x,y])+\eta([z,x])+\xi([y,z])=0$ for all $(z,\zeta)\in\mathfrak{D}$. $\square$

1.2.4 Invariance of Dirac structures under infinitesimal actions

Proposition 1.2.6 *Let E be a subbundle of the Pontryagin bundle $TM\times_M T^*M$ of a smooth manifold M. Let $Z\in\mathfrak{X}(M)$ be a smooth vector field on M and denote its flow by ϕ_t. If*

$$[(Z,0),(X,\alpha)]=\pounds_Z(X,\alpha)\in\Gamma(\mathsf{E})\quad\textit{for all}\quad(X,\alpha)\in\Gamma(\mathsf{E}),$$

where $[\cdot\,,\cdot]$ is the Courant-Dorfman bracket, then

$$\phi_t^*(X,\alpha)\in\Gamma(\mathsf{E})\quad\textit{for all}\quad(X,\alpha)\in\Gamma(\mathsf{E})\quad\textit{and}\quad t\in\mathbb{R}\textit{ where this makes sense.}$$

Corollary 1.2.7 *Let* D *be a Dirac structure on a smooth manifold* M*. If* $Z \in \mathfrak{X}(M)$ *is a vector field such that*

$$[(Z,0),d] \in \Gamma(\mathsf{D}) \quad \textit{for all} \quad d \in \Gamma(\mathsf{D}),$$

then $\phi_t^* d \in \Gamma(\mathsf{D})$ *for all* $d \in \Gamma(\mathsf{D})$ *and* $t \in \mathbb{R}$ *where this makes sense.*

Proof (of Proposition 1.2.6): The subbundle E of $\mathsf{P}_M = TM \times_M T^*M$ is an embedded submanifold of P_M. For each section σ of P_M, the smooth function $l_\sigma : TM \times_M T^*M \to \mathbb{R}$ is defined by $l_\sigma(v,\alpha) = \langle \sigma(q(v,\alpha)), (v,\alpha)\rangle$ for all $(v,\alpha) \in TM \times_M T^*M$, where $q : TM \times_M T^*M \to M$ is the projection. For all $e \in \mathsf{E}$, the tangent space $T_e\mathsf{E}$ of the submanifold E of P_M is equal to

$$\ker\left\{\mathbf{d}_e l_\sigma \mid \sigma \in \Gamma\left(\mathsf{E}^\perp\right)\right\}.$$

Consider the complete lift $\tilde{Z}$ to P_M of Z, i.e., the vector field $\tilde{Z} \in \mathfrak{X}(\mathsf{P}_M)$ defined by

$$\tilde{Z}(l_\sigma) = l_{[(Z,0),\sigma]} \qquad \text{and} \qquad \tilde{Z}(q^*f) = q^*(Z(f))$$

for all $\sigma \in \Gamma(\mathsf{P}_M)$ and $f \in C^\infty(M)$ (see Mackenzie (2005)).
Note that if $\sigma \in \Gamma\left(\mathsf{E}^\perp\right)$, then we have $\pounds_Z\sigma \in \Gamma\left(\mathsf{E}^\perp\right)$ since for all $\tau \in \Gamma(\mathsf{E})$:

$$\langle \pounds_Z\sigma, \tau\rangle = Z\left(\langle\sigma,\tau\rangle\right) - \langle\sigma, \pounds_Z\tau\rangle = 0.$$

Choose $e \in \mathsf{E}$ and $\sigma \in \Gamma\left(\mathsf{E}^\perp\right)$. Then we have

$$(\mathbf{d}_e l_\sigma)(\tilde{Z}(e)) = \left(\tilde{Z}(l_\sigma)\right)(e) = l_{[(Z,0),\sigma]}(e) = l_{\pounds_Z\sigma}(e) = 0.$$

Hence, the vector field $\tilde{Z}$ is tangent to E on E. As a consequence, its flow curves starting at points of e remain in the submanifold E.
We check that the flow Φ_t of the vector field $\tilde{Z}$ is equal to $(T\phi_t, (\phi_{-t})^*)$, i.e.,

$$\Phi_t(v_m,\alpha_m) = (T_m\phi_t(v_m), \alpha_m \circ T_{\phi_t(m)}\phi_{-t})$$

for all $(v_m,\alpha_m) \in \mathsf{P}_M(m)$. We have, for all $\sigma = (Y,\beta) \in \Gamma(\mathsf{P}_M)$:

$$\begin{aligned}
\frac{d}{dt} l_\sigma\left(\Phi_t(v_m,\alpha_m)\right) &= \frac{d}{dt}\left\langle \sigma(\phi_t(m)), (T_m\phi_t(v_m), \alpha_m \circ T_{\phi_t(m)}\phi_{-t})\right\rangle \\
&= \frac{d}{dt}\left\langle (Y(\phi_t(m)), \beta(\phi_t(m))), (T_m\phi_t(v_m), \alpha_m \circ T_{\phi_t(m)}\phi_{-t})\right\rangle \\
&= \frac{d}{dt}\left\langle (\phi_t^*(Y,\beta))(m), (v_m,\alpha_m)\right\rangle = \left\langle \phi_t^*(\pounds_Z(Y,\beta))(m), (v_m,\alpha_m)\right\rangle \\
&= l_{[(Z,0),\sigma]}\left(\Phi_t(v_m,\alpha_m)\right) = \tilde{Z}(l_\sigma)\left(\Phi_t(v_m,\alpha_m)\right).
\end{aligned}$$

In the same manner, we compute for any $f \in C^\infty(M)$:

$$\frac{d}{dt}(q^*f)\left(\Phi_t(v_m,\alpha_m)\right) = \frac{d}{dt}f(\phi_t(m)) = Z(f)(\phi_t(m)) = \tilde{Z}(q^*f)\left(\Phi_t(v_m,\alpha_m)\right).$$

Choose a section $(X,\alpha) \in \Gamma(\mathsf{E})$ and a point $m \in M$. We find

$$(\phi_t^*(X,\alpha))(m) = \left(T_{\phi_t(m)}\phi_{-t}X(\phi_t(m)), \alpha_{\phi_t(m)} \circ T_m\phi_t\right) = \Phi_{-t}\left((X,\alpha)(\phi_t(m))\right) \in \mathsf{E}(m)$$

since $(X,\alpha)(\phi_t(m)) \in \mathsf{E}(\phi_t(m))$. Thus, we have shown that $\phi_t^*(X,\alpha)$ is a section of E. $\square$

2 Multiplicative Poisson and Dirac structures

A *Poisson Lie group* is a Lie group endowed with a Poisson structure that is compatible with the Lie group structure. Poisson Lie groups were introduced by Drinfel′d (1983) and studied by Semenov-Tian-Shansky (1985). The systematic study of the geometry of Poisson Lie groups was started with the works of Lu and Weinstein (see for instance Lu and Weinstein (1989), Lu (1990), Lu and Weinstein (1990)).

There are two main reasons for the wide interest in Poisson Lie groups. First, the *Quantum Yang-Baxter Equation*, which is a basic equation in statistical mechanics and quantum field theory, has solutions that define *quantum groups* in the sense of Faddeev and Drinfel′d (Drinfel′d (1986)). Since one can take formally the *classical limit* of a quantum group to get a Poisson Lie group, Poisson Lie groups are good objects of study for the quantization of Poisson manifolds. Another motivation for studying Poisson Lie groups is to understand the Hamiltonian structures of the groups of dressing transformations of certain integrable systems. For some integrable systems, there is a Lie group called the *dressing transformation group*, that plays the role of *hidden symmetry group* of the system. The dressing transformation group does not in general preserve the Poisson structure on the phase space, but it carries a natural Poisson structure that makes it a Poisson Lie group, and the dressing action defines a Poisson action (Semenov-Tian-Shansky (1985)).

A *Poisson homogeneous space* of a Poisson Lie group is a homogeneous space of the Lie group that is endowed with a Poisson structure such that the left action of the Lie group on the homogeneous space is a Poisson map. Poisson homogeneous spaces of Poisson Lie groups are in correspondence with suitable subspaces of the direct sum of the Lie algebra with its dual.

The notion of Poisson Lie group was generalized to the notion of Poisson groupoid by Weinstein (1988), and the homogeneous spaces of Poisson groupoids were studied and classified in Liu et al. (1998).

In this chapter, we start by recalling the definitions of Poisson group(oid)s and their homogeneous spaces and state the theorems that will be generalized later on in a classification of Dirac homogeneous spaces of Dirac groupoids. Then we give the definitions of multiplicative Dirac structures on Lie groups and Lie groupoids, together with some examples. Finally, we define the action of $(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$ on $T(G/H) \times_{G/H} T^*(G/H)$ associated to the left action of a Lie groupoid $G \rightrightarrows P$ on a homogeneous space G/H. Using this, we define the notion of a Dirac homogeneous space of a Dirac groupoid. This definition turns out to generalize in a straightforward manner the one of Poisson homogeneous spaces of a Poisson group(oid).

2.1 Poisson Lie group(oid)s and Poisson homogeneous spaces

We recall in this section some background notions about Poisson Lie groups, Poisson groupoids and their homogeneous spaces. Most of the material here can be found for instance in Laurent-Gengoux et al. (2007).

Definition 2.1.1 *Let G be a Lie group endowed with a Poisson bivector π_G. Then (G, π_G) is a Poisson Lie group if the multiplication m is a Poisson map, where $G \times G$ is endowed with the product Poisson structure, or equivalently, if*

$$\pi_G(gh) = TR_h \pi_G(g) + TL_g \pi_G(h) \tag{2.1}$$

for all $g, h \in G$.

Equation (2.1) yields then automatically $\pi_G(e) = 0$ and

$$TR_{(gh)^{-1}} \pi_G(gh) = TR_{g^{-1}} \pi_G(g) + \mathrm{Ad}_g \left(TR_{h^{-1}} \pi_G(h) \right)$$

for all $g, h \in G$. The map

$$C_{\pi_G} : G \to \mathfrak{g} \wedge \mathfrak{g}$$

defined by

$$C_{\pi_G}(g) = TR_{g^{-1}} \pi_G(g)$$

for all $g \in G$ is hence a *group* 1*-cocycle*. By derivation at e, we get a *Lie algebra* 1*-cocycle*

$$\mathbf{d}_e C_{\pi_G} : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$$

and the fact that $[\pi_G, \pi_G] = 0$ is equivalent to

$$(\mathbf{d}_e C_{\pi_G})^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$$

being a Lie algebra bracket on $\mathfrak{g}^*$.

Definition 2.1.2 *A Lie bialgebra is a pair $(\mathfrak{g}, \nu)$ with a Lie algebra $\mathfrak{g}$ and a Lie algebra 1-cocycle $\nu : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ such that the dual map $\nu^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie algebra bracket on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$.*

Alternatively, one says that a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is a pair of dual Lie algebras such that the map $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ dual to the Lie bracket on $\mathfrak{g}^*$ is a Lie algebra 1-cocycle.
The following theorem of Drinfel′d shows that the construction above yields a classification of Poisson Lie groups.

Theorem 2.1.3 (Drinfel′d (1983)) *Let G be a connected and simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then the Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$ are in one-to-one correspondence with the multiplicative Poisson structures π_G on G.*

If $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, there is an induced Lie algebra structure on $\mathfrak{g} \times \mathfrak{g}^*$ that is given by

$$[(x,\xi),(y,\eta)] = \left([x,y] - \mathrm{ad}^*_\eta x + \mathrm{ad}^*_\xi y, [\xi,\eta] + \mathrm{ad}^*_x \eta - \mathrm{ad}^*_y \xi\right),$$

for all $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$. Note also that there is a natural non degenerate symmetric pairing on $\mathfrak{g} \times \mathfrak{g}^*$, given by $\langle (x,\xi),(y,\eta)\rangle_{\mathfrak{g}} = \xi(y) + \eta(x)$ for all $x, y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$. The Lie algebra $\mathfrak{g} \times \mathfrak{g}^*$ is called the *(Drinfel′d) double* of the Lie bialgebra.
The Lie algebra defined this way by the Lie bialgebra of a given Poisson Lie group has an important role in the classification of the homogeneous spaces of this Poisson Lie group.

Definition 2.1.4 *Let (G, π_G) be a Poisson Lie group and H a closed subgroup of G. A Poisson bivector field $\pi_{G/H}$ on G/H is π_G-homogeneous if the left action $\sigma : G \times G/H \to G/H$ of G on G/H is a Poisson map, where $G \times G/H$ is endowed with the product Poisson structure. The pair $(G/H, \pi_{G/H})$ is then a* Poisson homogeneous space of the Poisson Lie group *(G, π_G).*

If $(G/H, \pi_{G/H})$ is a Poisson homogeneous space of a Poisson Lie group (G, π_G), one can look at the pullback Dirac structure $q^*(\mathsf{D}_{\pi_{G/H}})$, where $q : G \to G/H$ is the surjective submersion and $\mathsf{D}_{\pi_{G/H}}$ is the graph of the vector bundle homomorphism ${\pi_{G/H}}^\sharp : T^*(G/H) \to T(G/H)$, as in Example 1.2.1. The fiber $\mathfrak{D} := (q^*(\mathsf{D}_{\pi_{G/H}}))(e) \subseteq \mathfrak{g} \times \mathfrak{g}^*$ over e of this Dirac structure satisfies then

1. $\mathfrak{h} \times \{0\} = \mathfrak{D} \cap (\mathfrak{g} \times \{0\})$,
2. $\mathfrak{D}$ is a subalgebra of $\mathfrak{g} \times \mathfrak{g}^*$,
3. $\mathfrak{D}$ is Lagrangian,

where $\mathfrak{h}$ is the Lie algebra of H. Drinfel′d classifies the Poisson homogeneous spaces of a Poisson Lie group in terms of such Lagrangian subalgebras of $\mathfrak{g} \times \mathfrak{g}^*$.

Theorem 2.1.5 (Drinfel′d (1993)) *Let (G, π_G) be a Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ and let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. There is a one-to-one correspondence between π_G-homogeneous Poisson structures on G/H and Lagrangian subalgebras $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ such that $\mathfrak{h} \times \{0\} = \mathfrak{D} \cap (\mathfrak{g} \times \{0\})$.*

This theorem is as the origin of this thesis. The first question that arises is if it is possible to drop the condition on the intersection of the Lagrangian subalgebra with $\mathfrak{g} \times \{0\}$. This leads us to the notion of "Dirac homogeneous spaces" of Poisson Lie groups, as we will see in Chapter 4. The proof of this theorem, which is rather short in Drinfel′d (1993) (see Diatta and Medina (1999), Lu (2008), Evens and Lu (2001) for more details about the proof, see also Liu et al. (1998)), will be natural in the general framework of Dirac Lie groups and their homogeneous spaces.

Recall that if (M, π_M) is a Poisson manifold, a smooth submanifold N of M is coisotropic if for all $n \in N$, we have $\pi_M(\alpha_n, \beta_n) = 0$ for all $\alpha_n, \beta_n \in T_nN^\circ \subseteq T^*_nM$. Equivalently, the

Hamiltonian vector field corresponding to any function that is constant on N is tangent to N (see Weinstein (1988)), i.e., $\pi_M^\sharp(\alpha_n) \in T_nN$ for all $\alpha_n \in T_nN^\circ \subseteq T_n^*M$.
We write $(M, -\pi_M)$ or shorter $\bar{M}$ for the manifold M endowed with the opposite Poisson structure and if (N, π_N) is another Poisson manifold, we write $(M \times N, \pi_M \oplus \pi_N)$ for the product Poisson manifold.
Consider now a Lie groupoid $G \rightrightarrows P$ endowed with a Poisson bivector π_G.

Definition 2.1.6 *The pair $(G \rightrightarrows P, \pi_G)$ is a Poisson groupoid if the graph*

$$\Lambda_G = \{(g, h, g \star h) \mid g, h \in G, \mathsf{s}(g) = \mathsf{t}(h)\} \subseteq G \times G \times G$$

of the multiplication map m is a coisotropic submanifold of $(G \times G \times G, \pi_G \oplus \pi_G \oplus (-\pi_G))$.

Poisson Lie groupoids were introduced in Weinstein (1988) and studied in Weinstein (1988), Xu (1995), Mackenzie and Xu (1994) among other, see also Mackenzie (2005).
It is shown in Mackenzie and Xu (1994) that $(G \rightrightarrows P, \pi_G)$ is a Poisson Lie groupoid if and only if the vector bundle map $\pi_G^\sharp : T^*G \to TG$ associated to π_G is a morphism of Lie groupoids over some map $a_* : A^*G \to TP$ (the restriction of $\pi_G^\sharp$ to A^*G). Equivalently, the graph $\mathsf{D}_{\pi_G} \subseteq \mathsf{P}_G$ (see Example 1.2.1) of the multiplicative Poisson structure π_G on $G \rightrightarrows P$ is a *subgroupoid* of the Pontryagin groupoid $(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$.
It is shown in Weinstein (1988) that the manifold P of identity elements of a Poisson groupoid G is coisotropic in G, and its conormal bundle $TP^\circ \subseteq T^*G|_P$ thereby acquires a Lie algebroid structure. This conormal bundle may be identified with A^*G, the dual vector bundle of AG, in a standard way. The Lie algebroid structure on the dual A^*G of the Lie algebroid $AG \to P$ of $G \rightrightarrows P$ is then such that (AG, A^*G) is a Lie bialgebroid (see Mackenzie and Xu (1994), Mackenzie (2005)). Hence, there is a Courant algebroid structure on $AG \times_P A^*G \to P$ as in Section 1.1.4.
Multiplicative Poisson structures on a t-connected and simply connected Lie groupoid are classified by the possible Lie bialgebroid structures on the Lie algebroid of the Lie groupoid.

Theorem 2.1.7 (Mackenzie and Xu (2000)) *Let (AG, A^*G) be a Lie bialgebroid, where AG is the Lie algebroid of a t-connected and simply connected Lie groupoid $G \rightrightarrows P$. Then there is a unique Poisson structure on G that makes G into a Poisson groupoid with Lie bialgebroid (AG, A^*G).*

Let now H be a wide t-connected Lie subgroupoid of $G \rightrightarrows P$ such that the quotient G/H is a smooth manifold and the projection $q : G \to G/H$ is a surjective submersion.

Definition 2.1.8 *Assume that G/H is endowed with a Poisson structure $\pi_{G/H}$. The pair $(G/H, \pi_{G/H})$ is a Poisson homogeneous space of the Poisson groupoid $(G \rightrightarrows P, \pi_G)$ if the graph*

$$\Lambda_{G/H} = \{(g, g'H, gg'H) \mid g, g' \in G, \mathsf{s}(g) = \mathsf{t}(g')\} \subseteq G \times (G/H) \times (G/H)$$

of the left action of $G \rightrightarrows P$ on G/H is a coisotropic submanifold of $(G \times (G/H) \times (G/H), \pi_G \oplus \pi_{G/H} \oplus (-\pi_{G/H}))$.

We will give in Section 2.3 an interpretation of this by means of an action of $\mathsf{P}_G \rightrightarrows TP \times_P A^*G$ on $\mathsf{P}_{G/H}$.
The following theorem classifies the Poisson homogeneous spaces $(G/H, \pi_{G/H})$ of a Poisson groupoid $(G \rightrightarrows P, \pi_G)$ in terms of Lagrangian subalgebroids $\mathfrak{D}$ of $AG \times_P A^*G$ such that $\mathfrak{D} \cap (AG \times_P 0_{A^*G}) = AH \times_P 0_{A^*G}$, where AH is the Lie algebroid of H, seen as a subalgebroid of AG.

Theorem 2.1.9 (Liu et al. (1998)) *For a Poisson groupoid $(G \rightrightarrows P, \pi_G)$, there is a one-to-one correspondence between Poisson homogeneous spaces $(G/H, \pi_{G/H})$ and regular Dirac structures $\mathfrak{D}$ of its tangent Lie bialgebroid, where H is the* t*-connected closed subgroupoid of G corresponding to the subalgebroid $\mathfrak{D} \cap (AG \times_P 0_{A^*G})$.*

2.2 Multiplicative Dirac structures on Lie group(oid)s

Definition 2.2.1 (Ortiz (2009)) *A Dirac groupoid is a Lie groupoid $G \rightrightarrows P$ endowed with a Dirac structure D_G such that $\mathsf{D}_G \subseteq TG \times_G T^*G$ is a Lie subgroupoid. The Dirac structure D_G is then said to be* multiplicative.

Note that in Ortiz (2009), Dirac manifolds are always integrable by definition. Here, we will also study non integrable Dirac structures.
The set of units of this Lie groupoid is then a subbundle of $(TG \times_G T^*G)\,|_P$, that will be written $\mathfrak{A}(\mathsf{D}_G) \to P$ in the following. It will be studied more carefully in Section 5.2.

In the case of a Lie group, we have the following equivalent definition. The proof of the equivalence can be done as the proof that we will give later for an analogous statement about homogeneous spaces (see Proposition 2.3.4).

Proposition 2.2.2 *A* Dirac Lie group *is a Lie group G endowed with a Dirac structure $\mathsf{D}_G \subseteq TG \times_G T^*G$ such that the group multiplication map*

$$\mathsf{m} : (G \times G, \mathsf{D}_G \oplus \mathsf{D}_G) \to (G, \mathsf{D}_G)$$

is a forward Dirac map.

We give here the standard examples of Dirac groupoids, which will illustrate the theory later on.

Example 2.2.3 As we have seen in the previous section, $(G \rightrightarrows P, \pi_G)$ is a Poisson groupoid if and only if $(G \rightrightarrows P, \mathsf{D}_{\pi_G})$ is an integrable Dirac groupoid. $\Diamond$

Example 2.2.4 Let $G \rightrightarrows P$ be a Lie groupoid. A 2-form ω_G on G is *multiplicative* if the partial multiplication map $\mathsf{m} : G \times_P G \to G$ satisfies $\mathsf{m}^*\omega_G = \mathrm{pr}_1^*\,\omega_G + \mathrm{pr}_2^*\,\omega_G$. The graph $\mathsf{D}_{\omega_G} = \mathrm{Graph}(\omega_G^\flat : TG \to T^*G) \subseteq \mathsf{P}_G$ is then multiplicative, and $(G \rightrightarrows P, \mathsf{D}_{\omega_G})$ is a Dirac groupoid, see Ortiz (2009), Bursztyn et al. (2009). The 2-form is closed if and only if the Dirac groupoid is integrable.

Conversely, if a Dirac groupoid $(G \rightrightarrows P, \mathsf{D}_G)$ is such that $\mathsf{G_1} = TG$, then D_G is the graph of the vector bundle homomorphism $TG \to T^*G$ induced by a multiplicative 2-form. If the set of smooth sections of D_G is closed under the Courant-Dorfman bracket, then the 2-form is closed.
Note that *presymplectic groupoids* have been studied in Bursztyn et al. (2004), Bursztyn and Crainic (2005). These are Lie groupoids endowed with closed, multiplicative 2-forms satisfying some additional non degeneracy properties that will be recalled in Example 5.1.13. ♢

Example 2.2.5 Let (M, D_M) be a smooth Dirac manifold. Recall from Example 1.1.22 the higher Lie groupoid structures on $T(M \times M) \rightrightarrows TM$ and $T^*(M \times M) \rightrightarrows T^*M$. We check that the Dirac structure $\mathsf{D}_M \ominus \mathsf{D}_M$, defined by

$$(\mathsf{D}_M \ominus \mathsf{D}_M)(m,n) = \left\{ ((v_m, -v_n), (\alpha_m, \alpha_n)) \in \mathsf{P}_{M \times M}(m,n) \;\middle|\; \begin{array}{c} (v_m, \alpha_m) \in \mathsf{D}_M(m) \\ \text{and } (v_n, \alpha_n) \in \mathsf{D}_M(n) \end{array} \right\}$$

for all $(m,n) \in M \times M$, is a multiplicative Dirac structure on $M \times M \rightrightarrows M$. This generalizes the fact that if (M, π_M) is a Poisson manifold, then $M \times M \rightrightarrows M$ endowed with $\pi_M \oplus (-\pi_M)$ is a Poisson groupoid.
If $((v_m, -v_n), (\alpha_m, \alpha_n)) \in (\mathsf{D}_M \ominus \mathsf{D}_M)(m,n)$, then, by Example 1.1.22, we have

$$\mathbb{T}\mathsf{t}((v_m, -v_n), (\alpha_m, \alpha_n)) = ((v_m, v_m), (\alpha_m, -\alpha_m)) \in (\mathsf{D}_M \ominus \mathsf{D}_M)(m,m)$$

and

$$\mathbb{T}\mathsf{s}((v_m, -v_n), (\alpha_m, \alpha_n)) = ((-v_n, -v_n), (-\alpha_n, \alpha_n)) \in (\mathsf{D}_M \ominus \mathsf{D}_M)(n,n)$$

by definition of $\mathsf{D}_M \ominus \mathsf{D}_M$. Choose a composable pair $((v_m, -v_n), (\alpha_m, \alpha_n)) \in (\mathsf{D}_M \ominus \mathsf{D}_M)(m,n)$, $((-v_n, -v_p), (-\alpha_n, \alpha_p)) \in (\mathsf{D}_M \ominus \mathsf{D}_M)(n,p)$. Then we have

$$((v_m, -v_n), (\alpha_m, \alpha_n)) \star ((-v_n, -v_p), (-\alpha_n, \alpha_p)) = ((v_m, -v_p), (\alpha_m, \alpha_p)),$$

which is an element of $(\mathsf{D}_M \ominus \mathsf{D}_M)(m,p)$. Finally, if $((v_m, -v_n), (\alpha_m, \alpha_n)) \in (\mathsf{D}_M \ominus \mathsf{D}_M)(m,n)$, then $((v_m, -v_n), (\alpha_m, \alpha_n))^{-1}$ equals $((-v_n, v_m), (-\alpha_n, -\alpha_m))$, which is also an element of $(\mathsf{D}_M \ominus \mathsf{D}_M)(n,m)$.

We call the Dirac groupoid $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ the *pair Dirac groupoid* associated to (M, D_M). It is integrable if and only if (M, D_M) is integrable. ♢

2.3 Dirac homogeneous spaces

To be able to define the notion of a homogeneous Dirac structure on a homogeneous space of a Lie groupoid, we have to prove the following proposition.

Proposition 2.3.1 *Let $G \rightrightarrows P$ be a Lie groupoid acting on a smooth manifold M with momentum map $\mathsf{J} : M \to P$. Then there is an induced action of $TG \rightrightarrows TP$ on $T\mathsf{J} : TM \to TP$.*

Assume that $M \simeq G/H$ is a smooth homogeneous space of G and let $q : G \to G/H$ be the projection. The map $\widehat{\mathsf{J}} : T^*(G/H) \to A^*G$, $\widehat{\mathsf{J}}(\alpha_{gH}) = \hat{\mathsf{t}}\left((T_g q)^* \alpha_{gH}\right)$ for all $gH \in G/H$ is well-defined and $\widehat{\Phi} : T^*G \times_{A^*G} T^*(G/H) \to T^*(G/H)$ given by

$$\left(\widehat{\Phi}(\alpha_{g'}, \alpha_{gH})\right)\left(T_{(g',gH)}\Phi(v_{g'}, v_{gH})\right) = \alpha_{g'}(v_{g'}) + \alpha_{gH}(v_{gH})$$

defines an action of $T^*G \rightrightarrows A^*G$ on $\widehat{\mathsf{J}} : T^*(G/H) \to A^*G$.

In the following, we often write $\alpha_g \cdot \alpha_{g'H}$ for $\widehat{\Phi}(\alpha_g, \alpha_{g'H})$.

Proof: We have to check first that $\widehat{\mathsf{J}}$ and $\widehat{\Phi}$ are well-defined. We verify that $\widehat{\mathsf{J}}(\alpha_{gH})$ doesn't depend on the choice of the representative g. Choose g and $g' \in G$ such that $g'H = gH$, and choose $\alpha_{gH} \in T^*_{gH}(G/H)$. Since $lgH = lg'H$ for all $l \in G$ such that $\mathsf{s}(l) = \mathsf{t}(g) = \mathsf{t}(g') =: p$, we know that $q \circ R_g$ and $q \circ R_{g'}$ coincide on their domain of definition $\mathsf{s}^{-1}(p)$ and we see that $\hat{\mathsf{t}}((T_g q)^* \alpha_{gH}) \in A^*_p G$ is given by

$$\begin{aligned}\hat{\mathsf{t}}((T_g q^*)\alpha_{gH})(u_p) &= \alpha_{gH}(T_g q \circ T_p R_g(u_p - T_p \mathsf{s} u_p)) = \alpha_{gH}(T_p(q \circ R_g)(u_p - T_p \mathsf{s} u_p)) \\ &= \alpha_{gH}(T_p(q \circ R_{g'})(u_p - T_p \mathsf{s} u_p)) = \hat{\mathsf{t}}((T_{g'} q)^* \alpha_{gH})(u_p)\end{aligned}$$

for all $u_p \in A_p G$.

Now assume that $\alpha_g \in T^*_g G$ and $\alpha_{g'H} \in T^*_{gH}(G/H)$ are such that $\hat{\mathsf{s}}(\alpha_g) = \widehat{\mathsf{J}}(\alpha_{g'H})$. We have to show that $\widehat{\Phi}(\alpha_g, \alpha_{g'H})$ is well-defined. Choose $v_{gg'H} = T_{(g,g'H)}\Phi(v_g, v_{g'H}) \in T_{gg'H}(G/H)$. Choose a bisection K through g, and a bisection K' through g'. Then $\mathcal{K}' := q \circ K'$ is a section of J through $g'H$ because $\mathsf{J} \circ \mathcal{K}' = \mathsf{J} \circ q \circ K' = \mathsf{t} \circ K' = \mathrm{Id}_P$. If we define $\Phi_K : G/H \to G/H$ by $\Phi_K(x) = \Phi(K((\mathsf{s} \circ K)^{-1}\mathsf{J}(x)), x)$ we have $\Phi_K \circ q = q \circ L_K$, and it is easy to check that $R_{\mathcal{K}'} = q \circ R_{K'}$. Then, by a formula in Liu et al. (1998), we have for any $v_{g'} \in T_{g'}G$ such that $T_{g'} q v_{g'} = v_{g'H}$:

$$\begin{aligned}v_{gg'H} &= T_g R_{\mathcal{K}'} v_g + T_{g'H}\Phi_K v_{g'H} - T_{g'H}(\Phi_K \circ R_{\mathcal{K}'} \circ \mathsf{J}) v_{g'H} \\ &= T_g(q \circ R_{K'}) v_g + T_{g'}(q \circ L_K) v_{g'} - T_{g'}(q \circ L_K \circ R_{K'} \circ \mathsf{t}) v_{g'} \\ &= T_{g'g} q\left(T_g R_{K'} v_g + T_{g'} L_K v_{g'} - T_{g'}(L_K \circ R_{K'} \circ \mathsf{t}) v_{g'}\right) = T_{gg'} q(v_g \star v_{g'}),\end{aligned}$$

which is defined since $T_{g'}\mathsf{t} v_{g'} = T_{g'}(\mathsf{J} \circ q) v_{g'} = T_{g'H}\mathsf{J}(v_{g'H}) = T_g \mathsf{s} v_g$. Note that the formula that we have used in the last equality was proven by Xu (1995). Conversely $T_{gg'} q(v_g \star v_{g'}) = v_g \cdot T_{g'} q v_{g'}$ for all composable pairs $(v_g, v_{g'}) \in T_{(g,g')}(G \times_P G)$ and $\widehat{\Phi}(\alpha_g, \alpha_{g'H})$ satisfies

$$(T_{gg'} q)^* \left(\widehat{\Phi}(\alpha_g, \alpha_{g'H})\right) = \alpha_g \star ((T_{g'} q)^* \alpha_{g'H}).$$

This holds for any choice of $g'' \in G$ such that $g''H = g'H$, since if $v_{g''} \in T_{g''}G$ is such that $T_{g''} q v_{g''} = T_{g'} q v_{g'}$, then it is easy to see from the computation above that $T_{gg''} q(v_g \star v_{g''}) = T_{gg'} q(v_g \star v_{g'})$. Since this defines $\widehat{\Phi}(\alpha_g, \alpha_{g'H})$ in an unique manner, $\widehat{\Phi}(\alpha_g, \alpha_{g'H})$ is well-defined.

We show next that $\widehat{\Phi}$ defines an action of $T^*G \rightrightarrows A^*G$ on $\widehat{\mathsf{J}} : T^*(G/H) \to A^*G$. For all $\alpha_g \in T_g^*G$ and $\alpha_{g'H} \in T_{g'H}^*(G/H)$ such that $\hat{\mathsf{s}}(\alpha_g) = \widehat{\mathsf{J}}(\alpha_{g'H})$, we have by the considerations above $\widehat{\mathsf{J}}(\alpha_g \cdot \alpha_{g'H}) = \hat{\mathsf{t}}((T_{gg'}q)^*(\alpha_g \cdot \alpha_{g'H})) = \hat{\mathsf{t}}(\alpha_g \star (T_{g'}q)^*\alpha_{g'H}) = \hat{\mathsf{t}}(\alpha_g)$. If $\alpha_l \in T_l^*G$ is such that $\hat{\mathsf{s}}(\alpha_l) = \hat{\mathsf{t}}(\alpha_g)$, the product $\alpha_l \cdot (\alpha_g \cdot \alpha_{g'H})$ is defined by

$$\begin{aligned}(T_{lgg'}q)^*(\alpha_l \cdot (\alpha_g \cdot \alpha_{g'H})) &= \alpha_l \star (T_{gg'}q)^*(\alpha_g \cdot \alpha_{g'H}) = \alpha_l \star \alpha_g \star (T_{g'}q)^*\alpha_{g'H}\\ &= (\alpha_l \star \alpha_g) \star (T_{g'}q)^*\alpha_{g'H} = (T_{lgg'}q)^*((\alpha_l \star \alpha_g) \cdot \alpha_{g'H}),\end{aligned}$$

and hence we get $\alpha_l \cdot (\alpha_g \cdot \alpha_{g'H}) = (\alpha_l \star \alpha_g) \cdot \alpha_{g'H}$ since q is a smooth surjective submersion. Finally, the product $\widehat{\mathsf{J}}(\alpha_{g'H}) \cdot \alpha_{g'H}$ satisfies

$$\begin{aligned}(T_{g'}q)^*\left(\widehat{\mathsf{J}}(\alpha_{g'H}) \cdot \alpha_{g'H}\right) &= \widehat{\mathsf{J}}(\alpha_{g'H}) \star (T_{g'}q)^*\alpha_{g'H}\\ &= \hat{\mathsf{t}}((T_{g'}q)^*\alpha_{g'H}) \star (T_{g'}q)^*\alpha_{g'H} = (T_{g'}q)^*\alpha_{g'H},\end{aligned}$$

which shows that $\widehat{\mathsf{J}}(\alpha_{g'H}) \cdot \alpha_{g'H} = \alpha_{g'H}$. $\square$

Corollary 2.3.2 *If G/H is a smooth homogeneous space of $G \rightrightarrows P$, there is an induced action $\mathbb{T}\Phi = (T\Phi, \widehat{\Phi})$ of*

$$(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$$

on

$$\mathbb{T}\mathsf{J} := T\mathsf{J} \times \widehat{\mathsf{J}} : (T(G/H) \times_{G/H} T^*(G/H)) \to (TP \times_P A^*G).$$

We will show that the following definition generalizes in a natural manner the notion of Poisson homogeneous space of a Poisson groupoid.

Definition 2.3.3 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid, and G/H a smooth homogeneous space of $G \rightrightarrows P$ endowed with a Dirac structure $\mathsf{D}_{G/H}$. The pair $(G/H, \mathsf{D}_{G/H})$ is a* Dirac homogeneous space *of the Dirac groupoid $(G \rightrightarrows P, \mathsf{D}_G)$ if the induced action of $(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$ on $\mathbb{T}\mathsf{J} : (T(G/H) \times_{G/H} T^*(G/H)) \to (TP \times_P A^*G)$ restricts to an action of*

$$\mathsf{D}_G \rightrightarrows \mathfrak{A}(\mathsf{D}_G) \qquad \textit{on} \qquad \mathbb{T}\mathsf{J}|_{\mathsf{D}_{G/H}} : \mathsf{D}_{G/H} \to \mathfrak{A}(\mathsf{D}_G).$$

Let (G, D_G) be a Dirac Lie group and H a closed connected Lie subgroup of G. Let $G/H = \{gH \mid g \in G\}$ be the homogeneous space defined as the quotient space by the right action of H on G. Let $q : G \to G/H$ be the quotient map. For $g \in G$, let $\sigma_g : G/H \to G/H$ be the map defined by $\sigma_g(g'H) = gg'H$. In Chapter 4, we will work mostly with the definition given by the following proposition.

Proposition 2.3.4 *Let (G, D_G) be a Dirac Lie group and H a closed connected Lie subgroup of G. Let G/H be endowed with a Dirac structure $\mathsf{D}_{G/H}$. The pair $(G/H, \mathsf{D}_{G/H})$ is a* Dirac homogeneous space of (G, D_G) *if and only if the left action*

$$\sigma : G \times G/H \to G/H, \quad \sigma_g(g'H) = gg'H$$

is a forward Dirac map, where $G \times G/H$ is endowed with the product Dirac structure $\mathsf{D}_G \oplus \mathsf{D}_{G/H}$.

This is easily shown to be equivalent to the following: for all $gH \in G/H$ and $(v_{gH}, \alpha_{gH}) \in \mathsf{D}_{G/H}(gH)$, there exist $(w_g, \beta_g) \in \mathsf{D}_G(g)$ and $(u_{eH}, \gamma_{eH}) \in \mathsf{D}_{G/H}(eH)$ such that

$$\beta_g = (T_g q)^*(\alpha_{gH}), \quad \gamma_{eH} = (T_{eH}\sigma_g)^*(\alpha_{gH}), \quad \text{and} \quad v_{gH} = T_g q w_g + T_{eH}\sigma_g u_{eH}. \tag{2.2}$$

Before we prove the proposition, we need to show that the distributions G_0 and P_1 associated to a multiplicative Dirac structure on G are left and right invariant. The following result has been shown independently by Ortiz (2008).

Proposition 2.3.5 *Let (G, D_G) be a Dirac Lie group. The associated codistribution (respectively distribution) P_1 (respectively G_0) has constant rank on G, and is given by $\mathsf{P}_1 = \mathfrak{p}_1^l = \mathfrak{p}_1^r$ (respectively $\mathsf{G}_0 = \mathfrak{g}_0^l = \mathfrak{g}_0^r$), where $\mathfrak{p}_1 = \mathsf{P}_1(e)$ and $\mathfrak{g}_0 = \mathsf{G}_0(e)$.*

Note that $\mathfrak{p}_1$ equals then automatically the set of units $\mathbb{T}\mathsf{t}(\mathsf{D}_G) = \mathbb{T}\mathsf{s}(\mathsf{D}_G)$ of D_G seen as a subgroupoid of $TG \times_G T^*G$.

Proof: If v_g is an element of $\mathsf{G}_0(g)$, we have $(v_g, 0_g) \in \mathsf{D}_G(g)$ and $\mathbb{T}\mathsf{t}(v_g, 0_g) = \mathbb{T}\mathsf{s}(v_g, 0_g) = 0 \in \mathfrak{g}^*$. Thus, since $(0_{g^{-1}}, 0_{g^{-1}}) \in \mathsf{D}_G(g^{-1})$, we have $(0_{g^{-1}}, 0_{g^{-1}}) \star (v_g, 0_g) \in \mathsf{D}_G(e)$ and $(v_g, 0_g) \star (0_{g^{-1}}, 0_{g^{-1}}) \in \mathsf{D}_G(e)$. But it is easy to see that $0_{g^{-1}} \star v_g = T_{(g^{-1},g)}\mathsf{m}(0_{g^{-1}}, v_g) = T_g L_{g^{-1}} v_g$ and $v_g \star 0_{g^{-1}} = T_g R_{g^{-1}} v_g$, and we get $(T_g L_{g^{-1}} v_g, 0_e) \in \mathsf{D}_G(e)$ and $(T_g R_{g^{-1}} v_g, 0_e) \in \mathsf{D}_G(e)$. We have thus shown that $T_g L_{g^{-1}} \mathsf{G}_0(g) \subseteq \mathfrak{g}_0$ and $T_g R_{g^{-1}} \mathsf{G}_0(g) \subseteq \mathfrak{g}_0$. Conversely, if $x \in \mathfrak{g}_0$, then $(x, 0) \in \mathsf{D}_G(e)$ and $(T_e L_g x, 0_g) = (0_g, 0_g) \star (x, 0) \in \mathsf{D}_G(g)$ and $(T_e R_g x, 0_g) = (x, 0) \star (0_g, 0_g) \in \mathsf{D}_G(g)$. Thus, we have shown the equalities $\mathsf{G}_0(g) = T_e L_g \mathfrak{g}_0 = T_e R_g \mathfrak{g}_0$ and G_0 has constant rank on G.
As a consequence, P_1 is the annihilator of G_0, has also constant rank on G and satisfies $\mathsf{P}_1 = \mathfrak{p}_1^r = \mathfrak{p}_1^l$. $\square$

Proof (of Proposition 2.3.4): Assume first that the map

$$\sigma : (G \times G/H, \mathsf{D}_G \oplus \mathsf{D}_{G/H}) \to (G/H, \mathsf{D}_{G/H}), \qquad (g', gH) \mapsto g'gH$$

is a forward Dirac map. We show that $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of (G, D_G) in the sense of Definition 2.3.3.
The map $\widehat{\mathsf{J}} : T^*(G/H) \to \mathfrak{g}^*$ is given here by $\widehat{\mathsf{J}}(\alpha_{gH}) = (T_e R_g)^*(T_g q)^*\alpha_{gH}$. By (2.2), $(T_g q)^*\alpha_{gH} \in \mathsf{P}_1(g)$ and since P_1 is left and right invariant, we find that $\widehat{\mathsf{J}}$ has image in $\mathfrak{p}_1$. Choose $(v_g, \alpha_g) \in \mathsf{D}_G(g)$ and $(v_{g'H}, \alpha_{g'H}) \in \mathsf{D}_{G/H}(gH)$ such that $(T_e L_g)^*\alpha_g = \hat{\mathsf{s}}(\alpha_g) = \widehat{\mathsf{J}}(\alpha_{g'H}) = (T_e(q \circ R_{g'}))^*\alpha_{g'H}$. We have to show that $(v_g, \alpha_g) \cdot (v_{g'H}, \alpha_{g'H})$ is an element of $\mathsf{D}_{G/H}(gg'H)$. Choose $(w_{gg'H}, \beta_{gg'H}) \in \mathsf{D}_{G/H}(gg'H)$. Then, since σ is a forward Dirac map, there exists $(u_g, \gamma_g) \in \mathsf{D}_G(g)$ and $(u_{g'H}, \gamma_{g'H}) \in \mathsf{D}_{G/H}(g'H)$ such that

$$\begin{aligned} w_{gg'H} &= T_{(g,g'H)}\sigma(u_g, u_{g'H}) = T_{g'H}\sigma_g u_{g'H} + T_{gg'}q(T_g R_{g'} u_g), \\ \gamma_g &= (T_g(q \circ R_{g'}))^*\beta_{gg'H} \qquad \text{and} \qquad \gamma_{g'H} = (T_{g'H}\sigma_g)^*\beta_{gg'H}. \end{aligned}$$

We can then compute

$$\begin{aligned} &\langle (v_g, \alpha_g) \cdot (v_{g'H}, \alpha_{g'H}), (w_{gg'H}, \beta_{gg'H}) \rangle \\ &= \langle (T_{(g,g'H)}\sigma(v_g, v_{g'H}), \alpha_g \cdot \alpha_{g'H}), (T_{(g,g'H)}\sigma(u_g, u_{g'H}), \beta_{gg'H}) \rangle \\ &= \alpha_g(u_g) + \alpha_{g'H}(u_{g'H}) + \beta_{gg'H}(T_{g'H}\sigma_g v_{g'H} + T_g(q \circ R_{g'})v_g) \\ &= \langle (u_g, T_g(q \circ R_{g'})^*\beta_{gg'H}), (v_g, \alpha_g) \rangle + \langle (u_{g'H}, (T_{g'H}\sigma_g)^*\beta_{gg'H}), (v_{g'H}, \alpha_{g'H}) \rangle = 0 + 0 = 0, \end{aligned}$$

since $(u_g, T_g(q\circ R_{g'})^*\beta_{gg'H}), (v_g,\alpha_g) \in \mathsf{D}_G(g)$ and $(u_{g'H}, (T_{g'H}\sigma_g)^*\beta_{gg'H}), (v_{g'H},\alpha_{g'H}) \in \mathsf{D}_{G/H}(g'H)$. Since $(w_{gg'H},\beta_{gg'H})\in \mathsf{D}_{G/H}(gg'H)$ was arbitrary, we have shown that

$$(v_g,\alpha_g)\cdot(v_{g'H},\alpha_{g'H}) \in D_{G/H}(gg'H)^{\perp} = D_{G/H}(gg'H).$$

Conversely, we assume that $(G/H,\mathsf{D}_{G/H})$ is a Dirac homogeneous space of (G,D_G) in the sense of Definition 2.3.3 and we show that σ is a forward Dirac map. Choose $(v_{gg'H},\alpha_{gg'H})\in \mathsf{D}_{G/H}(gg'H)$. Then we know that $(T_{gg'}q)^*\alpha_{gg'H}\in \mathsf{P}_1(gg')$ since $\widehat{\mathsf{J}}(\alpha_{gg'H}) = \hat{\mathsf{t}}((T_{gg'}q)^*\alpha_{gg'H}) = T_eR^*_{gg'}((T_{gg'}q)^*\alpha_{gg'H})$ and, by Lemma 2.3.5, P_1 is right invariant. Hence, there exists $v_{gg'}\in T_{gg'}G$ such that $(v_{gg'},(T_{gg'}q)^*\alpha_{gg'H})\in \mathsf{D}_G(gg')$. If we set $\alpha_g := (T_gR_{g'})^*(T_{gg'}q)^*\alpha_{gg'H} \in \mathsf{P}_1(g)$, there exists in the same manner $v_g\in T_gG$ such that $(v_g,(T_g(q\circ R_{g'}))^*\alpha_{gg'H})\in \mathsf{D}_G(g)$.
Let $\mathsf{i}: G\to G$ be the inversion map. Then

$$(v_g,(T_g(q\circ R_{g'}))^*\alpha_{gg'H})^{-1} = (T_g\mathsf{i}v_g,(T_{g^{-1}}(q\circ R_{gg'}\circ L_g))^*\alpha_{gg'H})\in \mathsf{D}_G(g^{-1})$$

and since

$$\hat{\mathsf{s}}\left(((T_g(q\circ R_{g'}))^*\alpha_{gg'H})^{-1}\right) = \hat{\mathsf{t}}\left((T_g(q\circ R_{g'}))^*\alpha_{gg'H}\right) = \hat{\mathsf{t}}\left((T_{gg'}q)^*\alpha_{gg'H}\right) = \widehat{\mathsf{J}}(\alpha_{gg'H}),$$

we have

$$(v_{g'H},\alpha_{g'H}) := (v_g,(T_g(q\circ R_{g'}))^*\alpha_{gg'H})^{-1}\cdot(v_{gg'H},\alpha_{gg'H})\in \mathsf{D}_{G/H}(g'H).$$

By definition, we have then $T_{(g,g'H)}\sigma(v_g,v_{g'H}) = v_{gg'H}$, $\alpha_{g'H} = (T_{g'H}\sigma_g)^*\alpha_{gg'H}$ and $\alpha_g = (T_g(q\circ R_{g'}))^*\alpha_{gg'H}$. □

Example 2.3.6 Consider a Poisson homogeneous space $(G/H,\pi)$ of a Poisson groupoid $(G\rightrightarrows P,\pi_G)$, i.e., the graph $\operatorname{Graph}(\Phi)\subseteq G\times G/H\times\overline{G/H}$ is a coisotropic submanifold (see Liu et al. (1998)).
Consider the Dirac groupoid $(G\rightrightarrows P,\mathsf{D}_{\pi_G})$ defined by $(G\rightrightarrows P,\pi_G)$ and the Dirac manifold $(G/H,\mathsf{D}_{G/H})$, defined by $\mathsf{D}_{G/H} = \operatorname{Graph}(\pi^\sharp: T^*(G/H)\to T(G/H))$. We verify that $(G/H,\mathsf{D}_{G/H})$ is a Dirac homogeneous space of the Dirac groupoid $(G\rightrightarrows P,\mathsf{D}_{\pi_G})$.
Choose $\alpha_{gH}\in T^*_{gH}(G/H)$ and set $p:=\mathsf{t}(g)=\mathsf{J}(gH)$. Then we have for all $v_p\in T_pG$ and $v_g\in T_gG$ such that $T_p\mathsf{s}v_p = T_g\mathsf{t}v_g = T_{gH}\mathsf{J}(T_gqv_g)$:

$$\begin{aligned}
&\widehat{\mathsf{J}}(\alpha_{gH})(v_p)+\alpha_{gH}(T_gqv_g)-\alpha_{gH}(T_{(p,gH)}\sigma(v_p,T_gqv_g))\\
&=\hat{\mathsf{t}}((T_gq)^*\alpha_{gH})(v_p)+\alpha_{gH}(T_gqv_g)-\alpha_{gH}(T_gq(v_p\star v_g))\\
&=\left(\hat{\mathsf{t}}((T_gq)^*\alpha_{gH})\star(T_gq)^*\alpha_{gH}\right)(v_p\star v_g)-((T_gq)^*\alpha_{gH})(v_p\star v_g)=0.
\end{aligned}$$

This shows that $\left(\widehat{\mathsf{J}}(\alpha_{gH}),\alpha_{gH},-\alpha_{gH}\right)\in\left(T_{(p,gH,gH)}\Lambda_{G/H}\right)^\circ$, which implies

$$T\mathsf{s}\left(\pi_G^\sharp\left(\widehat{\mathsf{J}}(\alpha_{gH})\right)\right) = T\mathsf{J}\left(\pi^\sharp(\alpha_{gH})\right)$$

and
$$\pi_G^\sharp\left(\widehat{\mathsf{J}}(\alpha_{gH})\right)\cdot\pi^\sharp(\alpha_{gH})=\pi^\sharp(\alpha_{gH})$$
since $\Lambda_{G/H}$ is coisotropic. Hence, we have shown that
$$\pi_G^\sharp\left(\widehat{\mathsf{J}}(\alpha_{gH})\right)=T\mathsf{J}\left(\pi^\sharp(\alpha_{gH})\right)=T\mathsf{s}\left(\pi_G^\sharp\left(\widehat{\mathsf{J}}(\alpha_{gH})\right)\right)$$
and hence
$$\mathbb{T}\mathsf{J}\left(\pi^\sharp(\alpha_{gH}),\alpha_{gH}\right)=\mathbb{T}\mathsf{s}\left(\pi_G^\sharp\left(\widehat{\mathsf{J}}(\alpha_{gH})\right),\widehat{\mathsf{J}}(\alpha_{gH})\right)\in\mathfrak{A}(\mathsf{D}_{\pi_G})(p)$$
for all $\alpha_{gH}\in T^*_{gH}(G/H)$. Let now $\alpha_{g'}\in T^*_{g'}G$ be such that $\hat{\mathsf{s}}\left(\alpha_{g'}\right)=\widehat{\mathsf{J}}(\alpha_{gH})$. Then we have $T\mathsf{s}\left(\pi_G^\sharp(\alpha_g')\right)=\pi_G^\sharp\left(\hat{\mathsf{s}}(\alpha_{g'})\right)=\pi_G^\sharp\left(\widehat{\mathsf{J}}(\alpha_{gH})\right)=T\mathsf{J}\left(\pi^\sharp(\alpha_{gH})\right)$ since π_G is multiplicative, and hence $\mathbb{T}\mathsf{s}\left(\pi^\sharp(\alpha_{g'}),\alpha_{g'}\right)=\mathbb{T}\mathsf{J}\left(\pi^\sharp(\alpha_{gH}),\alpha_{gH}\right)$. Since for all $v_{g'}\in T_{g'}G$ and $v_g\in T_gG$ such that $T\mathsf{s}(v_{g'})=T\mathsf{t}(v_g)=T\mathsf{J}(T_gqv_g)$, we have
$$\begin{aligned}&\alpha_{g'}(v_{g'})+\alpha_{gH}(T_gqv_g)-(\alpha_{g'}\cdot\alpha_{gH})(v_{g'}\cdot T_gqv_g)\\=&\,\alpha_{g'}(v_{g'})+\alpha_{gH}(T_gqv_g)-(\alpha_{g'}\cdot\alpha_{gH})T_{g'g}q(v_{g'}\star v_g)\\=&\,(\alpha_{g'}\star(T_gq)^*\alpha_{gH})(v_{g'}\star v_g)-(\alpha_{g'}\star(T_gq)^*\alpha_{gH})(v_{g'}\star v_g)=0,\end{aligned}$$
we get again
$$(\alpha_{g'},\alpha_{gH},-\alpha_{g'}\cdot\alpha_{gH})\in\left(T_{(g',gh,g'gH)}\Lambda_{G/H}\right)^\circ$$
and hence
$$\pi_G^\sharp(\alpha_g')\cdot\pi^\sharp(\alpha_{gH})=\pi^\sharp\left(\alpha_{g'}\cdot\alpha_{gH}\right).$$
Conversely, we show in a similar manner that if $\mathbb{T}\Phi$ restricts to an action of D_{π_G} on D_π, then the graph of the left action of G on G/H is coisotropic. $\Diamond$

Example 2.3.7 Let $(G\rightrightarrows P,\omega_G)$ be a presymplectic groupoid and H a wide subgroupoid of $G\rightrightarrows P$. Assume that G/H has a smooth manifold structure such that the projection $q:G\to G/H$ is a surjective submersion. Let ω be a closed 2-form on G/H such that the action $\Phi:G\times_P(G/H)\to G/H$ is a presymplectic groupoid action, i.e., $\Phi^*\omega=\mathrm{pr}^*_{G/H}\omega+\mathrm{pr}^*_G\omega_G$. Let D_ω be the graph of the vector bundle map $\omega^\flat:T(G/H)\to T^*(G/H)$ associated to ω. It is easy to check that the pair $(G/H,\mathsf{D}_\omega)$ is an integrable Dirac homogeneous space of the integrable Dirac groupoid $(G\rightrightarrows P,\mathsf{D}_{\omega_G})$, see Example 2.2.4. $\Diamond$

Example 2.3.8 Let $(G\rightrightarrows P,\mathsf{D}_G)$ be a Dirac groupoid. Then $(\mathsf{t}:G\to P,\mathsf{D}_G)$ is a Dirac homogeneous space of $(G\rightrightarrows P,\mathsf{D}_G)$. $\Diamond$

3 Multiplicative foliations

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and multiplication map $\mathsf{m} : G \times G \to G$. Then the tangent space TG of G is also a Lie group with unit $0_e \in \mathfrak{g}$ and with the multiplication map $T\mathsf{m} : TG \times TG \to TG$. A multiplicative distribution $S \subseteq TG$ is a distribution on G that is a subgroup of TG. The zero section of TG is contained in S and $T\mathsf{m}(0_g, v_h) = TL_g v_h$ for any $g, h \in G$ and $v_h \in T_hG$, where $L_g : G \to G$ is the left translation by g. Thus, the distribution S is left invariant and consequently a smooth left invariant vector bundle on G defined by $S(g) = \mathfrak{s}^l(g)$ with $\mathfrak{s}$ the vector subspace $S(e) = S \cap \mathfrak{g}$ of $\mathfrak{g}$. In the same manner, S is right invariant and we find that $\mathfrak{s}$ is invariant under the adjoint action of G on $\mathfrak{g}$. Hence $\mathfrak{s}$ is an ideal in $\mathfrak{g}$ and the subbundle $S \subseteq TG$ is completely integrable in the sense of Frobenius. Its leaf N through the unit element e of G is a normal subgroup of G and since the leaf space G/S of S is equal to G/N, it inherits a group structure from G such that the projection $G \to G/N$ is a homomorphism of groups. If N is closed in G, the foliation defined by S is regular and the leaf space $G/S = G/N$ is a Lie group such that the projection is a smooth surjective submersion.
If $G \rightrightarrows P$ is a Lie groupoid, we have seen in Subsection 1.1.5 that its tangent space TG is, in the same manner as in the Lie group case, a Lie groupoid over TP, the tangent space of the units. Hence, we can define more generally:

Definition 3.0.9 *Let $G \rightrightarrows P$ be a Lie groupoid and $TG \rightrightarrows TP$ its tangent prolongation. A subdistribution $S \subseteq TG$ is* multiplicative *if S is a (set) subgroupoid of $TG \rightrightarrows TP$.*

In this chapter, we show that if a smooth multiplicative *subbundle* $S \subseteq TG$ is involutive with a completeness condition, and its space of leaves G/S satisfies a certain compatibility condition with the left translations, then G/S inherits a groupoid structure over the leaf space of the intersection $TP \cap S$ in P. We use the theory about normal subgroupoid systems in Lie groupoids (see Mackenzie (2005)) to show the main theorem of this chapter (Theorem 3.3.11). It is also possible to show in a direct manner that, under the necessary conditions, one can define a groupoid structure on the leaf space. Yet, since our two leadings examples (Examples 3.1.1 and 3.1.2) illustrate the discussion in Mackenzie (2005) about kernels of Lie groupoid morphisms (see also Subsection 1.1.2), we choose to use the theory in this book. We show that there is a natural normal subgroupoid system in $G \rightrightarrows P$, that is associated to the multiplicative subbundle S. The induced quotient groupoid is then exactly the leaf space G/S of S.
If $S \subseteq TG$ is a multiplicative, involutive subbundle, then $\mathsf{D}_S = S \oplus S^\circ$ is an integrable, multiplicative Dirac structure on $G \rightrightarrows P$. Hence, since multiplicative foliations on Lie groupoids are in this manner particular examples of integrable Dirac groupoids, this chapter gives a first idea of the difference between the Dirac groups and the Dirac groupoids.

As an application of Theorem 3.3.11, we will show in Chapter 5 that, under rather strong regularity conditions on its characteristic distribution, a Dirac groupoid $(G\rightrightarrows P, \mathsf{D}_G)$ is the pullback of a multiplicative Poisson structure on a quotient groupoid of $G\rightrightarrows P$. In the group case, these required regularity conditions are much weaker. For example, we will see that they are always satisfied if the underlying Lie group is simply connected.

Outline of the chapter In the first section of this chapter, we start by giving two examples of integrable multiplicative distributions where the leaf spaces inherit groupoid structures. Then we study general properties of multiplicative subbundles of the tangent space TG of a Lie groupoid $G\rightrightarrows P$ (Section 3.2), before we show in the third section how all the structure maps of the Lie groupoid "descend" to the leaf space of an involutive multiplication foliation (Section 3.3). In Section 3.4, we give the Lie algebroid of the obtained Lie groupoid in the context of ideal systems.

3.1 Examples

The two examples in this section illustrate the general theory in the following sections. We study two completely integrable, multiplicative distributions on special classes of Lie groupoids and show that their spaces of leaves inherit groupoid structures. In general, we will have to assume that the distributions have constant rank, that the elements of special spanning families of vector fields are *complete* and that their leaves satisfy an additional condition on compatibility with the left translations. We will see that these two last regularity conditions are satisfied in both examples, although the studied distributions are here not necessarily subbundles of the tangent space.

Example 3.1.1 Let M be a smooth manifold and $M \times M \rightrightarrows M$ the pair groupoid. If $\mathcal{D}$ is a smooth subdistribution of TM, then $S := \mathcal{D} \times \mathcal{D} \subseteq TM \times TM \simeq T(M \times M)$ is a multiplicative subdistribution of $TM \times TM \rightrightarrows TM$. Its intersection with $T\Delta_M$ is $\Delta_{\mathcal{D}}$, where $\Delta_{\mathcal{D}}(m,m) = \{(v_m, v_m) \mid v_m \in \mathcal{D}(m)\}$, which is also a smooth subdistribution of $T\Delta_M$.
If $\mathcal{D}$ is completely integrable in the sense of Stefan and Sussmann, then S and $\Delta_{\mathcal{D}}$ are also completely integrable. Let $\mathrm{pr}_S : M \times M \to (M \times M)/S$ and $\mathrm{pr}_{\mathcal{D}} : M \to M/\mathcal{D}$ be the quotient maps. If N_m, respectively N_n is the leaf of $\mathcal{D}$ through m, respectively n, then the leaf of S through $(m,n) \in M \times M$ is $N_{(m,n)} = N_m \times N_n$. Thus, the space $(M \times M)/S$ of leaves of S coincides with $M/\mathcal{D} \times M/\mathcal{D}$ via the map $\Phi : (M \times M)/S \to M/\mathcal{D} \times M/\mathcal{D}$, $\mathrm{pr}_S(m,n) \mapsto (\mathrm{pr}_{\mathcal{D}}(m), \mathrm{pr}_{\mathcal{D}}(n))$. To see this, note that if (m,n) and (m',n') are in the same leaf of S, then there exists without loss of generality one smooth vector field $X = (X_1, X_2) \in \Gamma(\mathcal{D}\times\mathcal{D})$ such that $(m',n') = \phi_t^X(m,n) = (\phi_t^{X_1}(m), \phi_t^{X_2}(n))$ for some $t \in \mathbb{R}$, where $\phi^X, \phi^{X_1}, \phi^{X_2}$ are the flows of X, X_1, X_2, respectively. But then m, m' and respectively n, n' are in the same leaves of $\mathcal{D}$. We get $\mathrm{pr}_{\mathcal{D}}(m) = \mathrm{pr}_{\mathcal{D}}(m')$, $\mathrm{pr}_{\mathcal{D}}(n) = \mathrm{pr}_{\mathcal{D}}(n')$ and Φ is thus well-defined. It is obviously injective and surjective, and since the following

diagram commutes,

$$\begin{tikzcd} M \times M \arrow[d, "\mathrm{pr}_S"'] \arrow[dr, "\mathrm{pr}_{\mathcal{D}} \times \mathrm{pr}_{\mathcal{D}}"] & \\ (M \times M)/S \arrow[r, "\Phi"'] & M/\mathcal{D} \times M/\mathcal{D} \end{tikzcd}$$

it is easy to check that it is a homeomorphism.
The pair $(\mathrm{pr}_S, \mathrm{pr}_{\mathcal{D}})$ is a groupoid morphism, where $(M \times M)/S \simeq M/\mathcal{D} \times M/\mathcal{D}$ is endowed with the pair groupoid structure, $M/\mathcal{D} \times M/\mathcal{D} \rightrightarrows M/\mathcal{D}$. That is, the following diagram commutes

$$\begin{tikzcd} M \times M \arrow[r, "\mathrm{pr}_S"] \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] & (M \times M)/S \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ M \arrow[r, "\mathrm{pr}_{\mathcal{D}}"'] & M/\mathcal{D} \end{tikzcd}$$

and we have $\mathrm{pr}_S((m,n) \star (n,p)) = \mathrm{pr}_S(m,n) \star \mathrm{pr}_S(n,p)$ for all $m, n, p \in M$.

Consider now the union

$$N = \cup_{m \in M} N_{(m,m)} = \cup_{m \in M} (N_m \times N_m)$$

of leaves of S through the unities $(m,m) \in \Delta_M$. Note that $(m,n) \in N$ if and only if there exists $q \in M$ such that $\mathrm{pr}_S(m,n) = \mathrm{pr}_S(q,q)$, that is, $(\mathrm{pr}_{\mathcal{D}}(m), \mathrm{pr}_{\mathcal{D}}(n)) = (\mathrm{pr}_{\mathcal{D}}(q), \mathrm{pr}_{\mathcal{D}}(q))$ and hence $\mathrm{pr}_{\mathcal{D}}(m) = \mathrm{pr}_{\mathcal{D}}(q) = \mathrm{pr}_{\mathcal{D}}(n)$. Thus, we have

$$N = \{(m,n) \in M \times M \mid \mathrm{pr}_S(m,n) \in \Delta_{M/\mathcal{D}}\} = \ker(\mathrm{pr}_S, \mathrm{pr}_{\mathcal{D}})$$

and N is a normal subgroupoid of $M \times M$ in the sense of Mackenzie (1987).

Following Mackenzie (1987), we define the equivalence relations $\sim_\circ$ on $M \simeq \Delta_M$ and $\sim_N$ on $M \times M$ by

$$(m,m) \sim_\circ (n,n) \iff (m,n) \in N$$

and

$$(m,n) \sim_N (p,q) \iff (m,p), (n,q) \in N,$$

and we get a groupoid structure on $(M \times M)/\sim_N \rightrightarrows M/\sim_\circ$. But here, we see immediately that

$$\begin{aligned}(m,n) \sim_N (p,q) &\iff (m,p), (n,q) \in N \iff \mathrm{pr}_{\mathcal{D}}(m) = \mathrm{pr}_{\mathcal{D}}(p) \text{ and } \mathrm{pr}_{\mathcal{D}}(n) = \mathrm{pr}_{\mathcal{D}}(q) \\ &\iff \mathrm{pr}_S(m,n) = \mathrm{pr}_S(p,q)\end{aligned}$$

and

$$(m,m) \sim_\circ (n,n) \iff (m,n) \in N \iff \mathrm{pr}_{\mathcal{D}}(m) = \mathrm{pr}_{\mathcal{D}}(n).$$

Hence, we have $(M \times M)/S = (M \times M)/\sim_N$ and $M/\mathcal{D} = M/\sim_\circ$. Thus, the quotient groupoid that we obtain by the normal subgroupoid N is exactly the groupoid defined by the leaf space of the multiplicative distribution S.

Note that $(m,p)N \in (M \times M)/N$ is given by

$$(m,p)N = \{(m,p) \star (p,q) \mid (p,q) \in N\} = \{(m,q) \mid q \in N_p\} = \{m\} \times N_p.$$

If $\mathcal{R} := \{(m,n) \in M \times M \mid m \sim_{\mathcal{D}} n\}$ is the relation defined by $\mathcal{D}$ on $M \times M$, and $\theta : \mathcal{R} \times_M ((M \times M)/N) \to (M \times M)/N$, $\theta((n,m), \{m\} \times N_p) = \{n\} \times N_p$, then $\mathcal{N} = (N, \mathcal{R}, \theta)$ is a normal groupoid system in $M \times M \rightrightarrows M$ and the set of points in the orbit of $(m,p)N$ under θ is equal to

$$\bigcup_{\{n \in M \mid (n,m) \in \mathcal{R}\}} \theta((n,m), \{m\} \times N_p) = \bigcup_{\{n \in M \mid n \sim_{\mathcal{D}} m\}} \{n\} \times N_p = N_m \times N_p = N_{(m,p)}.$$

The set $\mathcal{S}_{\mathcal{N}}$ associated to $\mathcal{N} = (N, \mathcal{R}, \theta)$ is then

$$\begin{aligned}
\mathcal{S}_{\mathcal{N}} &= \left\{ ((m,n),(p,q)) \in (M \times M) \times (M \times M) \;\middle|\; \begin{array}{c} m \sim_{\mathcal{D}} p \text{ and} \\ \theta((m,p), \{p\} \times N_q) = \{m\} \times N_n \end{array} \right\} \\
&= \{((m,n),(p,q)) \in (M \times M) \times (M \times M) \mid m \sim_{\mathcal{D}} p \text{ and } n \sim_{\mathcal{D}} q\} \\
&= \{((m,n),(p,q)) \in (M \times M) \times (M \times M) \mid (m,n) \sim_S (p,q)\}
\end{aligned}$$

The groupoid structure induced by $(N, \mathcal{R}, \theta)$ on $(M \times M)/\mathcal{S}_{\mathcal{N}} \rightrightarrows M/\mathcal{R}$ is again the groupoid defined by the leaf space of the multiplicative distribution S.

If $\mathcal{D}$ is an involutive subbundle of TM such that the leaf space $M/\mathcal{D}$ is a smooth manifold and $\mathrm{pr}_{\mathcal{D}} : M \to M/\mathcal{D}$ a smooth surjective submersion, then the induced groupoid $(M \times M)/S \rightrightarrows M/\mathcal{D}$ is a Lie groupoid.

Note that the leaf through (m,n) of $S \cap T^{\mathsf{t}}(M \times M) = 0_{TM} \times \mathcal{D}$ equals $\{m\} \times N_n$, which is exactly the intersection of the leaf $N_m \times N_n$ of S through (m,n) with $\mathsf{t}^{-1}(\mathsf{t}(m,n)) = \{m\} \times M$. This implies that the compatibility condition (3.5) that we will have to assume in the general case is satisfied here (see also Remark 3.3.8):

$$L_{(m,n)}(N_{(n,n)} \cap (\{n\} \times M)) = L_{(m,n)}(\{n\} \times N_n) = \{m\} \times N_n = N_{(m,n)} \cap (\{m\} \times M)$$

for all $(m,n) \in M \times M$. $\Diamond$

In the next example, we get in the same manner a groupoid structure on the leaf space of the involutive multiplicative distribution that is considered. This time, it does not coincide with the quotient by the kernel N but rather with the quotient by the normal subgroupoid system defined by the union N of leaves through the units. This illustrates the different definitions of normal subgroupoids in the two references Mackenzie (1987) and Mackenzie (2005), and the fact that the notion of *normal subgroupoid systems of groupoids* is needed to generalize to groupoids the relation between the kernel of surjective homomorphisms and normal subgroups of a group (see Mackenzie (2005) and also Subsection 1.1.2).

Example 3.1.2 Let M be a smooth manifold and $\mathsf{p} : \mathbb{R}^k \times M \to M$ a trivial vector bundle over M. Then $\mathbb{R}^k \times M \rightrightarrows M$ has the structure of a Lie groupoid over the base M (see Example 1.1.7).

In the following, we identify $T(\mathbb{R}^k \times M)$ with $\mathbb{R}^k \times \mathbb{R}^k \times TM$, and we write $T_{(x,m)}(\mathbb{R}^k \times M) = \{x\} \times \mathbb{R}^k \times T_mM$. The source and target maps in $T(\mathbb{R}^k \times M) \rightrightarrows TM$ are then $T\mathsf{s}(x, v, v_m) = T\mathsf{t}(x, v, v_m) = v_m$ and the partial multiplication is given by $(x, v, v_m) \star (y, w, v_m) = (x + y, v + w, v_m)$. Hence, the groupoid $T(\mathbb{R}^k \times M) \rightrightarrows TM$ is a vector bundle groupoid $\mathbb{R}^{2k} \times TM \rightrightarrows TM$.
Consider a completely integrable distribution F in TM and a vector subspace $W \subseteq \mathbb{R}^k$. It is easy to see that $S := \mathbb{R}^k \times W \times F$ is an integrable multiplicative distribution in $T(\mathbb{R}^k \times M)$.
The leaf of S through (x, m) is the set $(x + W) \times L_m$, where L_m is the leaf of F through m. The intersection $TM \cap S$ equals F and we have a foliation M/F. It is easy to see that the quotient $(\mathbb{R}^k \times M)/S \simeq (\mathbb{R}^k/W) \times (M/F)$ inherits the structure of a groupoid over M/F such that the diagram

$$\begin{tikzcd} \mathbb{R}^k \times M \arrow[r, "\mathrm{pr}_S"] \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] & (\mathbb{R}^k/W) \times (M/F) \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ M \arrow[r, "\mathrm{pr}_F"'] & M/F \end{tikzcd}$$

commutes and $(\mathrm{pr}_S, \mathrm{pr}_F)$ is a morphism of groupoids. If (x, m) and (y, n) are in the same leaf of S, then $x - y \in W$ and $n \in L_m$. In particular, the elements $\mathsf{s}(x, m) = \mathsf{t}(x, m) = m$ and $\mathsf{s}(y, n) = \mathsf{t}(y, n) = n$ are in the same leaf of F. Hence, we can define maps s, t : $(\mathbb{R}^k/W) \times (M/F) \to M/F$ by $\mathsf{s}((x + W) \times L_m) = \mathsf{t}((x + W) \times L_m) = L_m$. The product of $(x + W) \times L_m$ with $(y + W) \times L_n$ is defined if and only if $L_n = L_m$ and it is then equal to $(x + y + W) \times L_m$.
The multiplication is hence defined as follows: if $\mathsf{s}(x, m) = m \sim_F n = \mathsf{t}(y, n)$, then there exists $z \in y + W$ (for instance $z = y$) such that $(z, m) \sim_S (y, n)$ and the product is the class of $(x, m) \star (z, m) = (x + z, m)$ in $(\mathbb{R}^k \times M)/S$, that is, $(x + z + W) \times L_m = (x + y + W) \times L_m$.

The set $N = \cup_{m \in M} L_{(0,m)}$ is given here by $N = \cup_{m \in M}(W \times L_m) = W \times (\cup_{m \in M} L_m) = W \times M$. It is a normal subgroupoid of $\mathbb{R}^k \times M$ and defines consequently equivalence relations on M and on $\mathbb{R}^k \times M$:

$$m \sim_\circ n \quad \Leftrightarrow \quad \exists (x, p) \in N \text{ such that } \mathsf{t}(x, p) = m \text{ and } \mathsf{s}(x, p) = n \quad \Leftrightarrow \quad m = n = p$$

and

$$\begin{aligned} (x, m) \sim_N (y, n) \ &\Leftrightarrow \exists (z, p) \in N = W \times M \text{ such that } \mathsf{t}(z, p) = m, \mathsf{s}(z, p) = n \\ &\qquad \text{and } (x, m) \star (z, p) = (y, n) \\ &\Leftrightarrow m = n \text{ and } x - y \in W. \end{aligned}$$

The quotient groupoid defined by N is hence $(\mathbb{R}^k/W) \times M \rightrightarrows M$, which is this time not isomorphic to $(\mathbb{R}^k \times M)/S \rightrightarrows M/F$, except if F is trivial. It seems that we loose too much information about S in the construction of the normal subgroupoid N.

We have hence to consider the normal subgroupoid system defined by S on $\mathbb{R}^k \times M \rightrightarrows M$. The normal subgroupoid system is the triple $\mathcal{N} = (N, \mathcal{R}_F, \theta)$, where $N = W \times M$ is the

wide subgroupoid, $\mathcal{R}_F$ is the wide subgroupoid of $M \times M \rightrightarrows M$ defined by $\mathcal{R}_F := \{(m,n) \in M \times M \mid m \sim_F n\} = \{(m,n) \in M \times M \mid \mathrm{pr}_F(m) = \mathrm{pr}_F(n)\}$ and θ is the action of $\mathcal{R}_F$ on $G/N = (\mathbb{R}^k/W) \times M$ given by $\theta(m,n)(x+W,n) = (x+W,m)$. Note that G/N is equal to $(\mathbb{R}^k/W) \times M$ because, by definition, $G/N = \{(x,m)N \mid (x,m) \in \mathbb{R}^k \times M\}$, with $(x,m)N = \{(x,m) \star (w,m) \mid w \in W\} = (x+W) \times \{m\}$.
Define $\mathcal{S} \subseteq (\mathbb{R}^k \times M) \times (\mathbb{R}^k \times M)$ by

$$\mathcal{S} = \{((x,m),(y,n)) \mid (m,n) \in \mathcal{R}_F \text{ and } \theta(m,n)(y+W,n) = (x+W,m)\}.$$

The quotient of M by the equivalence relation $\mathcal{R}_F$ is exactly the leaf space of F in M, and the quotient of $\mathbb{R}^k \times M$ by the equivalence relation $\mathcal{S}$ is the leaf space of S in $\mathbb{R}^k \times M$. We have indeed

$$(x,m) \sim_{\mathcal{S}} (y,n) \Leftrightarrow m \sim_F n \text{ and } x - y \in W \Leftrightarrow \mathrm{pr}_S(x,m) = \mathrm{pr}_S(y,n)$$

for all $(x,m),(y,n) \in \mathbb{R}^k \times M$.
The quotient groupoid defined by $\mathcal{N}$ is $(\mathbb{R}^k \times M)/\mathcal{S} \rightrightarrows M/\mathcal{R}_F$. Using the equalities $(\mathbb{R}^k \times M)/\mathcal{S} = (\mathbb{R}^k \times M)/S = (\mathbb{R}^k/W) \times (M/F)$ and $M/\mathcal{R}_F = M/F$, it is easy to see that all the groupoid maps are well-defined and

$$\begin{tikzcd} \mathbb{R}^k \times M \arrow[r, "\mathrm{pr}_S"] \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] & (\mathbb{R}^k \times M)/\mathcal{S} \arrow[d, shift right, "\mathsf{t}"'] \arrow[d, shift left, "\mathsf{s}"] \\ M \arrow[r, "\mathrm{pr}_F"'] & M/\mathcal{R}_F \end{tikzcd}$$

commutes. We have hence shown that the two groupoid structures on

$$(\mathbb{R}^k \times M)/S \rightrightarrows M/(S \cap TM)$$

coincide.

Note that the leaf of $S \cap T^{\mathsf{t}}(\mathbb{R}^k \times M) = S \cap (T\mathbb{R}^k \times 0_{TM}) = (\mathbb{R}^k \times W) \times 0_{TM}$ through (x,m) equals $(x+W) \times \{m\}$ and is exactly the intersection of the leaf $(x+W) \times L_m$ of S through (x,m) with $\mathsf{t}^{-1}(\mathsf{t}(x,m)) = \mathbb{R}^k \times \{m\}$. $\Diamond$

We will generalize these results to the leaf space of any involutive multiplicative *subbundle* S of $TG \rightrightarrows TP$ for an arbitrary Lie groupoid $G \rightrightarrows P$, under technical assumptions about the compatibility of the leaves of S with the left translations L_g, $g \in G$ and with the source and target maps.

3.2 Properties of multiplicative subbundles of TG

Note that the fact that $S \subseteq TG$ is a subgroupoid implies in particular that $\mathsf{s}(v_g) \in S(\mathsf{s}(g))$ and $\mathsf{t}(v_g) \in S(\mathsf{t}(g))$ for all $g \in G$ and $v_g \in S(g)$. This yields splittings of the subbundle over the set of units.

Lemma 3.2.1 *Let $G\rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a multiplicative* subbundle. *Then the intersection $S \cap TP$ has constant rank on P. Since it is the set of units of S seen as a subgroupoid of TG, the pair $S\rightrightarrows(S \cap TP)$ is a Lie groupoid.*
The bundle $S|_P$ splits as $S|_P = (S \cap TP) \oplus (S \cap AG)$. Furthermore, if we denote by S^{t} the intersection $S \cap T^{\mathsf{t}}G$ of vector bundles over G, we have $S^{\mathsf{t}}(g) = 0_g \star S^{\mathsf{t}}(\mathsf{s}(g)) = T_{\mathsf{s}(g)}L_g\left(S^{\mathsf{t}}(\mathsf{s}(g))\right)$ for all $g \in G$. In the same manner, $S^{\mathsf{s}}(g) = S^{\mathsf{s}}(\mathsf{t}(g)) \star 0_g$ for all $g \in G$. As a consequence, the intersections $S \cap T^{\mathsf{t}}G$ and $S \cap T^{\mathsf{s}}G$ have constant rank on G.

Note that these statements for multiplicatives tangent subbundles will be generalized in Theorem 5.1.5 and Lemma 5.1.7 to similar properties for arbitrary multiplicative Dirac structures. Also Corollary 3.2.2 is a result that holds in the more general situation of Dirac groupoids (see Proposition 5.2.1).

Proof: We start by showing that the intersection $S|_P \cap TP$ is smooth. If $p \in P$ and $v_p \in S(p) \cap T_pP$, then we find a smooth section X of S defined at p such that $X(p) = v_p$. The restriction of X to $\mathrm{Dom}(X) \cap P$ is then a smooth section of $S|_P$, and, since s is a smooth surjective submersion, and S is a subgroupoid of TG, the image $T\mathsf{s}(X|_S)$ is a smooth section of $S|_P \cap TP$. Furthermore, we have $T\mathsf{s}(X|_S)(p) = T\mathsf{s}(X(p)) = T\mathsf{s}(v_p) = v_p$ since $v_p \in T_pP$.
Since the intersection $S \cap TP$ is a smooth intersection of vector bundles over P, we know (for instance by Proposition 4.4 in Jotz et al. (2011b)) that it is a vector bundle on P. In particular, it is the set of units of S seen as a subgroupoid of TG.
Since S is a vector bundle on G, it has constant rank on G and in particular on P. For each $p \in P$, we can write $S(p) = (S(p) \cap T_pP) \oplus (S(p) \cap T_p^{\mathsf{t}}G) = (S(p) \cap T_pP) \oplus (S(p) \cap T_p^{\mathsf{s}}G)$. Indeed, if $v_p \in S(p)$, then we can write $v_p = T_p\mathsf{t}v_p + (v_p - T_p\mathsf{t}v_p) = T_p\mathsf{s}v_p + (v_p - T_p\mathsf{s}v_p)$. From this follows the fact that $S|_P \cap T_P^{\mathsf{t}}G$ and $S|_P \cap T_P^{\mathsf{s}}G$ have constant rank on P. If $v_g \in S^{\mathsf{t}}(g)$, then we have $T\mathsf{s}(0_{g^{-1}}) = 0_{\mathsf{t}(g)} = T\mathsf{t}(v_g)$, and since $S(g^{-1})$ is a vector subspace of $T_{g^{-1}}G$, we have $0_{g^{-1}} \in S(g^{-1})$. Since S is multiplicative, we find hence $0_{g^{-1}} \star v_g \in S(\mathsf{s}(g))$. But $0_{g^{-1}} \star v_g = \left.\frac{d}{dt}\right|_{t=0} g^{-1} \star c(t)$, with $c : (-\varepsilon, \varepsilon) \to \mathsf{t}^{-1}(\mathsf{t}(g))$ satisfying $c(0) = g$ and $\dot{c}(0) = v_g$. Thus, we find $0_{g^{-1}} \star v_g = T_gL_{g^{-1}}v_g$. We show the other inclusion and the equality $S^{\mathsf{s}}(g) = T_{\mathsf{t}(g)}R_g\left(S^{\mathsf{s}}(\mathsf{t}(g))\right)$ in the same manner.
Since $S|_P \cap AG$ and $S|_P \cap T_P^{\mathsf{s}}G$ have constant rank on P, we get from this that $S \cap T^{\mathsf{t}}G$ and $S \cap T^{\mathsf{s}}G$ have constant rank on G. □

Corollary 3.2.2 *Let $G\rightrightarrows P$ be a Lie groupoid and S a multiplicative* subbundle *of TG. The induced maps $T_g\mathsf{s} : S(g) \to S(\mathsf{s}(g)) \cap T_{\mathsf{s}(g)}P$ and $T_g\mathsf{t} : S(g) \to S(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P$ are surjective for each $g \in G$.*

Proof: The map $T\mathsf{s} : S/(S \cap T^{\mathsf{s}}G) \to S \cap TP$ is a well-defined injective vector bundle homomorphism over $\mathsf{s} : G \to P$. Since

$$\begin{aligned}\mathrm{rank}(S/(S \cap T^{\mathsf{s}}G)) &= \dim((S/(S \cap T^{\mathsf{s}}G))(g)) = \dim(S(g)) - \dim(S(g) \cap T_g^{\mathsf{s}}G)\\ &= \dim(S(\mathsf{t}(g))) - \dim(S(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}^{\mathsf{s}}G)\\ &= \dim(S(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P) = \mathrm{rank}(S \cap TP)\end{aligned}$$

for any $g \in G$, both vector bundles have the same rank, and the map is an isomorphism in every fiber. Thus, the claim follows. $\square$

The following corollary (see also a result in Mackenzie (2000) about *star-sections*) will be used often in the following.

Corollary 3.2.3 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a multiplicative subbundle. Let $\bar{X}$ be a section of $\Gamma(S \cap TP)$ defined on $\operatorname{Dom}(\bar{X}) =: \bar{U} \subseteq P$. Then there exist sections $X, Y \in \Gamma(S)$ defined on $U := \mathsf{s}^{-1}(\bar{U})$, respectively $V = \mathsf{t}^{-1}(\bar{U})$ such that $X \sim_{\mathsf{s}} \bar{X}$ and $Y \sim_{\mathsf{t}} \bar{X}$.*

PROOF: Since the induced map $T\mathsf{s} : S/(S \cap T^{\mathsf{s}}G) \to S \cap TP$ is a smooth isomorphism in every fiber, there exists a unique smooth section σ of $S/(S \cap T^{\mathsf{s}}G)$ defined on $\mathsf{s}^{-1}(\bar{U})$ such that $T\mathsf{s}(\sigma(g)) = \bar{X}(\mathsf{s}(g))$ for all $g \in U$. Choose a representative $X \in \Gamma(S)$ for σ, then we have $T_g\mathsf{s}X(g) = \bar{X}(\mathsf{s}(g))$ for all $g \in U$. $\square$

We say that a vector field $X \in \mathfrak{X}(G)$ is t- (respectively s-) descending if there exists $\bar{X} \in \mathfrak{X}(P)$ such that $X \sim_{\mathsf{t}} \bar{X}$ (respectively $X \sim_{\mathsf{s}} \bar{X}$), that is, for all $g \in \operatorname{Dom}(X)$, we have $T_g\mathsf{t}X(g) = \bar{X}(\mathsf{t}(g))$.

Corollary 3.2.4 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a multiplicative subbundle. Then*

1. *S is spanned by local t-descending sections and*
2. *S is spanned by local s-descending sections.*

PROOF: Choose $g \in G$ and smooth sections $\bar{X}_1, \ldots, \bar{X}_k$ of $S \cap TP$ spanning $S \cap TP$ on a neighborhood U_1 of $\mathsf{t}(g)$. Choose also $Y_1, \ldots, Y_m$ spanning $(S \cap T^{\mathsf{t}}G)|_P$ in a neighborhood U_2 of $\mathsf{s}(g)$. The vector fields $Y_1^l, \ldots, Y_m^l$ span $S \cap T^{\mathsf{t}}G$ on the neighborhood $\mathsf{s}^{-1}(U_2)$ of g and we find smooth t-descending sections $X_1, \ldots, X_k$ of S such that $X_i \sim_{\mathsf{t}} \bar{X}_i$ on $\mathsf{t}^{-1}(U_1)$. The sections $Y_1^l, \ldots, Y_m^l, X_1, \ldots, X_k$ are t-descending and span S on the neighborhood $U := \mathsf{s}^{-1}(U_2) \cap \mathsf{t}^{-1}(U_1)$ of g. $\square$

Let M be a smooth manifold and $F \subseteq TM$ a subbundle spanned by a family $\mathcal{F}$ of vector fields. If F is involutive, it is integrable in the sense of Frobenius and each of its leaves is an *accessible set* of $\mathcal{F}$, i.e., the leaf L_m of F through $m \in M$ is the set

$$L_m = \left\{ \phi_{t_1}^{X_1} \circ \ldots \circ \phi_{t_k}^{X_k}(m) \,\middle|\, \begin{array}{c} k \in \mathbb{N}, X_1, \ldots, X_k \in \mathcal{F}, t_1, \ldots, t_k \in \mathbb{R} \\ \text{and } \phi^{X_i} \text{ is a local flow of } X_i \end{array} \right\}$$

(see Ortega and Ratiu (2004), Stefan (1974), Sussmann (1973), Stefan (1980)). If the multiplicative subbundle $S \subseteq TG$ is involutive, we get the following corollary which will be very useful in the next section.

Corollary 3.2.5 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive multiplicative subbundle. Then S is completely integrable in the sense of Frobenius and its leaves are the accessible sets of each of the two following families of vector fields.*

$$\mathcal{F}_S^{\mathsf{t}} = \{X \in \Gamma(S) \mid \exists \bar{X} \in \mathfrak{X}(P) \text{ such that } X \sim_{\mathsf{t}} \bar{X}\} \tag{3.1}$$
$$\mathcal{F}_S^{\mathsf{s}} = \{X \in \Gamma(S) \mid \exists \bar{X} \in \mathfrak{X}(P) \text{ such that } X \sim_{\mathsf{s}} \bar{X}\}. \tag{3.2}$$

By Corollary 3.2.3, there exists for each section $\bar{X}$ of $TP \cap S$ defined on $U \subseteq P$ a section of S that is defined on $\mathsf{t}^{-1}(U)$ and t-related to $\bar{X}$. We can find a family of spanning sections of $S \cap TP$ that are all complete, but the corresponding sections of S are then not necessarily complete. In the following, we will have to assume that $S \cap TP$ is spanned by the following families of vector fields:

$$\bar{\mathcal{F}}_S^{\mathsf{t}} := \left\{ \bar{X} \in \Gamma(S \cap TP) \,\middle|\, \begin{array}{c} \exists X \in \Gamma(S) \text{ such that } X \sim_{\mathsf{t}} \bar{X} \\ \text{and } X, \bar{X} \text{ are complete ,} \\ \mathrm{Dom}(X) = \mathsf{t}^{-1}(\mathrm{Dom}(\bar{X})) \end{array} \right\} \tag{3.3}$$
$$\bar{\mathcal{F}}_S^{\mathsf{s}} := \left\{ \bar{X} \in \Gamma(S \cap TP) \,\middle|\, \begin{array}{c} \exists X \in \Gamma(S) \text{ such that } X \sim_{\mathsf{s}} \bar{X} \\ \text{and } X, \bar{X} \text{ are complete ,} \\ \mathrm{Dom}(X) = \mathsf{s}^{-1}(\mathrm{Dom}(\bar{X})) \end{array} \right\}. \tag{3.4}$$

Note that there exists a complete family $\bar{\mathcal{F}}_S^{\mathsf{t}}$ if and only if there exists a complete family $\bar{\mathcal{F}}_S^{\mathsf{s}}$, since the vector fields in $\mathcal{F}_S^{\mathsf{t}}$ are the inverses of the vector fields in $\mathcal{F}_S^{\mathsf{s}}$ and vice versa. We say that S *is complete* if it has this property. Also, note that the multiplicative distributions in Examples 3.1.1 and 3.1.2 are complete.

3.3 The leaf space of a regular multiplicative subbundle of TG.

Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive multiplicative subbundle. Then S is completely integrable in the sense of Frobenius. Let $\mathrm{pr} : G \to G/S$ be the projection on the space of leaves of S.
It is easy to check that the intersection $S \cap TP$ is an involutive subbundle of TP and hence itself also completely integrable. Let P/S be *the space of leaves of* $S \cap TP$ in P and $\mathrm{pr}_\circ$ the projection $\mathrm{pr}_\circ : P \to P/S$.
For $g, h \in G$, we will write $g \sim_S h$ if g and h lie in the same leaf of S and $[g] := \{h \in G \mid h \sim_S g\}$ for the leaf of S through $g \in G$. By the following proposition, we can use the same notation for the equivalence relation defined by the foliation by $S \cap TP$ on P. We will write $[p]_\circ$ for the leaf of $S \cap TP$ through $p \in P$.
Note that g and $h \in G$ lie in the same leaf of S if they can be joined by finitely many flow curves of vector fields lying in $\mathcal{F}_S^{\mathsf{t}}$ or $\mathcal{F}_S^{\mathsf{s}}$ (see Corollary 3.2.5). For simplicity, if $g \sim_S h$, we will often assume without loss of generality that g and h can be joined by *one* such integral curve.

Proposition 3.3.1 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive, multiplicative subbundle. If p and q lie in the same leaf of $S \cap TP$, then p and q seen as elements of G lie in the same leaf of S. Hence, there is a well defined map $[\epsilon] : P/S \to G/S$ such that* $\mathrm{pr} \circ \epsilon = [\epsilon] \circ \mathrm{pr}_\circ$.

PROOF: Choose p and $q \in P$ lying in the same leaf of $S \cap TP$. Then p and q can be joined by a concatenation of segments of integral curves of vector fields in $S \cap TP$. Without loss of generality, assume that $p = \phi_t^{\bar{X}}(q)$ for some $t \in \mathbb{R}$, where $\phi^{\bar{X}}$ is the flow of a section $\bar{X} \in \Gamma(S \cap TP)$. Then there exists a smooth section $X \in \Gamma(S)$ such that $X|_P = \bar{X}$, i.e., $\bar{X} \sim_\epsilon X$, and $p = \epsilon(p) = \epsilon(\phi_t^{\bar{X}}(q)) = \phi_t^X(\epsilon(q)) = \phi_t^X(q)$ shows that $p \sim_S q$. $\square$

We will see that under certain conditions, the space G/S inherits a groupoid structure over P/S. If the foliation is regular, we will get a Lie groupoid $G/S \rightrightarrows P/S$. We start by showing that the structure maps s, t, i induce maps on G/S.

Proposition 3.3.2 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive, multiplicative subbundle. Choose $g, h \in G$ such that $g \sim_S h$. Then $\mathsf{s}(g) \sim_S \mathsf{s}(h)$ and $\mathsf{t}(g) \sim_S \mathsf{t}(h)$ and we get induced maps $[\mathsf{s}], [\mathsf{t}] : G/S \to P/S$ defined by $[\mathsf{s}]([g]) = [\mathsf{s}(g)]_\circ$, $[\mathsf{t}]([g]) = [\mathsf{t}(g)]_\circ$ for all $g \in G$.*

PROOF: If g and $h \in G$ are in the same leaf of S, we find by Corollary 3.2.5 smooth s-descending vector fields $X_1, \ldots, X_k \in \Gamma(S)$ and $t_1, \ldots, t_k \in \mathbb{R}$ such that $h = \phi^k_{t_k} \circ \ldots \circ \phi^1_{t_1}(g)$, where ϕ^i is the flow of the vector field X_i for each $i = 1, \ldots, k$. There exist then smooth vector fields $\bar{X}_1, \ldots, \bar{X}_k \in \Gamma(S \cap TP)$ such that $X_i \sim_{\mathsf{s}} \bar{X}_i$, and hence, if $\bar{\phi}^i$ is the flow of the vector field $\bar{X}_i$, $\mathsf{s} \circ \phi^i = \bar{\phi}^i \circ \mathsf{s}$ for $i = 1, \ldots, k$. We compute then

$$\mathsf{s}(h) = \mathsf{s}\left(\phi^k_{t_k} \circ \ldots \circ \phi^1_{t_1}(g)\right) = \bar{\phi}^k_{t_k} \circ \ldots \circ \bar{\phi}^1_{t_1}(\mathsf{s}(g)),$$

which shows that $\mathsf{s}(h)$ and $\mathsf{s}(g)$ lie in the same leaf of $S \cap TP$. The map $[\mathsf{s}] : G/S \to P/S$, $[g] \mapsto [\mathsf{s}(g)]_\circ$ is consequently well-defined. We show in the same manner, but using this time the family (3.1), that $[\mathsf{t}] : G/S \to P/S$, $[g] \mapsto [\mathsf{t}(g)]_\circ$ is well-defined (note that we don't use here the completeness of $\mathcal{F}^{\mathsf{t}}_S$). $\square$

Proposition 3.3.3 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ an involutive, multiplicative subbundle. Choose $g, h \in G$ such that $g \sim_S h$. Then $g^{-1} \sim_S h^{-1}$. Hence, $[\mathsf{i}] : G/S \to G/S$, $[\mathsf{i}]([g]) = [\mathsf{i}(g)] = [g^{-1}]$ is well-defined.*

PROOF: If $g \sim_S h$, then there exists without loss of generality one smooth section $X \in \Gamma(S)$ and $\sigma \in \mathbb{R}$ such that $g = \phi^X_\sigma(h)$. Since $X(\phi^X_\tau(h)) \in S(\phi^X_\tau(h))$ for all $\tau \in [0, \sigma]$, the curve $c : [0, \sigma] \to G$, $c(\tau) = (\phi^X_\tau(h))^{-1}$ satisfies $\dot{c}(\tau) = T_{\phi^X_\tau(h)}\mathsf{i}\left(X(\phi^X_\tau(h))\right) \in S(\mathsf{i}(\phi^X_\tau(h)))$ for all $\tau \in [0, \sigma]$. The image of c lies hence in the leaf of S through $c(0) = h^{-1}$. Since $c(\sigma) = g^{-1}$, we have shown that $h^{-1} \sim_S g^{-1}$. $\square$

Hence, we have shown that the structure maps ϵ, s, t and i project to well-defined maps on P/S and G/S. For the multiplication, which cannot be defined in this straightforward manner, we will need the following technical lemmas. Note that we have not used the completeness condition until here.

Lemma 3.3.4 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ be a* complete *multiplicative involutive subbundle of TG. If $g \in G$ and $\mathsf{t}(g) \sim_S p \in P$, then there exists $h \in G$ such that $g \sim_S h$ and $\mathsf{t}(h) = p$. In the same manner, if $\mathsf{s}(g) \sim_S p \in P$, then there exists $h \in G$ such that $g \sim_S h$ and $\mathsf{s}(h) = p$.*

PROOF: Choose a vector field $\bar{X} \in \bar{\mathfrak{F}}_S^{\mathsf{t}}$ and $\sigma \in \mathbb{R}$ such that $p = \phi_\sigma^{\bar{X}}(\mathsf{t}(g))$. We find then a t-descending vector field $X \sim_{\mathsf{t}} \bar{X}$ defined at g. Since $\bar{X}$ and X can be taken complete by hypothesis (see (3.3)), the integral curve of X starting at g is defined at σ and we have $\mathsf{t}(\phi_\tau^X(g)) = \phi_\tau^{\bar{X}}(\mathsf{t}(g))$ for all $\tau \in [0, \sigma]$. Set $h := \phi_\sigma^X(g)$, then $h \sim_S g$ and $\mathsf{t}(h) = \phi_\sigma^{\bar{X}}(\mathsf{t}(g)) = p$. $\square$

Lemma 3.3.5 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a complete, involutive, multiplicative subbundle. Choose $g \in G$ and set $\mathsf{s}(g) =: p \in P$. By Proposition 3.3.2, the source map $\mathsf{s} : G \to P$ restricts to a map $\mathsf{s}_{[g]} : [g] \to [p]_\circ$. This map is a smooth surjective submersion. In the same manner, $\mathsf{t}_{[g]} : [g] \to [\mathsf{t}(g)]_\circ$ is a smooth surjective submersion.*

PROOF: The map $\mathsf{s}_{[g]}$ is surjective by Lemma 3.3.4. The leaf $[p]_\circ$ of $S \cap TP$ through p is an *initial submanifold* of P, that is, the inclusion $\iota_{[p]_\circ} : [p]_\circ \hookrightarrow P$ is an injective immersion such that for any smooth manifold Q, an arbitrary map $f : Q \to [p]_\circ$ is smooth if and only if $\iota_{[p]_\circ} \circ f : Q \to P$ is smooth (see for instance Ortega and Ratiu (2004)).
We have $\iota_{[p]_\circ} \circ \mathsf{s}_{[g]} = \mathsf{s} \circ \iota_{[g]}$, where $\iota_{[g]} : [g] \hookrightarrow G$ is the injective immersion. Since the right hand side of this equality is smooth, we find hence that $\mathsf{s}_{[g]}$ is a smooth map. For any $h \in [g]$, we have

$$T_h \mathsf{s}_{[g]} : T_h[g] \to T_{\mathsf{s}(h)}[p]_\circ.$$

Since $T_h[g] = S(h)$ and $T_{\mathsf{s}(h)}[p]_\circ = S(\mathsf{s}(h)) \cap T_{\mathsf{s}(h)}P$, we find using Corollary 3.2.2 that $T_h \mathsf{s}_{[g]}$ is surjective. $\square$

Theorem 3.3.6 *Let $G \rightrightarrows P$ be a Lie groupoid and $S \subseteq TG$ a complete, involutive, multiplicative subbundle. The subset $N := \cup_{p \in P}[p] \subseteq G$ is a wide subgroupoid of $G \rightrightarrows P$.*

PROOF: We have $P \subseteq N$ by definition.
First, we have to show that if $g, h \in G$ are in N and composable, then $g \star h \in N$. Assume that $g \in [p]$ for some $p \in P$. Since $g \sim_S p$, we have $\mathsf{s}(g) \sim_S \mathsf{s}(p) = p$ by Proposition 3.3.2 and hence $g \sim_S p \sim_S \mathsf{s}(g)$. If $h \in [q]$, then we find in the same manner that $h \sim_S q \sim_S \mathsf{t}(h)$, and since $\mathsf{s}(g) = \mathsf{t}(h)$, we find that we can take $p = q = \mathsf{s}(g) = \mathsf{t}(h)$ and that we have $g, h \in [p]$.
We find hence without loss of generality one t-descending vector field $X \in \Gamma(S)$ and $\sigma \in \mathbb{R}$ such that $h = \phi_\sigma^X(\mathsf{t}(h))$, where ϕ^X is the flow of X. Since X is t-descending, there exists $\bar{X} \in \Gamma(S \cap TP)$ such that $X \sim_{\mathsf{t}} \bar{X}$. We have then $\mathsf{t}(h) = \mathsf{t} \circ \phi_\sigma^X(\mathsf{t}(h)) = \phi_\sigma^{\bar{X}}(\mathsf{t}(h))$. Using Lemma 3.3.5, we find a curve $c : [0, \sigma] \to [g]$ such that $\mathsf{s}(c(\tau)) = \phi_\tau^{\bar{X}}(\mathsf{t}(h))$ for all $\tau \in [0, \sigma]$ and $c(\sigma) = g$. Set $g' := c(0)$. Then we have $g' \sim_S g \sim_S p$ and $\dot{c}(\tau) \in S(c(\tau))$ for all $\tau \in [0, \sigma]$. We get also for all $\tau \in [0, \sigma]$:

$$(\mathsf{s} \circ c)(\tau) = \phi_\tau^{\bar{X}}(\mathsf{t}(h)) = \left(\mathsf{t} \circ \phi_\tau^X\right)(\mathsf{t}(h)).$$

Hence, $c(\tau) \star \phi_\tau^X(\mathsf{t}(h))$ is defined for all τ. We have also

$$T_{c(\tau)}\mathsf{s}\left(\dot{c}(\tau)\right) = \bar{X}(\mathsf{t}(\phi_\tau^X(\mathsf{t}(h)))) = T_{\phi_\tau^X(\mathsf{t}(h))}\mathsf{t}\left(X(\phi_\tau^X(\mathsf{t}(h)))\right)$$

and, consequently, the product

$$\dot{c}(\tau) \star X(\phi_\tau^X(\mathsf{t}(h)))$$

is defined for all $\tau \in [0, \sigma]$, and takes values in $S(c(\tau) \star \phi_\tau^X(\mathsf{t}(h)))$ since S is multiplicative. Consider the curve $\gamma : [0, \sigma] \to G$, $\gamma(\tau) = c(\tau) \star \phi_\tau^X(\mathsf{t}(h))$. Then

$$\dot{\gamma}(\tau) = \frac{d}{d\tau}\left(c(\tau) \star \phi_\tau^X(\mathsf{t}(h))\right) = \dot{c}(\tau) \star X(\phi_\tau^X(\mathsf{t}(h))) \in S(\gamma(\tau))$$

for all $\tau \in [0, \sigma]$, and $\gamma(0) = g'$, $\gamma(\sigma) = g \star h$. Since $g' \in [p]$, this shows $g \star h \in [p]$.
Finally, we have to check that if $g \in N$, $g \in [p]$ for some $p \in P$, then $g^{-1} \in N$. This follows directly from Proposition 3.3.3: since $g \sim_S p$, we have $g^{-1} \sim_S p^{-1} = p$. Hence, $g^{-1} \in [p] \subseteq N$. □

Consider the wide subgroupoid $\mathcal{R}_S$ of the pair groupoid $P \times P \rightrightarrows P$ defined by

$$\mathcal{R}_S = \{(p, q) \in P \times P \mid p \sim_S q\} = \{(p, q) \in P \times P \mid \mathrm{pr}_\circ(p) = \mathrm{pr}_\circ(q)\}.$$

Consider also the quotient G/N of G by the subgroupoid N. That is, the elements of G/N are the sets $gN := \{g \star n \mid n \in N,\ \mathsf{s}(g) = \mathsf{t}(n)\}$, $g \in G$.

Lemma 3.3.7 *Let $G \rightrightarrows P$ be a Lie groupoid and S a complete involutive multiplicative subbundle of TG. Let N be the wide subgroupoid of $G \rightrightarrows P$ associated to S. Then, for any $p \in P$, we have $pN = [p] \cap \mathsf{t}^{-1}(p)$ and for any $g \in G$, $gN \subseteq [g] \cap \mathsf{t}^{-1}(\mathsf{t}(g))$.*

PROOF: Choose $p \in P$ and $n \in pN$. Then $n \in N$ and $\mathsf{t}(n) = p$. Since $n \in N$, $n \sim_S q$ for some $q \in P$ and thus $p = \mathsf{t}(n) \sim_S \mathsf{t}(q) = q$. This shows that $n \in [p] \cap \mathsf{t}^{-1}(p)$. Conversely, if $g \in [p] \cap \mathsf{t}^{-1}(p)$, then $g \in N$ by definition of N and $\mathsf{t}(g) = p$. Thus $g = p \star g \in pN$. This shows $pN = [p] \cap \mathsf{t}^{-1}(p)$ for all $p \in P$.
Choose $g \in G$ and set $p = \mathsf{s}(g)$. Choose $h \in [p] \cap \mathsf{t}^{-1}(p)$. Then $h \sim_S p$ and there exists without loss of generality one t-descending vector field $X \sim_{\mathsf{t}} \bar{X}$ and $\sigma \in \mathbb{R}$ such that $h = \phi_\sigma^X(p)$. Then $p = \mathsf{t}(h) = (\mathsf{t} \circ \phi_\sigma^X)(p) = \phi_\sigma^{\bar{X}}(p)$. As in the proof of the preceding theorem, we can choose a smooth curve $c : [0, \sigma] \to [g]$ such that $\mathsf{s}(c(\tau)) = \phi_\tau^{\bar{X}}(p)$ for all $\tau \in [0, \sigma]$ and $c(\sigma) = g$. Set $g' = c(0)$. Then $\mathsf{s}(g') = p$ and since $(\mathsf{s} \circ c)(\tau) = (\mathsf{t} \circ \phi_\tau^X)(p)$ for all $\tau \in [0, \sigma]$, we find that $\gamma : [0, \sigma] \to G$, $\gamma(\tau) = c(\tau) \star \phi_\tau^X(p)$ is well-defined and tangent to the leaf of S through $g' \star p = g'$. But since $\gamma(\sigma) = g \star h$, this yields $g \star h \sim_S g' \sim_S g$ and hence $g \star h \in [g]$. We have thus shown that $gN = g \star ([p] \cap \mathsf{t}^{-1}(p)) \subseteq [g] \cap \mathsf{t}^{-1}(\mathsf{t}(g))$. □

In the situation above, we have $gN = g \star \mathsf{s}(g)N = g \star ([\mathsf{s}(g)] \cap \mathsf{t}^{-1}(\mathsf{s}(g)))$. Hence, if

$$g \star ([\mathsf{s}(g)] \cap \mathsf{t}^{-1}(\mathsf{s}(g))) = [g] \cap \mathsf{t}^{-1}(\mathsf{t}(g)) \qquad \text{for all } g \in G, \tag{3.5}$$

then $gN = [g] \cap \mathsf{t}^{-1}(\mathsf{t}(g))$ for all $g \in G$.

Remark 3.3.8 The condition (3.5) is satisfied for instance if $[g]\cap \mathsf{t}^{-1}(\mathsf{t}(g))$ has one connected component and is hence the leaf of the involutive vector bundle $T^{\mathsf{t}}G\cap S$ through g. This is the case in Examples 3.1.1 and 3.1.2. △

Proposition 3.3.9 *Let $G\rightrightarrows P$ be a Lie groupoid and S a complete involutive multiplicative subbundle of TG such that the foliation of G by the leaves of S satisfies condition (3.5). Let N and $\mathcal{R}_S$ be the wide subgroupoids of $G\rightrightarrows P$ and $P\times P\rightrightarrows P$ associated to S. Define $\mathsf{J}: G/N\to P$ by $\mathsf{J}(gN)=\mathsf{t}(g)$ for all $g\in G$. Set*

$$\mathcal{R}_S\times_P(G/N)=\{((p,q),gN)\mid \mathsf{s}(p,q)=q=\mathsf{t}(g)=J(gN)\}$$

and

$$\theta:\mathcal{R}_S\times_P(G/N)\to G/N,\qquad \theta(p,q)(gN)=hN\quad \textit{if}\quad h\sim_S g\quad \textit{is such that}\quad \mathsf{t}(h)=p.$$

The map θ is an action of the groupoid $\mathcal{R}_S\rightrightarrows P$ on $\mathsf{J}:G/N\to P$. The orbit

$$\bigcup_{\{p\in P\mid (p,\mathsf{t}(g))\in\mathcal{R}_S\}}\theta(p,\mathsf{t}(g))(gN)$$

of gN by this action is, as a set, the leaf $[g]$ of S through g.

In other words, $G/S=(G/N)/\mathcal{R}_S$ and if $\mathrm{pr}_N: G\to G/N$, $\mathrm{pr}_\theta: G/N\to(G/N)/\mathcal{R}_S$ are the projections, then the following diagram is commutative.

$$\begin{tikzcd} G \arrow[r, "\mathrm{pr}_N"] \arrow[dr, "\mathrm{pr}"'] & G/N \arrow[d, "\mathrm{pr}_\theta"] \\ & G/S \end{tikzcd}$$

Proof: We show that θ is well-defined. The existence of h in the definition of $\theta((p,q),gN)$ is ensured by Lemma 3.3.4. If $gN\in G/N$, $\mathsf{J}(gN)=q$ and h,h' are such that $\mathsf{t}(h)=\mathsf{t}(h')=p\sim_S q$, and $h\sim_S g$, $h'\sim_S g$, then we have $h\sim_S h'$ and hence $[h]\cap\mathsf{t}^{-1}(\mathsf{t}(h))=[h']\cap\mathsf{t}^{-1}(\mathsf{t}(h'))$. The condition (3.5) yields then $h'N=hN$.
Next, we check that θ is an action of $\mathcal{R}_S\rightrightarrows P$ on $\mathsf{J}: G\to G/N$. Choose $gN\in G/N$ and set $p=\mathsf{t}(g)$. We have then $(p,p)\in\mathcal{R}_S$ and $\theta((p,p),gN)=gN$ since $g\sim_S g$ and $\mathsf{t}(g)=p$. By definition of θ, we have $\mathsf{J}(\theta((q,p),gN))=q=\mathsf{t}(q,p)$. Choose $(q,r),(r,p)\in\mathcal{R}_S$. Since $q\sim_S r\sim_S p$, we find h and l in G such that $h\sim_S l\sim_S g$ and $\mathsf{t}(h)=q$, $\mathsf{t}(l)=r$. We get then $\theta((q,r)\star(r,p),gN)=\theta((q,p),gN)=hN=\theta((q,r),lN)=\theta((q,r),\theta((r,p),gN))$.

It remains to show that for any $gN\in G/N$, the orbit

$$\bigcup_{\{p\in P\mid (p,\mathsf{t}(g))\in\mathcal{R}_S\}}\theta((p,\mathsf{t}(g)),gN)$$

is the leaf $[g]$ of S through g. We have

$$\bigcup_{\{p\in P\mid (p,\mathsf{t}(g))\in\mathcal{R}_S\}} \theta((p,\mathsf{t}(g)),gN) = \left\{ h\star n \;\middle|\; \begin{array}{c} n\in N, h\in G, h\sim_S g \\ \text{such that } (\mathsf{t}(h),\mathsf{t}(g))\in\mathcal{R}_S, \mathsf{s}(h)=\mathsf{t}(n) \end{array} \right\}$$
$$= \{h\star n \mid n\in N, h\in G, h\sim_S g \text{ such that } \mathsf{s}(h)=\mathsf{t}(n)\}$$

since $g\sim_S h$ implies $\mathsf{t}(g)\sim_S \mathsf{t}(h)$. For simplicity, we call this union M_g. We have $[g]\subseteq M_g$ since for all $h\in[g]$, we know that $\mathsf{t}(h)\sim_S \mathsf{t}(g)$ and hence $h\star\mathsf{s}(h)\in M_g$ since $\mathsf{s}(h)\in P\subseteq N$. Conversely, if $h\star n\in M_g$, then we have $h\sim_S g$ and since $hN=[h]\cap\mathsf{t}^{-1}(\mathsf{t}(h))$ by the condition (3.5), we find that $h\star n\in[h]=[g]$. □

Note that the map θ is well-defined if and only if the foliation defined by S on G satisfies (3.5).

Theorem 3.3.10 *Let $G\rightrightarrows P$ be a Lie groupoid and S a complete, multiplicative, involutive subbundle of TG such that the foliation of G by the leaves of S satisfies* (3.5). *Then the triple $\mathcal{N}=(N,\mathcal{R}_S,\theta)$ is a normal subgroupoid system on $G\rightrightarrows P$.*

Proof: Choose $(p,q)\in\mathcal{R}_S$ and $gN\in G/N$ such that $\mathsf{J}(gN)=q$. Then $\theta((p,q),gN)=hN$ for any $h\sim_S g$ such that $\mathsf{t}(h)=p$. We have then $\mathsf{s}(h)\sim_S\mathsf{s}(g)$ by Proposition 3.3.2, and hence $(\mathsf{s}(h),\mathsf{s}(g))\in\mathcal{R}_S$. We know also that $\theta((p,q),qN)=pN$ since $p\in G$ is such that $p\sim_S q$ and $\mathsf{t}(p)=p$.
Consider $(p,q)\in\mathcal{R}_S$ and $gN\in G/N$ such that $\mathsf{J}(gN)=q$. Choose $hN\in G/N$ such that $\mathsf{t}(h)=\mathsf{s}(g)$. Set $\theta((p,q),gN)=g'N$, then $g'\sim_S g$, hence $\mathsf{s}(g')\sim_S\mathsf{s}(g)=\mathsf{t}(h)$ and we can set $\theta((\mathsf{s}(g'),\mathsf{s}(g)),hN)=: h'N$. We have to show that $\theta((p,q),ghN)=g'h'N$. That is, we have to show that $g\star h\sim_S g'\star h'$. Since $g\sim_S g'$, there exists without loss of generality one s-descending vector field $X\in\Gamma(S)$ and $\sigma\in\mathbb{R}$ such that $g'=\phi^X_\sigma(g)$, where ϕ^X is the flow of X. Let $\bar X\in\Gamma(S\cap TP)$ be such that $X\sim_{\mathsf{s}}\bar X$, and choose a t-descending vector field $Y\in\Gamma(S)$ defined at h such that $Y\sim_{\mathsf{t}}\bar X$. Consider $h''=\phi^Y_\sigma(h)$ and the curve $c:[0,\sigma]\to G$, $c(t)=\phi^X_t(g)\star\phi^Y_t(h)$. The product $\phi^X_t(g)\star\phi^Y_t(h)$ is defined for all $t\in[0,\sigma]$ because

$$(\mathsf{s}\circ\phi^X_t)(g)=\phi^{\bar X}_t(\mathsf{s}(g))=\phi^{\bar X}_t(\mathsf{t}(h))=(\mathsf{t}\circ\phi^Y_t)(h)$$

and we have

$$\dot c(t)=X(\phi^X_t(g))\star Y(\phi^Y_t(h))\in S(c(t)).$$

Thus, the curve c is tangent to the leaf of S through $c(0)=g\star h$ and its endpoint is

$$c(\sigma)=\phi^X_\sigma(g)\star\phi^Y_\sigma(h)=g'\star h''.$$

This shows that $g\star h\sim_S g'\star h''$. Now since $h'\sim_S h\sim_S h''$ and $\mathsf{t}(h')=\mathsf{s}(g')=\mathsf{t}(h'')$ we have $h''\in[h']\cap\mathsf{t}^{-1}(\mathsf{t}(h'))=h'N$ by condition (3.5). This leads to $g'h''\in g'h'N\subseteq[g'h']$ and hence $g\star h\sim_S g'\star h''\sim_S g'\star h'$. □

We get hence the main theorem of this chapter.

Theorem 3.3.11 *Let $G \rightrightarrows P$ be a Lie groupoid and S a complete, multiplicative, involutive subbundle of TG such that the foliation of G by the leaves of S satisfies* (3.5)*. Then there is an induced groupoid structure on the leaf space $G/S \rightrightarrows P/S$ such that* $(\mathrm{pr}, \mathrm{pr}_\circ)$ *is a groupoid morphism.*

Remark 3.3.12 In the situation of the previous theorem, the groupoid structure on the leaf space of S is defined as follows. The object inclusion map is the map $[\epsilon]$ as in Proposition 3.3.1, the source and targets are the maps $[\mathsf{s}]$ and $[\mathsf{t}]$ as in Proposition 3.3.2 and the inversion is $[\mathsf{i}]$ as in Proposition 3.3.3.
If $[g], [h] \in G/S$ are such that $[\mathsf{s}]([g]) = [\mathsf{t}]([h])$, then there exists $h' \in G$ such that $h \sim_S h'$ and $\mathsf{s}(g) = \mathsf{t}(h')$. The product of $[g]$ and $[h]$ is given by $[g] \star [h] = [g \star h']$. △

Example 3.3.13 Consider a Lie groupoid $G \rightrightarrows P$ and let a connected Lie group H act freely and properly on $G \rightrightarrows P$ by Lie groupoid homomorphisms. Let $\Phi : H \times G \to G$ be the action. That is, for all $h \in H$, the map $\Phi_h : G \to G$ is a groupoid morphism over the map $\phi_h := \Phi_h|_P : P \to P$. Let $\mathcal{V} \subseteq TG$ be the vertical space of the action, i.e., $\mathcal{V}(g) = \{\xi_G(g) \mid \xi \in \mathfrak{h}\}$ for all $g \in G$, where $\mathfrak{h}$ is the Lie algebra of H.
We check that $\mathcal{V} \subseteq TG$ is multiplicative. Choose $\xi_G(g) \in \mathcal{V}(g)$. Then we have

$$T_g\mathsf{t}\left(\xi_G(g)\right) = \left.\frac{d}{dt}\right|_{t=0} \mathsf{t}(\Phi_{\exp(t\xi)}(g)) = \left.\frac{d}{dt}\right|_{t=0} \phi_{\exp(t\xi)}(\mathsf{t}(g)) = \xi_P(\mathsf{t}(g)) \in \mathcal{V}(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P$$

and in the same manner

$$T_g\mathsf{s}\left(\xi_G(g)\right) \in \mathcal{V}(\mathsf{s}(g)) \cap T_{\mathsf{s}(g)}P.$$

This shows also that $\mathcal{V} \cap T^{\mathsf{s}}G = \mathcal{V} \cap T^{\mathsf{t}}G = 0_{TM}$ in this example.
If $\xi_G(g) \in \mathcal{V}(g)$ and $\eta_G(g') \in \mathcal{V}(g')$ are such that

$$T_g\mathsf{s}\left(\xi_G(g)\right) = T_{g'}\mathsf{t}\left(\eta_G(g')\right),$$

then we have $\mathsf{s}(g) = \mathsf{t}(g') =: p$ and

$$\xi_P(p) = \eta_P(p),$$

which implies $\xi = \eta$ since the action is free. We get then

$$\begin{aligned} \xi_G(g) \star \eta_G(g') = \xi_G(g) \star \xi_G(g') &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(g) \star \Phi_{\exp(t\xi)}(g') \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp(t\xi)}(g \star g') = \xi_G(g \star g') \in \mathcal{V}(g \star g'). \end{aligned}$$

The inverse of $\xi_G(g)$ is then $\xi_G(g^{-1})$ for all $\xi \in \mathfrak{h}$ and $g \in G$.
It also follows from the considerations above that the vector fields ξ_G are t- and s-descending to ξ_P. Hence, $\mathcal{V}$ is spanned by complete vector fields, that are t-related to

complete vector fields. The leaf of $\mathcal{V}$ through $g \in G$ is equal to $\{\Phi_h(g) \mid h \in H\}$ and we find

$$\begin{aligned}[g] \cap \mathsf{t}^{-1}(\mathsf{t}(g)) &= \{\Phi_h(g) \mid h \in H, \phi_h(\mathsf{t}(g)) = \mathsf{t}(g)\} \\ &= \{g\} = g \star \big([\mathsf{s}(g)] \cap \mathsf{t}^{-1}(\mathsf{s}(g))\big)\end{aligned}$$

since the action of H on P is free.
Hence, we recover from Theorem 3.3.11 the fact that the quotient $G/H \rightrightarrows P/H$ has the structure of a Lie groupoid such that the projections $\mathrm{pr} : G \to G/H$, $\mathrm{pr}_\circ : P \to P/H$ form a Lie groupoid morphism. $\Diamond$

In the situation of Theorem 3.3.11, assume that the induced groupoid $G/S \rightrightarrows P/S$ is a Lie groupoid and pr, $\mathrm{pr}_\circ$ are smooth surjective submersions. Choose $v_{[g]} \in T_{[g]}(G/S)$ and $v_{[h]} \in T_{[h]}(G/S)$ such that $T_{[g]}[\mathsf{s}]v_{[g]} = T_{[h]}[\mathsf{t}]v_{[h]}$. Since $[\mathsf{s}(g)] = [\mathsf{s}][g] = [\mathsf{t}][h] = [\mathsf{t}(h)]$, there exists by Lemma 3.3.4 $h' \sim_S h$ such that $\mathsf{t}(h') = \mathsf{s}(g) =: p$. Assume without loss of generality that $h' = h$. Choose $v_g \in T_gG$ and $v_h \in T_hG$ such that $T_g \,\mathrm{pr}\, v_g = v_{[g]}$ and $T_h \,\mathrm{pr}\, v_h = v_{[h]}$. Since this yields

$$T_p \,\mathrm{pr}_\circ(T_g\mathsf{s}v_g) = T_{[g]}[\mathsf{s}](T_g \,\mathrm{pr}\, v_g) = T_{[g]}[\mathsf{s}]v_{[g]} = T_{[h]}[\mathsf{t}]v_{[h]} = T_p \,\mathrm{pr}_\circ(T_h\mathsf{t}v_h),$$

we find $T_g\mathsf{s}v_g - T_h\mathsf{t}v_h \in S(p) \cap T_pP$ and hence, by Corollary 3.2.2, a vector $w_g \in S(g)$ such that $T_g\mathsf{s}w_g = T_g\mathsf{s}v_g - T_h\mathsf{t}v_h$. We get then

$$v_{[g]} \star v_{[h]} = T_g \,\mathrm{pr}(v_g - w_g) \star T_g \,\mathrm{pr}\, v_h = T_{g\star h} \,\mathrm{pr}((v_g - w_g) \star v_h).$$

If $\alpha_{[g]} \in T^*_{[g]}(G/S)$ and $\alpha_{[h]} \in T^*_{[h]}(G/S)$ such that $\hat{[\mathsf{s}]}(\alpha_{[g]}) = \hat{[\mathsf{t}]}(\alpha_{[h]})$, then, as above, we can assume without loss of generality that $\mathsf{s}(g) = \mathsf{t}(h)$. Choose $u_{\mathsf{s}(g)} \in A_{\mathsf{s}(g)}G$. Then we have $T_{[\mathsf{s}(g)]}[\mathsf{t}](T_{\mathsf{s}(g)} \,\mathrm{pr}\, u_{\mathsf{s}(g)}) = T_{\mathsf{s}(g)} \,\mathrm{pr}(T_{\mathsf{s}(g)}\mathsf{t}u_{\mathsf{s}(g)}) = 0$ and hence $T_{\mathsf{s}(g)} \,\mathrm{pr}\, u_{\mathsf{s}(g)} \in A_{[\mathsf{s}(g)]}(G/S)$. We compute

$$\begin{aligned}\hat{\mathsf{s}}((T_g \,\mathrm{pr})^*\alpha_{[g]})(u_{\mathsf{s}(g)}) &= (T_g \,\mathrm{pr})^*\alpha_{[g]}(0_g \star u_{\mathsf{s}(g)}) = \alpha_{[g]}(T_g \,\mathrm{pr}(0_g \star u_{\mathsf{s}(g)})) \\ &= \alpha_{[g]}(0_{[g]} \star (T_{\mathsf{s}(g)} \,\mathrm{pr}\, u_{\mathsf{s}(g)})) \\ &= \hat{[\mathsf{s}]}(\alpha_{[g]})(T_{\mathsf{s}(g)} \,\mathrm{pr}\, u_{\mathsf{s}(g)}) = \hat{[\mathsf{t}]}(\alpha_{[h]})(T_{\mathsf{s}(g)} \,\mathrm{pr}\, u_{\mathsf{s}(g)}) \\ &= \alpha_{[h]}((T_{\mathsf{s}(g)} \,\mathrm{pr}\, u_{\mathsf{s}(g)} - T_{[\mathsf{s}(g)]}[\mathsf{s}](T_{\mathsf{s}(g)} \,\mathrm{pr}\, u_{\mathsf{s}(g)})) \star 0_{[h]}) \\ &= \alpha_{[h]}\left(T_h \,\mathrm{pr}\left((u_{\mathsf{s}(g)} - T_{\mathsf{s}(g)}\mathsf{s}u_{\mathsf{s}(g)}) \star 0_h\right)\right) = \hat{\mathsf{t}}((T_h \,\mathrm{pr})^*\alpha_{[h]})(u_{\mathsf{s}(g)}).\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}\hat{\mathsf{s}}((T_g \,\mathrm{pr})^*\alpha_{[g]}) &= (T_{\mathsf{s}(g)} \,\mathrm{pr})^*\hat{[\mathsf{s}]}(\alpha_{[g]}), \\ \hat{\mathsf{t}}((T_h \,\mathrm{pr})^*\alpha_{[h]}) &= (T_{\mathsf{t}(h)} \,\mathrm{pr})^*\hat{[\mathsf{t}]}(\alpha_{[h]}), \\ \hat{\mathsf{s}}((T_g \,\mathrm{pr})^*\alpha_{[g]}) &= \hat{\mathsf{t}}((T_h \,\mathrm{pr})^*\alpha_{[h]})\end{aligned}$$

and hence that $(T_g \,\mathrm{pr})^*\alpha_{[g]} \star (T_h \,\mathrm{pr})^*\alpha_{[h]}$ makes sense. Choose $v_g \in T_gG$ and $v_h \in T_hG$ such that $T_g\mathsf{s}v_g = T_h\mathsf{t}v_h$. Then we have

$$\begin{aligned}((T_g \,\mathrm{pr})^*\alpha_{[g]} \star (T_h \,\mathrm{pr})^*\alpha_{[h]})(v_g \star v_h) &= \alpha_{[g]}(T_g \,\mathrm{pr}\, v_g) + \alpha_{[h]}(T_h \,\mathrm{pr}\, v_h)\\ &= (\alpha_{[g]} \star \alpha_{[h]})(T_g \,\mathrm{pr}\, v_g \star T_h \,\mathrm{pr}\, v_h)\\ &= (\alpha_{[g]} \star \alpha_{[h]})(T_{g\star h} \,\mathrm{pr}(v_g \star v_h))\\ &= (T_{g\star h} \,\mathrm{pr})^*(\alpha_{[g]} \star \alpha_{[h]})(v_g \star v_h).\end{aligned}$$

We have thus shown the following lemma, which will be useful for the proof of Theorem 5.1.2.

Lemma 3.3.14 *In the setting of Theorem 3.3.11, if $G/S \rightrightarrows P/S$ is a Lie groupoid and $(\mathrm{pr}, \mathrm{pr}_\circ)$ a pair of smooth surjective submersions, choose $v_{[g]} \in T_{[g]}(G/S)$ and $v_{[h]} \in T_{[h]}(G/S)$ such that $T_{[g]}[\mathsf{s}]v_{[g]} = T_{[h]}[\mathsf{t}]v_{[h]}$. Then we can assume without loss of generality that $\mathsf{s}(g) = \mathsf{t}(h)$. If $v_g \in T_gG$ and $v_h \in T_hG$ are such that $T_g \,\mathrm{pr}\, v_g = v_{[g]}$ and $T_h \,\mathrm{pr}\, v_h = v_{[h]}$, then there exists $w_g \in S(g)$ such that $T_g\mathsf{s}(v_g - w_g) = T_h\mathsf{t}v_h$. We have then $T_{g\star h} \,\mathrm{pr}((v_g - w_g) \star v_h) = v_{[g]} \star v_{[h]}$.*
If $\alpha_{[g]} \in T^*_{[g]}(G/S)$ and $\alpha_{[h]} \in T^*_{[h]}(G/S)$ are such that $\hat{[\mathsf{s}]}(\alpha_{[g]}) = \hat{[\mathsf{t}]}(\alpha_{[h]})$, then

$$\hat{\mathsf{s}}((T_g \,\mathrm{pr})^*\alpha_{[g]}) = (T_{\mathsf{s}(g)} \,\mathrm{pr})^*\hat{\mathsf{s}}(\alpha_{[g]}), \qquad \hat{\mathsf{t}}((T_h \,\mathrm{pr})^*\alpha_{[h]}) = (T_{\mathsf{t}(h)} \,\mathrm{pr})^*\hat{\mathsf{t}}(\alpha_{[h]}),$$

hence $\hat{\mathsf{s}}\left((T_g \,\mathrm{pr})^*\alpha_{[g]}\right) = \hat{\mathsf{t}}\left((T_h \,\mathrm{pr})^*\alpha_{[h]}\right)$ and we have

$$((T_g \,\mathrm{pr})^*\alpha_{[g]}) \star ((T_h \,\mathrm{pr})^*\alpha_{[h]}) = (T_{g\star h} \,\mathrm{pr})^*(\alpha_{[g]} \star \alpha_{[h]}).$$

3.4 The Lie algebroid of the quotient Lie groupoid

We discuss here shortly the construction of the Lie algebroid of the quotient Lie groupoid.

Definition 3.4.1 (Mackenzie (2005)) *Let $\varphi : A \to A'$, $f : P \to P'$ be a morphism of Lie algebroids. Then (φ, f) is a fibration if both f and $\varphi^! : A \to f^!A'$ are surjective submersions, where $f^!A' \to P$ is the pullback of A' under f.*

That is, a morphism (φ, f) of Lie algebroids is a fibration if f is a surjective submersion and φ is a fiberwise surjection.

Definition 3.4.2 (Mackenzie (2005)) *Let A be a Lie algebroid on a manifold M with anchor map a. An* ideal system *of A is a triple $\mathcal{J} = (J, \mathcal{R}, \theta)$, where J is a wide Lie subalgebroid of A, $\mathcal{R} = \mathcal{R}(f) = \{(m, n) \in M \times M \mid f(m) = f(n)\}$ is a closed embedded, wide Lie subgroupoid of $M \times M \rightrightarrows M$ corresponding to a surjective submersion $f : M \to M'$, and where θ is a linear action of $\mathcal{R}$ on the vector bundle $A/J \to M$ such that*

1. *if $X, Y \in \Gamma(A)$ are θ-stable, i.e., $\theta((p, q), X(q) + J(q)) = X(p) + J(p)$ for all $(p, q) \in R$, then $[X, Y]$ is θ-stable,*

2. *if* $X \in \Gamma(J)$ *and* $Y \in \Gamma(A)$ *is* θ*-stable, then* $[X, Y] \in \Gamma(J)$,

3. *the anchor* $\mathsf{a} : A \to TM$ *maps* J *into* $T^f M = \ker Tf$,

4. *the induced map* $A/J \to TM/T^f M$ *is* $\mathcal{R}$*-equivariant with respect to* θ *and the canonical action* θ_0 *of* $\mathcal{R}$ *on* $TM/T^f M$.

Here, the canonical action θ_0 *of* $\mathcal{R}$ *on* $TM/T^f M$ *is given by* $\theta_0\big((p,q), v_q + T^f_q M\big) = v_p + T^f_p M$ *if* $T_p f v_p = T_q f v_q$.

In the situation of Theorems 3.3.10 and 3.3.11, if the normal subgroupoid system $\mathcal{N}$ is regular, then the quotient $G/S \rightrightarrows P/S$ is a Lie groupoid and $(\mathrm{pr}, \mathrm{pr}_\circ)$ is a fibration. Consider the vector bundle $A^S := S \cap AG$ over P and the subgroupoid $\mathcal{R}(\mathrm{pr}_\circ)$ of $P \times P \rightrightarrows P$. Then there is an induced action θ of $\mathcal{R}(\mathrm{pr}_\circ)$ on AG/A^S defined by $\theta((p,q), u_q + A^S(q)) = u_p + A^S(p)$ if $T_p \,\mathrm{pr}\, u_p = T_q \,\mathrm{pr}\, u_q$. Since $T\mathsf{s}$ sends A^S into $S \cap TP = T^{\mathrm{pr}_\circ} P$, one can check that $\mathcal{J}_S := (A^S, \mathcal{R}(\mathrm{pr}_\circ), \theta)$ is an ideal system of AG. It is the *kernel system* of the fibration of Lie algebroids $AG \to A(G/S)$ defined by the fibration $(\mathrm{pr}, \mathrm{pr}_\circ)$ of Lie groupoids (see Mackenzie (2005) for more details).

Theorem 3.4.3 (Mackenzie (2005)) *If* $(A \to M, \mathsf{a}, [\cdot\,,\cdot])$ *is a Lie algebroid and* $\mathcal{J} = (J, \mathcal{R}(f), \theta)$, $f : M \to M'$, *an ideal system of* A*, then there exists a unique Lie algebroid structure on the quotient vector bundle* $A' = A/\mathcal{J} \to M'$ *such that the natural map* $\sharp : A \to A'$, $f : M \to M'$ *is a morphism of Lie algebroids with* kernel system $\mathcal{J}$.

The induced Lie algebroid $AG/\mathcal{J}_S \to P/S$ is the Lie algebroid $A(G/S)$ of the Lie groupoid $G/S \rightrightarrows P/S$.

4 The group case

Let (G, π_G) be a Poisson Lie group. A theorem of Drinfel′d (see Theorem 2.1.5) relates the π_G-homogeneous *Poisson* structures on G/H to *Dirac* subspaces of the double Lie algebra $\mathfrak{g} \times \mathfrak{g}^*$ defined by π_G, with characteristic subspace equal to $\mathfrak{h}$, the Lie algebra of H.
Hence, it appears natural to ask which kind of objects would correspond to arbitrary Dirac structures in $\mathfrak{g} \times \mathfrak{g}^*$ via this correspondence, or an extension of it. The answer that appears the most reasonable is that we get *Dirac homogeneous spaces* of the Poisson Lie group. Since we pass to the category of Dirac manifolds, it is then also natural to study not only the Dirac homogeneous spaces of Poisson Lie groups, but also of Dirac Lie groups. This is done in this chapter, where we show how the theorem of Drinfel′d generalizes to the more general setting of Dirac manifolds.
Dirac Lie groups have been defined independently by Ortiz (2008). His approach uses the theory about Poisson Lie groups for the definition of the Lie bialgebra of a Dirac Lie group. Here, we choose to phrase everything in the Dirac setting so that we get the known results, such as the definition of the Lie bialgebra of a Poisson Lie group, as corollaries in the class of examples given by the Poisson Lie groups (i.e. with $\mathsf{G}_0 = 0_{TG}$).
The reason why we prefer this approach is because the situation is quite different in the case of a Dirac groupoid, where the characteristic distribution G_0 can be more complicated. The involved geometry is not necessarily induced by an underlying Poisson groupoid anymore.
For the generalization of the results in this chapter to Dirac homogeneous spaces of Dirac groupoids, we will need to construct in Chapter 5 the object that will play the role of the Courant algebroid in this setting. There, the results known for Poisson groupoids will be the guidelines, but it will not be possible to use them as it is done in Ortiz (2008) in the particular case of Dirac Lie groups. Because of the technicality of the constructions for the general groupoid case, we choose to give in this chapter the proofs for all the results in the more easy Lie group case, even if most of them will be corollaries of the more general results later on. In this manner, this chapter stays as self-contained as possible. Also, the understanding of the constructions in the group case is helpful for the general constructions in the groupoid case.
The last section of this chapter is about the Poisson Lie group underlying a regular Dirac Lie group, that is also a Poisson homogeneous space of the Dirac Lie group. We will see that this has no counterpart in the more general theory of Dirac groupoids.

Outline of the chapter Geometric properties of Dirac Lie groups are studied in Section 4.1 and the construction of the Lie bialgebra of a Dirac Lie group is given, as well as the

definition of the induced action of G on it.
Dirac homogeneous spaces of Dirac Lie groups are studied in Section 4.2. The main theorem of this chapter, about the correspondence between (integrable) Dirac homogeneous spaces of an (integrable) Dirac Lie group and Lagrangian subspaces (subalgebras) of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{g}_0^\circ$, is proved in this section.
In Section 4.3, we study the special class of Dirac Lie groups where the characteristic subgroup N is closed in the Lie group G, and the corresponding Dirac homogeneous spaces.

4.1 Geometric properties of Dirac Lie groups

Recall that a *Dirac Lie group* is a Lie group G endowed with a Dirac structure $\mathsf{D}_G \subseteq TG \times_G T^*G$ such that the group multiplication map

$$\mathsf{m} : (G \times G, \mathsf{D}_G \oplus \mathsf{D}_G) \to (G, \mathsf{D}_G)$$

is a *forward Dirac map*, or, equivalently, such that $\mathsf{D}_G \subseteq TG \times_G T^*G$ is a Lie subgroupoid. More explicitly, there exist for all $g, h \in G$ and pairs (v_{gh}, α_{gh}) in $\mathsf{D}_G(gh)$, two pairs $(w_g, \beta_g) \in \mathsf{D}_G(g)$ and $(u_h, \gamma_h) \in \mathsf{D}_G(h)$ such that

$$T_{(g,h)}\mathsf{m}(w_g, u_h) = v_{gh} \quad \text{and} \quad (\beta_g, \gamma_h) = (T_{(g,h)}\mathsf{m})^*\alpha_{gh}.$$

That is, we have

$$T_h L_g u_h + T_g R_h w_g = v_{gh} \in T_{gh}G, \quad \beta_g = (T_g R_h)^*\alpha_{gh} \quad \text{and} \quad \gamma_h = (T_h L_g)^*\alpha_{gh}.$$

In this section and the following, (G, D_G) will always be a Dirac Lie group. We denote by $\mathfrak{g}_1 := \mathsf{G}_1(e)$, $\mathfrak{g}_0 := \mathsf{G}_0(e)$, $\mathfrak{p}_1 := \mathsf{P}_1(e)$ and $\mathfrak{p}_0 := \mathsf{P}_0(e)$ the fibers of the characteristic distributions over the neutral element e of G.
The following results are immediate corollaries of Proposition 2.3.5.

Corollary 4.1.1 *The subspaces $\mathfrak{g}_0 \subseteq \mathfrak{g}$ and $\mathfrak{p}_1 \subseteq \mathfrak{g}^*$ satisfy $\mathrm{Ad}_g^* \mathfrak{p}_1 = \mathfrak{p}_1$, $\mathrm{Ad}_g \mathfrak{g}_0 = \mathfrak{g}_0$ for all $g \in G$. Consequently, we have $\mathrm{ad}_x^* \mathfrak{p}_1 \subseteq \mathfrak{p}_1$ for all $x \in \mathfrak{g}$ and $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$.*

PROOF: We have $\mathsf{P}_1 = \mathfrak{p}_1^r = \mathfrak{p}_1^l$ and $\mathsf{G}_0 = \mathfrak{g}_0^r = \mathfrak{g}_0^l$ by Proposition 2.3.5. Then, for all $g \in G$ and $\xi \in \mathfrak{p}_1$, the covector $(T_g L_{g^{-1}})^*\xi$ is an element of $\mathsf{P}_1(g)$ and there exists $\eta \in \mathfrak{p}_1$ such that $(T_g L_{g^{-1}})^*\xi = (T_g R_{g^{-1}})^*\eta$. This yields $\mathrm{Ad}_{g^{-1}}^* \xi = \eta \in \mathfrak{p}_1$ and $\mathfrak{p}_1$ is consequently $\mathrm{Ad}_{g^{-1}}^*$-invariant for all $g \in G$. In the same manner, we show that $\mathfrak{g}_0$ is Ad_g-invariant for all $g \in G$.
This yields by derivation $\mathrm{ad}_x^* \xi \in \mathfrak{p}_1$ for all $\xi \in \mathfrak{p}_1$ and $\mathrm{ad}_x z \in \mathfrak{g}_0$ for all $z \in \mathfrak{g}_0$ and $x \in \mathfrak{g}$, i.e., $[\mathfrak{g}, \mathfrak{g}_0] \subseteq \mathfrak{g}_0$. □

If G is a simple Lie group, the ideal $\mathfrak{g}_0$ is either trivial or equal to $\mathfrak{g}$ and we get the following corollary.

Corollary 4.1.2 *If (G, D_G) is a simple Dirac Lie group, the Dirac structure D_G is either the graph of the vector bundle homomorphism $T^*G \to TG$ induced by a multiplicative bivector field on G, or the trivial tangent Dirac structure $\mathsf{D}_G = TG \times_G 0_{TG}$.*

We have also the following proposition.

Proposition 4.1.3 *Let (G, D_G) be a Dirac Lie group. Then we have $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1 \subseteq \mathfrak{g} \times \mathfrak{g}^*$. Consequently, the equality $\alpha(e)(Y(e)) = 0 = \beta(e)(X(e))$ holds for all sections (X, α) and (Y, β) of D_G defined on a neighborhood of the neutral element e.*

Proof: Choose $(x, \xi) \in \mathsf{D}_G(e)$. Then we have $\mathbb{T}\mathsf{s}(x, \xi) = \xi \in \mathfrak{g}^*$ and hence $\mathbb{T}\epsilon(\xi) = (0, \xi) = (x, \xi)^{-1} \star (x, \xi) \in \mathsf{D}_G(e)$. Thus, $(x, 0) = (x, \xi) - (0, \xi)$ is also an element of $\mathsf{D}_G(e)$ and $x \in \mathfrak{g}_0$. This shows that $\mathsf{D}_G(e) \subseteq \mathfrak{g}_0 \times \mathfrak{p}_1$ and also $\mathfrak{p}_1 = \mathfrak{p}_0$. Because of this last equality, the inclusion $\mathfrak{g}_0 \times \mathfrak{p}_1 \subseteq \mathsf{D}_G(e)$ is obvious. $\square$

We see from this that there are no nontrivial multiplicative 2-forms on a Lie group.

Corollary 4.1.4 *Let G be a Lie group. Then $\omega = 0$ is the only multiplicative 2-form on G.*

Proof: If ω is a multiplicative 2-form on G, then the associated multiplicative Dirac structure D_ω satisfies $\mathsf{D}_\omega(e) = \operatorname{Graph}(\omega^\flat|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}^*)$. Hence the ideal $\mathfrak{g}_0 \subseteq \mathfrak{g}$ such that $\mathsf{D}_\omega(e) = \mathfrak{g}_0 \times \mathfrak{g}_0^\circ$ has to be equal to $\mathfrak{g}$. Thus, the kernel of ω is $\mathsf{G}_0 = \mathfrak{g}^l = TG$ and ω is trivial. $\square$

Remark 4.1.5 Since $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$, the left and right invariant vector bundle $\mathsf{G}_0 = \mathfrak{g}_0^l = \mathfrak{g}_0^r$ is completely integrable in the sense of Frobenius and its integral leaf N through $e \in G$ is the normal subgroup of G integrating the ideal $\mathfrak{g}_0$ of $\mathfrak{g}$. If N is in addition closed in G, its (left or right) action on G is proper.
We will see later that in certain cases (for example when the Dirac Lie group is integrable), the induced action of N on (G, D_G) is canonical. Also, since $\mathcal{V}_N := \mathfrak{g}_0^l = \mathfrak{g}_0^r$ is the vertical space of the action of N on G, it is easy to see that $\mathsf{D}_G \cap \mathcal{K}_N^\perp = \mathsf{D}_G$ and this intersection has consequently constant rank on G (recall the paragraph about regular reduction of symmetric Dirac structures in Section 1.2). If N is closed in G, we can hence build the quotient $q_N : G \to G/N$ and $(G/N, q_N(\mathsf{D}_G))$ will be shown later to be a Dirac manifold with $q_N(\mathsf{D}_G)$ the graph of a skew-symmetric multiplicative bivector field on G/N. In particular, if (G, D_G) is integrable, the quotient $(G/N, q_N(\mathsf{D}_G))$ will be a Poisson Lie group. $\triangle$

Definition 4.1.6 *A Dirac Lie group (G, D_G) is said to be* regular *if the group integrating $\mathfrak{g}_0$ is closed in G.*

Consider a Lie group G and $\tilde{p} : \tilde{G} \to G$ its universal covering. Then there exists a discrete normal subgroup Γ of $\tilde{G}$ such that $G = \tilde{G}/\Gamma$ (see Knapp (2002)). The following proposition is easy to prove.

Proposition 4.1.7 *Let D_G be a multiplicative (integrable) Dirac structure on G. Then the pullback Dirac structure $\tilde{\mathsf{D}_G} := \tilde{p}^*\mathsf{D}_G$ is an (integrable) multiplicative Dirac structure on $\tilde{G}$.*

Remark 4.1.8 The integral leaf $\tilde{N}$ through $e \in \tilde{G}$ of the characteristic distribution $\tilde{\mathsf{G}_0}$ defined by $\tilde{\mathsf{D}_G}$ on $\tilde{G}$ is normal in $\tilde{G}$ and hence closed since $\tilde{G}$ is simply connected (see Hilgert and Neeb (1991)). Hence, the quotient $\tilde{G}/\tilde{N}$ is here always well-defined and the Dirac Lie group $(\tilde{G}, \tilde{\mathsf{D}_G})$ is regular. $\triangle$

Example 4.1.9 Let G be a connected Lie group. The Lie algebra $\mathfrak{g}$ of G can be Levi-decomposed as the semi-direct product $\mathfrak{g} = \mathfrak{s} \oplus_\phi \operatorname{rad}\mathfrak{g}$ with $\mathfrak{s}$ semi-simple and $\phi : \mathfrak{s} \to \operatorname{Der}(\operatorname{rad}\mathfrak{g})$ a Lie algebra homomorphism (see for instance Knapp (2002)).
The ideal $\operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ is a solvable ideal of $\mathfrak{g}$ and its integral leaf R is closed in G (see Hilgert and Neeb (1991)). The quotient G/R is then a semi-simple Lie group. Let $q_R : G \to G/R$ be the projection and π be the standard multiplicative Poisson structure on the semi-simple Lie group G/R (see Etingof and Schiffmann (2002) and Lu (1990)). The pullback $q_R^*\mathsf{D}_\pi$ is an integrable Dirac structure on G. Its characteristic distribution is the left or right invariant image of the ideal $\mathfrak{g}_0 = \operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ and the action of the integral leaf R of G_0 on $(G, q_R^*\mathsf{D}_\pi)$ is canonical, the Poisson Lie group associated to this Dirac Lie group as in Remark 4.1.5 is obviously $(G/R, \pi)$. $\lozenge$

The following lemma will be useful for many proofs in this chapter. We will always use the following notation. If ξ is an element of the subspace $\mathfrak{p}_1 \subseteq \mathfrak{g}^*$, then the one-form $\xi^l \in \Omega^1(G)$, defined by $\xi^l(g) = \xi \circ T_gL_{g^{-1}}$ for all $g \in G$, is a section of P_1 by Proposition 2.3.5. We denote by $X_\xi \in \mathfrak{X}(G)$ a vector field satisfying $(X_\xi, \xi^l) \in \Gamma(\mathsf{D}_G)$. The vector field X_ξ is not necessarily unique: all $Y \in X_\xi + \Gamma(\mathsf{G}_0)$ satisfy then the condition $(Y, \xi^l) \in \Gamma(\mathsf{D}_G)$. Note that the pair $(X_\xi, \xi^l) \in \Gamma(\mathsf{D}_G)$ is such that $\mathbb{T}\mathsf{s}(X_\xi, \xi^l)(g) = \xi$ for all $g \in G$.

Lemma 4.1.10 *Choose $\xi \in \mathfrak{p}_1$ and corresponding vector fields X_ξ and $X_{\operatorname{Ad}^*_{h^{-1}}\xi}$ for $h \in G$. Then the inclusion*

$$X_\xi(gh) \quad \in \quad T_hL_gX_\xi(h) + T_gR_hX_{\operatorname{Ad}^*_{h^{-1}}\xi}(g) + \mathsf{G}_0(gh) \tag{4.1}$$

holds for all $g \in G$.

Remark 4.1.11 If Y_ξ and $Y_{\operatorname{Ad}^*_h\xi} \in \mathfrak{X}(G)$ are such that $(Y_\xi, \xi^r), (Y_{\operatorname{Ad}^*_h\xi}, (\operatorname{Ad}^*_h\xi)^r) \in \Gamma(\mathsf{D}_G)$, then we can show in the same manner

$$Y_\xi(hg) \quad \in \quad T_gL_hY_{\operatorname{Ad}^*_h\xi}(g) + T_hR_gY_\xi(h) + \mathsf{G}_0(hg)$$

for all $g \in G$. $\triangle$

Proof (of Lemma 4.1.10): By Example 1.1.21, we have

$$\mathbb{T}\mathsf{t}(X_\xi(h), \xi^l(h)) = \operatorname{Ad}^*_{h^{-1}}\xi = \mathbb{T}\mathsf{s}(X_{\operatorname{Ad}^*_{h^{-1}}\xi}(g), (\operatorname{Ad}^*_{h^{-1}}\xi)^l(g)).$$

Hence, the product $(X_\xi(h), \xi^l(h)) \star (X_{\mathrm{Ad}^*_{h^{-1}}\xi}(g), (\mathrm{Ad}^*_{h^{-1}}\xi)^l(g)) \in \mathsf{D}_G(gh)$ is defined and equals

$$\begin{aligned}&\Big(T_{(g,h)}\mathsf{m}(X_{\mathrm{Ad}^*_{h^{-1}}\xi}(g), X_\xi(h)), (T_{gh}L_{g^{-1}})^*\xi^l(h)\Big)\\ =& \Big(T_hL_gX_\xi(h) + T_gR_hX_{\mathrm{Ad}^*_{h^{-1}}\xi}(g), \xi^l(gh)\Big),\end{aligned}$$

which concludes the proof. □

Proposition 4.1.12 *Let ξ and η be elements of $\mathfrak{p}_1$ and $X_\xi, X_\eta \in \mathfrak{X}(G)$ corresponding vector fields. The one-form $\pounds_{X_\xi}\eta^l - \mathbf{i}_{X_\eta}\mathbf{d}\xi^l$ is left invariant and equal to $\big(\mathbf{d}_e\big(\eta^l(X_\xi)\big)\big)^l$.*

Proof: Choose $x \in \mathfrak{g}$ and, using the preceding lemma and the notation $(\mathrm{ad}^*_x\xi)(y) = \xi([y,x])$ for all $\xi \in \mathfrak{g}^*$, $x, y \in \mathfrak{g}$, compute

$$\begin{aligned}
&\left(\pounds_{X_\xi}\eta^l - \mathbf{i}_{X_\eta}\mathbf{d}\xi^l\right)(x^l)(g)\\
=\;& X_\xi(\eta^l(x^l))(g) + \eta^l(\pounds_{x^l}X_\xi)(g) - X_\eta(\xi^l(x^l))(g) + x^l(\xi^l(X_\eta))(g) - \xi^l(\pounds_{x^l}X_\eta)(g)\\
=\;& \eta^l(g)\left(\left.\frac{d}{dt}\right|_{t=0} T_{g\exp(tx)}R_{\exp(-tx)}X_\xi(g\exp(tx))\right) + (\pounds_{x^l}\xi^l)(X_\eta)(g)\\
\overset{(4.1)}{=}\;& \left.\frac{d}{dt}\right|_{t=0}\eta^l(g)(X_{\mathrm{Ad}^*_{\exp(-tx)}\xi}(g)) + \eta\left(\left.\frac{d}{dt}\right|_{t=0} T_{\exp(tx)}R_{\exp(-tx)}X_\xi(\exp(tx))\right)\\
&\qquad + (\mathrm{ad}^*_x\xi)^l(X_\eta)(g)\\
=\;& -\left.\frac{d}{dt}\right|_{t=0}(\mathrm{Ad}^*_{\exp(-tx)}\xi)^l(g)(X_\eta(g)) + \eta(\pounds_{x^l}X_\xi(e)) + (\mathrm{ad}^*_x\xi)^l(X_\eta)(g)\\
=\;& -(\mathrm{ad}^*_x\xi)^l(X_\eta)(g) + \pounds_{x^l}\left(\eta^l(X_\xi)\right)(e) - (\pounds_{x^l}\eta^l)(X_\xi)(e) + (\mathrm{ad}^*_x\xi)^l(X_\eta)(g)\\
=\;& \mathbf{d}_e\left(\eta^l(X_\xi)\right)(x),
\end{aligned}$$

where we have used Proposition 4.1.3 and $\pounds_{x^l}\eta^l = (\mathrm{ad}^*_x\eta)^l \in \Gamma(\mathsf{P}_1)$ by Corollary 4.1.1. □

Definition 4.1.13 *Let (G, D_G) be a Dirac Lie group. Define the bilinear, antisymmetric bracket*

$$[\cdot\,,\cdot] : \mathfrak{p}_1 \times \mathfrak{p}_1 \to \mathfrak{g}^* \quad \text{by} \quad [\xi,\eta] = \mathbf{d}_e\big(\eta^l(X_\xi)\big),$$

where $X_\xi \in \mathfrak{X}(G)$ is such that $(X_\xi, \xi^l) \in \Gamma(\mathsf{D}_G)$. That is, we set the notation $\pounds_{X_\xi}\eta^l - \mathbf{i}_{X_\eta}\mathbf{d}\xi^l =: [\xi,\eta]^l$ and hence $[(X_\xi,\xi^l),(X_\eta,\eta^l)] = ([X_\xi,X_\eta],[\xi,\eta]^l)$ for all $\xi,\eta \in \mathfrak{p}_1$.

Note that $[\xi,\eta]$ does not depend on the choice of the vector field X_ξ. Indeed, if $Y_\xi \in \mathfrak{X}(G)$ is an other vector field such that $(Y_\xi, \xi^l) \in \Gamma(\mathsf{D}_G)$, we have $Y_\xi - X_\xi \in \Gamma(\mathsf{G}_0)$ and hence $\eta^l(X_\xi) = \eta^l((X_\xi - Y_\xi) + Y_\xi)) = \eta^l(Y_\xi)$ since $\eta^l \in \Gamma(\mathsf{P}_1)$.
The bilinearity of the bracket is obvious. For the antisymmetry, choose $\xi,\eta \in \mathfrak{p}_1$. Then we have $\xi^l(X_\eta) + \eta^l(X_\xi) = 0$ since (X_ξ,ξ^l) and (X_η,η^l) are sections of D_G, and this leads to

$$[\xi,\eta] = \mathbf{d}_e\left(\eta^l(X_\xi)\right) = \mathbf{d}_e\left(-\xi^l(X_\eta)\right) = -[\eta,\xi].$$

As a direct corollary of Proposition 4.1.12 we recover the generalization to the Dirac case of the fact that every multiplicative Poisson structure on a torus is trivial.

Corollary 4.1.14 *Consider an* Abelian *Dirac Lie group* (G, D_G) *and choose* x *in the Lie algebra* $\mathfrak{g}$. *Then the equality*

$$\left(\eta^l(X_\xi)\right)(g \cdot \exp(tx)) = [\xi, \eta](tx) + \left(\eta^l(X_\xi)\right)(g)$$

holds for all $g \in G$ *and* $t \in \mathbb{R}$.
As a consequence, if $\mathsf{D}_{\mathbb{T}^n}$ *is a multiplicative Dirac structure on the* n*-torus* $\mathbb{R}^n/\mathbb{Z}^n$, *then* $\mathsf{D}_{\mathbb{T}^n}$ *is a direct sum*

$$\mathsf{D}_{\mathbb{T}^n} = \mathsf{G}_0 \times_{\mathbb{T}^n} \mathsf{P}_1.$$

PROOF: We have shown in Proposition 4.1.12 that $\pounds_{X_\xi}\eta^l - \mathbf{i}_{X_\eta}\xi^l = [\xi, \eta]^l$ is a left invariant one-form on G. We have for all $\xi, \eta \in \mathfrak{p}_1$ and $x \in \mathfrak{g}$:

$$\begin{aligned}
[\xi, \eta]^l(x^l) &= (\pounds_{X_\xi}\eta^l - \mathbf{i}_{X_\eta}\mathbf{d}\xi^l)(x^l) = \eta^l(\pounds_{x^l}X_\xi) + (\pounds_{x^l}\xi^l)(X_\eta) \qquad (4.2)\\
&\qquad\qquad\qquad\qquad \text{(see the proof of Proposition 4.1.12)}\\
&= x^l(\eta^l(X_\xi)) - (\pounds_{x^l}\eta^l)(X_\xi) + (\pounds_{x^l}\xi^l)(X_\eta)\\
&= x^l(\eta^l(X_\xi)) - (\mathrm{ad}^*_x \eta)^l(X_\xi) + (\mathrm{ad}^*_x \xi)^l(X_\eta) = x^l(\eta^l(X_\xi)) \qquad (4.3)
\end{aligned}$$

since $\mathrm{ad}^*_x \xi = \mathrm{ad}^*_x \eta = 0$ because $\mathfrak{g}$ is Abelian. We get $\mathbf{d}(\eta^l(X_\xi)) = [\xi, \eta]^l$ and the equality $\frac{d}{dt}R^*_{\exp(tx)}f = R^*_{\exp(tx)}(\pounds_{x^l}f)$ for all $f \in C^\infty(G)$ yields

$$\frac{d}{dt}\left(\eta^l(X_\xi)\right)(g\exp(tx)) = R^*_{\exp(tx)}\left(x^l(\eta^l(X_\xi))\right)(g) \overset{(4.3)}{=} R^*_{\exp(tx)}([\xi, \eta](x)) = [\xi, \eta](x)$$

for all $g \in G$ and $t \in \mathbb{R}$. We get

$$\left(\eta^l(X_\xi)\right)(g\exp(tx)) = [\xi, \eta](x) \cdot t + \left(\eta^l(X_\xi)\right)(g) = [\xi, \eta](tx) + \left(\eta^l(X_\xi)\right)(g).$$

On the n-dimensional torus $\mathbb{T}^n$, we have $\exp(tx) = tx + \mathbb{Z}^n$ for all $x \in \mathfrak{g} = \mathbb{R}^n$ and all $t \in \mathbb{R}$. This yields

$$\left(\eta^l(X_\xi)\right)(\exp(tx)) = [\xi, \eta](tx) + \left(\eta^l(X_\xi)\right)(0) = [\xi, \eta](tx)$$

for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$. But since the function $\left(\eta^l(X_\xi)\right)$ is well-defined on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, the equality $[\xi, \eta](tx) = [\xi, \eta](tx + z)$ has to hold for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $z \in \mathbb{Z}^n$. This leads to $[\xi, \eta] = 0$ and hence $\eta^l(X_\xi)$ is constant and equal to its value at the neutral element; $\eta^l(X_\xi)(0) = \eta(X_\xi(0)) = 0$ for all $\xi, \eta \in \mathfrak{p}_1$ by Proposition 4.1.3. Thus, each spanning vector field X_ξ, $\xi \in \mathfrak{p}_1$, of G_1 is annihilated by P_1 and is consequently a section of G_0. □

The next proposition shows that the value of $[\xi, \eta]$, for $\xi, \eta \in \mathfrak{p}_1$, can be computed with any two one-forms in $\Gamma(\mathsf{P}_1)$ taking value ξ, η in e.

Proposition 4.1.15 *Let $\alpha, \beta \in \Gamma(\mathsf{P}_1)$ be such that $\alpha(e) = \xi$ and $\beta(e) = \eta \in \mathfrak{p}_1$. Then we have*

$$[\xi, \eta] = \mathbf{d}_e(\beta(X_\alpha)),$$

where $X_\alpha \in \mathfrak{X}(G)$ is such that $(X_\alpha, \alpha) \in \Gamma(\mathsf{D}_G)$.

PROOF: First, we show that $\pounds_{x^r}\alpha \in \Gamma(\mathsf{P}_1)$ for all $\alpha \in \Gamma(\mathsf{P}_1)$. For all $g, h \in G$ we have $(L_g^*\alpha)(h) = \alpha_{gh} \circ T_h L_g = (T_h L_g)^*\alpha_{gh}$. This is an element of $\mathsf{P}_1(h)$ since $\alpha_{gh} \in \mathsf{P}_1(gh)$ and P_1 is left invariant by Proposition 2.3.5. Thus, we get

$$(\pounds_{x^r}\alpha)(h) = \left.\frac{d}{dt}\right|_{t=0} (L^*_{\exp(tx)}\alpha)(h) \in \mathsf{P}_1(h)$$

since $(L^*_{\exp(tx)}\alpha)(h) \in \mathsf{P}_1(h)$ for all t. Choose $x \in \mathfrak{g}$ and compute

$$\begin{aligned}
[\xi, \eta](x) &= x^r(\eta^l(X_\xi))(e) = \eta(\pounds_{x^r}X_\xi(e)) = \beta(e)(\pounds_{x^r}X_\xi(e)) \\
&= x^r(\beta(X_\xi)) - (\pounds_{x^r}\beta)(X_\xi)(e) = -x^r(\xi^l(X_\beta))(e) \\
&= -\xi(\pounds_{x^r}X_\beta(e)) = -\alpha(e)(\pounds_{x^r}X_\beta(e)) \\
&= -\pounds_{x^r}(\alpha(X_\beta))(e) + (\pounds_{x^r}\alpha)(X_\beta)(e) = \mathbf{d}_e(\beta(X_\alpha))(x)
\end{aligned}$$

In the fifth and ninth equalities, we have used the fact that $\pounds_{x^r}\alpha, \pounds_{x^r}\beta \in \Gamma(\mathsf{P}_1)$ and Proposition 4.1.3. $\square$

The next lemma holds for integrable Dirac Lie groups, and is in general not true if the Dirac Lie group (G, D_G) is not integrable, as shows the example following it. Recall that N is the normal subgroup of G defined by the integral leaf through e of the integrable subbundle $\mathsf{G}_0 \subseteq TG$.

Lemma 4.1.16 *If (G, D_G) is integrable, then we have*

$$(\pounds_{x^l}X, \pounds_{x^l}\alpha), \quad (\pounds_{x^r}X, \pounds_{x^r}\alpha) \in \Gamma(\mathsf{D}_G)$$

for all $x \in \mathfrak{g}_0$ and $(X, \alpha) \in \Gamma(\mathsf{D}_G)$, and the pairs $(R_n^*X, R_n^*\alpha)$ and $(L_n^*X, L_n^*\alpha)$ are also elements of $\Gamma(\mathsf{D}_G)$ for all $n \in N$.

PROOF: The right and left invariant vector fields x^r and x^l defined on G by an element of $\mathfrak{g}_0$ are sections of $\mathsf{G}_0 = \mathfrak{g}_0^r = \mathfrak{g}_0^l$. If (G, D_G) is integrable, we have $(\pounds_{x^l}X, \pounds_{x^l}\alpha) = \left[(x^l, 0), (X, \alpha)\right]$ and $(\pounds_{x^r}X, \pounds_{x^r}\alpha) = [(x^r, 0), (X, \alpha)] \in \Gamma(\mathsf{D}_G)$ for all $(X, \alpha) \in \Gamma(\mathsf{D}_G)$.
For each $x \in \mathfrak{g}_0$, the flow of x^l is $R_{\exp(tx)}$ and the flow of x^r is $L_{\exp(tx)}$. Thus, by Corollary 1.2.7, we have $(R^*_{\exp(tx)}X, R^*_{\exp(tx)}\alpha)$ and $(L^*_{\exp(tx)}X, L^*_{\exp(tx)}\alpha) \in \Gamma(\mathsf{D}_G)$ for all $(X, \alpha) \in \Gamma(\mathsf{D}_G)$. This yields the claim since N is generated as a group by the elements $\exp(tx)$, $x \in \mathfrak{g}_0$ and small t. $\square$

Example 4.1.17 Consider the Dirac structure $\mathsf{D}_{\mathbb{R}^3}$ defined on the Lie group $\mathbb{R}^3$ as the span of the sections

$$(\partial_z, 0), \quad (z\partial_x, \mathbf{d}y), \quad (-z\partial_y, \mathbf{d}x)$$

of $\mathsf{P}_{\mathbb{R}^3}$. It is easy to show that $(\mathbb{R}^3, \mathsf{D}_{\mathbb{R}^3})$ is a Dirac Lie group (see also Corollary 4.1.14 for a description of the multiplicative Dirac structures on $\mathbb{R}^n$). It is not integrable because, for instance, the bracket of $(\partial_z, 0)$ and $(z\partial_x, \mathbf{d}y)$ is equal to $(\partial_x, 0)$, which is not a section of $\mathsf{D}_{\mathbb{R}^3}$. The Dirac structure is obviously not invariant under the action of $N = \{(0,0)\} \times \mathbb{R}$ on $\mathbb{R}^3$. $\Diamond$

The following theorem shows how to decide if the action of N on (G, D_G) is canonical.

Theorem 4.1.18 *The Dirac Lie group (G, D_G) is N-invariant if and only if the bracket $[\cdot,\cdot]$ defined in Definition 4.1.13 has image in $\mathfrak{p}_1$.*

Example 4.1.19 Consider again Example 4.1.17. The bracket on

$$\mathfrak{p}_1 = \operatorname{span}\{\mathbf{d}x(0), \mathbf{d}y(0)\}$$

is given by $[\mathbf{d}y(0), \mathbf{d}x(0)] = \mathbf{d}_0\left((\mathbf{d}x)(z\partial_x)\right) = \mathbf{d}z(0) \notin \mathfrak{p}_1$. $\Diamond$

For the proof of Theorem 4.1.18, we need to introduce a new notation and show a lemma, that will also be useful in the following.

Definition 4.1.20 *Choose $\xi \in \mathfrak{p}_1$ and $x \in \mathfrak{g}$. Then the elements*

$$\operatorname{ad}_x^* \xi \in \mathfrak{p}_1 \qquad \textit{and} \qquad \operatorname{ad}_\xi^* x \in \mathfrak{p}_1^* = \mathfrak{g}/\mathfrak{g}_0$$

are defined by $(\operatorname{ad}_x^* \xi)(y) = \xi([y, x])$ for all $y \in \mathfrak{g}$, and $(\operatorname{ad}_\xi^* x)(\eta) = [\eta, \xi](x)$ for all $\eta \in \mathfrak{p}_1$.

Note that $\operatorname{ad}_x^* \xi$ is an element of $\mathfrak{p}_1$ by Corollary 4.1.1.

Lemma 4.1.21 *Choose $\xi \in \mathfrak{p}_1$ and $X_\xi \in \mathfrak{X}(G)$ such that $(X_\xi, \xi^l) \in \Gamma(\mathsf{D}_G)$. Then we have for all $x \in \mathfrak{g}$:*

$$(\pounds_{x^l} X_\xi)(e) + \mathfrak{g}_0 = -\operatorname{ad}_\xi^* x \ \in \mathfrak{g}/\mathfrak{g}_0$$

and consequently

$$\pounds_{x^l} X_\xi \in (-\operatorname{ad}_\xi^* x)^l + X_{\operatorname{ad}_x^* \xi} + \Gamma(\mathsf{G}_0) \tag{4.4}$$

for all $g \in G$.

Proof: Choose $\eta \in \mathfrak{p}_1$ and compute

$$\eta\left((\pounds_{x^l} X_\xi)(e)\right) = \pounds_{x^l}\left(\eta^l(X_\xi)\right)(e) - \left(\pounds_{x^l}\eta^l\right)(X_\xi)(e) = [\xi, \eta](x) - 0 = \eta\left(-\operatorname{ad}_\xi^* x\right).$$

This yields the first equality. Using this and the proof of Proposition 4.1.12, we get:

$$\begin{aligned}\eta^l(\pounds_{x^l} X_\xi) &= -(\operatorname{ad}_x^* \xi)^l(X_\eta) + \eta\left(\pounds_{x^l} X_\xi(e)\right) \\ &= \eta^l\left(X_{\operatorname{ad}_x^* \xi}\right) + \eta^l\left((-\operatorname{ad}_\xi^* x)^l\right) = \eta^l\left(X_{\operatorname{ad}_x^* \xi} + (-\operatorname{ad}_\xi^* x)^l\right).\end{aligned}$$

Since the left invariant one-forms η^l, for all $\eta \in \mathfrak{p}_1$, span $\Gamma(\mathsf{P}_1)$ as a $C^\infty(G)$-module, we have $\alpha\left(\pounds_{x^l} X_\xi\right) = \alpha\left(X_{\operatorname{ad}_x^* \xi} - (\operatorname{ad}_\xi^* x)^l\right)$ for all $\alpha \in \Gamma(\mathsf{P}_1)$, and hence we are done using $\mathsf{G}_0 = {\mathsf{P}_1}^\circ$. $\square$

Remark 4.1.22 Note that if x is an element of $\mathfrak{g}_0$, we have

$$(\pounds_{x^l} X_\xi, \pounds_{x^l}\xi^l) = \left[\left(x^l, 0\right), \left(X_\xi, \xi^l\right)\right] \in \Gamma(\mathsf{D}_G)$$

if D_G is integrable. Since x lies in the ideal $\mathfrak{g}_0$ and $\xi \in \mathfrak{p}_1 = \mathfrak{g}_0^\circ$, we have $\operatorname{ad}_x^* \xi = 0$, thus $\pounds_{x^l}\xi^l = (\operatorname{ad}_x^* \xi)^l = 0$ and we get $\pounds_{x^l} X_\xi \in \Gamma(\mathsf{G}_0)$.
With Lemma 4.1.21, we can show that this is true without the assumption that D_G is integrable; we need only the hypothesis that the bracket on $\mathfrak{p}_1$ has image in $\mathfrak{p}_1$. We have then

$$\pounds_{x^l} X_\xi \in X_{\operatorname{ad}_x^* \xi} - (\operatorname{ad}_\xi^* x)^l + \Gamma(\mathsf{G}_0) = X_0 - (\operatorname{ad}_\xi^* x)^l + \Gamma(\mathsf{G}_0) = \Gamma(\mathsf{G}_0).$$

The vector field X_0 is indeed an element of $\Gamma(\mathsf{G}_0)$ by definition, and for all $\eta \in \mathfrak{p}_1$, we have $(\operatorname{ad}_\xi^* x)(\eta) = [\eta, \xi](x) = 0$ since $[\eta, \xi] \in \mathfrak{p}_1$ and $x \in \mathfrak{g}_0$, which shows that $\operatorname{ad}_\xi^* x$ is trivial in $\mathfrak{g}/\mathfrak{g}_0$ and thus $(\operatorname{ad}_\xi^* x)^l \in \Gamma(\mathsf{G}_0)$. △

Proof (of Theorem 4.1.18): If the right action of N on (G, D_G) is canonical, we have $(R_n^* X_\xi, R_n^* \xi^l) \in \Gamma(\mathsf{D}_G)$ for all $n \in N$ and $\xi \in \mathfrak{p}_1$. This yields $(\pounds_{x^l} X_\xi, \pounds_{x^l}\xi^l) \in \Gamma(\mathsf{D}_G)$ for all $x \in \mathfrak{g}_0$. Since $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$, we get $\pounds_{x^l} X_\xi(e) \in \mathfrak{g}_0$. Hence, we have $[\xi, \eta](x) = x^l(\eta^l(X_\xi))(e) = (\operatorname{ad}_x^* \eta)(X_\xi(e)) + \eta(\pounds_{x^l} X_\xi(e)) = 0$ for all $\xi, \eta \in \mathfrak{p}_1$ and $x \in \mathfrak{g}_0$ and consequently $[\xi, \eta] \in \mathfrak{p}_1$.
Conversely, if $[\xi, \eta] \in \mathfrak{p}_1$ for all $\xi, \eta \in \mathfrak{p}_1$, we get $\pounds_{x^l} X_\xi \in \Gamma(\mathsf{G}_0)$ by Remark 4.1.22. Hence, recalling that $\operatorname{ad}_x^* \xi = 0$ for $x \in \mathfrak{g}_0$ and $\xi \in \mathfrak{p}_1$, we can compute

$$\begin{aligned}
&\frac{d}{dt} \left\langle (R^*_{\exp(tx)} X_\xi, R^*_{\exp(tx)}\xi^l), (X_\eta, \eta^l) \right\rangle (g)\\
=&\frac{d}{dt} \left(\eta^l(g)(R^*_{\exp(tx)} X_\xi)(g) + (R^*_{\exp(tx)}\xi^l)(g)(X_\eta(g))\right)\\
=&\eta^l \left(R^*_{\exp(tx)}(\pounds_{x^l} X_\xi)\right)(g) + (R^*_{\exp(tx)}(\operatorname{ad}_x^* \xi)^l)(X_\eta)(g)\\
=&\eta \circ T_g L_{g^{-1}} \circ T_{g\exp(tx)} R_{\exp(-tx)} \left(\pounds_{x^l} X_\xi\right)(g \exp(tx))\\
=&(\operatorname{Ad}^*_{\exp(tx)} \eta)^l \left(\pounds_{x^l} X_\xi\right)(g \exp(tx)) = 0
\end{aligned}$$

since $\operatorname{Ad}^*_{\exp(tx)} \eta \in \mathfrak{p}_1$ and $\pounds_{x^l} X_\xi \in \Gamma(\mathsf{G}_0)$. But this yields

$$\begin{aligned}
\left\langle (R^*_{\exp(tx)} X_\xi, R^*_{\exp(tx)}\xi^l), (X_\eta, \eta^l) \right\rangle (g) &= \left\langle (R^*_{\exp(0\cdot x)} X_\xi, R^*_{\exp(0\cdot x)}\xi^l), (X_\eta, \eta^l) \right\rangle (g)\\
&= \left\langle (X_\xi, \xi^l), (X_\eta, \eta^l) \right\rangle (g) = 0
\end{aligned}$$

for all $t \in \mathbb{R}$, which shows that $(R^*_{\exp(tx)} X_\xi, R^*_{\exp(tx)}\xi^l)$ is a section of D_G for all $t \in \mathbb{R}$. Hence, since N is generated as a group by the elements $\exp(tx)$, for $x \in \mathfrak{g}_0$ and small $t \in \mathbb{R}$, the proof is finished. □

The following theorem will be useful in the next subsection about integrable Dirac Lie groups.

Theorem 4.1.23 *The equality*

$$[\xi, \eta]([x, y]) = (\operatorname{ad}_y^* \xi)(\operatorname{ad}_\eta^* x) - (\operatorname{ad}_x^* \xi)(\operatorname{ad}_\eta^* y) + (\operatorname{ad}_x^* \eta)(\operatorname{ad}_\xi^* y) - (\operatorname{ad}_y^* \eta)(\operatorname{ad}_\xi^* x) \tag{4.5}$$

holds for all $\xi, \eta \in \mathfrak{p}_1$ *and* $x, y \in \mathfrak{g}$.

Proof: By Definition 4.1.13, we have $[\xi,\eta]([x,y]) = [x,y]^l\left(\eta^l(X_\xi)\right)(e)$ for any $x,y \in \mathfrak{g}$ and $\xi,\eta \in \mathfrak{p}_1$. Hence we can compute

$$\begin{aligned}
[\xi,\eta]([x,y]) =& [x,y]^l\left(\eta^l(X_\xi)\right)(e) = \pounds_{x^l}\pounds_{y^l}\left(\eta^l(X_\xi)\right)(e) - \pounds_{y^l}\pounds_{x^l}\left(\eta^l(X_\xi)\right)(e)\\
=& \pounds_{x^l}\left(\pounds_{y^l}\eta^l(X_\xi) + \eta^l(\pounds_{y^l}X_\xi)\right)(e) - \pounds_{y^l}\left(\pounds_{x^l}\eta^l(X_\xi) + \eta^l(\pounds_{x^l}X_\xi)\right)(e)\\
=& \pounds_{x^l}\left((\operatorname{ad}_y^*\eta)^l(X_\xi) + \eta^l(-\operatorname{ad}_\xi^* y)^l + \eta^l\left(X_{\operatorname{ad}_y^*\xi}\right)\right)(e)\\
&- \pounds_{y^l}\left((\operatorname{ad}_x^*\eta)^l(X_\xi) + \eta^l(-\operatorname{ad}_\xi^* x)^l + \eta^l\left(X_{\operatorname{ad}_x^*\xi}\right)\right)(e).
\end{aligned}$$

Since $\eta^l(-\operatorname{ad}_\xi^* y)^l$ and $\eta^l(-\operatorname{ad}_\xi^* x)^l$ are constant functions on G, we get hence:

$$\begin{aligned}
&[\xi,\eta]([x,y])\\
=& \pounds_{x^l}\left((\operatorname{ad}_y^*\eta)^l(X_\xi) + \eta^l\left(X_{\operatorname{ad}_y^*\xi}\right)\right)(e) - \pounds_{y^l}\left((\operatorname{ad}_x\eta)^l(X_\xi) + \eta^l\left(X_{\operatorname{ad}_x^*\xi}\right)\right)(e)\\
=& \Big(\pounds_{x^l}(\operatorname{ad}_y^*\eta)^l(X_\xi) + (\operatorname{ad}_y^*\eta)^l(\pounds_{x^l}X_\xi) + (\pounds_{x^l}\eta^l)\left(X_{\operatorname{ad}_y^*\xi}\right) + \eta^l\left(\pounds_{x^l}X_{\operatorname{ad}_y^*\xi}\right)\Big)(e)\\
&- \Big(\pounds_{y^l}(\operatorname{ad}_x^*\eta)^l(X_\xi) + (\operatorname{ad}_x^*\eta)^l(\pounds_{y^l}X_\xi) + (\pounds_{y^l}\eta^l)\left(X_{\operatorname{ad}_x^*\xi}\right) + \eta^l\left(\pounds_{y^l}X_{\operatorname{ad}_x^*\xi}\right)\Big)(e)\\
=& (\operatorname{ad}_x^*\operatorname{ad}_y^*\eta)(X_\xi(e)) - (\operatorname{ad}_y^*\eta)(\operatorname{ad}_\xi^* x) + (\operatorname{ad}_x^*\eta)\left(X_{\operatorname{ad}_y^*\xi}(e)\right) - \eta\left(\operatorname{ad}^*_{\operatorname{ad}_y^*\xi} x\right)\\
&- (\operatorname{ad}_y^*\operatorname{ad}_x^*\eta)(X_\xi(e)) + (\operatorname{ad}_x^*\eta)(\operatorname{ad}_\xi^* y) - (\operatorname{ad}_y^*\eta)\left(X_{\operatorname{ad}_x^*\xi}(e)\right) + \eta\left(\operatorname{ad}^*_{\operatorname{ad}_x^*\xi} y\right).
\end{aligned}$$

Since $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ and $\mathfrak{p}_1$ is ad_x^*-invariant for all $x \in \mathfrak{g}$, the first, third, fifth and seventh terms of this sum vanish. Thus, we get

$$\begin{aligned}
[\xi,\eta]([x,y]) &= -(\operatorname{ad}_y^*\eta)(\operatorname{ad}_\xi^* x) - \eta\left(\operatorname{ad}^*_{\operatorname{ad}_y^*\xi} x\right) + (\operatorname{ad}_x^*\eta)(\operatorname{ad}_\xi^* y) + \eta\left(\operatorname{ad}^*_{\operatorname{ad}_x^*\xi} y\right)\\
&= -(\operatorname{ad}_y^*\eta)(\operatorname{ad}_\xi^* x) + [\operatorname{ad}_y^*\xi,\eta](x) + (\operatorname{ad}_x^*\eta)(\operatorname{ad}_\xi^* y) - [\operatorname{ad}_x^*\xi,\eta](y)\\
&= -(\operatorname{ad}_y^*\eta)(\operatorname{ad}_\xi^* x) + (\operatorname{ad}_y^*\xi)(\operatorname{ad}_\eta^* x) + (\operatorname{ad}_x^*\eta)(\operatorname{ad}_\xi^* y) - (\operatorname{ad}_x^*\xi)(\operatorname{ad}_\eta^* y). \quad \square
\end{aligned}$$

Remark 4.1.24 Equation (4.5) is equivalent to either one of the following equations for all $x,y \in \mathfrak{g}$ and $\xi,\eta \in \mathfrak{p}_1$:

$$\operatorname{ad}_x^*([\xi,\eta]) = [\operatorname{ad}_x^*\xi,\eta] - [\operatorname{ad}_x^*\eta,\xi] + \operatorname{ad}^*_{\operatorname{ad}_\eta^* x}\xi - \operatorname{ad}^*_{\operatorname{ad}_\xi^* x}\eta, \tag{4.6}$$

$$\operatorname{ad}_\xi^*([x,y]) = [\operatorname{ad}_\xi^* x,y] - [\operatorname{ad}_\xi^* y,x] - \operatorname{ad}^*_{\operatorname{ad}_x^*\xi} y + \operatorname{ad}^*_{\operatorname{ad}_y^*\xi} x \;\in\; \mathfrak{p}_1^* = \mathfrak{g}/\mathfrak{g}_0 \tag{4.7}$$

$\triangle$

Let (G,D_G) be a Dirac Lie group. Then the space $\bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ is a G-module via

$$g\cdot((x+\mathfrak{g}_0)\wedge(y+\mathfrak{g}_0)) = (\operatorname{Ad}_g x+\mathfrak{g}_0)\wedge(\operatorname{Ad}_g y+\mathfrak{g}_0)$$

by Proposition 2.3.5, and by derivation, it is a $\mathfrak{g}$-module via

$$z\cdot((x+\mathfrak{g}_0)\wedge(y+\mathfrak{g}_0)) = ([z,x]+\mathfrak{g}_0)\wedge(y+\mathfrak{g}_0) + (x+\mathfrak{g}_0)\wedge([z,y]+\mathfrak{g}_0).$$

Theorem 4.1.23 states then that the map $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ defined as the dual map of $[\cdot,\cdot] : \bigwedge^2 \mathfrak{p}_1 \to \mathfrak{g}^*$ is a Lie algebra 1-cocycle, that is, we have

$$\delta([x,y]) = x \cdot \delta(y) - y \cdot \delta(x)$$

for all $x, y \in \mathfrak{g}$. Hence, we can associate to each Dirac Lie group (G, D_G) an ideal $\mathfrak{g}_0$ and a Lie algebra 1-cocycle $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$. If (G, D_G) is a Dirac Lie group, then the map $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ defined by $C(g)(\xi,\eta) = \eta^r(Y_\xi)(g) = -\xi^r(Y_\eta)(g)$, where $Y_\xi \in \mathfrak{X}(G)$ is a vector field satisfying $(Y_\xi, \xi^r) \in \Gamma(\mathsf{D}_G)$, is a Lie group 1-cocycle, i.e., it satisfies $C(gh) = C(g) + \mathrm{Ad}_g C(h)$ for all $g, h \in G$. The proof of this uses Remark 4.1.11. We have $C(e) = 0 \in \mathfrak{g}/\mathfrak{g}_0 \wedge \mathfrak{g}/\mathfrak{g}_0$ by Proposition 4.1.3 and

$$\begin{aligned}(\mathbf{d}_e C)(x)(\xi,\eta) &= \left.\frac{d}{dt}\right|_{t=0} C(\exp(tx))(\xi,\eta)\\ &= \left.\frac{d}{dt}\right|_{t=0} \eta^r(Y_\xi)(\exp(tx)) \overset{\text{Prop. 4.1.15}}{=} [\xi,\eta](x) = \delta(x)(\xi,\eta)\end{aligned}$$

for all $\xi, \eta \in \mathfrak{p}_1$.
Note that if G is connected and $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ is a Lie group 1-cocycle integrating δ, that is, with $C(e) = 0$ and $\mathbf{d}_e C = \delta$, then C is unique (see Lu (1990)) and D_G is consequently given on G by

$$\mathsf{D}_G(g) = \left\{ (T_e R_g(C(g)^\sharp(\xi) + x), \xi^r(g)) \mid \xi \in \mathfrak{p}_1, x \in \mathfrak{g}_0 \right\} \tag{4.8}$$

for all $g \in G$, where $C(g)^\sharp : \mathfrak{g}/\mathfrak{g}_0 \to \mathfrak{g}$ is defined as follows. Choose a vector subspace $W \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus W$, then we have an isomorphism $\phi_W : W \to \mathfrak{g}/\mathfrak{g}_0$, $w \mapsto w + \mathfrak{g}_0$. Set $C(g)^\sharp(\xi) = \phi_W^{-1}(C(g)(\xi,\cdot))$ for all $\xi \in \mathfrak{p}_1 = \mathfrak{g}_0^\circ$. Note that by definition, (4.8) does not depend on the choice of W.
Conversely, let G be a connected and simply connected Lie group and $\mathfrak{g}_0$ an ideal in $\mathfrak{g}$. Choose a Lie algebra 1-cocycle $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$. Then there exists a unique Lie group 1-cocycle $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ integrating δ (see for instance Dufour and Zung (2005)). Define $\mathsf{D}_G \subseteq \mathsf{P}_G$ by (4.8). Then it is easy to check that D_G is a multiplicative Dirac structure on G. We have shown the following classification theorem, that generalizes Drinfel′d's classification of Poisson Lie groups (Drinfel′d (1983)).

Theorem 4.1.25 *Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then we have a one-to-one correspondence*

$$\left\{ \left(\mathfrak{g}_0,\, \delta : \mathfrak{g} \to \bigwedge\nolimits^2 \mathfrak{g}/\mathfrak{g}_0\right) \,\middle|\, \begin{array}{c} \mathfrak{g}_0 \subseteq \mathfrak{g} \textit{ ideal },\\ \delta \textit{ Lie algebra } 1\textit{-cocycle} \end{array} \right\} \overset{1:1}{\leftrightarrow} \left\{ \begin{array}{c} \textit{multiplicative Dirac}\\ \textit{structures on } G \end{array} \right\}.$$

We will see in the next subsection that the *integrable* multiplicative Dirac structures on G correspond via this bijection to the pairs $\left(\mathfrak{g}_0,\, \delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0\right)$ such that the dual $[\cdot,\cdot] := \delta^* : \bigwedge^2 \mathfrak{g}_0^\circ \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}_0^\circ =: \mathfrak{p}_1$.

Example 4.1.26 1. Consider $G = \mathbb{R}^n$. Then any vector subspace $V \subseteq \mathbb{R}^n \simeq T_0\mathbb{R}^n$ is an ideal in $\mathfrak{g} = \mathbb{R}^n$ and any $\delta : \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n/V$ is a cocycle since the cocycle condition is trivial in this particular case. The cocycle C integrating δ is then the unique linear map $C : \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n/V$ with $\mathbf{d}_0 C = \delta$, that is, C is equal to δ if we identify $G = \mathbb{R}^n$ with $\mathfrak{g} = T_0\mathbb{R}^n$ via the exponential map. This shows that each multiplicative Dirac structure on $\mathbb{R}^n$ is given by

$$\mathsf{D}_{\mathbb{R}^n}(r) = \left\{(\delta(r)^\sharp(\xi) + x, \xi) \mid \xi \in V^\circ, x \in V\right\} \subseteq T_r\mathbb{R}^n \times T_r^*\mathbb{R}^n,$$

with V a vector subspace of $\mathbb{R}^n$, $\delta : \mathbb{R}^n \to \bigwedge^2 \mathbb{R}^n/V$ a linear map and $\delta(r)^\sharp$ defined as in (4.8) with a complement W of V in $\mathbb{R}^n$.

2. Let $G \subseteq \mathrm{GL}_n(\mathbb{R})$ be the set of upper triangular matrices with non-vanishing determinant. The Lie algebra $\mathfrak{g}$ of G is then the set of upper triangular matrices. Its commutator $\mathfrak{g}_0 := [\mathfrak{g}, \mathfrak{g}]$ is the set of strictly upper triangular matrices, and integrates to the normal subgroup $N \subseteq G$ of upper triangular matrices with all entries on the diagonal equal to 1. Note that G is not connected. The connected component of the neutral element $e \in G$ is the set of upper triangular matrices with strictly positive diagonal entries.

The quotient $\mathfrak{g}/\mathfrak{g}_0$ is isomorphic to the set of diagonal matrices in $\mathfrak{g}$. Since $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$, it is easy to see that the cocycle condition is satisfied for a linear map $\delta : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ if and only if $\delta|_{\mathfrak{g}_0}$ vanishes, that is, if and only if δ factors to $\bar{\delta} : \mathfrak{g}/\mathfrak{g}_0 \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$. In other words, the dual map $[\cdot\,,\cdot]$ of a Lie algebra 1-cocycle δ has necessarily image in $\mathfrak{p}_1 := \mathfrak{g}_0^\circ$. Hence, if $C : G \to \bigwedge^2 \mathfrak{g}/\mathfrak{g}_0$ is the Lie group 1-cocycle associated to a multiplicative Dirac structure D_G on G, then the dual of its derivative at e has image in $\mathfrak{p}_1$, that is, the bracket on $\mathfrak{p}_1$ defined in Definition 4.1.13 has automatically image in $\mathfrak{p}_1$. Since N is closed in G, this shows by Remark 4.1.5 and Theorem 4.1.18 that any multiplicative Dirac structure on G with $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$ is automatically the pullback to G under $q : G \to G/N$ of the graph of a multiplicative bivector field on G/N.

More generally, this result holds for any Dirac Lie group (G, D_G) such that $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$. $\diamondsuit$

4.1.1 Integrable Dirac Lie groups: induced Lie bialgebra

In continuation of the results in the preceding subsection, we can show that the integrability of (G, D_G) depends only on the properties of the bracket defined in Definition 4.1.13.

Theorem 4.1.27 *The Dirac Lie group (G, D_G) is integrable if and only if the bracket $[\cdot\,,\cdot]$ on $\mathfrak{p}_1 \times \mathfrak{p}_1$ defined in Definition 4.1.13 is a Lie bracket on $\mathfrak{p}_1$.*

In this case, Theorem 4.1.23 implies that the pair $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ is a Lie bialgebra.
Of course, with Theorem 4.1.18, we could show this theorem by considering the Lie bialgebra structure defined on $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ by the multiplicative Poisson structure on $\tilde{G}/\tilde{N}$ (recall Proposition 4.1.7 and Remarks 4.1.5 and 4.1.8), but, in preparation of the more general case of Lie groupoids where this easier method is not possible, we prefer to do that in the setting of Dirac manifolds.
For the proof of the theorem, we will need the following lemmas about the tensor $\mathsf{T}_{\mathsf{D}_G}$ (see Section 1.2 about Dirac manifolds).

Lemma 4.1.28 *Let (G, D_G) be a Dirac Lie group. The tensor $\mathsf{T}_{\mathsf{D}_G}$ is given by*

$$\begin{aligned} 2\,\mathsf{T}_{\mathsf{D}_G}\Big((X_\xi,\xi^l),(X_\eta,\eta^l),(X_\zeta,\zeta^l)\Big) =& X_\zeta(\eta^l(X_\xi)) + X_\xi(\zeta^l(X_\eta)) + X_\eta(\xi^l(X_\zeta)) \\ & - [\zeta,\eta]^l(X_\xi) - [\xi,\zeta]^l(X_\eta) - [\eta,\xi]^l(X_\zeta) \end{aligned} \tag{4.9}$$

for all $\xi, \eta, \zeta \in \mathfrak{p}_1$ and corresponding $X_\xi, X_\eta, X_\zeta \in \mathfrak{X}(G)$, and in particular

$$\mathsf{T}_{\mathsf{D}_G}(e)((x,\xi),(y,\eta),(z,\zeta)) = [\xi,\eta](z) + [\eta,\zeta](x) + [\zeta,\xi](y) \tag{4.10}$$

for any $x, y, z \in \mathfrak{g}_0$.

Proof: Choose $\xi, \eta, \zeta \in \mathfrak{p}_1$ and corresponding vector fields $X_\xi, X_\eta, X_\zeta \in \mathfrak{X}(G)$. Then (1.11) yields

$$\begin{aligned} \mathsf{T}_{\mathsf{D}_G}\Big((X_\xi,\xi^l),(X_\eta,\eta^l),(X_\zeta,\zeta^l)\Big) =& \zeta^l([X_\xi,X_\eta]) + \xi^l([X_\eta,X_\zeta]) + \eta^l([X_\zeta,X_\xi]) \\ & + X_\zeta(\xi^l(X_\eta)) + X_\xi(\eta^l(X_\zeta)) + X_\eta(\zeta^l(X_\xi)). \end{aligned}$$

Using the definition of the bracket, we have also

$$\begin{aligned} & [\zeta,\eta]^l(X_\xi) + [\xi,\zeta]^l(X_\eta) + [\eta,\xi]^l(X_\zeta) \\ =& (\pounds_{X_\zeta}\eta^l - \mathbf{i}_{X_\eta}\mathbf{d}\zeta^l)(X_\xi) + (\pounds_{X_\xi}\zeta^l - \mathbf{i}_{X_\zeta}\mathbf{d}\xi^l)(X_\eta) + (\pounds_{X_\eta}\xi^l - \mathbf{i}_{X_\xi}\mathbf{d}\eta^l)(X_\zeta) \\ =& 2\left(\zeta^l([X_\eta,X_\xi]) + \eta^l([X_\xi,X_\zeta]) + \xi^l([X_\zeta,X_\eta])\right) \\ & + 3\left(X_\zeta(\eta^l(X_\xi)) + X_\xi(\zeta^l(X_\eta)) + X_\eta(\xi^l(X_\zeta))\right) \\ =& -2\,\mathsf{T}_{\mathsf{D}_G}\Big((X_\xi,\xi^l),(X_\eta,\eta^l),(X_\zeta,\zeta^l)\Big) + X_\zeta(\eta^l(X_\xi)) + X_\xi(\zeta^l(X_\eta)) + X_\eta(\xi^l(X_\zeta)). \end{aligned}$$

Evaluated at e, this leads to

$$\begin{aligned} & \mathsf{T}_{\mathsf{D}_G}(e)((X_\xi(e),\xi),(X_\eta(e),\eta),(X_\zeta(e),\zeta)) \\ =& \frac{1}{2}\Big(\mathbf{d}_e(\eta^l(X_\xi))(X_\zeta(e)) + \mathbf{d}_e(\zeta^l(X_\eta))(X_\xi(e)) + \mathbf{d}_e(\xi^l(X_\zeta))(X_\eta(e)) \\ & \qquad - [\zeta,\eta](X_\xi(e)) - [\xi,\zeta](X_\eta(e)) - [\eta,\xi](X_\zeta(e))\Big) \\ =& [\xi,\eta](X_\zeta(e)) + [\eta,\zeta](X_\xi(e)) + [\zeta,\xi](X_\eta(e)). \end{aligned}$$

□

Lemma 4.1.29 *Assume that the bracket on $\mathfrak{p}_1 \times \mathfrak{p}_1$ has image in $\mathfrak{p}_1$. Then,*

$$\mathsf{T}_{\mathsf{D}_G}\Big((X_\xi, \xi^l), (X_\eta, \eta^l), (X_\zeta, \zeta^l)\Big)$$

is independent of the choice of the vector fields $X_\xi, X_\eta, X_\zeta \in \mathfrak{X}(G)$. The tensor $\mathsf{T}_{\mathsf{D}_G}$ defines in this case a tensor $S_{\mathsf{D}_G} \in \Gamma(\bigwedge^3 \mathsf{P}_1{}^*)$ by

$$S_{\mathsf{D}_G}(\xi^l, \eta^l, \zeta^l) = \mathsf{T}_{\mathsf{D}_G}\Big((X_\xi, \xi^l), (X_\eta, \eta^l), (X_\zeta, \zeta^l)\Big)$$

for all $\xi, \eta, \zeta \in \mathfrak{p}_1$ and (G, D_G) is integrable if and only if S_{D_G} vanishes on G.

Proof: Consider (4.9). Since $[\xi, \eta] \in \mathfrak{p}_1$, we have $[\xi, \eta]^l(X_\zeta + Z) = [\xi, \eta]^l(X_\zeta)$ for all $Z \in \Gamma(\mathsf{G}_0)$. Thus, we have only to show that $X_\xi\left(\eta^l(X_\zeta)\right)$ is independent of the choices of X_ξ, X_ζ. Choose Z and $W \in \Gamma(\mathsf{G}_0)$ and compute

$$\begin{aligned}(X_\xi + Z)(\eta^l(X_\zeta + W)) &= X_\xi(\eta^l(X_\zeta)) + Z(\eta^l(X_\zeta)) + (X_\xi + Z)(\eta^l(W)) \\ &= X_\xi(\eta^l(X_\zeta) + Z(\eta^l(X_\zeta))\end{aligned}$$

since $\eta^l(W) = 0$. For any $x \in \mathfrak{g}_0$, we have $x^l(\eta^l(X_\zeta)) = (\mathrm{ad}_x^* \eta)^l(X_\zeta) + \eta^l(\pounds_{x^l} X_\zeta) = 0$ since $\mathrm{ad}_x^* \eta = 0$ and $\pounds_{x^l} X_\zeta \in \Gamma(\mathsf{G}_0)$ by Remark 4.1.22. Since $\Gamma(\mathsf{G}_0)$ is spanned as a $C^\infty(G)$-module by $\{x^l \mid x \in \mathfrak{g}_0\}$, we are done.
Recall that the pairs $(x^l, 0)$ and (X_ξ, ξ^l), for all $x \in \mathfrak{g}_0$ and $\xi \in \mathfrak{p}_1$ span the Dirac bundle D_G. Hence, to prove the integrability of D_G, we have only to show that the Courant bracket of two sections of D_G of this type is a section of D_G. We have already $\left[\left(x_1^l, 0\right), \left(x_2^l, 0\right)\right] \in \Gamma(\mathsf{D}_G)$ for all $x_1, x_2 \in \mathfrak{g}_0$ since $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$ and $\left[\left(x^l, 0\right), \left(X_\xi, \xi^l\right)\right] = \left(\pounds_{x^l} X_\xi, (\mathrm{ad}_x^* \xi)^l\right) = (\pounds_{x^l} X_\xi, 0) \in \Gamma(\mathsf{D}_G)$ by Remark 4.1.22 . Thus, we have only to show that $([X_\xi, X_\eta], [\xi, \eta]^l)$ is a section of D_G for all $\xi, \eta \in \mathfrak{p}_1$. Since $[\xi, \eta] \in \mathfrak{p}_1$, we have $\left\langle (x^l, 0), ([X_\xi, X_\eta], [\xi, \eta]^l) \right\rangle = [\xi, \eta](x) = 0$ for all $x \in \mathfrak{g}_0$. The Dirac structure D_G is thus integrable if and only if $\left\langle ([X_\xi, X_\eta], [\xi, \eta]^l), (X_\zeta, \zeta^l) \right\rangle = 0$ for all $\xi, \eta, \zeta \in \mathfrak{p}_1$, that is, if and only if $S_{\mathsf{D}_G} = 0$. □

Proof (of Theorem 4.1.27): We have to show that the bracket has image in $\mathfrak{p}_1$ and satisfies the Jacobi identity if and only if the Dirac Lie group (G, D_G) is integrable.
Assume first that (G, D_G) is integrable. The tensor $\mathsf{T}_{\mathsf{D}_G}$ vanishes identically on G and

$$\left[(X_\xi, \xi^l), (X_\eta, \eta^l)\right] = \left([X_\xi, X_\eta], \pounds_{X_\xi}\eta^l - \mathbf{i}_{X_\eta}\mathbf{d}\xi^l\right) = \left([X_\xi, X_\eta], [\xi, \eta]^l\right)$$

is a section of D_G for any $\xi, \eta \in \mathfrak{p}_1$. Hence the covector $[\xi, \eta] = \mathbf{d}_e(\eta^l(X_\xi))$ is an element of $\mathfrak{p}_1$ for all $\xi, \eta \in \mathfrak{p}_1$. We get then using (4.9)

$$\begin{aligned}&[\xi, [\zeta, \eta]] + [\eta, [\xi, \zeta]] + [\zeta, [\eta, \xi]] \\ =& \,\mathbf{d}_e\left([\zeta, \eta]^l(X_\xi)\right) + \mathbf{d}_e\left([\xi, \zeta]^l(X_\eta)\right) + \mathbf{d}_e\left([\eta, \xi]^l(X_\zeta)\right) \\ =& -2\mathbf{d}_e\left(\mathsf{T}_{\mathsf{D}_G}\Big((X_\xi, \xi^l), (X_\eta, \eta^l), (X_\zeta, \zeta^l)\Big)\right) + \mathbf{d}_e\left(X_\zeta(\eta^l(X_\xi)) + X_\xi(\zeta^l(X_\eta)) + X_\eta(\xi^l(X_\zeta))\right)\end{aligned}$$

for all $\xi, \zeta, \eta \in \mathfrak{p}_1$. We have for any $x \in \mathfrak{g}$:

$$\begin{aligned} &\mathbf{d}_e \left(X_\zeta(\eta^l(X_\xi))\right)(x) = x^l \left(X_\zeta(\eta^l(X_\xi))\right)(e) \\ \overset{(4.4)}{=} &\mathbf{d}_e \left(\eta^l(X_\xi)\right)(-\operatorname{ad}^*_\zeta x) + \mathbf{d}_e \left((\operatorname{ad}^*_x \eta)^l(X_\xi)\right)(X_\zeta(e)) + X_\zeta \left(\eta^l \left((-\operatorname{ad}^*_\xi x)^l + X_{\operatorname{ad}^*_x \xi}\right)\right)(e) \\ = &\;[\xi, \eta](-\operatorname{ad}^*_\zeta x) + [\xi, \operatorname{ad}^*_x \eta](X_\zeta(e)) + [\operatorname{ad}^*_x \xi, \eta](X_\zeta(e)) = [\zeta, [\xi, \eta]](x). \end{aligned}$$

We have used the equality $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ and $[\xi, \operatorname{ad}^*_x \eta], [\operatorname{ad}^*_x \xi, \eta] \in \mathfrak{p}_1$ as we have seen above. This leads to

$$[\xi, [\zeta, \eta]] + [\eta, [\xi, \zeta]] + [\zeta, [\eta, \xi]] = -\mathbf{d}_e \left(T_{\mathsf{D}_G}\Big((X_\xi, \xi^l), (X_\eta, \eta^l), (X_\zeta, \zeta^l)\Big)\right) = 0.$$

For the converse implication, we know by Lemma 4.1.29 and the hypothesis that the Lie bracket has image in $\mathfrak{p}_1$ that we have only to show the equality $S_{\mathsf{D}_G} = 0$. We compute $\pounds_{x^l}(S_{\mathsf{D}_G})$ for any $x \in \mathfrak{g}$. It is given for any $g \in G$ and $\xi, \eta, \zeta \in \mathfrak{p}_1$ by

$$\begin{aligned} &\left(\pounds_{x^l} S_{\mathsf{D}_G}\right)(g)(\xi^l(g), \eta^l(g), \zeta^l(g)) \\ =&\pounds_{x^l}\left(S_{\mathsf{D}_G}(\xi^l, \eta^l, \zeta^l)\right)(g) - S_{\mathsf{D}_G}((\operatorname{ad}^*_x \xi)^l, \eta^l, \zeta^l)(g) \\ &\quad - S_{\mathsf{D}_G}(\xi^l, (\operatorname{ad}^*_x \eta)^l, \zeta^l)(g) - S_{\mathsf{D}_G}(\xi^l, \eta^l, (\operatorname{ad}^*_x \zeta)^l)(g). \end{aligned}$$

Using the definition of S_{D_G}, (4.9), (4.4) and (4.6), one can show that

$$\begin{aligned} \pounds_{x^l}\left(S_{\mathsf{D}_G}(\xi^l, \eta^l, \zeta^l)\right) =& S_{\mathsf{D}_G}((\operatorname{ad}^*_x \xi)^l, \eta^l, \zeta^l) + S_{\mathsf{D}_G}(\xi^l, (\operatorname{ad}^*_x \eta)^l, \zeta^l) \\ &+ S_{\mathsf{D}_G}(\xi^l, \eta^l, (\operatorname{ad}^*_x \zeta)^l) + \Big([[\zeta, \eta], \xi] + [[\xi, \zeta], \eta] + [[\eta, \xi], \zeta]\Big)(x). \end{aligned}$$

Since $[\cdot, \cdot] : \mathfrak{p}_1 \times \mathfrak{p}_1 \to \mathfrak{p}_1$ satisfies the Jacobi identity by hypothesis, this shows that $\pounds_{x^l} S_{\mathsf{D}_G} = 0$ for all $x \in \mathfrak{g}$ and consequently that S_{D_G} is right invariant. Thus, we get

$$\begin{aligned} S_{\mathsf{D}_G}(g)\left(\xi^r(g), \zeta^r(g), \eta^r(g)\right) =&\; S_{\mathsf{D}_G}(e)(\xi, \zeta, \eta) \\ &\overset{(4.10)}{=} [\zeta, \eta](X_\xi(e)) + [\xi, \zeta](X_\eta(e)) + [\eta, \xi](X_\zeta(e)) = 0 \end{aligned}$$

since $[\cdot, \cdot]$ has image in $\mathfrak{p}_1$ and $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$. Hence, we have shown that S_{D_G} vanishes identically on G and the Dirac Lie group (G, D_G) is consequently integrable by Lemma 4.1.29. $\square$

Remark 4.1.30

1. We can see from the last proof that

$$(\pounds_{x^l} S_{\mathsf{D}_G})(\xi^l, \zeta^l, \eta^l) = [[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta]$$

if the bracket on $\mathfrak{p}_1 \times \mathfrak{p}_1$ has image in $\mathfrak{p}_1$. This shows that $\pounds_{x^l} S_{\mathsf{D}_G}$ is left invariant and we can see using (4.10) and $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ that $S_{\mathsf{D}_G}(e) = 0$. Thus, $S_{\mathsf{D}_G} \in \Gamma(\bigwedge^3 \mathsf{P}_1{}^*)$ is multiplicative (see for instance Lu (1990)).

2. Note that if (G, D_G) is integrable, then $[\xi, \eta] \in \mathfrak{p}_1$ and we get $[X_\xi, X_\eta] \in X_{[\xi,\eta]} + \Gamma(\mathsf{G}_0)$ for all $\xi, \eta \in \mathfrak{p}_1$ and any vector field $X_{[\xi,\eta]}$ such that $\left(X_{[\xi,\eta]}, [\xi, \eta]^l\right) \in \Gamma(\mathsf{D}_G)$. $\triangle$

Recall that since $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{g}_0$ has a canonical Lie algebra structure given by $[x+\mathfrak{g}_0, y+\mathfrak{g}_0] = [x,y]+\mathfrak{g}_0$ for all $x,y\in\mathfrak{g}$. Recall from Theorems 4.1.23 and 4.1.27 that if (G,D_G) is integrable, then the pair $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ is a Lie bialgebra. The Lie algebra structures on $\mathfrak{g}/\mathfrak{g}_0$ and $\mathfrak{p}_1$ induce then a double Lie algebra structure on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$, with bracket given by

$$[(x+\mathfrak{g}_0,\xi),(y+\mathfrak{g}_0,\eta)] = \left([x,y]-\mathrm{ad}^*_\eta x+\mathrm{ad}^*_\xi y+\mathfrak{g}_0, [\xi,\eta]+\mathrm{ad}^*_x\eta-\mathrm{ad}^*_y\xi\right), \qquad (4.11)$$

for all $x,y\in\mathfrak{g}$ and $\xi,\eta\in\mathfrak{p}_1$ (see for instance Chapter 2 or Lu and Weinstein (1990)).

4.1.2 The action of G on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$

Theorem 4.1.31 *Let (G,D_G) be a Dirac Lie group. Define*

$$A: G\times(\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1)\to\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$$

by

$$A(g,(x+\mathfrak{g}_0,\xi)) = \left(\mathrm{Ad}_g x + T_gR_{g^{-1}}X_\xi(g)+\mathfrak{g}_0, \mathrm{Ad}^*_{g^{-1}}\xi\right)$$

for all $g\in G$, where $X_\xi\in\mathfrak{X}(G)$ is a vector field such that $(X_\xi,\xi^l)\in\Gamma(\mathsf{D}_G)$. The map A is a well-defined action of G on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$.

Proof: We prove first the fact that the action is well-defined, that is, that it doesn't depend on the choices of x and X_ξ. Choose $x'\in\mathfrak{g}$ such that $x'+\mathfrak{g}_0 = x+\mathfrak{g}_0$. Then $x'-x =: x_0\in\mathfrak{g}_0$ and

$$\mathrm{Ad}_g x' = \mathrm{Ad}_g(x+x_0) = \mathrm{Ad}_g x+\mathrm{Ad}_g x_0 \in \mathrm{Ad}_g x+\mathfrak{g}_0$$

for all $g\in G$, since $\mathfrak{g}_0$ is Ad_g-invariant for all $g\in G$.
Next, if X_ξ and $X'_\xi\in\mathfrak{X}(G)$ are such that (X_ξ,ξ^l) and $(X'_\xi,\xi^l)\in\Gamma(\mathsf{D}_G)$, the difference $X'_\xi - X_\xi$ is a section of G_0 and hence we can write $(X'_\xi-X_\xi)(g) = T_eR_g y_0$ with $y_0\in\mathfrak{g}_0$. This leads to

$$T_gR_{g^{-1}}X'_\xi(g) = T_gR_{g^{-1}}X_\xi(g)+y_0\in T_gR_{g^{-1}}X_\xi(g)+\mathfrak{g}_0.$$

The map A is hence shown to be well-defined. We show next that A is an action of G on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$. We have to show that

$$A\left(g', A(g,(x+\mathfrak{g}_0,\xi))\right) = A(g'g,(x+\mathfrak{g}_0,\xi))$$

for all $g,g'\in G$, $x\in\mathfrak{g}$ and $\xi\in\mathfrak{p}_1$.
We have with the same arguments as above

$$\begin{aligned}
&A_{g'g}(x+\mathfrak{g}_0,\xi)\\
=\ &\Big(\mathrm{Ad}_{g'}(\mathrm{Ad}_g x)+T_{g'g}R_{g^{-1}g'^{-1}}X_\xi(g'g)+\mathfrak{g}_0, \mathrm{Ad}^*_{g'^{-1}}(\mathrm{Ad}^*_{g^{-1}}\xi)\Big)\\
\overset{(4.1)}{=}\ &\Big(\mathrm{Ad}_{g'}(\mathrm{Ad}_g x)+T_{g'g}R_{g^{-1}g'^{-1}}(T_gL_{g'}X_\xi(g)+T_{g'}R_gX_{\mathrm{Ad}^*_{g^{-1}}\xi}(g'))+\mathfrak{g}_0, \mathrm{Ad}^*_{g'^{-1}}(\mathrm{Ad}^*_{g^{-1}}\xi)\Big)\\
=\ &\Big(\mathrm{Ad}_{g'}(\mathrm{Ad}_g x+T_gR_{g^{-1}}X_\xi(g))+T_{g'}R_{g'^{-1}}X_{\mathrm{Ad}^*_{g^{-1}}\xi}(g')+\mathfrak{g}_0, \mathrm{Ad}^*_{g'^{-1}}(\mathrm{Ad}^*_{g^{-1}}\xi)\Big)\\
=\ &A_{g'}\left(\mathrm{Ad}_g x+T_gR_{g^{-1}}X_\xi(g)+\mathfrak{g}_0, \mathrm{Ad}^*_{g^{-1}}\xi\right) = A_{g'}\left(A_g(x+\mathfrak{g}_0,\xi)\right).
\end{aligned}$$

□

Remark 4.1.32 Assume that the bracket on $\mathfrak{p}_1 \times \mathfrak{p}_1$ has image in $\mathfrak{p}_1$.
We have $N \subseteq G_{(x+\mathfrak{g}_0,\xi)}$, where $G_{(x+\mathfrak{g}_0,\xi)}$ is the isotropy group of $(x+\mathfrak{g}_0,\xi) \in \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.
Indeed, for $n \in N$, we have $\mathrm{Ad}_n x \in x + \mathfrak{g}_0$, for all $x \in \mathfrak{g}$ and hence $\mathrm{Ad}^*_{n^{-1}} \xi = \xi$ for all $\xi \in \mathfrak{p}_1$. The proof of this is easy, see also Ortega and Ratiu (2004), Lemma 2.1.13. Since $\left(X_\xi, \xi^l\right) \in \Gamma(\mathsf{D}_G)$ and $n \in N$, we know by Theorem 4.1.18 that $\left(R_n^* X_\xi, R_n^* \xi^l\right) \in \Gamma(\mathsf{D}_G)$. Hence, we have $R_n^* X_\xi(e) \in \mathfrak{g}_0$ because $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$, that is, $T_n R_{n^{-1}} X_\xi(n) \in \mathfrak{g}_0$. Using this and $\mathrm{Ad}_n x \in x + \mathfrak{g}_0$, we get $\mathrm{Ad}_n x + T_n R_{n^{-1}} X_\xi(n) \in x + \mathfrak{g}_0$.
Thus, we get a well-defined action $\bar{A}$ of G/N on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$, that is given by $\bar{A}(gN,(x+\mathfrak{g}_0,\xi)) = A(g,(x+\mathfrak{g}_0,\xi))$ for all $g \in G$. △

If (G,D_G) is integrable and N is closed in G, the next theorem shows that $\bar{A}$ is the adjoint action of G/N on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ integrating the adjoint action of $\mathfrak{g}/\mathfrak{g}_0$ defined by the bracket on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.

Theorem 4.1.33 *Assume that (G,D_G) is an integrable Dirac Lie group. The adjoint action of $\mathfrak{g}/\mathfrak{g}_0 \simeq \mathfrak{g}/\mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ "integrates" to the action A of G on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ in the sense that*

$$\left.\frac{d}{dt}\right|_{t=0} A\left(\exp(ty),(x+\mathfrak{g}_0,\xi)\right) = [(y+\mathfrak{g}_0,0),(x+\mathfrak{g}_0,\xi)]$$

for all $y \in \mathfrak{g}$ and $(x+\mathfrak{g}_0,\xi) \in \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.

Proof: Choose $x, y \in \mathfrak{g}$ and $\xi \in \mathfrak{p}_1$ and compute

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} A\left(\exp(ty),(x+\mathfrak{g}_0,\xi)\right)\\
=&\left.\frac{d}{dt}\right|_{t=0} \left(\mathrm{Ad}_{\exp(ty)} x + T_{\exp(ty)} R_{\exp(-ty)} X_\xi(\exp(ty)) + \mathfrak{g}_0, \mathrm{Ad}^*_{\exp(-ty)} \xi\right)\\
=&\left([y,x] + \pounds_{y^l} X_\xi(e) + \mathfrak{g}_0, \mathrm{ad}^*_y \xi\right) \overset{(4.4)}{=} \left([y,x] - \mathrm{ad}^*_\xi y + \mathfrak{g}_0, \mathrm{ad}^*_y \xi\right)\\
=&\left[(y+\mathfrak{g}_0,0),(x+\mathfrak{g}_0,\xi)\right].
\end{aligned}$$

□

4.2 Dirac homogeneous spaces

4.2.1 Properties of Dirac homogeneous spaces

Let (G,D_G) be a Dirac Lie group and H a closed subgroup of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ the Lie algebra of H. Recall from Proposition 2.3.4 that $(G/H,\mathsf{D}_{G/H})$ is a Dirac homogeneous space of (G,D_G) if the left action σ of G on G/H is a forward Dirac map.

We show first that the characteristic distribution of a D_G-homogeneous Dirac structure is left invariant. In order to simplify the notation, we write G_0 and P_1, respectively, for both the distributions, respectively codistributions, defined by D_G on G and by $\mathsf{D}_{G/H}$ on G/H. It will always be clear from the context which object is referred to.

Lemma 4.2.1 *Let (G, D_G) be a Dirac Lie group. Let G/H be a homogeneous space of G endowed with a D_G-homogeneous Dirac structure $\mathsf{D}_{G/H}$. Then the codistribution P_1 has constant rank on G/H. Consequently,the distribution G_0 defined by $\mathsf{D}_{G/H}$ on G/H is also a subbundle of $T(G/H)$. More explicitly, the distribution G_0 and the codistribution P_1 are given by*

$$\mathsf{G}_0(gH) = T_{eH}\sigma_g \mathsf{G}_0(eH) \quad \text{and} \quad \mathsf{P}_1(gH) = (T_{gH}\sigma_{g^{-1}})^* \mathsf{P}_1(eH)$$

for all $gH \in G/H$.

PROOF: We show that $\mathsf{P}_1(gH) = (T_{gH}\sigma_{g^{-1}})^*\mathsf{P}_1(eH)$ for all $g \in G$: choose first $\bar{\xi} \in \mathsf{P}_1(eH)$, then there exists $\bar{x} \in T_{eH}(G/H)$ such that $(\bar{x}, \bar{\xi}) \in \mathsf{D}_{G/H}(eH)$ and hence $w_{g^{-1}} \in T_{g^{-1}}G$ and $u_{gH} \in T_{gH}(G/H)$ such that

$$(w_{g^{-1}}, (T_{g^{-1}}(q \circ R_g))^* \bar{\xi}) \in \mathsf{D}_G(g^{-1}),$$

$$(u_{gH}, (T_{gH}\sigma_{g^{-1}})^* \bar{\xi}) \in \mathsf{D}_{G/H}(gH)$$

and

$$(T_e q \circ T_{g^{-1}} R_g)(w_{g^{-1}}) + T_{gH}\sigma_{g^{-1}}(u_{gH}) = \bar{x}.$$

This yields immediately

$$(T_{gH}\sigma_{g^{-1}})^* \mathsf{P}_1(eH) \subseteq \mathsf{P}_1(gH).$$

The other inclusion is a direct consequence of (2.2). Thus, the codistribution P_1 is a subbundle of $T^*(G/H)$ and its annihilator is equal to G_0, which is consequently given by $\mathsf{G}_0(gH) = T_{eH}\sigma_g \mathsf{G}_0(eH)$ for all $g \in G$. □

The fact that σ is a forward Dirac map yields immediately: for all $h \in H$ and $(v_{eH}, \alpha_{eH}) \in \mathsf{D}_{G/H}(eH) = \mathsf{D}_{G/H}(hH)$, there exist $(w_h, \beta_h) \in \mathsf{D}_G(h)$ and $(u_{eH}, \gamma_{eH}) \in \mathsf{D}_{G/H}(eH)$ such that $\beta_h = (T_h q)^*(\alpha_{eH})$, $\gamma_{eH} = (T_{eH}\sigma_h)^*(\alpha_{eH})$, and $v_{eH} = T_h q w_h + T_{eH}\sigma_h u_{eH}$.

Definition 4.2.2 *Let (G, D_G) be a Dirac Lie group and H a closed connected Lie subgroup of G. We say that a subspace $S \subseteq \mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ has property $(*)$ if for all $h \in H$ and $(\bar{x}, \bar{\xi}) \in S$, there exist $(w_h, \beta_h) \in \mathsf{D}_G(h)$ and $(\bar{y}, \bar{\eta}) \in S$ such that $\beta_h = (T_h q)^*(\bar{\xi})$, $\bar{\eta} = (T_{eH}\sigma_h)^*(\bar{\xi})$, and $\bar{x} = T_h q w_h + T_{eH}\sigma_h \bar{y}$.*

By the considerations above, if $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of the Dirac Lie group (G, D_G), then $\mathsf{D}_{G/H}(e)$ has property $(*)$. This leads to the following lemma.

Lemma 4.2.3 *Let $\mathfrak{L}$ be a Dirac subspace of $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ with the property $(*)$. Then the inclusions $(T_e q)^* \bar{\mathfrak{p}_1} \subseteq \mathfrak{p}_1$ and $T_e q \mathfrak{g}_0 \subseteq \bar{\mathfrak{g}_0}$ hold, where $\bar{\mathfrak{p}_1} \subseteq (\mathfrak{g}/\mathfrak{h})^*$ and $\bar{\mathfrak{g}_0} \subseteq \mathfrak{g}/\mathfrak{h}$ are the subspaces defined by $\mathfrak{L}$.*

PROOF: Choose $\alpha \in \bar{\mathfrak{p}_1}$, then there exists $v \in \mathfrak{g}/\mathfrak{h}$ such that $(v, \alpha) \in \mathfrak{L}$. By $(*)$, there exist $(w_e, \beta_e) \in \mathsf{D}_G(e)$ and $(u_{eH}, \gamma_{eH}) \in \mathfrak{L}$ such that $\beta_e = (T_e q)^*\alpha$, $\gamma_{eH} = (T_{eH}\sigma_h)^*\alpha$, and $v = T_e q w_e + u_{eH}$. The covector $\beta_e = (T_e q)^*\alpha$ is an element of $\mathfrak{p}_1$. The second inclusion is a consequence of the first. □

We call in the following $\mathfrak{D} := (T_e q)^*\mathfrak{L} \subseteq \mathfrak{g}\times\mathfrak{g}^*$ the pullback Dirac subspace of $\mathfrak{L} \subseteq \mathfrak{g}/\mathfrak{h}\times(\mathfrak{g}/\mathfrak{h})^*$ with Property $(*)$, that is

$$\mathfrak{D} = \left\{(x,\xi) \mid \exists\bar{\xi} \in (\mathfrak{g}/\mathfrak{h})^* \text{ such that } (T_e q)^*\bar{\xi} = \xi \text{ and } (T_e q x, \bar{\xi}) \in \mathfrak{L}\right\}.$$

Lemma 4.2.4 *Let $\mathfrak{p}_1' \subseteq \mathfrak{g}^*$ and $\mathfrak{g}_0' \subseteq \mathfrak{g}$ be the vector subspaces associated to the Dirac subspace $\mathfrak{D} \subseteq \mathfrak{g}\times\mathfrak{g}^*$. Then we have the inclusions*

$$\mathfrak{g}_0 + \mathfrak{h} \subseteq \mathfrak{g}_0' \quad \textit{and} \quad \mathfrak{p}_1' \subseteq \mathfrak{p}_1 \cap \mathfrak{h}^\circ.$$

Hence, we have $\mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g}\times\mathfrak{p}_1$ and the vector space $\bar{\mathfrak{D}} := \mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ can be seen as a subset of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$.

Proof: We know from Lemma 4.2.3 that $T_e q\mathfrak{g}_0 \subseteq \bar{\mathfrak{g}_0}$ and $(T_e q)^*\bar{\mathfrak{p}_1} \subseteq \mathfrak{p}_1$. The inclusions here follow directly from this and the definition of $\mathfrak{D}$. □

4.2.2 The pullback to G of a homogeneous Dirac structure

Consider a Dirac Lie group (G, D_G) and let $\mathfrak{D}$ be a Dirac subspace of $\mathfrak{g}\times\mathfrak{g}^*$ satisfying

$$\mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g}\times\mathfrak{p}_1. \qquad (**)$$

We denote by $\mathfrak{g}_0' \subseteq \mathfrak{g}$ and $\mathfrak{p}_1' \subseteq \mathfrak{g}^*$ the subspaces defined by $\mathfrak{D}$. We have $\mathfrak{g}_0 \subseteq \mathfrak{g}_0'$ and $\mathfrak{p}_1' \subseteq \mathfrak{p}_1$ and we can define the generalized distribution $\mathsf{D} \subseteq \mathsf{P}_G$ by

$$\mathsf{D}(g) := \left\{ \left(x^l(g) + v_g, \xi^l(g)\right) \in \mathsf{P}_G(g) \middle| (x,\xi) \in \mathfrak{D} \text{ and } \left(v_g, \xi^l(g)\right) \in \mathsf{D}_G(g)\right\} \qquad (4.12)$$

for all $g \in G$. Note that D is smooth since it is spanned by the smooth sections $(X_\xi + x^l, \xi^l)$, with $(x,\xi) \in \mathfrak{D}$ and $X_\xi \in \mathfrak{X}(G)$ such that $(X_\xi, \xi^l) \in \Gamma(\mathsf{D}_G)$.

Proposition 4.2.5 *Let (G, D_G) be a Dirac Lie group and $\mathfrak{D}$ a Dirac subspace of $\mathfrak{g}\times\mathfrak{g}^*$ satisfying $(**)$. The induced subset $\mathsf{D} \subseteq TG \times_G T^*G$ as in (4.12) is a Dirac structure on G.*

The construction of the Dirac structure D is inspired by Diatta and Medina (1999). Note that the codistribution $\mathsf{P}_1{}'$ induced by D on G is equal to $\mathsf{P}_1{}' = {\mathfrak{p}_1'}^l$ by definition and consequently the distribution $\mathsf{G}_0{}'$ induced by D on G is equal to $\mathsf{G}_0{}'$. We have $\mathsf{G}_0 \subseteq \mathsf{G}_0{}'$ and $\mathsf{P}_1{}' \subseteq \mathsf{P}_1$.

Proof: Choose $g \in G$ and $(x^l(g) + v_g, \xi^l(g))$, $(y^l(g) + w_g, \eta^l(g)) \in \mathsf{D}(g)$, i.e., with $(x,\xi), (y,\eta) \in \mathfrak{D}$ and $(v_g, \xi^l(g)), (w_g, \eta^l(g)) \in \mathsf{D}_G(g)$. We have

$$\begin{aligned}&\left\langle \left(x^l(g) + v_g, \xi^l(g)\right), \left(y^l(g) + w_g, \eta^l(g)\right)\right\rangle \\ &= \left\langle \left(v_g, \xi^l(g)\right), \left(w_g, \eta^l(g)\right)\right\rangle + \langle (x,\xi), (y,\eta)\rangle = 0,\end{aligned}$$

since $\mathsf{D}_G = \mathsf{D}_G^\perp$ and $\mathfrak{D} = \mathfrak{D}^\perp$. This shows $\mathsf{D}(g) \subseteq \mathsf{D}(g)^\perp$.

Conversely, let $(u_g, \gamma_g) \in \mathsf{P}_G(g)$ be an element of $\mathsf{D}(g)^\perp$. Then we have $\gamma_g(x^l(g)) = \langle (x^l(g), 0), (u_g, \gamma_g)\rangle = 0$ for all $x \in \mathfrak{g}_0'$, and γ_g is thus an element of $\mathsf{P}_1{}'(g) \subseteq \mathsf{P}_1(g)$. Choose $v_g \in T_gG$ such that $(v_g, \gamma_g) \in \mathsf{D}_G(g)$ and set $w_g = u_g - v_g$. Then we get for any $(x, \xi) \in \mathfrak{D}$:

$$\begin{aligned}\langle (w_g, \gamma_g), (x^l, \xi^l)(g)\rangle &= \gamma_g(x^l(g)) + \xi^l(g)(w_g)\\ &= \gamma_g(x^l(g)) + \xi^l(g)(w_g) + \xi^l(g)(v_g) + \gamma_g(X_\xi(g))\end{aligned}$$

where $X_\xi \in \mathfrak{X}(G)$ is such that (X_ξ, ξ^l) is a section of D_G that is defined at g. We have used the identity $\xi^l(v_g) + \gamma_g(X_\xi(g)) = 0$ which holds because $(X_\xi, \xi^l) \in \Gamma(\mathsf{D}_G)$ and $(v_g, \gamma_g) \in \mathsf{D}_G(g)$. But this equals

$$\langle (v_g + w_g, \gamma_g), (X_\xi + x^l, \xi^l)(g)\rangle = \langle (u_g, \gamma_g), (X_\xi + x^l, \xi^l)(g)\rangle = 0,$$

since $(X_\xi + x^l, \xi^l)$ is by definition a section of D and $(u_g, \gamma_g) \in \mathsf{D}(g)^\perp$. This shows that $(w_g, \gamma_g) \in \mathfrak{D}^l(g)^\perp = \mathfrak{D}^l(g)$. Hence, we have shown $(u_g, \gamma_g) = (w_g + v_g, \gamma_g) \in \mathsf{D}(g)$. $\square$

Remark 4.2.6 If (Z, α) is a section of D, then we have $(Z, \alpha) = (X_\alpha + Y_\alpha, \alpha)$ with X_α and $Y_\alpha \in \mathfrak{X}(G)$ such that $(X_\alpha, \alpha) \in \Gamma(\mathsf{D}_G)$ and $(Y_\alpha, \alpha) \in \Gamma(\mathfrak{D}^l)$. Hence, we have

$$(Z(e), \alpha(e)) = (Y_\alpha(e), \alpha(e)) + (X_\alpha(e), 0) \in \mathfrak{D} + (\mathfrak{g}_0 \times \{0\}) = \mathfrak{D}$$

because $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ and $\mathfrak{g}_0 \times \{0\} \subseteq \mathfrak{D}$. Since $\mathfrak{D}$ and $\mathsf{D}(e)$ are Lagrangian, this shows that $\mathfrak{D} = \mathsf{D}(e)$. $\triangle$

Let now H be a closed subgroup of G with Lie algebra $\mathfrak{h}$, and denote by $q : G \to G/H$ the smooth surjective submersion. Let $\mathfrak{D}_{G/H} \subseteq \mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ be a Dirac subspace, such that $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ defined by $\mathfrak{D} = (T_eq)^*\mathfrak{D}_{G/H}$ satisfies $(**)$. Recall that property $(*)$ has been defined in Definition 4.2.2.

Theorem 4.2.7 *The following are equivalent for $\mathfrak{D}_{G/H}$ and $\mathfrak{D}$ as above and D as in* (4.12).

1. *$\mathfrak{D}_{G/H}$ has property $(*)$*
2. *$A_h\left(\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})\right) \subseteq \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ for all $h \in H$*
3. *D is invariant under the right action of H on G*
4. *(G, D) projects under q to a Dirac homogeneous space $(G/H, \mathsf{D}_{G/H})$ such that $\mathsf{D}_{G/H}(eH) = \mathfrak{D}_{G/H}$.*

Proof: Assume first that $\mathfrak{D}_{G/H}$ satisfies $(*)$ and choose $(x + \mathfrak{g}_0, \xi) \in \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$. We have then $(x, \xi) \in \mathfrak{D}$ and there exists $\bar{\xi} \in (\mathfrak{g}/\mathfrak{h})^*$ such that $(T_eq)^*\bar{\xi} = \xi$ and $(T_eqx, \bar{\xi}) \in \mathfrak{D}_{G/H}$. By $(*)$ for $(T_eqx, \bar{\xi}) \in \mathfrak{D}_{G/H}$ and $h^{-1} \in H$, there exists $(w_{h^{-1}}, \beta_{h^{-1}}) \in \mathsf{D}_G(h^{-1})$

and $(\bar y,\bar\eta)\in\mathfrak{D}_{G/H}$ such that $\beta_{h^{-1}}=(T_{h^{-1}}q)^*\bar\xi$, $\bar\eta=(T_{eH}\sigma_{h^{-1}})^*\bar\xi$ and $T_eqx=T_{eH}\sigma_{h^{-1}}\bar y+T_{h^{-1}}qw_{h^{-1}}$. We compute

$$(T_eL_{h^{-1}})^*\beta_{h^{-1}}=(T_{h^{-1}}q\circ T_eL_{h^{-1}})^*\bar\xi=(T_eq\circ T_{h^{-1}}R_h\circ T_eL_{h^{-1}})^*\bar\xi=\mathrm{Ad}_{h^{-1}}^*\xi$$

and also

$$\mathrm{Ad}_{h^{-1}}^*\xi=(T_{h^{-1}}q\circ T_eL_{h^{-1}})^*\bar\xi=(T_{eH}\sigma_{h^{-1}}\circ T_eq)^*\bar\xi=(T_eq)^*\bar\eta=:\eta.$$

This yields also $\eta\in\mathfrak{p}_1$, and there exists a vector field $X_\eta\in\mathfrak{X}(G)$ such that $(X_\eta,\eta^l)\in\Gamma(\mathsf{D}_G)$ and $X_\eta(h^{-1})=w_{h^{-1}}$. We have

$$X_\eta(e)=X_{\mathrm{Ad}_{h^{-1}}^*\xi}(hh^{-1})\overset{(4.1)}{=}T_{h^{-1}}L_hX_{\mathrm{Ad}_{h^{-1}}^*\xi}(h^{-1})+T_hR_{h^{-1}}X_\xi(h)+z$$

with $z\in\mathfrak{g}_0$. We get

$$\begin{aligned}\bar y&=T_{eH}\sigma_hT_eqx-T_{eH}\sigma_hT_{h^{-1}}qw_{h^{-1}}=T_hqT_eL_hx-T_eqT_{h^{-1}}L_hX_\eta(h^{-1})\\&=T_eq\left(T_hR_{h^{-1}}T_eL_hx+T_hR_{h^{-1}}X_\xi(h)-X_{\mathrm{Ad}_{h^{-1}}^*\xi}(e)+z\right).\end{aligned}$$

Since $(\bar y,\bar\eta)$ is an element of $\mathfrak{D}_{G/H}$, we have $(y,\eta)\in\mathfrak{D}$ for any $y\in\mathfrak{g}$ such that $T_eqy=\bar y$. Hence, the pair $(\mathrm{Ad}_h x+T_hR_{h^{-1}}X_\xi(h)-X_{\mathrm{Ad}_{h^{-1}}^*\xi}(e)+z,\eta)=(\mathrm{Ad}_h x+T_hR_{h^{-1}}X_\xi(h)-X_{\mathrm{Ad}_{h^{-1}}^*\xi}(e)+z,\mathrm{Ad}_{h^{-1}}^*\xi)$ is an element of $\mathfrak{D}$. With $X_{\mathrm{Ad}_{h^{-1}}^*\xi}(e)+z\in\mathfrak{g}_0$, this shows that $A(h,(x+\mathfrak{g}_0,\xi))=(\mathrm{Ad}_h x+T_hR_{h^{-1}}X_\xi(h)+\mathfrak{g}_0,\mathrm{Ad}_{h^{-1}}^*\xi)\in\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$.

Assume next that $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ is H-invariant and choose a spanning section $\left(X_\xi+x^l,\xi^l\right)$ of D, hence with $(x,\xi)\in\mathfrak{D}$ and $X_\xi\in\mathfrak{X}(G)$ a vector field satisfying $\left(X_\xi,\xi^l\right)\in\Gamma(\mathsf{D}_G)$. Since $(x+\mathfrak{g}_0,\xi)$ is an element of $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ we get for an arbitrary $h\in H$

$$A_h(x+\mathfrak{g}_0,\xi)=(\mathrm{Ad}_h x+T_hR_{h^{-1}}X_\xi(h)+\mathfrak{g}_0,\mathrm{Ad}_{h^{-1}}^*\xi)\in\mathfrak{D}/(\mathfrak{g}_0\times\{0\}),$$

and hence

$$(\mathrm{Ad}_h x+T_hR_{h^{-1}}X_\xi(h),\mathrm{Ad}_{h^{-1}}^*\xi)\in\mathfrak{D}.\tag{4.13}$$

Then we can compute for $g\in G$:

$$\begin{aligned}&\left(R_h^*(X_\xi+x^l)(g),R_h^*(\xi^l)(g)\right)\\&=(T_{gh}R_{h^{-1}}T_eL_{gh}x+T_{gh}R_{h^{-1}}X_\xi(gh),\xi\circ T_{gh}L_{h^{-1}g^{-1}}\circ T_gR_h)\\&\overset{(4.1)}{=}\left(T_eL_g\,\mathrm{Ad}_h x+T_{gh}R_{h^{-1}}(T_gR_hX_{\mathrm{Ad}_{h^{-1}}^*\xi}(g)+T_hL_gX_\xi(h)+T_gR_hT_eL_gz),(\mathrm{Ad}_{h^{-1}}^*\xi)^l(g)\right)\end{aligned}$$

for some $z\in\mathfrak{g}_0$. Thus, we get

$$\begin{aligned}&\left(R_h^*(X_\xi+x^l)(g),R_h^*(\xi^l)(g)\right)\\&=\left((\mathrm{Ad}_h x)^l(g)+X_{\mathrm{Ad}_{h^{-1}}^*\xi}(g)+\left(T_hR_{h^{-1}}X_\xi(h)\right)^l(g)+z^l(g),(\mathrm{Ad}_{h^{-1}}^*\xi)^l(g)\right)\\&=\left(\left(\mathrm{Ad}_h x+T_hR_{h^{-1}}X_\xi(h)\right)^l(g)+X_{\mathrm{Ad}_{h^{-1}}^*\xi}(g),(\mathrm{Ad}_{h^{-1}}^*\xi)^l(g)\right)+(z^l,0)(g).\end{aligned}$$

By the definition of D and (4.13), we get consequently that

$$\big(R_h^*(X_\xi + x^l)(g), R_h^*(\xi^l)(g)\big) \in \mathsf{D}(g)$$

(note that z^l is a section of $\mathsf{G}_0 \subseteq \mathsf{G}_0'$), and hence that the right action of H on (G, D) is canonical.

Assume that the right action of H on (G, D) is canonical. We use Theorem 1.2.2. The vertical space $\mathcal{V}_H$ of the right action of H on G is $\mathcal{V}_H = \mathfrak{h}^l \subseteq \mathsf{G}_0{}'$ since by definition of $\mathfrak{D}$ and $\mathfrak{g}_0'$, we have $\mathfrak{h} \subseteq \mathfrak{g}_0'$. Thus, we have $\mathsf{P}_1{}' \subseteq \mathcal{V}_H{}^\circ$ and hence, $\mathsf{D} \cap \mathcal{K}_H^\perp = \mathsf{D}$. The reduced Dirac structure $\mathsf{D}_{G/H}$ is then given by

$$\mathsf{D}_{G/H} := q(\mathsf{D}) = \left(\frac{\mathsf{D} + \mathcal{K}_H}{\mathcal{K}_H}\right) \Big/ H = \frac{\mathsf{D}}{\mathcal{K}_H} \Big/ H.$$

We have to show that this defines a Dirac homogeneous space of (G, D_G). Note first that if $(\bar{x}, \bar{\xi}) \in \mathsf{D}_{G/H}(eH)$, then there exists $(x, \xi) \in \mathsf{D}(e) = \mathfrak{D}$ (see Remark 4.2.6) such that $T_e q x = \bar{x}$ and $(T_e q)^* \bar{\xi} = \xi$. But then $(\bar{x}, \bar{\xi})$ is an element of $\mathfrak{D}_{G/H}$. The other inclusion can be shown in the same manner and we get $\mathsf{D}_{G/H}(eH) = \mathfrak{D}_{G/H}$. Choose then $gH \in G/H$ and $(\bar{v}, \bar{\alpha}) \in \mathsf{D}_{G/H}(gH)$, that is, $(\bar{v}, \bar{\alpha}) \in T_{gH}(G/H) \times T_{gH}(G/H)^*$ such that there exists $v \in T_g G$ with $T_g q v = \bar{v}$ and $(v, (T_g q)^* \bar{\alpha}) \in \mathsf{D}(g)$. Then we can write v as a sum $v = w + u$ with $w, u \in T_g G$ such that $(w, (T_g q)^* \bar{\alpha}) \in \mathsf{D}_G(g)$ and $(u, (T_g q)^* \bar{\alpha}) \in \mathfrak{D}^l(g)$, i.e., $(T_g L_{g^{-1}} u, (T_e L_g)^* \circ (T_g q)^* \bar{\alpha}) \in \mathfrak{D}$. Since $(T_e L_g)^* \circ (T_g q)^* \bar{\alpha} = (T_e q)^* (T_{eH} \sigma_g)^* \bar{\alpha}$, we get $(T_e q T_g L_{g^{-1}} u, (T_{eH} \sigma_g)^* \bar{\alpha}) \in \mathfrak{D}_{G/H} = \mathsf{D}_{G/H}(eH)$. Set $\bar{u} := T_e q T_g L_{g^{-1}} u$, then we have $T_{eH} \sigma_g \bar{u} = T_g q u$ and hence

$$\bar{v} = T_g q w + T_g q u = T_g q w + T_{eH} \sigma_g \bar{u}.$$

The proof of the last implication $4 \Rightarrow 1$ is given by (2.2). $\square$

We have immediately the following corollary, which, together with the preceding theorem, classifies the Dirac structures on G/H that make $(G/H, \mathsf{D}_{G/H})$ a Dirac homogeneous space of (G, D_G).

Corollary 4.2.8 *Let (G, D_G) be a Dirac Lie group, H a closed Lie subgroup of G and $(G/H, \mathsf{D}_{G/H})$ a Dirac homogeneous space of (G, D_G). The Dirac structure $\mathsf{D}_{G/H}$ on G/H is then uniquely determined by $\mathsf{D}_{G/H}(eH)$ and (G, D_G).*

Proof: Since $(G/H, \mathsf{D}_{G/H})$ is a Dirac homogeneous space of (G, D_G), the subspace $\mathsf{D}_{G/H}(eH)$ satisfies $(*)$ by (2.2), and $\mathfrak{D} = T_e q^* \mathsf{D}_{G/H}(eH)$ satisfies $(**)$ by Lemma 4.2.4. Define D as above. Then, by the preceding theorem, we get that D is right H-invariant and projects under q to a Dirac structure $q(\mathsf{D})$. It is easy to check that $q(\mathsf{D}) = \mathsf{D}_{G/H}$. $\square$

Remark 4.2.9 1. Since the vertical space of the right action of H on (G, D) denoted here by $\mathcal{V}_H$ is equal to $\mathfrak{h}^l$ and hence contained in $\mathsf{G}_0{}'$, the Dirac structure D is the backward Dirac image of $\mathsf{D}_{G/H}$ under q (see Section 1.2).

2. The quotient $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is easily shown to be a Lagrangian subspace of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ if and only if $\mathfrak{D}_{G/H}$ is a Lagrangian subspace of $\mathfrak{g}/\mathfrak{h} \times (\mathfrak{g}/\mathfrak{h})^*$ satisfying $(**)$. △

Corollary 4.2.10 *The Dirac homogeneous space* $(G/H, \mathsf{D}_{G/H})$ *is integrable if and only if the smooth Dirac manifold* (G, D) *defined by* $\mathsf{D}_{G/H}(eH)$ *as in* (4.12) *is integrable.*

Proof: It is known by the theory about Dirac reduction that if an integrable Dirac manifold (M, D) is acted upon in a free and proper canonical way by a Lie group H, then the quotient Dirac manifold $(M/H, q(\mathsf{D}))$ is also integrable. Hence, if (G, D) is integrable, then $(G/H, \mathsf{D}_{G/H})$ is also integrable.
For the converse implication, we deduce from the proof of Theorem 4.2.7 that $(\pounds_{\xi^l} X, \pounds_{\xi^l} \alpha)$ is an element of $\Gamma(\mathsf{D})$ for all sections (X, α) of D and Lie algebra elements $\xi \in \mathfrak{h}$. This yields that $\mathsf{D} = \mathsf{D} \cap \mathcal{K}_H^{\perp}$ satisfies

$$[\Gamma(\mathcal{K}_H), \Gamma(\mathsf{D})] \subseteq \Gamma(\mathsf{D} + \mathcal{K}_H).$$

We get from a result in Jotz et al. (2011a) that D is spanned by *right H-descending sections* $(X, \alpha) \in \Gamma(\mathsf{D})$, that is, with $[X, \Gamma(\mathcal{V}_H)] \subseteq \Gamma(\mathcal{V}_H)$ and $\alpha \in \Gamma(\mathcal{V}_H{}^\circ)^H$. Hence, it suffices to show that if (X, α) and (Y, β) are such elements of $\Gamma(\mathsf{D})$, then their bracket $[(X, \alpha), (Y, \beta)]$ is a section of D.
Since (X, α) and (Y, β) are H-descending and $(G/H, \mathsf{D}_{G/H})$ is the Dirac quotient space of (G, D), we find $(\bar{X}, \bar{\alpha})$ and $(\bar{Y}, \bar{\beta}) \in \Gamma(\mathsf{D}_{G/H})$ such that $X \sim_q \bar{X}$, $Y \sim_q \bar{Y}$, $\alpha = q^* \bar{\alpha}$ and $\beta = q^* \bar{\beta}$.
We have then $[X, Y] \sim_q [\bar{X}, \bar{Y}]$ and $\pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha = q^*(\pounds_{\bar{X}} \bar{\beta} - \mathbf{i}_{\bar{Y}} \mathbf{d}\bar{\alpha})$. If $(G/H, \mathsf{D}_{G/H})$ is integrable, the pair $[(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta})] = ([\bar{X}, \bar{Y}], \pounds_{\bar{X}} \bar{\beta} - \mathbf{i}_{\bar{Y}} \mathbf{d}\bar{\alpha})$ is a section of $\mathsf{D}_{G/H}$. By construction of the Dirac quotient of a Dirac manifold by a smooth Dirac action, there exists a smooth vector field $Z \in \mathfrak{X}(G)$ such that $(Z, q^*(\pounds_{\bar{X}} \bar{\beta} - \mathbf{i}_{\bar{Y}} \mathbf{d}\bar{\alpha}))$ is an element of $\Gamma(\mathsf{D})$ and $Z \sim_q [\bar{X}, \bar{Y}]$. But then there exists a smooth section $V \in \Gamma(\mathcal{V}_H) = \Gamma(\mathfrak{h}^l)$ such that $Z + V = [X, Y]$. Since $\mathfrak{h}^l \subseteq \mathsf{G_0}'$, this yields

$$[(X, \alpha), (Y, \beta)] = ([X, Y], \pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha) = (Z, \pounds_X \beta - \mathbf{i}_Y \mathbf{d}\alpha) + (V, 0) \in \Gamma(\mathsf{D})$$

and thus the Dirac manifold (G, D) is integrable. □

Remark 4.2.11 1. If $(G/H, \mathsf{D}_{G/H})$ is integrable, the Dirac structure D is also integrable as we have seen above and the subbundle $\mathsf{G_0}' = {\mathfrak{g}_0'}^l$ is integrable in the sense of Frobenius. The vector subspace $\mathfrak{g}_0' \subseteq \mathfrak{g}$ is then a subalgebra and the integral leaf of $\mathsf{G_0}'$ through e is a Lie subgroup of G, which will be called J in the following. As in the proof of Lemma 4.1.16, we can show that the right action of J on G is canonical on (G, D). Theorem 4.2.7 yields then

$$A_j\left(\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})\right) \subseteq \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$$

for all $j \in J$.

2. We can also show that if (G, D_G) is N-invariant, then (G, D) is N-invariant. By Theorem 4.1.18, the bracket on $\mathfrak{p}_1 \times \mathfrak{p}_1$ defined in Definition 4.1.13 has image in $\mathfrak{p}_1$, and by Remark 4.1.32, we know then that the action of N on $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ is trivial. Hence, we have $A_n(\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})) = \mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ for all $n \in N$, and we can apply Theorem 4.2.7. $\triangle$

4.2.3 Integrable Dirac homogeneous spaces

We consider here an *integrable* Dirac Lie group (G, D_G), a closed Lie subgroup H of G (with Lie algebra $\mathfrak{h}$) and a Dirac homogeneous space $(G/H, \mathsf{D}_{G/H})$ of (G, D_G). As above, we consider the backward image $\mathfrak{D} = (T_e q)^* \mathsf{D}_{G/H}(eH)$ of $\mathsf{D}_{G/H}(eH)$ under $T_e q$, i.e.,

$$\mathfrak{D} = \left\{ (x, (T_e q)^* \bar{\xi}) \mid x \in \mathfrak{g}, \bar{\xi} \in (\mathfrak{g}/\mathfrak{h})^* \text{ such that } (T_e q x, \bar{\xi}) \in \mathsf{D}_{G/H}(eH) \right\},$$

and the Dirac structure D defined by $\mathfrak{D}$ on G as in (4.12) and Proposition 4.2.5.

Theorem 4.2.12 *The quotient $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is a subalgebra of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ if and only if (G, D) (or equivalently $(G/H, \mathsf{D}_{G/H})$) is integrable.*

PROOF: Choose $(x, \xi), (y, \eta)$ in $\mathfrak{D}$, then the pairs $(X_\xi + x^l, \xi^l)$ and $(X_\eta + y^l, \eta^l)$ are sections of D. We have by Proposition 4.1.12, Definition 4.1.13, Proposition 4.1.21 and Remark 4.1.30:

$$\begin{aligned}
&\left[(X_\xi + x^l, \xi^l), (X_\eta + y^l, \eta^l) \right] \qquad (4.14)\\
&= \left([X_\xi, X_\eta] + \pounds_{x^l} X_\eta - \pounds_{y^l} X_\xi + [x, y]^l, \pounds_{X_\xi} \eta^l - \mathbf{i}_{X_\eta} \mathbf{d} \xi^l + \pounds_{x^l} \eta^l - \mathbf{i}_{y^l} \mathbf{d} \xi^l \right)\\
&\overset{(4.4)}{=} \left(X_{[\xi,\eta]} + X_{\operatorname{ad}_x^* \eta} - (\operatorname{ad}_\eta^* x)^l - X_{\operatorname{ad}_y^* \xi} + (\operatorname{ad}_\xi^* y)^l + [x, y]^l, ([\xi, \eta] + \operatorname{ad}_x^* \eta - \operatorname{ad}_y^* \xi)^l \right)\\
&\quad + (Z, 0) \quad \text{ for some } Z \in \Gamma(\mathsf{G}_0)\\
&= \left(X_{[\xi,\eta] + \operatorname{ad}_x^* \eta - \operatorname{ad}_y^* \xi} + ([x, y] - \operatorname{ad}_\eta^* x + \operatorname{ad}_\xi^* y)^l, ([\xi, \eta] + \operatorname{ad}_x^* \eta - \operatorname{ad}_y^* \xi)^l \right), \qquad (4.15)
\end{aligned}$$

where we have chosen the vector field $X_{[\xi,\eta] + \operatorname{ad}_x^* \eta - \operatorname{ad}_y^* \xi} := X_{[\xi,\eta]} - X_{\operatorname{ad}_y^* \xi} - X_{\operatorname{ad}_y^* \xi} + Z$.
If (G, D) is integrable, we have $\left[(X_\xi + x^l, \xi^l), (X_\eta + y^l, \eta^l) \right] \in \Gamma(\mathsf{D})$, and hence its value at the neutral element e is an element of $\mathfrak{D}$ by Remark 4.2.6. But since $X_{[\xi,\eta] + \operatorname{ad}_x^* \eta - \operatorname{ad}_y^* \xi}(e)$ is an element of $\mathfrak{g}_0$, (4.15) yields

$$\begin{aligned}
&\left[(X_\xi + x^l, \xi^l), (X_\eta + y^l, \eta^l) \right](e)\\
&\quad \in \left([x, y] - \operatorname{ad}_\eta^* x + \operatorname{ad}_\xi^* y, [\xi, \eta] + \operatorname{ad}_x^* \eta - \operatorname{ad}_y^* \xi \right) + (\mathfrak{g}_0 \times \{0\}).
\end{aligned}$$

This leads to

$$\begin{aligned}
&[(x + \mathfrak{g}_0, \xi), (y + \mathfrak{g}_0, \eta)]\\
&= ([x, y] - \operatorname{ad}_\eta^* x + \operatorname{ad}_\xi^* y + \mathfrak{g}_0, [\xi, \eta] + \operatorname{ad}_x^* \eta - \operatorname{ad}_y^* \xi) \in \mathfrak{D}/(\mathfrak{g}_0 \times \{0\}).
\end{aligned}$$

For the converse implication, it is sufficient to show that for all $(x,\xi),(y,\eta)\in\mathfrak{D}$, we have

$$\left[(X_\xi + x^l, \xi^l), (X_\eta + y^l, \eta^l)\right] \in \Gamma(\mathsf{D})$$

since D is spanned by these sections. By hypothesis, we have

$$\begin{aligned}&[(x+\mathfrak{g}_0,\xi),(y+\mathfrak{g}_0,\eta)]\\ &=([x,y]-\mathrm{ad}^*_\eta x+\mathrm{ad}^*_\xi y+\mathfrak{g}_0,[\xi,\eta]+\mathrm{ad}^*_x\eta-\mathrm{ad}^*_y\xi)\in\mathfrak{D}/(\mathfrak{g}_0\times\{0\})\end{aligned} \tag{4.16}$$

for all $(x,\xi),(y,\eta)\in\mathfrak{D}$ and the claim follows using (4.15). □

We have proved the following theorem which is a generalization of the theorem in Drinfel′d (1993).

Theorem 4.2.13 *Let (G,D_G) be a Dirac Lie group and H a closed subgroup of G with Lie algebra $\mathfrak{h}$. The assignment*

$$\mathsf{D}_{G/H}\mapsto\mathfrak{D}=(T_eq)^*\mathsf{D}_{G/H}(eH)$$

gives a one-to-one correspondence between (G,D_G)-Dirac homogeneous structures on G/H and Dirac subspaces $\mathfrak{D}\subseteq\mathfrak{g}\times\mathfrak{g}^*$ such that

1. $(\mathfrak{g}_0+\mathfrak{h})\times\{0\}\subseteq\mathfrak{D}\subseteq\mathfrak{g}\times(\mathfrak{p}_1\cap\mathfrak{h}^\circ)$,
2. $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ *is Lagrangian in* $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$, *and*
3. $A_h\left(\mathfrak{D}/(\mathfrak{g}_0\times\{0\})\right)\subseteq\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ *for all* $h\in H$.

If the Dirac Lie group is integrable, then $(G/H,\mathsf{D}_{G/H})$ is integrable if and only if $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ is a subalgebra *of $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$.*

Example 4.2.14
1. Let (G,D_G) be a Dirac Lie group and $(G/H,\mathsf{D}_{G/H})$ a Dirac homogeneous space of (G,D_G) corresponding by Theorem 4.2.13 to the Lagrangian subspace $\mathfrak{D}\subseteq\mathfrak{g}\times\mathfrak{g}^*$. Then, again by Theorem 4.2.13 applied to the Lie subgroup $\{e\}$ of G and the Dirac subspace $\mathfrak{D}\subseteq\mathfrak{g}\times\mathfrak{g}^*$, we get that (G,D) is a Dirac homogeneous space of (G,D_G). If (G,D_G) is integrable, we recover the fact that (G,D) is integrable if and only if $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ is a *subalgebra* of $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$, that is, if and only if $(G/H,\mathsf{D}_{G/H})$ is integrable.
2. Choose a Dirac Lie group (G,D_G) and assume that the corresponding bracket on $\mathfrak{p}_1$ has image in $\mathfrak{p}_1$ and that the Lie subgroup N is closed in G. The Lagrangian subspace $\mathfrak{g}_0\times\mathfrak{p}_1$ of $\mathfrak{g}\times\mathfrak{g}^*$ satisfies $(**)$ and the corresponding Dirac structure D is equal to D_G by definition. Since N corresponds to the Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ and fixes $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ pointwise by Remark 4.2.11, we get from Theorem 4.2.7 that the quotient $(G/N,q_N(\mathsf{D}_\mathsf{G}))$ is a Dirac homogeneous space of the Dirac Lie group (G,D_G). We will study this particular Dirac homogeneous space in section 4.3. ◊

Remark 4.2.15 The previous theorem does not reduce, in the case of Poisson Lie groups, to the same theorem but with $\mathfrak{g}_0$ set to be $\{0\}$, as in many previous statements in this chapter. Indeed, the theorem of Drinfel′d (Drinfel′d (1993)) gives a correspondence between homogeneous *Poisson* structures on a homogeneous space G/H of a *Poisson* Lie group $(G, \{\cdot\,,\cdot\})$ and Lagrangian subalgebras $\mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{g}^*$ satisfying $A_h\mathfrak{D} \subseteq \mathfrak{D}$ for all $h \in H$ and *the equality* $\mathfrak{D} \cap (\mathfrak{g} \times \{0\}) = \mathfrak{h} \times \{0\}$ (see Drinfel′d (1993)), that is, $\mathfrak{g}_0' = \mathfrak{h}$. Here, we have $\mathfrak{D} \cap (\mathfrak{g} \times \{0\}) = \mathfrak{g}_0' \times \{0\}$ and we get the following cases.

1. $\mathfrak{g}_0 \subseteq \mathfrak{h} = \mathfrak{g}_0'$: the Dirac homogeneous space is a Poisson homogeneous space of the Dirac Lie group (G, D_G). Furthermore, if $\mathfrak{g}_0 = \{0\}$, the Dirac Lie group is a Poisson Lie group and we are in the situation of Drinfel′d's theorem. The case $\mathfrak{g}_0 = \mathfrak{h}$ (see the second part of Example 4.2.14) will be studied in section 4.3.

2. $\mathfrak{h} \subsetneq \mathfrak{g}_0'$: the Dirac homogeneous space has non-trivial G_0-distribution and is hence not a Poisson homogeneous space of (G, D_G). Therefore, in the case where $\mathfrak{g}_0 = \{0\}$, we obtain a Dirac homogeneous space of a Poisson Lie group. △

Example 4.2.16 Consider an n-dimensional torus $G := \mathbb{T}^n$. In Corollary 4.1.14, we have recovered the fact that the only multiplicative Poisson structure on $\mathbb{T}^n$ is the trivial Poisson structure $\pi = 0$, that is $\mathsf{D}_\pi = 0_{\mathbb{T}^n} \times T^*\mathbb{T}^n$. The Lie algebra structure on $\mathfrak{g} \times \mathfrak{g}^*$ is given by $[(x,\xi),(y,\eta)] = ([x,y], \mathrm{ad}_x^*\eta - \mathrm{ad}_y^*\xi) = (0,0)$ since the Lie group $\mathbb{T}^n$ is Abelian. Hence, every Dirac subspace of $\mathfrak{g} \times \mathfrak{g}^*$ is a Lagrangian subalgebra. Indeed, it is easy to verify that each left invariant Dirac structure on $\mathbb{T}^n$ is an integrable Dirac homogeneous space of the trivial Poisson Lie group $(\mathbb{T}^n, \pi = 0)$.
In general, if $(G, \pi = 0)$ is a trivial Poisson Lie group, the Lie algebra structure on $\mathfrak{g} \times \mathfrak{g}^*$ is given by $[(x,\xi),(y,\eta)] = ([x,y], \mathrm{ad}_x^*\eta - \mathrm{ad}_y^*\xi)$. The $(G, \pi = 0)$-homogeneous Dirac structures on G are here the left invariant Dirac structures $\mathfrak{D}^l$ on G. Hence, the integrable homogeneous Dirac structures on G are the left invariant Dirac structures $\mathfrak{D}^l$ such that $\mathfrak{D}$ is a subalgebra of $\mathfrak{g} \times \mathfrak{g}^*$. But $\mathfrak{D}$ is a subalgebra of $\mathfrak{g} \times \mathfrak{g}^*$ if and only if

$$\langle ([x,y], \mathrm{ad}_x^*\eta - \mathrm{ad}_y^*\xi), (z,\zeta)\rangle = \xi([y,z]) + \eta([z,x]) + \zeta([x,y]) = 0$$

for all (x,ξ), (y,η) and $(z,\zeta) \in \mathfrak{D}$. We recover Proposition 1.2.5 about the integrability of a left invariant Dirac structure on G, see also Milburn (2007). ◊

4.3 The regular case

We will see in this section that if (G, D_G) is an integrable regular Dirac Lie group, i.e, such that the leaf N of the involutive subbundle G_0 through the neutral element e is a *closed* normal subgroup of G, then the Lie bialgebra $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1) \simeq (\mathfrak{g}/\mathfrak{g}_0, (\mathfrak{g}/\mathfrak{g}_0)^*)$ arises from a natural multiplicative Poisson structure π on the quotient G/N, that makes $(G/N, \pi)$ a Poisson homogeneous space of (G, D_G).

Theorem 4.3.1 *Let (G, D_G) be a regular Dirac Lie group such that the bracket on $\mathfrak{p}_1$ has image in $\mathfrak{p}_1$. The reduced Dirac structure $\mathsf{D}_{G/N} = q_N(\mathsf{D}_G)$ on G/N (that is a homogeneous Dirac structure of (G, D_G), see Example 4.2.14) is the graph of a multiplicative skew-symmetric bivector field π on G/N. If (G, D_G) is integrable, the quotient $(G/N, \mathsf{D}_{G/N}) =: (G/N, \pi)$ is a Poisson Lie group, and the induced Lie bialgebra $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1) \simeq (\mathfrak{g}/\mathfrak{g}_0, (\mathfrak{g}/\mathfrak{g}_0)^*)$ as in Remark 4.1.30 is the Lie bialgebra defined by (G, D_G) as in Theorems 4.1.23 and 4.1.27.*

Since each normal subgroup of a simply connected Lie group G is closed (see Hilgert and Neeb (1991)), we have the following immediate corollary.

Corollary 4.3.2 *Let (G, D_G) be an integrable, simply connected Dirac Lie group. Then D_G is the pullback Dirac structure defined on G by $q_N : G \to G/N$ and a multiplicative Poisson bracket on G/N.*

The proof of Theorem 4.3.1 will be repeated in the more general situation of regular Dirac groupoids (Theorem 5.1.2). We chose to do it here in the group case in order to let this chapter as self-contained as possible.

Proof (of Theorem 4.3.1): Since $\mathfrak{g}_0$ is an ideal in $\mathfrak{g}$, the Lie subgroup N is normal in G. If it is closed in G, the left or right action of N on G is free and proper and the reduced space G/N is a Lie group. Let $q_N : G \to G/N$ be the projection.
The vertical distribution $\mathcal{V}_N$ of the left (right) action of N on G is the span of the right invariant vector fields x^r, for all $x \in \mathfrak{g}_0$, that is, $\mathcal{V}_N = \mathsf{G}_0$. This yields $\mathcal{K}_N = \mathcal{V}_N \times_G 0_{T^*G} = \mathsf{G}_0 \times_G 0_{T^*G}$, and hence $\mathcal{K}_N^\perp = TG \times_G \mathsf{P}_1$. The intersection $\mathsf{D}_G \cap \mathcal{K}_N^\perp$ is consequently equal to D_G and has constant rank on G. Hence, D_G pushes forward under q to a Dirac structure $\mathsf{D}_{G/N}$ on G/N. The set of smooth local sections of $\mathsf{D}_{G/N}$ is given by

$$\left\{ (\bar{X}, \bar{\alpha}) \in \mathfrak{X}(G/N) \times \Omega^1(G/N) \;\middle|\; \begin{array}{c} \exists X \in \mathfrak{X}(G) \text{ such that } X \sim_{q_N} \bar{X} \\ \text{and } (X, q_N^* \bar{\alpha}) \in \Gamma(\mathsf{D}_G) \end{array} \right\}.$$

Since N lets (G, D_G) invariant by Theorem 4.1.18 and is connected by definition, we have $[\Gamma(\mathcal{K}_N), \Gamma(\mathsf{D}_G)] \subseteq \Gamma(\mathsf{D}_G)$ and we get using a result in Jotz et al. (2011a) that D_G is spanned by its N-descending sections, that is, the pairs $(X, \alpha) \in \Gamma(\mathsf{D}_G)$ with $[X, \Gamma(\mathcal{V}_N)] \subseteq \Gamma(\mathcal{V}_N)$ and $\alpha \in \Gamma(\mathcal{V}_N^\circ)^N$ (see Jotz et al. (2011b) or Jotz et al. (2011a)). The vector subbundle $\mathsf{P}_1 = \mathcal{V}_N^\circ$ of T^*G is spanned by its descending sections since $\mathcal{V}_N$ is a smooth integrable subbundle of TG (see Jotz et al. (2011a)), and the push-forwards of the descending sections of $\mathcal{V}_N^\circ$ are exactly the sections of the cotangent space $T^*(G/N)$ of G/N. Since D_G is spanned by its descending sections $(X, \alpha) \in \Gamma(\mathsf{D}_G)$, P_1 is in particular spanned by descending sections belonging to descending pairs in $\Gamma(\mathsf{D}_G)$. This shows that the cotangent distribution $\bar{\mathsf{P}_1}$ defined by $\mathsf{D}_{G/N}$ on G/N is equal to $T^*(G/N)$, and $(T_g q_N)^*(T^*_{gN} G/N) = \mathsf{P}_1(g)$ for all $g \in G$. This yields that $\mathsf{D}_{G/N}$ is the graph of a skew-symmetric bivector field π on G/N. Thus, if we show that $(G/N, \mathsf{D}_{G/N})$ is a Dirac Lie group, we will have simultaneously proved that $(G/N, \pi)$ is multiplicative.

We show that $\mathsf{D}_{G/N}$ is multiplicative. Choose a product $gNg'N = gg'N \in G/N$ and $(\bar{v}_{gg'N}, \bar{\alpha}_{gg'N}) \in \mathsf{D}_{G/N}(gg'N)$. Then there exists a pair $(v_{gg'}, \alpha_{gg'}) \in \mathsf{D}_G(gg')$ such that

$$T_{gg'}q_N v_{gg'} = \bar{v}_{gg'N} \quad \text{and} \quad (T_{gg'}q_N)^*\bar{\alpha}_{gg'N} = \alpha_{gg'}.$$

Since D_G is multiplicative, we can find $w_g \in T_gG$ and $u_{g'} \in T_{g'}G$ such that

$$T_gR_{g'}w_g + T_{g'}L_gu_{g'} = v_{gg'}, \quad (w_g, (T_gR_{g'})^*\alpha_{gg'}) \in \mathsf{D}_G(g),$$

and

$$(u_{g'}, (T_{g'}L_g)^*\alpha_{gg'}) \in \mathsf{D}_G(g').$$

We have $\gamma_{g'} := (T_{g'}L_g)^*\alpha_{gg'} \in \mathsf{P}_1(g')$, $\beta_g := (T_gR_{g'})^*\alpha_{gg'} \in \mathsf{P}_1(g)$ and hence, by the considerations above, there exist $\bar{\beta}_{gN} \in \bar{\mathsf{P}_1}(gN)$ and $\bar{\gamma}_{g'N} \in \bar{\mathsf{P}_1}(g'N)$ satisfying $(T_gq_N)^*\bar{\beta}_{gN} = \beta_g$ and $(T_{g'}q_N)^*\bar{\gamma}_{g'N} = \gamma_{g'}$. By construction of $\mathsf{D}_{G/N}$, we have then $(T_gq_Nw_g, \bar{\beta}_{gN}) \in \mathsf{D}_{G/N}(gN)$ and $(T_{g'}q_Nu_{g'}, \bar{\gamma}_{g'N}) \in \mathsf{D}_{G/N}(g'N)$.
We compute

$$\begin{aligned} T_{g'N}L_{gN}T_{g'}q_Nu_{g'} + T_{gN}R_{g'N}T_gq_Nw_g &= T_{g'g}q_N\left(T_{g'}L_gu_{g'} + T_gR_{g'}w_g\right) \\ &= T_{gg'}q_Nv_{gg'} = \bar{v}_{gg'N}, \end{aligned}$$

$$\begin{aligned} (T_{g'}q_N)^*\left((T_{g'N}L_{gN})^*\bar{\alpha}_{gg'N}\right) &= (T_{g'}L_g)^*\left((T_{gg'}q_N)^*\bar{\alpha}_{gg'N}\right) \\ &= (T_{g'}L_g)^*\,\alpha_{gg'} = \gamma_{g'} = (T_{g'}q_N)^*\bar{\gamma}_{g'N}, \end{aligned}$$

and in the same manner $(T_gq_N)^*\left((T_{gN}R_{g'N})^*\bar{\alpha}_{gg'N}\right) = \beta_g = (T_gq_N)^*\bar{\beta}_{gN}$. This leads to

$$(T_{g'N}L_{gN})^*(\bar{\alpha}_{gg'N}) = \bar{\gamma}_{g'N} \quad \text{and} \quad (T_{gN}R_{g'N})^*(\bar{\alpha}_{gg'N}) = \bar{\beta}_{gN}$$

since q_N is a smooth surjective submersion. Hence, we have shown that $(G/N, \mathsf{D}_{G/N})$ is a Dirac Lie group which we will also write $(G/N, \pi)$ in the following since $\mathsf{D}_{G/N}$ is the graph of a multiplicative skew-symmetric bivector field π on G/N.
The last statement is obvious with the considerations above and Proposition 4.1.15. □

Furthermore, we can show that each Dirac homogeneous structure on G/H, H a closed subgroup of G, can be assigned to a unique Dirac homogeneous space of the Poisson Lie group $(G/N, \pi)$ if the product $N \cdot H$ remains closed in G.
Let $(G/H, \mathsf{D}_{G/H})$ be a Dirac homogeneous space. We assume that the Lie subgroup $N \cdot H$ (with Lie algebra $\mathfrak{g}_0 + \mathfrak{h}$) is closed in G. The Lie group N acts by smooth left actions given by $n \cdot gH = ngH$ for all $n \in N$ and $g \in G$ on the homogeneous space G/H. This is well-defined since if $g^{-1}g' \in H$, we have $g^{-1}n^{-1}ng' \in H$ and hence $ngH = ng'H$. It is easy to check that the quotient of G/H by the left action of N is equal to the quotient of G by the right action of $N \cdot H$. Indeed, the class of gH in $(G/H)/N$ is the set $\{ngH \mid n \in N\} = NgH$. But since N is normal in G, this class is equal to gNH, which is the class of the element $g \in G$ in the quotient by the right action of $N \cdot H$ on G. Since

$G/(N\cdot H)$ has the structure of a smooth regular quotient manifold and the maps q_H and $q_{N\cdot H}$ are smooth surjective submersions, the projection $q_{N,H}: G/H \to (G/H)/N$ is also a smooth surjective submersion.
In the second diagram, we have $(G/N)/(NH/N) \simeq G/(N\cdot H) \simeq (G/H)/N$.

$$\begin{tikzcd} N\times G \arrow[r, "m|_{N\times G}"] \arrow[d, "\mathrm{Id}_N\times q_H"'] & G \arrow[d, "q_H"] \\ N\times G/H \arrow[r] & G/H \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, "q_H"] \arrow[d, "q_N"'] \arrow[drr, "q_{NH}"] & & G/H \arrow[d, "q_{N,H}"] \\ G/N \arrow[rr, "q_{N,NH}"'] & & G/(N\cdot H) \end{tikzcd}$$

We have the following theorem. We assume here for simplicity that the Dirac Lie group (G,D_G) is integrable, but analogous results can be shown for a Dirac Lie group that is just invariant under the action of the induced Lie subgroup N.

Theorem 4.3.3 *Let $(G/H,\mathsf{D}_{G/H})$ be an integrable Dirac homogeneous space of the integrable Dirac Lie group (G,D_G) such that N and $N\cdot H$ are closed in G.
The Lie group N acts smoothly on the left on $(G/H,\mathsf{D}_{G/H})$ by Dirac actions, and the Lie group $N\cdot H$ acts smoothly on the right on the Dirac manifold (G,D) by Dirac actions.
The quotient Dirac structures on $G/(N\cdot H)\simeq (G/H)/N$ are equal and will be called $\mathsf{D}_{G/(NH)}$. The pair $(G/(N\cdot H),\mathsf{D}_{G/(NH)})$ is a Dirac homogeneous space of the Poisson Lie group $(G/N,\pi)$ and of the Dirac Lie group (G,D_G).
Conversely, if $(G/(NH),\mathsf{D}_{G/(NH)})$ is a Dirac homogeneous space of the Poisson Lie group $(G/N,\pi)$, then the pullbacks $(G/H, q_{N,H}^*(\mathsf{D}_{G/(NH)}))$ and $(G, q_{NH}^*(\mathsf{D}_{G/(NH)}))$ are Dirac homogeneous spaces of the Dirac Lie group (G,D_G).*

Proof: Consider again the Dirac subspace $\mathfrak{D}=(T_eq_H)^*\mathsf{D}_{G/H}(eH)\subseteq \mathfrak{g}\times\mathfrak{p}_1$. We write $\bar{\mathfrak{D}}$ for the quotient $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$. Since (G,D_G) and $(G/H,\mathsf{D}_{G/H})$ are integrable, we get from Theorem 4.2.13 that $\bar{\mathfrak{D}}$ is a subalgebra of $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ and from Remark 4.2.11 that $\bar{\mathfrak{D}}$ is N-invariant. We have then $A_{nh}\bar{\mathfrak{D}}\subseteq\bar{\mathfrak{D}}$ for all $nh\in N\cdot H$ and, by Theorem 4.2.13, the group $N\cdot H$ acts on (G,D) by Dirac actions and the quotient $(G/(N\cdot H), q_{NH}(\mathsf{D})) =: (G/(N\cdot H),\mathsf{D}_{G/NH})$ is an integrable Dirac homogeneous space of the Dirac Lie group (G,D_G).
Next we show that the left action Φ of N on $(G/H,\mathsf{D}_{G/H})$ is canonical. Let $(\bar{X},\bar{\alpha})$ be a section of $\mathsf{D}_{G/H}$. Then there exists $(X,\alpha)\in\Gamma(\mathsf{D})$ such that $X\sim_{q_H}\bar{X}$ and $\alpha=q_H^*\bar{\alpha}$. We have $q_H\circ L_n=\Phi_n\circ q_H$ for all $n\in N$ and hence $L_n^*X\sim_{q_H}\Phi_n^*\bar{X}$ and $L_n^*\alpha=q_H^*\Phi_n^*\bar{\alpha}$. Since the action of N on (G,D') is canonical, we have $(L_n^*X, L_n^*\alpha)\in\Gamma(\mathsf{D})$ and the pair $(\Phi_n^*\bar{X},\Phi_n^*\bar{\alpha})$ is consequently a section of $\mathsf{D}_{G/H}$.
Let $\mathcal{V}$ be the vertical space of the action Φ of N on G/H and $\mathcal{K}=\mathcal{V}\times_{G/H}0_{T^*(G/H)}$. The subbundle $\mathcal{V}$ of $T(G/H)$ is spanned by the projections to G/H of the right invariant vector fields x^r on G, for all $x\in\mathfrak{g}_0$, and $\mathcal{V}^\circ$ is spanned by the push-forwards of the one-forms ξ^r, for all $\xi\in\mathfrak{p}_1\cap\mathfrak{h}^\circ$. But since $\mathsf{D}\cap\mathcal{K}_H^\perp=\mathsf{D}\subseteq TG\times_G(\mathfrak{p}_1\cap\mathfrak{h}^\circ)^r$, and $\mathsf{D}_{G/H}=q_H(\mathsf{D})$, we get easily $\mathsf{D}_{G/H}\cap\mathcal{K}^\perp=\mathsf{D}_{G/H}$, which has consequently constant dimensional fibers on G/H. Thus, by the regular reduction theorem for Dirac manifolds, the quotient $((G/H)/N, q_{N,H}(\mathsf{D}_{G/H}))$ is a smooth Dirac manifold.

We have then to show that the quotient Dirac structure $q_{N,H}(\mathsf{D}_{G/H})$ is equal to $\mathsf{D}_{G/(NH)}$. If $(\tilde{X},\tilde{\alpha})$ is a section of $q_{N,H}(\mathsf{D}_{G/H})$, then there exists $(\bar{X},\bar{\alpha})$ in $\Gamma(\mathsf{D}_{G/H})$ such that $\bar{X} \sim_{q_{N,H}} \tilde{X}$ and $q_{N,H}^*\tilde{\alpha} = \bar{\alpha}$. But then there exists $(X,\alpha) \in \Gamma(\mathsf{D})$ such that $X \sim_{q_H} \bar{X}$ and $\alpha = q_H^*\bar{\alpha}$. Then we have $\alpha = q_H^* q_{N,H}^*\tilde{\alpha} = q_{NH}^*\tilde{\alpha}$, $X \sim_{q_{NH}} \tilde{X}$ and $(\tilde{X},\tilde{\alpha})$ is a section of $\mathsf{D}_{G/NH}$. This shows $q_{N,H}(\mathsf{D}_{G/H}) \subseteq \mathsf{D}_{G/NH}$ and hence equality since both Dirac structures have the same rank.
Finally, we show that $(G/(N\cdot H), \mathsf{D}_{G/(NH)})$ is a Dirac homogeneous space of the Poisson Lie group $(G/N,\pi)$. The Lie bialgebra of the Poisson Lie group $(G/N,\pi)$ is $(\mathfrak{g}/\mathfrak{g}_0, \mathfrak{p}_1)$ with the bracket as in (4.11). We have

$$\begin{aligned}(T_{eN}q_{N,NH})^*\mathsf{D}_{G/(NH)}(eNH) &= (T_e q_N)\left((T_e q_{NH})^*\mathsf{D}_{G/(NH)}(eNH)\right)\\ &= \mathfrak{D}/(\mathfrak{g}_0\times\{0\}) \subseteq \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1.\end{aligned}$$

By Remark 4.1.32, the action of G on $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ induces an action of G/N on $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$; this is exactly the action of G/N defined by the Poisson Lie group $(G/N,\pi)$ on its Lie bialgebra. Since $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ is NH-invariant, it is NH/N-invariant under $\bar{A}$. Since $\mathfrak{D}/(\mathfrak{g}_0\times\{0\})$ is a Lagrangian subalgebra of $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ and $(\mathfrak{g}_0+\mathfrak{h})/\mathfrak{g}_0\times\{0\} \subseteq \mathfrak{D}/(\mathfrak{g}_0\times\{0\}) \subseteq \mathfrak{g}/\mathfrak{g}_0\times\mathfrak{h}^\circ\cap\mathfrak{p}_1$, we are done by Theorem 4.2.13.

For the converse statement, we use Remark 4.1.32 about the action $\bar{A}$ of G/N on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ and apply the first part of Example 4.2.14 to the Dirac Lie group $(G/N,\pi)$ and the closed subgroup NH/N of G/N and to the Dirac Lie group (G,D_G) and the closed subgroup NH of G. □

Now choose an integrable Dirac homogeneous space $(G/H,\mathsf{D}_{G/H})$ of (G,D_G) and let (G,D) be the Dirac structure on G defined as in the preceding section. Since D is integrable and G_0' is left invariant, it is an involutive subbundle of TG which is consequently integrable in the sense of Frobenius. Then the integral leaf J of $\mathsf{G_0}'$ through the neutral element e, which was defined in Remark 4.2.11, is a Lie subgroup of G.

Lemma 4.3.4 *If the Lie subgroup J is closed in G, it acts properly on the right on (G,D') by Dirac actions. The intersection $\mathsf{D}\cap\mathcal{K}_J^\perp$, with $\mathcal{K}_J = \mathcal{V}_J\times_G 0_{T^*G} = {\mathfrak{g}_0'}^l \times_G 0_{T^*G}$, is equal to D' by definition of J and we can build the quotient $(G/J, q_J(\mathsf{D}))$, where $q_J : G\to G/J$ is the projection.*
Furthermore, since $N\subseteq J$ is a normal subgroup, the quotient J/N is a Lie group if N is closed in G. It acts properly on the right on G/N and we can see that $G/J \simeq (G/N)/(J/N)$ as a homogeneous space of G/N.

Proof: We have seen in Remark 4.2.11 that if (G,D) is integrable, we have $A_j\bar{\mathfrak{D}} = \bar{\mathfrak{D}}$ for all $j\in J$. By Theorem 4.2.7 (note that all the hypotheses are satisfied since $\mathfrak{g}_0'\times\{0\}\subseteq\mathfrak{D}\subseteq\mathfrak{g}\times\mathfrak{p}_1'$ and the Dirac subspace $\mathfrak{D}$ is equal to the pullback $\mathfrak{D} = T_e q_J^*(T_e q_J\mathfrak{D})$), we get that the Dirac manifold (G,D) is right J-invariant. □

In the following diagram, the dashed arrows join Dirac Lie groups to their Dirac homogeneous spaces.

$$\begin{tikzcd}
(G, \mathsf{D}_G) \arrow[rrr, dashed] \arrow[drr, dashed] \arrow[dd, "q_N"'] \arrow[dd, dashed, shift left] \arrow[ddr, dashed] \arrow[ddd, dashed] & & & (G, \mathsf{D}) \arrow[dl, "q_H"'] \arrow[ddl, "q_{NH}"] \arrow[dddll, bend left, "q_J"] \\
& & (G/H, \mathsf{D}_{G/H}) \arrow[d, "q_{N,H}"] & \\
& (G/N, \pi) \arrow[r, dashed] \arrow[d, dashed] & \big(G/(N \cdot H), \mathsf{D}_{G/(NH)}\big) & \\
& (G/J, \pi_{G/J}) & &
\end{tikzcd}$$

Theorem 4.3.5 *Under the hypotheses of the preceding lemma, the pair $(G/J, q_J(\mathsf{D})) =: (G/J, \pi_{G/J})$ is a Poisson homogeneous space of the Dirac Lie group (G, D_G) and of the Poisson Lie group $(G/N, \pi)$.*

Proof: By Theorem 4.2.13 and Lemma 4.3.4, $(G/J, \pi_{G/J})$ is a Dirac homogeneous space of the integrable Dirac Lie group (G, D_G). The quotient $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is a Lagrangian subalgebra of $\mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1$ with $(\mathfrak{g}_0 + \mathfrak{g}_0') \times \{0\} = \mathfrak{g}_0' \times \{0\} \subseteq \mathfrak{D} \subseteq \mathfrak{g} \times \mathfrak{p}_1' = \mathfrak{g} \times ((\mathfrak{g}_0')^\circ \cap \mathfrak{p}_1)$. Furthermore, $\mathfrak{D}/(\mathfrak{g}_0 \times \{0\})$ is A_j-invariant and hence also $\bar{A}_{jN}$-invariant by Remark 4.1.32 for all $j \in J$ (see also the proof of the preceding theorem). Hence, $(G/J, \pi_{G/J})$ is also a Dirac homogeneous space of the Poisson Lie group $(G/N, \pi)$.
Note that since $\mathsf{G_0}' = \mathcal{V}_J$, the quotient Dirac structure $\mathsf{D}_{G/J} := q_J(\mathsf{D})$ has vanishing characteristic distribution and is hence a Poisson manifold (see the proof of Theorem 4.3.1) $\square$

Finally, we give examples where it is not possible to build the diverse quotients as above. Let $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ be the universal covering of the Lie group $\mathrm{SL}_2(\mathbb{R})$.

Example 4.3.6 1. Consider the Lie group

$$G = \left(\mathbb{T}^2 \times \widetilde{\mathrm{SL}_2(\mathbb{R})}\right)/\Gamma,$$

where Γ is the group homomorphism $Z(\widetilde{\mathrm{SL}_2(\mathbb{R})}) \simeq \mathbb{Z} \to \mathbb{T}^2$ given by $\Gamma(z) = (e^{i\sqrt{2}z}, e^{iz})$ for all $z \in Z(\widetilde{\mathrm{SL}_2(\mathbb{R})})$ (or more generally a group homomorphism with dense image in $\mathbb{T}^2$). The graph of Γ is a discrete normal subgroup of $\mathbb{T}^2 \times \widetilde{\mathrm{SL}_2(\mathbb{R})}$

and hence the quotient G is a Lie group. The Lie algebra of G is equal to the direct sum of Lie algebras $\mathfrak{g} = \mathbb{R}^2 \oplus \mathfrak{sl}_2(\mathbb{R})$ and has hence $\mathfrak{g}_0 := \mathfrak{sl}_2(\mathbb{R})$ as an ideal. The corresponding Lie subgroup N of G corresponds to the images of the elements of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ in G and is hence by construction not closed in G. Let G be endowed with the trivial Dirac structure such that $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$; the quotient Poisson Lie group $(G/N, \pi)$ does not exist here.

2. Consider $G = \mathbb{T}^4 \times \mathbb{R}$ with coordinates s_1, s_2, s_3, s_4, t and the integrable Dirac structure given by $\mathfrak{g}_0 = \operatorname{span}\{x_1\}$, $\mathfrak{p}_1 = \operatorname{span}\{\xi_2, \xi_3, \xi_4, \xi_5\}$ and D_G the (necessarily) trivial multiplicative Dirac structure

$$\mathsf{D}_G = \mathfrak{g}_0^l \times_G \mathfrak{p}_1^l = \operatorname{span}\left\{(x_1^l, 0), (0, \xi_2^l), (0, \xi_3^l), (0, \xi_4^l), (0, \xi_5^l)\right\},$$

where

$$\begin{aligned} \xi_2 &= \sqrt{2}\mathbf{d}s_1(0) - \mathbf{d}s_2(0) \qquad & x_1 &= \partial_{s_1}(0) + \sqrt{2}\partial_{s_2}(0) + \partial_t(0) \\ \xi_3 &= \mathbf{d}s_1(0) - \mathbf{d}t(0) \\ \xi_4 &= \mathbf{d}s_3(0) \\ \xi_5 &= \mathbf{d}s_4(0) \end{aligned}$$

The group N is then equal to $N = \left\{ \left(e^{ti}, e^{\sqrt{2}ti}, 1, 1, t\right) \,\middle|\, t \in \mathbb{R} \right\}$ and is closed in G as the graph of a smooth map $\mathbb{R} \to \mathbb{T}^4$. The quotient $(G/N, \pi)$ is a torus $\mathbb{T}^4$ with trivial Poisson Lie group structure. Consider the subgroup $H = \{(e^{2ti}, e^{2\sqrt{2}ti}, 1, 1, t) \mid t \in \mathbb{R}\}$ of G. Then H is closed in G and each Dirac subspace $\mathfrak{D} \subseteq \mathbb{R}^5 \times \mathbb{R}^{5*}$ with $(\mathfrak{h} + \mathfrak{g}_0) \times \{0\} \subseteq \mathfrak{g} \times (\mathfrak{h}^\circ \cap \mathfrak{p}_1)$ induces a homogeneous Dirac structure on $G/H \simeq \mathbb{T}^4$. The subgroup $N \cdot H$ of G is dense in $\mathbb{T}^2 \times \{1\}^2 \times \mathbb{R} \subseteq G$ and is hence not closed in G.

Consider now the Dirac subspace

$$\mathfrak{D} := \operatorname{span}\left\{ \begin{array}{c} (\partial_{s_1}(0), 0), \quad (\partial_{s_2}(0), 0), \quad (\sqrt{3}\partial_{s_3}(0) + \partial_{s_4}(0), 0), \\ (\partial_t(0), 0), \quad (X, \mathbf{d}s_3(0) - \sqrt{3}\mathbf{d}s_4(0)) \end{array} \right\} \text{ of } \mathfrak{g} \times \mathfrak{g}^*$$

with $X \in \mathfrak{g}$ an arbitrary vector satisfying $(\mathbf{d}s_3(0) - \sqrt{3}\mathbf{d}s_4(0))(X) = 0$. The Dirac structure $\mathsf{D} = \mathfrak{D}^l$ defines a Dirac homogeneous space structure of (G, D_G) since $\mathfrak{g}_0 + \mathfrak{h} \subseteq \mathfrak{g}_0'$, but the leaf J of $\mathsf{G}_0{}'$ through the neutral element 0 is equal to $J = \{(e^{\theta i}, e^{\phi i}, e^{\sqrt{3}ti}, e^{ti}, s) \mid \theta, \phi, t, s \in \mathbb{R}\}$ and thus dense in G. $\Diamond$

5 The geometry of Dirac groupoids

Recall from Theorem 2.1.9 that the Poisson homogeneous spaces of a Poisson groupoid $(G\rightrightarrows P, \pi_G)$ are classified in terms of Dirac structures in the Courant algebroid $AG\times_P A^*G$. To generalize this to a classification of the Dirac homogeneous spaces of a Dirac groupoid, one needs hence to determine the object that will play the role of the Lie bialgebroid in this more general situation. We show in this chapter that an integrable multiplicative Dirac structure on $G\rightrightarrows P$ defines a Courant algebroid on P. In the case of a Poisson groupoid, we recover the Courant algebroid structure on $AG \times_P A^*G$ as in (1.5), and in the case of a Lie groupoid endowed with a closed multiplicative 2-form, we find simply the standard Courant bracket on $TP \times_P T^*P$. This new approach shows how to see the Courant algebroid structure on $AG \times_P A^*G$ as induced by the ambient Courant algebroid structure on $TG \times_G T^*G$.

The geometry is more involved in the Lie groupoid case than in the Lie group case, where the Lie bialgebra of the Dirac Lie group can be defined using the theory that is already known about Poisson Lie groups. By Theorem 4.3.1 and Remark 4.1.8, we can always construct the Lie bialgebra of a Dirac Lie group by considering a Poisson Lie group that is canonically associated to the Dirac Lie group. As we will see, the reason for this is that the characteristic distribution of a Dirac Lie group is exactly the kernel of the restriction to D_G of the source and target maps $\mathbb{T}\mathsf{s}$ and $\mathbb{T}\mathsf{t}$ and this is not the case in the general situation of Dirac groupoids. The next issue is the fact that multiplicative foliations on Lie groups are much more easy to handle with than multiplicative foliations on Lie groupoids, as we have seen in Chapter 3. We will see in the first section of this chapter that, under strong regularity conditions on the characteristic distribution of a Lie groupoid, there is a smooth surjective forward Dirac submersion on a Poisson groupoid (see Theorem 5.1.2). Yet, the hypotheses that have to be made on G_0 together with the technicalities in Chapter 3 give a flavor of the difficulties in the groupoid case. There is no way of defining a Lie bialgebroid by this associated Poisson groupoid, since it doesn't exist in general. The constructions that we have made to recover the Lie bialgebra in Chapter 4 are nevertheless useful guidelines for the general case, that will be treated in the same spirit.
Along the way, we will define several Lie algebroids and a Courant algebroid associated to an integrable Dirac groupoid. These objects turn out to generalize in a sense the infinitesimal data known in the presymplectic and Poisson cases. In particular, we give an integrability criterion for Dirac groupoids in terms of these algebroids. In Mackenzie and Xu (2000) the multiplicative Poisson bivector field on $G\rightrightarrows P$ integrating a given Lie bialgebroid is constructed. It seems that the main difficulty is to show that the bivector field is Poisson. Our integrability criterion gives an alternative method for this.

Although the work in this chapter was originally a preparation for the classification of the homogeneous spaces of a Dirac groupoid, it has become independently interesting since we find new results on the infinitesimal data of Dirac groupoids.

Outline of the chapter In Section 5.1, we show a theorem on the quotient of a "regular" Dirac groupoid by its characteristic foliation. Then we give (under some hypotheses) a generalization to Dirac groupoids of a theorem in Weinstein (1988) about the induced Poisson structure on the units of a Poisson groupoid.
In Section 5.2, we study the set $\mathfrak{A}(\mathsf{D}_G)$ of units of a multiplicative Dirac structure, seen as a subgroupoid of $(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$. We show that if the Dirac Lie groupoid $(G \rightrightarrows P, \mathsf{D}_G)$ is integrable, then there is a Lie algebroid structure on this vector bundle over P. In Section 5.3, we show that the integrability of the Dirac groupoid is completely encoded in the Lie algebroids that we find in the previous section. In the next section 5.4, we define a vector bundle over P that is associated to the Dirac structure D_G. We prove the existence of a Courant algebroid structure on this vector bundle if the Dirac structure is integrable. In Section 5.5, we prove that there is an induced action of the bisections of $G \rightrightarrows P$ on the vector bundle defined in Section 5.4. In Sections 5.2, 5.4 and 5.5, each one of the main results is illustrated by the three special examples of Poisson Lie groupoids, multiplicative 2-forms on Lie groupoids and pair Dirac groupoids.

5.1 General facts

First, we study the characteristic distribution of an arbitrary Dirac groupoid. The results here illustrate how the situation in the case of Dirac groupoids is different from the case of Dirac Lie groups.

Proposition 5.1.1 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid. Then the subbundle $\mathsf{G}_0 \subseteq TG$ is a (set) subgroupoid over $TP \cap \mathsf{G}_0$.*

PROOF: Choose $(g,h) \in G \times_P G$ and $v_g \in \mathsf{G}_0(g)$, $v_h \in \mathsf{G}_0(h)$ such that $T\mathsf{s}(v_g) = T\mathsf{t}(v_h)$. Then we have $(v_g, 0_g) \in \mathsf{D}_G(g)$, $(v_h, 0_h) \in \mathsf{D}_G(h)$ such that $\mathbb{T}\mathsf{s}(v_g, 0_g) = (T\mathsf{s}(v_g), 0_{\mathsf{s}(g)}) = (T\mathsf{t}(v_h), 0_{\mathsf{t}(h)}) = \mathbb{T}\mathsf{t}(v_h, 0_h)$ and hence

$$\mathbb{T}\mathsf{t}(v_g, 0_g) \in \mathsf{D}_G(\mathsf{t}(g)), \qquad \mathbb{T}\mathsf{s}(v_g, 0_g) \in \mathsf{D}_G(\mathsf{s}(g)),$$

$$(v_g, 0_g)^{-1} \in \mathsf{D}_G(g^{-1})$$

and

$$(v_g, 0_g) \star (v_h, 0_h) \in \mathsf{D}_G(g \star h).$$

Since $(v_g, 0_g)^{-1} = (v_g^{-1}, 0_{g^{-1}})$ and $(v_g, 0_g) \star (v_h, 0_h) = (v_g \star v_h, 0_{g \star h})$, this shows that $T\mathsf{s}(v_g) \in \mathsf{G}_0(\mathsf{s}(g))$, $T\mathsf{t}(v_g) \in \mathsf{G}_0(\mathsf{t}(g))$, $v_g^{-1} \in \mathsf{G}_0(g^{-1})$ and $v_g \star v_h \in \mathsf{G}_0(g \star h)$. $\square$

Unlike the special group case, the distribution G_0 doesn't need here to be smooth in general. If G_0 associated to an *integrable* Dirac groupoid $(G\rightrightarrows P, \mathsf{D}_G)$ is *assumed* to be a vector bundle on G, then we are in the same situation as in the group case. Yet, we know by the considerations in Chapter 3 that, even if it is regular, the quotient G/G_0 doesn't necessarily inherit a groupoid structure. If G_0 is complete, the leaf space G/G_0 inherits a multiplication map if and only if the leaves of G_0 satisfy the technical condition (3.5). Hence, this is an additional (topological) obstruction to the existence of a Poisson groupoid structure on the quotient G/G_0.
Since each manifold can be seen as a (trivial) groupoid over itself (i.e., with $\mathsf{t} = \mathsf{s} = \mathrm{Id}_M$), any Dirac manifold can be seen as a Dirac groupoid, which will, in general not satisfy these conditions. Thus, trivial Dirac groupoids and pair Dirac groupoids yield already many examples of Dirac groupoids that do not have these properties and the class of Dirac groupoids described in Theorem 5.1.2 seems to be a small class of examples.

Theorem 5.1.2 *Let $(G\rightrightarrows P, \mathsf{D}_G)$ be an integrable Dirac groupoid. Assume that G_0 is a subbundle of TG, that it is complete and that the leaves of G_0 satisfy* (3.5). *If the leaf spaces G/G_0 and P/G_0 have smooth manifold structures such that the projections are submersions, then there is an induced multiplicative Poisson structure on the Lie groupoid $G/\mathsf{G}_0 \rightrightarrows P/\mathsf{G}_0$, such that the projection $\mathrm{pr} : G \to G/\mathsf{G}_0$ is a forward Dirac map.*

PROOF: Recall that since D_G is integrable, the vector bundle G_0 is involutive. Since G_0 is multiplicative by Proposition 5.1.1 and all the hypotheses for Theorem 3.3.11 are satisfied, we get that $G/\mathsf{G}_0\rightrightarrows P/\mathsf{G}_0$ has the structure of a Lie groupoid such that if $\mathrm{pr} : G \to G/\mathsf{G}_0$ and $\mathrm{pr}_\circ : P \to P/\mathsf{G}_0$ are the projections, then $(\mathrm{pr}, \mathrm{pr}_\circ)$ is a Lie groupoid morphism.
We have $\mathsf{D}_G \cap (TG \times_G {\mathsf{G}_0}^\circ) = \mathsf{D}_G \cap (TG \times_G \mathsf{P}_1) = \mathsf{D}_G$ and $[\Gamma(\mathsf{G}_0 \times_G 0_{T^*G}), \Gamma(\mathsf{D}_G)] \subseteq \Gamma(\mathsf{D}_G)$ because D_G is integrable. Hence, by a result in Jotz et al. (2011a), we find that the Dirac structure pushes-forward to the quotient G/G_0. The Dirac structure $\mathrm{pr}(\mathsf{D}_G)$ is given by

$$\mathrm{pr}(\mathsf{D}_G)([g]) = \left\{ (v_{[g]}, \alpha_{[g]}) \in \mathsf{P}_{G/\mathsf{G}_0}([g]) \;\middle|\; \begin{array}{c} \exists v_g \in T_gG \text{ such that } (v_g, (T_g\,\mathrm{pr})^*\alpha_{[g]}) \in \mathsf{D}_G(g) \\ \text{and } T_g\,\mathrm{pr}\, v_g = v_{[g]} \end{array} \right\}$$

for all $g \in G$. The integrability of $\mathrm{pr}(\mathsf{D}_G)$ follows from the integrability of D_G. If $(v_{[g]}, 0) \in \mathrm{pr}(\mathsf{D}_G)([g])$, there exists $v_g \in T_gG$ such that $T_g\,\mathrm{pr}\, v_g = v_{[g]}$ and $(v_g, 0) \in \mathsf{D}_G(g)$. But then we get $v_g \in \mathsf{G}_0(g)$ and hence $v_{[g]} = T_g\,\mathrm{pr}\, v_g = 0_{[g]}$. This shows that the characteristic distribution G_0 associated to the Dirac structure $\mathrm{pr}(\mathsf{D}_G)$ is trivial, and since it is integrable, $\mathrm{pr}(\mathsf{D}_G)$ is the graph of the vector bundle homomorphism $T^*(G/\mathsf{G}_0) \to T(G/\mathsf{G}_0)$ associated to a Poisson bivector on G/G_0.
We have then to show that the Dirac structure $\mathrm{pr}(\mathsf{D}_G)$ on G/G_0 is multiplicative. Choose $(v_{[g]}, \alpha_{[g]}) \in \mathrm{pr}(\mathsf{D}_G)([g])$ and $(v_{[h]}, \alpha_{[h]}) \in \mathrm{pr}(\mathsf{D}_G)([h])$ such that $\mathbb{T}\mathsf{s}(v_{[g]}, \alpha_{[g]}) = \mathbb{T}\mathsf{t}(v_{[h]}, \alpha_{[h]})$. We can then assume without loss of generality that $\mathsf{s}(g) = \mathsf{t}(h)$ (see Lemma 3.3.14). By the definition of $\mathrm{pr}(\mathsf{D}_G)$, we find then $v_g \in T_gG$ and $v_h \in T_hG$ such that $T_g\,\mathrm{pr}\, v_g = v_{[g]}$, $T_h\,\mathrm{pr}\, v_h = v_{[h]}$ and $(v_g, (T_g\,\mathrm{pr})^*\alpha_{[g]}) \in \mathsf{D}_G(g)$, $(v_h, (T_h\,\mathrm{pr})^*\alpha_{[h]}) \in \mathsf{D}_G(h)$. Since

$$\mathbb{T}\mathsf{s}(v_g, (T_g\,\mathrm{pr})^*\alpha_{[g]}) \in \mathsf{D}_G(\mathsf{s}(g))$$

and

$$T_{\mathsf{s}(g)}\operatorname{pr}(T_g\mathsf{s}v_g) = T_{[g]}[\mathsf{s}]v_{[g]},$$
$$(T_{\mathsf{s}(g)}\operatorname{pr})^*\left(\hat{[\mathsf{s}]}(\alpha_{[g]})\right) = \hat{\mathsf{s}}((T_g\operatorname{pr})^*\alpha_{[g]}),$$

we find that

$$\mathbb{T}[\mathsf{s}](v_{[g]},\alpha_{[g]}) \in \operatorname{pr}(\mathsf{D}_G)([\mathsf{s}][g])$$

(see Lemma 3.3.14). In the same manner, we get that $\mathbb{T}[\mathsf{t}](v_{[g]},\alpha_{[g]}) \in \operatorname{pr}(\mathsf{D}_G)([\mathsf{t}][g])$ and the Dirac structure is closed under the source and target maps on $\mathsf{P}_{G/\mathsf{G}_0}$.
By Lemma 3.3.14, we find $w_g \in \mathsf{G}_0(g)$ such that $T_g\mathsf{s}(v_g - w_g) = T_h\mathsf{t}v_h$ and $T_{g\star h}\operatorname{pr}((v_g - w_g)\star v_h) = v_{[g]}\star v_{[h]}$. By the same Lemma, we have $\hat{\mathsf{s}}((T_g\operatorname{pr})^*\alpha_{[g]}) = \hat{\mathsf{t}}((T_h\operatorname{pr})^*\alpha_{[h]})$ and $(T_g\operatorname{pr})^*\alpha_{[g]}\star(T_h\operatorname{pr})^*\alpha_{[h]} = (T_{g\star h}\operatorname{pr})^*(\alpha_{[g]}\star\alpha_{[h]})$. The pairs $(v_g - w_g, (T_g\operatorname{pr})^*\alpha_{[g]}) \in \mathsf{D}_G(g)$ and $(v_h, (T_h\operatorname{pr})^*\alpha_{[h]}) \in \mathsf{D}_G(h)$ are hence compatible and their product,

$$((v_g - w_g)\star v_h, (T_{g\star h}\operatorname{pr})^*(\alpha_{[g]}\star\alpha_{[h]}))$$

is an element of $\mathsf{D}_G(g\star h)$. Since it pushes forward to $(v_{[g]}\star v_{[h]}, \alpha_{[g]}\star\alpha_{[h]})$, we find that $(v_{[g]}\star v_{[h]}, \alpha_{[g]}\star\alpha_{[h]}) \in \operatorname{pr}(\mathsf{D}_G)([g]\star[h])$.
It remains to show that the inverse $(v_{[g]},\alpha_{[g]})^{-1}$ is an element of $\operatorname{pr}(\mathsf{D}_G)([g]^{-1})$. Recall that $[g]^{-1} = [g^{-1}]$. We have $(v_g, (T_g\operatorname{pr})^*\alpha_{[g]})^{-1} \in \mathsf{D}_G(g^{-1})$. Since $\left((T_g\operatorname{pr})^*\alpha_{[g]}\right)\star\left((T_{g^{-1}}\operatorname{pr})^*\alpha_{[g]}^{-1}\right) = \hat{\mathsf{t}}((T_g\operatorname{pr})^*\alpha_{[g]})$ and in the same manner $\left((T_{g^{-1}}\operatorname{pr})^*\alpha_{[g]}^{-1}\right)\star\left((T_g\operatorname{pr})^*\alpha_{[g]}\right) = \hat{\mathsf{s}}((T_g\operatorname{pr})^*\alpha_{[g]})$, we find that $(T_{g^{-1}}\operatorname{pr})^*\alpha_{[g]}^{-1} = ((T_g\operatorname{pr})^*\alpha_{[g]})^{-1}$. Since $(\operatorname{pr},\operatorname{pr}_\circ)$ is a morphism of Lie groupoids, we find also that $T_{g^{-1}}\operatorname{pr}(v_g^{-1}) = (T_g\operatorname{pr}v_g)^{-1} = v_{[g]}^{-1}$. Thus, $(v_g, (T_g\operatorname{pr})^*\alpha_{[g]})^{-1} \in \mathsf{D}_G(g^{-1})$ pushes forward to $(v_{[g]},\alpha_{[g]})^{-1}$, which is consequently an element of $\operatorname{pr}(\mathsf{D}_G)([g]^{-1})$. $\square$

Remark 5.1.3 In the Lie group case, the Poisson Lie group $(G/N, q(\mathsf{D}_G))$ associated to an integrable Dirac Lie group (G,D_G) satisfying the necessary regularity assumptions was also a *Poisson homogeneous space* of the Dirac Lie group. Here, the Poisson Lie groupoid associated to the Dirac groupoid is, in general, *not* a Poisson homogeneous space of the Dirac groupoid since the quotient G/G_0 is not a homogeneous space of the Lie groupoid $G\rightrightarrows P$. $\triangle$

For the sake of completeness, we show next how the result in Weinstein (1988) about the induced Poisson structure on the units of a Poisson Lie groupoid can be generalized to the situation of Dirac groupoids. For that, we need to study the units of the Dirac groupoid. It is natural to ask what the set of units of D_G is, when seen as a subgroupoid of $(TG\times_G T^*G)\rightrightarrows(TP\times_P A^*G)$. It is easy to see that D_G is a Lie groupoid over $\mathsf{D}_G\cap(TP\times_P A^*G)$. We will write $\mathfrak{A}(\mathsf{D}_G) := \mathsf{D}_G\cap(TP\times_P A^*G)$ for the set of units of D_G. Here, we will show that it is a vector bundle over P.

Definition 5.1.4 *1. Let $(G\rightrightarrows P,\mathsf{D}_G)$ be a Dirac groupoid and $\mathfrak{A}(\mathsf{D}_G)$ the set of units of D_G, i.e., the subdistribution $\mathsf{D}_G\cap(TP\times_P A^*G)$ of $TP\times_P A^*G$. We write $\mathsf{a}_\star : \mathfrak{A}(\mathsf{D}_G)\to TP$ for the map defined by $\mathsf{a}_\star(v_p,\alpha_p) = v_p$ for all $p\in P$, $(v_p,\alpha_p)\in\mathfrak{A}_p(\mathsf{D}_G)$.*

2. *We write* $\ker \mathbb{T}\mathsf{s}$*, respectively* $\ker \mathbb{T}\mathsf{t}$ *for the kernel* $T^{\mathsf{s}}G \times_G (T^{\mathsf{t}}G)^\circ$ *(respectively* $T^{\mathsf{t}}G \times_G (T^{\mathsf{s}}G)^\circ$*) of the source map* $\mathbb{T}\mathsf{s} : \mathsf{P}_G \to TP \times_P A^*G$ *(respectively the target map* $\mathbb{T}\mathsf{t} : \mathsf{P}_G \to TP \times_P A^*G$*). We denote by* $I^{\mathsf{s}}(\mathsf{D}_G)$ *the restriction to* P *of* $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s}$*, i.e.,*

$$I^{\mathsf{s}}(\mathsf{D}_G) := \mathsf{D}_G \cap (T^{\mathsf{s}}_P G \times_G (T^{\mathsf{t}}_P G)^\circ) = (\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})|_P.$$

In the same manner, we write $I^{\mathsf{t}}(\mathsf{D}_G) := \mathsf{D}_G \cap (T^{\mathsf{t}}_P G \times_G (T^{\mathsf{s}}_P G)^\circ) = (\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t})|_P$.

Theorem 5.1.5 *Let* $(G \rightrightarrows P, \mathsf{D}_G)$ *be a Dirac groupoid. Then the Dirac subspace* $\mathsf{D}_G|_P$ *splits as a direct sum*

$$\begin{aligned}\mathsf{D}_G|_P &= (\mathsf{D}_G \cap (TP \times_P (TP)^\circ)) \oplus (\mathsf{D}_G \cap (T^{\mathsf{t}}_P G \times_P (T^{\mathsf{s}}_P G)^\circ)) \\ &= \mathfrak{A}(\mathsf{D}_G) \oplus I^{\mathsf{t}}(\mathsf{D}_G)\end{aligned}$$

and in the same manner

$$\begin{aligned}\mathsf{D}_G|_P &= (\mathsf{D}_G \cap (TP \times_P (TP)^\circ)) \oplus (\mathsf{D}_G \cap (T^{\mathsf{s}}_P G \times_P (T^{\mathsf{t}}_P G)^\circ)) \\ &= \mathfrak{A}(\mathsf{D}_G) \oplus I^{\mathsf{s}}(\mathsf{D}_G).\end{aligned}$$

The three intersections are smooth and have constant rank on P*.*

Proof: Choose $p \in P$ and $(v_p, \alpha_p) \in \mathsf{D}_G(p)$. Then we have $\mathbb{T}\mathsf{t}(v_p, \alpha_p) \in \mathsf{D}_G(p)$ and hence also $(v_p, \alpha_p) - \mathbb{T}\mathsf{t}(v_p, \alpha_p) \in \mathsf{D}_G(p)$. We find that $v_p - T_p\mathsf{t} v_p \in T^{\mathsf{t}}_p G$ and $T_p\mathsf{t}(v_p) \in T_pP$, and in the same manner $\hat{\mathsf{t}}(\alpha_p) \in A^*_pG = (T_pP)^\circ$, by definition, and $\alpha_p - \hat{\mathsf{t}}(\alpha_p) \in (T^{\mathsf{s}}_p G)^\circ$.
Since

$$(v_p, \alpha_p) = \mathbb{T}\mathsf{t}(v_p, \alpha_p) + ((v_p, \alpha_p) - \mathbb{T}\mathsf{t}(v_p, \alpha_p)),$$

we have shown the first equality. The second formula can be shown in the same manner, using the map $\mathbb{T}\mathsf{s} : \mathsf{D}_G(p) \to \mathsf{D}_G(p) \cap (T_pP \times A^*_pG)$.

Next, we show that the intersection of D_G with $TP \times_P A^*G$ is smooth. Choose $p \in P$ and $(v_p, \alpha_p) \in \mathsf{D}_G(p) \cap (T_pP \times A^*_pG)$. Since D_G is a smooth vector bundle on G, we find a section $(X, \alpha) \in \Gamma(\mathsf{D}_G)$ defined on a neighborhood of p such that $(X, \alpha)(p) = (v_p, \alpha_p)$. The restriction $(X, \alpha)|_P$ is then a smooth section of $\mathsf{D}_G|_P$. We have $\mathbb{T}\mathsf{s}((X, \alpha)|_P) \in \Gamma(\mathsf{D}_G \cap (TP \times_P A^*G))$ and $\mathbb{T}\mathsf{s}(X, \alpha)(p) = (T_p\mathsf{s} v_p, \alpha_p|_{T^{\mathsf{t}}_p G}) = (v_p, \alpha_p)$ since $v_p \in T_pP$ and $\alpha_p \in A^*_pG = (T_pP)^\circ$.
Thus, we have found a smooth section of $\mathsf{D}_G \cap (TP \times_P A^*G)$ defined on a neighborhood of p in P and taking value (v_p, α_p) at p.

Since $(\mathsf{D}_G|_P)^\perp = \mathsf{D}_G|_P$ and $TP \times_P A^*G = (TP \times_P A^*G)^\perp$ are smooth subbundles of $\mathsf{P}_G|_P$, we get from Proposition 4.4 in Jotz et al. (2011b) that $\mathsf{D}_G \cap (TP \times_P A^*G)$ has constant rank on P. By the splittings shown above and the fact that $\mathsf{D}_G|_P$ has constant rank on P, we find that the two other intersections have constant rank on P, and are thus smooth. $\square$

In the case of a Dirac Lie group, the bundle $I^{\mathsf{s}}(\mathsf{D}_G) \to P$ is $\mathfrak{g}_0 \to \{e\}$, as shows the next example. We will see later that $I^{\mathsf{s}}(\mathsf{D}_G)$ has a crucial role in the construction of the Courant algebroid associated to a Dirac groupoid $(G\rightrightarrows P, \mathsf{D}_G)$. The fact that the left and right invariant images of this subspace are exactly the characteristic distribution of the Dirac structure is a very special and convenient feature in the group case, that makes the Dirac Lie groups much easier to understand than arbitrary Dirac groupoids (see Chapter 4).

Example 5.1.6 If (G, D_G) is a Dirac Lie group, we have $P = \{e\}$ (the neutral element of G),

$$\mathsf{D}_G(e) \cap (T_eP \times (T_eP)^\circ) = \mathsf{D}_G(e) \cap (\{0\} \times \mathfrak{g}^*) = \{0\} \times \mathfrak{p}_1$$

and

$$\mathsf{D}_G(e) \cap (T_e^{\mathsf{s}}G \times (T_e^{\mathsf{t}}G)^\circ) = \mathsf{D}_G(e) \cap (\mathfrak{g} \times \{0\}) = \mathfrak{g}_0 \times \{0\}.$$

We recover hence the equality $\mathsf{D}_G(e) = \mathfrak{g}_0 \times \mathfrak{p}_1$ (Proposition 4.1.3).
In this particular case, D_G is a Poisson structure if and only if $\mathsf{D}_G(e)$ is equal to the set of units of $TG \times_G T^*G \rightrightarrows \{0\} \times \mathfrak{g}^*$, i.e., $\mathfrak{g}_0 = \{0\}$ and $\mathfrak{p}_1 = \mathfrak{g}^*$. In the general case, this is not true since the intersection with $TP \times_P A^*G$ of the graph of a Poisson bivector π_G on a Lie groupoid G is equal to the graph of the restriction of $\pi_G^\sharp$ to A^*G. ♢

Lemma 5.1.7 *Let $(G\rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid. For all $g \in G$, we have*

$$\mathsf{D}_G(g) \cap \ker \mathbb{T}\mathsf{t} = (0_g, 0_g) \star I^{\mathsf{t}}_{\mathsf{s}(g)}(\mathsf{D}_G)$$

and

$$\mathsf{D}_G(g) \cap \ker \mathbb{T}\mathsf{s} = I^{\mathsf{s}}_{\mathsf{t}(g)}(\mathsf{D}_G) \star (0_g, 0_g).$$

The intersections $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t}$ and $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s}$ have consequently constant rank on G.

PROOF: Choose $g \in G$ and $v_{\mathsf{s}(g)} \in T^{\mathsf{t}}_{\mathsf{s}(g)}G$, $\alpha_{\mathsf{s}(g)} \in T^*_{\mathsf{s}(g)}P$ such that $(v_{\mathsf{s}(g)}, (T_{\mathsf{s}(g)}\mathsf{s})^*\alpha_{\mathsf{s}(g)}) \in \mathsf{D}_G(\mathsf{s}(g))$. Then we have $\mathbb{T}\mathsf{t}(u_{\mathsf{s}(g)}, (T_{\mathsf{s}(g)}\mathsf{s})^*\alpha_{\mathsf{s}(g)}) = (0_{\mathsf{s}(g)}, 0_{\mathsf{s}(g)})$ and $(0_g, 0_g) \in \mathsf{D}_G(g)$ with $\mathbb{T}\mathsf{s}(0_g, 0_g) = (0_{\mathsf{s}(g)}, 0_{\mathsf{s}(g)})$. Thus, the product

$$(0_g, 0_g) \star (u_{\mathsf{s}(g)}, (T_{\mathsf{s}(g)}\mathsf{s})^*\alpha_{\mathsf{s}(g)})$$

makes sense and is an element of $\mathsf{D}_G(g) \cap \ker \mathbb{T}\mathsf{t}$.

Conversely, choose $(v_g, \alpha_g) \in \mathsf{D}_G(g) \cap (T^{\mathsf{t}}_gG \times_G (T^{\mathsf{s}}_gG)^\circ)$. Since $\mathbb{T}\mathsf{t}(v_g, \alpha_g) = (0_{\mathsf{t}(g)}, 0_{\mathsf{t}(g)}) = \mathbb{T}\mathsf{s}(0_{g^{-1}}, 0_{g^{-1}})$ and $(0_{g^{-1}}, 0_{g^{-1}}) \in \mathsf{D}_G(g^{-1})$, the composition

$$(0_{g^{-1}}, 0_{g^{-1}}) \star (v_g, \alpha_g)$$

makes sense and is an element of $\mathsf{D}_G(\mathsf{s}(g)) \cap \ker \mathbb{T}\mathsf{t} = I^{\mathsf{t}}_{\mathsf{s}(g)}(\mathsf{D}_G)$. Since $(0_{g^{-1}}, 0_{g^{-1}}) = (0_g, 0_g)^{-1}$, we have shown that $(v_g, \alpha_g) \in (0_g, 0_g) \star I^{\mathsf{t}}_{\mathsf{s}(g)}(\mathsf{D}_G)$.

There is hence an isomorphism

$$\mathsf{D}_G(\mathsf{s}(g)) \cap \left(T^{\mathsf{t}}_{\mathsf{s}(g)}G \times (T^{\mathsf{s}}_{\mathsf{s}(g)}G)^\circ\right) \leftrightarrow \mathsf{D}_G(g) \cap \left(T^{\mathsf{t}}_gG \times (T^{\mathsf{s}}_gG)^\circ\right).$$

As a consequence, $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t}$ has constant rank along s-fibers. Since $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t}$ has constant rank on P by Theorem 5.1.5, it has hence constant rank on the whole of G. □

Example 5.1.8 If $(G \rightrightarrows P, \pi_G)$ is a Poisson Lie groupoid, then $\pi_G^\sharp(\mathbf{d}(\mathsf{s}^*f)) \in \Gamma(T^{\mathsf{t}}G)$ for all $f \in C^\infty(P)$ (see Weinstein (1988)). The intersection $\mathsf{D}_{\pi_G} \cap \ker \mathbb{T}\mathsf{t}$ is hence spanned by the sections $(\pi_G^\sharp(\mathbf{d}(\mathsf{s}^*f)), \mathbf{d}(\mathsf{s}^*f))$, with $f \in C^\infty(P)$, and has constant rank. The intersection $\mathsf{D}_{\pi_G} \cap \ker \mathbb{T}\mathsf{s}$ is spanned by the sections $(\pi_G^\sharp(\mathbf{d}(\mathsf{t}^*f)), \mathbf{d}(\mathsf{t}^*f))$ with $f \in C^\infty(P)$.$\Diamond$

Using this, we will show the next main theorem of this section. We will need the following lemma.

Lemma 5.1.9 *Let $G \rightrightarrows P$ be a Lie groupoid. Choose $g \in G$ and set $p = \mathsf{t}(g)$. Then, for all $\alpha_p \in T_p^*P$, we have*

$$-(T_{g^{-1}}\mathsf{s})^*\alpha_p = ((T_g\mathsf{t})^*\alpha_p)^{-1}.$$

PROOF: Compute for any $u_p \in A_pG$, :

$$\begin{aligned}\hat{\mathsf{t}}((T_g\mathsf{t})^*\alpha_p)(u_p) &= ((T_g\mathsf{t})^*\alpha_p)(T_pR_g(u_p - T_p\mathsf{s}u_p)) = \alpha_p(T_g\mathsf{t}(T_pR_g(u_p - T_p\mathsf{s}u_p)))\\ &= \alpha_p(T_p\mathsf{t}(u_p - T_p\mathsf{s}u_p)) = \alpha_p(T_p\mathsf{t}u_p - T_p\mathsf{s}u_p) = -((T_p\mathsf{s})^*\alpha_p)(u_p)\end{aligned}$$

and

$$\begin{aligned}\hat{\mathsf{s}}(-(T_{g^{-1}}\mathsf{s})^*\alpha_p)(u_p) &= -((T_{g^{-1}}\mathsf{s})^*\alpha_p)(T_pL_{g^{-1}}u_p)\\ &= -\alpha_p(T_{g^{-1}}\mathsf{s}(T_pL_{g^{-1}}u_p)) = -\alpha_p(T_p\mathsf{s}u_p) = -((T_p\mathsf{s})^*\alpha_p)(u_p).\end{aligned}$$

In the same manner, we set $q = \mathsf{t}(g^{-1}) = \mathsf{s}(g)$ and compute for any $u_q \in A_qG$:

$$\begin{aligned}\hat{\mathsf{t}}(-(T_{g^{-1}}\mathsf{s})^*\alpha_p)(u_q) &= -((T_{g^{-1}}\mathsf{s})^*\alpha_p)(T_qR_{g^{-1}}(u_q - T_q\mathsf{s}u_q))\\ &= -\alpha_p(T_{g^{-1}}\mathsf{s}(T_qR_{g^{-1}}(u_q - T_p\mathsf{s}u_q))) = 0\end{aligned}$$

and

$$\hat{\mathsf{s}}((T_g\mathsf{t})^*\alpha_p)(u_q) = ((T_g\mathsf{t})^*\alpha_p)(T_qL_gu_q) = \alpha_p(T_g\mathsf{t}(T_qL_gu_q)) = 0.$$

Hence, we can compute $((T_g\mathsf{t})^*\alpha_p) \star (-(T_{g^{-1}}\mathsf{s})^*\alpha_p)$ and $(-(T_{g^{-1}}\mathsf{s})^*\alpha_p) \star ((T_g\mathsf{t})^*\alpha_p)$. We choose for any $u_q \in T_qG$ two vectors $u_{g^{-1}} \in T_{g^{-1}}G$ and $u_g \in T_gG$ such that $u_q = u_{g^{-1}} \star u_g$ and we get

$$\begin{aligned}\left((-(T_{g^{-1}}\mathsf{s})^*\alpha_p) \star ((T_g\mathsf{t})^*\alpha_p)\right)(u_q) &= (-(T_{g^{-1}}\mathsf{s})^*\alpha_p)(u_{g^{-1}}) + ((T_g\mathsf{t})^*\alpha_p)(u_g)\\ &= -\alpha_p(T_{g^{-1}}\mathsf{s}u_{g^{-1}}) + \alpha_p(T_g\mathsf{t}u_g)\\ &= -\alpha_p(T_g\mathsf{t}u_g) + \alpha_p(T_g\mathsf{t}u_g) = 0,\end{aligned}$$

which shows that $(-(T_{g^{-1}}\mathsf{s})^*\alpha_p) \star ((T_g\mathsf{t})^*\alpha_p) = 0_q = \hat{\mathsf{s}}((T_g\mathsf{t})^*\alpha_p)$. For any $w_p = w_g \star w_{g^{-1}} \in T_pG$, we compute in the same manner

$$\begin{aligned}\left(((T_g\mathsf{t})^*\alpha_p) \star (-(T_{g^{-1}}\mathsf{s})^*\alpha_p)\right)(w_p) &= \alpha_p(T_g\mathsf{t}w_g) - \alpha_p(T_{g^{-1}}\mathsf{s}w_{g^{-1}})\\ &= \alpha_p(T_p\mathsf{t}w_p) - \alpha_p(T_p\mathsf{s}w_p) = \alpha_p \circ (T_p\mathsf{t} - T_p\mathsf{s})(w_p).\end{aligned}$$

Thus, $((T_g\mathsf{t})^*\alpha_p) \star (-(T_{g^{-1}}\mathsf{s})^*\alpha_p) = \hat{\mathsf{t}}((T_g\mathsf{t})^*\alpha_p)$. $\square$

Remark 5.1.10 If $(v_p, (T_p\mathsf{s})^*\alpha_p)$ is such that $T_p\mathsf{t}v_p = 0_p$, then $\mathbb{T}\mathsf{t}(v_p, (T_p\mathsf{s})^*\alpha_p) = (0_p, 0_p)$. If $g \in G$ is such that $\mathsf{s}(g) = p$, then $(0_g, 0_g) \star (v_p, (T_p\mathsf{s})^*\alpha_p) = (T_pL_gv_p, (T_g\mathsf{s})^*\alpha_p)$ for all $g \in \mathsf{s}^{-1}(p)$.
To see this, let $c : (-\varepsilon, \varepsilon) \to \mathsf{t}^{-1}(p)$ be a curve such that $c(0) = p$ and $\dot{c}(0) = v_p$. We can then compute $0_g \star v_p = T_{(g,p)}\mathsf{m}(0_g, v_p) = \frac{d}{d\sigma}\big|_{\sigma=0} g \star c(\sigma) = T_pL_gv_p$. If $v_g \in T_gG$, the equality $v_g = v_g \star (T_g\mathsf{s}v_g)$ yields $(0_g \star (T_p\mathsf{s})^*\alpha_p)(v_g) = 0_g(v_g) + ((T_p\mathsf{s})^*\alpha_p)(T_g\mathsf{s}v_g) = \alpha_p(T_g\mathsf{s}v_g) = ((T_g\mathsf{s})^*\alpha_p)(v_g)$. $\triangle$

Now we can prove a generalization of the fact that the units of a Poisson groupoid inherit a Poisson structure such that the target map if a Poisson map and the source is anti-Poisson (see Weinstein (1988)).

Theorem 5.1.11 *Assume that $(G \rightrightarrows P, \mathsf{D}_G)$ is a Dirac groupoid such that $TP \cap \mathsf{G}_0$ is smooth. Define the subspace D_P of P_P by*

$$\mathsf{D}_P(p) = \left\{ (v_p, \alpha_p) \in \mathsf{P}_P(p) \;\middle|\; \begin{array}{c} \exists (w_p, (T_p\mathsf{t})^*\alpha_p) \in \mathsf{D}_G(p) \cap (T_pG \times (A_pG)^\circ) \\ \textit{such that } v_p = T_p\mathsf{t}w_p \end{array} \right\} \tag{5.1}$$

for all $p \in P$. Then D_P is a Dirac structure on P.
Furthermore, if for all $g \in G$, the restriction to $\mathsf{G}_0(g)$ of the target map $T_g\mathsf{t} : \mathsf{G}_0(g) \to \mathsf{G}_0(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P$ is surjective, then the maps $\mathsf{t} : (G, \mathsf{D}_G) \to (P, \mathsf{D}_P)$ and $\mathsf{s} : (G, \mathsf{D}_G) \to (P, -\mathsf{D}_P)$ are forward Dirac maps, where $-\mathsf{D}_P$ is the Dirac structure defined on P by $\mathsf{D}_P(p) = \{(-v_p, \alpha_p) \in \mathsf{P}_P(p) \mid (v_p, \alpha_p) \in \mathsf{D}_P(p)\}$.

The characteristic distribution ${\mathsf{G}_0}^P$ of (P, D_P) is then equal to the intersection $\mathsf{G}_0 \cap TP$. Note that this theorem generalizes Theorem 4.2.3 in Weinstein (1988) (see also Weinstein (1987), Coste et al. (1987) for the special case of symplectic groupoids), since in the Poisson case, we have $\mathsf{G}_0 = 0_{TG}$ and the hypotheses are consequently trivially satisfied. If all conditions are satisfied, the Dirac structure on P is just the push forward of the Dirac structure on G under the quotient map $\mathsf{t} : G \to G/G \simeq P$ (see Example 1.1.24).

Proof: First note that $I^{\mathsf{s}}(\mathsf{D}_G) \oplus ((\mathsf{G}_0 \cap TP) \times_P 0_{A^*G}) = \mathsf{D}_G \cap (T_PG \times_P AG^\circ)$. Indeed, we have obviously $I^{\mathsf{s}}(\mathsf{D}_G) \oplus ((\mathsf{G}_0 \cap TP) \times_P 0_{A^*G}) \subseteq \mathsf{D}_G \cap (T_PG \times_P AG^\circ)$, and conversely, if $(v_p, (T_p\mathsf{t})^*\alpha_p) \in \mathsf{D}_G(p) \cap (T_pG \times A_pG^\circ)$, we have $\mathbb{T}\mathsf{s}(v_p, (T_p\mathsf{t})^*\alpha_p) = (T_p\mathsf{s}v_p, 0) \in \mathsf{D}_G(p)$ and hence $(v_p, (T_p\mathsf{t})^*\alpha_p) = (v_p - T_p\mathsf{s}v_p, (T_p\mathsf{t})^*\alpha_p) + (T_p\mathsf{s}v_p, 0) \in I_p^{\mathsf{s}}(\mathsf{D}_G) + ((\mathsf{G}_0(p) \cap T_pP) \times \{0\})$. By the hypothesis on $\mathsf{G}_0 \cap TP$, the intersection $\mathsf{D}_G \cap (T_PG \times_P AG^\circ)$ is hence smooth and has consequently constant rank on P by a Proposition in Jotz et al. (2011b). The space D_P is smooth since it is spanned by the smooth sections of $\mathsf{G}_0 \cap TP$ and the smooth sections $(T\mathsf{t}X, \alpha)$ for all $(X^r, \mathsf{t}^*\alpha) \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$.
We show that $\mathsf{D}_P(p) = \mathsf{D}_P(p)^\perp$ for all $p \in P$. If $(T_p\mathsf{t}v_p, \alpha_p), (T_p\mathsf{t}w_p, \beta_p) \in \mathsf{D}_P(p)$, that is, with $(v_p, (T_p\mathsf{t})^*\alpha_p), (w_p, (T_p\mathsf{t})^*\beta_p) \in \mathsf{D}_G(p) \cap (T_pG \times (A_pG)^\circ)$, we have

$$\langle (T_p\mathsf{t}v_p, \alpha_p), (T_p\mathsf{t}w_p, \beta_p) \rangle = \langle (v_p, (T_p\mathsf{t})^*\alpha_p), (w_p, (T_p\mathsf{t})^*\beta_p) \rangle = 0$$

since $(v_p, (T_p\mathsf{t})^*\alpha_p), (w_p, (T_p\mathsf{t})^*\beta_p) \in \mathsf{D}_G(p)$. This shows the inclusion $\mathsf{D}_P(p) \subseteq \mathsf{D}_P(p)^\perp$.

Conversely, if $(v_p, \alpha_p) \in \mathsf{D}_P(p)^\perp \subseteq \mathsf{P}_P(p)$ and $y_p \in T_pG$ is chosen such that $T_p\mathsf{t}y_p = v_p$, then we have

$$\langle (w_p, (T_p\mathsf{t})^*\beta_p), (y_p, (T_p\mathsf{t})^*\alpha_p\rangle = \langle (T_p\mathsf{t}w_p, \beta_p), (v_p, \alpha_p)\rangle = 0$$

for all $(w_p, (T_p\mathsf{t})^*\beta_p) \in \mathsf{D}_G(p) \cap (T_pG \times A_pG^\circ)$. Hence, we get

$$(y_p, (T_p\mathsf{t})^*\alpha_p) \in (\mathsf{D}_G(p) \cap (T_pG \times (A_pG)^\circ))^\perp = (\mathsf{D}_G(p) + A_pG \times \{0_p\})$$

and consequently $(y_p, (T_p\mathsf{t})^*\alpha_p) = (y_p', (T_p\mathsf{t})^*\alpha_p) + (u_p, 0)$ for some $(y_p', (T_p\mathsf{t})^*\alpha_p) \in \mathsf{D}_G(p)$ and $u_p \in A_pG$. But then $T_p\mathsf{t}y_p' = T_p\mathsf{t}y_p = v_p$ and $(v_p, \alpha_p) \in \mathsf{D}_P(p)$.

Assume that the target map $T_g\mathsf{t} : \mathsf{G}_0(g) \to \mathsf{G}_0(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P$ is surjective for all $g \in G$. We show that $\mathsf{t} : (G, \mathsf{D}_G) \to (P, \mathsf{D}_P)$ is a forward Dirac map. Choose $p \in P$, $g \in \mathsf{t}^{-1}(p)$ and $(v_p, \alpha_p) \in \mathsf{D}_P(p)$. We have to prove that there exists $(v_g, \alpha_g) \in \mathsf{D}_G(g)$ such that $\alpha_g = (T_g\mathsf{t})^*\alpha_p$ and $T_g\mathsf{t}v_g = v_p$. By definition of D_P and the considerations above, there exists $u_p \in T_p^{\mathsf{s}}G$ and $z_p \in \mathsf{G}_0(p) \cap T_pP$ such that $T_p\mathsf{t}u_p + z_p = v_p$ and $(u_p, (T_p\mathsf{t})^*\alpha_p) \in I_p^{\mathsf{s}}(\mathsf{D}_G)$. Then the pair $(T_pR_gu_p, (T_g\mathsf{t})^*\alpha_p) = (u_p, (T_p\mathsf{t})^*\alpha_p) \star (0_g, 0_g)$ is an element of $\mathsf{D}_G(g)$ (this equality can be shown as in Remark 5.1.10) and by hypothesis, we find $z_g \in \mathsf{G}_0(g)$ such that $T_g\mathsf{t}z_g = z_p$. The pair $(T_pR_gu_p + z_g, (T_g\mathsf{t})^*\alpha_p)$ is then an element of $\mathsf{D}_G(g)$ and $T_g\mathsf{t}(T_pR_gu_p + z_g) = T_p\mathsf{t}u_p + z_p = v_p$.
It remains to prove that $\mathsf{s} : (G, \mathsf{D}_G) \to (P, -\mathsf{D}_P)$ is also a forward Dirac map. Choose $p \in P$, $g \in \mathsf{s}^{-1}(p)$ and $(v_p, \alpha_p) \in -\mathsf{D}_P(p)$. Then $(-v_p, \alpha_p) \in \mathsf{D}_P(p)$ and, since $\mathsf{t}(g^{-1}) = \mathsf{s}(g) = p$, there exists by the considerations above $w_g \in T_{g^{-1}}G$ such that $T_{g^{-1}}\mathsf{t}w_{g^{-1}} = v_p$ and $(-w_{g^{-1}}, (T_{g^{-1}}\mathsf{t})^*\alpha_p) \in \mathsf{D}_G(g^{-1})$. But by Lemma 5.1.9, we have then $((-w_{g^{-1}})^{-1}, -(T_g\mathsf{s})^*\alpha_p)$ $\in \mathsf{D}_G(g)$. This leads to $((w_{g^{-1}})^{-1}, (T_g\mathsf{s})^*\alpha_p) \in \mathsf{D}_G(g)$ and since $T_g\mathsf{s}((w_{g^{-1}})^{-1}) = T_{g^{-1}}\mathsf{t}w_{g^{-1}} = v_p$, the proof is finished. □

Note that the hypotheses on the distribution G_0 in Theorem 5.1.11 are rather strong. The following example shows that this theorem can hold under weaker hypotheses.

Example 5.1.12 Assume that (M, D_M) is a smooth Dirac manifold such that G_0 is a singular distribution. Then, the induced pair Dirac groupoid $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ as in Example 2.2.5 doesn't satisfy the conditions for Theorem 5.1.11. The space $I^{\mathsf{s}}(\mathsf{D}_M \ominus \mathsf{D}_M)$ is here given by

$$I^{\mathsf{s}}_{(m,m)}(\mathsf{D}_M \ominus \mathsf{D}_M) = \{(v_m, 0_m, \alpha_m, 0_m) \mid (v_m, \alpha_m) \in \mathsf{D}_M(m)\}$$

for all $m \in M$, and the space $\mathsf{G}_0 \cap T\Delta_M$ is given by

$$\mathsf{G}_0(m, m) \cap T_{(m,m)}\Delta_M = \{(v_m, v_m) \mid (v_m, 0_m) \in \mathsf{D}_M(m)\}$$

for all $m \in M$. Hence, we have

$$\begin{aligned} &I^{\mathsf{s}}_{(m,m)}(\mathsf{D}_M \ominus \mathsf{D}_M) + (\mathsf{G}_0(m, m) \cap T_{(m,m)}\Delta_M) \times_{M\times M} \{0\} \\ &= \{(v_m, w_m, \alpha_m, 0_m) \mid (v_m, \alpha_m) \in \mathsf{D}_M(m), (w_m, 0_m) \in \mathsf{D}_M(m)\} \end{aligned}$$

and we find that the same construction as in Theorem 5.1.11 defines a Dirac structure on $M \simeq \Delta_M$, which equals the original Dirac structure D_M on M since its fiber over $m \in M$ is given by $\{(v_m, \alpha_m) \mid (v_m, w_m, \alpha_m, 0_m) \in I^{\mathsf{s}}_{(m,m)}(\mathsf{D}_M \ominus \mathsf{D}_M) + \mathsf{G}_0(m, m) \cap T_{(m,m)}\Delta_M\}$. ◊

Example 5.1.13 Let $G \rightrightarrows P$ be a Lie groupoid and $\phi \in \Omega^3(P)$ a closed 3-form. A Dirac structure on P is said to be ϕ-twisted if it is closed under the *ϕ-twisted Courant bracket* defined by $[(X,\alpha),(Y,\beta)] = ([X,Y], \pounds_X\beta - \mathbf{i}_Y\mathbf{d}\alpha + \mathbf{i}_Y\mathbf{i}_X\phi)$ for all $(X,\alpha),(Y,\beta) \in \Gamma(\mathsf{P}_P)$. Let $\omega_G \in \Omega^2(G)$ be a 2-form on G. Then $(G \rightrightarrows P, \omega_G)$ is a ϕ-twisted presymplectic groupoid if ω_G is multiplicative and satisfies the following conditions:

1. $d\omega_G = \mathsf{s}^*\phi - \mathsf{t}^*\phi$,
2. $\dim G = 2\dim P$ and
3. $(\ker \omega_G)(p) \cap T_p^{\mathsf{t}}G \cap T_p^{\mathsf{s}}G = \{0_p\}$ for all $p \in P$.

Presymplectic groupoids were introduced by Bursztyn et al. (2004). It is shown there that if $(G \rightrightarrows P, \omega_G)$ is a ϕ-twisted presymplectic groupoid, then there exists a ϕ-twisted Dirac structure D_P on P such that the target map $\mathsf{t} : (G, \mathsf{D}_G) \to (P, \mathsf{D}_P)$ is a forward Dirac map.
Note that $(\ker \omega_G)(p) = \mathsf{G}_0(p)$, if G_0 is the characteristic distribution associated to the Dirac groupoid $(G \rightrightarrows P, \mathsf{D}_{\omega_G})$, see Example 2.2.4. In Bursztyn et al. (2004), a multiplicative 2-form is said to be *of Dirac type* if it has a property that is shown to be equivalent to our surjectivity condition on the restriction to the characteristic distribution $\mathsf{G}_0 = \ker \omega_G$ of the target map $T\mathsf{t}$. It is shown that if the bundle of Dirac structures defined as in (5.1) by the multiplicative Dirac structure D_{ω_G} associated to a multiplicative 2-form ω_G of Dirac type is smooth, then D_P is a Dirac structure on P such that the target map $\mathsf{t} : (G, \mathsf{D}_G) \to (P, \mathsf{D}_P)$ is a forward Dirac map. Thus, we recover here their two conditions since we made the hypothesis on smoothness of $\mathsf{G}_0 \cap TP$ to ensure the smoothness of D_P. It is shown in Bursztyn et al. (2004) that presymplectic groupoids satisfy automatically these conditions. $\Diamond$

Remark 5.1.14 In the situation of Theorem 5.1.2, the multiplicative subbundle G_0 of TG has constant rank on G. In particular, the intersection $TP \cap \mathsf{G}_0$ is a smooth vector bundle over P and for each $g \in G$, the restriction to $\mathsf{G}_0(g)$ of the target map, $T_g\mathsf{t} : \mathsf{G}_0(g) \to \mathsf{G}_0(\mathsf{t}(g)) \cap T_{\mathsf{t}(g)}P$, is surjective (see Lemma 3.2.1). By Theorem 5.1.11, there exists then a Dirac structure D_P on P such that $\mathsf{t} : (G, \mathsf{D}_G) \to (P, \mathsf{D}_P)$ is a forward Dirac map. Since $(G/\mathsf{G}_0 \rightrightarrows P/\mathsf{G}_0, \mathrm{pr}(\mathsf{D}_G))$ is a Poisson Lie groupoid, we know also by a theorem in Weinstein (1988) that there is a Poisson structure $\{\cdot\,,\cdot\}_{P/\mathsf{G}_0}$ on P/G_0 such that $[\mathsf{t}] : (G/\mathsf{G}_0, \mathrm{pr}(\mathsf{D}_G)) \to (P/\mathsf{G}_0, \{\cdot\,,\cdot\}_{P/\mathsf{G}_0})$ is a Poisson map. It is easy to check that the map $\mathrm{pr}_\circ : (P, \mathsf{D}_P) \to (P/\mathsf{G}_0, \{\cdot\,,\cdot\}_{P/\mathsf{G}_0})$ is then also a forward Dirac map, i.e., the graph of the vector bundle homomorphism $T^*(P/\mathsf{G}_0) \to T(P/\mathsf{G}_0)$ defined by the Poisson structure is the forward Dirac image of D_P under $\mathrm{pr}_\circ$. $\triangle$

5.2 The units of a Dirac groupoid

In this section, we discuss further properties of the set of units $\mathfrak{A}(\mathsf{D}_G)$ of a multiplicative Dirac structure.

Proposition 5.2.1 *Let $\bar\xi=(\bar X_\xi,\bar\theta_\xi)$ be a section of $\mathsf{D}_G\cap(TP\times_P A^*G)=\mathfrak{A}(\mathsf{D}_G)$. Then there exists a smooth section $\xi=(X_\xi,\theta_\xi)$ of D_G such that $\xi|_P=\bar\xi$ and $\mathbb{T}\mathsf{s}(\xi(g))=\bar\xi(\mathsf{s}(g))$ for all $g\in\mathsf{s}^{-1}\left(\mathrm{Dom}\left(\bar\xi\right)\right)$.*

We say that then that the section ξ of D_G is s-*descending* and we write $\xi\sim_\mathsf{s}\bar\xi$. Indeed, since $\bar X_\xi\in\Gamma(TP)$ and $T_g\mathsf{s}X_\xi(g)=\bar X_\xi(\mathsf{s}(g))$ for all $g\in G$ where this makes sense, the vector fields X_ξ and $\bar X_\xi$ are s-related, $X_\xi\sim_\mathsf{s}\bar X_\xi$. Following Mackenzie (2000), the pair $(\xi,\bar\xi)$ can also be called a *star section* of $\mathsf{D}_G\rightrightarrows\mathfrak{A}(\mathsf{D}_G)$. Note that outside of P, ξ is defined modulo sections of $\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s}$.
Consider the smooth section $\xi^{-1}\in\Gamma(\mathsf{D}_G)$ defined by $\xi^{-1}(g)=(\xi(g^{-1}))^{-1}$ for all $g\in G$. Then ξ^{-1} is a t-*descending section* of D_G, $\xi^{-1}\sim_\mathsf{t}\bar\xi$.

Proof: We have shown in Lemma 5.1.7 that $\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s}$ is a subbundle of D_G. Hence, we can consider the smooth vector bundle $\mathsf{D}_G/(\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})$ over G. Since D_G is a Lie subgroupoid of $\mathsf{P}_G\rightrightarrows(TP\times_P A^*G)$, we can consider the restriction to D_G of the source map, $\mathbb{T}\mathsf{s}:\mathsf{D}_G\to\mathfrak{A}(\mathsf{D}_G)$. Since $\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s}$ is the kernel of this map, we have an induced smooth vector bundle homomorphism $\overline{\mathbb{T}\mathsf{s}}:\mathsf{D}_G/(\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})\to\mathfrak{A}(\mathsf{D}_G)$ over the source map $\mathsf{s}:G\to P$, that is bijective in every fiber. Hence, there exists a unique smooth section $[\xi]$ of $\mathsf{D}_G/(\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})$ such that $\overline{\mathbb{T}\mathsf{s}}([\xi](g))=\bar\xi(\mathsf{s}(g))$ for all $g\in G$. If $\xi\in\Gamma(\mathsf{D}_G)$ is a representative of $[\xi]$ such that $\xi|_P=\bar\xi$, then $\mathbb{T}\mathsf{s}(\xi(g))=\bar\xi(\mathsf{s}(g))$ for all $g\in G$. $\square$

Lemma 5.2.2 *Choose $\bar\xi,\bar\eta\in\Gamma(\mathfrak{A}(\mathsf{D}_G))$ and s-descending sections $\xi\sim_\mathsf{s}\bar\xi$, $\eta\sim_\mathsf{s}\bar\eta$ of D_G. Then, if $\xi=(X_\xi,\theta_\xi)$ and $\eta=(X_\eta,\theta_\eta)$, the identity*

$$\theta_\eta(\pounds_{Z^l}X_\xi)+(\pounds_{Z^l}\theta_\xi)(X_\eta)=\mathsf{s}^*\Big(\big(\theta_\eta(\pounds_{Z^l}X_\xi)+(\pounds_{Z^l}\theta_\xi)(X_\eta)\big)|_P\Big)\tag{5.2}$$

holds for any section $Z\in\Gamma(AG)$.

Proof: Choose $g\in G$ and set $p=\mathsf{s}(g)$. For all $t\in(-\varepsilon,\varepsilon)$ for a small ε, we have

$$\xi(g\star\mathrm{Exp}(tZ)(p))=\big(\xi(g\star\mathrm{Exp}(tZ)(p))\big)\star\big(\xi(\mathrm{Exp}(tZ)(p))\big)^{-1}\star\big(\xi(\mathrm{Exp}(tZ)(p))\big).$$

The pair

$$\big(\xi(g\star\mathrm{Exp}(tZ)(p))\big)\star\big(\xi(\mathrm{Exp}(tZ)(p))\big)^{-1}$$

is an element of $\mathsf{D}_G(g)$ for all $t\in(-\varepsilon,\varepsilon)$ and will be written $\delta_t(g)$ to simplify the notation. Note that we have

$$\begin{aligned}&\big(T_{R_{\mathrm{Exp}(tZ)}(g)}R_{\mathrm{Exp}(-tZ)}X_\xi(g\star\mathrm{Exp}(tZ)(p)),\theta_\xi(g\star\mathrm{Exp}(tZ)(p))\circ T_gR_{\mathrm{Exp}(tZ)}\big)\\&=\delta_t(g)\star\big(T_{\mathrm{Exp}(tZ)}R_{\mathrm{Exp}(-tZ)}X_\xi(\mathrm{Exp}(tZ)(p)),\theta_\xi(\mathrm{Exp}(tZ)(p))\circ T_pR_{\mathrm{Exp}(tZ)}\big).\end{aligned}\tag{5.3}$$

We compute

$$\begin{aligned}
&(\theta_\eta(\pounds_{Z^l}X_\xi)+(\pounds_{Z^l}\theta_\xi)(X_\eta))\,(g)\\
=&\left\langle \eta(g),\,\left.\frac{d}{dt}\right|_{t=0}\left(\left(R^*_{\mathrm{Exp}(tZ)}X_\xi\right)(g),\left(R^*_{\mathrm{Exp}(tZ)}\theta_\xi\right)(g)\right)\right\rangle\\
\overset{(5.3)}{=}&\left.\frac{d}{dt}\right|_{t=0}\left\langle \eta(g)\star\bar\eta(p),\,\delta_t(g)\star\left(\left(R^*_{\mathrm{Exp}(tZ)}X_\xi\right)(p),\left(R^*_{\mathrm{Exp}(tZ)}\theta_\xi\right)(p)\right)\right\rangle\\
=&\left.\frac{d}{dt}\right|_{t=0}\langle\eta(g),\delta_t(g)\rangle+\left.\frac{d}{dt}\right|_{t=0}\left\langle\bar\eta(p),\left(\left(R^*_{\mathrm{Exp}(tZ)}X_\xi\right)(p),\left(R^*_{\mathrm{Exp}(tZ)}\theta_\xi\right)(p)\right)\right\rangle\\
=&\left(\left.\frac{d}{dt}\right|_{t=0}0\right)+\big\langle\bar\eta,(\pounds_{Z^l}X_\xi,\pounds_{Z^l}\theta_\xi)\big\rangle(p)=\left(\bar\theta_\eta(\pounds_{Z^l}X_\xi)+(\pounds_{Z^l}\theta_\xi)(\bar X_\eta)\right)(\mathsf{s}(g)).\qquad\square
\end{aligned}$$

Proposition 5.2.3 *Let $(G\rightrightarrows P,\mathsf{D}_G)$ be a Dirac groupoid. Choose $\bar\xi,\bar\eta\in\Gamma(\mathfrak{A}(\mathsf{D}_G))$ and s-descending sections $\xi\sim_{\mathsf{s}}\bar\xi$, $\eta\sim_{\mathsf{s}}\bar\eta$ of D_G, as in Proposition 5.2.1. Then the Courant-Dorfman bracket*

$$[\xi,\eta]=\left([X_\xi,X_\eta],\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)$$

is s-descending and its values on P are elements of $TP\times_P A^*G$.

Proof: Since $X_\xi\sim_{\mathsf{s}}\bar X_\xi$ and $X_\eta\sim_{\mathsf{s}}\bar X_\eta$, we know that $[X_\xi,X_\eta]\sim_{\mathsf{s}}[\bar X_\xi,\bar X_\eta]$. Since $X_\xi|_P=\bar X_\xi$, $X_\eta|_P=\bar X_\eta$ the value of $[X_\xi,X_\eta]$ on points in P is equal to the value of $[\bar X_\xi,\bar X_\eta]\in\mathfrak{X}(P)$. We check that for all $p\in P$, we have $\hat{\mathsf{s}}\left((\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi)(g)\right)=(\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi)(p)$ for any $g\in\mathsf{s}^{-1}(p)$.
We have for any $Z\in\Gamma(AG)$:

$$\hat{\mathsf{s}}\left(\left(\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)(g)\right)(Z(p))=\left(\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)\left(Z^l\right)(g).$$

Hence, we compute with (5.2)

$$\begin{aligned}
&\left(\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)(Z^l)\\
&=X_\xi(\theta_\eta(Z^l))+\theta_\eta(\pounds_{Z^l}X_\xi)-X_\eta(\theta_\xi(Z^l))+Z^l(\theta_\xi(X_\eta))-\theta_\xi(\pounds_{Z^l}X_\eta)\\
&=X_\xi(\mathsf{s}^*(\bar\theta_\eta(Z)))+\theta_\eta(\pounds_{Z^l}X_\xi)-X_\eta(\mathsf{s}^*(\bar\theta_\xi(Z)))+(\pounds_{Z^l}\theta_\xi)(X_\eta)\\
&=\mathsf{s}^*\Big(\bar X_\xi(\bar\theta_\eta(Z))+\bar\theta_\eta(\pounds_{Z^l}X_\xi)-\bar X_\eta(\bar\theta_\xi(Z))+(\pounds_{Z^l}\theta_\xi)(\bar X_\eta)\Big).
\end{aligned}$$

We have then also for $p\in P$:

$$\left(\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)(Z(p))=\left(\bar X_\xi(\bar\theta_\eta(Z))+\bar\theta_\eta(\pounds_{Z^l}X_\xi)-\bar X_\eta(\bar\theta_\xi(Z))+(\pounds_{Z^l}\theta_\xi)(\bar X_\eta)\right)(p).$$

Choose $X\in\Gamma(TP)$, then

$$\begin{aligned}
\left(\pounds_{X_\xi}\theta_\eta-\mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)(X(p))=&\ \bar X_\xi(\bar\theta_\eta(X))(p)+\bar\theta_\eta([X,\bar X_\xi])(p)-\bar X_\eta(\bar\theta_\xi(X))(p)\\
&+X(\bar\theta_\xi(\bar X_\eta))(p)-\bar\theta_\xi([X,\bar X_\eta])(p)=0
\end{aligned}$$

since $\bar{\theta}_\eta, \bar{\theta}_\xi \in \Gamma(TP^\circ)$ and $X, [X, \bar{X}_\xi], \bar{X}_\eta, [X, \bar{X}_\eta] \in \Gamma(TP)$.
Thus, we have shown that $\left(\pounds_{X_\xi}\theta_\eta - \mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)|_P$ is a section of $A^*G = TP^\circ$ and

$$\hat{\mathsf{s}}\left(\left(\pounds_{X_\xi}\theta_\eta - \mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)(g)\right) = \left(\pounds_{X_\xi}\theta_\eta - \mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)(\mathsf{s}(g))$$

for all $g \in G$. □

Theorem 5.2.4 *Let $(G\rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid. Then there is an induced antisymmetric bracket*

$$[\cdot\,,\cdot]_\star \quad : \Gamma(\mathfrak{A}(\mathsf{D}_G)) \times \Gamma(\mathfrak{A}(\mathsf{D}_G)) \quad \to \quad \Gamma(TP \times_P A^*G)$$

defined by $[\bar{\xi}, \bar{\eta}]_\star = [\xi, \eta]|_P$ for any choice of s-descending sections $\xi \sim_{\mathsf{s}} \bar{\xi}$, $\eta \sim_{\mathsf{s}} \bar{\eta}$ of D_G. If $(G\rightrightarrows P, \mathsf{D}_G)$ is integrable, then $(\mathfrak{A}(\mathsf{D}_G), [\cdot\,,\cdot]_\star, \mathsf{a}_\star)$ is a Lie algebroid over P.

Proof: By Proposition 5.2.3, if $\xi \sim_{\mathsf{s}} \bar{\xi}$, $\eta \sim_{\mathsf{s}} \bar{\eta}$ then

$$[(X_\xi, \theta_\xi), (X_\eta, \theta_\eta)] \sim_{\mathsf{s}} \left([\bar{X}_\xi, \bar{X}_\eta], \left(\pounds_{X_\xi}\theta_\eta - \mathbf{i}_{X_\eta}\mathbf{d}\theta_\xi\right)|_P\right).$$

Thus, we have first to show that the right-hand side of this equation doesn't depend on the choice of the sections ξ and η. Choose a s-descending section $\nu \sim_{\mathsf{s}} 0$ of D_G, i.e., $\nu \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$ with $\nu|_P = 0$. For any $Z \in \Gamma(AG)$, we find as in the proof of Proposition 5.2.3

$$\begin{aligned}
\left(\pounds_{X_\nu}\theta_\xi - \mathbf{i}_{X_\xi}\mathbf{d}\theta_\nu\right)\left(Z^l\right) &= \mathsf{s}^*(\bar{X}_\nu(\bar{\theta}_\xi(Z))) + \theta_\xi(\pounds_{Z^l}X_\nu) - \mathsf{s}^*(\bar{X}_\xi(\bar{\theta}_\nu(Z))) + (\pounds_{Z^l}\theta_\nu)(X_\xi)\\
&= \theta_\xi(\pounds_{Z^l}X_\nu) + (\pounds_{Z^l}\theta_\nu)(X_\xi) \qquad \text{since } \bar{X}_\nu = 0 \text{ and } \bar{\theta}_\nu = 0\\
&= \pounds_{Z^l}\langle(X_\nu, \theta_\nu), (X_\xi, \theta_\xi)\rangle - \langle\pounds_{Z^l}(X_\xi, \theta_\xi), (X_\nu, \theta_\nu)\rangle\\
&= -\langle\pounds_{Z^l}(X_\xi, \theta_\xi), (X_\nu, \theta_\nu)\rangle.
\end{aligned}$$

Hence, at any $p \in P$, we find

$$\begin{aligned}
(\pounds_{X_\nu}\theta_\xi - \mathbf{i}_{X_\xi}\mathbf{d}\theta_\nu)(Z(p)) &= -\langle\pounds_{Z^l}(X_\xi, \theta_\xi)(p), (\bar{X}_\nu, \bar{\theta}_\nu)(p)\rangle\\
&= -\langle\pounds_{Z^l}(X_\xi, \theta_\xi)(p), (0_p, 0_p)\rangle = 0.
\end{aligned}$$

Thus, we find $(\pounds_{X_\nu}\theta_\xi - \mathbf{i}_{X_\xi}\mathbf{d}\theta_\nu)(p) = 0_p$ since we know by the previous proposition that $(\pounds_{X_\nu}\theta_\xi - \mathbf{i}_{X_\xi}\mathbf{d}\theta_\nu)(p) \in A_p^*G = T_pP^\circ$. We get hence

$$[(X_\nu, \theta_\nu), (X_\xi, \theta_\xi)]\,(p) = \left([\bar{X}_\nu, \bar{X}_\xi], \pounds_{X_\nu}\theta_\xi - \mathbf{i}_{X_\xi}\mathbf{d}\theta_\nu\right)(p) = ([0, \bar{X}_\xi]_p, 0_p) = (0_p, 0_p).$$

This shows that the bracket on $\Gamma(\mathfrak{A}(\mathsf{D}_G))$ is well-defined. It is antisymmetric because the Courant-Dorfman bracket on sections of D_G is antisymmetric.

If D_G is integrable, then for all s-descending $\xi, \eta \in \Gamma(\mathsf{D}_G)$, the bracket $[\xi, \eta]$ is also a section of D_G and its restriction to P is a section of $\mathfrak{A}(\mathsf{D}_G)$ since it is a section of $TP \times_P A^*G$.

The Jacobi identity is satisfied by $[\cdot\,,\cdot]_\star$ because the Courant-Dorfman bracket on sections of D_G satisfies the Jacobi identity. For any $\bar\xi,\bar\eta\in\Gamma(\mathfrak{A}(\mathsf{D}_G))$ and $f\in C^\infty(P)$, we have

$$\mathsf{a}_\star\left[\bar\xi,\bar\eta\right]_\star=\left[\bar X_\xi,\bar X_\eta\right]=\left[\mathsf{a}_\star(\bar\xi),\mathsf{a}_\star(\bar\eta)\right]$$

and

$$\begin{aligned}\left[\bar\xi,f\cdot\bar\eta\right]_\star(p)&=\left[(X_\xi,\theta_\xi),(\mathsf{s}^*f)(X_\eta,\theta_\eta)\right](p)\\&=X_\xi(\mathsf{s}^*f)(X_\eta,\theta_\eta)(p)+(\mathsf{s}^*f)\left[(X_\xi,\theta_\xi),(X_\eta,\theta_\eta)\right](p)\\&=\bar X_\xi(f)(p)\cdot(\bar X_\eta,\bar\theta_\eta)(p)+f(p)\cdot\left[\bar\xi,\bar\eta\right]_\star(p)\\&=\mathsf{a}_\star(\bar\xi)(f)(p)\cdot\bar\eta(p)+f(p)\cdot\left[\bar\xi,\bar\eta\right]_\star(p)\end{aligned}$$

for all $p\in P$. $\square$

If the Dirac structure D_G is integrable, we get the structure of a $\mathcal{L}A$-*groupoid* on D_G (Mackenzie (2000)). Let $G\rightrightarrows P$ be a Lie groupoid, $TG\rightrightarrows TP$ its tangent prolongation and $(A\to P,\mathsf{a},[\cdot\,,\cdot]_{\mathsf{a}})$ a Lie algebroid over P. Let Ω be a smooth manifold. The quadruple $(\Omega;G,A;P)$ is a $\mathcal{L}A$-groupoid if Ω has both a Lie groupoid structure over A and a Lie algebroid structure over G such that the two structures on Ω commute in the sense that the maps defining the groupoid structure are all Lie algebroid morphisms. (The bracket on sections of $\mathfrak{A}(\mathsf{D}_G)$ can be defined in the same manner with the target map, and the fact that the multiplication in $T^*G\times_G TG$ is a Lie algebroid morphism is shown in Ortiz (2009).) The double source map $(\tilde q,\tilde{\mathsf{s}}):\Omega\to G\times_P A$ has furthermore to be a surjective submersion. Recall from Courant (1990) that if D_G is integrable, then $\mathsf{D}_G\to G$ has the structure of a Lie algebroid with the Courant-Dorfman bracket and the projection on TM as anchor. Thus, the previous theorem shows that the quadruple $(\mathsf{D}_G;G,\mathfrak{A}(\mathsf{D}_G);P)$ is a $\mathcal{L}A$-groupoid (see also Ortiz (2009)):

$$\begin{tikzcd}
\mathsf{D}_G \arrow[rr, shift left, "\mathbb{T}\mathsf{s}"] \arrow[rr, shift right, "\mathbb{T}\mathsf{t}"'] \arrow[dr, "\pi_{TG}"] \arrow[dd, "q"'] & & \mathfrak{A}(\mathsf{D}_G) \arrow[dr, "\mathsf{a}_\star"] \arrow[dd, dashed] & \\
& TG \arrow[rr, shift left, "T\mathsf{s}"] \arrow[rr, shift right, "T\mathsf{t}"'] \arrow[dl] & & TP \arrow[dl] \\
G \arrow[rr, shift left, "\mathsf{s}"] \arrow[rr, shift right, "\mathsf{t}"'] & & P &
\end{tikzcd}$$

Then, in the terminology of Mackenzie (2000), our s-descending sections of D_G are the *star sections* of $(\mathsf{D}_G;G,\mathfrak{A}(\mathsf{D}_G);P)$. It is shown in Mackenzie (2000) (see also Mackenzie (1992)), that the bracket of two star sections is again a star section. Here, we have shown this fact in Proposition 5.2.3 and get as a consequence the fact that $\mathfrak{A}(\mathsf{D}_G)$ has the structure of a Lie algebroid over P.

The next interesting object in Mackenzie (2000) is the *core* K of Ω. It is defined as the pullback vector bundle across $\epsilon:P\hookrightarrow G$ of the kernel $\ker(\tilde{\mathsf{s}}:\Omega\to A)$. Hence, it is

here exactly the vector bundle $I^{\mathsf{s}}(\mathsf{D}_G)$ over P. It comes equipped with the vector bundle morphisms $\delta_{\mathfrak{A}(\mathsf{D}_G)} : I^{\mathsf{s}}(\mathsf{D}_G) \to \mathfrak{A}(\mathsf{D}_G)$, $(v_p, \alpha_p) \mapsto \mathbb{T}\mathsf{t}(v_p, \alpha_p)$ and $\delta_{\widetilde{AG}} : I^{\mathsf{s}}(\mathsf{D}_G) \to \widetilde{AG}$, $(v_p, \alpha_p) \mapsto v_p$. We have then $\tilde{\mathsf{a}} \circ \delta_{\widetilde{AG}} = \mathsf{a}_\star \circ \delta_{\mathfrak{A}(\mathsf{D}_G)} =: \mathsf{k}$. Furthermore, there is an induced bracket $[\cdot\,,\cdot]_{I^{\mathsf{s}}(\mathsf{D}_G)}$ on sections of $I^{\mathsf{s}}(\mathsf{D}_G)$ such that $(I^{\mathsf{s}}(\mathsf{D}_G), [\cdot\,,\cdot]_{I^{\mathsf{s}}(\mathsf{D}_G)}, \mathsf{k})$ is a Lie algebroid over P. We prove this fact for our special situation in the following proposition.
Recall that if (v_p, α_p), $p \in P$, is an element of $I^{\mathsf{s}}_p(\mathsf{D}_G)$, then α_p can be written $(T_p\mathsf{t})^*\beta_p$ with some $\beta_p \in T_p^*P$. Furthermore, if σ is a section of $I^{\mathsf{s}}(\mathsf{D}_G) \subseteq (T^{\mathsf{s}}G \times_G (T^{\mathsf{t}}G)^\circ)|_P$, then σ^r defined by $\sigma^r(g) = \sigma(\mathsf{t}(g)) \star (0_g, 0_g)$ for all $g \in G$ is a section of $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s}$ by Lemma 5.1.7 and Remark 5.1.10. We write $\sigma^r = (X_\sigma^r, \mathsf{t}^*\alpha_\sigma)$ with some $X_\sigma \in \Gamma\left(\widetilde{AG}\right)$ and $\alpha_\sigma \in \Omega^1(P)$.

Proposition 5.2.5 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Lie groupoid. Define $[\cdot\,,\cdot]_{I^{\mathsf{s}}(\mathsf{D}_G)} : \Gamma(I^{\mathsf{s}}(\mathsf{D}_G)) \times \Gamma(I^{\mathsf{s}}(\mathsf{D}_G)) \to \Gamma((\ker \mathbb{T}\mathsf{s})|_P)$ by*

$$\left([\sigma, \tau]_{I^{\mathsf{s}}(\mathsf{D}_G)}\right)^r = [\sigma^r, \tau^r]$$

for all sections $\sigma, \tau \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$, i.e.,

$$[\sigma, \tau]_{I^{\mathsf{s}}(\mathsf{D}_G)} = \left([X_\sigma, X_\tau]_{\widetilde{AG}}, (\mathsf{t}^*(\pounds_{\tilde{\mathsf{a}}(X_\sigma)}\alpha_\tau - \mathbf{i}_{\tilde{\mathsf{a}}(X_\tau)}\mathbf{d}\alpha_\sigma))|_P\right).$$

If D_G is integrable, this bracket has image in $\Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$ and $I^{\mathsf{s}}(\mathsf{D}_G)$ has the structure of a Lie algebroid over P with the anchor map k defined by $\mathsf{k}(v_p, \alpha_p) = T_p\mathsf{t}v_p$ for all $(v_p, \alpha_p) \in I^{\mathsf{s}}_p(\mathsf{D}_G)$, $p \in P$.

Note that this bracket on $I^{\mathsf{s}}(\mathsf{D}_G)$ is the restriction to $I^{\mathsf{s}}(\mathsf{D}_G)$ of a bracket defined in the same manner on the sections of $(\ker \mathbb{T}\mathsf{s})|_P$. Note also that, if D_G is integrable, the space $I^{\mathsf{t}}(\mathsf{D}_G)$ has in the same manner the structure of an algebroid over P.

PROOF: Choose $\sigma, \tau \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$ and assume that D_G is integrable. The bracket

$$[\sigma^r, \tau^r] = [(X_\sigma^r, \mathsf{t}^*\alpha_\sigma), (X_\tau^r, \mathsf{t}^*\alpha_\tau)]$$

is then itself a section of D_G. The identity

$$[(X_\sigma^r, \mathsf{t}^*\alpha_\sigma), (X_\tau^r, \mathsf{t}^*\alpha_\tau)] = \left(\left([X_\sigma, X_\tau]_{\widetilde{AG}}\right)^r, \mathsf{t}^*(\pounds_{\tilde{\mathsf{a}}(X_\sigma)}\alpha_\tau - \mathbf{i}_{\tilde{\mathsf{a}}(X_\tau)}\mathbf{d}\alpha_\sigma)\right)$$

shows hence that $[\sigma^r, \tau^r] \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$ is right invariant and consequently

$$[\sigma, \tau]_{I^{\mathsf{s}}(\mathsf{D}_G)} = [(X_\sigma^r, \mathsf{t}^*\alpha_\sigma), (X_\tau^r, \mathsf{t}^*\alpha_\tau)]\,|_P \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G)).$$

The bracket $[\cdot\,,\cdot]_{I^{\mathsf{s}}(\mathsf{D}_G)}$ satisfies then the Jacobi identity because the Courant bracket on sections of D_G satisfies it. The Leibniz rule is easy to check. □

As in Mackenzie (2000), we have thus four Lie algebroids over P:

$$
\begin{tikzcd}
I^{\mathsf{s}}(\mathsf{D}_G) \arrow[rr, "\delta_{\mathfrak{A}(\mathsf{D}_G)}"] \arrow[dd, "\delta_{\widetilde{AG}}"'] \arrow[dr] & & \mathfrak{A}(\mathsf{D}_G) \arrow[dd, "\mathsf{a}_\star"] \arrow[dl] \\
& P & \\
\widetilde{AG} \arrow[rr, "\tilde{\mathsf{a}}"'] \arrow[ur] & & TP \arrow[ul]
\end{tikzcd}
$$

The anchors $\tilde{\mathsf{a}}$, $\mathsf{a}_\star$ and the map $\delta_{\widetilde{AG}}$ are obviously Lie algebroid morphisms and the theory in Mackenzie (1992), Mackenzie (2000) yields that $\delta_{\mathfrak{A}(\mathsf{D}_G)}$ is also a Lie algebroid morphism.

Next, we compute the Lie algebroid $\mathfrak{A}(\mathsf{D}_G) \to P$ for our three "standard" examples.

Example 5.2.6 Let $(G \rightrightarrows P, \pi_G)$ be a Poisson groupoid and D_{π_G} the graph of the vector bundle homomorphism $\pi_G^\sharp : T^*G \to TG$ associated to π_G. The pair $(G \rightrightarrows P, \mathsf{D}_{\pi_G})$ is an integrable Dirac groupoid. The set of units $\mathfrak{A}(\mathsf{D}_{\pi_G})$ of D_{π_G} equals here $\operatorname{Graph}\left(\pi_G^\sharp\big|_{A^*G} : A^*G \to TP\right)$ and is hence isomorphic to A^*G as a vector bundle, via the maps $\Theta := \mathrm{pr}_{A^*G} : \mathfrak{A}(\mathsf{D}_{\pi_G}) \to A^*G$ and $\Theta^{-1} = \left(\pi_G^\sharp\big|_{A^*G}, \mathrm{Id}_{A^*G}\right) : A^*G \to \mathfrak{A}(\mathsf{D}_{\pi_G})$ over Id_P.
The vector bundle A^*G has the structure of a Lie algebroid over P with anchor map given by $A^*G \to TP$, $\alpha_p \mapsto \pi_G^\sharp(\alpha_p) \in T_pP$ and with bracket the restriction to A^*G of the bracket $[\cdot\,,\cdot]_{\pi_G}$ on $\Omega^1(G)$ defined by π_G: $[\alpha, \beta]_{\pi_G} = \pounds_{\pi_G^\sharp(\alpha)}\beta - \pounds_{\pi_G^\sharp(\beta)}\alpha - \mathbf{d}\pi_G(\alpha, \beta)$ for all $\alpha, \beta \in \Omega^1(G)$ (Coste et al. (1987)). Thus, A^*G with this Lie algebroid structure and $\mathfrak{A}(\mathsf{D}_{\pi_G})$ are isomorphic as Lie algebroids via Θ and Θ^{-1}. $\Diamond$

Example 5.2.7 Let ω_G be a multiplicative closed 2-form on a Lie groupoid $G \rightrightarrows P$ and consider the associated multiplicative Dirac structure D_{ω_G} on G. The Lie algebroid $\mathfrak{A}(\mathsf{D}_{\omega_G}) \to P$ is here equal to

$$\mathfrak{A}(\mathsf{D}_{\omega_G}) = \operatorname{Graph}\left(\omega_G^\flat|_{TP} : TP \to A^*G\right)$$

with anchor map $\mathsf{a}_\star : \mathfrak{A}(\mathsf{D}_{\omega_G}) \to TP$ given by $\mathsf{a}_\star\left(v_p, \omega_G^\flat(v_p)\right) = v_p$. The bracket of two sections $\left(\bar{X}, \omega_G^\flat(\bar{X})\right), \left(\bar{Y}, \omega_G^\flat(\bar{Y})\right) \in \Gamma(\mathfrak{A}(\mathsf{D}_{\omega_G}))$ is simply given by

$$\left[\left(\bar{X}, \omega_G^\flat(\bar{X})\right), \left(\bar{Y}, \omega_G^\flat(\bar{Y})\right)\right] = \left([\bar{X}, \bar{Y}], \omega_G^\flat\left([\bar{X}, \bar{Y}]\right)\right).$$

The Lie algebroid $\mathfrak{A}(\mathsf{D}_{\omega_G})$ is obviously isomorphic to the tangent Lie algebroid $TP \to P$ of P, via the maps $\mathrm{pr}_{TP} : \mathfrak{A}(\mathsf{D}_{\omega_G}) \to TP$ (the anchor map) and $\left(\mathrm{Id}_{TP}, \omega_G^\flat|_{TP}\right) : TP \to \mathfrak{A}(\mathsf{D}_{\omega_G})$.
Note that if $(G \rightrightarrows P, \omega)$ is a presymplectic groupoid, then $\mathfrak{A}(\mathsf{D}_\omega)$ is the graph of the dual of the map $\sigma_\omega : AG \to T^*P$ in Bursztyn et al. (2009). $\Diamond$

Example 5.2.8 Let (M, D_M) be a smooth Dirac manifold and $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ the associated pair Dirac groupoid as in Example 2.2.5. The set $\mathfrak{A}(\mathsf{D}_M \ominus \mathsf{D}_M)$ is defined here by

$$\mathfrak{A}(\mathsf{D}_M \ominus \mathsf{D}_M)_{(m,m)} = \mathbb{T}\mathsf{t}((\mathsf{D}_M \ominus \mathsf{D}_M)(m,m)) = \{(v_m, v_m, \alpha_m, -\alpha_m) \mid (v_m, \alpha_m) \in \mathsf{D}_M(m)\}$$

for all $m \in M$. Hence, we have an isomorphism $\mathfrak{A}(\mathsf{D}_M \ominus \mathsf{D}_M) \to \mathsf{D}_M$ over the map $\mathrm{pr}_1 : \Delta_M \to M$. Sections of $\mathfrak{A}(\mathsf{D}_M \ominus \mathsf{D}_M)$ are exactly the sections $(X, X, \alpha, -\alpha)|_{\Delta_M}$ for sections $(X, \alpha) \in \mathsf{D}_M$. The section $(X, X, \alpha, -\alpha)$ of $\mathsf{D}_M \ominus \mathsf{D}_M$ defined on $M \times M$ by $(X, X, \alpha, -\alpha)(m,n) = (X(m), X(n), \alpha(m), -\alpha(n))$ for all $(m,n) \in M \times M$ is then easily shown to be s-descending to $(X, X, \alpha, -\alpha)|_{\Delta_M}$. Using this, one can check that, if (M, D_M) is integrable, the Lie algebroid structure on $\mathfrak{A}(\mathsf{D}_M \ominus \mathsf{D}_M)$ corresponds to the Lie algebroid structure on (M, D_M) (see Courant (1990)). $\Diamond$

5.3 Integrability criterion

The main theorem of this section shows that the integrability of the Dirac groupoid is completely encoded in its square of Lie algebroids. The proof is very technical. We begin by showing a derivation formula for s-descending sections, that will also be useful later.

Theorem 5.3.1 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid, $\xi \sim_{\mathsf{s}} \bar{\xi}$ a s-descending section of D_G and $Z \in \Gamma(AG)$. Then the derivative $\pounds_{Z^l}(X_\xi, \theta_\xi)$ can be written as a sum*

$$\pounds_{Z^l}(X_\xi, \theta_\xi) = (X_{\mathcal{L}_Z\xi}, \theta_{\mathcal{L}_Z\xi}) + \left(Y^l_{\xi,Z}, \mathsf{s}^*\alpha_{\xi,Z}\right) =: \mathcal{L}_Z\xi + (\sigma_{\xi,Z})^l \tag{5.4}$$

with $Y_{\xi,Z} \in \Gamma(AG)$, $\alpha_{\xi,Z} \in \Omega^1(P)$ and $\mathcal{L}_Z\xi := (X_{\mathcal{L}_Z\xi}, \theta_{\mathcal{L}_Z\xi})$ a s-descending section of D_G. We have $\mathcal{L}_Z\xi \sim_{\mathsf{s}} \mathbb{T}\mathsf{t}\left(\pounds_{Z^l}(X_\xi, \theta_\xi)|_P\right)$ in the sense that

$$\mathbb{T}\mathsf{s}\left(X_{\mathcal{L}_Z\xi}(g), \theta_{\mathcal{L}_Z\xi}(g)\right) = \mathbb{T}\mathsf{t}\left(\pounds_{Z^l}(X_\xi, \theta_\xi)(\mathsf{s}(g))\right)$$

for all $g \in G$.
In addition, if $(X_\nu, \theta_\nu) \sim_s (0,0)$, then $\pounds_{Z^l}(X_\nu, \theta_\nu) \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$. In particular, its restriction to P is a section of $I^{\mathsf{s}}(\mathsf{D}_G)$.

Recall that in the Dirac Lie group case, we had $\pounds_{x^l}(X_\xi, \xi^l) = (X_{\mathrm{ad}^*_x \xi} + (\mathrm{ad}^*_\xi x)^l, (\mathrm{ad}^*_x \xi)^l)$ for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{p}_1$, with $X_{\mathrm{ad}^*_x \xi}$ defined modulo sections of G_0 (see (4.4)). The following lemma will be useful for the proof of this theorem.

Lemma 5.3.2 *Let $G \rightrightarrows P$ be a Lie groupoid. Choose $(X, (\mathsf{t}^*\alpha)|_P) \in \Gamma((\ker \mathbb{T}\mathsf{s})|_P)$ and $Z \in \Gamma(AG)$. Then we have*

$$\pounds_{Z^l}(X^r, \mathsf{t}^*\alpha) = 0.$$

Proof: By the considerations following Proposition 1.1.18, we have $\pounds_{Z^l} X^r = 0$. Since $Z^l \sim_{\mathsf{t}} 0$, we have $\pounds_{Z^l}\mathsf{t}^*\alpha = \mathsf{t}^*(\pounds_0 \alpha) = 0$. Thus, $\pounds_{Z^l}(X^r, \mathsf{t}^*\alpha) = (\pounds_{Z^l}X^r, \pounds_{Z^l}(\mathsf{t}^*\alpha)) = (0,0)$. $\square$

Proof (of Theorem 5.3.1): Note first that, in general $\pounds_{Z^l}(X_\xi,\theta_\xi)$ is a section of $\mathsf{D}_G+\ker\mathbb{T}\mathsf{t}$: for all $\sigma^r=(X^r_\sigma,\mathsf{t}^*\alpha_\sigma)\in\Gamma(\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})$, we have

$$\langle \pounds_{Z^l}(X_\xi,\theta_\xi),\sigma^r\rangle=\pounds_{Z^l}\left(\langle(X_\xi,\theta_\xi),(X^r_\sigma,\mathsf{t}^*\alpha_\sigma)\rangle\right)-\langle(X_\xi,\theta_\xi),\pounds_{Z^l}(X^r_\sigma,\mathsf{t}^*\alpha_\sigma)\rangle=0$$

using $\mathsf{D}_G=\mathsf{D}_G^\perp$ and Lemma 5.3.2. This leads to $\pounds_{Z^l}(X_\xi,\theta_\xi)\in\Gamma\left((\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})^\perp\right)=\Gamma(\mathsf{D}_G+\ker\mathbb{T}\mathsf{t})$. Choose $g\in G$. Then

$$T_g\mathsf{s}(\pounds_{Z^l}X_\xi)(g)=T_g\mathsf{s}\left[Z^l,X_\xi\right](g)=\left[\mathsf{a}(Z),\bar{X}_\xi\right](\mathsf{s}(g))$$

and for any $W\in\Gamma(AG)$

$$\begin{aligned}\hat{\mathsf{s}}(\pounds_{Z^l}\theta_\xi(g))(W(\mathsf{s}(g)))&=(\pounds_{Z^l}\theta_\xi)(W^l)(g)=(Z^l(\mathsf{s}^*(\bar{\theta}_\xi(W)))-\mathsf{s}^*(\bar{\theta}_\xi([Z,W]_{AG})))(g)\\&=\left((\mathsf{a}(Z))(\bar{\theta}_\xi(W))-\bar{\theta}_\xi([Z,W]_{AG})\right)(\mathsf{s}(g)).\end{aligned}$$

This shows that $\mathbb{T}\mathsf{s}\left(\pounds_{Z^l}\left(X_\xi,\theta_\xi\right)\right)(g)$ depends only on the values of $Z,\bar{X}_\xi,\bar{\theta}_\xi$ at $\mathsf{s}(g)$.
Set

$$(Y^l_{\xi,Z},\mathsf{s}^*\alpha_{\xi,Z})(g):=(0_g,0_g)\star\Big((\pounds_{Z^l}X_\xi,\pounds_{Z^l}\theta_\xi)(\mathsf{s}(g))-\mathbb{T}\mathsf{t}\left((\pounds_{Z^l}X_\xi,\pounds_{Z^l}\theta_\xi)(\mathsf{s}(g))\right)\Big)$$

and

$$(X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})(g):=\left(\pounds_{Z^l}X_\xi,\pounds_{Z^l}\theta_\xi\right)(g)-\left(Y^l_{\xi,Z},\mathsf{s}^*\alpha_{\xi,Z}\right)(g)$$

for all $g\in G$. Then $\left(Y^l_{\xi,Z},\mathsf{s}^*\alpha_{\xi,Z}\right)$ is a smooth section of $\ker\mathbb{T}\mathsf{t}$ satisfying

$$\mathbb{T}\mathsf{s}\left(\left(Y^l_{\xi,Z},\mathsf{s}^*\alpha_{\xi,Z}\right)(g)\right)=\mathbb{T}\mathsf{s}\left(\left(Y^l_{\xi,Z},\mathsf{s}^*\alpha_{\xi,Z}\right)(\mathsf{s}(g))\right)$$

by construction for all $g\in G$ and $\mathcal{L}_Z\xi=(X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})$ is consequently s-descending if we can show that

$$\mathbb{T}\mathsf{s}((X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})(g))=(X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})(\mathsf{s}(g))$$

for all $g\in G$. Using the computations above for $\mathbb{T}\mathsf{s}\left(\pounds_{Z^l}(X_\xi,\theta_\xi)\right)$, it is easy to see that, for $g\in G$, we have

$$\mathbb{T}\mathsf{s}\left((X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})(g)\right)=\mathbb{T}\mathsf{t}\left(\pounds_{Z^l}X_\xi,\pounds_{Z^l}\theta_\xi\right)(\mathsf{s}(g)),$$

which, by definition, is equal to $(X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})(\mathsf{s}(g))$.
It remains hence to show that $(X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})$ is a section of D_G. The equality

$$\langle\sigma^r,(X_{\mathcal{L}_Z\xi},\theta_{\mathcal{L}_Z\xi})\rangle=\langle\sigma^r,(\pounds_{Z^l}X_\xi,\pounds_{Z^l}\theta_\xi)\rangle-\left\langle\sigma^r,(Y^l_{\xi,Z},\mathsf{s}^*\alpha_{\xi,Z})\right\rangle=0-0$$

holds for all $\sigma^r\in\Gamma(\ker\mathbb{T}\mathsf{s}\cap\mathsf{D}_G)$, and for all s-descending sections (X_η,θ_η) of D_G, we compute

$$
\begin{aligned}
&\langle (X_\eta,\theta_\eta), (X_{\pounds_Z\xi},\theta_{\pounds_Z\xi})\rangle (g)\\
&= \left\langle (X_\eta,\theta_\eta)(g), (\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(g) - \left(Y^l_{\xi,Z}, \mathsf{s}^*\alpha_{\xi,Z}\right)(g)\right\rangle\\
&= \langle (X_\eta,\theta_\eta)(g), (\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(g)\rangle\\
&\quad - \Big\langle (X_\eta,\theta_\eta)(g)\star\left(\bar{X}_\eta,\bar{\theta_\eta}\right)(\mathsf{s}(g)),\\
&\qquad\qquad (0_g,0_g)\star\left((\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(\mathsf{s}(g)) - \mathbb{T}\mathsf{t}\,(\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(\mathsf{s}(g))\right)\Big\rangle\\
&\overset{(5.2)}{=} \left\langle \left(\bar{X}_\eta,\bar{\theta_\eta}\right)(\mathsf{s}(g)), (\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(\mathsf{s}(g))\right\rangle\\
&\quad - \left\langle \left(\bar{X}_\eta,\bar{\theta_\eta}\right)(\mathsf{s}(g)), (\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(\mathsf{s}(g)) - \mathbb{T}\mathsf{t}\,(\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(\mathsf{s}(g))\right\rangle\\
&= \left\langle \left(\bar{X}_\eta,\bar{\theta_\eta}\right)(\mathsf{s}(g)), \mathbb{T}\mathsf{t}\,(\pounds_{Z^l}X_\xi, \pounds_{Z^l}\theta_\xi)(\mathsf{s}(g))\right\rangle = 0
\end{aligned}
$$

since $TP\times_P A^*G = (TP\times_P A^*G)^\perp$. Thus, we have shown that $(X_{\pounds_Z\xi},\theta_{\pounds_Z\xi})\in\Gamma(\mathsf{D}_G{}^\perp) = \Gamma(\mathsf{D}_G)$.

For the proof of the second statement, assume that (X_ν,θ_ν) is a smooth section of D_G that is s-descending to $(\bar{X}_\nu,\bar{\theta_\nu}) = (0,0)$. For all left invariant sections $(Y^l,\mathsf{s}^*\gamma)$ of $\ker\mathbb{T}\mathsf{t}$, we have

$$
\begin{aligned}
\left\langle \pounds_{Z^l}(X_\nu,\theta_\nu), (Y^l,\mathsf{s}^*\gamma)\right\rangle &= \pounds_{Z^l}\left\langle (X_\nu,\theta_\nu), (Y^l,\mathsf{s}^*\gamma)\right\rangle - \left\langle (X_\nu,\theta_\nu), \pounds_{Z^l}(Y^l,\mathsf{s}^*\gamma)\right\rangle\\
&= \pounds_{Z^l}\left(\mathsf{s}^*(\gamma(\bar{X}_\nu) + \bar{\theta_\nu}(Y))\right) - \mathsf{s}^*\left(\bar{\theta_\nu}([Z,Y]_{AG}) + (\pounds_{\mathsf{a}(Z)}\gamma)(\bar{X}_\nu)\right)\\
&= 0
\end{aligned}
$$

since $\bar{X}_\nu = 0$ and $\bar{\theta}_\nu = 0$. Choose any s-descending section $\xi = (X_\xi,\theta_\xi)$ of D_G. Then

$$
\langle \pounds_{Z^l}(X_\nu,\theta_\nu), (X_\xi,\theta_\xi)\rangle = \pounds_{Z^l}\langle (X_\nu,\theta_\nu), (X_\xi,\theta_\xi)\rangle - \langle (X_\nu,\theta_\nu), \pounds_{Z^l}(X_\xi,\theta_\xi)\rangle = 0
$$

since $\pounds_{Z^l}(X_\xi,\theta_\xi)\in\Gamma\left((\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})^\perp\right)$. We have also $\pounds_{Z^l}(X_\nu,\theta_\nu)\in\Gamma\left((\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})^\perp\right)$ and, because the s-descending sections of D_G and the sections of $\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s}$ span D_G, this shows that $\pounds_{Z^l}(X_\nu,\theta_\nu)\in\Gamma\left((\mathsf{D}_G+\ker\mathbb{T}\mathsf{t})^\perp\right) = \Gamma(\mathsf{D}_G\cap\ker\mathbb{T}\mathsf{s})$. $\square$

We have also for any s-descending section ξ of D_G, any section $\sigma\in\Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$ and $Z\in\Gamma(AG)$:

$$
\begin{aligned}
\frac{d}{dt}\left\langle R^*_{\mathrm{Exp}(tZ)}\xi,\sigma^r\right\rangle(g) &= \frac{d}{dt}\left\langle \xi, R^*_{\mathrm{Exp}(-tZ)}\sigma^r\right\rangle(R_{\mathrm{Exp}(tZ)}(g))\\
&= \frac{d}{dt}\langle \xi,\sigma^r\rangle(R_{\mathrm{Exp}(tZ)}(g)) = 0
\end{aligned}
$$

since $R^*_K\sigma^r = \sigma^r$ for all bisections $K\in\mathcal{B}(G)$. (Recall that, by convention, we consider the bisections satisfying $\mathsf{t}\circ K = \mathrm{Id}_P$.) Hence, we get

$$
\left\langle R^*_{\mathrm{Exp}(tZ)}\xi,\sigma^r\right\rangle(g) = \left\langle R^*_{\mathrm{Exp}(0\cdot Z)}\xi,\sigma^r\right\rangle(g) = \langle\xi,\sigma^r\rangle(g) = 0
$$

for all $g\in G$, $\sigma\in\Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$ and $t\in\mathbb{R}$ where this makes sense and we find consequently $R^*_{\mathrm{Exp}(tZ)}\xi\in\Gamma(\mathsf{D}_G+\ker\mathbb{T}\mathsf{t})$. If the s-fibers of $G\rightrightarrows P$ are connected, the set of bisections of

G is generated as a group by the bisections $\operatorname{Exp}(tZ)$, $t \in \mathbb{R}$ small enough and $Z \in \Gamma(AG)$ (see Mackenzie and Xu (2000)). We know then that $R_K^*\xi \in \Gamma(\mathsf{D}_G + \ker \mathbb{T}\mathsf{t})$ for any bisection $K \in \mathcal{B}(G)$.

We denote here by $\mathsf{S}(\mathsf{D}_G)$ the set of s-descending sections of D_G. Note that D_G is spanned on $G \setminus P$ by the values of the elements of $\mathsf{S}(\mathsf{D}_G)$, since $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s}$ is spanned there by the values of the s-descending sections that vanish on P.

Consider the vector bundle $\mathsf{E} = \mathsf{E}(\mathsf{D}_G) = \mathsf{D}_G/(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t}) \simeq (\mathsf{D}_G + \ker \mathbb{T}\mathsf{t})/\ker \mathbb{T}\mathsf{t}$ over G. Since the fiber $\mathsf{D}_G(g)$ over g of the Dirac structure is spanned for each $g \in G \setminus P$ by the values of the elements of $\mathsf{S}(\mathsf{D}_G)$ at g and, for each $p \in P$, the vector space $\mathsf{E}(p)$ is spanned by the classes $\bar{\xi}(p) + I_p^{\mathsf{t}}(\mathsf{D}_G)$ for all s-descending sections ξ of D_G, we find that the vector bundle E is spanned at each point $g \in G$ by the elements $\xi(g) + (\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t})(g)$ for all $\xi \in \mathsf{S}(\mathsf{D}_G)$. To simplify the notation, we write $\tilde{\xi}$ for the image of the section $\xi \in \mathsf{S}(\mathsf{D}_G)$ in E, and $\widetilde{\mathsf{S}(\mathsf{D}_G)}$ for the set of these special sections of E. By the considerations above, for any $K \in \mathcal{B}(G)$ and $\xi \in \mathsf{S}(\mathsf{D}_G)$, we can define $R_K^*\tilde{\xi} := \widetilde{R_K^*\xi}$. If we set in the same manner

$$\pounds_{Z^l}\tilde{\xi} = \widetilde{\pounds_{Z^l}\xi} \overset{(5.4)}{=} \widetilde{\mathcal{L}_Z\xi + \sigma_{\xi,Z}}^{\,l} = \widetilde{\mathcal{L}_Z\xi} \in \widetilde{\mathsf{S}(\mathsf{D}_G)}$$

for all $\xi \in \mathsf{S}(\mathsf{D}_G)$ and $Z \in \Gamma(AG)$, we find for any $g \in G$:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \left(R^*_{\operatorname{Exp}(tZ)}\tilde{\xi}\right)(g) &= \left.\frac{d}{dt}\right|_{t=0} \widetilde{R^*_{\operatorname{Exp}(tZ)}\xi}(g) \\ &= \widetilde{\left.\frac{d}{dt}\right|_{t=0} R^*_{\operatorname{Exp}(tZ)}\xi(g)} \\ &= \widetilde{\pounds_{Z^l}\xi}(g) = \widetilde{\mathcal{L}_Z\xi}(g) = \pounds_{Z^l}\tilde{\xi}(g). \end{aligned}$$

Assume here that the bracket on sections of $\mathfrak{A}(\mathsf{D}_G)$ induced by D_G as in Theorem 5.2.4 has image in $\Gamma(\mathfrak{A}(\mathsf{D}_G))$. Recall from (1.10) the definition of the Courant 3-tensor $\mathsf{T}_{\mathsf{D}_G}$ on sections of D_G. We show that $\mathsf{T} = \mathsf{T}_{\mathsf{D}_G}$ induces a tensor $\tilde{\mathsf{T}} \in \Gamma\left(\bigwedge^3 \mathsf{E}^*\right)$. By the considerations above, we can define our 3-tensor $\tilde{\mathsf{T}}$ by its values on the elements of $\widetilde{\mathsf{S}(\mathsf{D}_G)}$. Set

$$\tilde{\mathsf{T}}\left(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}\right) = \mathsf{T}(\xi, \eta, \zeta)$$

for all $\xi, \eta, \zeta \in \mathsf{S}(\mathsf{D}_G)$. To see that $\tilde{\mathsf{T}}$ is well-defined, choose $g \in G$ and $\sigma_g \in (\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t})(g)$. Then there exists $\sigma \in \Gamma(I^{\mathsf{t}}(\mathsf{D}_G))$ such that $\sigma^l(g) = \sigma_g$. Then, since $[\xi, \eta]$ is s-descending to $[\bar{\xi}, \bar{\eta}]_\star$, we have:

$$\mathsf{T}\left(\xi(g), \eta(g), \sigma_g\right) = \mathsf{T}\left(\xi, \eta, \sigma^l\right)(g) = \left\langle [\xi, \eta], \sigma^l \right\rangle(g) = \left\langle [\bar{\xi}, \bar{\eta}]_\star, \sigma \right\rangle(\mathsf{s}(g)) = 0$$

since $[\bar{\xi}, \bar{\eta}]_\star(\mathsf{s}(g)) \in \mathsf{D}_G(\mathsf{s}(g))$ by hypothesis.

For any bisection $K \in \mathcal{B}(G)$, we can define the 3-tensor $R_K^*\tilde{\mathsf{T}}$ by

$$\left(R_K^*\tilde{\mathsf{T}}\right)\left(\tilde{\xi},\tilde{\eta},\tilde{\zeta}\right) = R_K^*\left(\tilde{\mathsf{T}}\left(R_{K^{-1}}^*\tilde{\xi}, R_{K^{-1}}^*\tilde{\eta}, R_{K^{-1}}^*\tilde{\zeta}\right)\right)$$

for all $\xi,\eta,\zeta \in \mathsf{S}(\mathsf{D}_G)$. For $Z \in \Gamma(AG)$, we can thus define $\pounds_{Z^l}\tilde{\mathsf{T}}$ by

$$\pounds_{Z^l}\tilde{\mathsf{T}} = \left.\frac{d}{dt}\right|_{t=0} R_{\mathrm{Exp}(tZ)}^*\tilde{\mathsf{T}}.$$

We have

$$R_{\mathrm{Exp}(tZ)}^*\left(\tilde{\mathsf{T}}\left(\tilde{\xi},\tilde{\eta},\tilde{\zeta}\right)\right) = \left(R_{\mathrm{Exp}(tZ)}^*\tilde{\mathsf{T}}\right)\left(R_{\mathrm{Exp}(tZ)}^*\tilde{\xi}, R_{\mathrm{Exp}(tZ)}^*\tilde{\eta}, R_{\mathrm{Exp}(tZ)}^*\tilde{\zeta}\right)$$

for all $\xi,\eta,\zeta \in \mathsf{S}(\mathsf{D}_G)$, which yields

$$\begin{aligned}
\pounds_{Z^l}\left(\mathsf{T}\left(\xi,\eta,\zeta\right)\right) &= \pounds_{Z^l}\left(\tilde{\mathsf{T}}\left(\tilde{\xi},\tilde{\eta},\tilde{\zeta}\right)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} R_{\mathrm{Exp}(tZ)}^*\left(\tilde{\mathsf{T}}\left(\tilde{\xi},\tilde{\eta},\tilde{\zeta}\right)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0}\left(R_{\mathrm{Exp}(tZ)}^*\tilde{\mathsf{T}}\right)\left(R_{\mathrm{Exp}(tZ)}^*\tilde{\xi}, R_{\mathrm{Exp}(tZ)}^*\tilde{\eta}, R_{\mathrm{Exp}(tZ)}^*\tilde{\zeta}\right)\\
&= \left(\pounds_{Z^l}\tilde{\mathsf{T}}\right)\left(\tilde{\xi},\tilde{\eta},\tilde{\zeta}\right) + \tilde{\mathsf{T}}\left(\widetilde{\mathcal{L}_Z\xi},\tilde{\eta},\tilde{\zeta}\right) + \tilde{\mathsf{T}}\left(\tilde{\xi},\widetilde{\mathcal{L}_Z\eta},\tilde{\zeta}\right) + \tilde{\mathsf{T}}\left(\tilde{\xi},\tilde{\eta},\widetilde{\mathcal{L}_Z\zeta}\right)\\
&= \left(\pounds_{Z^l}\tilde{\mathsf{T}}\right)\left(\tilde{\xi},\tilde{\eta},\tilde{\zeta}\right) + \mathsf{T}(\mathcal{L}_Z\xi,\eta,\zeta) + \mathsf{T}(\xi,\mathcal{L}_Z\eta,\zeta) + \mathsf{T}(\xi,\eta,\mathcal{L}_Z\zeta). \quad (5.5)
\end{aligned}$$

Assume that $G\rightrightarrows P$ is t-connected. If $\pounds_{Z^l}\tilde{\mathsf{T}} = 0$ for all $Z \in \Gamma(AG)$, then we have $R_K^*\tilde{\mathsf{T}} = \tilde{\mathsf{T}}$ for all $K \in \mathcal{B}(G)$. Thus, if $\pounds_{Z^l}\tilde{\mathsf{T}} = 0$ we find that $\tilde{\mathsf{T}} = 0$ on G. This implies $\mathsf{T} = 0$ on $G \setminus P$. If in addition $\mathsf{T} = 0$ on P, we can then conclude that $\mathsf{T} = 0$ on G. We will use this method in the proof of the main theorem of this section. We first need a lemma.

Lemma 5.3.3 *Let $(G\rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid. Consider three s-descending sections $\xi \sim_{\mathsf{s}} \bar{\xi}$, $\eta \sim_{\mathsf{s}} \bar{\eta}$ and $\zeta \sim_{\mathsf{s}} \bar{\zeta}$ of D_G. Then, if $[\bar{\xi},\bar{\eta}]_\star \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$, we have*

$$\left[\bar{\zeta}, \left[\bar{\xi},\bar{\eta}\right]_\star\right]_\star = [\zeta,[\xi,\eta]]|_P$$

where the bracket on the right-hand side is the Courant-Dorfman bracket on sections of $TG \times_G T^*G$.

Proof: If $\bar{\tau} := [\bar{\xi},\bar{\eta}]_\star \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$, then there exists a s-descending section τ of D_G such that $\tau \sim_{\mathsf{s}} \bar{\tau}$. Since $[\xi,\eta]|_P = [\bar{\xi},\bar{\eta}]_\star = \tau|_P$ and $\mathbb{T}\mathsf{s}\left([\xi,\eta](g)\right) = [\bar{\xi},\bar{\eta}]_\star(\mathsf{s}(g)) = \mathbb{T}\mathsf{s}\left(\tau(g)\right)$ for all $g \in G$, there exists then a section χ of $\ker\mathbb{T}\mathsf{s}$ that is vanishing on P such that $\tau - [\xi,\eta] = \chi$. Choose $p \in P$. Then, on a neighborhood U of p in G, the section χ of $\ker\mathbb{T}\mathsf{s}$ can be written $\chi = \sum_{i=1}^n f_i\sigma_i^r$ with functions $f_1,\dots,f_n \in C^\infty(U)$ that vanish on $P \cap U$ and basis sections $\sigma_1,\dots,\sigma_n$ of $(\ker\mathbb{T}\mathsf{s})|_P$ on $U \cap P$. We have then

$$\left[\bar{\zeta}, \left[\bar{\xi},\bar{\eta}\right]_\star\right]_\star = [\zeta,\tau]|_P = [\zeta,[\xi,\eta]+\chi]|_P = [\zeta,[\xi,\eta]]|_P + [\zeta,\chi]|_P.$$

If we write $\zeta = (X_\zeta, \omega_\zeta)$, we can compute using (1.9)

$$[\zeta, \chi] = \sum_{i=1}^{n} \left(f_i \,[\zeta, \sigma_i^r] + X_\zeta(f_i)\sigma_i^r\right)$$

Since X_ζ is tangent to P on P and $f_1, \ldots, f_n$ vanish on P, we have $X_\zeta(f_i)|_P = 0$. This shows that $[\zeta, \chi](p) = 0$ for all $p \in P$. Hence, we have $\left[\bar\zeta, \left[\bar\xi, \bar\eta\right]_\star\right]_\star(p) = [\zeta, [\xi, \eta]](p)$. $\square$

Now we can show the main theorem of this section.

Theorem 5.3.4 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid. Assume that $G \rightrightarrows P$ is t-connected. Then the Dirac structure D_G is integrable if and only if:*

1. *the induced bracket as in Theorem 5.2.4 has image in $\Gamma(\mathfrak{A}(\mathsf{D}_G))$ and satisfies the Jacobi identity*

$$\left[\bar\zeta, \left[\bar\xi, \bar\eta\right]_\star\right]_\star + \left[\bar\eta, \left[\bar\zeta, \bar\xi\right]_\star\right]_\star + \left[\bar\xi, \left[\bar\eta, \bar\zeta\right]_\star\right]_\star = 0 \qquad \text{for all } \bar\xi, \bar\eta, \bar\zeta \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$$

   and

2. *the induced bracket on sections of $I^{\mathsf{s}}(\mathsf{D}_G)$ as in Proposition 5.2.5 has image in $\Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$.*

Proof: We have shown in Theorem 5.2.4 and Proposition 5.2.5 that the integrability of D_G implies 1) and 2).

Conversely, assume that 1) and 2) hold. We will show that D_G is integrable. First choose $p \in P$. The fiber $\mathsf{D}_G(p)$ of D_G over p is spanned by the values of the sections in $I^{\mathsf{s}}(\mathsf{D}_G)$ defined at p, and the values at p of the s-descending sections of D_G. Since the brackets on sections of $I^{\mathsf{s}}(\mathsf{D}_G)$ and $\mathfrak{A}(\mathsf{D}_G)$ have values in $\Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$, and respectively $\Gamma(\mathfrak{A}(\mathsf{D}_G))$, we find for all $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$ and $\xi_1, \xi_2, \xi_3 \in \mathsf{S}(\mathsf{D}_G)$:

$$\begin{aligned}
\mathsf{T}(\sigma_1^r, \sigma_2^r, \sigma_3^r)(p) &= \langle [\sigma_1, \sigma_2]^r, \sigma_3^r\rangle(p) = \langle [\sigma_1, \sigma_2]_{I^{\mathsf{s}}(\mathsf{D}_G)}, \sigma_3\rangle(p) = 0\\
\mathsf{T}(\sigma_1^r, \sigma_2^r, \xi_3)(p) &= \langle [\sigma_1, \sigma_2]_{I^{\mathsf{s}}(\mathsf{D}_G)}, \bar\xi_3\rangle(p) = 0\\
\mathsf{T}(\xi_1, \xi_2, \sigma_3^r)(p) &= \langle [\bar\xi_1, \bar\xi_2]_\star, \sigma_3\rangle(p) = 0\\
\mathsf{T}(\xi_1, \xi_2, \xi_3)(p) &= \langle [\bar\xi_1, \bar\xi_2]_\star, \bar\xi_3\rangle(p) = 0.
\end{aligned}$$

Hence, T vanishes over points in P.

Consider the 3-tensor $\tilde{\mathsf{T}}$ induced on the sections of $\mathsf{E} = \mathsf{D}_G/(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{t})$ by T and choose $\xi_1, \xi_2, \xi_3 \in \mathsf{S}(\mathsf{D}_G)$ and $Z \in \Gamma(AG)$. We show that $(\pounds_{Z^l}\tilde{\mathsf{T}})(\xi_1, \xi_2, \xi_3) = 0$. We have by

(1.11):

$$
\begin{aligned}
& Z^l(\mathsf{T}(\xi_1,\xi_2,\xi_3)) \\
&= Z^l\Big(\omega_{\xi_1}([X_{\xi_2},X_{\xi_3}]) + X_{\xi_1}(\omega_{\xi_2}(X_{\xi_3})) + \text{c.p.}\Big) \\
&= \left(\omega_{\mathcal{L}_Z\xi_1} + \mathsf{s}^*\alpha_{\xi_1,Z}\right)([X_{\xi_2},X_{\xi_3}]) + \omega_{\xi_1}\left(\left[\left(X_{\mathcal{L}_Z\xi_2} + Y^l_{\xi_2,Z}\right), X_{\xi_3}\right]\right) \\
&\quad + \omega_{\xi_1}\left(\left[X_{\xi_2}, \left(X_{\mathcal{L}_Z\xi_3} + Y^l_{\xi_3,Z}\right)\right]\right) + \left(X_{\mathcal{L}_Z\xi_1} + Y^l_{\xi_1,Z}\right)\left(\omega_{\xi_2}\left(X_{\xi_3}\right)\right) \\
&\quad + X_{\xi_1}\left(\left(\omega_{\mathcal{L}_Z\xi_2} + \mathsf{s}^*\alpha_{\xi_2,Z}\right)\left(X_{\xi_3}\right)\right) + X_{\xi_1}\left(\omega_{\xi_2}\left(X_{\mathcal{L}_Z\xi_3} + Y^l_{\xi_3,Z}\right)\right) \\
&\quad + \text{c.p.} \\
&= \left(\omega_{\mathcal{L}_Z\xi_1} + \mathsf{s}^*\alpha_{\xi_1,Z}\right)([X_{\xi_2},X_{\xi_3}]) + \omega_{\xi_1}\left(\left[\left(X_{\mathcal{L}_Z\xi_2} + Y^l_{\xi_2,Z}\right), X_{\xi_3}\right]\right) \\
&\quad + \omega_{\xi_1}\left(\left[X_{\xi_2}, \left(X_{\mathcal{L}_Z\xi_3} + Y^l_{\xi_3,Z}\right)\right]\right) + X_{\mathcal{L}_Z\xi_1}\left(\omega_{\xi_2}\left(X_{\xi_3}\right)\right) \\
&\quad + X_{\xi_1}\left(\left(\omega_{\mathcal{L}_Z\xi_2} + \mathsf{s}^*\alpha_{\xi_2,Z}\right)\left(X_{\xi_3}\right)\right) + X_{\xi_1}\left(\omega_{\xi_2}\left(X_{\mathcal{L}_Z\xi_3} + Y^l_{\xi_3,Z}\right)\right) \\
&\quad + \left(\omega_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_2} + \mathsf{s}^*\alpha_{\xi_2,Y_{\xi_1,Z}}\right)\left(X_{\xi_3}\right) + \omega_{\xi_2}\left(X_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_3} + Y^l_{\xi_3,Y_{\xi_1,Z}}\right) \\
&\quad + \text{c.p.} \\
&= \mathsf{T}\left(\mathcal{L}_Z\xi_1,\xi_2,\xi_3\right) + \mathsf{s}^*\alpha_{\xi_1,Z}\left([X_{\xi_2},X_{\xi_3}]\right) + \omega_{\xi_1}\left(\left[Y^l_{\xi_2,Z}, X_{\xi_3}\right]\right) + \omega_{\xi_1}\left(\left[X_{\xi_2}, Y^l_{\xi_3,Z}\right]\right) \\
&\quad + X_{\xi_1}\left(\mathsf{s}^*\alpha_{\xi_2,Z}\left(X_{\xi_3}\right)\right) + X_{\xi_1}\left(\omega_{\xi_2}\left(Y^l_{\xi_3,Z}\right)\right) \\
&\quad + \left(\omega_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_2} + \mathsf{s}^*\alpha_{\xi_2,Y_{\xi_1,Z}}\right)\left(X_{\xi_3}\right) + \omega_{\xi_2}\left(X_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_3} + Y^l_{\xi_3,Y_{\xi_1,Z}}\right) \\
&\quad + \text{c.p.} \\
&= \mathsf{T}\left(\mathcal{L}_Z\xi_1,\xi_2,\xi_3\right) + \mathsf{s}^*\left(\alpha_{\xi_1,Z}\left(\left[\bar{X}_{\xi_2},\bar{X}_{\xi_3}\right]\right)\right) \\
&\quad + \omega_{\xi_1}\left(X_{\mathcal{L}_{Y_{\xi_2,Z}}\xi_3} + Y^l_{\xi_3,Y_{\xi_2,Z}}\right) - \omega_{\xi_1}\left(X_{\mathcal{L}_{Y_{\xi_3,Z}}\xi_2} + Y^l_{\xi_2,Y_{\xi_3,Z}}\right) \\
&\quad + \mathsf{s}^*\left(\bar{X}_{\xi_1}\left(\alpha_{\xi_2,Z}\left(\bar{X}_{\xi_3}\right)\right)\right) + \mathsf{s}^*\left(\bar{X}_{\xi_1}\left(\omega_{\xi_2}\left(Y_{\xi_3,Z}\right)\right)\right) \\
&\quad + \left(\omega_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_2} + \mathsf{s}^*\alpha_{\xi_2,Y_{\xi_1,Z}}\right)\left(X_{\xi_3}\right) + \omega_{\xi_2}\left(X_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_3} + Y^l_{\xi_3,Y_{\xi_1,Z}}\right) \\
&\quad + \text{c.p.} \\
&= \mathsf{T}\left(\mathcal{L}_Z\xi_1,\xi_2,\xi_3\right) + \mathsf{s}^*\left(\alpha_{\xi_1,Z}\left(\left[\bar{X}_{\xi_2},\bar{X}_{\xi_3}\right]\right)\right) + \mathsf{s}^*\left(\bar{\omega}_{\xi_1}\left(Y_{\xi_3,Y_{\xi_2,Z}} - Y_{\xi_2,Y_{\xi_3,Z}}\right)\right) \\
&\quad + \mathsf{s}^*\left(\bar{X}_{\xi_1}\left(\alpha_{\xi_2,Z}\left(\bar{X}_{\xi_3}\right)\right)\right) + \mathsf{s}^*\left(\bar{X}_{\xi_1}\left(\bar{\omega}_{\xi_2}\left(Y_{\xi_3,Z}\right)\right)\right) \\
&\quad + \mathsf{s}^*\left(\alpha_{\xi_2,Y_{\xi_1,Z}}\left(\bar{X}_{\xi_3}\right)\right) + \mathsf{s}^*\left(\bar{\omega}_{\xi_2}\left(Y_{\xi_3,Y_{\xi_1,Z}}\right)\right) \\
&\quad + \text{c.p.}.
\end{aligned}
$$

In the last equality, we use

$$
\begin{aligned}
&\omega_{\xi_1}\left(X_{\mathcal{L}_{Y_{\xi_2,Z}}\xi_3}\right) - \omega_{\xi_1}\left(X_{\mathcal{L}_{Y_{\xi_3,Z}}\xi_2}\right) + \omega_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_2}\left(X_{\xi_3}\right) + \omega_{\xi_2}\left(X_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_3}\right) \\
&+ \omega_{\xi_2}\left(X_{\mathcal{L}_{Y_{\xi_3,Z}}\xi_1}\right) - \omega_{\xi_2}\left(X_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_3}\right) + \omega_{\mathcal{L}_{Y_{\xi_2,Z}}\xi_3}\left(X_{\xi_1}\right) + \omega_{\xi_3}\left(X_{\mathcal{L}_{Y_{\xi_2,Z}}\xi_1}\right) \\
&+ \omega_{\xi_3}\left(X_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_2}\right) - \omega_{\xi_3}\left(X_{\mathcal{L}_{Y_{\xi_2,Z}}\xi_1}\right) + \omega_{\mathcal{L}_{Y_{\xi_3,Z}}\xi_1}\left(X_{\xi_2}\right) + \omega_{\xi_1}\left(X_{\mathcal{L}_{Y_{\xi_3,Z}}\xi_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \omega_{\xi_1}\left(X_{\mathcal{L}_{Y_{\xi_2,Z}}\xi_3}\right) + \omega_{\mathcal{L}_{Y_{\xi_2,Z}}\xi_3}(X_{\xi_1}) + \omega_{\xi_2}\left(X_{\mathcal{L}_{Y_{\xi_3,Z}}\xi_1}\right) + \omega_{\mathcal{L}_{Y_{\xi_3,Z}}\xi_1}(X_{\xi_2}) \\
&\quad + \omega_{\xi_3}\left(X_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_2}\right) + \omega_{\mathcal{L}_{Y_{\xi_1,Z}}\xi_2}(X_{\xi_3}) = 0,
\end{aligned}
$$

since $(X_{\xi_i}, \omega_{\xi_i})$ and $\left(X_{\mathcal{L}_{Y_{\xi_i,Z}}\xi_j}, X_{\mathcal{L}_{Y_{\xi_i,Z}}\xi_j}\right) \in \Gamma(\mathsf{D}_G)$ for $i,j = 1,2,3$, $i \neq j$. By (5.5), this yields for all $g \in G$:

$$
\begin{aligned}
(\pounds_{Z^l}\tilde{\mathsf{T}})\left(\tilde{\xi_1}, \tilde{\xi_2}, \tilde{\xi_3}\right)(g) = &\, \mathsf{s}^*\left(\alpha_{\xi_1,Z}\left([\bar{X}_{\xi_2}, \bar{X}_{\xi_3}]\right)\right) + \mathsf{s}^*\left(\bar{\omega}_{\xi_1}\left(Y_{\xi_3,Y_{\xi_2,Z}} - Y_{\xi_2,Y_{\xi_3,Z}}\right)\right) \\
&+ \mathsf{s}^*\left(\bar{X}_{\xi_1}\left(\alpha_{\xi_2,Z}\left(\bar{X}_{\xi_3}\right)\right)\right) + \mathsf{s}^*\left(\bar{X}_{\xi_1}\left(\bar{\omega}_{\xi_2}\left(Y_{\xi_3,Z}\right)\right)\right) \\
&+ \mathsf{s}^*\left(\alpha_{\xi_2,Y_{\xi_1,Z}}\left(\bar{X}_{\xi_3}\right)\right) + \mathsf{s}^*\left(\bar{\omega}_{\xi_2}\left(Y_{\xi_3,Y_{\xi_1,Z}}\right)\right) \\
&+ \text{c.p.} \\
= &\left(\pounds_{Z^l}\tilde{\mathsf{T}}\right)\left(\tilde{\xi_1}, \tilde{\xi_2}, \tilde{\xi_3}\right)(\mathsf{s}(g)).
\end{aligned}
$$

But since T vanishes on the units by hypothesis, we find by (5.5) that

$$
(\pounds_{Z^l}\tilde{\mathsf{T}})(\tilde{\xi_1}, \tilde{\xi_2}, \tilde{\xi_3})(\mathsf{s}(g)) = Z^l(\mathsf{T}(\xi_1, \xi_2, \xi_3))(\mathsf{s}(g)).
$$

Since we know that the cotangent part of $[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]]$ is equal to $\mathbf{d}(\mathsf{T}(\xi_1, \xi_2, \xi_3))$, we find finally, using Lemma 5.3.3, that

$$
\begin{aligned}
(\pounds_{Z^l}\tilde{\mathsf{T}})(\tilde{\xi_1}, \tilde{\xi_2}, \tilde{\xi_3})(g) &= Z^l\left(\mathsf{T}(\xi_1, \xi_2, \xi_3)\right)(\mathsf{s}(g)) \\
&= \left\langle \left([\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]]\right)(\mathsf{s}(g)), (Z, 0)(\mathsf{s}(g)) \right\rangle \\
&= \left\langle \left([\xi_1, [\xi_2, \xi_3]_\star]_\star + [\xi_2, [\xi_3, \xi_1]_\star]_\star + [\xi_3, [\xi_1, \xi_2]_\star]_\star\right)(\mathsf{s}(g)), (Z, 0)(\mathsf{s}(g)) \right\rangle \\
&= 0
\end{aligned}
$$

since by condition 1), $[\cdot\,,\cdot]_\star$ satisfies the Jacobi identity.

Hence, we have shown that $\pounds_{Z^l}\tilde{\mathsf{T}} = 0$ for all $Z \in \Gamma(AG)$. This yields that $R^*_{\mathrm{Exp}(tZ)}\tilde{\mathsf{T}} = \tilde{\mathsf{T}}$ for all $Z \in \Gamma(AG)$ and $t \in \mathbb{R}$ where this makes sense and hence, since G is t-connected and $\tilde{\mathsf{T}}$ vanishes on the units, we find $\tilde{\mathsf{T}} = 0$. Thus, $\mathsf{T} = 0$ on G and the proof is finished.□

Remark 5.3.5 For $Z \in \Gamma(AG)$, define $\nabla_Z : \Gamma(\mathsf{E}) \to \Gamma(\mathsf{E})$ by $\nabla_Z\tilde{\xi} = \widetilde{\mathcal{L}_Z\xi}$ for all $\tilde{\xi} \in \widetilde{\mathsf{S}(\mathsf{D}_G)}$, and $\nabla_Z\left(\sum_{i=1}^n f_i\tilde{\xi}_i\right) = \sum_{i=1}^n\left(Z^l(f_i)\tilde{\xi}_i + f_i\nabla_Z\tilde{\xi}_i\right)$ for all $f_1, \ldots, f_n \in C^\infty(G)$ and $\tilde{\xi}_1, \ldots, \tilde{\xi}_n \in \widetilde{\mathsf{S}(\mathsf{D}_G)}$. Then ∇_Z is a derivative endomorphism of E over Z^l. The map $\Gamma(AG) \to \Gamma(\mathfrak{D}(\mathsf{E}))$, $Z \to \nabla_Z$ is a *derivative representation of AG on E associated to the action of AG on* $\mathsf{s} : G \to P$, $Z \in \Gamma(AG) \to Z^l$ (see Kosmann-Schwarzbach and Mackenzie (2002)). △

5.4 The Courant algebroid associated to an integrable Dirac groupoid

The dual space of $\mathfrak{A}(\mathsf{D}_G)$ can be identified with $\mathsf{P}_G|_P/\mathfrak{A}(\mathsf{D}_G)^\perp$. Since

$$\mathfrak{A}(\mathsf{D}_G)^\perp = \mathsf{D}_G|_P + (TP \times_P A^*G) = I^{\mathsf{t}}(\mathsf{D}_G) \oplus (TP \times_P A^*G)$$

and

$$\mathsf{P}_G|_P = (TP \times_P A^*G) + \ker \mathbb{T}\mathsf{t}|_P,$$

we have

$$(\mathfrak{A}(\mathsf{D}_G))^* \simeq \frac{\ker \mathbb{T}\mathsf{t}|_P}{I^{\mathsf{t}}(\mathsf{D}_G)}.$$

Since $\mathsf{D}_G|_P \subseteq \mathfrak{A}(\mathsf{D}_G) \oplus \ker \mathbb{T}\mathsf{t}|_P$, we have $I^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{A}(\mathsf{D}_G) \oplus \ker \mathbb{T}\mathsf{t}|_P$ and the quotient

$$\mathfrak{B}(\mathsf{D}_G) := \frac{\mathfrak{A}(\mathsf{D}_G) \oplus \ker \mathbb{T}\mathsf{t}|_P}{I^{\mathsf{s}}(\mathsf{D}_G)}$$

is a smooth vector bundle over P. Consider the map

$$\Psi : \ker \mathbb{T}\mathsf{t}|_P \oplus \mathfrak{A}(\mathsf{D}_G) \to \mathfrak{B}(\mathsf{D}_G),$$

$$\Psi\left(\sigma + \bar{\xi}\right) = \sigma + \bar{\xi} + I^{\mathsf{s}}(\mathsf{D}_G)$$

for all $\sigma \in \Gamma(\ker \mathbb{T}\mathsf{t}|_P)$ and $\bar{\xi} \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$. If $\Psi(\sigma + \bar{\xi}) = I^{\mathsf{s}}(\mathsf{D}_G)$, then we have $\sigma + \bar{\xi} \in \Gamma(\mathsf{D}_G|_P)$ and hence $\sigma \in \Gamma(\mathsf{D}_G|_P)$ since $\bar{\xi} \in \Gamma(\mathsf{D}_G|_P)$. This yields $\sigma \in \Gamma(I^{\mathsf{t}}(\mathsf{D}_G))$ and the map Ψ factors to a vector bundle homomorphism

$$\bar{\Psi} : (\mathfrak{A}(\mathsf{D}_G))^* \oplus \mathfrak{A}(\mathsf{D}_G) \to \mathfrak{B}(\mathsf{D}_G)$$

over the identity Id_P.
Set $r = \operatorname{rank} I^{\mathsf{s}}(\mathsf{D}_G)$, $n = \dim G$. Then we have also $r = \operatorname{rank} I^{\mathsf{t}}(\mathsf{D}_G)$ and we can compute $\operatorname{rank} \mathfrak{B}(\mathsf{D}_G) = \operatorname{rank}(\ker \mathbb{T}\mathsf{t}) + \operatorname{rank}(\mathfrak{A}(\mathsf{D}_G)) - \operatorname{rank} I^{\mathsf{s}}(\mathsf{D}_G) = n + (n-r) - r = 2n - 2r$. We have also $\operatorname{rank}((\mathfrak{A}(\mathsf{D}_G))^* \oplus \mathfrak{A}(\mathsf{D}_G)) = n - r + n - r = 2n - 2r$ and since $\bar{\Psi}$ is surjective, it is hence a vector bundle isomorphism.

Since $(\ker \mathbb{T}\mathsf{t}|_P \oplus \mathfrak{A}(\mathsf{D}_G))^\perp = (\ker \mathbb{T}\mathsf{t}|_P + \mathsf{D}_G|_P)^\perp = I^{\mathsf{s}}(\mathsf{D}_G)$, the bracket $\langle \cdot\,, \cdot \rangle$ restricts to a non degenerate symmetric bracket on $\mathfrak{B}(\mathsf{D}_G)$, that will also be written $\langle \cdot\,, \cdot \rangle$ in the following.

Recall from Example 5.2.6 that if $(G \rightrightarrows P, \mathsf{D}_{\pi_G})$ is a Poisson Lie groupoid, the bundle $\mathfrak{A}(\mathsf{D}_{\pi_G})$ is equal to $\operatorname{Graph}(\pi_G^\sharp|_{A^*G}) \simeq A^*G$, $\mathsf{a}_\star(\xi) = \pi_G^\sharp(\xi)$ for all $\xi \in \Gamma(A^*G)$ and the bracket on sections of $\mathfrak{A}(\mathsf{D}_G)$ is the bracket induced by the Poisson structure. In the same manner, we have $(\mathfrak{A}(\mathsf{D}_G))^* = \ker \mathbb{T}\mathsf{t}|_P / I^{\mathsf{t}}(\mathsf{D}_G) = \ker \mathbb{T}\mathsf{t}|_P / \operatorname{Graph}\left(\pi_G^\sharp|_{(T_P^{\mathsf{s}}G)^\circ}\right)$ which is isomorphic as a vector bundle to AG. The vector bundle $\mathfrak{B}(\mathsf{D}_{\pi_G})$ is thus the vector bundle underlying the Courant algebroid associated to $(G \rightrightarrows P, \pi)$ We will study this example in

more detail in Example 5.4.2, where we will show that $\mathfrak{B}(\mathsf{D}_{\pi_G})$ carries a natural Courant algebroid structure that makes it isomorphic as a Courant algebroid to $AG \times_P A^*G$. In this case, we have $r = \operatorname{rank} I^{\mathsf{t}}(\mathsf{D}_G) = \dim P$, since the rank of AG is $\dim G - \dim P$.
We show here that if the Dirac groupoid $(G\rightrightarrows P, \mathsf{D}_G)$ is integrable, the vector bundle $\mathfrak{B}(\mathsf{D}_G) \to P$ always inherits the structure of a Courant algebroid from the ambient standard Courant algebroid structure of P_G (see Example 1.1.20).
Because of the special case of Poisson Lie groupoids, we have chosen the notation $\mathfrak{B}(\mathsf{D}_G)$: this Courant algebroid will play the role of the "Lie bialgebroid of the Dirac groupoid $(G\rightrightarrows P, \mathsf{D}_G)$".

Theorem 5.4.1 *Let $(G\rightrightarrows P, \mathsf{D}_G)$ be an integrable Dirac groupoid and*

$$\mathfrak{B}(\mathsf{D}_G) = \frac{\mathfrak{A}(\mathsf{D}_G) \oplus \ker \mathbb{T}\mathsf{t}|_P}{I^{\mathsf{s}}(\mathsf{D}_G)} \to P$$

the associated vector bundle over P. Set $\mathsf{b} : \mathfrak{B}(\mathsf{D}_G) \to TP$, $\mathsf{b}(v_p, \alpha_p) = T_p\mathsf{s} v_p$. Define

$$[\cdot\,,\cdot] : \Gamma(\mathfrak{B}(\mathsf{D}_G)) \times \Gamma(\mathfrak{B}(\mathsf{D}_G)) \to \Gamma(\mathfrak{B}(\mathsf{D}_G))$$

by

$$\left[\bar\xi + \sigma + I^{\mathsf{s}}(\mathsf{D}_G), \bar\eta + \tau + I^{\mathsf{s}}(\mathsf{D}_G)\right] = \left[\xi + \sigma^l, \eta + \tau^l\right]\big|_P + I^{\mathsf{s}}(\mathsf{D}_G)$$

for all $\sigma, \tau \in \Gamma(\ker \mathbb{T}\mathsf{t}|_P)$, $\bar\xi, \bar\eta \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$ and s-descending sections $\xi \sim_{\mathsf{s}} \bar\xi$, $\eta \sim_{\mathsf{s}} \bar\eta$ of D_G, where the bracket on the right-hand side of this equation is the Courant bracket on sections of the Courant algebroid P_G. This bracket is well-defined and $(\mathfrak{B}(\mathsf{D}_G), \mathsf{b}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle)$ is a Courant algebroid.

PROOF: The map b is well-defined since $T_p\mathsf{s} v_p = 0_p$ for all $(v_p, \alpha_p) \in I^{\mathsf{s}}(\mathsf{D}_G)$. We show that the bracket on sections of $\mathfrak{B}(\mathsf{D}_G)$ is well-defined, that is, that it has image in $\Gamma(\mathfrak{B}(\mathsf{D}_G))$ and doesn't depend on the choice of the sections $\bar\xi+\sigma$ and $\bar\eta+\tau$ representing $\bar\xi+\sigma+I^{\mathsf{s}}(\mathsf{D}_G)$ and $\bar\eta + \tau + I^{\mathsf{s}}(\mathsf{D}_G)$. We have, writing $\sigma^l = \left(X^l, \mathsf{s}^*\alpha\right)$ and $\tau^l = \left(Y^l, \mathsf{s}^*\beta\right)$,

$$\begin{aligned}
&\left[(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta)\right] \\
&= [(X_\xi, \theta_\xi), (X_\eta, \theta_\eta)] + \pounds_{X^l}(X_\eta, \theta_\eta) - \pounds_{Y^l}(X_\xi, \theta_\xi) \\
&\quad + \Bigg([X, Y]^l_{AG}, \pounds_{X_\xi + X^l}(\mathsf{s}^*\beta) - \pounds_{X_\eta + Y^l}(\mathsf{s}^*\alpha) \\
&\qquad + \frac{1}{2}\mathbf{d}\big((\mathsf{s}^*\alpha)\left(X_\eta + Y^l\right) + \theta_\xi\left(Y^l\right) - (\mathsf{s}^*\beta)\left(X_\xi + X^l\right) - \theta_\eta\left(X^l\right)\big)\Bigg) \\
&= [(X_\xi, \theta_\xi), (X_\eta, \theta_\eta)] + \pounds_{X^l}(X_\eta, \theta_\eta) - \pounds_{Y^l}(X_\xi, \theta_\xi) \\
&\quad + \Bigg([X, Y]^l_{AG}, \mathsf{s}^*\left(\pounds_{\bar X_\xi + \mathsf{a}(X)}\beta - \pounds_{\bar X_\eta + \mathsf{a}(Y)}\alpha\right) \\
&\qquad + \mathsf{s}^*\left(\frac{1}{2}\mathbf{d}\big(\alpha(\bar X_\eta + \mathsf{a}(Y)) + \bar\theta_\xi(Y) - \beta(\bar X_\xi + \mathsf{a}(X)) - \bar\theta_\eta(X)\big)\right)\Bigg).
\end{aligned} \tag{5.6}$$

By Theorems 5.2.4 and 5.3.1, the restriction of this to P is a section of $\mathfrak{A}(\mathsf{D}_G) \oplus \ker \mathbb{T}\mathsf{t}|_P$ and depends on the choice of the s-descending sections (X_ξ, θ_ξ), (X_η, θ_η) only by sections of $I^{\mathsf{s}}(\mathsf{D}_G)$.
Choose $\sigma \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$. Then we have for all $(Y^l, \mathsf{s}^*\beta) \in \Gamma(\ker \mathbb{T}\mathsf{t})$:

$$\left[\sigma^r, (Y^l, \mathsf{s}^*\beta)\right] = \left(0, \pounds_{X^r_\sigma}(\mathsf{s}^*\beta) - \pounds_{Y^l}(\mathsf{t}^*\alpha_\sigma) + \frac{1}{2}\mathbf{d}\left((\mathsf{t}^*\alpha_\sigma)(Y^l) - (\mathsf{s}^*\beta)(X^r_\sigma)\right)\right) = (0,0).$$

We have used Lemma 5.3.2. If (X_ν, θ_ν) is a section of D_G that is s-descending to $(\bar{X}_\nu, \bar{\theta}_\nu) = (0,0)$, then we have $(X_\nu, \theta_\nu) \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$ and we find smooth sections $\sigma_1^r, \ldots, \sigma_k^r \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$ and functions $f_1, \ldots, f_k \in C^\infty(G)$ such that $(X_\nu, \theta_\nu) = \sum_{i=1}^k f_i \sigma_i^r$. Then we get for all $(Y^l, \mathsf{s}^*\beta) \in \Gamma(\ker \mathbb{T}\mathsf{t})$

$$\begin{aligned}
\left[(X_\nu, \theta_\nu), \left(Y^l, \mathsf{s}^*\beta\right)\right] &= \sum_{i=1}^k \left(f_i \left[\sigma_i^r, (Y^l, \mathsf{s}^*\beta)\right] - Y^l(f_i)\sigma_i^r + \frac{1}{2}\left\langle \sigma_i^r, (Y^l, \mathsf{s}^*\beta)\right\rangle (0, \mathbf{d}f_i)\right) \\
&= -\sum_{i=1}^k Y^l(f_i)\sigma_i^r,
\end{aligned}$$

which is a section of $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s}$. Hence, the restriction to P of $\left[(X_\nu, \theta_\nu), \left(Y^l, \mathsf{s}^*\beta\right)\right]$ is a section of $I^{\mathsf{s}}(\mathsf{D}_G)$.
In the same manner, for $i = 1, \ldots, k$ and any s-descending section $(X_\xi, \theta_\xi) \sim_{\mathsf{s}} (\bar{X}_\xi, \bar{\theta}_\xi)$,

$$[\sigma_i^r, (X_\xi, \theta_\xi)] = \left(\left[X^r_{\sigma_i}, X_\xi\right], \pounds_{X^r_{\sigma_i}}\theta_\xi - \mathbf{i}_{X_\xi}\mathbf{d}(\mathsf{t}^*\alpha_{\sigma_i})\right)$$

is a section of D_G since D_G is integrable Since $X_\xi \sim_{\mathsf{s}} \bar{X}_\xi$ and $X^r_{\sigma_i} \sim_{\mathsf{s}} 0$, we have $\left[X^r_{\sigma_i}, X_\xi\right] \sim_{\mathsf{s}} [0, \bar{X}_\xi] = 0$ and we compute for any $Y \in \Gamma(AG)$, using the equality $(\mathsf{t}^*\alpha_{\sigma_i})(X_\xi) = -\theta_\xi(X^r_{\sigma_i})$:

$$\begin{aligned}
\left(\pounds_{X^r_{\sigma_i}}\theta_\xi - \mathbf{i}_{X_\xi}\mathbf{d}(\mathsf{t}^*\alpha_{\sigma_i})\right)\left(Y^l\right) &= X^r_{\sigma_i}\left(\theta_\xi\left(Y^l\right)\right) - \theta_\xi\left(\left[X^r_{\sigma_i}, Y^l\right]\right) - X_\xi\left((\mathsf{t}^*\alpha_{\sigma_i})\left(Y^l\right)\right) \\
&\quad + Y^l\left((\mathsf{t}^*\alpha_{\sigma_i})(X_\xi)\right) + (\mathsf{t}^*\alpha_{\sigma_i})\left(\left[X_\xi, Y^l\right]\right) \\
&= X^r_{\sigma_i}(\mathsf{s}^*(\bar{\theta}_\xi(Y))) + (\pounds_{Y^l}(\mathsf{t}^*\alpha_{\sigma_i}))(X_\xi) = 0.
\end{aligned}$$

This shows that $\left[\left(X^r_{\sigma_i}, \mathsf{t}^*\alpha_{\sigma_i}\right), (X_\xi, \theta_\xi)\right]$ is a section of $\ker \mathbb{T}\mathsf{s} \cap \mathsf{D}_G$ for $i = 1, \ldots, k$. Then we get as above

$$\begin{aligned}
[(X_\nu, \theta_\nu), (X_\xi, \theta_\xi)] &= \sum_{i=1}^k \left(f_i\left[\sigma_i^r, (X_\xi, \theta_\xi)\right] - X_\xi(f_i)\sigma_i^r + \frac{1}{2}\langle \sigma_i^r, (X_\xi, \theta_\xi)\rangle(0, \mathbf{d}f_i)\right) \\
&= \sum_{i=1}^k \left(f_i\left[\sigma_i^r, (X_\xi, \theta_\xi)\right] - X_\xi(f_i)\sigma_i^r\right),
\end{aligned}$$

which is a section of $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s}$ by the considerations above. Hence, its restriction to P is a section of $I^{\mathsf{s}}(\mathsf{D}_G)$. If $\bar{\xi} + \sigma \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$, then as above, we find that $\sigma \in \Gamma(I^{\mathsf{t}}(\mathsf{D}_G))$.

The section $\xi+\sigma^l-(\bar\xi+\sigma)^r$ is then a section of D_G that is s-descending to 0. Since by the considerations above, we know that

$$\left[\xi+\sigma^l-(\bar\xi+\sigma)^r,\eta+\tau^l\right]\big|_P\in\Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$$

and

$$\left[(\bar\xi+\sigma)^r,\eta+\tau^l\right]\big|_P\in\Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$$

for all s-descending sections $\eta\sim_{\mathsf{s}}\bar\eta$ of D_G and $\tau\in\Gamma((\ker(\mathbb{T}\mathsf{t}))|_P)$, we have shown that the bracket doesn't depend on the choice of the representatives for $(\bar X_\xi,\bar\theta_\xi)+(X,(\mathsf{s}^*\alpha)|_P)+I^{\mathsf{s}}(\mathsf{D}_G)$ and $(\bar X_\eta,\bar\theta_\eta)+(Y,(\mathsf{s}^*\beta)|_P)+I^{\mathsf{s}}(\mathsf{D}_G)$.

We show now that $(\mathfrak{B}(\mathsf{D}_G),\mathsf{b},[\cdot\,,\cdot],\langle\cdot\,,\cdot\rangle)$ is a Courant algebroid. The map

$$\mathcal{D}:C^\infty(P)\to\Gamma(\mathfrak{B}(\mathsf{D}_G))$$

is simply given by

$$\mathcal{D}f=\frac{1}{2}(0,\mathsf{s}^*\mathbf{d}f)+I^{\mathsf{s}}(\mathsf{D}_G)$$

since

$$\left\langle\mathcal{D}f,\overline{(v_p,\alpha_p)}\right\rangle=\frac{1}{2}\mathsf{b}\left(\overline{(v_p,\alpha_p)}\right)(f)=\frac{1}{2}T_p\mathsf{s}v_p(f)$$

for all $\overline{(v_p,\alpha_p)}\in\mathfrak{B}_p(\mathsf{D}_G)$. We check all the Courant algebroid axioms. Choose

$$(\bar X_\xi+X,\bar\theta_\xi+(\mathsf{s}^*\alpha)|_P)+I^{\mathsf{s}}(\mathsf{D}_G),\qquad(\bar X_\eta+Y,\bar\theta_\eta+(\mathsf{s}^*\beta)|_P)+I^{\mathsf{s}}(\mathsf{D}_G)$$

and

$$(\bar X_\tau+Z,\bar\theta_\tau+(\mathsf{s}^*\gamma)|_P)+I^{\mathsf{s}}(\mathsf{D}_G)\in\Gamma(\mathfrak{B}(\mathsf{D}_G))$$

and let f be an arbitrary element of $C^\infty(P)$.

1. By (5.6), the bracket

$$\left[(X_\xi+X^l,\theta_\xi+\mathsf{s}^*\alpha),(X_\eta+Y^l,\theta_\eta+\mathsf{s}^*\beta)\right]$$

can be taken as the section extending

$$\left[(\bar X_\xi+X,\bar\theta_\xi+(\mathsf{s}^*\alpha)|_P)+I^{\mathsf{s}}(\mathsf{D}_G),(\bar X_\eta+Y,\bar\theta_\eta+(\mathsf{s}^*\beta)|_P)+I^{\mathsf{s}}(\mathsf{D}_G)\right]$$

to compute its bracket with $(\bar X_\tau+Z,\bar\theta_\tau+(\mathsf{s}^*\gamma)|_P)+I^{\mathsf{s}}(\mathsf{D}_G)$. Since P_G is a Courant algebroid, we have

$$\begin{aligned}&\left[\left[(X_\xi+X^l,\theta_\xi+\mathsf{s}^*\alpha),(X_\eta+Y^l,\theta_\eta+\mathsf{s}^*\beta)\right],(X_\tau+Z^l,\theta_\tau+\mathsf{s}^*\gamma)\right]+\text{c.p.}\\=&\frac{1}{6}\left(0,\mathbf{d}\left(\left\langle\left[\left(X_\xi+X^l,\theta_\xi+\mathsf{s}^*\alpha\right),\left(X_\eta+Y^l,\theta_\eta+\mathsf{s}^*\beta\right)\right],\left(X_\tau+Z^l,\theta_\tau+\mathsf{s}^*\gamma\right)\right\rangle\right)\right)\\&+\text{c.p.}.\end{aligned}$$

To simplify the notation in the following computation, we define $F \in C^\infty(P)$,

$$F := \frac{1}{2}\left(\alpha(\bar{X}_\eta + \mathsf{a}(Y)) + \bar{\theta}_\xi(Y) - \beta(\bar{X}_\xi + \mathsf{a}(X)) - \bar{\theta}_\eta(X)\right)$$

We have then:

$$\begin{aligned}
&\left\langle \left[(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta)\right], (X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma)\right\rangle \\
\overset{(5.6)}{=} &\left\langle [(X_\xi, \theta_\xi), (X_\eta, \theta_\eta)], (X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma)\right\rangle \\
&+ \left\langle \pounds_{X^l}(X_\eta, \theta_\eta) - \pounds_{Y^l}(X_\xi, \theta_\xi), (X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma)\right\rangle \\
&+ \left\langle \left([X, Y]^l_{AG}, \mathsf{s}^*\left(\pounds_{\bar{X}_\xi + \mathsf{a}(X)}\beta - \pounds_{\bar{X}_\eta + \mathsf{a}(Y)}\alpha + \mathbf{d}F\right)\right), (X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma)\right\rangle \\
= &\ \mathsf{s}^*\left\langle [(\bar{X}_\xi, \bar{\theta}_\xi), (\bar{X}_\eta, \bar{\theta}_\eta)]_\star, (Z, \gamma)\right\rangle \\
&+ \left\langle \mathcal{L}_X \eta + (Y^l_{\eta,X}, \mathsf{s}^*\alpha_{\eta,X}), (X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma)\right\rangle \\
&- \left\langle \mathcal{L}_Y \xi + (Y^l_{\xi,Y}, \mathsf{s}^*\alpha_{\xi,Y}), (X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma)\right\rangle \\
&+ \mathsf{s}^*\left\langle \left([X, Y]_{AG}, \pounds_{\bar{X}_\xi + \mathsf{a}(X)}\beta - \pounds_{\bar{X}_\eta + \mathsf{a}(Y)}\alpha + \mathbf{d}F\right), (\bar{X}_\tau, \bar{\theta}_\tau)\right\rangle \\
&+ \mathsf{s}^*\left\langle \left(\mathsf{a}\left([X, Y]_{AG}\right), \pounds_{\bar{X}_\xi + \mathsf{a}(X)}\beta - \pounds_{\bar{X}_\eta + \mathsf{a}(Y)}\alpha + \mathbf{d}F\right), (\mathsf{a}(Z), \gamma)\right\rangle \\
= &\ \mathsf{s}^*\Big\langle \left[(\bar{X}_\xi + X, \bar{\theta}_\xi + \mathsf{s}^*\alpha) + I^\mathsf{s}(\mathsf{D}_G), (\bar{X}_\eta + Y, \bar{\theta}_\eta + \mathsf{s}^*\beta) + I^\mathsf{s}(\mathsf{D}_G)\right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\bar{X}_\tau + Z, \bar{\theta}_\tau + \mathsf{s}^*\gamma) + I^\mathsf{s}(\mathsf{D}_G)\Big\rangle.
\end{aligned}$$

Hence, if we write $e_{\xi,X,\alpha}$ for $(\bar{X}_\xi + X, \bar{\theta}_\xi + (\mathsf{s}^*\alpha)|_P) + I^\mathsf{s}(\mathsf{D}_G)$, etc, we have shown that

$$\begin{aligned}
&[e_{\xi,X,\alpha}, [e_{\eta,Y,\beta}, e_{\tau,Z,\gamma}]] + [e_{\eta,Y,\beta}, [e_{\tau,Z,\gamma}, e_{\xi,X,\alpha}]] + [e_{\tau,Z,\gamma}, [e_{\xi,X,\alpha}, e_{\eta,Y,\beta}]] \\
=&\frac{1}{3}\mathcal{D}\left(\langle [e_{\xi,X,\alpha}, e_{\eta,Y,\beta}], e_{\tau,Z,\gamma}\rangle + \langle [e_{\eta,Y,\beta}, e_{\tau,Z,\gamma}], e_{\xi,X,\alpha}\rangle + \langle [e_{\tau,Z,\gamma}, e_{\xi,X,\alpha}], e_{\eta,Y,\beta}\rangle\right).
\end{aligned}$$

2. We have

$$\begin{aligned}
\mathsf{b}[e_{\xi,X,\alpha}, e_{\eta,Y,\beta}] &= T\mathsf{s}\left[X_\xi + X^l, X_\eta + Y^l\right]|_P = \left[T\mathsf{s}\left(X_\xi + X^l\right), T\mathsf{s}\left(X_\eta + Y^l\right)\right] \\
&= \left[\bar{X}_\xi + \mathsf{a}(X), \bar{X}_\eta + \mathsf{a}(Y)\right] = [\mathsf{b}(e_{\xi,X,\alpha}), \mathsf{b}(e_{\eta,Y,\beta})].
\end{aligned}$$

3. We compute

$$\begin{aligned}
&[e_{\xi,X,\alpha}, f \cdot e_{\eta,Y,\beta}] \\
= &\ \left[(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (\mathsf{s}^* f) \cdot (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta)\right]\Big|_P + I^\mathsf{s}(\mathsf{D}_G) \\
= &\ \Big((\mathsf{s}^* f)\left[(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta)\right] \\
&\qquad + (X_\xi + X^l)(\mathsf{s}^* f) \cdot \left(X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta\right)
\end{aligned}$$

$$- \left\langle (X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta) \right\rangle \cdot \frac{1}{2}(0, \mathbf{d}(\mathsf{s}^* f)) \Big) \Big|_P + I^{\mathsf{s}}(\mathsf{D}_G)$$

$$\begin{aligned} &= f[e_{\xi,X,\alpha}, e_{\eta,Y,\beta}] + \mathsf{b}(e_{\xi,X,\alpha})(f) \cdot e_{\eta,Y,\beta} \\ &\qquad - \left\langle \left(\bar{X}_\xi + X, \bar{\theta}_\xi + \mathsf{s}^*\alpha\right), \left(\bar{X}_\eta + Y, \bar{\theta}_\eta + \mathsf{s}^*\beta\right) \right\rangle \mathcal{D} f \\ &= f\left[e_{\xi,X,\alpha}, e_{\eta,Y,\beta}\right] + \mathsf{b}\left(e_{\xi,X,\alpha}\right)(f) \cdot e_{\eta,Y,\beta} - \left\langle e_{\xi,X,\alpha}, e_{\eta,Y,\beta}\right\rangle \mathcal{D} f. \end{aligned}$$

4. We have obviously $\mathsf{b} \circ \mathcal{D} = 0$.

5. Finally, since P_G is a Courant algebroid, we have the equality

$$\begin{aligned} &(X_\xi + X^l) \left(\left\langle \left(X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta\right), \left(X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma\right)\right\rangle\right) \\ &= \left\langle \left[\left(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha\right), \left(X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta\right)\right], \left(X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma\right)\right\rangle \\ &\quad + \frac{1}{2}\left\langle \left(0, \mathbf{d}\left\langle \left(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha\right), \left(X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta\right)\right\rangle\right), \left(X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma\right)\right\rangle \\ &\quad + \left\langle \left[\left(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha\right), \left(X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma\right)\right], \left(X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta\right)\right\rangle \\ &\quad + \frac{1}{2}\left\langle \left(0, \mathbf{d}\left\langle \left(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha\right), \left(X_\tau + Z^l, \theta_\tau + \mathsf{s}^*\gamma\right)\right\rangle\right), \left(X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta\right)\right\rangle, \end{aligned}$$

which yields easily, with the same computations as in the previous points

$$\begin{aligned} \mathsf{b}(e_{\xi,X,\alpha})\langle e_{\eta,Y,\beta}, e_{\tau,Z,\gamma}\rangle = &\left\langle [e_{\xi,X,\alpha}, e_{\eta,Y,\beta}] + \mathcal{D}\langle e_{\xi,X,\alpha}, e_{\eta,Y,\beta}\rangle, e_{\tau,Z,\gamma}\right\rangle \\ &+ \left\langle e_{\eta,Y,\beta}, [e_{\xi,X,\alpha}, e_{\tau,Z,\gamma}] + \mathcal{D}\langle e_{\xi,X,\alpha}, e_{\tau,Z,\gamma}\rangle\right\rangle. \qquad \square \end{aligned}$$

Example 5.4.2 We show in this example that in the special case of a Poisson groupoid $(G \rightrightarrows P, \mathsf{D}_{\pi_G})$, the obtained Courant algebroid is isomorphic to the Courant algebroid defined by the Lie bialgebroid associated to $(G \rightrightarrows P, \pi_G)$, see Liu et al. (1997), Liu et al. (1998). This shows how the Courant algebroid structure on $AG \times_P A^*G$ induced by the Lie bialgebroid of the Poisson Lie groupoid $(G \rightrightarrows P, \pi_G)$ can be related to the standard Courant algebroid structure on $\mathsf{P}_G = TG \times_G T^*G$.
In this example, we will write sections of $\mathfrak{A}(\mathsf{D}_{\pi_G})$ as pairs $(\pi_G^\sharp(\xi), \xi)$, with $\xi \in \Gamma(A^*G)$. A section of D_{π_G} that is s-descending to $(\pi_G^\sharp(\xi), \xi)$ will be written (X_ξ, θ_ξ).
Recall that the Courant algebroid $\mathsf{E}_{\pi_G} = AG \times_P A^*G$ associated to the Lie bialgebroid (AG, A^*G) of $(G \rightrightarrows P, \pi_G)$ is endowed with the anchor $\rho : AG \times_P A^*G \to TP$ defined by $\rho(v_p, \alpha_p) = \mathsf{a}(v_p) + \pi_G^\sharp(\alpha_p)$ for all $p \in P$ and $(v_p, \alpha_p) \in A_pG \times A_p^*G$ and the symmetric bracket $\langle \cdot\,, \cdot \rangle$ defined by $\langle (v_p, \alpha_p), (w_p, \beta_p)\rangle = \alpha_p(w_p) + \beta_p(v_p)$ for all $p \in P$ and $(v_p, \alpha_p), (w_p, \beta_p) \in A_pG \times A_p^*G$. Its Courant bracket is given by

$$\begin{aligned} [(X,\xi),(Y,\eta)] = \Big(&[X,Y]_{AG} + \mathfrak{L}_\xi^* Y - \mathfrak{L}_\eta^* X - \frac{1}{2}\mathsf{d}_*(\xi(Y) - \eta(X)), \\ &[\xi,\eta]_\star + \mathfrak{L}_X \eta - \mathfrak{L}_Y \xi + \frac{1}{2}\mathsf{d}(\xi(Y) - \eta(X))\Big), \end{aligned} \tag{5.7}$$

where, if $\bar{X}_\xi := \pi_G^\sharp(\xi) \in \Gamma(TP)$ for $\xi \in \Gamma(A^*G)$,

$$\mathfrak{L}_\xi^* Y \in \Gamma(AG), \quad \tau(\mathfrak{L}_\xi^* Y) = \bar{X}_\xi(\tau(Y)) - [\xi,\tau](Y) \qquad \forall \tau \in \Gamma(A^*G)$$
$$\mathfrak{L}_X \eta \in \Gamma(A^*G), \quad (\mathfrak{L}_X\eta)(Z) = \mathsf{a}(X)(\eta(Z)) - \eta([X,Z]_{AG}) \qquad \forall Z \in \Gamma(AG)$$

and, for any $f \in C^\infty(P)$,

$$(\mathsf{d}_* f)(\tau) = \bar{X}_\tau(f) = X_\tau(\mathsf{s}^* f)|_P = -\tau(X_{\mathsf{s}^* f}), \qquad \text{hence } \mathsf{d}_* f = -X_{\mathsf{s}^* f}|_P$$
$$(\mathsf{d} f)(Z) = \mathsf{a}(Z)(f) = Z(\mathsf{s}^*(f)), \qquad \text{hence } \mathsf{d} f = \mathbf{d}(\mathsf{s}^* f)|_{AG} = \hat{\mathsf{s}}(\mathbf{d}(\mathsf{s}^* f)).$$

The isomorphism $\Psi : AG \times_P A^*G \to \mathfrak{B}(\mathsf{D}_{\pi_G})$ is given by

$$\Psi(X(p),\xi(p)) = (X + \bar{X}_\xi, \xi)(p) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G}),$$

with inverse

$$\Psi^{-1}((v_p,\alpha_p) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G})) = (v_p - \pi_G^\sharp(\alpha_p), \hat{\mathsf{s}}(\alpha_p)).$$

We will check that $\Psi^{-1} \circ \Psi = \mathrm{Id}_{\mathsf{E}_{\pi_G}}$ and $\Psi \circ \Psi^{-1} = \mathrm{Id}_{\mathfrak{B}(\mathsf{D}_{\pi_G})}$. Note first that Ψ^{-1} is well-defined: if $(v_p,\alpha_p) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G}) \in \mathfrak{B}_p(\mathsf{D}_G)$, then $\mathbb{T}\mathsf{t}(v_p,\alpha_p) \in \mathfrak{A}_p(\mathsf{D}_{\pi_G})$ by definition of $\mathfrak{B}(\mathsf{D}_{\pi_G})$. This yields $T_p\mathsf{t} v_p = \pi_G^\sharp\left(\hat{\mathsf{t}}(\alpha_p)\right)$. We get $v_p - \pi_G^\sharp(\alpha_p) = v_p - T_p\mathsf{t} v_p + \pi_G^\sharp\left(\hat{\mathsf{t}}(\alpha_p) - \alpha_p\right)$. But $v_p - T_p\mathsf{t}v_p \in A_pG$ and $\hat{\mathsf{t}}(\alpha_p) - \alpha_p \in (T_p^{\mathsf{s}}G)^\circ$ yields $\pi_G^\sharp\left(\hat{\mathsf{t}}(\alpha_p) - \alpha_p\right) \in T_p^{\mathsf{t}}G = A_pG$ (see Weinstein (1988) or the results in Section 5.2). Furthermore, elements of $I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G})$ are of the form $\left(\pi_G^\sharp(\mathbf{d}(\mathsf{t}^*f)_p), \mathbf{d}(\mathsf{t}^*f)_p\right)$ for some $f \in C^\infty(P)$ and we have

$$\begin{aligned}
&\Psi^{-1}\left((v_p,\alpha_p) + \left(\pi_G^\sharp(\mathbf{d}(\mathsf{t}^*f)_p), \mathbf{d}(\mathsf{t}^*f)_p\right) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G})\right)\\
=&\left(v_p - \pi_G^\sharp(\alpha_p) + \left(\pi_G^\sharp(\mathbf{d}(\mathsf{t}^*f)_p) - \pi_G^\sharp(\mathbf{d}(\mathsf{t}^*f)_p)\right), \hat{\mathsf{s}}(\alpha_p) + \hat{\mathsf{s}}(\mathbf{d}(\mathsf{t}^*f)_p)\right)\\
=&\left(v_p - \pi_G^\sharp(\alpha_p), \hat{\mathsf{s}}(\alpha_p)\right).
\end{aligned}$$

This shows that Ψ^{-1} doesn't depend on the choice of the representative (v_p,α_p) of $(v_p,\alpha_p) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G})$.
Choose $p \in P$, $X(p) \in A_pG$ and $\xi(p) \in A_p^*G$. Then we have

$$\begin{aligned}
(\Psi^{-1}\circ\Psi)(X(p),\xi(p)) &= \Psi^{-1}((X + \bar{X}_\xi, \xi)(p) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G}))\\
&= ((X + \bar{X}_\xi)(p) - \pi_G^\sharp(\xi(p)), \hat{\mathsf{s}}(\xi(p)))\\
&= ((X + \bar{X}_\xi)(p) - \bar{X}_\xi(p), \xi(p)) = (X(p),\xi(p)).
\end{aligned}$$

In the same manner, if $\overline{(v_p,\alpha_p)} = (v_p,\alpha_p) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G}) \in \mathfrak{B}_p(\mathsf{D}_{\pi_G})$, then

$$\begin{aligned}
&(\Psi\circ\Psi^{-1})\left(\overline{(v_p,\alpha_p)}\right)\\
=&\Psi(v_p - \pi_G^\sharp(\alpha_p), \hat{\mathsf{s}}(\alpha_p))\\
=&(v_p - \pi_G^\sharp(\alpha_p) + \pi_G^\sharp(\hat{\mathsf{s}}(\alpha_p)), \hat{\mathsf{s}}(\alpha_p)) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G})\\
=&\left(v_p + \pi_G^\sharp(\hat{\mathsf{s}}(\alpha_p) - \alpha_p), \hat{\mathsf{s}}(\alpha_p)\right) + \left(\pi_G^\sharp(\alpha_p - \hat{\mathsf{s}}(\alpha_p)), \alpha_p - \hat{\mathsf{s}}(\alpha_p)\right) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G})\\
=&\left(v_p,\alpha_p\right) + I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G}) = \overline{(v_p,\alpha_p)},
\end{aligned}$$

where we use $\left(\pi_G^\sharp(\alpha_p - \hat{\mathsf{s}}(\alpha_p)), \alpha_p - \hat{\mathsf{s}}(\alpha_p)\right) \in I_p^{\mathsf{s}}(\mathsf{D}_{\pi_G})$. We compute also

$$\begin{aligned}(\rho\circ\Psi^{-1})\left(\overline{(v_p,\alpha_p)}\right) &= \rho(v_p - \pi_G^\sharp(\alpha_p), \hat{\mathsf{s}}(\alpha_p)) = T_p\mathsf{s}(v_p - \pi_G^\sharp(\alpha_p)) + \pi_G^\sharp(\hat{\mathsf{s}}(\alpha_p))\\ &= T_p\mathsf{s}v_p = \mathsf{b}\left(\overline{(v_p,\alpha_p)}\right)\end{aligned}$$

and

$$(\mathsf{b}\circ\Psi)(X(p),\xi(p)) = T_p\mathsf{s}X(p) + \bar{X}_\xi(p) = \rho(X(p),\xi(p)).$$

Now we have to show that the two Courant brackets correspond to each other via Ψ and Ψ^{-1}. For this, we have to study (5.7) in more detail for $X, Y \in \Gamma(AG)$ and $\xi,\eta\in\Gamma(A^*G)$. We begin with the AG-component. Choose $\tau\in\Gamma(A^*G)$, and compute for any $p\in P$, using the proof of Proposition 5.2.3:

$$\begin{aligned}&\tau\left(\mathfrak{L}_\xi^* Y - \mathfrak{L}_\eta^* X\right)(p)\\ &= \bar{X}_\xi(\tau(Y))(p) - [\xi,\tau](Y)(p) - \bar{X}_\eta(\tau(X))(p) + [\eta,\tau](X)(p)\\ &= \bar{X}_\xi(\tau(Y))(p) - \bar{X}_\xi(\tau(Y))(p) - \tau(\pounds_{Y^l}X_\xi)(p) + \bar{X}_\tau(\xi(Y))(p) - (\pounds_{Y^l}\theta_\xi)(\bar{X}_\tau)(p)\\ &\quad - \bar{X}_\eta(\tau(X))(p) + \bar{X}_\eta(\tau(X))(p) + \tau(\pounds_{X^l}X_\eta)(p) - \bar{X}_\tau(\eta(X))(p) + (\pounds_{X^l}\theta_\eta)(\bar{X}_\tau)(p)\\ &= \tau(\pounds_{X^l}X_\eta - \pounds_{Y^l}X_\xi)(p) + (\mathbf{d}(\xi(Y)) - \mathbf{d}(\eta(X)) - \pounds_{Y^l}\theta_\xi + \pounds_{X^l}\theta_\eta)(\bar{X}_\tau)(p)\\ &= \tau(\pounds_{X^l}X_\eta - \pounds_{Y^l}X_\xi)(p) + (\mathsf{s}^*\mathbf{d}(\xi(Y) - \eta(X)) + \pounds_{X^l}\theta_\eta - \pounds_{Y^l}\theta_\xi)(X_\tau)(p)\\ &= \tau(\pounds_{X^l}X_\eta - \pounds_{Y^l}X_\xi)(p) + \tau\Big(X_{\mathsf{s}^*(\eta(X)-\xi(Y))} + \pi_G^\sharp(\pounds_{Y^l}\theta_\xi - \pounds_{X^l}\theta_\eta)\Big)(p).\end{aligned}$$

This shows that

$$\mathfrak{L}_\xi^* Y - \mathfrak{L}_\eta^* X = (\mathrm{Id} - T\mathsf{t})\Big(\pounds_{X^l}X_\eta - \pounds_{Y^l}X_\xi + \pi_G^\sharp\left(\pounds_{Y^l}\theta_\xi - \pounds_{X^l}\theta_\eta\right)\Big) - X_{\mathsf{s}^*(\xi(Y)-\eta(X))}|_P.$$

Hence, we get that the left-hand side of (5.7) is

$$[X,Y]_{AG} + (\mathrm{Id} - T\mathsf{t})\Big(\pounds_{X^l}X_\eta - \pounds_{Y^l}X_\xi + \pi_G^\sharp(\pounds_{Y^l}\theta_\xi - \pounds_{X^l}\theta_\eta)\Big) - \frac{1}{2}X_{\mathsf{s}^*(\xi(Y)-\eta(X))}|_P.$$

For the A^*G-component, choose $Z\in\Gamma(AG)$ and compute for any $p\in P$:

$$\begin{aligned}\hat{\mathsf{s}}\left(\pounds_{X^l}\theta_\eta - \pounds_{Y^l}\theta_\xi\right)(p)(Z(p)) &= \left(\pounds_{X^l}\theta_\eta - \pounds_{Y^l}\theta_\xi\right)(Z^l)(p)\\ &= \left(X(\mathsf{s}^*(\eta(Z))) - \theta_\eta([X,Z]^l_{AG}) - Y(\mathsf{s}^*(\xi(Z))) + \theta_\xi([Y,Z]^l)\right)(p)\\ &= \left(\mathsf{a}(X)(\eta(Z)) - \eta([X,Z]_{AG}) - \mathsf{a}(Y)(\xi(Z)) + \xi([Y,Z]_{AG})\right)(p)\\ &= \left(\mathfrak{L}_X\eta - \mathfrak{L}_Y\xi\right)(Z)(p).\end{aligned}$$

Thus, we have shown that

$$\mathfrak{L}_X\eta - \mathfrak{L}_Y\xi = \hat{\mathsf{s}}\left(\left(\pounds_{X^l}\theta_\eta - \pounds_{Y^l}\theta_\xi\right)|_P\right)$$

and the right-hand side of (5.7) is consequently equal to

$$[\xi,\eta]_\star + \hat{\mathsf{s}}\left((\pounds_{X^l}\theta_\eta - \pounds_{Y^l}\theta_\xi)|_P\right) + \frac{1}{2}\hat{\mathsf{s}}\left(\mathsf{s}^*\mathbf{d}(\xi(Y)-\eta(X))\right).$$

Recall that for $p\in P$, $\overline{(v_p,\alpha_p)}\in\mathfrak{B}_p(\mathsf{D}_{\pi_G})$, we have $v_p-\pi_G^\sharp(\alpha_p) = v_p - T_p\mathsf{t}v_p + \pi_G^\sharp(\hat{\mathsf{t}}(\alpha_p)-\alpha_p)$. Hence, using the identities

$$\begin{aligned}
(\mathrm{Id}-T\mathsf{t})([X,Y]_{AG}) &= [X,Y]_{AG},\\
(\mathrm{Id}-T\mathsf{t})([\bar{X}_\xi,\bar{X}_\eta]) &= 0,\\
\pi_G^\sharp\left(\hat{\mathsf{t}}(\pounds_{X^l}\theta_\eta-\pounds_{Y^l}\theta_\xi) - (\pounds_{X^l}\theta_\eta-\pounds_{Y^l}\theta_\xi)\right) &= (\mathrm{Id}-T\mathsf{t})\left(\pi_G^\sharp(\pounds_{Y^l}\theta_\xi-\pounds_{X^l}\theta_\eta)\right),\\
\pi_G^\sharp\left(\hat{\mathsf{t}}(\mathsf{s}^*\mathbf{d}(\xi(Y)-\eta(X))) - \mathsf{s}^*\mathbf{d}(\xi(Y)-\eta(X))\right) &= -\pi_G^\sharp\left(\mathsf{s}^*\mathbf{d}(\xi(Y)-\eta(X))\right)\\
&= -X_{\mathsf{s}^*(\xi(Y)-\eta(X))}
\end{aligned}$$

on P, one gets for $(X,\xi),(Y,\eta)\in\Gamma(AG\times_P A^*G)$:

$$\begin{aligned}
&\Psi^{-1}[\Psi(X,\xi),\Psi(Y,\eta)]\\
&=\Psi^{-1}[e_{\xi,X,0},e_{\eta,Y,0}] = \Psi^{-1}\left([(X_\xi+X^l,\theta_\xi),(X_\eta+Y^l,\theta_\eta)]|_P + I^{\mathsf{s}}(\mathsf{D}_{\pi_G})\right)\\
&=\Psi^{-1}\Bigg([(X_\xi,\theta_\xi),(X_\eta,\theta_\eta)]|_P + \pounds_{X^l}(X_\eta,\theta_\eta)|_P - \pounds_{Y^l}(X_\xi,\theta_\xi)|_P\\
&\qquad + \left([X,Y]_{AG},\, \frac{1}{2}\mathsf{s}^*\mathbf{d}\left(\xi(Y)-\eta(X)\right)\Big|_P\right) + I^{\mathsf{s}}(\mathsf{D}_{\pi_G})\Bigg)\\
&=\Psi^{-1}\Bigg(\bigg([X,Y]_{AG} + [\bar{X}_\xi,\bar{X}_\eta] + (\pounds_{X^l}X_\eta - \pounds_{Y^l}X_\xi)|_P,\\
&\qquad [\xi,\eta]_\star + \left(\frac{1}{2}\mathsf{s}^*\mathbf{d}\left(\xi(Y)-\eta(X)\right) + (\pounds_{X^l}\theta_\eta - \pounds_{Y^l}\theta_\xi)\right)\Big|_P\bigg) + I^{\mathsf{s}}(\mathsf{D}_{\pi_G})\Bigg)\\
&=\Bigg([X,Y]_{AG} + (\mathrm{Id}-T\mathsf{t})\left(\pounds_{X^l}X_\eta - \pounds_{Y^l}X_\xi + \pi_G^\sharp(\pounds_{Y^l}\theta_\xi - \pounds_{X^l}\theta_\eta)\right) - \frac{1}{2}X_{\mathsf{s}^*(\xi(Y)-\eta(X))},\\
&\qquad [\xi,\eta]_\star + \frac{1}{2}\hat{\mathsf{s}}\left(\mathsf{s}^*\mathbf{d}\left(\xi(Y)-\eta(X)\right)\right) + \hat{\mathsf{s}}((\pounds_{X^l}\theta_\eta - \pounds_{Y^l}\theta_\xi)|_P)\Bigg)\\
&=[(X,\xi),(Y,\eta)].
\end{aligned}$$

$\Diamond$

Example 5.4.3 Consider a Lie groupoid $G\rightrightarrows P$ endowed with a closed multiplicative 2-form $\omega_G\in\Omega^2(G)$. The Courant algebroid $\mathfrak{B}(\mathsf{D}_{\omega_G})$ is given here by

$$\mathfrak{B}(\mathsf{D}_{\omega_G}) = \left(\mathrm{Graph}(\omega_G^\flat|_{TP}: TP\to A^*G) + \ker \mathbb{T}\mathsf{t}|_P\right) / \,\mathrm{Graph}\left(\omega_G^\flat|_{T_P^{\mathsf{s}}G}: T_P^{\mathsf{s}}G \to (T_P^{\mathsf{t}}G)^\circ\right).$$

We show that it is isomorphic as a Courant algebroid to the standard Courant algebroid $\mathsf{P}_P = TP \times_P T^*P$. For this, consider the maps

$$\Lambda : \mathfrak{B}(\mathsf{D}_{\omega_G}) \to TP \times_P T^*P, \qquad \Lambda\left(\overline{(v_p, \alpha_p)}\right) = (T_p\mathsf{s}v_p, \beta_p),$$

where $(T_p\mathsf{s})^*\beta_p = \alpha_p - \omega_G^\flat(v_p)$, and

$$\Lambda^{-1} : TP \times_P T^*P \to \mathfrak{B}(\mathsf{D}_{\omega_G}), \qquad \Lambda^{-1}(v_p, \alpha_p) = \overline{(\epsilon(v_p), (T_p\mathsf{s})^*\alpha_p + \omega_G^\flat(\epsilon(v_p)))}.$$

Note that Λ is well-defined: if $(v_p, \alpha_p) \in \mathfrak{A}_p(\mathsf{D}_{\omega_G}) + \ker \mathbb{T}\mathsf{t}$, then $\hat{\mathsf{t}}(\alpha_p) = \omega_G^\flat(T_p\mathsf{t}v_p)$ and hence $\hat{\mathsf{t}}(\alpha_p - \omega_G^\flat(v_p)) = \hat{\mathsf{t}}(\alpha_p) - \omega_G^\flat(T_p\mathsf{t}v_p) = 0_p$. Thus, the covector $\alpha_p - \omega_G^\flat(v_p)$ can be written $(T_p\mathsf{s})^*\beta_p$ with some $\beta_p \in T_p^*P$. (For simplicity, we will identify elements β_p of T_p^*P with $(T_p\mathsf{s})^*\beta_p \in (T_p^\mathsf{s}G)^\circ \subseteq T_p^*G$ and $v_p \in T_pP$ with $\epsilon(v_p) \in T_pP \subseteq T_pG$ in the following.) Furthermore, if $u_p \in T_p^\mathsf{s}G$, we have $\Lambda\left(\overline{(u_p, \omega_G^\flat(u_p))}\right) = (T_p\mathsf{s}u_p, \omega_G^\flat(u_p) - \omega_G^\flat(u_p)) = (0_p, 0_p)$. The map Λ^{-1} has image in $\mathfrak{B}(\mathsf{D}_{\omega_G})$ because for any $(v_p, \alpha_p) \in \mathsf{P}_P(p)$, we have

$$\left(v_p, (T_p\mathsf{s})^*\alpha_p + \omega_G^\flat(v_p)\right) = \mathbb{T}\mathsf{t}\left(v_p, \omega_G^\flat(v_p)\right) + (0, (T_p\mathsf{s})^*\alpha_p) \in \mathfrak{A}_p(\mathsf{D}_{\omega_G}) + (\ker \mathbb{T}\mathsf{t})_p.$$

Choose now $p \in P$, $\overline{(v_p, \alpha_p)} \in \mathfrak{B}_p(\mathsf{D}_{\omega_G})$ and compute

$$\begin{aligned}(\Lambda^{-1} \circ \Lambda)\left(\overline{(v_p, \alpha_p)}\right) &= \Lambda^{-1}(T_p\mathsf{s}v_p, \alpha_p - \omega_G^\flat(v_p))\\ &= \overline{(T_p\mathsf{s}v_p, \alpha_p - \omega_G^\flat(v_p) + \omega_G^\flat(T_p\mathsf{s}v_p))}\\ &= \overline{(v_p, \alpha_p) + (T_p\mathsf{s}v_p - v_p, \omega_G^\flat(T_p\mathsf{s}v_p - v_p))} = \overline{(v_p, \alpha_p)}.\end{aligned}$$

In the same manner, if $(v_p, \alpha_p) \in \mathsf{P}_P(p)$, we have

$$(\Lambda \circ \Lambda^{-1})(v_p, \alpha_p) = \Lambda\left(\overline{(v_p, \alpha_p + \omega_G^\flat(v_p))}\right) = (T_p\mathsf{s}v_p, \alpha_p + \omega_G^\flat(v_p) - \omega_G^\flat(v_p)) = (v_p, \alpha_p)$$

since $v_p \in T_pP$. The equality $\mathsf{b} \circ \Lambda^{-1} = \mathrm{pr}_{TP}$ is immediate.
Now if $(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \Gamma(\mathsf{P}_P)$, we choose $X, Y \in \mathfrak{X}(G)$ such that $X \sim_\mathsf{s} \bar{X}$, $X|_P = \bar{X}$, $Y \sim_\mathsf{s} \bar{Y}$, $Y|_P = \bar{Y}$ and we compute

$$\begin{aligned}&\Lambda\left[\Lambda^{-1}(\bar{X}, \bar{\alpha}), \Lambda^{-1}(\bar{Y}, \bar{\beta})\right]\\ &= \Lambda\left(\left([X, Y], \omega_G^\flat([X, Y]) + \pounds_X(\mathsf{s}^*\bar{\beta}) - \pounds_Y(\mathsf{s}^*\bar{\alpha}) + \frac{1}{2}\mathbf{d}((\mathsf{s}^*\bar{\alpha})(Y) - (\mathsf{s}^*\bar{\beta})(X))\right)\Big|_P + I^\mathsf{s}(\mathsf{D}_{\omega_G})\right)\\ &= \Lambda\left(\overline{\left([\bar{X}, \bar{Y}], \omega_G^\flat([\bar{X}, \bar{Y}]) + \mathsf{s}^*\left(\pounds_{\bar{X}}\bar{\beta} - \pounds_{\bar{Y}}\bar{\alpha} + \frac{1}{2}\mathbf{d}(\bar{\alpha}(\bar{Y}) - \bar{\beta}(\bar{X}))\right)\right)}\right)\\ &= \left([\bar{X}, \bar{Y}], \pounds_{\bar{X}}\bar{\beta} - \pounds_{\bar{Y}}\bar{\alpha} + \frac{1}{2}\mathbf{d}(\bar{\alpha}(\bar{Y}) - \bar{\beta}(\bar{X}))\right),\end{aligned}$$

and we recover the standard Courant bracket of the two sections $(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \Gamma(\mathsf{P}_P)$.$\Diamond$

Example 5.4.4 Consider the pair Dirac groupoid $(M\times M \rightrightarrows M, \mathsf{D}_M\ominus\mathsf{D}_M)$ associated to an integrable Dirac manifold (M,D_M) (see Example 2.2.5). The vector bundle $\mathfrak{B}(\mathsf{D}_M\ominus\mathsf{D}_M)\to\Delta_M$ is defined here by

$$\mathfrak{B}_{(m,m)}(\mathsf{D}_M\ominus\mathsf{D}_M)=\frac{\mathfrak{A}_{(m,m)}(\mathsf{D}_M\ominus\mathsf{D}_M)+\{0\}\times T_mM\times\{0\}\times T_m^*M}{\{(v_m,0_m,\alpha_m,0_m)\mid(v_m,\alpha_m)\in\mathsf{D}_M(m)\}}$$

for all $m\in M$ (recall that we have computed $\mathfrak{A}(\mathsf{D}_M\ominus\mathsf{D}_M)$ in Example 5.2.8). Hence, we get an isomorphism

$$\Pi:\mathfrak{B}(\mathsf{D}_M\ominus\mathsf{D}_M)\to TM\times_M T^*M,\quad \overline{(v_m,w_m,\alpha_m,\beta_m)}\mapsto(w_m,\beta_m) \tag{5.8}$$

over $\mathrm{pr}_1:\Delta_M\to M$, with inverse

$$\Pi^{-1}:TM\times_M T^*M\to\mathfrak{B}(\mathsf{D}_M\ominus\mathsf{D}_M),\quad (w_m,\beta_m)\mapsto\overline{(0_m,w_m,0_m,\beta_m)}.$$

The Courant bracket on $\mathfrak{B}(\mathsf{D}_M\ominus\mathsf{D}_M)$ is easily seen to correspond via this isomorphism to the standard Courant bracket on $\mathsf{P}_M=TM\times_M T^*M$ (and hence, doesn't depend on D_M). ◊

Remark 5.4.5 Consider a Dirac groupoid as in Theorem 5.1.2. Set $N:=G/\mathsf{G}_0$ and $Q=:P/(TP\cap\mathsf{G}_0)$. Then the Courant algebroid $TG\times_G T^*G$ projects under pr to the Courant algebroid $TN\times_N T^*N$. We have a map $\mathbb{T}\,\mathrm{pr}:TG\times_G\mathsf{P}_1\to TN\times_N T^*N$, $(v_g,\alpha_g)\mapsto(T_g\,\mathrm{pr}(v_g),\alpha_{[g]})$, where $\alpha_{[g]}$ is such that $\alpha_g=(T_g\,\mathrm{pr})^*\alpha_{[g]}$. By definition of the reduced Dirac structure $\mathrm{pr}(\mathsf{D}_G)=D_\pi$ and the results in Theorem 3.3.14, the restriction of this map to $(\mathfrak{A}(\mathsf{D}_G)\oplus\ker(\mathbb{T}\mathsf{t})|_P)\cap(TG\times_G\mathsf{P}_1)$ has image $\mathfrak{A}(\mathsf{D}_\pi)+(\ker\mathbb{T}\mathsf{t}_N)|_Q$. Furthermore, we find that $(v_p,(T_p\,\mathrm{pr})^*\alpha_{[p]})\in I_p^{\mathsf{s}}(\mathsf{D}_G)+\mathsf{G}_0(p)\times 0_p$ if and only if $(T_p\,\mathrm{pr}\,v_p,\alpha_{[p]})\in I_{[p]}^{\mathsf{s}_N}(\mathsf{D}_\pi)$. Hence, the map $\mathbb{T}\,\mathrm{pr}$ factors to a map $\mathfrak{B}(\mathsf{D}_G)\to\mathfrak{B}(\mathsf{D}_\pi)$. It is straightforward to check that this is a morphism of Courant algebroids. △

5.5 Induced action of the group of bisections on $\mathfrak{B}(\mathsf{D}_G)$

We give here the correct generalization of the action of G on $\mathfrak{g}/\mathfrak{g}_0\times\mathfrak{p}_1$ in Theorem 4.1.31. In this section, the Dirac groupoids that we consider are not necessarily integrable. Hence, the vector bundle $\mathfrak{B}(\mathsf{D}_G)$ exists, but doesn't necessarily have a Courant algebroid structure.
We begin with a lemma, which will also be useful in the following section about Dirac homogeneous spaces.

Lemma 5.5.1 *Let $(G\rightrightarrows P,\mathsf{D}_G)$ be a Dirac groupoid and $(v_p,\alpha_p)\in\mathfrak{A}_p(\mathsf{D}_G)\oplus(\ker\mathbb{T}\mathsf{t})|_P$ for some $p\in P$. If $\mathbb{T}\mathsf{t}(v_p,\alpha_p)=(\bar X_\xi(p),\bar\theta_\xi(p))\in\mathfrak{A}_p(\mathsf{D}_G)$, then $(v_p,\alpha_p)=(\bar X_\xi(p),\bar\theta_\xi(p))+(u_p,(T_p\mathsf{s})^*\gamma_p)$ with some $u_p\in A_pG$ and $\gamma_p\in T_p^*P$ and*

$$((X_\xi,\theta_\xi)(g))\star(v_p,\alpha_p)=(X_\xi(g)+T_pL_gu_p,\theta_\xi(g)+(T_g\mathsf{s})^*\gamma_p)$$

for any $g\in\mathsf{s}^{-1}(p)$ and $(X_\xi,\theta_\xi)\in\Gamma(\mathsf{D}_G)$ such that $(X_\xi,\theta_\xi)\sim_{\mathsf{s}}(\bar X_\xi,\bar\theta_\xi)$.

PROOF: If $(v_p,\alpha_p)\in\mathfrak{A}_p(\mathsf{D}_G)\oplus(\ker\mathbb{T}\mathsf{t})|_P$, then $\mathbb{T}\mathsf{t}(v_p,\alpha_p)\in\mathfrak{A}_p(\mathsf{D}_G)$ and hence $\mathbb{T}\mathsf{t}(v_p,\alpha_p)=(\bar{X}_\xi,\bar{\theta}_\xi)(p)$ for some section $(\bar{X}_\xi,\bar{\theta}_\xi)\in\Gamma(\mathfrak{A}_p(\mathsf{D}_G))$. The difference $(v_p,\alpha_p)-(\bar{X}_\xi,\bar{\theta}_\xi)(p)=(v_p,\alpha_p)-\mathbb{T}\mathsf{t}(v_p,\alpha_p)$ is then an element of $(\ker\mathbb{T}\mathsf{t})|_P$ and there exists $\gamma_p\in T_p^*P$ such that $(v_p,\alpha_p)-(\bar{X}_\xi,\bar{\theta}_\xi)(p)=(u_p,(T_p\mathsf{s})^*\gamma_p)$ if we set $u_p=v_p-T_p\mathsf{t}v_p$.
Since the section $(X_\xi,\theta_\xi)\in\Gamma(\mathsf{D}_G)$ is a pair that is s-descending to $(\bar{X}_\xi,\bar{\theta}_\xi)$, the product $(X_\xi,\theta_\xi)(g)\star(v_p,\alpha_p)$ is defined for any $g\in\mathsf{s}^{-1}(p)$.
We compute, using a bisection K through g,

$$X_\xi(g)\star v_p=X_\xi(g)+T_pL_Kv_p-T_pL_K(T_p\mathsf{t}v_p)=X_\xi(g)+T_pL_K(u_p)=X_\xi(g)+T_pL_g(u_p).$$

For the first equality, we have used the formula proved in Xu (1995), see also Mackenzie (2005).
We have also, for any $v_g=v_g\star(T_g\mathsf{s}v_g)\in T_gG$

$$\begin{aligned}(\theta_\xi(g)\star\alpha_p)(v_g)&=(\theta_\xi(g)\star\alpha_p)(v_g\star(T_g\mathsf{s}v_g))=\theta_\xi(g)(v_g)+\alpha_p(T_g\mathsf{s}v_g)\\&=\theta_\xi(g)(v_g)+(\alpha_p-\hat{\mathsf{t}}(\alpha_p))(T_g\mathsf{s}v_g)=\theta_\xi(g)(v_g)+(T_p\mathsf{s}^*\gamma_p)(T_g\mathsf{s}v_g)\\&=\left(\theta_\xi(g)+(T_g\mathsf{s}^*\gamma_p)\right)(v_g).\end{aligned}$$

□

Theorem 5.5.2 *Let $(G\rightrightarrows P,\mathsf{D}_G)$ be a Dirac groupoid. Choose a bisection $K\in\mathcal{B}(G)$ and consider*

$$r_K:\mathfrak{A}(\mathsf{D}_G)\oplus\ker\mathbb{T}\mathsf{t}|_P\to\mathfrak{B}(\mathsf{D}_G)$$

$$r_K(v_p,\alpha_p)=\left(T_{K(p)^{-1}}R_K(v_{K(p)^{-1}}\star v_p),\left(T_{(\mathsf{s}\circ K)(p)}R_K^{-1}\right)^*(\alpha_{K(p)^{-1}}\star\alpha_p)\right)+I^{\mathsf{s}}_{(\mathsf{s}\circ K)(p)}(\mathsf{D}_G),$$

where $(v_{K(p)^{-1}},\alpha_{K(p)^{-1}})\in\mathsf{D}_G(K(p)^{-1})$ is such that

$$\mathbb{T}\mathsf{s}\left(v_{K(p)^{-1}},\alpha_{K(p)^{-1}}\right)=\mathbb{T}\mathsf{t}(v_p,\alpha_p).$$

The map r_K is well-defined and induces the right translation by K,

$$\begin{array}{rccc}\rho_K: & \mathfrak{B}(\mathsf{D}_G) & \to & \mathfrak{B}(\mathsf{D}_G)\\ & (v_p,\alpha_p)+I^{\mathsf{s}}_p(\mathsf{D}_G) & \mapsto & r_K(v_p,\alpha_p).\end{array}$$

The map $\rho:\mathcal{B}(G)\times\Gamma(\mathfrak{B}(\mathsf{D}_G))\to\Gamma(\mathfrak{B}(\mathsf{D}_G))$ is a right action.

For the proof of this theorem, we will need the following lemma:

Lemma 5.5.3 *Let $G\rightrightarrows P$ be a Lie groupoid. Choose $g,h\in G$ and $K\in\mathcal{B}(G)$. Choose $(v_h,\alpha_h)\in\mathsf{P}_G(h)$, $(v_g,\alpha_g)\in\mathsf{P}_G(g)$ such that $\mathbb{T}\mathsf{s}\,(v_g,\alpha_g)=\mathbb{T}\mathsf{t}\,(v_h,\alpha_h)$. Then*

$$T_{g\star h}R_K(v_g\star v_h)=v_g\star(T_hR_Kv_h)\tag{5.9}$$

and

$$\alpha_g\star\left(\left(T_{R_K(h)}R_K^{-1}\right)^*\alpha_h\right)=\left(T_{R_K(g\star h)}R_K^{-1}\right)^*(\alpha_g\star\alpha_h).\tag{5.10}$$

PROOF: Choose $g:(-\varepsilon,\varepsilon)\to G$, $\dot g(0)=v_g$ and $h:(-\varepsilon,\varepsilon)\to G$, $\dot h(0)=v_h$ such that $\mathsf{s}(g(t))=\mathsf{t}(h(t))$ for all $t\in(-\varepsilon,\varepsilon)$. Then we have

$$T_{g\star h}R_K(v_g\star v_h)=\left.\frac{d}{dt}\right|_{t=0} g(t)\star h(t)\star K(\mathsf{s}(h(t)))=v_g\star(T_hR_Kv_h)\,.$$

Choose $v\in T_{R_K(g\star h)}G$ and any $w_g\in T_gG$ such that $T_g\mathsf{t}(w_g)=T_{R_K(g\star h)}\mathsf{t}(v)$. Then we get using (5.9)

$$\begin{aligned}\left(\left(T_{R_K(g\star h)}R_K^{-1}\right)^*(\alpha_g\star\alpha_h)\right)(v)&=(\alpha_g\star\alpha_h)\left(T_{R_K(g\star h)}R_K^{-1}v\right)\\&=(\alpha_g\star\alpha_h)\left(w_g\star w_g^{-1}\star\left(T_{R_K(g\star h)}R_K^{-1}v\right)\right)\\&=\alpha_g(w_g)+\alpha_h\left(w_g^{-1}\star\left(T_{R_K(g\star h)}R_K^{-1}v\right)\right)\\&=\alpha_g(w_g)+\left(T_{R_K(h)}R_K^{-1}\right)^*\alpha_h\left(w_g^{-1}\star v\right)\\&=\left(\alpha_g\star\left(T_{R_K(h)}R_K^{-1}\right)^*\alpha_h\right)\left(w_g\star w_g^{-1}\star v\right)\\&=\left(\alpha_g\star\left(T_{R_K(h)}R_K^{-1}\right)^*\alpha_h\right)(v).\end{aligned}$$

This proves (5.10). □

PROOF (OF THEOREM 5.5.2): First, we check that the map r_K is well-defined, that is, that it has image in $\mathfrak{B}(\mathsf{D}_G)$ and doesn't depend on the choices made.
Choose $p\in P$, $(v_p,\alpha_p)\in\mathfrak{A}_p(\mathsf{D}_G)\oplus(\ker\mathbb{T}\mathsf{t})_p$ and $K\in\mathcal{B}(G)$. Set $K(p)=g$. Since the map r_K is linear in every fiber of $\mathfrak{A}(\mathsf{D}_G)\oplus(\ker\mathbb{T}\mathsf{t})|_P$, it suffices to show that the image of $(0_p,0_p)$ is $I^{\mathsf{s}}_{\mathsf{s}(g)}(\mathsf{D}_G)$ for any choice of $(v_{g^{-1}},\alpha_{g^{-1}})\in\mathsf{D}_G(g^{-1})$ such that $\mathbb{T}\mathsf{s}(v_{g^{-1}},\alpha_{g^{-1}})=(0_p,0_p)$ to prove that it is well-defined. Using (5.9) and (5.10), we get

$$\begin{gathered}T_{g^{-1}}R_K(v_{g^{-1}}\star 0_p)=v_{g^{-1}}\star(T_pR_K0_p)=v_{g^{-1}}\star 0_g\\(\alpha_{g^{-1}}\star 0_p)\circ T_{\mathsf{s}(g)}R_{K^{-1}}=\alpha_{g^{-1}}\star(0_p\circ T_gR_{K^{-1}})=\alpha_{g^{-1}}\star 0_g.\end{gathered}$$

Thus, we have shown that

$$r_K(0_p,0_p)=(v_{g^{-1}},\alpha_{g^{-1}})\star(0_g,0_g)\in\mathsf{D}_G(\mathsf{s}(g))\cap\ker\mathbb{T}\mathsf{s}=I^{\mathsf{s}}_{\mathsf{s}(g)}(\mathsf{D}_G).$$

Choose next $(v_p,\alpha_p)\in\mathfrak{A}(\mathsf{D}_G)\oplus(\ker\mathbb{T}\mathsf{t})|_P$ such that $(v_p,\alpha_p)\in I^{\mathsf{s}}_p(\mathsf{D}_G)$, that is, such that $\overline{(v_p,\alpha_p)}=0$ in $\mathfrak{B}_p(\mathsf{D}_G)$. Choose $(v_{g^{-1}},\alpha_{g^{-1}})\in\mathsf{D}_G(g^{-1})$ such that $\mathbb{T}\mathsf{s}(v_{g^{-1}},\alpha_{g^{-1}})=\mathbb{T}\mathsf{t}(v_p,\alpha_p)$. Then we have $T_{g^{-1}}R_K(v_{g^{-1}}\star v_p)=T_{g^{-1}}R_K(v_{g^{-1}}\star v_p\star 0_p)=v_{g^{-1}}\star v_p\star 0_g$, since $T_p\mathsf{s}v_p=0$. We have also $\hat{\mathsf{s}}(\alpha_p)=0$, and by (5.10):

$$(T_{\mathsf{s}(g)}R_K^{-1})^*(\alpha_{g^{-1}}\star\alpha_p)=(T_{\mathsf{s}(g)}R_K^{-1})^*(\alpha_{g^{-1}}\star\alpha_p\star 0_p)=\alpha_{g^{-1}}\star\alpha_p\star 0_g.$$

Thus, $r_K(v_p,\alpha_p)=(v_{g^{-1}},\alpha_{g^{-1}})\star(v_p,\alpha_p)\star(0_g,0_g)\in I^{\mathsf{s}}_{\mathsf{s}(K(p))}(\mathsf{D}_G)$. The map $\rho_K:\mathfrak{B}(\mathsf{D}_G)\to\mathfrak{B}(\mathsf{D}_G)$ is consequently well-defined.

We show now that $\rho:\mathcal{B}(G)\times\mathfrak{B}(\mathsf{D}_G)\to\mathfrak{B}(\mathsf{D}_G)$ defines an action of the group of bisections of $G\rightrightarrows P$ on $\Gamma(\mathfrak{B}(\mathsf{D}_G))$.

Choose $K, L \in \mathcal{B}(G)$, $p \in P$ and $\overline{(v_p, \alpha_p)}$ in $\mathfrak{B}_p(\mathsf{D}_G)$. Set $K(p) = g$ and choose a pair $(v_{g^{-1}}, \alpha_{g^{-1}}) \in \mathsf{D}_G(g^{-1})$ such that $\mathbb{T}\mathsf{s}(v_{g^{-1}}, \alpha_{g^{-1}}) = \mathbb{T}\mathsf{t}(v_p, \alpha_p)$. Set also $h := L(\mathsf{s}(g))$ and choose $(w_{h^{-1}}, \beta_{h^{-1}}) \in \mathsf{D}_G(h^{-1})$ such that $\mathbb{T}\mathsf{s}(w_{h^{-1}}, \beta_{h^{-1}}) = \mathbb{T}\mathsf{t}\,(v_{g^{-1}}, \alpha_{g^{-1}})$. Then we have $(K \star L)(p) = g \star h$ and we compute, using (5.9) and (5.10):

$$
\begin{aligned}
&\rho_L\left(\rho_K\left(\overline{(v_p, \alpha_p)}\right)\right)\\
&= \rho_L\left(\left(T_{g^{-1}}R_K, \left(T_{\mathsf{s}(g)}R_K^{-1}\right)^*\right)\left((v_{g^{-1}}, \alpha_{g^{-1}}) \star (v_p, \alpha_p)\right)\right)\\
&= \left(T_{(g\star h)^{-1}}R_{K\star L}, (T_{\mathsf{s}(g\star h)}R_{K\star L}^{-1})^*\right)\left((w_{h^{-1}}, \beta_{h^{-1}}) \star (v_{g^{-1}}, \alpha_{g^{-1}}) \star (v_p, \alpha_p)\right) + I^{\mathsf{s}}_{\mathsf{s}(h)}(\mathsf{D}_G)\\
&= \rho_{K\star L}\left(\overline{(v_p, \alpha_p)}\right)
\end{aligned}
$$

since $(w_{h^{-1}}, \beta_{h^{-1}}) \star (v_{g^{-1}}, \alpha_{g^{-1}})$ is an element of $\mathsf{D}_G((g\star h)^{-1})$ satisfying

$$\mathbb{T}\mathsf{s}\left((w_{h^{-1}}, \beta_{h^{-1}}) \star (v_{g^{-1}}, \alpha_{g^{-1}})\right) = \mathbb{T}\mathsf{s}\,(v_{g^{-1}}, \alpha_{g^{-1}}) = (v_p, \alpha_p).$$

□

Example 5.5.4 Consider a Poisson Lie groupoid $(G \rightrightarrows P, \pi_G)$. We will compute the action of the bisections $\mathcal{B}(G)$ on $\mathfrak{B}(\mathsf{D}_{\pi_G}) \simeq AG \times_P A^*G$.
Choose $K \in \mathcal{B}(G)$, $p \in P$ and $\Psi(X(p), \xi(p)) \in \mathfrak{B}_p(\mathsf{D}_{\pi_G})$, $(X(p), \xi(p)) \in A_pG \times A_p^*G$ (recall that Ψ has been defined in Example 5.4.2). If $K(p) = g$ and $\theta_\xi \in \Omega^1(G)$ is s-descending to ξ, then $\rho_K(\Psi(X(p), \xi(p)))$ is given by

$$\rho_K(\Psi(X(p), \xi(p))) = \left(T_{g^{-1}}R_K(\pi_G^\sharp(\theta_\xi(g^{-1})) + X^l(g^{-1})), ((R_K^{-1})^*\theta_\xi)(\mathsf{s}(g))\right)$$

and corresponds to

$$
\begin{aligned}
&(\Psi^{-1} \circ \rho_K \circ \Psi)(X(p), \xi(p))\\
&= \left(T_{g^{-1}}R_K(\pi_G^\sharp(\theta_\xi(g^{-1})) + X^l(g^{-1})) - \pi_G^\sharp((R_K^{-1})^*\theta_\xi(\mathsf{s}(g))), \hat{\mathsf{s}}\left(((R_K^{-1})^*\theta_\xi)(\mathsf{s}(g))\right)\right)\\
&= \Big(T_p(L_{g^{-1}} \circ R_K)X(p) + T_{g^{-1}}R_K(\pi_G^\sharp(\theta_\xi(g^{-1}))) - \pi_G^\sharp((R_K^{-1})^*\theta_\xi(\mathsf{s}(g))),\\
&\qquad\qquad (T_{\mathsf{s}(g)}(L_g \circ R_{K^{-1}}))^*\xi(p)\Big).
\end{aligned}
$$

Note that by the general theorem, this doesn't depend on the choice of θ_ξ. In the case of a Poisson Lie group, we recover the action of G on the Lie bialgebroid, see Drinfel′d (1993). In the case of a trivial Poisson groupoid, i.e., with $\pi_G = 0$, this is simply the pair of maps on AG and A^*G generalizing Ad and Ad* in the Lie group case. ♢

Example 5.5.5 Consider a Lie groupoid $G \rightrightarrows P$ endowed with a closed multiplicative 2-form $\omega_G \in \Omega^2(G)$. We will compute the action of the bisections $\mathcal{B}(G)$ on $\mathfrak{B}(\mathsf{D}_{\omega_G}) \simeq TP \times_P T^*P$.
Choose a bisection $K \in \mathcal{B}(G)$, a vector $\Lambda^{-1}(v_p, \alpha_p) = \overline{(v_p, (T_p\mathsf{s})^*\alpha_p + \omega_G^\flat(v_p))} \in \mathfrak{B}_p(\mathsf{D}_{\omega_G})$, $(v_p, \alpha_p) \in \mathsf{P}_P(p)$ (recall Example 5.4.3) and set $K(p) = g \in G$. Then we have

$$\mathbb{T}\mathsf{t}(v_p, (T_p\mathsf{s})^*\alpha_p + \omega_G^\flat(v_p)) = (v_p, \omega_G^\flat(v_p)) \in \mathfrak{A}_p(\mathsf{D}_{\omega_G}).$$

Hence, any vector $v_{g^{-1}} \in T_{g^{-1}}G$ such that $T_{g^{-1}}\mathsf{s}v_{g^{-1}} = v_p$ leads to $(v_{g^{-1}}, \omega_G^\flat(v_{g^{-1}})) \in \mathsf{D}_{\omega_G}(v_{g^{-1}})$ and $\mathbb{T}\mathsf{s}(v_{g^{-1}}, \pi_G^\flat(v_{g^{-1}})) = \mathbb{T}\mathsf{t}(v_p, (T_p\mathsf{s})^*\alpha_p + \omega_G^\flat(v_p))$.
The vector $\rho_K\left(\Lambda^{-1}(v_p, \alpha_p)\right) \in \mathfrak{B}_{\mathsf{s}(g)}(\mathsf{D}_{\omega_G})$ is then given by

$$\begin{aligned}
&\rho_K\left(\Lambda^{-1}(v_p,\alpha_p)\right)\\
&= \left(T_{g^{-1}}R_K(v_{g^{-1}}\star v_p), (T_{\mathsf{s}(g)}R_K^{-1})^*\left(\omega_G^\flat(v_{g^{-1}})\star\left((T_p\mathsf{s})^*\alpha_p+\omega_G^\flat(v_p)\right)\right)\right)\\
&= \left(T_{g^{-1}}R_K v_{g^{-1}}, (T_{\mathsf{s}(g)}R_K^{-1})^*\left(\omega_G^\flat(v_{g^{-1}})+(T_{g^{-1}}\mathsf{s})^*\alpha_p\right)\right).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&(\Lambda\circ\rho_K\circ\Lambda^{-1})(v_p,\alpha_p)\\
&= \left(T_{g^{-1}}(\mathsf{s}\circ R_K)v_{g^{-1}}, (T_{\mathsf{s}(g)}R_K^{-1})^*\left(\omega_G^\flat(v_{g^{-1}})+(T_{g^{-1}}\mathsf{s})^*\alpha_p\right)-\omega_G^\flat\left(T_{g^{-1}}R_K v_{g^{-1}}\right)\right)\\
&= \left(T_p(s\circ K)v_p, \left(T_{\mathsf{s}(g)}(\mathsf{s}\circ K)^{-1}\right)^*\alpha_p+(T_{\mathsf{s}(g)}R_K^{-1})^*\omega_G^\flat(v_{g^{-1}})-\omega_G^\flat\left(T_{g^{-1}}R_K v_{g^{-1}}\right)\right).
\end{aligned}$$

Again, by the general theorem, this doesn't depend on the choice of $v_{g^{-1}}$. In the trivial case $\omega_G = 0$, ρ_K is simply the map $(T(\mathsf{s}\circ K), (T(\mathsf{s}\circ K)^{-1})^*)$ induced by the diffeomorphism $s\circ K$ on P_P. $\Diamond$

Example 5.5.6 Let (M, D) be a smooth Dirac manifold and consider the pair Dirac groupoid $(M\times M \rightrightarrows M, \mathsf{D}\ominus\mathsf{D})$ associated to it. Recall from Example 1.1.11 that the set of bisections of $M\times M\rightrightarrows M$ is equal to $\mathcal{B}(M\times M) = \{\mathrm{Id}_M\}\times \mathrm{Diff}(M)$. Choose $K = (\mathrm{Id}_M, \phi_K) \in \mathcal{B}(M\times M)$, $p := (m,m) \in \Delta_M$ and $\overline{(v_m, w_m, \alpha_m, \beta_m)} \in \mathfrak{B}_p(\mathsf{D}_M\ominus\mathsf{D}_M)$. Then we have $\mathbb{T}\mathsf{t}(v_m, w_m, \alpha_m, \beta_m) = (v_m, v_m, \alpha_m, -\alpha_m) \in \mathfrak{A}_p(\mathsf{D}_M\ominus\mathsf{D}_M)$. Set $n := \phi_K(m)$. Then we have $K(p)^{-1} = (n,m)$ and $(0_n, v_m, 0_n, -\alpha_m) \in (\mathsf{D}_M\ominus\mathsf{D}_M)(n,m)$ is such that $\mathbb{T}\mathsf{s}(0_n, v_m, 0_n, -\alpha_m) = (v_m, v_m, \alpha_m, -\alpha_m) = \mathbb{T}\mathsf{t}(v_m, w_m, \alpha_m, \beta_m)$.
The vector $\rho_K\left(\overline{(v_m, w_m, \alpha_m, \beta_m)}\right)$ is thus given by

$$\begin{aligned}
\rho_K\left(\overline{(v_m,w_m,\alpha_m,\beta_m)}\right) &= \overline{(T_{(n,m)}R_K(0_n,w_m), (T_{(n,n)}R_K^{-1})^*(0_n,\beta_m))}\\
&= \overline{(0_n, T_m\phi_K w_m, 0_n, (T_n\phi_K^{-1})^*\beta_m)}.
\end{aligned}$$

Recall that $\mathfrak{B}(\mathsf{D}_M\ominus\mathsf{D}_M)$ is isomorphic to P_M via (5.8). It is easy to see that the action of $\mathcal{B}(M\times M)$ on $\mathfrak{B}(\mathsf{D}_M\ominus\mathsf{D}_M)$ corresponds via this identification to the action of $\mathrm{Diff}(M)$ on P_M given by $\phi\cdot(v_m,\alpha_m) = (T_m\phi v_m, (T_{\phi(m)}\phi^{-1})^*\alpha_m)$ for all $\phi\in\mathrm{Diff}(M)$ and $(v_m,\alpha_m)\in\mathsf{P}_M(m)$. $\Diamond$

6 Dirac homogeneous spaces and the classification

In this chapter, we show that the Courant algebroid found in Chapter 5 is the right ambient Courant algebroid for the classification of the Dirac homogeneous spaces of a Dirac groupoid.
We prove our main theorem (Theorem 6.3.4) about the correspondence between (integrable) Dirac homogeneous spaces of an (integrable) Dirac groupoid and Lagrangian subspaces (subalgebroids) of the vector bundle (Courant algebroid) $\mathfrak{B}(\mathsf{D}_G)$. This result generalizes the result of Drinfel′d (1993) about the Poisson homogeneous spaces of Poisson Lie groups, of Liu et al. (1998) about Poisson homogeneous spaces of Poisson groupoids and the result in Chapter 4 about the Dirac homogeneous spaces of Dirac Lie groups.
For the sake of completeness, we start by studying the counterpart in the Dirac setting of some properties that the moment map has in the Poisson case.

6.1 The moment map J

In the Poisson case, if $(G/H, \mathsf{D}_{G/H})$ is a Poisson homogeneous space of a Poisson groupoid $(G \rightrightarrows P, \pi_G)$, then the map $\mathsf{J} : G/H \to P$ is a Poisson map (see Liu et al. (1998)). This is also true here under some regularity conditions on the characteristic distributions of the involved Dirac structures.

Theorem 6.1.1 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid such that Theorem 5.1.11 holds and $(G/H, \mathsf{D}_{G/H})$ a Dirac homogeneous space of $(G \rightrightarrows P, \mathsf{D}_G)$. Assume that the map $\mathsf{J}|_{{\mathsf{G}_0}^{G/H}} : {\mathsf{G}_0}^{G/H} \to \mathsf{G}_0 \cap TP$ is surjective in every fiber, where ${\mathsf{G}_0}^{G/H}$ is the characteristic distribution defined on G/H by $\mathsf{D}_{G/H}$. Then the map $\mathsf{J} : (G/H, \mathsf{D}_{G/H}) \mapsto (P, \mathsf{D}_P)$ is a forward Dirac map.*

Proof: Choose $(v_p, \alpha_p) \in \mathsf{D}_P(p)$, for some $p \in P$ and $gH \in G/H$ such that $\mathsf{t}(g) = p$. Then there exists $w_p \in T_p^{\mathsf{s}}G$ and $u_p \in \mathsf{G}_0(p) \cap T_pP$ such that $(w_p, (T_p\mathsf{t})^*\alpha_p) \in I_p^{\mathsf{s}}(\mathsf{D}_G)$ and $v_p = u_p + T_p\mathsf{t}w_p$. Since $\mathbb{T}\mathsf{s}(w_p, (T_p\mathsf{t})^*\alpha_p) = (0_p, 0_p)$, and D_G acts on $\mathsf{D}_{G/H}$, we find that $(w_p, (T_p\mathsf{t})^*\alpha_p) \cdot (0_{gH}, 0_{gH}) \in \mathsf{D}_{G/H}(gH)$. We have $w_p \cdot 0_{gH} = T_p(q \circ R_g)w_p$ and, for all $v_{gH} \in T_{gH}(G/H)$:

$$\begin{aligned}
\widehat{\Phi}\left((T_p\mathsf{t})^*\alpha_p, 0_{gH}\right)(v_{gH}) &= \widehat{\Phi}\left((T_p\mathsf{t})^*\alpha_p, 0_{gH}\right)(T_{gH}\mathsf{J}v_{gH} \cdot v_{gH}) \\
&= ((T_p\mathsf{t})^*\alpha_p)(T_{gH}\mathsf{J}(v_{gH})) + 0_{gH}(v_{gH}) \\
&= \alpha_p(T_{gH}(\mathsf{t} \circ \mathsf{J})v_{gH}) = \alpha_p(T_{gH}\mathsf{J}\, v_{gH}) = ((T_{gH}\mathsf{J})^*\alpha_p)(v_{gH}).
\end{aligned}$$

Thus, we have shown that $(T_p(q\circ R_g)w_p, (T_{gH}\mathsf{J})^*\alpha_p) \in \mathsf{D}_{G/H}(gH)$. We have $T_{gH}\mathsf{J}(T_p(q\circ R_g)w_p) = T_p(\mathsf{J}\circ q\circ R_g)w_p = T_p(\mathsf{t}\circ R_g)w_p = T_p\mathsf{t}w_p = v_p - u_p$. Choose $u_{gH} \in \mathsf{G_0}^{G/H}(gH)$ such that $T_{gH}\mathsf{J}(u_{gH}) = u_p$. Then the pair $(T_p(q\circ R_g)w_p + u_{gH}, (T_{gH}\mathsf{J})^*\alpha_p) \in \mathsf{D}_{G/H}(gH)$ is such that $T_{gH}\mathsf{J}\left(T_p(q\circ R_g)w_p + u_{gH}\right) = v_p$. □

6.2 The homogeneous Dirac structure on the classes of the units

Let $G\rightrightarrows P$ be a Lie groupoid and G/H a smooth homogeneous space of $G\rightrightarrows P$ endowed with a Dirac structure $\mathsf{D}_{G/H}$. Consider the Dirac bundle $\mathfrak{D} = q^*(\mathsf{D}_{G/H})|_P \subseteq \mathsf{P}_G|_P$ over the units P. More explicitly, we have

$$\mathfrak{D}(p) = \left\{ (v_p, \alpha_p) \in T_pG \times T_p^*G \;\middle|\; \begin{array}{c} \exists (v_{pH}, \alpha_{pH}) \in \mathsf{D}_{G/H}(pH) \quad \text{such that} \\ \alpha_p = (T_pq)^*\alpha_{pH} \text{ and } T_pqv_p = v_{pH} \end{array} \right\} \tag{6.1}$$

for all $p \in P$. We have then the following proposition, that generalizes the analogous fact in the Lie group case (Lemma 4.2.4).

Proposition 6.2.1 *Let $(G\rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid and $(G/H, \mathsf{D}_{G/H})$ a Dirac homogeneous space of $(G\rightrightarrows P, \mathsf{D}_G)$. Then $\mathfrak{D} \subseteq \mathsf{P}_G|_P$ defined as in (6.1) satisfies*

$$I^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{D} \subseteq \mathfrak{A}(\mathsf{D}_G) \oplus (\ker \mathbb{T}\mathsf{t})|_P. \tag{6.2}$$

Thus, the quotient $\bar{\mathfrak{D}} = \mathfrak{D}/I^{\mathsf{s}}(\mathsf{D}_G)$ is a smooth subbundle of $\mathfrak{B}(\mathsf{D}_G)$. We have by definition $AH \times_P \{0\} \subseteq \mathfrak{D}$.

Proof: Choose $p \in P$ and $(v_p, \alpha_p) \in I^{\mathsf{s}}_p(\mathsf{D}_G) = \mathsf{D}_G(p) \cap \ker \mathbb{T}\mathsf{s}$. Then $\mathbb{T}\mathsf{s}(v_p, \alpha_p) = (0_p, 0_p)$ and the product $(v_p, \alpha_p)\cdot(0_{pH}, 0_{pH})$ makes sense. Since $(0_{pH}, 0_{pH}) \in \mathsf{D}_{G/H}(pH)$, we have then $(T_pqv_p, \alpha_p \cdot 0_{pH}) = (v_p, \alpha_p)\cdot(0_{pH}, 0_{pH}) \in \mathsf{D}_{G/H}(pH)$. But $\alpha_p \cdot 0_{pH}$ is such that $(T_pq)^*(\alpha_p \cdot 0_{pH}) = \alpha_p \star ((T_pq)^*0_{pH}) = \alpha_p$, and we have hence $(v_p, \alpha_p) \in \mathfrak{D}(p)$ by definition of $\mathfrak{D}$.
The inclusion $I^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{D}$ yields immediately $\mathfrak{D} = \mathfrak{D}^\perp \subseteq (I^{\mathsf{s}}(\mathsf{D}_G))^\perp = \mathsf{D}_G|_P + (\ker \mathbb{T}\mathsf{t})|_P = \mathfrak{A}(\mathsf{D}_G) \oplus (\ker \mathbb{T}\mathsf{t})|_P$. □

Theorem 6.2.2 *Let $(G\rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid and $\mathfrak{D}$ a Dirac subspace of $\mathsf{P}_G|_P$ satisfying (6.2). Then the set $\mathsf{D} = \mathsf{D}_G \cdot \mathfrak{D} \subseteq \mathsf{P}_G$ defined by*

$$\mathsf{D}(g) = \left\{ (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \;\middle|\; \begin{array}{c} (v_g, \alpha_g) \in \mathsf{D}_G(g), \\ (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g)), \\ \mathbb{T}\mathsf{s}(v_g, \alpha_g) = \mathbb{T}\mathsf{t}(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \end{array} \right\}$$

is a Dirac structure on G and (G, D) is a Dirac homogeneous space of $(G\rightrightarrows P, \mathsf{D}_G)$.

By Lemma 5.5.1, this is exactly the same construction as in (4.12).
Note that $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s} \subseteq \mathsf{D}$ by construction: for all $(v_g, \alpha_g) \in \mathsf{D}_G(g) \cap \ker \mathbb{T}\mathsf{s}$, we have $\mathbb{T}\mathsf{s}(v_g, \alpha_g) = (0_{\mathsf{s}(g)}, 0_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g))$ and hence $(v_g, \alpha_g) = (v_g, \alpha_g) \star (0_{\mathsf{s}(g)}, 0_{\mathsf{s}(g)}) \in \mathsf{D}(g)$.

Proof: By Lemma 5.5.1, D is spanned by sections $\xi + \sigma^l$ such that $\bar{\xi} + \sigma$ is a section of $\mathfrak{D}$ (with $\bar{\xi} \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$ and $\sigma \in \Gamma((\ker \mathbb{T}\mathsf{t})|_P)$) and all the sections of $\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s}$. This shows that D is smooth.

Choose $(v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)})$ and $(w_g, \beta_g) \star (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \in \mathsf{D}(g)$, that is, with $(v_g, \alpha_g), (w_g, \beta_g) \in \mathsf{D}_G(g)$ and $(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}), (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g))$. We have then

$$\begin{aligned} &\left\langle (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}), (w_g, \beta_g) \star (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \right\rangle \\ &= \alpha_g(w_g) + \alpha_{\mathsf{s}(g)}(w_{\mathsf{s}(g)}) + \beta_g(v_g) + \beta_{\mathsf{s}(g)}(v_{\mathsf{s}(g)}) \\ &= \langle (v_g, \alpha_g), (w_g, \beta_g) \rangle + \langle (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}), (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \rangle = 0. \end{aligned}$$

This shows $\mathsf{D} \subseteq \mathsf{D}^\perp$.

For the converse inclusion, choose $(w_g, \beta_g) \in \mathsf{D}(g)^\perp$. Then

$$(w_g, \beta_g) \in (\mathsf{D}_G(g) \cap \ker \mathbb{T}\mathsf{s})^\perp = (\mathsf{D}_G + \ker \mathbb{T}\mathsf{t})(g)$$

and consequently, we get the fact that $\mathbb{T}\mathsf{t}(w_g, \beta_g) \in \mathbb{T}\mathsf{t}(\mathsf{D}_G(g)) = \mathfrak{A}_{\mathsf{t}(g)}(\mathsf{D}_G)$. We write $\mathsf{t}(g) = p$ and $\mathbb{T}\mathsf{t}(w_g, \beta_g) = \bar{\xi}(p)$ for some section $\bar{\xi} \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$. Consider a section $\xi \in \Gamma(\mathsf{D}_G)$ such that $\xi \sim_{\mathsf{s}} \bar{\xi}$. Then we have for all $(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g))$ and $(v_g, \alpha_g) \in \mathsf{D}_G(g)$ such that $\mathbb{T}\mathsf{t}(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) = \mathbb{T}\mathsf{s}(v_g, \alpha_g)$:

$$\begin{aligned} \left\langle \xi(g^{-1}) \star (w_g, \beta_g), \left(v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}\right) \right\rangle &= \left\langle \xi(g^{-1}) \star (w_g, \beta_g), (v_g, \alpha_g)^{-1} \star (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \right\rangle \\ &= \langle (w_g, \beta_g), (v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \rangle + \langle \xi(g^{-1}), (v_g, \alpha_g)^{-1} \rangle \\ &= 0, \end{aligned}$$

since $(v_g, \alpha_g) \star (v_{\mathsf{s}(g)}, \alpha_{\mathsf{s}(g)}) \in \mathsf{D}(g)$ and $(v_g, \alpha_g)^{-1} \in \mathsf{D}_G(g^{-1})$. This proves that

$$\xi(g^{-1}) \star (w_g, \beta_g) \in \mathfrak{D}(\mathsf{s}(g))^\perp = \mathfrak{D}(\mathsf{s}(g)),$$

and hence, if we write $\xi(g^{-1}) \star (w_g, \beta_g) = (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \in \mathfrak{D}(\mathsf{s}(g))$,

$$(w_g, \beta_g) = \left(\xi(g^{-1})\right)^{-1} \star (w_{\mathsf{s}(g)}, \beta_{\mathsf{s}(g)}) \in \mathsf{D}(g).$$

The second claim is obvious since the restriction to D of the map $\mathbb{T}\mathsf{J}$ has image in $\mathbb{T}\mathsf{t}(\mathsf{D}_G) = \mathfrak{A}(\mathsf{D}_G)$ and, by construction of D, the map $\mathsf{D}_G \times_{\mathfrak{A}(\mathsf{D}_G)} \mathsf{D}$, $((v_g, \alpha_g), (v_h, \alpha_h)) \mapsto (v_g, \alpha_g) \star (v_h, \alpha_h)$ is a well-defined Lie groupoid action. $\square$

Theorem 6.2.3 *In the situation of the preceding theorem, if $\mathfrak{D}$ is the restriction to P of the pullback $q^*(\mathsf{D}_{G/H})$ (as in (6.1)) for some Dirac homogeneous space $(G/H, \mathsf{D}_{G/H})$ of $(G \rightrightarrows P, \mathsf{D}_G)$, then $\mathsf{D} = q^*(\mathsf{D}_{G/H})$.*

Proof: Choose $(v_g, \alpha_g) \in q^*\left(\mathsf{D}_{G/H}\right)(g)$. Then α_g is equal to $(T_g q)^* \alpha_{gH}$ for some $\alpha_{gH} \in T^*_{gH}(G/H)$ such that $(T_g q v_g, \alpha_{gH}) \in \mathsf{D}_{G/H}(gH)$. Then $\mathbb{T}\mathsf{J}(T_g q v_g, \alpha_{gH}) = \mathbb{T}\mathsf{t}(v_g, \alpha_g) \in \mathfrak{A}_{\mathsf{t}(g)}(\mathsf{D}_G)$ and there exists $(w_{g^{-1}}, \beta_{g^{-1}}) \in \mathsf{D}_G(g^{-1})$ such that

$$\mathbb{T}\mathsf{s}(w_{g^{-1}}, \beta_{g^{-1}}) = \mathbb{T}\mathsf{J}(T_g q v_g, \alpha_{gH}).$$

Set $p = \mathsf{s}(g)$ and consider $(u_{pH}, \gamma_{pH}) := (w_{g^{-1}}, \beta_{g^{-1}}) \cdot (T_g q v_g, \alpha_{gH}) \in \mathsf{D}_{G/H}(pH)$. Then we have

$$(T_p q)^* \gamma_{pH} = \beta_{g^{-1}} \star ((T_g q)^* \alpha_{gH}) = \beta_{g^{-1}} \star \alpha_g$$

by Proposition 2.3.1 about the action of $T^*G \rightrightarrows A^*G$ on $\widehat{\mathsf{J}} : T^*(G/H) \to A^*G$, and

$$u_{pH} = w_{g^{-1}} \cdot (T_g q v_g) = T_{(g^{-1}, gH)} \Phi(w_{g^{-1}}, T_g q v_g) = T_p q(w_{g^{-1}} \star v_g).$$

Thus, $(u_p, \gamma_p) := (w_{g^{-1}}, \beta_{g^{-1}}) \star (v_g, \alpha_g)$ is an element of $\mathfrak{D}(p)$, and we have $(v_g, \alpha_g) = (w_{g^{-1}}, \beta_{g^{-1}})^{-1} \star (u_p, \gamma_p)$. Since D_G is multiplicative and $(w_{g^{-1}}, \beta_{g^{-1}}) \in \mathsf{D}_G(g^{-1})$, the pair $(w_{g^{-1}}, \beta_{g^{-1}})^{-1}$ is an element of $\mathsf{D}_G(g)$ and we have shown that $(v_g, \alpha_g) \in \mathsf{D}(g)$.
Since $q^*(\mathsf{D}_{G/H}) \subseteq \mathsf{D}$ is an inclusion of Dirac structures, we have then equality. □

Remark 6.2.4 Note that Theorem 6.2.3 shows that if $(G \rightrightarrows P, \mathsf{D}_G)$ is a Dirac groupoid, a D_G-homogeneous Dirac structure on G/H is uniquely determined by its restriction to $q(P) \subseteq G/H$. △

Example 6.2.5 We have seen in Example 2.3.8 that if $(G \rightrightarrows P, \mathsf{D}_G)$ is a Dirac groupoid, then $(\mathsf{t} : G \to P, \mathsf{D}_G)$ is a Dirac homogeneous space of $(G \rightrightarrows P, \mathsf{D}_G)$.
The space $\mathfrak{D}$ is here the direct sum $I^{\mathsf{s}}(\mathsf{D}_G) \oplus \mathfrak{A}(\mathsf{D}_G)$. The corresponding Dirac structure D is equal to D_G by the last theorem. This can also be seen directly from the definition of D, since D is spanned by the sections (X_ξ, θ_ξ) for $(\bar{X}_\xi, \bar{\theta}_\xi) \in \Gamma(\mathfrak{A}(\mathsf{D}_G))$ and the sections σ^r for all $\sigma \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$, which are spanning sections for D_G. ◊

6.3 The Theorem of Drinfel'd

Recall that if $(G \rightrightarrows P, \mathsf{D}_G)$ is a Dirac groupoid, then there is an induced action of the set of bisections $\mathcal{B}(G)$ of G on the vector bundle $\mathfrak{B}(\mathsf{D}_G)$ associated to D_G (see Theorem 5.5.2). If H is a wide Lie subgroupoid of $G \rightrightarrows P$, this action restricts to an action of $\mathcal{B}(H)$ on $\mathfrak{B}(\mathsf{D}_G)$. We use this action to characterize D_G-homogeneous Dirac structures on G/H.

Theorem 6.3.1 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid, H a t-connected wide subgroupoid of G such that the homogeneous space G/H has a smooth manifold structure and $q : G \to G/H$ is a smooth surjective submersion. Let $\mathfrak{D}$ be a Dirac subspace of $\mathsf{P}_G|_P$ satisfying (6.2) and such that $AH \times_P \{0\} \subseteq \mathfrak{D}$. Then the following are equivalent:*

1. *$\mathfrak{D}$ is the pullback $q^*(\mathsf{D}_{G/H})|_P$ as in (6.1), where $\mathsf{D}_{G/H}$ is some D_G-homogeneous Dirac structure on G/H.*
2. *$\bar{\mathfrak{D}} = \mathfrak{D}/I^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{B}(\mathsf{D}_G)$ is invariant under the induced action of $\mathcal{B}(H)$ on $\mathfrak{B}(\mathsf{D}_G)$.*
3. *The D_G-homogeneous Dirac structure $\mathsf{D} = \mathsf{D}_G \cdot \mathfrak{D} \subseteq \mathsf{P}_G$ as in Theorem 6.2.2 pushes-forward to a (D_G-homogeneous) Dirac structure on the quotient G/H.*

Note that, together with Theorem 6.2.3, this shows that a Dirac structure $\mathsf{D}_{G/H}$ on G/H is D_G-homogeneous if and only if $I^{\mathsf{s}}(\mathsf{D}_G) \subseteq (q^*\mathsf{D}_{G/H})|_P$ and $q^*\mathsf{D}_{G/H} = \mathsf{D}_G \cdot (q^*\mathsf{D}_{G/H})|_P$, that is, $(G/H, \mathsf{D}_{G/H})$ is $(G \rightrightarrows P, \mathsf{D}_G)$-homogeneous if and only if $(G, q^*\mathsf{D}_{G/H})$ is.
For the proof of Theorem 6.3.1, we will need the following Lemma.

Lemma 6.3.2 *In the situation of Theorem 6.2.2, we have* $\mathfrak{D} = \mathsf{D}|_P$.

PROOF: Choose $p \in P$ and $(v_p, \alpha_p) \in \mathfrak{D}(p)$. Then $\mathbb{T}\mathsf{t}(v_p, \alpha_p) \in \mathfrak{A}_p(\mathsf{D}_G) \subseteq \mathsf{D}_G(p)$ and $(v_p, \alpha_p) = \mathbb{T}\mathsf{t}(v_p, \alpha_p) \star (v_p, \alpha_p) \in \mathsf{D}(p)$. This shows $\mathfrak{D} \subseteq \mathsf{D}|_P$ and we are done since both vector bundles have the same rank. $\square$

PROOF (OF THEOREM 6.3.1): Assume first that $\mathfrak{D} = q^*(\mathsf{D}_{G/H})|_P$ for some D_G-homogeneous Dirac structure $\mathsf{D}_{G/H}$ on G/H and choose $K \in \mathcal{B}(H)$ and $(v_p, \alpha_p) \in \mathfrak{D}(p)$, $p \in P$. Then there exists $\alpha_{pH} \in T^*_{pH}(G/H)$ such that $\alpha_p = (T_pq)^*\alpha_{pH}$ and $(T_pqv_p, \alpha_{pH}) \in \mathsf{D}_{G/H}(pH)$. If we set $K(p) =: h \in H$ and write $\overline{(v_p, \alpha_p)}$ for $(v_p, \alpha_p) + I^{\mathsf{s}}_p(\mathsf{D}_G) \in \bar{\mathfrak{D}}(p) \subseteq \mathfrak{B}_p(\mathsf{D}_G)$, we have

$$\rho_K\left(\overline{(v_p, \alpha_p)}\right) = \left(T_{h^{-1}}R_K(v_{h^{-1}} \star v_p), (T_{\mathsf{s}(h)}R_K^{-1})^*(\alpha_{h^{-1}} \star \alpha_p)\right) + I^{\mathsf{s}}_{\mathsf{s}(h)}(\mathsf{D}_G)$$

for any $(v_{h^{-1}}, \alpha_{h^{-1}}) \in \mathsf{D}_G(h^{-1})$ satisfying $\mathbb{T}\mathsf{s}(v_{h^{-1}}, \alpha_{h^{-1}}) = \mathbb{T}\mathsf{t}(v_p, \alpha_p)$. Since

$$\mathbb{T}\mathsf{J}(T_pqv_p, \alpha_{pH}) = \mathbb{T}\mathsf{t}(v_p, \alpha_p) = \mathbb{T}\mathsf{s}(v_{h^{-1}}, \alpha_{h^{-1}}),$$

the product $(v_{h^{-1}}, \alpha_{h^{-1}}) \cdot (T_pqv_p, \alpha_{pH})$ makes sense and is an element of $\mathsf{D}_{G/H}(\mathsf{s}(h)H)$. Note that since $K \in \mathcal{B}(H)$, we have $q \circ R_K = q$. The pair $(T_{h^{-1}}R_K(v_{h^{-1}} \star v_p), (T_{\mathsf{s}(h)}R_K^{-1})^*(\alpha_{h^{-1}} \star \alpha_p))$ satisfies $T_{\mathsf{s}(h)}q(T_{h^{-1}}R_K(v_{h^{-1}} \star v_p)) \in T_{\mathsf{s}(h)H}(G/H)$,

$$T_{\mathsf{s}(h)}q(T_{h^{-1}}R_K(v_{h^{-1}} \star v_p)) = T_{h^{-1}}(q \circ R_K)(v_{h^{-1}} \star v_p) = T_{h^{-1}}q(v_{h^{-1}} \star v_p) = v_{h^{-1}} \cdot (T_pqv_p)$$

and

$$\begin{aligned}(T_{\mathsf{s}(h)}R_K^{-1})^*(\alpha_{h^{-1}} \star \alpha_p) &= (T_{\mathsf{s}(h)}R_K^{-1})^*(\alpha_{h^{-1}} \star (T_pq)^*\alpha_{pH}) \\ &= (T_{\mathsf{s}(h)}R_K^{-1})^*\left((T_{h^{-1}}q)^*(\alpha_{h^{-1}} \cdot \alpha_{pH})\right) = (T_{\mathsf{s}(h)}q)^*(\alpha_{h^{-1}} \cdot \alpha_{pH}).\end{aligned}$$

Thus, $(T_{h^{-1}}R_K(v_{h^{-1}} \star v_p), (T_{\mathsf{s}(h)}R_K^{-1})^*(\alpha_{h^{-1}} \star \alpha_p))$ is an element of $\mathfrak{D}(\mathsf{s}(h))$ and $\rho_K\left(\overline{(v_p, \alpha_p)}\right)$ is an element of $\bar{\mathfrak{D}}(\mathsf{s}(h))$. This shows (1) $\Rightarrow$ (2).

Assume now that $\bar{\mathfrak{D}}$ is invariant under the action of $\mathcal{B}(H)$ on $\mathfrak{B}(\mathsf{D}_G)$. Recall the backgrounds about Dirac reduction in Section 1.2 and also Subsection 1.1.6. Set $\mathcal{K} = \mathcal{H} \times_G 0_{T^*G}$, and hence $\mathcal{K}^\perp = TG \times_G \mathcal{H}^\circ$.
We have $AH \times_P \{0\} \subseteq \mathfrak{D}$ by hypothesis. By definition of D and $\mathcal{H}$, this yields immediately $\mathcal{K} = \mathcal{H} \times_G \{0\} \subseteq \mathsf{D}$, hence $\mathsf{D} \subseteq \mathcal{K}^\perp$ and $\mathsf{D} \cap \mathcal{K}^\perp = \mathsf{D}$ has constant rank on G. By (1.13), we have to show that D is invariant under the right action of $\mathcal{B}(H)$ on G. We will use the fact that D is spanned by the sections $\sigma^r \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$ for all $\sigma \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$ and $(X_\xi, \theta_\xi) + (X^l, \mathsf{s}^*\alpha)$ for all sections $(\bar{X}_\xi, \bar{\theta}_\xi) + (X, (\mathsf{s}^*\alpha)|_P) \in \Gamma(\mathfrak{D}) \subseteq \Gamma(\mathfrak{A}(\mathsf{D}_G) \oplus (\ker \mathbb{T}\mathsf{t})|_P)$.

Choose $K \in \mathcal{B}(H)$. It is easy to verify that

$$(R_K^* Z^r, R_K^*(\mathsf{t}^*\gamma)) = (Z^r, \mathsf{t}^*\gamma) \quad \text{for all} \quad (Z, (\mathsf{t}^*\gamma)|_P) \in \Gamma(\ker \mathbb{T}\mathsf{s}|_P).$$

Choose a section $(X_\xi, \theta_\xi) + (X^l, \mathsf{s}^*\alpha)$ of D. We want to show that $(R_K^*(X_\xi + X^l), R_K^*(\theta_\xi + \mathsf{s}^*\alpha))$ is then also a section of D. Choose $g \in G$ and set for simplicity $h = K(\mathsf{s}(g)) \in H$, $p = \mathsf{s}(h)$, $q = \mathsf{t}(h) = \mathsf{s}(g)$ and $(\bar{X}_\xi + X, \bar{\theta}_\xi + \mathsf{s}^*\alpha)(p) =: (u_p, \gamma_p) \in \mathfrak{D}(p)$. Then $\left((X_\xi, \theta_\xi) + (X^l, \mathsf{s}^*\alpha)\right)(g \star h) = (X_\xi, \theta_\xi)(g \star h) \star (u_p, \gamma_p)$ and we can compute

$$\begin{aligned} &\left(R_K^*(X_\xi + X^l), R_K^*(\theta_\xi + \mathsf{s}^*\alpha)\right)(g) \\ =& \left(T_{g\star h} R_K^{-1}(X_\xi + X^l)(g \star h), (T_g R_K)^*((\theta_\xi + \mathsf{s}^*\alpha)(g \star h))\right) \\ =& \left(T_{g\star h} R_K^{-1}\left(X_\xi(gh) \star u_p\right), (T_g R_K)^*\left(\theta_\xi(gh) \star \gamma_p\right)\right). \end{aligned}$$

Choose $(v_g, \alpha_g) \in \mathsf{D}_G(g)$ such that $\mathbb{T}\mathsf{t}(v_g, \alpha_g) = \mathbb{T}\mathsf{t}(X_\xi(gh), \theta_\xi(gh))$. Then the product $(w_h, \alpha_h) := (v_g, \alpha_g)^{-1} \star (X_\xi(gh), \theta_\xi(gh))$ is an element of $\mathsf{D}_G(h)$ such that $\mathbb{T}\mathsf{s}(w_h, \alpha_h) = (\bar{X}_\xi, \bar{\theta}_\xi)(p)$ and we have

$$\begin{aligned} &\left(R_K^*(X_\xi + X^l), R_K^*(\theta_\xi + \mathsf{s}^*\alpha)\right)(g) \\ =&(v_g, \alpha_g) \star \left(T_h R_K^{-1}\left(v_g^{-1} \star X_\xi(gh) \star u_p\right), (T_g R_K)^*\left(\alpha_g^{-1} \star \theta_\xi(gh) \star \gamma_p\right)\right) \\ =&(v_g, \alpha_g) \star \left(T_h R_K^{-1}\left(w_h \star u_p\right), (T_q R_K)^*\left(\beta_h \star \gamma_p\right)\right). \end{aligned}$$

But since $\bar{\mathfrak{D}}$ is invariant under the action of $\mathcal{B}(H)$ on $\mathfrak{B}(\mathsf{D}_G)$ and $\overline{(u_p, \gamma_p)} = (u_p, \gamma_p) + I_p^{\mathsf{s}}(\mathsf{D}_G)$ is an element of $\bar{\mathfrak{D}}(p)$, we have

$$\left(T_h R_K^{-1}\left(w_h \star u_p\right), (T_q R_K)^*\left(\beta_h \star \gamma_p\right)\right) + I_q^{\mathsf{s}}(\mathsf{D}_G) = \rho_{K^{-1}}\left(\overline{(u_p, \gamma_p)}\right) \in \bar{\mathfrak{D}}(q).$$

Because $I_q^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{D}(q)$, we have consequently

$$\left(T_h R_K^{-1}\left(w_h \star u_p\right), (T_q R_K)^*\left(\beta_h \star \gamma_p\right)\right) \in \mathfrak{D}(q)$$

and hence

$$\left(R_K^*(X_\xi + X^l), R_K^*(\theta_\xi + \mathsf{s}^*\alpha)\right)(g) = (v_g, \alpha_g) \star \left(T_h R_K^{-1}\left(w_h \star u_p\right), (T_q R_K)^*\left(\beta_h \star \gamma_p\right)\right) \in \mathsf{D}(g)$$

since $(v_g, \alpha_g) \in \mathsf{D}_G(g)$.

We show then that the push-forward $q(\mathsf{D})$ is a D_G-homogeneous Dirac structure on G/H. By definition of $\mathbb{T}\mathsf{J}$, we have $\mathbb{T}\mathsf{J}(q(\mathsf{D})) = \mathbb{T}\mathsf{t}(\mathsf{D}) \subseteq \mathbb{T}\mathsf{t}(\mathsf{D}_G) = \mathfrak{A}(\mathsf{D}_G)$. Choose $(v_{gH}, \alpha_{gH}) \in q(\mathsf{D})(gH)$ and $(w_{g'}, \beta_{g'}) \in \mathsf{D}_G(g')$ such that $\mathbb{T}\mathsf{s}(w_{g'}, \beta_{g'}) = \mathbb{T}\mathsf{J}(v_{gH}, \alpha_{gH})$. Then there exists $v_g \in T_g G$ such that $T_g q v_g = v_{gH}$ and $(v_g, (T_g q)^*\alpha_{gH}) \in \mathsf{D}(g)$. The pair $(v_g, (T_g q)^*\alpha_{gH})$ satisfies then $\mathbb{T}\mathsf{t}(v_g, (T_g q)^*\alpha_{gH}) = \mathbb{T}\mathsf{J}(v_{gH}, \alpha_{gH}) = \mathbb{T}\mathsf{s}(w_{g'}, \beta_{g'})$ and since (G, D) is a Dirac homogeneous space of $(G \rightrightarrows P, \mathsf{D}_G)$, we have $(w_{g'}, \beta_{g'}) \star (v_g, (T_g q)^*\alpha_{gH}) \in \mathsf{D}(g' \star g)$ and the identities $(T_{g'\star g} q)^*(\beta_{g'} \cdot \alpha_{gH}) = \beta_{g'} \star (T_g q)^*\alpha_{gH}$ and $T_{g'\star g} q(w_{g'} \star v_g) = w_{g'} \cdot (T_g q v_g) = w_{g'} \cdot v_{gH}$. Thus, the pair $(w_{g'}, \beta_{g'}) \cdot (v_{gH}, \alpha_{gH})$ is an element of $q(\mathsf{D})(gg'H)$ and $q(\mathsf{D})$ is shown to be D_G-homogeneous. Hence, we have shown (2) $\Rightarrow$ (3).

To show that (3) implies (1), we have then just to show that the vector bundle $\mathfrak{D} \to P$ is the restriction to P of the pullback $q^*(q(\mathsf{D}))$. Since $\mathsf{D}|_P = \mathfrak{D}$ by Lemma 6.3.2, we can show that $\mathsf{D} = q^*(q(\mathsf{D}))$. This follows from the inclusion $\mathcal{H} \times_G 0_{T^*G} \subseteq \mathsf{D}$. Choose $(v_g, \alpha_g) \in \mathsf{D}(g)$. Then $\alpha_g \in \mathcal{H}(g)^\circ$. Thus, there exists $\alpha_{gH} \in T^*_{gH}(G/H)$ such that $\alpha_g = (T_g q)^* \alpha_{gH}$ and, by definition of $q(\mathsf{D})$, the pair $(T_g q v_g, \alpha_{gH})$ is an element of $q(\mathsf{D})(gH)$. But then $(v_g, \alpha_g) \in q^*(q(\mathsf{D}))(g)$. Conversely, if $(v_g, \alpha_g) \in q^*(q(\mathsf{D}))(g)$, then $\alpha_g = (T_g q)^* \alpha_{gH}$ for some $\alpha_{gH} \in T^*_{gH}(G/H)$ satisfying $(T_g q v_g, \alpha_{gH}) \in q(\mathsf{D})(gH)$. By definition of $q(\mathsf{D})(gH)$, there exists then $u_g \in T_g G$ such that $(u_g, (T_g q)^* \alpha_{gH}) = (u_g, \alpha_g) \in \mathsf{D}(g)$ and $T_g q u_g = T_g q v_g$. But this yields that $v_g - u_g \in \mathcal{H}(g)$ and hence $(v_g, \alpha_g) = (u_g, \alpha_g) + (v_g - u_g, 0_g) \in \mathsf{D}(g) + (\mathcal{H}(g) \times \{0_g\}) = \mathsf{D}(g)$ since $(\mathcal{H}(g) \times \{0_g\}) \subseteq \mathsf{D}(g)$. □

Theorem 6.3.3 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be an integrable Dirac groupoid. In the situation of the previous theorem, the following are equivalent*

1. *The Dirac structure $q(\mathsf{D}) = \mathsf{D}_{G/H}$ is integrable.*
2. *The Dirac structure D is integrable.*
3. *The set of sections of $\bar{\mathfrak{D}} \subseteq \mathfrak{B}(\mathsf{D}_G)$ is closed under the bracket on the sections of the Courant algebroid $\mathfrak{B}(\mathsf{D}_G)$.*

Proof: If D is integrable, then $q(\mathsf{D})$ is integrable by a Theorem in Zambon (2008) about Dirac reduction by foliations (see the generalities about Dirac reduction in Section 1.2). Conversely, assume that $q(\mathsf{D})$ is integrable. Since $\mathsf{D} \subseteq TG \times_G \mathcal{H}^\circ$ and by the proof of Theorem 6.3.1, the Dirac structure D is spanned by q-descending sections, that is, sections (X, α) such that $\alpha \in \Gamma(\mathcal{H}^\circ)$ and $R_K^*(X, \alpha) = (X, \alpha)$ for all $K \in \mathcal{B}(H)$. Choose two descending sections $(X, \alpha), (Y, \beta)$ of D. Choose $(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \Gamma(q(\mathsf{D}))$ such that $(X, \alpha) \sim_q (\bar{X}, \bar{\alpha})$ and $(Y, \beta) \sim_q (\bar{Y}, \bar{\beta})$. Then the bracket $[(X, \alpha), (Y, \beta)]$ descends to $[(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta})]$ which is a section of $q(\mathsf{D})$ since $(G/H, q(\mathsf{D}))$ is integrable. But since $\mathcal{H} \times_G 0_{T^*G} \subseteq \mathsf{D}$, we have $\mathsf{D} = q^*(q(\mathsf{D}))$ (recall the proof of Theorem 6.3.1). Since $[(X, \alpha), (Y, \beta)]$ is a section of $q^*(q(\mathsf{D}))$, we have shown that $[(X, \alpha), (Y, \beta)] \in \Gamma(\mathsf{D})$. This proves (1) $\iff$ (2).

Assume that (G, D) is integrable and choose two sections $e_{\xi, X, \alpha} = (\bar{X}_\xi + X, \bar{\theta}_\xi + \mathsf{s}^* \alpha) + I^{\mathsf{s}}(\mathsf{D}_G), e_{\eta, Y, \beta} = (\bar{X}_\eta + Y, \bar{\theta}_\eta + \mathsf{s}^* \beta) + I^{\mathsf{s}}(\mathsf{D}_G)$ of $\bar{\mathfrak{D}} \subseteq \mathfrak{B}(\mathsf{D}_G)$. Then the two pairs $(X_\xi + X^l, \theta_\xi + \mathsf{s}^* \alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^* \beta)$ are smooth sections of D by construction and since (G, D) is integrable, we have $[(X_\xi + X^l, \theta_\xi + \mathsf{s}^* \alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^* \beta)] \in \Gamma(\mathsf{D})$. But since $\mathfrak{D} = \mathsf{D}|_P$ and $[e_{\xi, X, \alpha}, e_{\eta, Y, \beta}] = [(X_\xi + X^l, \theta_\xi + \mathsf{s}^* \alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^* \beta)]|_P + I^{\mathsf{s}}(\mathsf{D}_G)$, this yields $[e_{\xi, X, \alpha}, e_{\eta, Y, \beta}] \in \Gamma(\bar{\mathfrak{D}})$.
Conversely, assume that $\Gamma(\bar{\mathfrak{D}})$ is closed under the Courant bracket on sections of $\mathfrak{B}(\mathsf{D}_G)$ and choose two spanning sections $(X_\xi + X^l, \theta_\xi + \mathsf{s}^* \alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^* \beta)$ of D corresponding to $(\bar{X}_\xi + X, \bar{\theta}_\xi + \mathsf{s}^* \alpha|_P)$ and $(\bar{X}_\eta + Y, \bar{\theta}_\eta + \mathsf{s}^* \beta|_P) \in \Gamma(\mathfrak{D}) \subseteq \Gamma(\mathfrak{A}(\mathsf{D}_G) \oplus (\ker \mathbb{T}\mathsf{t})|_P)$. Since $[e_{\xi, X, \alpha}, e_{\eta, Y, \beta}]$ is then an element of $\Gamma(\bar{\mathfrak{D}})$ and $I^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{D}$, we have

$$[(X_\xi + X^l, \theta_\xi + \mathsf{s}^* \alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^* \beta)]|_P \in \Gamma(\mathfrak{D})$$

by definition of the bracket on the sections of $\mathfrak{B}(\mathsf{D}_G)$. By Theorem 5.3.1, Lemma 5.5.1 and (5.6), the value of $[(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta)]$ at $g \in G$ equals

$$\Big(([(X_\xi, \theta_\xi), (X_\eta, \theta_\eta)] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi)\,(g)\Big)\star \\ \big([(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta)](\mathsf{s}(g))\big)$$

and we find that $[(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), (X_\eta + Y^l, \theta_\eta + \mathsf{s}^*\beta)]$ is a section of D, since the first factor is an element of $\mathsf{D}_G(g)$ and the second an element of $\mathfrak{D}(\mathsf{s}(g))$. Recall also that, by the proof of Theorem 5.4.1, we know that $[(X_\xi + X^l, \theta_\xi + \mathsf{s}^*\alpha), \sigma^r] \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$ for all $\sigma \in \Gamma(I^{\mathsf{s}}(\mathsf{D}_G))$. Finally, since D_G is integrable, we know that $[\sigma_1^r, \sigma_2^r] \in \Gamma(\mathsf{D}_G)$ for all $\sigma_1, \sigma_2 \in \Gamma(\mathsf{D}_G \cap \ker \mathbb{T}\mathsf{s})$. Thus, by the Leibniz identity for the restriction to $\Gamma(\mathsf{D})$ of the Courant bracket on P_G, we have shown that (G, D) is integrable. □

As a corollary of the Theorems 6.2.2, 6.2.3, 6.3.1, and 6.3.3 we get our main result, that generalizes the correspondence theorems in Drinfel′d (1993), Liu et al. (1998) and Theorem 4.2.13.

Theorem 6.3.4 *Let $(G \rightrightarrows P, \mathsf{D}_G)$ be a Dirac groupoid. Let H be a wide Lie subgroupoid of G such that the quotient G/H is a smooth manifold and the map $q : G \to G/H$ a smooth surjective submersion. There is a one-one correspondence between D_G-homogeneous Dirac structures on G/H and Dirac subspaces $\mathfrak{D}$ of $\mathsf{P}_G|_P$ such that $AH \times \{0\} + I^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{D} \subseteq \mathfrak{A}(\mathsf{D}_G) \oplus (\ker \mathbb{T}\mathsf{t})|_P$ and $\bar{\mathfrak{D}} := \mathfrak{D}/I^{\mathsf{s}}(\mathsf{D}_G)$ is a $\mathcal{B}(H)$-invariant Dirac subspace of $\mathfrak{B}(\mathsf{D}_G)$. If $(G \rightrightarrows P, \mathsf{D}_G)$ is integrable, then integrable D_G-homogeneous Dirac structures on G/H correspond in this way to Lagrangian subalgebroids $\bar{\mathfrak{D}}$ of $\mathfrak{B}(\mathsf{D}_G)$.*

Remark 6.3.5 Assume that $(G \rightrightarrows P, \mathsf{D}_G)$ is an integrable Dirac groupoid, $\mathfrak{D} \subseteq \mathsf{P}_G|_P$ a Dirac subspace satisfying (6.2) and $AH \times \{0\} \subseteq \mathfrak{D}$ for some t-connected wide Lie subgroupoid H of $G \rightrightarrows P$, and such that $\mathfrak{D}/I^{\mathsf{s}}(\mathsf{D}_G) \subseteq \mathfrak{B}(\mathsf{D}_G)$ is closed under the bracket on $\mathfrak{B}(\mathsf{D}_G)$. It is easy to check (as in the proof of Theorem 6.3.3) that we have then $\big[(X^l, 0), (X, \alpha)\big] \in \Gamma(\mathsf{D}_G \cdot \mathfrak{D})$ for all $(X, \alpha) \in \Gamma(\mathsf{D}_G \cdot \mathfrak{D})$ and $X \in \Gamma(AH)$. Since H is t-connected, we get then the fact that $(R_K^* X, R_K^* \alpha) \in \Gamma(\mathsf{D}_G \cdot \mathfrak{D})$ for all bisections $K \in \mathcal{B}(H)$ and the Dirac structure $\mathsf{D}_G \cdot \mathfrak{D}$ projects to a Dirac structure on G/H, that is D_G-homogeneous. The quotient $\mathfrak{D}/I^{\mathsf{s}}(\mathsf{D}_G)$ is then automatically invariant under the induced action of the bisections $\mathcal{B}(H)$ on $\mathfrak{B}(\mathsf{D}_G)$ and this shows that the condition 2 of Theorem 6.3.1 is always satisfied if D_G is integrable, $\mathfrak{D}/I^{\mathsf{s}}(\mathsf{D}_G)$ is closed under the Courant bracket on sections of $\mathfrak{B}(\mathsf{D}_G)$ and H is t-connected. △

Example 6.3.6 In Liu et al. (1998), it is shown that for a Poisson groupoid $(G \rightrightarrows P, \pi_G)$, there is a one to one correspondence between π_G-homogeneous Poisson structures on smooth homogeneous spaces G/H and regular integrable Dirac structures L of the Courant algebroid $AG \times_P A^*G$, such that H is the t-connected subgroupoid of G corresponding to the subalgebroid $L \cap (AG \times 0_{A^*G})$. Since pullbacks to G of Poisson structures on G/H correspond to integrable Dirac structures on G with characteristic distribution $\mathcal{H}$,

we recover this result as a special case of Theorem 6.3.4, using Remark 6.3.5 and the isomorphism in Example 5.4.2.
Note that in this particular situation of a Poisson Lie groupoid, Theorem 6.3.4 classifies not only the Poisson homogeneous spaces of $(G \rightrightarrows P, \pi_G)$, but all its (not necessarily integrable) Dirac homogeneous spaces. $\Diamond$

Example 6.3.7 Let $(G \rightrightarrows P, \pi_G)$ be a Poisson groupoid and H a wide subgroupoid of G. Assume that the Poisson structure descends to the quotient G/H, i.e., that π_G is invariant under the action of the bisections of H. Let π be the induced structure on G/H. We show that $(G, q^*\mathsf{D}_\pi)$ is a Dirac homogeneous space of $(G \rightrightarrows P, \pi_G)$. This is equivalent to the fact that $(G/H, \pi)$ is a Poisson homogeneous space of $(G \rightrightarrows P, \pi_G)$.
The Dirac structure $q^*\mathsf{D}_\pi$ is equal to $(\mathcal{H} \times_G 0_{T^*G}) \oplus \operatorname{Graph}\left(\pi_G^\sharp\big|_{\mathcal{H}^\circ} : \mathcal{H}^\circ \to TG\right)$. Since $\mathcal{H} \subseteq T^{\mathsf{t}}G$, the inclusion $\mathbb{T}\mathsf{t}\,(q^*\mathsf{D}_\pi) \subseteq \mathfrak{A}(\mathsf{D}_{\pi_G})$ is obvious. Choose $(v_g, \alpha_g) \in (q^*\mathsf{D}_\pi)(g)$ and $\alpha_h \in T_h^*G$ such that $\mathbb{T}\mathsf{s}\left(\pi_G^\sharp(\alpha_h), \alpha_h\right) = \mathbb{T}\mathsf{t}(v_g, \alpha_g)$. Then we have $(v_g, \alpha_g) = \left(u_g + \pi_G^\sharp(\alpha_g), \alpha_g\right)$ with some $u_g \in \mathcal{H}(g)$ and the product $\left(\pi_G^\sharp(\alpha_h), \alpha_h\right) \star (v_g, \alpha_g)$ is equal to

$$
\begin{aligned}
(\pi_G^\sharp(\alpha_h), \alpha_h) \star (u_g + \pi_G^\sharp(\alpha_g), \alpha_g) &= \left(\pi_G^\sharp(\alpha_h) \star \pi_G^\sharp(\alpha_g) + 0_h \star u_g, \alpha_g \star \alpha_h\right) \\
&= \left(\pi_G^\sharp(\alpha_g \star \alpha_h) + T_g L_h u_g, \alpha_g \star \alpha_h\right)
\end{aligned}
$$

since π_G is multiplicative. The vector $T_g L_h u_g$ is an element of $\mathcal{H}$ by definition and consequently, $(\pi_G^\sharp(\alpha_h), \alpha_h) \star (u_g + \pi_G^\sharp(\alpha_g), \alpha_g)$ is an element of $q^*(\mathsf{D}_\pi)$, which is shown to be π_G-homogeneous. It corresponds to the Lagrangian subalgebroid $(AH \times 0_{T^*P}) \oplus \operatorname{Graph}\left(\pi_G^\sharp\big|_{AH^\circ} : AH^\circ \to TP\right) + I^{\mathsf{s}}(D_{\pi_G})$ of $\mathfrak{B}(\mathsf{D}_{\pi_G})$, or more simply, to the Lagrangian subalgebroid $AH \times_P AH^\circ$ in the Courant algebroid $AG \times_P A^*G$.
Thus, Theorem 6.3.4 together with the isomorphism in Example 5.4.2 shows that the multiplicative Poisson structure on G descends to G/H if and only if the Lagrangian subspace $AH \times_P AH^\circ$ is a subalgebroid of the Courant algebroid $AG \times_P A^*G$.
The Poisson homogeneous space that corresponds in this way to the Lagrangian subalgebroid $AG \times_P 0_{A^*G}$ is the Poisson manifold (P, π_P), where π_P is the Poisson structure induced on P by π_G, see Weinstein (1988) and also Theorem 5.1.11. Note that the other trivial Dirac structure $0_{AG} \times_P A^*G$ corresponds to (G, π_G) seen as a Poisson homogeneous space of $(G \rightrightarrows P, \pi_G)$ (see Example 1.1.24).
In the same manner, we can show that if a Dirac groupoid $(G \rightrightarrows P, \mathsf{D}_G)$ is invariant under the action of a wide subgroupoid H, and the Dirac structure descends to the quotient G/H, then $(G/H, q(\mathsf{D}_G))$ is $(G \rightrightarrows P, \mathsf{D}_G)$-homogeneous. For that, we use the formula $q^*(q(\mathsf{D}_G)) = \mathcal{K}_H + \mathsf{D}_G \cap \mathcal{K}_H^\perp$. In particular, the Dirac structure on P obtained under some regularity conditions in Theorem 5.1.11 is D_G-homogeneous. As in the Poisson case, we find hence that the Dirac structure descends to G/H if and only if

$$
\overline{AH \times_P 0_{T^*P} \oplus \mathfrak{A}(\mathsf{D}_G) \cap (TP \times_P AH^\circ)} \subseteq \mathfrak{B}(\mathsf{D}_G)
$$

is invariant under the induced action of $\mathcal{B}(H)$. $\Diamond$

Example 6.3.8 Let (M, D_M) be a smooth Dirac manifold and $(M \times M \rightrightarrows M, \mathsf{D}_M \ominus \mathsf{D}_M)$ the pair Dirac groupoid associated to it.

The wide Lie subgroupoids of $M \times M \rightrightarrows M$ are the equivalence relations $R \subseteq M \times M$, and the corresponding homogeneous spaces are the products $M \times M/R$. For instance, if $\Phi : G \times M \to M$ is an action of a Lie group G (with Lie algebra $\mathfrak{g}$) on M, the subset $R_G = \{(m, \Phi_g(m)) \mid m \in M, g \in G\}$ is a wide subgroupoid of $M \times M$, and $(M \times M)/R_G$ is easily seen to equal $M \times M/G$. Hence, if the action is free and proper, the homogeneous space $(M \times M)/R_G$ has a smooth manifold structure such that the projection $q : M \times M \to M \times M/G$ is a smooth surjective submersion.

In this case, the bisections of R_G are the diffeomorphisms of M that leave the orbits of G invariant. For instance, for every $g \in G$, the map $K_g : \Delta_M \simeq M \to M \times M$, $m \to (m, \Phi_g(m))$ is a bisection of R_G.

Choose $(m, m) \in \Delta_M$. A vector $(v_m, w_m) \in T_{(m,m)}(M \times M)$ is an element of $T^{\mathsf{t}}_{(m,m)} R_G$ if and only if $v_m = 0_m$ and $(0_m, w_m) \in T_{(m,m)} R_G$, that is if and only if $v_m = 0_m$ and $w_m \in T_m(G \cdot m)$. Thus, if $\mathcal{V}$ is the vertical space of the action, we find that $AR_G = (\{0\} \oplus \mathcal{V})|_{\Delta_M}$.

By Theorem 6.3.1, $\mathsf{D}_M \ominus \mathsf{D}_M$-homogeneous Dirac structures on $M \times M/G$ are in one-one correspondence with Lagrangian subspaces $\mathfrak{D}$ of $\mathsf{P}_G|_P$ such that $I^{\mathsf{s}}(\mathsf{D}_M \ominus \mathsf{D}_M) + (AR_G \times \{0\}) \subseteq \mathfrak{D}$ and such that $\mathfrak{D}/I^{\mathsf{s}}(\mathsf{D}_M \ominus \mathsf{D}_M)$ is invariant under the induced action of $\mathcal{B}(R_G)$ on $\mathfrak{B}(\mathsf{D}_M \ominus \mathsf{D}_M)$. But since $I^{\mathsf{s}}_{(m,m)}(\mathsf{D}_M \ominus \mathsf{D}_M) + A_{(m,m)} R_G = \{(v_m, \xi_M(m), \alpha_m, 0_m) \mid (v_m, \alpha_m) \in \mathsf{D}_M(m),\ \xi \in \mathfrak{g}\}$, we find, using the isomorphism in Example 5.4.4 and the considerations in Example 5.5.6, that $(\mathsf{D}_M \ominus \mathsf{D}_M) \cdot \mathfrak{D}$ is a product of Dirac structures $\mathsf{D}_M \oplus \mathsf{D}$ such that $\mathcal{V} \times_G 0_{T^*G} \subseteq \mathsf{D}$ and $(\Phi_g^* X, \Phi_g^* \alpha) \in \Gamma(\mathsf{D})$ for all $g \in G$ and $(X, \alpha) \in \Gamma(\mathsf{D})$. But Dirac structures D satisfying these conditions are exactly the pullbacks to M of Dirac structures on M/G and we find that the $\mathsf{D}_M \ominus \mathsf{D}_M$-homogeneous Dirac structures on $M \times M/G$ are of the form $\mathsf{D}_M \oplus \bar{\mathsf{D}} := \mathsf{D}_M \oplus q_G(\mathsf{D})$, where $q_G : M \to M/G$ is the canonical projection.

With Example 5.4.4 and Theorem 6.3.3, we get hence that $\bar{\mathsf{D}}$ is integrable if and only if D is integrable. $\Diamond$

Example 6.3.9 Recall from Example 4.2.16 that the left invariant Dirac structures on a Lie group G are the homogeneous structures relative to the trivial Poisson bracket on G. Hence, if we consider this example in the groupoid situation, we should recover the "right" definition for left invariant Dirac structures on a Lie groupoid. We say that a Dirac structure D on a Lie groupoid $G \rightrightarrows P$ is *left-invariant* if the action $\mathbb{T}\Phi$ of $TG \times_G T^*G$ on $\mathbb{T}\mathsf{t} : TG \times_G T^*G \to TP \times_P A^*G$ restricts to an action of $0_{TG} \times_G T^*G$ on D, i.e.,

$$(0_{TG} \times_G T^*G) \cdot \mathsf{D} = \mathsf{D}.$$

In Liu et al. (1998), a Dirac structure on a Lie groupoid $G \rightrightarrows P$ is said to be left-invariant if it is the pullback under the map

$$\begin{array}{rcll} \Phi : T^{\mathsf{t}}G \times_G T^*G & \to & AG \times_P A^*G & \\ (v_g, \alpha_g) & \mapsto & (T_g L_{g^{-1}} v_g, \hat{\mathsf{s}}(\alpha_g)) & \in A_{\mathsf{s}(g)}G \times A^*_{\mathsf{s}(g)}G \end{array}$$

of a Dirac structure in $AG \times_P A^*G$. These two definitions are easily seen to be equivalent, the inclusion $0_{TG} \times_G (T^{\mathsf{t}}G)^\circ \subseteq \mathsf{D}$ is immediate and it is easy to check that D is invariant under the lifted right actions of the bisections if and only if the corresponding Dirac structure in $\mathfrak{B}(T^*G)$ is invariant under the induced action of $\mathcal{B}(G)$ on $\mathfrak{B}(T^*G)$ (compare with Proposition 6.2 in Liu et al. (1998)).

The result in Theorem 6.3.3 implies that a left-invariant Dirac structure D is integrable if and only if the corresponding Dirac structure $\Phi(\mathsf{D}|_P) \subseteq AG \times_P A^*G$ is a subalgebroid. $\Diamond$

Conclusion and open problem

In this thesis, we have classified the (integrable) Dirac homogeneous spaces of an (integrable) Dirac Lie group via Lagrangian subbundles (subalgebras) of the double of its Lie bialgebra in a first step, and then generalized this result to a classification of the Dirac homogeneous spaces of Dirac groupoids.
Doing this, we have studied properties of multiplicative structures on Lie groupoids that were necessary for the classification, but are also of independent interest.

We also have shown how the leaf space of a multiplicative, involutive subbundle of the tangent space of a Lie groupoid inherits under certain regularity conditions a groupoid structure such that the projection is a groupoid morphism. This is a fact that can easily be seen in the case of a simply-connected Lie group, since a multiplicative distribution is then automatically the left- and right-invariant image of an ideal in the Lie algebra of the Lie group.

We have observed that an (integrable) Dirac groupoid gives rise to infinitesimal objects (Sections 5.2, 5.3, 5.4). A natural question is in what sense such infinitesimal objects determine infinitesimal invariants of a multiplicative Dirac structure.
The infinitesimal data of Dirac Lie groupoids has been identified in Ortiz (2009) in the following manner. Since a multiplicative Dirac structure D_G on $G \rightrightarrows P$ is by definition a subgroupoid of $(TG \times_G T^*G) \rightrightarrows (TP \times_P A^*G)$, its Lie algebroid is a subalgebroid of $A(TG \times_G T^*G) \simeq T(AG) \times_{AG} T^*(AG)$. Ortiz shows that $A(\mathsf{D}_G)$ is a Dirac structure on AG, that is integrable (as a Dirac structure) if and only if D_G is. The Dirac structures on AG are thus identified as the infinitesimal objects in one-to-one correspondence with the multiplicative Dirac structures on $G \rightrightarrows P$. Yet, in the case of a Poisson groupoid, this doesn't correspond to the Lie bialgebroids (AG, A^*G) that are in one-one correspondence with the Poisson groupoids $(G \rightrightarrows P, \pi_G)$ and hence known as their infinitesimal data, but to an intermediate step in the reconstruction of the multiplicative Poisson structure from the compatible Lie algebroids in duality (see Mackenzie and Xu (2000)).
In Chapter 5, we have shown that if D_G is a multiplicative Dirac structure on a Lie groupoid $G \rightrightarrows P$, then is restriction to the submanifold of units P splits as the direct sum of two vector bundles $\mathfrak{A}(\mathsf{D}_G)$ and $I^{\mathsf{s}}(\mathsf{D}_G)$ over P (the units and the *core* of D_G), that inherit Lie algebroid structures from D_G if D_G is integrable. The Lie algebroid structure that we find on the units $\mathfrak{A}(\mathsf{D}_G)$ was predicted by Ortiz (2009) and generalizes the fact that the dual bundle $A^*G \to P$ of the Lie algebroid associated to a Lie groupoid endowed with a multiplicative Poisson structure inherits the structure of a Lie algebroid over P. Indeed, the data that we find in the Poisson case is the Lie bialgebroid of the Poisson groupoid; $\mathfrak{A}(\mathsf{D}_{\pi_G}) = \operatorname{graph}(\mathsf{a}_* : A^*G \to TP)$. If $(G \rightrightarrows P, \mathsf{D}_G)$ is an integrable Dirac Lie

groupoid, the integrability of the Dirac structure is completely encoded in the square of morphisms of Lie algebroids over P that we have found:

$$\begin{tikzcd}
I^{\mathsf{s}}(\mathsf{D}_G) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathfrak{A}(\mathsf{D}_G) \arrow[dd, "\mathsf{a}_\star"] \arrow[dl] \\
& P & \\
AG \arrow[rr, "\mathsf{a}"'] \arrow[ur] & & TP \arrow[ul]
\end{tikzcd}$$

(see Section 5.3). In the Poisson and presymplectic groupoid cases, we know by the results in Mackenzie and Xu (2000) and Bursztyn et al. (2004) that the whole data of the multiplicative structures is encoded in this square, together with some properties.
It would be interesting to study to what extend a more general Dirac groupoid can be recovered from this square of lie algebroids, and, if possible, to prove a general classification theorem for Dirac groupoids in terms of these Lie algebroids, that would have the results in the Poisson and presymplectic cases as corollaries. The deeper understanding of the Courant algebroid $\mathfrak{B}(\mathsf{D}_G)$, which is still rather mysterious although it plays a crucial role as the ambient object for the classification theorem in Chapter 6, should be helpful in doing this.

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Poisson reduction by distributions (with Tudor Ratiu) “Letters in Mathematical Physics” (2009), Volume 87, numbers 1–2.

Singular reduction of Dirac structures (with Tudor Ratiu and Jedrzej Sniatycki), “Transactions of the AMS” (2011), Volume 363, 2967-3013.

Induced Dirac structure on isotropy type manifolds (with Tudor Ratiu), “Transformation groups” (2011), Volume 16, Number 1, 175-191.

Dirac Lie groups, Dirac homogeneous spaces and the Theorem of Drinfel'd, to appear in “Indiana University Mathematics Journal”.

Invariant frames for vector bundles and applications (with Tudor Ratiu and Marco Zambon), to appear in “Geometriae Dedicata”.

All available on `http://cag.epfl.ch/page31999.html` or arxiv.org.

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A classification theorem for Dirac homogeneous spaces of Dirac groupoids (2010).

The leaf space of a multiplicative foliation (2010).

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