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COMPUTING PERIODIC DEFLATING SUBSPACES ASSOCIATED WITH A SPECIFIED SET OF EIGENVALUES*,**

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Abstract.

We present a direct method for reordering eigenvalues in the generalized periodic real Schur form of a regular K-cyclic matrix pair sequence (A_k, E_k). Following and generalizing existing approaches, reordering consists of consecutively computing the solution to an associated Sylvester-like equation and constructing K pairs of orthogonal matrices. These pairs define an orthogonal K-cyclic equivalence transformation that swaps adjacent diagonal blocks in the Schur form. An error analysis of this swapping procedure is presented, which extends existing results for reordering eigenvalues in the generalized real Schur form of a regular pair (A, E). Our direct reordering method is used to compute periodic deflating subspace pairs corresponding to a specified set of eigenvalues. This computational task arises in various applications related to discrete-time periodic descriptor systems. Computational experiments confirm the stability and reliability of the presented eigenvalue reordering method.

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1 Introduction.

Discrete-time periodic descriptor systems of the form

$$\begin{aligned} E_k x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k + D_k u_k, \end{aligned} \tag{1.1}$$

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with $A_k = A_{k+K}$, $E_k = E_{k+K} \in \mathbb{R}^{n\times n}$, $B_k = B_{k+K} \in \mathbb{R}^{n\times m}$, $C_k = C_{k+K} \in \mathbb{R}^{r\times n}$ and $D_k = D_{k+K} \in \mathbb{R}^{r\times m}$ for some period $K \geq 1$ arise naturally from processes that exhibit seasonal or periodic behavior [6]. Design and analysis problems of such systems (see, e.g., [31, 32, 39]) are conceptually studied in terms of *state transition matrices* [39] $\Phi_{E^{-1}A}(j,i) = E_{j-1}^{-1}A_{j-1}E_{j-2}^{-1}A_{j-2}\dots E_i^{-1}A_i \in \mathbb{R}^{n\times n}$, with the convention $\Phi_{E^{-1}A}(i,i) = I_n$. A state transition matrix over a complete period $\Phi_{E^{-1}A}(j+K,j)$ is the *monodromy matrix* of (1.1) at time j. Its eigenvalues are called the *characteristic multipliers* and are independent of the time j. Specifically, the monodromy matrix at time $j = 0$ corresponds to the matrix product

$$E_{K-1}^{-1}A_{K-1}E_{K-2}^{-1}A_{K-2}\cdots E_1^{-1}A_1E_0^{-1}A_0. \tag{1.2}$$

Matrix products of the general form (1.2) are studied, e.g., in [3, 5, 26, 40].

We study the K-cyclic matrix pair sequence (A_k, E_k) with $A_k, E_k \in \mathbb{R}^{n\times n}$ from (1.1) via the generalized periodic Schur decomposition [8, 18]: there exists a K-cyclic orthogonal matrix pair sequence (Q_k, Z_k) with $Q_k, Z_k \in \mathbb{R}^{n\times n}$ such that, given $k \oplus 1 = (k+1) \bmod K$, we have

$$\begin{cases} S_k = Q_k^T A_k Z_k, \\ T_k = Q_k^T E_k Z_{k\oplus 1}, \end{cases} \tag{1.3}$$

where all matrices S_k, except for some fixed index j with $0 \leq j \leq K-1$, and all matrices T_k are upper triangular. The matrix S_j is upper quasi-triangular; typically j is chosen to be 0 or $K-1$. The sequence (S_k, T_k) is the *generalized periodic real Schur form* (GPRSF) of (A_k, E_k), $k = 0, 1, \dots, K-1$. The decomposition (1.3) is a *K-cyclic equivalence transformation* of the matrix pair sequence (A_k, E_k).

Computing the GPRSF is the standard method for solving the *generalized periodic (product) eigenvalue problem* (GPEVP)

$$E_{K-1}^{-1}A_{K-1}E_{K-2}^{-1}A_{K-2}\cdots E_1^{-1}A_1E_0^{-1}A_0x = \lambda x, \tag{1.4}$$

where all matrices in the pairs (A_k, E_k) are general and dense. For $K = 1$, (1.4) corresponds to a generalized eigenvalue problem $Ax = \lambda Ex$ with (A, E) regular (see, e.g., [12]). Using the GPRSF to solve a GPEVP for $K \geq 1$ means that we do not need to compute any matrix products in (1.4) explicitly, which avoids numerical instabilities and allows to handle singular factors E_k.

The 1×1 and 2×2 blocks on the diagonal of a GPRSF define $t \leq n$ K-cyclic diagonal block pairs $(S_{ii}^{(k)}, T_{ii}^{(k)})$, corresponding to *real eigenvalues* and *complex conjugate pairs of eigenvalues*, respectively.

A real eigenvalue is simply given by

$$\lambda_i = \frac{S_{ii}^{(K-1)}}{T_{ii}^{(K-1)}}\frac{S_{ii}^{(K-2)}}{T_{ii}^{(K-2)}}\cdots\frac{S_{ii}^{(0)}}{T_{ii}^{(0)}}.$$

This eigenvalue is called *infinite* if $\prod_{k=0}^{K-1} T_{ii}^{(k)} = 0$ but $\prod_{k=0}^{K-1} S_{ii}^{(k)} \neq 0$. If there are 1×1 blocks for which both $\prod_{k=0}^{K-1} S_{ii}^{(k)} = 0$ and $\prod_{k=0}^{K-1} T_{ii}^{(k)} = 0$ then the K-cyclic matrix pair sequence (A_k, E_k) is called *singular*, otherwise the sequence (A_k, E_k) is called *regular*. In the degenerate singular case, the eigenvalues become ill-defined and other tools [28, 37] need to be used to study the periodic eigenvalue problem. For the rest of the paper, it is therefore assumed that (A_k, E_k) is regular.

For two complex conjugate eigenvalues $\lambda_i, \bar{\lambda}_i$, all matrices $T_{ii}^{(k)}$ are nonsingular and

$$\lambda_i, \bar{\lambda}_i \in \lambda\big(T_{ii}^{(K-1)^{-1}} S_{ii}^{(K-1)} T_{ii}^{(K-2)^{-1}} S_{ii}^{(K-2)} \cdots T_{ii}^{(0)^{-1}} S_{ii}^{(0)}\big),$$

where $\lambda(M)$ denotes the set of eigenvalues of a matrix M. In finite precision arithmetic, great care has to be exercised to avoid underflow and overflow in the explicit eigenvalue computation, especially when it involves 2×2 blocks [35].

For every l with $1 \leq l \leq n$ such that no 2×2 block resides in $S_j(l : l+1, l : l+1)$, the first l pairs of columns of (Q_0, Z_0) span a *deflating subspace pair* corresponding to the first l eigenvalues of the matrix product (1.2). More generally, the first l pairs of columns of (Q_k, Z_k) span a *left and right periodic (or cyclic) deflating subspace pair* sequence associated with the first l eigenvalues of the matrix product (1.2) [5].

The decomposition (1.3) is computed via the periodic QZ algorithm (see, e.g., [8, 18, 24, 25]), which consists of an initial reduction to generalized periodic Hessenberg form and a subsequent iterative process to generalized periodic Schur form. In [38], the generalized periodic Schur form is extended to periodic matrix pairs with time-varying and possibly rectangular dimensions. This includes a preprocessing step that truncates parts corresponding to spurious characteristic values, which then yields square system matrices of constant dimensions.

1.1 Ordered GPRSF and periodic deflating subspaces.

In many applications, it is desirable to have the eigenvalues along the diagonal of the GPRSF in a certain order. If the generalized periodic Schur form has its eigenvalues ordered in a certain way as in (1.5), it is called an *ordered GPRSF*. For example, if we have

$$S_k = \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix}, \quad T_k = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix}, \tag{1.5}$$

with $S_{11}^{(k)}$, $T_{11}^{(k)} \in \mathbb{R}^{l\times l}$ such that the upper left part sequence $(S_{11}^{(k)}, T_{11}^{(k)})$ contains all eigenvalues in the open unit disc, then (S_k, T_k) is an ordered GPRSF and the first l columns of the sequence Z_k span *stable* right periodic deflating subspaces. For initial states $x_0 \in \operatorname{span}(Z_0 e_1, \ldots, Z_0 e_l)$ with e_i being the

ith unit vector, the states of the homogeneous system $E_k x_{k+1} = A_k x_k$ satisfy $x_k \in \text{span}(Z_k e_1, \ldots, Z_k e_l)$ and 0 is an asymptotically stable equilibrium.

Other important applications relating to periodic discrete-time systems include the *stable-unstable spectral separation* for computing the numerical solution of the discrete-time periodic Riccati equation [38] in LQ-optimal control, which we illustrate in Section 2, and pole placement where the goal is to move some or all of the poles to desired locations in the complex plane [29, 15]. In [4], ordered Schur forms are used for solving generalized Hamiltonian eigenvalue problems.

In this paper, we extend the work in [2, 14, 21, 25, 15] to perform eigenvalue reordering in a regular periodic matrix pair sequence in GPRSF.

The rest of the paper is organized as follows. In Section 2, we illustrate how an ordered GPRSF can be used to solve the discrete-time periodic Riccati equation that arises in an LQ-optimal control problem. Section 3 presents our direct method for reordering eigenvalues of a periodic (cyclic) matrix pair sequence (A_k, E_k) in GPRSF. To compute an ordered GPRSF, a method for reordering adjacent K-cyclic diagonal block pairs is combined with a bubble-sort like procedure in an LAPACK-style [1, 2, 23] fashion. The proposed method for swapping adjacent diagonal block pair sequences relies on orthogonal K-cyclic equivalence transformations and the core step consists of computing the solution to an associated periodic generalized coupled Sylvester equation, which is discussed in Section 3.4. An error analysis of the direct reordering method is presented in Section 5, which extends and generalizes results from [21, 14]. In Section 6, we discuss some implementation issues regarding the solution of small-sized periodic generalized coupled Sylvester equations and how we control and guarantee stability of the reordering. Some examples and computational results are presented and discussed in Section 7. Finally, in Section 8 we discuss some extensions of the reordering method.

2 LQ-optimal control and periodic deflating subspaces.

Given the system (1.1), the aim of *linear quadratic* (LQ) *optimal control* is to find a feedback sequence u_k which stabilizes the system and minimizes the functional

$$\frac{1}{2}\sum_{k=0}^{\infty}\left(x_k^T H_k x_k + u_k^T N_k u_k\right),$$

with $H_k \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite and $N_k \in \mathbb{R}^{m\times m}$ symmetric positive definite. Moreover, we suppose that the weighting matrices are K-periodic, i.e., $H_{k+K} = H_k$ and $N_{k+K} = N_k$. Under mild assumptions [7], the optimal feedback is linear and unique. For each k, it can be expressed as

$$u_k^\star = -\left(N_k + B_k^T X_{k+1} B_k\right)^{-1} B_k^T X_{k+1} A_k x_k,$$

where $X_k = X_{k+K}$ is the unique symmetric positive semidefinite solution of the *discrete-time periodic Riccati equation* (DPRE) [18]

$$\begin{aligned} 0 = {} & C_k^T H_k C_k - E_{k-1}^T X_k E_{k-1} + A_k^T X_{k+1} A_k \\ & - A_k^T X_{k+1} B_k \big(N_k + B_k^T X_{k+1} B_k\big)^{-1} B_k^T X_{k+1} A_k, \end{aligned} \tag{2.1}$$

provided that all E_k are invertible. The $2n \times 2n$ periodic matrix pair

$$(L_k, M_k) = \left(\begin{bmatrix} A_k & 0 \\ -C_k^T H_k C_k & E_{k-1}^T \end{bmatrix}, \begin{bmatrix} E_{k-1} & B_k N_k^{-1} B_k^T \\ 0 & A_k^T \end{bmatrix} \right)$$

is closely associated with (2.1). Similarly as for the case $E_k = I_n$ [18], it can be shown that this pair has exactly n eigenvalues inside the unit disk under the assumption that (1.1) is *d-stabilizable* and *d-detectable*. By reordering the GPRSF of (L_k, M_k) we can compute a periodic deflating subspace defined by the orthogonal matrices $U_k, V_k \in \mathbb{R}^{2n \times 2n}$ with $U_{k+K} = U_k$, $V_{k+K} = V_k$ such that

$$U_k^T L_k V_k = \begin{bmatrix} S_{11}^{(k)} & S_{12}^{(k)} \\ 0 & S_{22}^{(k)} \end{bmatrix}, \quad U_k^T M_k V_{k+1} = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} \\ 0 & T_{22}^{(k)} \end{bmatrix},$$

where the $n \times n$ periodic matrix pair $(S_{11}^{(k)}, T_{11}^{(k)})$ contains all eigenvalues inside the unit disk. If we partition

$$U_k = \begin{bmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{bmatrix}$$

with $U_{ij}^{(k)} \in \mathbb{R}^{n \times n}$, then

$$U_{21}^{(k)} \left[U_{11}^{(k)} \right]^{-1} = X_k E_{k-1},$$

from which X_k can be computed. The proof of this relation is similar as for the case $K = 1$, see, e.g., [27]. We note that if N_k is not well-conditioned then it is better to work with $(2n + m) \times (2n + m)$ matrix pairs, as described in [27].

3 Direct method for eigenvalue reordering in GPRSF.

Given a regular K-cyclic matrix pair sequence (A_k, E_k) in GPRSF, our method to compute an ordered GPRSF (1.5) with respect to a set of specified eigenvalues reorders 1×1 and 2×2 diagonal blocks in the GPRSF such that the selected set of eigenvalues appears in the matrix pair sequence $(S_{11}^{(k)}, T_{11}^{(k)})$. Following LAPACK, we assume that the set of specified eigenvalues are provided as an index vector for the blocks of eigenvalue pairs that should appear in $(S_{11}^{(k)}, T_{11}^{(k)})$. The procedure is now to swap adjacent diagonal blocks in the GPRSF in a bubble-sort fashion such that the specified eigenvalue ordering is satisfied [1, 2, 23]. In the following, we focus on the K-cyclic swapping of diagonal blocks using orthogonal transformations.

3.1 Swapping of K-cyclic diagonal block matrix pairs.

Consider a regular K-cyclic matrix pair sequence (A_k, E_k) in GPRSF

$$(A_k, E_k) = \left(\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix}, \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} \right) \tag{3.1}$$

with $A_{11}^{(k)}, E_{11}^{(k)} \in \mathbb{R}^{p_1 \times p_1}$ and $A_{22}^{(k)}, E_{22}^{(k)} \in \mathbb{R}^{p_2 \times p_2}$, for $k = 0, 1, \ldots, K-1$.

Swapping consists of computing orthogonal matrices U_k, V_k such that

$$\begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ 0 & \hat{A}_{22}^{(k)} \end{bmatrix} = U_k^T \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} V_k, \tag{3.2}$$

$$\begin{bmatrix} \hat{E}_{11}^{(k)} & \hat{E}_{12}^{(k)} \\ 0 & \hat{E}_{22}^{(k)} \end{bmatrix} = U_k^T \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} V_{k \oplus 1}, \tag{3.3}$$

for $k = 0, \ldots, K-1$, and

$$\lambda(\hat{\Pi}_{11}) = \lambda(\Pi_{22}), \quad \lambda(\hat{\Pi}_{22}) = \lambda(\Pi_{11}), \tag{3.4}$$

where

$$\Pi_{ii} = \left[E_{ii}^{(K-1)}\right]^{-1} A_{ii}^{(K-1)} \cdots \left[E_{ii}^{(0)}\right]^{-1} A_{ii}^{(0)}, \tag{3.5}$$

$$\hat{\Pi}_{ii} = \left[\hat{E}_{ii}^{(K-1)}\right]^{-1} \hat{A}_{ii}^{(K-1)} \cdots \left[\hat{E}_{ii}^{(0)}\right]^{-1} \hat{A}_{ii}^{(0)}. \tag{3.6}$$

If some of the $E_{ii}^{(k)}$ are singular then the products (3.5) and (3.6) should only be understood in a formal sense, with their finite and infinite eigenvalues defined via the GPRSF. The relation (3.4) means that all eigenvalues of Π_{22} are transferred to $\hat{\Pi}_{11}$ and all eigenvalues of Π_{11} to $\hat{\Pi}_{22}$. For our purpose, $A_{ii}^{(k)}, E_{ii}^{(k)} \in \mathbb{R}^{p_i \times p_i}$ are the diagonal blocks of a GPRSF and it can thus be assumed that $p_i \in \{1, 2\}$.

The K-cyclic swapping is performed in two main steps. First, the sequence (A_k, E_k) in (3.1) is block diagonalized by a nonorthogonal K-cyclic equivalence transformation. Second, orthogonal transformation matrices are computed from this matrix pair sequence that perform the required K-cyclic swapping.

3.2 Swapping by block diagonalization and permutation.

Let us consider a K-cyclic matrix pair sequence (L_k, R_k), with $L_k, R_k \in \mathbb{R}^{p_1 \times p_2}$, which solves the *periodic generalized coupled Sylvester equation* (PGCSY)

$$\begin{cases} A_{11}^{(k)} R_k - L_k A_{22}^{(k)} = -A_{12}^{(k)}, \\ E_{11}^{(k)} R_{k \oplus 1} - L_k E_{22}^{(k)} = -E_{12}^{(k)}. \end{cases} \tag{3.7}$$

Then (L_k, R_k) defines an equivalence transformation that block diagonalizes the K-cyclic matrix pair sequence (A_k, E_k) in (3.1):

$$\begin{aligned}
\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} &= \begin{bmatrix} I_{p_1} & L_k \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} A_{11}^{(k)} & 0 \\ 0 & A_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I_{p_1} & -R_k \\ 0 & I_{p_2} \end{bmatrix}, \\
\begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} &= \begin{bmatrix} I_{p_1} & L_k \\ 0 & I_{p_2} \end{bmatrix} \begin{bmatrix} E_{11}^{(k)} & 0 \\ 0 & E_{22}^{(k)} \end{bmatrix} \begin{bmatrix} I_{p_1} & -R_{k\oplus 1} \\ 0 & I_{p_2} \end{bmatrix},
\end{aligned} \tag{3.8}$$

for $k = 0, 1, \ldots, K-1$.

The diagonal blocks of the block diagonal matrices in (3.8) are swapped by a simple equivalence permutation:

$$\begin{aligned}
&\begin{bmatrix} 0 & I_{p_2} \\ I_{p_1} & 0 \end{bmatrix} \left(\begin{bmatrix} A_{11}^{(k)} & 0 \\ 0 & A_{22}^{(k)} \end{bmatrix}, \begin{bmatrix} E_{11}^{(k)} & 0 \\ 0 & E_{22}^{(k)} \end{bmatrix} \right) \begin{bmatrix} 0 & I_{p_1} \\ I_{p_2} & 0 \end{bmatrix} \\
&\qquad = \left(\begin{bmatrix} A_{22}^{(k)} & 0 \\ 0 & A_{11}^{(k)} \end{bmatrix}, \begin{bmatrix} E_{22}^{(k)} & 0 \\ 0 & E_{11}^{(k)} \end{bmatrix} \right).
\end{aligned} \tag{3.9}$$

Altogether, by defining the matrices

$$\boldsymbol{X}_k = \begin{bmatrix} L_k & I_{p_1} \\ I_{p_2} & 0 \end{bmatrix}, \quad \boldsymbol{Y}_k = \begin{bmatrix} 0 & I_{p_2} \\ I_{p_1} & -R_k \end{bmatrix}, \quad k = 0, \ldots, K-1, \tag{3.10}$$

we obtain a *non-orthogonal* K-cyclic equivalence transformation such that

$$\begin{aligned}
\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} &= \boldsymbol{X}_k \begin{bmatrix} A_{22}^{(k)} & 0 \\ 0 & A_{11}^{(k)} \end{bmatrix} \boldsymbol{Y}_k, \\
\begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} &= \boldsymbol{X}_k \begin{bmatrix} E_{22}^{(k)} & 0 \\ 0 & E_{11}^{(k)} \end{bmatrix} \boldsymbol{Y}_{k\oplus 1}.
\end{aligned} \tag{3.11}$$

It remains to show the existence of a solution to (3.7).

LEMMA 3.1. *Let the K-cyclic matrix sequences $(A_{11}^{(k)}, B_{11}^{(k)})$ and $(A_{22}^{(k)}, B_{22}^{(k)})$ be regular. Then the PGCSY* (3.7) *has a unique solution if and only if*

$$\lambda(\Pi_{11}) \cap \lambda(\Pi_{22}) = \emptyset, \tag{3.12}$$

where Π_{ii} is the formal matrix product defined in (3.5).

PROOF. Since (3.7) is a system of $2p_1p_2K$ linear equations in $2p_1p_2K$ variables, it suffices to show that the corresponding linear operator $\mathcal{L} : (\mathbb{R}^{p_1\times p_2})^{2K} \to (\mathbb{R}^{p_1\times p_2})^{2K}$, defined by

$$\mathcal{L} : (L_k, R_k)_{k=0}^{K-1} \mapsto \left(A_{11}^{(k)} R_k - L_k A_{22}^{(k)}, E_{11}^{(k)} R_{k\oplus 1} - L_k E_{22}^{(k)}\right)_{k=0}^{K-1} \tag{3.13}$$

has a trivial kernel if and only if (3.12) is satisfied.

1. Let $\lambda \in \lambda(\Pi_{11}) \cap \lambda(\Pi_{22})$ and assume $\lambda \neq \infty$ (the case $\lambda = \infty$ can be treated analogously by switching the roles of E and A, and reversing the index k). By the complex periodic Schur decomposition, there are sequences of nonzero, right and left eigenvectors $x_0, \ldots, x_{K-1} \in \mathbb{C}^{p_1}$, $y_0, \ldots, y_{K-1} \in \mathbb{C}^{p_2}$ satisfying

$$\lambda_k E_{11}^{(k)} x_{k\oplus 1} = A_{11}^{(k)} x_k, \quad \mu_k y_k^H E_{22}^{(k\ominus 1)} = y_{k\oplus 1}^H A_{22}^{(k)}, \tag{3.14}$$

for $k = 0, \ldots, K-1$, where

$$\lambda = \lambda_0 \cdots \lambda_{K-1} = \mu_0 \cdots \mu_{K-1} \tag{3.15}$$

and $E_{11}^{(k)} x_{k\oplus 1} \neq 0, y_k^H E_{22}^{(k\ominus 1)} \neq 0$. Here, $k \ominus 1$ denotes $(k-1)$ mod K. The relation (3.15) implies the existence of a sequence $\gamma_0, \ldots, \gamma_{K-1} \in \mathbb{C}$ such that

$$\gamma_k \lambda_k = \gamma_{k\oplus 1} \mu_k, \quad k = 0, \ldots, K-1, \tag{3.16}$$

with at least one γ_k being nonzero. Defining

$$R_k = \gamma_k x_k y_k^H E_{22}^{(k\ominus 1)}, \quad L_k = \gamma_{k\oplus 1} E_{11}^{(k)} x_{k\oplus 1} y_{k\oplus 1}^H,$$

this guarantees that at least one of the matrices R_k and L_k is nonzero. Moreover, (3.14) and (3.16) yield

$$\begin{aligned} A_{11}^{(k)} R_k - L_k A_{22}^{(k)} &= \gamma_k A_{11}^{(k)} x_k y_k^H E_{22}^{(k\ominus 1)} - \gamma_{k\oplus 1} E_{11}^{(k)} x_{k\oplus 1} y_{k\oplus 1}^H A_{22}^{(k)} \\ &= (\gamma_k \lambda_k - \gamma_{k\oplus 1} \mu_k) E_{11}^{(k)} x_{k\oplus 1} y_k^H E_{22}^{(k\ominus 1)} = 0, \\ E_{11}^{(k)} R_{k\oplus 1} - L_k E_{22}^{(k)} &= \gamma_{k\oplus 1} E_{11}^{(k)} x_{k\oplus 1} y_{k\oplus 1}^H E_{22}^{(k)} - \gamma_{k\oplus 1} E_{11}^{(k)} x_{k\oplus 1} y_{k\oplus 1}^H E_{22}^{(k)} \\ &= 0. \end{aligned}$$

Hence, the kernel of $\mathcal{L}$ is nonzero if (3.12) is not satisfied.

2. For the other direction of the proof, assume that (3.12) is satisfied. We first treat the case when all coefficient matrices are of order 1, i.e., we consider

$$\begin{cases} \alpha_1^{(k)} r_k - l_k \alpha_2^{(k)} = 0, \\ \beta_1^{(k)} r_{k\oplus 1} - l_k \beta_2^{(k)} = 0, \end{cases} \tag{3.17}$$

with scalars $\alpha_j^{(k)}$ and $\beta_j^{(k)}$. Because of (3.12), one of the products $\beta_1^{(0)} \cdots \beta_1^{(K-1)}$ or $\beta_2^{(0)} \cdots \beta_2^{(K-1)}$ must be nonzero. Without loss of generality, we may assume that $\beta_2^{(0)} \cdots \beta_2^{(K-1)} \neq 0$. Then (3.17) implies

$$\alpha_1^{(k)} r_k = \frac{\alpha_2^{(k)} \beta_1^{(k)}}{\beta_2^{(k)}} r_{k\oplus 1}, \quad k = 0, \ldots, K-1. \tag{3.18}$$

Recursively substituting r_k and $r_{k\oplus 1}$ yields

$$\left(\alpha_1^{(0)}\cdots\alpha_1^{(K-1)}\right)r_0 = \frac{\left(\alpha_2^{(0)}\cdots\alpha_2^{(K-1)}\right)\left(\beta_1^{(0)}\cdots\beta_1^{(K-1)}\right)}{\left(\beta_2^{(0)}\cdots\beta_2^{(K-1)}\right)}r_0.$$

The regularity assumption implies that one of $\alpha_1^{(0)}\cdots\alpha_1^{(K-1)}$ or $\beta_1^{(0)}\cdots\beta_1^{(K-1)}$ is nonzero. Together with (3.12), this implies $r_0 = 0$, and in combination with (3.18) we get $r_k = 0$ for all $k = 0,\ldots,K-1$. In addition, from (3.17) we have

$$l_k = \frac{\beta_1^{(k)}}{\beta_2^{(k)}}r_{k\oplus 1}, \quad k = 0,\ldots,K-1,$$

which in turn results in $l_k = 0$.

For coefficient matrices of larger order, we proceed by induction. By the complex Schur decomposition, we may assume that $A_{jj}^{(k)}$ and $E_{jj}^{(k)}$ are upper triangular. Conformably partition

$$L^{(k)} = \begin{bmatrix} L_{11}^{(k)} & L_{12}^{(k)} \\ L_{21}^{(k)} & L_{22}^{(k)} \end{bmatrix}, \quad R^{(k)} = \begin{bmatrix} R_{11}^{(k)} & R_{12}^{(k)} \\ R_{21}^{(k)} & R_{22}^{(k)} \end{bmatrix},$$

$$A_{11}^{(k)} = \begin{bmatrix} \bar{A}_{11}^{(k)} & \bar{A}_{12}^{(k)} \\ 0 & \bar{A}_{22}^{(k)} \end{bmatrix}, \quad A_{22}^{(k)} = \begin{bmatrix} \bar{A}_{33}^{(k)} & \bar{A}_{34}^{(k)} \\ 0 & \bar{A}_{44}^{(k)} \end{bmatrix},$$

and in an analogous manner $E_{jj}^{(k)}$. Then (3.7) with the right hand sides replaced by zero yields

$$\bar{A}_{22}^{(k)}R_{21}^{(k)} - L_{21}^{(k)}\bar{A}_{33}^{(k)} = 0, \quad \bar{E}_{22}^{(k)}R_{21}^{(k\oplus 1)} - L_{21}^{(k)}\bar{E}_{33}^{(k)} = 0.$$

By the induction assumption, we have $L_{21}^{(k)} = R_{21}^{(k)} = 0$ for all k. Subsequently, analogous periodic PGCSYs of smaller order can be found for $(L_{11}^{(k)}, R_{11}^{(k)})$, $(L_{22}^{(k)}, R_{22}^{(k)})$, and $(L_{12}^{(k)}, R_{12}^{(k)})$, see also [13], eventually showing that $L^{(k)} = R^{(k)} = 0$. This completes the proof. □

Related periodic Sylvester equations were also studied in, e.g., [30, 36] and an overview was given in [39]. For a recursive solution method based on the last part of the proof of Lemma 3.1, see [13].

3.3 Swapping by orthogonal transformation matrices.

From the definition (3.10), it can be observed that the first block column of $\boldsymbol{X}_k$ and the last block row of $\boldsymbol{Y}_k$ have full column and row ranks, respectively. Hence, if we choose orthogonal matrices Q_k and Z_k from QR and RQ factorizations such that

$$\begin{bmatrix} L_k \\ I_{p_2} \end{bmatrix} = Q_k \begin{bmatrix} T_L^{(k)} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} I_{p_1} & -R_k \end{bmatrix} = \begin{bmatrix} 0 & T_R^{(k)} \end{bmatrix} Z_k^T, \tag{3.19}$$

then $T_L^{(k)} \in \mathbb{R}^{p_2 \times p_2}, T_R^{(k)} \in \mathbb{R}^{p_1 \times p_1}$ are not only upper triangular but also nonsingular for $k = 0, 1, \ldots, K-1$.

Partitioning Q_k and Z_k in conformity with $\boldsymbol{X}_k$ and $\boldsymbol{Y}_k$ as

$$Q_k = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} \\ Q_{21}^{(k)} & Q_{22}^{(k)} \end{bmatrix}, \quad Z_k = \begin{bmatrix} Z_{11}^{(k)} & Z_{12}^{(k)} \\ Z_{21}^{(k)} & Z_{22}^{(k)} \end{bmatrix},$$

we obtain

$$Q_k^T \boldsymbol{X}_k = \begin{bmatrix} T_L^{(k)} & {Q_{12}^{(k)}}^T \\ 0 & {Q_{22}^{(k)}}^T \end{bmatrix}, \quad \boldsymbol{Y}_k Z_k = \begin{bmatrix} Z_{21}^{(k)} & Z_{22}^{(k)} \\ 0 & T_R^{(k)} \end{bmatrix}. \tag{3.20}$$

By applying (Q_k, Z_k) as an *orthogonal* K-cyclic equivalence transformation to (A_k, E_k) we obtain

$$\begin{aligned} &\left(Q_k^T A_k Z_k, Q_k^T E_k Z_{k\oplus 1}\right) \\ &\quad = \left(Q_k^T \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} Z_k, Q_k^T \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} Z_{k\oplus 1} \right) \\ &\quad = \left(Q_k^T \boldsymbol{X}_k \begin{bmatrix} A_{22}^{(k)} & 0 \\ 0 & A_{11}^{(k)} \end{bmatrix} \boldsymbol{Y}_k Z_k, Q_k^T \boldsymbol{X}_k \begin{bmatrix} E_{22}^{(k)} & 0 \\ 0 & E_{11}^{(k)} \end{bmatrix} \boldsymbol{Y}_{k\oplus 1} Z_{k\oplus 1} \right) \\ &\quad \equiv \left(\begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ 0 & \hat{A}_{22}^{(k)} \end{bmatrix}, \begin{bmatrix} \hat{E}_{11}^{(k)} & \hat{E}_{12}^{(k)} \\ 0 & \hat{E}_{22}^{(k)} \end{bmatrix} \right), \end{aligned}$$

where

$$\begin{cases} \hat{A}_{11}^{(k)} = T_L^{(k)} A_{22}^{(k)} Z_{21}^{(k)}, \\ \hat{A}_{12}^{(k)} = T_L^{(k)} A_{22}^{(k)} Z_{22}^{(k)} + {Q_{11}^{(k)}}^T A_{11}^{(k)} T_R^{(k)}, \\ \hat{A}_{22}^{(k)} = {Q_{12}^{(k)}}^T A_{11}^{(k)} T_R^{(k)}, \end{cases} \tag{3.21}$$

and

$$\begin{cases} \hat{E}_{11}^{(k)} = T_L^{(k)} E_{22}^{(k)} Z_{21}^{(k\oplus 1)}, \\ \hat{E}_{12}^{(k)} = T_L^{(k)} E_{22}^{(k)} Z_{22}^{(k\oplus 1)} + {Q_{11}^{(k)}}^T E_{11}^{(k)} T_R^{(k\oplus 1)}, \\ \hat{E}_{22}^{(k)} = {Q_{12}^{(k)}}^T E_{11}^{(k)} T_R^{(k\oplus 1)}. \end{cases} \tag{3.22}$$

Note that (3.19) implies the nonsingularity of $Q_{12}^{(k)}$ and $Z_{21}^{(k)}$. Hence, from the equations above, we see that $(A_{11}^{(k)}, E_{11}^{(k)})$ and $(A_{22}^{(k)}, E_{22}^{(k)})$ are K-*cyclic equivalent* to $(\hat{A}_{22}^{(k)}, \hat{E}_{22}^{(k)})$ and $(\hat{A}_{11}^{(k)}, \hat{E}_{11}^{(k)})$, respectively. In other words, the eigenvalues of the K-cyclic matrix pair sequence (A_k, E_k) have been reordered as desired.

We remark that $(\hat{A}_{11}^{(k)}, \hat{E}_{11}^{(k)})$ and $(\hat{A}_{22}^{(k)}, \hat{E}_{22}^{(k)})$ are generally not in GPRSF after the K-cyclic swapping and have to be further transformed by orthogonal transformations to restore the GPRSF of the matrix pair sequence (A_k, E_k) (see Section 5.2).

3.4 Matrix representation of the PGCSY.

The key step of the reordering method is to solve the associated PGCSY (3.7). Using Kronecker products this problem can be rewritten as a linear system of equations

$$Z_{\mathrm{PGCSY}}x = c, \tag{3.23}$$

where Z_{PGCSY} is a $2Kp_1p_2 \times 2Kp_1p_2$ matrix representation of the associated linear operator (3.13):

$$Z_{\mathrm{PGCSY}} = \begin{bmatrix} -A_{22}^{(0)^T} \otimes I_{p_1} & & & & & & I_{p_2} \otimes A_{11}^{(0)} \\ -E_{22}^{(0)^T} \otimes I_{p_1} & I_{p_2} \otimes E_{11}^{(0)} & & & & & \\ & I_{p_2} \otimes A_{11}^{(1)} & -A_{22}^{(1)^T} \otimes I_{p_1} & & & & \\ & & -E_{22}^{(1)^T} \otimes I_{p_1} & \ddots & & & \\ & & & \ddots & & & \\ & & & & -A_{22}^{(K-1)^T} \otimes I_{p_1} & & \\ & & & & -E_{22}^{(K-1)^T} \otimes I_{p_1} & I_{p_2} \otimes E_{11}^{(K-1)} \end{bmatrix},$$

and x and c are $2Kp_1p_2 \times 1$ vector representations of the assembled unknowns and right hand sides, respectively:

$$x = \begin{bmatrix} \mathrm{vec}(L_0) \\ \mathrm{vec}(R_1) \\ \mathrm{vec}(L_1) \\ \mathrm{vec}(R_2) \\ \vdots \\ \mathrm{vec}(R_{K-1}) \\ \mathrm{vec}(L_{K-1}) \\ \mathrm{vec}(R_0) \end{bmatrix}, \quad c = \begin{bmatrix} \mathrm{vec}\big(-A_{12}^{(0)}\big) \\ \mathrm{vec}\big(-E_{12}^{(0)}\big) \\ \mathrm{vec}\big(-A_{12}^{(1)}\big) \\ \mathrm{vec}\big(-E_{12}^{(1)}\big) \\ \vdots \\ \mathrm{vec}\big(-A_{12}^{(K-1)}\big) \\ \mathrm{vec}\big(-E_{12}^{(K-1)}\big) \end{bmatrix}.$$

Here, the operator $\mathrm{vec}(M)$ stacks the columns of a matrix M on top each other into one long vector. Note also that only the nonzero blocks of Z_{PGCSY} are displayed explicitly above. The sparsity structure of Z_{PGCSY} can be exploited when using Gaussian elimination with partial pivoting (GEPP) or a QR factorization to solve (3.23), see Section 5 for more details.

By Lemma 3.1, the matrix Z_{PGCSY} is invertible if and only if the eigenvalue condition (3.12) is fulfilled. Throughout the rest of this paper we assume that this condition holds. If the condition is violated then, since (A_k, E_k) is in GPRSF, the eigenvalues of Π_{11} and Π_{22} are actually equal and there is in principle no need for swapping.

The invertibility of Z_{PGCSY} is equivalent to

$$\mathrm{sep}[\mathrm{PGCSY}] = \sigma_{\min}(Z_{\mathrm{PGCSY}}) \neq 0. \tag{3.24}$$

As for deflating subspaces of regular matrix pairs (see, e.g., [33, 23]), the quantity sep[PGCSY] measures the sensitivity of the periodic deflating subspace pair of the GPRSF [5, 26, 34]. If K, p_1 or p_2 become large this quantity is very expensive to compute explicitly. By using the well-known estimation technique described in [17, 19, 22, 23], reliable sep[PGCSY]-estimates can be computed at the cost of solving a few PGCSYs.

4 Error analysis of K-cyclic equivalence swapping of diagonal blocks.

In this section, we present an error analysis of the direct method described in Section 3 by extending the results in [21] to the case of periodic matrix pairs. We sometimes omit the index range $k = 0, 1, \ldots, K-1$, assuming that it is implicitly understood.

In finite precision arithmetic, the transformed matrix pair sequence will be affected by roundoff errors, resulting in a computed sequence $(\tilde{A}_k, \tilde{E}_k)$. We express the computed transformed matrix pairs as

$$(\tilde{A}_k, \tilde{E}_k) = (\hat{A}_k + \Delta A_k, \hat{E}_k + \Delta E_k),$$

where $(\hat{A}_k, \hat{E}_k)$ for $k = 0, \ldots K-1$ correspond to the exact matrix pairs in the reordered GPRSF of (A_k, B_k). Our task is to derive explicit expressions and upper bounds for the error matrices ΔA_k and ΔE_k. Most critical are of course the subdiagonal blocks of a 2×2 block partioned sequence $(\Delta A_k, \Delta E_k)$. These must be negligible in order to guarantee numerical backward stability for the swapping of diagonal blocks.

Let $(\tilde{L}_k, \tilde{R}_k) = (L_k + \Delta L_k, R_k + \Delta R_k)$ denote the computed solution to the associated PGCSY. The residual pair sequence of the computed solution is then given by $(Y_1^{(k)}, Y_2^{(k)})$, where

$$\begin{cases} Y_1^{(k)} \equiv A_{11}^{(k)} \tilde{R}_k - \tilde{L}_k A_{22}^{(k)} + A_{12}^{(k)}, \\ Y_2^{(k)} \equiv E_{11}^{(k)} \tilde{R}_{k\oplus 1} - \tilde{L}_k E_{22}^{(k)} + E_{12}^{(k)}. \end{cases} \tag{4.1}$$

In addition, let $\tilde{Q}_k$, $\tilde{T}_L^{(k)}$ denote the computed factors of the kth QR factorization

$$\tilde{G}_L^{(k)} \equiv \begin{bmatrix} \tilde{L}_k \\ I_{p_2} \end{bmatrix} = \tilde{Q}_k \begin{bmatrix} \tilde{T}_L^{(k)} \\ 0 \end{bmatrix}, \tag{4.2}$$

where $\tilde{Q}_k = Q_k + \Delta Q_k$, $\tilde{T}_L^{(k)} = T_L^{(k)} + \Delta T_L^{(k)}$ and Q_k, $T_L^{(k)}$ are the exact factors. Similarly, let $\tilde{Z}_k$, $\tilde{T}_R^{(k)}$ denote the computed factors of the kth RQ factorization

$$\tilde{G}_R^{(k)} \equiv \begin{bmatrix} I_{p_1} & -\tilde{R}_k \end{bmatrix} = \begin{bmatrix} 0 & \tilde{T}_R^{(k)} \end{bmatrix} \tilde{Z}_k^T, \tag{4.3}$$

where $\tilde{Z}_k = Z_k + \Delta Z_k$, $\tilde{T}_R^{(k)} = T_R^{(k)} + \Delta T_R^{(k)}$ and Z_k, $T_R^{(k)}$ are the exact factors. If Householder transformations are used to compute the factorizations (4.2)–(4.3), $\tilde{Q}_k$ and $\tilde{Z}_k$ are orthogonal to machine precision [41]. The error matrices ΔQ_k and ΔZ_k are essentially bounded by the condition numbers of $\tilde{G}_L^{(k)}$ and $\tilde{G}_R^{(k)}$, respectively, times the relative errors in these matrices (e.g., see [33, 20]).

We transform (A_k, E_k) using the computed $(\tilde{Q}_k, \tilde{Z}_k)$ in a K-cyclic equivalence transformation giving

$$\tilde{Q}^T(A_k, E_k)\tilde{Z}_k = (\hat{A}_k + \Delta A_k, \hat{E}_k + \Delta E_k), \tag{4.4}$$

where $(\hat{A}_k, \hat{E}_k)$ is the exact reordered GPRSF of the periodic (A_k, B_k) sequence. Our aim is to derive explicit expressions and norm bounds for blocks of $(\Delta A_k, \Delta E_k)$. First,

$$\begin{aligned}\tilde{Q}^T A_k \tilde{Z}_k &= (Q_k + \Delta Q_k)^T A_k (Z_k + \Delta Z_k) \\ &= Q_k^T A_k Z_k + \Delta Q_k^T A_k Z_k + Q_k^T A_k \Delta Z_k + \Delta Q_k^T A_k \Delta Z_k,\end{aligned} \tag{4.5}$$

and by dropping the second order term and using $\hat{A}_k = Q_k^T A_k Z_k$ and $\Delta Q_k^T Q_k = -Q_k \Delta Q_k^T$ up to first order we get

$$\begin{aligned}&\tilde{Q}^T A_k \tilde{Z}_k = \hat{A}_k + \hat{A}_k\big(Z_k^T \Delta Z_k\big) + \big(-Q_k \Delta Q_k^T\big)\hat{A}_k = \hat{A}_k + \Delta A_k, \\ &\text{with } \Delta A_k \equiv \hat{A}_k U_k + W_k \hat{A}_k, \text{ where } U_k = Z_k^T \Delta Z_k \text{ and } W_k = -Q_k \Delta Q_k^T.\end{aligned} \tag{4.6}$$

Similarly, we get

$$\tilde{Q}^T B_k \tilde{Z}_{k\oplus 1} = \hat{E}_k + \Delta E_k \quad \text{with } \Delta E_k \equiv \hat{E}_k U_{k\oplus 1} + W_k \hat{E}_k. \tag{4.7}$$

After partitioning U_k, $U_{k\oplus 1}$, W_k and $(\Delta A_k, \Delta E_k)$ in conformity with $(\hat{A}_k, \hat{E}_k)$ and doing straightforward block matrix multiplications we get

$$\begin{aligned}\Delta A_{11}^{(k)} &= \hat{A}_{11} U_{11}^{(k)} + W_{11}^{(k)} \hat{A}_{11}^{(k)} + \hat{A}_{12}^{(k)} U_{21}^{(k)}, \\ \Delta A_{12}^{(k)} &= \hat{A}_{11}^{(k)} U_{12}^{(k)} + \hat{A}_{12}^{(k)} U_{22}^{(k)} + W_{11}^{(k)} \hat{A}_{12}^{(k)} + W_{12}^{(k)} \hat{A}_{22}^{(k)}, \\ \Delta A_{21}^{(k)} &= \hat{A}_{22}^{(k)} U_{21}^{(k)} + W_{21}^{(k)} \hat{A}_{11}^{(k)}, \\ \Delta A_{22}^{(k)} &= \hat{A}_{22}^{(k)} U_{22}^{(k)} + W_{22}^{(k)} \hat{A}_{22}^{(k)} + W_{21}^{(k)} \hat{A}_{12}^{(k)},\end{aligned}$$

and

$$\begin{aligned}\Delta E_{11}^{(k)} &= \hat{E}_{11} U_{11}^{(k\oplus 1)} + W_{11}^{(k)} \hat{E}_{11}^{(k)} + \hat{E}_{12}^{(k)} U_{21}^{(k\oplus 1)}, \\ \Delta E_{12}^{(k)} &= \hat{E}_{11}^{(k)} U_{12}^{(k\oplus 1)} + \hat{E}_{12}^{(k)} U_{22}^{(k\oplus 1)} + W_{11}^{(k)} \hat{E}_{12}^{(k)} + W_{12}^{(k)} \hat{E}_{22}^{(k)}, \\ \Delta E_{21}^{(k)} &= \hat{E}_{22}^{(k)} U_{21}^{(k\oplus 1)} + W_{21}^{(k)} \hat{E}_{11}^{(k)}, \\ \Delta E_{22}^{(k)} &= \hat{E}_{22}^{(k)} U_{22}^{(k\oplus 1)} + W_{22}^{(k)} \hat{E}_{22}^{(k)} + W_{21}^{(k)} \hat{E}_{12}^{(k)}.\end{aligned}$$

Observe that $\Delta A_{11}^{(k)}$, $\Delta A_{22}^{(k)}$, $\Delta E_{11}^{(k)}$, $\Delta E_{22}^{(k)}$ affect the reordered K-cyclic diagonal block pairs and possibly the eigenvalues, while $\Delta A_{21}^{(k)}$ and $\Delta E_{21}^{(k)}$ are even more critical since they affect the eigenvalues as well as the stability of the reordering; these are the perturbations of interest that we investigate further. The analysis in [21] applied to (4.2)–(4.3), results in the following expressions for

blocks of U_k and W_k:

$$
\begin{aligned}
U_{11}^{(k)} &= -{Z_{21}^{(k)}}^{-1} Z_{22}^{(k)} {T_R^{(k)}}^{-1} \Delta R_k Z_{21}^{(k)}, \\
U_{21}^{(k)} &= {T_R^{(k)}}^{-1} \Delta R_k Z_{21}^{(k)}, \\
U_{22}^{(k)} &= {T_R^{(k)}}^{-1} \Delta R_k Z_{22}^{(k)},
\end{aligned}
$$

and

$$
\begin{aligned}
W_{11}^{(k)} &= -{Q_{11}^{(k)}}^T \Delta L_k {T_L^{(k)}}^{-1}, \\
W_{21}^{(k)} &= -{Q_{12}^{(k)}}^T \Delta L_k {T_L^{(k)}}^{-1}, \\
W_{22}^{(k)} &= {Q_{12}^{(k)}}^T {T_L^{(k)}}^{-1} {Q_{11}^{(k)}}^T {Q_{12}^{(k)}}^{-T},
\end{aligned}
$$

up to first order perturbations. By substituting the expressions for $U_{ij}^{(k)}$ and $W_{ij}^{(k)}$ in $\Delta A_{ij}^{(k)}, \Delta E_{ij}^{(k)}$ we obtain

$$
\Delta A_{11}^{(k)} = {Q_{11}^{(k)}}^T Y_1^{(k)} Z_{21}^{(k)}, \tag{4.8}
$$

$$
\Delta A_{21}^{(k)} = {Q_{12}^{(k)}}^T Y_1^{(k)} Z_{21}^{(k)}, \tag{4.9}
$$

$$
\Delta A_{22}^{(k)} = {Q_{12}^{(k)}}^T Y_1^{(k)} Z_{22}^{(k)}, \tag{4.10}
$$

and

$$
\Delta E_{11}^{(k)} = {Q_{11}^{(k)}}^T Y_2^{(k)} Z_{21}^{(k\oplus 1)}, \tag{4.11}
$$

$$
\Delta E_{21}^{(k)} = {Q_{12}^{(k)}}^T Y_2^{(k)} Z_{21}^{(k\oplus 1)}, \tag{4.12}
$$

$$
\Delta E_{22}^{(k)} = {Q_{12}^{(k)}}^T Y_2^{(k)} Z_{22}^{(k\oplus 1)}, \tag{4.13}
$$

with the residuals $(Y_1^{(k)}, Y_2^{(k)})$ as in (4.1). From the QR and RQ factorizations (3.19) we have

$$
Q_{21}^{(k)} = {T_L^{(k)}}^{-1}, \quad {T_L^{(k)}}^T T_L^{(k)} = I_{p_2} + L_k^T L_k, \tag{4.14}
$$

and

$$
{Z_{12}^{(k)}}^T = {T_R^{(k)}}^{-1}, \quad T_R^{(k)} {T_R^{(k)}}^T = I_{p_1} + R_k R_k^T. \tag{4.15}
$$

From (4.14)–(4.15) we obtain the following relations between the singular values of $T_L^{(k)}, T_R^{(k)}, L_k$ and R_k:

$$
\sigma^2\big(T_L^{(k)}\big) = 1 + \sigma^2(L_k), \quad \sigma^2\big(T_R^{(k)}\big) = 1 + \sigma^2(R_k). \tag{4.16}
$$

Further, from the CS decomposition (see, e.g., [12]) of Q_k and Z_k, respectively, we obtain the relations

$$
\begin{aligned}
\big\|{Q_{12}^{(k)}}^T\big\|_2 &= \big\|Q_{21}^{(k)}\big\|_2, \quad \big\|Q_{22}^{(k)}\big\|_2 = \big\|Q_{11}^{(k)}\big\|_2, \\
\big\|{Z_{12}^{(k)}}^T\big\|_2 &= \big\|Z_{21}^{(k)}\big\|_2, \quad \big\|Z_{22}^{(k)}\big\|_2 = \big\|Z_{11}^{(k)}\big\|_2.
\end{aligned}
$$

Combining these results, we get

$$\left\|Q_{12}^{(k)^T}\right\|_2 = \left\|T_L^{(k)^{-1}}\right\|_2 = \frac{1}{\sigma_{\min}\left(T_L^{(k)}\right)} = \frac{1}{\left(1+\sigma_{\min}^2(L_k)\right)^{1/2}},$$

$$\left\|Q_{11}^{(k)}\right\|_2 = \frac{\sigma_{\max}(L_k)}{\left(1+\sigma_{\max}^2(L_k)\right)^{1/2}},$$

and

$$\left\|Z_{21}^{(k)}\right\|_2 = \left\|T_R^{(k)^{-1}}\right\|_2 = \frac{1}{\sigma_{\min}\left(T_R^{(k)}\right)} = \frac{1}{\left(1+\sigma_{\min}^2(R_k)\right)^{1/2}},$$

$$\left\|Z_{22}^{(k)}\right\|_2 = \frac{\sigma_{\max}(R_k)}{\left(1+\sigma_{\max}^2(R_k)\right)^{1/2}},$$

and we have proved the following theorem by applying the submultiplicativity of matrix norms to (4.8)–(4.13).

THEOREM 4.1. *After applying the computed transformation matrices $\tilde{Q}_k, \tilde{Z}_k$ from* (4.2)–(4.3) *in a K-cyclic equivalence transformation of (A_k, E_k) defined in* (3.1)*, we get*

$$\tilde{Q}_k^T A_k \tilde{Z}_k = \tilde{A}_k, \quad \text{where } \tilde{A}_k \equiv \hat{A}_k + \Delta A_k = \begin{bmatrix} \hat{A}_{11}^{(k)} & \hat{A}_{12}^{(k)} \\ 0 & \hat{A}_{22}^{(k)} \end{bmatrix} + \begin{bmatrix} \Delta A_{11}^{(k)} & \Delta A_{12}^{(k)} \\ \Delta A_{21}^{(k)} & \Delta A_{22}^{(k)} \end{bmatrix},$$

$$\tilde{Q}_k^T E_k \tilde{Z}_{k\oplus 1} = \tilde{E}_k, \quad \text{where } \tilde{E}_k \equiv \hat{E}_k + \Delta E_k = \begin{bmatrix} \hat{E}_{11}^{(k)} & \hat{E}_{12}^{(k)} \\ 0 & \hat{E}_{22}^{(k)} \end{bmatrix} + \begin{bmatrix} \Delta E_{11}^{(k)} & \Delta E_{12}^{(k)} \\ \Delta E_{21}^{(k)} & \Delta E_{22}^{(k)} \end{bmatrix}.$$

The critical blocks of the error matrix pair $(\Delta A_k, \Delta E_k)$ satisfy the following error bounds, up to first order perturbations:

$$\left\|\Delta A_{11}^{(k)}\right\|_2 \le \frac{\sigma_{\max}(L_k)}{\left(1+\sigma_{\max}^2(L_k)\right)^{1/2}} \cdot \frac{1}{\left(1+\sigma_{\min}^2(R_k)\right)^{1/2}} \cdot \left\|Y_1^{(k)}\right\|_F,$$

$$\left\|\Delta A_{21}^{(k)}\right\|_2 \le \frac{1}{\left(1+\sigma_{\min}^2(L_k)\right)^{1/2}} \cdot \frac{1}{\left(1+\sigma_{\min}^2(R_k)\right)^{1/2}} \cdot \left\|Y_1^{(k)}\right\|_F,$$

$$\left\|\Delta A_{22}^{(k)}\right\|_2 \le \frac{1}{\left(1+\sigma_{\min}^2(L_k)\right)^{1/2}} \cdot \frac{\sigma_{\max}(R_k)}{\left(1+\sigma_{\max}^2(R_k)\right)^{1/2}} \cdot \left\|Y_1^{(k)}\right\|_F,$$

and

$$\left\|\Delta E_{11}^{(k)}\right\|_2 \le \frac{\sigma_{\max}(L_k)}{\left(1+\sigma_{\max}^2(L_k)\right)^{1/2}} \cdot \frac{1}{\left(1+\sigma_{\min}^2(R_{k\oplus 1})\right)^{1/2}} \cdot \left\|Y_2^{(k)}\right\|_F,$$

$$\left\|\Delta E_{21}^{(k)}\right\|_2 \le \frac{1}{\left(1+\sigma_{\min}^2(L_k)\right)^{1/2}} \cdot \frac{1}{\left(1+\sigma_{\min}^2(R_{k\oplus 1})\right)^{1/2}} \cdot \left\|Y_2^{(k)}\right\|_F,$$

$$\left\|\Delta E_{22}^{(k)}\right\|_2 \le \frac{1}{\left(1+\sigma_{\min}^2(L_k)\right)^{1/2}} \cdot \frac{\sigma_{\max}(R_{k\oplus 1})}{\left(1+\sigma_{\max}^2(R_{k\oplus 1})\right)^{1/2}} \cdot \left\|Y_2^{(k)}\right\|_F,$$

for $k = 0, 1, \ldots, K-1$. *Moreover, the matrix pair sequences* $(\hat{A}_{11}^{(k)}, \hat{E}_{11}^{(k)})$, $(A_{22}^{(k)}, E_{22}^{(k)})$ *and* $(A_{11}^{(k)}, E_{11}^{(k)}), (\hat{A}_{22}^{(k)}, \hat{E}_{22}^{(k)})$ *are* K*-cyclic equivalent and have the same generalized eigenvalues, respectively.*

REMARK 4.1. Theorem 4.1 shows that the stability and accuracy of the reordering method is governed mainly by the conditioning and accuracy of the solution to the associated PGCSY. The errors $\|\Delta A_{ij}^{(k)}\|_2$ and $\|\Delta E_{ij}^{(k)}\|_2$ can be as large as the norm of the residuals $\|Y_1^{(k)}\|_F$ and $\|Y_2^{(k)}\|_F$, respectively. Indeed, this happens when the smallest singular values of the exact sequences L_k and R_k are tiny, indicating an ill-conditioned underlying PGCSY equation. We have experimental evidence that $\|Y_1^{(k)}\|_F$ and $\|Y_2^{(k)}\|_F$ can be large for large-normed (ill-conditioned) solutions of the associated PGCSY. In the next section, we show how we handle such situations and guarantee backward stability of the periodic reordering method.

REMARK 4.2. For period $K = 1$, Theorem 4.1 reduces to the main theorem of [21] on the perturbation of the generalized eigenvalues under eigenvalue reordering in the generalized real Schur form of a regular matrix pencil.

5 Algorithms and implementation aspects.

In this section, we address some implementation issues of the direct method for reordering eigenvalues in a generalized periodic real Schur form described and analyzed in the previous sections.

5.1 Algorithms for solving the PGCSY.

The linear system (3.23) that arises from the PGCSY (3.7) has a particular structure that needs to be exploited in order to keep the cost of the overall algorithm linear in K. The matrix Z_{PGCSY} in (3.23) belongs to the class of *bordered almost block diagonal* (BABD) matrices, which takes the more general form

$$Z = \begin{bmatrix} Z_{0,0} & & & & & & & Z_{0,2K-1} \\ Z_{1,0} & Z_{1,1} & & & & & & \\ & Z_{2,1} & Z_{2,2} & & & & & \\ & & Z_{3,2} & Z_{3,3} & & & & \\ & & & Z_{4,3} & \ddots & & & \\ & & & & \ddots & & & \\ & & & & & Z_{2K-2,2K-2} & & \\ & & & & & Z_{2K-1,2K-2} & Z_{2K-1,2K-1} \end{bmatrix}, \tag{5.1}$$

where each nonzero block $Z_{i,j}$ is $m \times m$ (note that $m = p_1 p_2$ for the matrix Z_{PGCSY} of size $2Km \times 2Km$). An overview of numerical methods that address linear systems with such BABD structure is given in [10]. Gaussian elimination with partial pivoting, for example, preserves much of the structure of Z and can be implemented very efficiently. Unfortunately, matrices with BABD structure

happen to be among the rare examples of practical relevance which may lead to numerical instabilities because of excessive pivot growth [43]. Gaussian elimination with complete pivoting avoids this phenomenom but is too expensive both in terms of cost and storage space. In contrast, structured variants of the QR factorization are both numerically stable and efficient [11, 42]. In the following, we describe such a structured QR factorization in more detail.

To solve a linear system $Zx = y$, we first reduce the matrix Z in (5.1) to upper triangular form. For this purpose, we successively apply Householder transformations to reduce each block $[Z_{k,k}^T,\ Z_{k+1,k}^T]^T,\ k = 0, 1, \ldots, 2K-2$, to upper trapezoidal form, and the block $Z_{2K-1,2K-1}$ to upper triangular form. Each computed Householder transformation is applied to the corresponding block row (as well as the right hand side y of the equation, which is blocked in conformity with Z) before the next transformation is computed. The factorization procedure is outlined in Algorithm 5.1, where for simplicity of presentation the Householder transformations are accumulated into orthogonal transformation matrices $\bar{Q}_k$.

Algorithm 5.1 Overlapping QR factorization of the BABD-system $Zx = y$

Input:	Matrix $Z \in \mathbb{R}^{2Km \times 2Km}$, right hand side vector $y \in \mathbb{R}^{2Km}$.
Output:	Orthogonal transformations $\bar{Q}_k \in \mathbb{R}^{2m \times 2m}$, $k = 0, 1, \ldots, 2K-2$, $\bar{Q}_{2K-1} \in \mathbb{R}^{m \times m}$, triangular factor $\bar{R} \in \mathbb{R}^{2Km \times 2Km}$ with structure as in Equation (5.2), vector $\bar{y} \in \mathbb{R}^{2Km}$ such that $\bar{R}x = \bar{y}$.

for $k = 0$ up to $2K-2$ **do**
  QR factorize: $\bar{Q}_k \bar{R}_k = \left[Z_{k,k}^T,\ Z_{k+1,k}^T\right]^T$
  Update: $\left[Z_{k,k+1}^T,\ Z_{k+1,k+1}^T\right]^T = \bar{Q}_k^T \left[Z_{k,k+1}^T,\ Z_{k+1,k+1}^T\right]^T$
  Update: $\left[Z_{k,K-1}^T,\ Z_{k+1,K-1}^T\right]^T = \bar{Q}_k^T \left[Z_{k,K-1}^T,\ Z_{k+1,K-1}^T\right]^T$
  Update right hand side: $\bar{y}_k = \bar{Q}_k^T y_k$
end for
QR factorize: $\bar{Q}_{2K-1} \bar{R}_{2K-1} = Z_{2K-1,2K-1}$
Update right hand side: $\bar{y}_{2K-1} = \bar{Q}_{2K-1}^T y_{2K-1}$

It is straightforward to see that this procedure of computing overlapping orthogonal factorizations produces the same amount of fill-in elements in the rightmost block columns of Z as would GEPP produce in the worst case, see also Figure 5.1. More formally, the QR factorization reduces the matrix Z into the following form:

$$
\begin{bmatrix}
\bar{R}_0 & G_0 & & & & & & & F_0 \\
 & \bar{R}_1 & \bar{L}_0 & & & & & & F_1 \\
 & & \bar{R}_2 & G_1 & & & & & F_2 \\
 & & & \bar{R}_3 & \bar{L}_1 & & & & F_3 \\
 & & & & \ddots & & & & \vdots \\
 & & & & \ddots & & & & \vdots \\
 & & & & & \bar{R}_{2K-4} & G_{K-2} & & F_{2K-4} \\
 & & & & & & \bar{R}_{2K-3} & \bar{L}_{K-2} & F_{2K-3} \\
 & & & & & & & \bar{R}_{2K-2} & G_{K-1} \\
 & & & & & & & & \bar{R}_{2K-1}
\end{bmatrix}, \tag{5.2}
$$

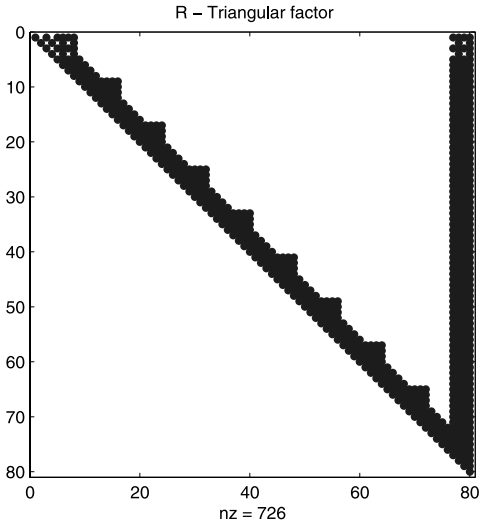

Figure 5.1: The resulting R-factor from applying overlapping QR factorizations to the matrix Z_{PGCSY} for $K = 10$, $p_1 = p_2 = 2$, visualized by the Matlab `spy` command. The "sawtooth" above the main block diagonal is typical for the PGCSY and does not occur in the case of periodic matrix reordering [14].

with $\bar{R}_k, \bar{L}_k, F_k, G_k \in \mathbb{R}^{m\times m}$: $\bar{R}_k$ $(k = 0, 1, \ldots, 2K-1)$ are upper triangular, whereas $\bar{L}_k$ $(k = 0, 1, \ldots, K-2)$, G_k, $(k = 0, 1, \ldots, K-1)$, and F_k $(k = 0, 1, \ldots, 2K-3)$ are dense matrices. Moreover, the blocks $\bar{L}_k$ are lower triangular provided that $Z_{2,2}, Z_{4,4}, \ldots, Z_{2K-2,2K-2}$ and $Z_{2,1}, Z_{4,3}, \ldots, Z_{2K-2,2K-1}$ in (5.1) are lower and upper triangular, respectively, which is the case for if the quasi-triangular factor is placed at position $k = K$. To compute x we employ backward substitution on this structure, as outlined in Algorithm 5.2. All updates of the right hand side vector $\bar{y}$ in Algorithm 5.2 are general matrix-vector

Algorithm 5.2 Backward substitution for solving $\bar{R}x = \bar{y}$

Input:	Matrix $\bar{R} \in \mathbb{R}^{2Km\times 2Km}$, with the upper triangular BABD structure of (5.2), right hand side vector $\bar{y}\epsilon \in \mathbb{R}^{2Km}$ partitioned in conformity with the structure of $\bar{R}$.
Output:	Solution vector $x \in \mathbb{R}^{2Km}$ such that $\bar{R}x = \bar{y}$.

Solve: $\bar{R}_{2K-1}x_{2K-1} = \bar{y}_{2K-1}$
Update and solve: $\bar{R}_{2K-2}x_{2K-2} = \bar{y}_{2K-2} - G_{K-1}x_{2K-1}$
for $i = 0$ **to** $2K-3$ **do**
  Update: $\bar{y}_i = \bar{y}_i - F_i x_{2K-1}$
end for
for $i = K-2$ **down to** 0 **do**
  Update and solve: $\bar{R}_{2i+1}x_{2i+1} = \bar{y}_{2i+1} - \bar{L}_i x_{2i+2}$
  Update and solve: $\bar{R}_{2i}x_{2i} = \bar{y}_{2i} - G_i x_{2i+1}$
end for

multiply and add (GEMV) operations, except the updates involving $\bar{L}_i$, which are triangular matrix-vector multipy (TRMV) operations. All triangular solves are level 2 TRMSV operations.

We remark that the new algorithms described here for solving small block-sized PGCSY equations can be used as kernel solvers in recursive blocked algorithms [13] for solving large-scale problems.

REMARK 5.1. Solving a linear system with QR factorization yields a small norm-wise backward error [20], i.e., the computed solution $\hat{x}$ is the exact solution of a slightly perturbed system $(Z + \triangle Z)\hat{x} = y$, where $\|\triangle Z\|_F = O(\mathbf{u}\|Z\|_F)$ with $\mathbf{u}$ denoting the unit roundoff. However, the standard implementation of the QR factorization is not row-wise backward stable, i.e., the norm of a row in $\triangle Z$ may not be negligible compared to the norm of the corresponding row in Z. This may cause instabilities if the norms of the coefficient matrices A_k, E_k differ significantly. To avoid this effect, we scale each A_k and E_k to Frobenius norm 1 before solving (3.7). Then each block row in Z_{PGCSY} has Frobenius norm at most $\sqrt{2}$ and $\|Z_{\mathrm{PGCSY}}\|_F \leq 2\sqrt{K}$. The resulting swapping transformation is applied to the original unscaled K-cyclic matrix pair sequence. The corresponding residuals satisfy

$$\left\|Y_1^{(k)}\right\|_F = O(\mathbf{u}\|A_k\|_F\|(L_k, R_k)\|_F), \quad \left\|Y_2^{(k)}\right\|_F = O(\mathbf{u}\|E_k\|_F\|(L_k, R_{k\oplus 1})\|_F).$$

Combined with Theorem 4.1, this shows that the backward error of the developed reordering method is norm-wise small for each coefficient A_k and E_k, unless (3.7) is too ill-conditioned.

5.2 K-cyclic equivalence swapping algorithm with stability tests.

Considering the error analysis in Section 4 and in the spirit of [23, 14], we formulate stability test criteria for deciding whether a K-cyclic equivalence swap should be accepted or not.

From Equation (3.19) and the following partition of the transformation matrix sequences Q_k and Z_k, we obtain the relations

$$L_k Q_{21}^{(k)} - Q_{11}^{(k)} = 0, \quad {Z_{12}^{(k)}}^T R_k + {Z_{22}^{(k)}}^T = 0, \tag{5.3}$$

which can be computed before the swapping is performed. We use computed quantities of these relations to define the *weak stability criterion*:

$$R_{\mathrm{weak}} = \max_{0\leq k\leq K-1} \max\left(\frac{\left\|\tilde{L}_k \tilde{Q}_{21}^{(k)} - \tilde{Q}_{11}^{(k)}\right\|_F}{\|\tilde{L}_k\|_F}, \frac{\left\|{\tilde{Z}_{12}^{(k)}}^T \tilde{R}_k + {\tilde{Z}_{22}^{(k)}}^T\right\|_F}{\|\tilde{R}_k\|_F} \right). \tag{5.4}$$

We remark that the relative criterion R_{weak} should be small even for ill-conditioned PGSCY equations with large normed solutions L_k and R_k (see also Remarks 5.1 and 6.1). After the swap has been performed, the maximum residual

over the whole K-period defines a *strong stability criterion*:

$$R_{\text{strong}} = \max_{0\le k\le K-1} \max\left(\frac{\left\|A_k - \tilde{Q}_k \tilde{A}_k \tilde{Z}_k^T\right\|_F}{\|A_k\|_F}, \frac{\left\|E_k - \tilde{Q}_k \tilde{E}_k \tilde{Z}_{k\oplus 1}^T\right\|_F}{\|E_k\|_F} \right). \tag{5.5}$$

If both R_{weak} and R_{strong} are less than a specified tolerance ε_{u} (a small constant times the machine precision), the swap is accepted, otherwise it is rejected. In this way, backward stability is guaranteed for the K-cyclic equivalence swapping.

In summary, we have the following algorithm for swapping two matrix pair sequences of diagonal blocks in the GPRSF of a regular K-cyclic matrix pair (A_k, B_k) of size $(p_1 + p_2) \times (p_1 + p_2)$:

1. *Compute K-cyclic matrix pair sequence $(\tilde{L}_k, \tilde{R}_k)$ by solving the scaled PGCSY* (3.7) *using Algorithm 5.1 and Algorithm 5.2.*

2. *Compute K-cyclic orthogonal matrix sequence $\tilde{Q}_k$ using QR factorizations:*

$$\begin{bmatrix} \tilde{L}_k \\ I_{p_2} \end{bmatrix} = \tilde{Q}_k \begin{bmatrix} \tilde{T}_L^{(k)} \\ 0 \end{bmatrix}, \quad k = 0, 1, \ldots, K-1.$$

3. *Compute K-cyclic orthogonal matrix sequence $\tilde{Z}_k$ using RQ factorizations:*

$$\begin{bmatrix} I_{p_1} & -\tilde{R}_k \end{bmatrix} = \begin{bmatrix} 0 & \tilde{T}_R^{(k)} \end{bmatrix} \tilde{Z}_k^T, \quad k = 0, 1, \ldots, K-1.$$

4. *Compute $(\tilde{A}, \tilde{E}) = (\tilde{Q}_k^T A_k \tilde{Z}_k, \tilde{Q}_k^T E_k \tilde{Z}_{k\oplus 1})$ for $k = 0, 1, \ldots, K-1$, i.e., an orthogonal K-cyclic equivalence transformation of (A_k, E_k):*

$$\tilde{A} \equiv \begin{bmatrix} \tilde{A}_{11}^{(k)} & \tilde{A}_{12}^{(k)} \\ \tilde{A}_{21}^{(k)} & \tilde{A}_{22}^{(k)} \end{bmatrix} = \tilde{Q}_k^T \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{bmatrix} \tilde{Z}_k,$$

$$\tilde{E} \equiv \begin{bmatrix} \tilde{E}_{11}^{(k)} & \tilde{E}_{12}^{(k)} \\ \tilde{E}_{21}^{(k)} & \tilde{E}_{22}^{(k)} \end{bmatrix} = \tilde{Q}_k^T \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ 0 & E_{22}^{(k)} \end{bmatrix} \tilde{Z}_{k\oplus 1}.$$

5. *If $R_{\text{weak}} < \varepsilon_{\text{u}} \wedge R_{\text{strong}} < \varepsilon_{\text{u}}$,* accept swap *and*

5a. *set $\tilde{A}_{21}^{(k)} = \tilde{E}_{21}^{(k)} = 0$,*

5b. *restore GPRSF of $(\tilde{A}_{11}^{(k)}, \tilde{E}_{11}^{(k)})$ and $(\tilde{A}_{22}^{(k)}, \tilde{E}_{22}^{(k)})$ by applying the periodic QZ algorithm to the two diagonal block matrix pair sequences;*

otherwise reject swap.

The stability tests in step 5 for accepting a K-cyclic swap guarantee that the subdiagonal blocks $\tilde{A}_{21}^{(k)}$ and $\tilde{E}_{21}^{(k)}$ are negligible compared to the rest of the matrices. Step 5b can be performed by a fixed number of operations for adjacent diagonal blocks in the GPRSF, i.e., for $p_i \in \{1, 2\}$ (see [14] for the standard periodic matrix case).

Properly implemented, this algorithm requires $O(K)$ floating point operations (flops), where K is the period. When it is used to reorder two adjacent diagonal blocks in a larger $n \times n$ periodic matrix pair in GPRSF then the off-diagonal parts are updated by the transformation matrices $\tilde{Q}_k$ and $\tilde{Z}_k$, which additionally requires $O(Kn)$ flops.

There are several other important implementation issues to be considered for a completely reliable implementation. For example, iterative refinement in extended precision arithmetic can be used to improve the accuracy of the PGCSY solution and avoid the possibility of rejection (see, e.g., [20]). Our experiences so far concern iterative refinement in standard precision arithmetic and (as expected) the results show no substantial improvements.

6 Computational experiments.

The direct reordering algorithm described in the previous sections has been implemented in MATLAB. A more robust and efficient Fortran implementation will be included in a forthcoming software toolbox for periodic eigenvalue problems [16]. In this section, we present some numerical results using our prototype implementation. All experiments were carried out in double precision ($\epsilon_{\text{mach}} \approx 2.2 \times 10^{-16}$).

The test examples range from well-conditioned to ill-conditioned problems, including matrix pair sequences of small and large period. In Table 6.1, we display some problem characteristics[1]: problem dimension n (2, 3 or 4 corresponding to swapping a mix of 1×1 and 2×2 blocks), period K, the computed value of sep[PGCSY] $= \sigma_{\min}(Z_{\text{PGCSY}})$ (see Section 3.4) and

$$s = 1/\sqrt{1 + \|(L_0, R_0)\|_F^2},$$

where (L_0, R_0) are the first solution components of the associated PGCSY (3.7). The quantities s and sep[PGCSY] partly govern the sensitivity of the selected eigenvalues and associated periodic deflating subspaces, see [5, 26, 34].

Table 6.1: Problem characteristics.

Example	n	K	sep[PGCSY]	s
I	2	2	1.1E−8	1.4E−4
II	4	10	3.3E−2	4.9E−1
III	4	100	1.4E−3	1.9E−1
IV	4	100	1.4E−14	6.1E−7
V	3	5	7.1E−2	6.2E−1
VI	2	50	1.6E−2	5.8E−1

The results from the periodic reordering are presented in Table 6.2. These include the weak (R_{weak}) and strong (R_{strong}) stability tests, the residual norms for the GPRSF before (R_{gprsf}) and after (R_{reord}) the reordering computed as in Equation (5.5), a relative orthogonality check of the accumulated transformations after (R_{orth}) the reordering computed as

$$R_{\text{orth}} = \frac{\max_k \left(\left\| I_{n_k} - \tilde{W}_k^T \tilde{W}_k \right\|_F, \left\| I_{n_k} - \tilde{W}_k \tilde{W}_k^T \right\|_F \right)}{\epsilon_{\text{mach}}},$$

[1] The test examples used are available at `http://www.cs.umu.se/~granat/gpreord/examples.m`.

Table 6.2: Reordering results using QR factorization to solve the associated PGCSY.

Example	R_{weak}	R_{strong}	R_{gprsf}	R_{reord}	R_{orth}	R_{eig}
I	6.3E−17	5.0E−16	0	5.0E−16	2.0	3.2E−9
II	1.6E−16	9.0E−16	4.8E−15	5.6E−15	7.5	4.6E−15
III	1.8E−16	1.3E−15	2.2E−16	3.2E−15	8.3	3.3E−14
IV	8.3E−17	1.0E−15	2.2E−16	2.4E−15	7.6	3.8E−14
V	1.3E−16	7.0E−16	8.3E−17	9.1E−16	2.8	1.8E−15
VI	3.8E−16	8.2E−16	0	9.8E−16	2.0	1.1E−16

where the maximum is taken over the period K for all transformation matrices $\tilde{Q}_k$ and $\tilde{Z}_k$. The last column displays the maximum relative change of the eigenvalues after the periodic reordering

$$R_{\mathrm{eig}} = \max_k \frac{|\lambda_k - \tilde{\lambda}_k|}{|\lambda_k|}, \quad \lambda_k \in \lambda(\Phi_{E^{-1}A}(K,0)).$$

Notice that we normally do not compute λ_i explicitly but keep it as an eigenvalue pair (α_i, β_i) to avoid losing information because of roundoff errors. This is especially important for tiny and large values of α_i and/or β_i.

The eigenvalues before and after reordering are shown in full precision under each example. For 2×2 matrix sequences, we compute the generalized eigenvalues via unitary transformations in the GPRSF as is done in LAPACK's `DTGSEN` [1].

Example I. Consider the following sequence with $n = 2, K = 2$:

$$A_1 = \begin{bmatrix} 2\epsilon^{1/2} & -1 \\ 0 & -2\epsilon^{1/2} \end{bmatrix}, \quad A_2 = E_1 = E_2 = \begin{bmatrix} \epsilon^{1/2} & 1 \\ 0 & \epsilon^{1/2} \end{bmatrix}.$$

This product has the (α, β)-pairs

$$(\alpha_1, \beta_1) = (4.4408920985006, 2.2204460492503) \times 10^{-16},$$
$$(\alpha_2, \beta_2) = (-4.4408920985006, -2.2204460492503) \times 10^{-16},$$

which correspond to well-defined eigenvalues $\lambda_1 = 2.0$ and $\lambda_2 = -2.0$. But all α_i and β_i are at the machine precision level and this fact signals an obvious risk for losing accuracy after the reordering:

$$(\tilde{\alpha}_1, \tilde{\beta}_1) = (9.5161972853921, -4.7580986273341) \times 10^{-16},$$
$$(\tilde{\alpha}_2, \tilde{\beta}_2) = (-2.0724163126336, -1.0362081563168) \times 10^{-16},$$

which define the eigenvalues

$$\tilde{\lambda}_1 = -2.00000000645717 \quad \text{and} \quad \tilde{\lambda}_2 = 2.00000000000000.$$

Example II. Consider reordering the eigenvalues $\lambda_{1,2} = 2 \pm 2i$ and $\lambda_{3,4} = 1 \pm i$ in a matrix pair sequence with dimension $n = 4$ and period $K = 10$. The computed eigenvalues from the GPRSF are correct to full machine precision.

After reordering we get the following (α, β)-pairs:

$$(\tilde{\alpha}_1, \tilde{\beta}_1) = (-6.69743899940721 - 6.69743899940718i, -6.69743899940718),$$
$$(\tilde{\alpha}_2, \tilde{\beta}_2) = (1.03550511685258 - 1.03550511685258i, 1.03550511685258),$$
$$(\tilde{\alpha}_3, \tilde{\beta}_3) = (1.93142454580911 + 1.93142454580911i, 0.96571227290455),$$
$$(\tilde{\alpha}_4, \tilde{\beta}_4) = (0.29862160747967 - 0.29862160747967i, 0.14931080373983).$$

A quick check reveals that these pairs correspond to a reordering at almost full machine precision.

EXAMPLE III. The eigenvalue pair $\cos\frac{\pi}{4} \pm \sin\frac{\pi}{4}i$ is located on the unit circle. In LQ-optimal control (see Section 2) we want to compute a periodic deflating subspace corresponding to the *stable* eigenvalues, i.e., the eigenvalues *inside* the unit disc.

For illustration, consider reordering the eigenvalues $\lambda_{1,2} = (\cos\frac{\pi}{4} + \delta) \pm (\sin\frac{\pi}{4} + \delta)i$ and $\lambda_{3,4} = (\cos\frac{\pi}{4} - \delta) \pm (\sin\frac{\pi}{4} - \delta)i$, where $\delta \in [0, 1]$, in a matrix pair sequence of period $K = 100$ arising, for example, from performing multi-rate sampling of a continuous-time system. At first, let $\delta = 10^{-1}$. The matrix product has the computed (α, β)-pairs

$$(\alpha_1, \beta_1) = (0.80710678118654 + 0.80710678118654i, 1.00000000000002),$$
$$(\alpha_2, \beta_2) = (0.80710678118654 - 0.80710678118654i, 1.00000000000002),$$
$$(\alpha_3, \beta_3) = (-0.60710678118655 - 0.60710678118655i, -0.99999999999999),$$
$$(\alpha_4, \beta_4) = (-0.60710678118655 + 0.60710678118655i, -1.00000000000000),$$

which correspond to the eigenvalues

$$\lambda_{1,2} = 0.80710678118652 \pm 0.80710678118652i,$$
$$\lambda_{3,4} = 0.60710678118655 \pm 0.60710678118655i.$$

After reordering we have

$$(\tilde{\alpha}_1, \tilde{\beta}_1) = (-1.53524924293502 - 1.53524924293503i, -2.52879607098851),$$
$$(\tilde{\alpha}_2, \tilde{\beta}_2) = (-6.49961741950939 + 6.49961741950943i, -10.70588835592705),$$
$$(\tilde{\alpha}_3, \tilde{\beta}_3) = (-0.07538905267396 - 0.07538905267396i, -0.09340654103182),$$
$$(\tilde{\alpha}_4, \tilde{\beta}_4) = (0.31916641695471 - 0.31916641695471i, 0.39544509400044),$$

which define the eigenvalues $\tilde{\lambda}_{1,2} = 0.60710678118654 \pm 0.60710678118655i$ and $\tilde{\lambda}_{3,4} = 0.80710678118654 \pm 0.80710678118654i$.

EXAMPLE IV. We consider Example III again, now with $\delta = 10^{-12}$ and $K = 100$ as before. The matrix product has the computed (α, β)-pairs

$$(\alpha_1, \beta_1) = (-0.70710678118754 - 0.70710678118754i, -1.00000000000002),$$
$$(\alpha_2, \beta_2) = (-0.70710678118755 + 0.70710678118755i, -0.99999999999999),$$

$$(\alpha_3, \beta_3) = (0.70710678118555 + 0.70710678118555i, 1.00000000000000),$$
$$(\alpha_4, \beta_4) = (-0.70710678118555 + 0.70710678118555i, -1.00000000000000),$$

which define the eigenvalues $\lambda_{1,2} = 0.70710678118755 \pm 0.70710678118754i$ and $\lambda_{3,4} = 0.70710678118555 \pm 0.70710678118555i$. After reordering we have

$$(\tilde{\alpha}_1, \tilde{\beta}_1) = (-0.70710678121274 - 0.70710678121274i, -1.00000000003845),$$
$$(\tilde{\alpha}_2, \tilde{\beta}_2) = (0.70710678121274 - 0.70710678121274i, 1.00000000003845),$$
$$(\tilde{\alpha}_3, \tilde{\beta}_3) = (-0.70710678116035 - 0.70710678116036i, -0.99999999996155),$$
$$(\tilde{\alpha}_4, \tilde{\beta}_4) = (-0.70710678116036 + 0.70710678116036i, -0.99999999996155),$$

which correspond to the eigenvalues

$$\tilde{\lambda}_{1,2} = 0.70710678118555 \pm 0.70710678118555i,$$
$$\tilde{\lambda}_{3,4} = 0.70710678118754 \pm 0.70710678118755i.$$

The eigenvalues outside and inside the unit disc come closer and closer with a decreasing δ and the problem gets more ill-conditioned but we are still able to reorder the eigenvalues with satisfying accuracy. We illustrate the situation in Figure 6.1.

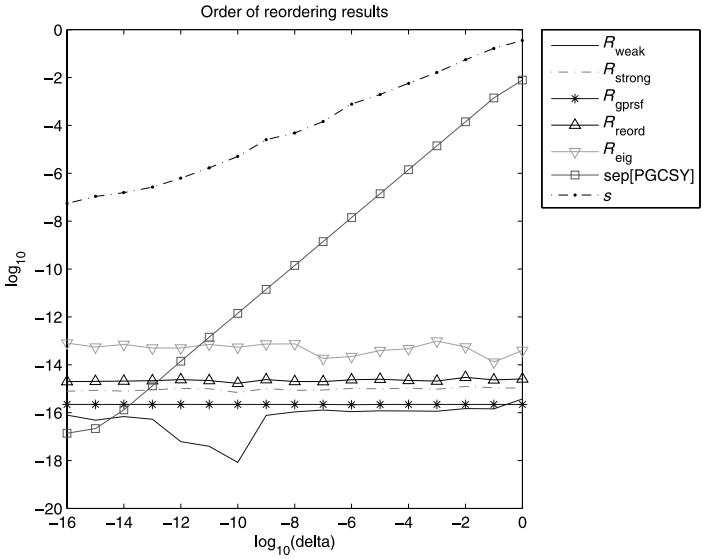

Figure 6.1: Results from reordering the eigenvalues of Examples III and IV with $\delta \in [0, 1]$. The displayed quantities are the same as in Tables 6.1–6.2. The horizontal axis shows the logarithm of the parameter δ and the vertical axis displays the logarithm of the computed quantities.

EXAMPLE V. Consider reordering the following single eigenvalue $\lambda_1 = \sqrt{3}$ with the eigenvalue pair $\lambda_{2,3} = \frac{\sqrt{3}}{2} \pm \frac{1}{\sqrt{7}}i$ and period $K = 5$. The original (α, β)-pairs

are

$$(\alpha_1, \beta_1) = (1.73205080756888, 1.00000000000000),$$
$$(\alpha_2, \beta_2) = (-0.86602540378444 - 0.37796447300923i, -1.00000000000000),$$
$$(\alpha_3, \beta_3) = (0.86602540378444 - 0.37796447300923i, 1.00000000000000).$$

After reordering we have

$$(\tilde{\alpha}_1, \tilde{\beta}_1) = (2.97791477286351 + 1.29966855807374i, 3.43859979147302),$$
$$(\tilde{\alpha}_2, \tilde{\beta}_2) = (-1.43573050214952 + 0.62660416225306i, -1.65783878379957),$$
$$(\tilde{\alpha}_3, \tilde{\beta}_3) = (0.30383422966230, 0.17541877428455),$$

which define eigenvalues $\tilde{\lambda}_{1,2} = 0.86602540378444 \pm 0.37796447300923i$ and $\tilde{\lambda}_3 = 1.73205080756888$.

EXAMPLE VI. Consider reordering the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \infty$ and period $K = 6$. The original (α, β)-pairs are

$$(\alpha_1, \beta_1) = (-0.9999999999999986, 1.000000000000000),$$
$$(\alpha_2, \beta_2) = (1.000000000000000, 0.000000000000000).$$

After reordering we have

$$(\tilde{\alpha}_1, \tilde{\beta}_1) = (-1.564941642946474\text{E}{-5}, 0.000000000000000),$$
$$(\tilde{\alpha}_2, \tilde{\beta}_2) = (6.390014634138052\text{E}{+4}, 6.390014634138062\text{E}{+4}),$$

which correspond to the eigenvalues $\tilde{\lambda}_1 = -\infty$ and $\tilde{\lambda}_2 = 0.9999999999999985$.

7 Remarks.

In this section, we give some closing remarks on the developed reordering method by presenting a comparison with existing methods and describing an extension to more general matrix products.

7.1 Comparison with existing methods.

Hench and Laub [18, Sec. II.F] proposed to swap the diagonal blocks in (3.1) by first explicitely computing the $(n_1 + n_2) \times (n_1 + n_2)$ matrix product

$$E_{K-1}^{-1} A_{K-1} \cdots E_0^{-1} A_0.$$

Then, in exact arithmetic, the standard swapping technique [2] applied to this product yields the outer orthogonal transformation matrix Z_0. The inner orthogonal matrices $Q_0, \ldots, Q_K$, $Z_1, \ldots, Z_K$ are obtained by propagating Z_0 through the triangular factors, using QR and RQ factorizations. In finite-precision arithmetic, however, such an approach can be expected to perform poorly if any of

the matrices E_k is nearly singular, see [21] for the case $K = 1$. Also for very well-conditioned E_k (e.g., identity matrices), there are serious numerical difficulties to be expected for long products as the computed entries become prone to under- and overflow. Further numerical instabilities arise from the fact that triangular matrix-matrix multiplication is in general not a numerically backward stable operation, unless $n_1 = n_2 = 1$ [12].

Benner et al. [3] developed collapsing techniques that can be used to improve the above approach by avoiding all explicit inversions of E_k. Instead of a single product, two $n \times n$ matrices $\bar{E}$ and $\bar{A}$ are computed such that $\bar{E}^{-1}\bar{A}$ has the same eigenvalues. The generalized swapping technique [21, 23] applied to the pair $(\bar{E}, \bar{A})$ yields Z_0. Again, the other orthogonal matrices are successively computed from QR and RQ factorizations. Although this approach avoids difficulties associated with (nearly) singular matrices E_k, it may still become numerically unstable, see [15] for an example.

Bojanczyk and Van Dooren [9] carefully modified the approach by Hench and Laub for the case $n_1 = n_2 = 1$ to avoid underflow, overflow, and numerical instabilities. This variant has been observed to perform remarkably well in finite-precision arithmetic. Unfortunately, its extension to $n_1 = 2$ and/or $n_2 = 2$ is not clear. Thus, only real matrix products having real eigenvalues can be addressed. For complex eigenvalues one could in principle work with the complex periodic Schur decomposition, which has no 2×2 blocks. Both, the swapping technique described in [9] and the one proposed in this paper, extend to the complex case in a straightforward manner. The obvious drawback of using complex arithmetic for real input data is the increased computational complexity. Moreover, real eigenvalues and complex conjugate eigenvalue pairs will not be preserved in finite-precision arithmetic. For example, if we apply [9] to Example I we obtain the following swapped eigenvalues:

$$\tilde{\lambda}_1 = -1.87282049572853 + 0.58861866785157i,$$
$$\tilde{\lambda}_2 = 1.74709648107590 + 0.47770138864644i.$$

The realness of the original eigenvalues is completely lost. Somewhat unexpectedly, our algorithm also achieves significantly higher accuracy for this particular example.

7.2 Reordering in even more general matrix products.

Reordering can also be considered in matrix products of the form

$$A_{K-1}^{s_{K-1}} A_{K-2}^{s_{K-2}} \cdots A_0^{s_0}, \quad s_0, \ldots, s_{K-1} \in \{1, -1\}, \tag{7.1}$$

which is needed, e.g., in [4]. This could be accomplished by the method described in this paper after inserting identity matrices into the matrix sequence such that the exponent structure has the same structure as in Equation (1.2), i.e., every second matrix is an inverse. It turns out that this trick is actually not needed. All techniques developed in this paper can be extended to work directly with (7.1).

For example, the associated periodic Sylvester-like matrix equation takes the form

$$\begin{cases} A_{11}^{(k)} X_k - X_{k+1} A_{22}^{(k)} = -A_{12}^{(k)}, & \text{for } s_k = 1, \\ A_{11}^{(k)} X_{k+1} - X_k A_{22}^{(k)} = -A_{12}^{(k)}, & \text{for } s_k = -1, \end{cases}$$

which can be addressed by the methods in Section 5. See [16] for more details.

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