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STRUCTURED EIGENVALUE CONDITION NUMBERS*

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Abstract. This paper investigates the effect of structure-preserving perturbations on the eigenvalues of linearly and nonlinearly structured eigenvalue problems. Particular attention is paid to structures that form Jordan algebras, Lie algebras, and automorphism groups of a scalar product. Bounds and computable expressions for structured eigenvalue condition numbers are derived for these classes of matrices, which include complex symmetric, pseudo-symmetric, persymmetric, skew-symmetric, Hamiltonian, symplectic, and orthogonal matrices. In particular we show that under reasonable assumptions on the scalar product, the structured and unstructured eigenvalue condition numbers are equal for structures in Jordan algebras. For Lie algebras, the effect on the condition number of incorporating structure varies greatly with the structure. We identify Lie algebras for which structure does not affect the eigenvalue condition number.

Key words. structured eigenvalue problem, condition number, Jordan algebra, Lie algebra, automorphism group, symplectic, perplectic, pseudo-orthogonal, pseudo-unitary, complex symmetric, persymmetric, perskew-symmetric, Hamiltonian, skew-Hamiltonian, structure preservation

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1. Introduction.

There is a growing interest in structured perturbation analysis due to the substantial development of algorithms for structured problems. When these algorithms preserve structure (see, for example, [2], [4], [13], and the literature cited therein) it is often appropriate to consider condition numbers that measure the sensitivity to structured perturbations. In this paper we investigate the effect of structure-preserving perturbations on linearly and nonlinearly structured eigenvalue problems.

Suppose that $\mathbb{S}$ is a class of structured matrices and define the (absolute) *structured condition number* of a simple eigenvalue λ of $A \in \mathbb{S}$ by

$$\kappa(A,\lambda;\mathbb{S}) = \lim_{\epsilon\to 0}\sup\Big\{\frac{|\widehat{\lambda}-\lambda|}{\epsilon} : \widehat{\lambda}\in \mathrm{Sp}(A+E),\ A+E\in\mathbb{S},\ \|E\|\le\epsilon\Big\}, \tag{1.1}$$

where $\mathrm{Sp}(A+E)$ denotes the spectrum of $A+E$ and $\|\cdot\|$ is an arbitrary matrix norm. Let x and y be the normalized right and left eigenvectors associated with λ, i.e.,

$$Ax = \lambda x, \quad y^*A = \lambda y^*, \quad \|x\|_2 = \|y\|_2 = 1.$$

Moreover, let $\kappa(A,\lambda) \equiv \kappa(A,\lambda;\mathbb{C}^{n\times n})$ denote the standard unstructured eigenvalue condition number, where n is the dimension of A. Clearly,

$$\kappa(A,\lambda;\mathbb{S}) \le \kappa(A,\lambda).$$

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If this inequality is not always close to being attained, then $\kappa(A, \lambda)$ may severely overestimate the worst case effect of structured perturbations. Note that the standard eigenvalue condition number allows complex perturbations even if A is real. Our definition in (1.1) automatically forces the perturbations to be real when A is real and $\mathbb{S} \subset \mathbb{R}^{n\times n}$.

In this paper we consider the case where $\mathbb{S}$ is a smooth manifold. This covers linear structures and some nonlinear structures, such as orthogonal, unitary, and symplectic structures. We show that for such $\mathbb{S}$, the structured problem in (1.1) simplifies to a linearly constrained optimization problem. We obtain an explicit expression for $\kappa(A, \lambda; \mathbb{S})$, thereby extending Higham and Higham's work [11] for linear structures in $\mathbb{C}^{n\times n}$.

Associated with a scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$ are three important classes of structured matrices: an automorphism group, a Lie algebra, and a Jordan algebra. We specialize our results to each of these three classes, starting with the linear structures. We show that under mild assumptions on the scalar product, the structured and unstructured eigenvalue condition numbers are equal for structures in Jordan algebras. For example, this equality holds for real and complex symmetric matrices, pseudo-symmetric, persymmetric, Hermitian, and J-Hermitian matrices. For Lie algebras, the effect on the condition number of incorporating structure varies greatly with the structure. We identify Lie algebras for which structure does not affect the eigenvalue condition number, such as skew-Hermitian structures, and Lie algebras for which the ratio between the unstructured and structured eigenvalue condition number can be large, such as skew-symmetric or perskew-symmetric structures. Our treatment extends and unifies recent work on these classes of matrices by Graillat [9] and Rump [17].

Finally we show how to compute structured eigenvalue condition numbers when $\mathbb{S}$ is the automorphism group of a scalar product. This includes the classes of unitary, complex orthogonal, and symplectic matrices. We provide bounds for the ratio between the structured and unstructured condition number. In particular we show that for unitary matrices this ratio is always equal to 1. This latter result also holds for orthogonal matrices with one exception: when λ is real and simple, the structured eigenvalue condition number is zero.

Note that for $\lambda \neq 0$ a relative condition number, on both data and output spaces, can also be defined, which is just $\kappa(A, \lambda; \mathbb{S})\|A\|/|\lambda|$. Our results comparing structured and unstructured absolute condition numbers clearly apply to relative condition numbers without change.

The rest of this paper is organized as follows. Section 2 provides the definition and a computable expression for the structured eigenvalue condition number of a nonlinearly structured matrix. In section 3, we introduce the scalar products and the associated structures to be considered. We treat linear structures (Jordan and Lie algebras) in section 4 and investigate the corresponding structured condition numbers. Nonlinear structures (automorphism groups) are discussed in section 5.

2. Structured condition number.

It is well known that simple eigenvalues $\lambda \in \mathrm{Sp}(A)$ depend analytically on the entries of A in a sufficiently small open neighborhood $\mathcal{B}_A$ of A [18]. To be more specific, there exists a uniquely defined analytic function $f_\lambda : \mathcal{B}_A \to \mathbb{C}$ so that $\lambda = f_\lambda(A)$ and $\widehat{\lambda} = f_\lambda(A+E)$ is an eigenvalue of $A+E$ for every $A + E \in \mathcal{B}_A$. Moreover, one has the expansion

$$\hat{\lambda} = \lambda + \frac{1}{|y^*x|} y^*Ex + O(\|E\|^2). \tag{2.1}$$

Combined with (1.1) this yields

$$\kappa(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|}\lim_{\epsilon\to 0}\sup\Big\{\frac{|y^*Ex|}{\epsilon}:\ A+E\in\mathbb{S},\ \|E\|\le\epsilon\Big\}. \tag{2.2}$$

The difficulty in obtaining an explicit expression for the supremum in (2.2) depends on the nature of $\mathbb{S}$ and the matrix norm $\|\cdot\|$. For example, when $\|\cdot\|$ is the Frobenius norm or the matrix 2-norm and for unstructured perturbations (i.e., $\mathbb{S}=\mathbb{C}^{n\times n}$), the supremum in (2.2) is attained by $E=\epsilon yx^*$, which implies the well-known formula [20]

$$\kappa_\nu(A,\lambda) = 1/|y^*x|, \quad \nu = 2, F.$$

Note that $\kappa_\nu(A,\lambda)\ge 1$ always, but $\kappa_\nu(A,\lambda;\mathbb{S})$ can be less than 1 for $\nu=2,F$.

When $\mathbb{S}$ is a smooth manifold (see [12] for an introduction to smooth manifolds), the task of computing the supremum (2.2) simplifies to a linearly constrained optimization problem.

THEOREM 2.1. *Let λ be a simple eigenvalue of $A\in\mathbb{S}$, where $\mathbb{S}$ is a smooth real or complex submanifold of $\mathbb{K}^{n\times n}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$). Then for any norm $\|\cdot\|$ on $\mathbb{K}^{n\times n}$ the structured condition number of λ with respect to $\mathbb{S}$ is given by*

$$\kappa(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|}\max\left\{|y^*Hx|:\ H\in T_A\mathbb{S},\ \|H\|=1\right\}, \tag{2.3}$$

where $T_A\mathbb{S}$ is the tangent space of $\mathbb{S}$ at A.

Proof. We show that $\lim_{\epsilon\to 0}\beta_\epsilon=\phi$, where

$$\begin{aligned}\beta_\epsilon &:= \sup\left\{\frac{|y^*Ex|}{\epsilon}:\ A+E\in\mathbb{S},\ \|E\|\le\epsilon\right\}, \qquad \epsilon>0,\\ \phi &:= \max\left\{|y^*Hx|:\ H\in T_A\mathbb{S},\ \|H\|=1\right\}.\end{aligned}$$

Let d denote the real dimension of $\mathbb{S}$. By definition of a smooth submanifold of a finite dimensional vector space there exist open neighborhoods $\mathcal{U}\subset\mathbb{R}^d$ of $0\in\mathbb{R}^d$ and $\mathcal{V}\subset\mathbb{K}^{n\times n}$ of A and a continuously differentiable map $F:\mathcal{U}\to\mathcal{V}$ with the following properties.

(i) $F(\mathcal{U})=\mathbb{S}\cap\mathcal{V}$.

(ii) F is a homeomorphism between $\mathcal{U}$ and $\mathbb{S}\cap\mathcal{V}$.

(iii) If $D_0F:\mathbb{R}^d\to\mathbb{K}^{n\times n}$ denotes the differential of F at $0\in\mathbb{R}^d$, then

(a) for all $\xi\in\mathcal{U}$, $F(\xi)=A+D_0F(\xi)+R(\xi)$ and the map $R:\mathcal{U}\to\mathbb{K}^{n\times n}$ satisfies

$$\lim_{\xi\to 0}\|R(\xi)\|/|\!|\!|\xi|\!|\!| = 0, \tag{2.4}$$

where $|\!|\!|\cdot|\!|\!|$ is an arbitrary norm on $\mathbb{R}^d$,

(b) D_0F is an injective linear map, i.e., $0<s:=\min_{|\!|\!|\xi|\!|\!|=1}\|D_0F(\xi)\|$,

(c) $T_A\mathbb{S}=\mathrm{range}(D_0F)$.

A map F with all the properties (i)–(iii) is called a local parametrization of $\mathbb{S}$ at the point A. The neighborhoods $\mathcal{U}$ and $\mathcal{V}$ can be chosen such that

(d) $\|R(\xi)\|\le\frac{1}{2}s\,|\!|\!|\xi|\!|\!|$ for all $\xi\in\mathcal{U}$.

Suppose now that $A+E \in \mathbb{S}$ and $0 < \|E\| \le \epsilon$. If ϵ is small enough, then, by (i) and (ii), there is a unique nonzero $\xi \in \mathcal{U}$ such that $A+E = F(\xi) = A + D_0F(\xi) + R(\xi)$. Hence

$$E = D_0F(\xi) + R(\xi). \tag{2.5}$$

By (b) and (d),

$$\epsilon \ge \|E\| \ge \|D_0F(\xi)\| - \|R(\xi)\| \ge \frac{s}{2}\,\|\xi\|. \tag{2.6}$$

This implies

$$\frac{\|D_0F(\xi)\|}{\epsilon} \le \frac{\|E\|}{\epsilon} + \frac{\|R(\xi)\|}{\epsilon} \le 1 + \frac{2}{s}\,\frac{\|R(\xi)\|}{\|\xi\|} \tag{2.7}$$

and

$$\frac{|y^*R(\xi)x|}{\epsilon} \le \frac{2}{s}\,\frac{|y^*R(\xi)x|}{\|\xi\|} \le \frac{2\,c}{s}\,\frac{\|R(\xi)\|}{\|\xi\|}, \tag{2.8}$$

where $c := \max\{|y^*Mx| \,:\, M \in \mathbb{K}^{n\times n},\ \|M\| = 1\}$. Using (2.5), (2.7), (2.8), and (c) we obtain the estimate

$$\begin{aligned} \frac{|y^*Ex|}{\epsilon} &\le \frac{|y^*D_0F(\xi)x|}{\epsilon} + \frac{|y^*R(\xi)x|}{\epsilon} \\ &\le \left(1 + \frac{2}{s}\,\frac{\|R(\xi)\|}{\|\xi\|}\right)\frac{|y^*D_0F(\xi)x|}{\|D_0F(\xi)\|} + \frac{2\,c}{s}\,\frac{\|R(\xi)\|}{\|\xi\|} \\ &\le \left(1 + \frac{2}{s}\,\frac{\|R(\xi)\|}{\|\xi\|}\right)\phi + \frac{2\,c}{s}\,\frac{\|R(\xi)\|}{\|\xi\|}. \end{aligned} \tag{2.9}$$

The relations (2.4) and (2.9) yield $\lim_{\epsilon\to 0}\beta_\epsilon \le \phi$. In order to show equality let $\widehat{H} \in T_A\mathbb{S}$ be such that $\|\widehat{H}\| = 1$ and $|y^*\widehat{H}x| = \phi$. By (c) there exists a $\hat{\xi} \in \mathbb{R}^d$ with $D_0F(\hat{\xi}) = \widehat{H}$. For $t \ge 0$ let $E_t = D_0F(t\,\hat{\xi}) + R(t\,\hat{\xi})$ and $\epsilon_t = \|E_t\|$. Then $A + E_t = F(t\,\hat{\xi}) \in \mathbb{S}$, $\lim_{t\to 0}\epsilon_t = 0$, and $\lim_{t\to 0}|y^*E_tx|/\epsilon_t = |y^*\widehat{H}x| = \phi$. Thus, $\lim_{\epsilon\to 0}\beta_\epsilon \ge \phi$, and the proof is complete. $\square$

It is convenient to introduce the notation

$$\phi(x,y;\mathbb{S}) = \max\Big\{|y^*Ex| :\ E \in \mathbb{S},\ \|E\| = 1\Big\} \tag{2.10}$$

so that (2.3) can be rewritten as

$$\kappa(A,\lambda;\mathbb{S}) = \phi(x,y;T_A\mathbb{S})/|y^*x|. \tag{2.11}$$

In a similar way to [19], an explicit expression for $\kappa(A,\lambda;\mathbb{S})$ can be obtained if one further assumes that the matrix norm $\|\cdot\|$ under consideration is the Frobenius norm $\|\cdot\|_F$. Let us rewrite

$$y^*Ex = \operatorname{vec}(y^*Ex) = (x^T \otimes y^*)\operatorname{vec}(E) = (\overline{x} \otimes y)^*\operatorname{vec}(E),$$

where $\otimes$ denotes the Kronecker product and vec denotes the operator that stacks the columns of a matrix into one long vector [8, p. 180]. Note that $T_A\mathbb{S}$ is a linear vector space of dimension $m \leq n^2$. Hence, there is an $n^2 \times m$ matrix B such that for every $E \in T_A\mathbb{S}$ there exists a uniquely defined parameter vector p with

$$\operatorname{vec}(E) = Bp, \quad \|E\|_F = \|p\|_2. \tag{2.12}$$

Any matrix B satisfying these properties is called a *pattern matrix* for $T_A\mathbb{S}$; see also [10], [19], and [6]. The relationships in (2.12) together with (2.10) yield

$$\phi_F(x, y; T_A\mathbb{S}) = \max\left\{|(\overline{x} \otimes y)^* Bp| : \ \|p\|_2 = 1, \ p \in \mathbb{K}^m\right\}, \tag{2.13}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will use the subscripts F and 2 to refer to the use of the Frobenius and matrix 2-norm in (2.10).

When $\mathbb{K} = \mathbb{C}$ the supremum is taken over all $p \in \mathbb{C}^m$ and consequently, from (2.11),

$$\kappa_F(A, \lambda; \mathbb{S}) = \frac{1}{|y^*x|}\|(\overline{x} \otimes y)^* B\|_2. \tag{2.14}$$

Complications arise if $\mathbb{K} = \mathbb{R}$ but λ is a complex eigenvalue or if B is a complex matrix. In this case, the supremum is also taken over all $p \in \mathbb{R}^m$ but $(\overline{x} \otimes y)^* B$ may be a complex vector. In a similar way as in [5] for the standard eigenvalue condition number we can show that the real structured eigenvalue condition number is within a small factor of the complex one in (2.14). To be more specific,

$$\frac{1}{\sqrt{2}|y^*x|}\|(\overline{x} \otimes y)^* B\|_2 \leq \kappa_F(A, \lambda; \mathbb{S}) \leq \frac{1}{|y^*x|}\|(\overline{x} \otimes y)^* B\|_2; \tag{2.15}$$

see also [9], [17]. To obtain an exact expression for the real structured eigenvalue condition number, let us consider the relation

$$|(\overline{x} \otimes y)^* Bp|^2 = \left|\operatorname{Re}\left((\overline{x} \otimes y)^* B\right)p\right|^2 + \left|\operatorname{Im}\left((\overline{x} \otimes y)^* B\right)p\right|^2,$$

which together with (2.13) implies

$$\kappa_F(A, \lambda; \mathbb{S}) = \frac{1}{|y^*x|}\left\|\begin{bmatrix}\operatorname{Re}\left((\overline{x} \otimes y)^* B\right) \\ \operatorname{Im}\left((\overline{x} \otimes y)^* B\right)\end{bmatrix}\right\|_2. \tag{2.16}$$

For a real pattern matrix B, this formula can be rewritten as

$$\kappa_F(A, \lambda; \mathbb{S}) = \frac{1}{|y^*x|}\|[x_R \otimes y_R + x_I \otimes y_I, \ x_I \otimes y_R - x_R \otimes y_I]^T B\|_2, \tag{2.17}$$

where $x = x_R + \imath x_I$ and $y = y_R + \imath y_I$ with $x_R, x_I, y_R, y_I \in \mathbb{R}^n$. If additionally λ is real, we can choose x and y real and (2.17) reduces to (2.14).

The difficulty in computing (2.14), (2.16), or (2.17) lies in characterizing the tangent space $T_A\mathbb{S}$ and building the pattern matrix B. We show in section 5 how these tasks can be achieved when $\mathbb{S}$ is an automorphism group.

It is difficult to compare the explicit formula for $\kappa_F(A, \lambda; \mathbb{S})$ in (2.14) or (2.16) to that of the standard condition number $\kappa_F(A, \lambda) = 1/|y^*x|$ unless $\mathbb{S}$ has some special structure. Noschese and Pasquini [16] show that for perturbations having an assigned zero structure (or sparsity pattern), (2.14) reduces to

$$\kappa_F(A, \lambda; \mathbb{S}) = \|(yx^*)|_{\mathbb{S}}\|_F / |y^*x|,$$

where $(yx^*)|_{\mathbb{S}}$ means the restriction of the rank-one matrix yx^* to the sparsity structure of $\mathbb{S}$. For example if the perturbation is upper triangular, then $(yx^*)|_{\mathbb{S}}$ is the upper triangular part of yx^*.

Starting from (2.11) we compare in sections 4 and 5 the structured condition number to the unstructured one for structured matrices belonging to the Jordan algebra, Lie algebra, or automorphism group of a scalar product.

3. Structured matrices in scalar product spaces. In this paper a *scalar product* refers to any nondegenerate bilinear or sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A real or complex bilinear form $\langle \cdot, \cdot \rangle$ has a unique matrix representation given by $\langle \cdot, \cdot \rangle = x^T M y$, while a sesquilinear form can be represented by $\langle \cdot, \cdot \rangle = x^* M y$, where the matrix M is nonsingular. We will denote $\langle \cdot, \cdot \rangle$ by $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ as needed. A bilinear form is symmetric if $\langle x, y \rangle = \langle y, x \rangle$, and skew-symmetric if $\langle x, y \rangle = -\langle y, x \rangle$. Hence for a symmetric form $M = M^T$ and for a skew-symmetric form $M = -M^T$. A sesquilinear form is Hermitian if $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and skew-Hermitian if $\langle x, y \rangle = -\overline{\langle y, x \rangle}$. The matrices associated with such forms are Hermitian and skew-Hermitian, respectively.

The *adjoint* $A^\star$ of $A \in \mathbb{K}^{n \times n}$ with respect to $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is the unique matrix satisfying

$$\langle Ax, y \rangle_{\mathrm{M}} = \langle x, A^\star y \rangle_{\mathrm{M}} \quad \text{for all } x, y \in \mathbb{K}^n.$$

It can be shown that the adjoint is given explicitly by

$$A^\star = \begin{cases} M^{-1} A^T M & \text{for bilinear forms,} \\ M^{-1} A^* M & \text{for sesquilinear forms.} \end{cases}$$

It is well known [1] that the set of self-adjoint matrices

$$\mathbb{J} = \left\{ S \in \mathbb{K}^{n \times n} : \langle Sx, y \rangle_{\mathrm{M}} = \langle x, Sy \rangle_{\mathrm{M}} \right\} = \left\{ S \in \mathbb{K}^{n \times n} : S^\star = S \right\}$$

forms a Jordan algebra, while the set of skew-adjoint matrices

$$\mathbb{L} = \left\{ L \in \mathbb{K}^{n \times n} : \langle Lx, y \rangle_{\mathrm{M}} = -\langle x, Ly \rangle_{\mathrm{M}} \right\} = \left\{ L \in \mathbb{K}^{n \times n} : L^\star = -L \right\}$$

forms a Lie algebra. The sets $\mathbb{L}$ and $\mathbb{J}$ are linear subspaces, but they are not closed under multiplication. A third class of matrices associated with $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ are those preserving the form, i.e.,

$$\mathbb{G} = \left\{ G \in \mathbb{K}^{n \times n} : \langle Gx, Gy \rangle_{\mathrm{M}} = \langle x, y \rangle_{\mathrm{M}} \right\} = \{ G \in \mathbb{K}^{n \times n} : \; G^\star = G^{-1} \}.$$

They form a Lie group under multiplication. We refer to $\mathbb{G}$ as an automorphism group. Table 3.1 shows a sample of well-known structured matrices in $\mathbb{J}$, $\mathbb{L}$, or $\mathbb{G}$ associated with some scalar products. In the rest of this paper we concentrate on structures belonging to at least one of these three classes.

The eigenvalues of matrices in $\mathbb{J}$, $\mathbb{L}$, and $\mathbb{G}$ have interesting pairing properties as shown by the following theorem.

THEOREM 3.1 ([14, Thms. 7.2 and 7.6]). *Let $A \in \mathbb{L}$ or $A \in \mathbb{J}$. Then the eigenvalues of A occur in pairs as shown below, with the same Jordan structure for each eigenvalue in a pair.*

	Bilinear	Sesquilinear
$A \in \mathbb{J}$	“no pairing”	$\lambda, \overline{\lambda}$
$A \in \mathbb{L}$	$\lambda, -\lambda$	$\lambda, -\overline{\lambda}$
$A \in \mathbb{G}$	$\lambda, 1/\lambda$	$\lambda, 1/\overline{\lambda}$

Table 3.1

A sampling of structured matrices associated with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{M}}$, where M is the matrix defining the scalar product.

Space	M	**Automorphism group** $\mathbb{G}=\{G : G^{\star}=G^{-1}\}$	**Jordan algebra** $\mathbb{J}=\{S : S^{\star}=S\}$	**Lie algebra** $\mathbb{L}=\{K : K^{\star}=-K\}$
Bilinear forms				
$\mathbb{R}^n$	I	Real orthogonals	Symmetrics	Skew-symmetrics
$\mathbb{C}^n$	I	Complex orthogonals	Complex symmetrics	Cplx skew-symmetrics
$\mathbb{R}^n$	$\Sigma_{p,q}$	Pseudo-orthogonals	Pseudo-symmetrics	Pseudo–skew-symmetrics
$\mathbb{C}^n$	$\Sigma_{p,q}$	Cplx pseudo-orthogonals	Cplx pseudo-symm.	Cplx pseudo skew-symm.
$\mathbb{R}^n$	R	Real perplectics	Persymmetrics	Perskew-symmetrics
$\mathbb{R}^{2n}$	J	Real symplectics	Skew-Hamiltonians	Hamiltonians
$\mathbb{C}^{2n}$	J	Complex symplectics	Cplx J-skew-symm.	Complex J-symmetrics
Sesquilinear forms				
$\mathbb{C}^n$	I	Unitaries	Hermitian	Skew-Hermitian
$\mathbb{C}^n$	$\Sigma_{p,q}$	Pseudo-unitaries	Pseudo-Hermitian	Pseudo–skew-Hermitian
$\mathbb{C}^{2n}$	J	Conjugate symplectics	J-skew-Hermitian	J-Hermitian

Here, $R=\left[\begin{smallmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{smallmatrix}\right]$ and $\Sigma_{p,q}=\left[\begin{smallmatrix} I_p & 0\\ 0 & -I_q\end{smallmatrix}\right]\in\mathbb{R}^{n\times n}$ are symmetric and $J=\left[\begin{smallmatrix} 0 & I_n\\ -I_n & 0\end{smallmatrix}\right]$ is skew-symmetric.

There is no eigenvalue structure property that holds for Jordan algebras of all bilinear forms. However, for certain special classes of $\mathbb{J}$ there may be additional structure in the eigenvalues. For example, it is known that the eigenvalues of any real or complex skew-Hamiltonian matrix all have even multiplicity [7]. More generally we have the following result.

Proposition 3.2 ([14, Prop. 7.7]). *Let $\mathbb{J}$ be the Jordan algebra of any skew-symmetric bilinear form on $\mathbb{K}^n$. Then for any $A\in\mathbb{J}$, the eigenvalues of A all have even multiplicity. Furthermore, all Jordan blocks of a fixed size appear an even number of times.*

Hence we will not consider matrices in these algebras since they cannot have simple eigenvalues.

Many of the results presented in the next two sections require the scalar product defining the structure to be unitary and orthosymmetric: a scalar product $\langle\cdot,\cdot\rangle_{M}$ is *unitary* if αM is unitary for some $\alpha>0$; a scalar product is said to be *orthosymmetric* if

$$M=\begin{cases}\beta M^T,\ \beta=\pm 1 & \text{for bilinear forms,}\\ \beta M^*,\ |\beta|=1 & \text{for sesquilinear forms.}\end{cases}$$

We refer to [14, Definitions A.4 and A.6] for a list of equivalent properties. Note that the classes of structured matrices listed in Table 3.1 are all associated with a scalar product which is both unitary and orthosymmetric with $\alpha=1$ and $\beta\pm 1$.

Remark 3.3. For a sesquilinear form $\langle x,y\rangle_{M}=x^*My$, orthosymmetry means that $M=\beta M^*$ for some $\beta\in\mathbb{C}$ with $|\beta|=1$. Then the matrix $H:=\bar{\beta}^{1/2}M$ is Hermitian and $\langle x, y \rangle_H = \bar{\beta}^{1/2} \langle x, y \rangle_M$, for all $x, y \in \mathbb{C}^n$. Hence

$$\langle Ax, y \rangle_H = \langle x, Ay \rangle_H \quad \Leftrightarrow \quad \bar{\beta}^{1/2} \langle Ax, y \rangle_M = \bar{\beta}^{1/2} \langle x, Ay \rangle_M \quad \Leftrightarrow \quad \langle Ax, y \rangle_M = \langle x, Ay \rangle_M$$

showing that the Jordan algebra of $\langle \cdot, \cdot \rangle_H$ is identical to the Jordan algebra of $\langle \cdot, \cdot \rangle_M$. Similarly the Lie algebras of $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_M$ are identical. Consequently, results for orthosymmetric sesquilinear forms just need to be established for Hermitian sesquilinear forms.

4. Jordan and Lie algebras.

Let $\mathbb{S}$ be the Jordan algebra or Lie algebra of a scalar product on $\mathbb{K}^n$. Since $\mathbb{S}$ is a linear subspace of $\mathbb{K}^{n \times n}$, the tangent space at $A \in \mathbb{S}$ is $\mathbb{S}$ itself. Hence (2.11) becomes

$$\kappa(A, \lambda; \mathbb{S}) = \frac{1}{|y^* x|} \phi(x, y; \mathbb{S}) = \frac{1}{|y^* x|} \max \{ |y^* E x| : \; E \in \mathbb{S}, \; \|E\| = 1 \}. \tag{4.1}$$

Clearly, if there exists $E \in \mathbb{S}$ such that $Ex = y$ and $\|E\| = 1$, then $\kappa(A, \lambda; \mathbb{S}) = \kappa(A, \lambda)$. When $\mathbb{S}$ is the Lie or Jordan algebra of an orthosymmetric scalar product, the next theorem gives necessary and sufficient conditions on two given vectors x and b for there to exist $E \in \mathbb{S}$ mapping x to b .

THEOREM 4.1 ([15, Thm. 3.2]). *Let $\mathbb{S}$ be the Lie algebra $\mathbb{L}$ or Jordan algebra $\mathbb{J}$ of an orthosymmetric scalar product $\langle \cdot, \cdot \rangle_M$ on $\mathbb{K}^n$. Then for any given pair of vectors x, $b \in \mathbb{K}^n$ with $x \neq 0$, there exists $E \in \mathbb{S}$ such that $Ex = b$ if and only if the conditions given in the following table hold:*

$\mathbb{S}$	**Bilinear forms**		**Sesquilinear forms**
	Symmetric	Skew-symmetric	Hermitian
$\mathbb{J}$	always	$b^T M x = 0$	$b^* M x \in \mathbb{R}$
$\mathbb{L}$	$b^T M x = 0$	always	$b^* M x \in \imath\mathbb{R}$

Mackey, Mackey, and Tisseur show that when the scalar product is both orthosymmetric and unitary and $\mathcal{S} = \{E \in \mathbb{S} : \; Ex = b\} \neq \emptyset$ then $\min_{E \in \mathcal{S}} \|E\|_2 = \|b\|_2 / \|x\|_2$ [15, Thm. 5.10]. The minimal 2-norm structured mapping in $\mathcal{S}$ is in general not unique. An explicit characterization of the set $\mathcal{M} = \{E \in \mathcal{S} : \; \|E\|_2 = \min_{A \in \mathcal{S}} \|A\|_2\}$ is given in [15, Thm. 5.10] and it is shown that $\min_{E \in \mathcal{M}} \|E\|_F \leq \sqrt{2} \|b\|_2 / \|x\|_2$. The next result follows.

LEMMA 4.2. *Let $\mathbb{S}$ be the Lie or Jordan algebra of a scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ which is both orthosymmetric and unitary and let $x, b \in \mathbb{K}^n$ of unit 2-norm be such that the relevant condition in Theorem* 4.1 *is satisfied. Then there exists $E \in \mathbb{S}$ such that $Ex = b$ with $\|E\|_2 = 1$ and $\|E\|_F \leq \sqrt{2}$.*

The next lemma will also be useful when $\mathbb{S} \subset \mathbb{R}^{n \times n}$ is a real algebra but the right and left eigenvectors are complex.

LEMMA 4.3 ([17, Lem. 2.5]). *Let $x \in \mathbb{C}^n$ with $\|x\|_2 = 1$ be given. Then there exists a real symmetric matrix S such that $Sx = \mu \overline{x}$ with $\mu \in \mathbb{C}$, $|\mu| = 1$ and $\|S\|_2 = 1$, $\|E\|_F = \sqrt{2}$.*

4.1. Jordan algebras.

Graillat [9] and Rump [17] show that for the structures symmetric, complex symmetric, persymmetric, complex persymmetric, and Hermitian, the structured and unstructured eigenvalue condition numbers are equal for the 2-norm. These are examples of Jordan algebras (see Table 3.1). The next theorem extends these results to all Jordan algebras of a unitary and orthosymmetric scalar product. Unlike the proofs in [9] and [17], our unifying proof does not need to consider each Jordan algebra individually.

THEOREM 4.4. *Let λ be a simple eigenvalue of $A \in \mathbb{J}$, where $\mathbb{J}$ is the Jordan algebra of an orthosymmetric and unitary scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ on $\mathbb{K}^n$. Then, for the 2-norm,*

$$\kappa_2(A, \lambda; \mathbb{J}) = \kappa_2(A, \lambda).$$

Proof. Since the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is unitary, αM is unitary for some $\alpha > 0$. Let x and y be right and left eigenvectors of A associated with λ normalized so that $\|x\|_2 = \|y\|_2 = 1$. From (4.1) and since $\phi_2(x, y; \lambda) \le 1$, we just need to find $E \in \mathbb{J}$ of unit 2-norm such that $|y^* E x| = 1$.

For bilinear forms, orthosymmetry of $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ means that $M = \pm M^T$. Suppose first that $M = M^T$, that is, the bilinear form is symmetric. When $\mathbb{K} = \mathbb{C}$, Lemma 4.2 says that there exists $E \in \mathbb{J}$ such that $Ex = y$ and $\|E\|_2 = 1$. Hence $|y^* E x| = |y^* y| = 1$.

When $\mathbb{K} = \mathbb{R}$, A is real but if λ is complex, then $x, y \in \mathbb{C}^n$ and we cannot use Lemma 4.2 to say that there exists a real $E \in \mathbb{J}$ of unit 2-norm sending x to y. However, $A \in \mathbb{J}$ implies $A = A^\star = M^{-1} A^T M$ so that

$$Ax = \lambda x \Longleftrightarrow \overline{x}^* A^T = \lambda \overline{x}^* \Longleftrightarrow \overline{x}^* M A = \lambda \overline{x}^* M$$

so that we can take $y = (\alpha M)^* \overline{x}$ as a normalized left eigenvector for A associated with λ. From Lemma 4.3 we know there exists a real symmetric S such that $Sx = \mu \overline{x}$, $|\mu| = 1$, and $\|S\|_2 = 1$. Let $E = \alpha M S \in \mathbb{R}^{n \times n}$. Since αM is real orthogonal and $M = M^T$ we have

$$E^\star = M^{-1} E^T M = (\alpha M)^{-1} S (\alpha M)^T (\alpha M) = \alpha M S = E$$

showing that $E \in \mathbb{J}$. Moreover $\|E\|_2 = \|\alpha M S\|_2 = \|S\|_2 = 1$ and $Ex = \alpha M S x = \mu \alpha M \overline{x}$ so that $|y^* E x| = |\mu x^T (\alpha M)^T (\alpha M) \overline{x}| = |\mu x^T \overline{x}| = 1$.

We do not need to consider the skew-symmetric bilinear case ($M = -M^T$) since from Proposition 3.2 the eigenvalues of matrices in Jordan algebras of skew-symmetric bilinear forms all have even multiplicity.

When $\langle \cdot, \cdot \rangle$ is an orthosymmetric sesquilinear form, Remark 3.3 says that we just need to establish the result for $M = M^*$, that is, for Hermitian sesquilinear forms. Let $\mu \in \mathbb{C}$, $|\mu| = 1$ be such that $(\mu y)^* M x \in \mathbb{R}$. Then from Lemma 4.2 there exists $E \in \mathbb{J}$ such that $Ex = \mu y$ and $\|E\|_2 = 1$. ∎

The proof above also shows that for the Frobenius norm,

$$\frac{1}{\sqrt{2}}\, \kappa_F(A, \lambda) \le \kappa_F(A, \lambda; \mathbb{J}) \le \kappa_F(A, \lambda).$$

For Jordan algebras $\mathbb{J}$ of sesquilinear forms, eigenvalues come in pairs λ and $\bar{\lambda}$ and if λ is simple so is $\bar{\lambda}$ (see Theorem 3.1). For unitary scalar products, αM is unitary for some $\alpha > 0$, and, if x and y are normalized right and left eigenvectors associated with λ, then $\alpha M y$ and $\alpha M x$ are normalized right and left eigenvectors associated with $\bar{\lambda}$. Hence, $|(\alpha M x)^* (\alpha M y)| = |x^* y|$ so that

$$\kappa(A, \lambda; \mathbb{J}) = \kappa(A, \bar{\lambda}; \mathbb{J}).$$

4.2. Lie algebras. We show that, with the exception of symmetric bilinear forms, incorporating structure does not affect the eigenvalue condition number for matrices in Lie algebras of scalar products that are both orthosymmetric and unitary. These include as special cases the skew-symmetric, complex skew-symmetric, and skew-Hermitian matrices considered by Rump [17].

THEOREM 4.5. *Let λ be a simple eigenvalue of $A \in \mathbb{L}$, where $\mathbb{L}$ is the Lie algebra of an orthosymmetric and unitary scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ on $\mathbb{C}^n$.*

- *For symmetric bilinear forms,*

$$\kappa_2(A, \lambda; \mathbb{L}) = \left(\max_{\substack{b \in (\overline{Mx})^\perp \\ \|b\|_2 = 1}} |y^* b| \right) \kappa_2(A, \lambda),$$

- *For skew-symmetric bilinear forms or sesquilinear forms,*

$$\kappa_2(A, \lambda; \mathbb{L}) = \kappa_2(A, \lambda).$$

Proof. Since the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is unitary, αM is unitary for some $\alpha > 0$. Let x and y be right and left eigenvectors of A associated with λ normalized so that $\|x\|_2 = \|y\|_2 = 1$.

For bilinear forms, orthosymmetry implies $M = \pm M^T$. Suppose first that $M = M^T$, that is, $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is a symmetric bilinear form. From (4.1) we just need to show that

$$\eta := \max \left\{ |y^* b| : \ b \in (\overline{Mx})^\perp, \ \|b\|_2 = 1 \right\}$$

is equal to $\phi_2(x, y; \mathbb{L})$. Let $E \in \mathbb{L}$ be of unit 2-norm and such that $|y^* E x| = \phi_2(x, y; \mathbb{L})$. Let $b = Ex$. Theorem 4.1 implies that $b^T M x = 0$, i.e., $b \in (\overline{Mx})^\perp$. Also, $\|b\|_2 = \|Ex\|_2 \le 1$. Hence $\phi_2(x, y; \mathbb{L}) \le \eta$. Let $b \in (\overline{Mx})^\perp$ be of unit 2-norm and such that $|y^* b| = \eta$. Lemma 4.2 then implies that there exists $E \in \mathbb{L}$ such that $Ex = b$ and $\|E\|_2 = 1$. Hence $\phi_2(x, y; \mathbb{L}) \ge |y^* E x| = |y^* b| = \eta$.

Now for skew-symmetric bilinear forms, Lemma 4.2 implies that there exists $E \in \mathbb{L}$ such that $Ex = y$ and $\|E\|_2 = 1$ so that $|y^* E x| = |y^* y| = 1$ and equality between the structured and unstructured eigenvalue condition numbers follows.

Finally when $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is an orthosymmetric sesquilinear form, Remark 3.3 says that we just need to prove the result for an Hermitian sesquilinear form ($M = M^*$). Let $\mu \in \mathbb{C}$, $|\mu| = 1$ be such that $\langle \mu y, x \rangle_{\mathrm{M}} = \bar{\mu} y^* M x \in \imath \mathbb{R}$. Then from Lemma 4.2 there exists $E \in \mathbb{L}$ such that $Ex = \mu y$ and $\|E\|_2 = 1$. Hence $|y^* E x| = |\mu y^* y| = 1$. The result follows then from (4.1). $\square$

With a very similar proof we can show that for Lie algebras of orthosymmetric and unitary scalar products and for perturbations measured in the Frobenius norm,

$$\frac{1}{\sqrt{2}} \gamma_{\mathbb{L}} \, \kappa_F(A, \lambda) \le \kappa_F(A, \lambda; \mathbb{L}) \le \gamma_{\mathbb{L}} \, \kappa_F(A, \lambda),$$

where $\gamma_{\mathbb{L}} = \max_{\substack{b \in (\overline{Mx})^\perp \\ \|b\|_2 = 1}} |y^* b|$ for symmetric bilinear forms and $\gamma_{\mathbb{L}} = 1$ otherwise.

Note that Theorem 4.5 deals with complex perturbations only. However, for real bilinear forms the results still hold when λ is real. For complex λ, in view of (2.15) we know that the real structured eigenvalue condition number is within a small factor of the complex one.

Now suppose that $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is symmetric bilinear. For $A \in \mathbb{L}$ we have $A^\star = -A$ and

$$\lambda \langle x, x \rangle_{\mathrm{M}} = \langle \lambda x, x \rangle_{\mathrm{M}} = \langle Ax, x \rangle_{\mathrm{M}} = \langle x, A^\star x \rangle_{\mathrm{M}} = -\langle x, Ax \rangle_{\mathrm{M}} = -\lambda \langle x, x \rangle_{\mathrm{M}}$$

so that if $\lambda \neq 0$, $\langle x, x\rangle_{\mathrm{M}} = (Mx)^T x = 0$, that is, $x \in (\overline{Mx})^{\perp}$. Hence for $\lambda \neq 0$,

$$|y^*x| \leq \max_{\substack{b\in(\overline{Mx})^{\perp}\\ \|b\|_2=1}} |y^*b| \leq 1.$$

When $\lambda = 0$ is an eigenvalue of $A \in \mathbb{L}$,

$$Ax = 0 \Longleftrightarrow -A^{\star}x = 0 \Longleftrightarrow M^{-1}A^T Mx = 0 \Longleftrightarrow (Mx)^T A = 0$$

so that we can take $y = \overline{Mx}$ as a left eigenvector of $\lambda = 0$. Hence if $\lambda = 0$ is simple,

$$\kappa_2(A, 0; \mathbb{L}) = 0 < \kappa_2(A, 0).$$

This result may be surprising but from Theorem 3.1 we know that eigenvalues of Lie algebras of bilinear forms come in pairs $\lambda, -\lambda$ so that for odd dimensions n, $\lambda = 0$ has to be an eigenvalue. Any perturbation of A leaves a simple 0 eigenvalue unchanged. For the special case where $M = I$, i.e., when $\mathbb{L}$ is the set of complex skew-symmetric matrices, Rump [17] exhibits a 3×3 example showing that the ratio $\kappa_2(A, \lambda; \mathbb{L})/\kappa_2(A, \lambda)$ for $\lambda \neq 0$ can be arbitrarily small. Our result shows that this ratio can be arbitrarily small for all Lie algebras of symmetric bilinear forms on $\mathbb{K}^n$.

Since the eigenvalues of matrices in $\mathbb{L}$ come in pairs $\lambda, -\lambda$ for bilinear forms and $\lambda, -\bar{\lambda}$ for sesquilinear forms (see Theorem 3.1) then if $0 \neq \lambda$ is simple so is $-\lambda$ (or $-\bar{\lambda}$). We can show that for unitary scalar products,

$$\kappa(A, \lambda; \mathbb{L}) = \begin{cases} \kappa(A, -\lambda; \mathbb{L}) & \text{for bilinear forms,} \\ \kappa(A, -\bar{\lambda}; \mathbb{L}) & \text{for sesquilinear forms.} \end{cases}$$

5. Automorphism groups.

We now consider structured condition numbers for automorphism groups $\mathbb{G}$ associated with the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{M}}$,

$$\mathbb{G} = \{A \in \mathbb{K}^{n\times n} : A^{\star} = A^{-1}\}.$$

This includes the groups of symplectic matrices ($M = J$), real and complex orthogonal matrices ($M = I$), as well as Lorentz transformations ($M = \mathrm{diag}(1, 1, 1, -1)$). We first show how to compute $\kappa_F(A, \lambda; \mathbb{G})$ in (2.14) and (2.16), then consider properties of the structured condition number, and finally provide lower bounds for $\kappa_2(A, \lambda; \mathbb{G})$.

5.1. Computation of $\kappa_F(A, \lambda; \mathbb{G})$.

An automorphism group $\mathbb{G}$ forms a smooth manifold. The Jacobian of the function

$$\Phi(A) = \begin{cases} A^T MA - M & \text{for bilinear forms,} \\ A^* MA - M & \text{for sesquilinear forms} \end{cases}$$

at $A \in \mathbb{K}^{n\times n}$ can be represented as the linear function

$$J_A(X) = \begin{cases} A^T MX + X^T MA & \text{for bilinear forms,} \\ A^* MX + X^* MA & \text{for sesquilinear forms.} \end{cases}$$

The tangent space $T_A\mathbb{G}$ at $A \in \mathbb{G}$ coincides with the kernel of this Jacobian,

$$T_A\mathbb{G} = \{X \in \mathbb{K}^{n\times n} : J_A(X) = 0\} = \{AH \in \mathbb{K}^{n\times n} : H^{\star} = -H\} = A\cdot\mathbb{L}, \tag{5.1}$$

where $\mathbb{L}$ is the Lie algebra of $\langle \cdot, \cdot \rangle_{\mathrm{M}}$.

TABLE 5.1
Pattern matrices L_M for $M\cdot\mathbb{L} = \mathrm{Sym}(\mathbb{K})$, $\mathrm{Skew}(\mathbb{K})$, or $\mathrm{Herm}(\mathbb{C})$. L_M is such that for any $H \in M\cdot\mathbb{L}$ there exists a uniquely defined parameter vector q with $\mathrm{vec}(H) = L_M q$, $\|H\|_F = \|q\|_2$. Here $n = 2$.

$M\cdot\mathbb{L}$	Sym($\mathbb{K}$)	Skew($\mathbb{K}$)	Herm($\mathbb{C}$)
L_{M}	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & 0\\ 0 & 1/\sqrt{2} & 0\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}0\\ 1/\sqrt{2}\\ -1/\sqrt{2}\\ 0\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1/\sqrt{2} & -\imath/\sqrt{2} & 0\\ 0 & 1/\sqrt{2} & \imath/\sqrt{2} & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

As the Lie algebra $\mathbb{L}$ in (5.1) is independent of A, it is often simple to explicitly construct a pattern matrix L such that for every $H \in \mathbb{L}$ there exists a uniquely defined parameter vector q with $\mathrm{vec}(H) = Lq$. To obtain a pattern matrix B for $A\cdot\mathbb{L}$ in the sense of (2.12), we can compute a QR decomposition $(I \otimes A)L = BR$, where the columns of B form an orthonormal basis for the space spanned by the columns of L, and R is an upper triangular matrix. Hence,

$$\mathrm{vec}(AH) = (I \otimes A)\,\mathrm{vec}(H) = (I \otimes A)Lq = Bp,$$

where $p = Rq$, and $\|AH\|_F = \|\,\mathrm{vec}(AH)\|_2 = \|p\|_2$.

According to (2.14) we have

$$\kappa_F(A, \lambda; \mathbb{G}) = \frac{1}{|y^*x|}\|(\overline{x} \otimes y)^* B\|_2 = \frac{|\lambda|}{|y^*x|}\|(\overline{x} \otimes y)^* LR^{-1}\|_2 \tag{5.2}$$

if $\mathbb{K} = \mathbb{C}$ or if $\mathbb{K} = \mathbb{R}$ with λ real. Otherwise, when $\mathbb{K} = \mathbb{R}$ and λ is complex or, when B is complex, (2.16) implies that

$$\kappa_F(A, \lambda; \mathbb{G}) = \frac{1}{|y^*x|}\left\|\begin{bmatrix}\mathrm{Re}\left(\lambda(\overline{x} \otimes y)^* LR^{-1}\right)\\ \mathrm{Im}\left(\lambda(\overline{x} \otimes y)^* LR^{-1}\right)\end{bmatrix}\right\|_2. \tag{5.3}$$

It is shown in [15, Lem. 5.9] that when the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ defining the structure is orthosymmetric, left multiplication by M is a bijection from $\mathbb{K}^{n\times n}$ to $\mathbb{K}^{n\times n}$ that maps $\mathbb{L}$ and $\mathbb{J}$ to $\mathrm{Skew}(\mathbb{K})$ and $\mathrm{Sym}(\mathbb{K})$ for bilinear forms and a scalar multiple of $\mathrm{Herm}(\mathbb{C})$ for sesquilinear forms, where

$$\mathrm{Skew}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : A^T = -A\}, \quad \mathrm{Sym}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : A^T = A\}$$

are the sets of symmetric and skew-symmetric matrices on $\mathbb{K}^{n\times n}$ and $\mathrm{Herm}(\mathbb{C})$ is the set of Hermitian matrices. More precisely, for bilinear forms on $\mathbb{K}^n$, ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) write,

$$M\cdot\mathbb{L} = \begin{cases}\mathrm{Skew}(\mathbb{K}) & \text{if } M = M^T,\\ \mathrm{Sym}(\mathbb{K}) & \text{if } M = -M^T,\end{cases} \tag{5.4}$$

and for sesquilinear forms on $\mathbb{C}^n$,

$$M\cdot\mathbb{L} = \beta^{1/2}\,\imath\,\mathrm{Herm}(\mathbb{C}), \tag{5.5}$$

where, by orthosymmetry, β is such that $M = \beta M^*$, $|\beta| = 1$. For any $H \in \mathbb{L}$, $MH \in M\cdot\mathbb{L}$ and if L_{M} is pattern matrix for $M\cdot\mathbb{L}$, that is, $\mathrm{vec}(MH) = L_{\mathrm{M}}q$ where q is a uniquely defined vector of parameters, then

$$\mathrm{vec}(H) = \mathrm{vec}(M^{-1}MH) = (I \otimes M^{-1})\,\mathrm{vec}(MH) = (I \otimes M^{-1})L_{\mathrm{M}}q$$

so that $L := (I \otimes M^{-1})L_{\mathrm{M}}$ is a pattern matrix for $\mathbb{L}$. An advantage of using left multiplication by M is that pattern matrices for $\mathrm{Sym}(\mathbb{K})$, $\mathrm{Skew}(\mathbb{K})$, and $\mathrm{Herm}(\mathbb{C})$ are easy to construct (see Table 5.1 for examples of such matrices).

5.2. Properties of $\kappa(A,\lambda;\mathbb{G})$. The eigenvalues of $A \in \mathbb{G}$ come in pairs λ and $1/\lambda$ for bilinear forms, and in pairs λ and $1/\overline{\lambda}$ for sesquilinear forms. In both cases these pairs have the same Jordan structure, and hence the same algebraic and geometric multiplicities (see Theorem 3.1). Hence if λ is simple so is $1/\lambda$ or $1/\bar{\lambda}$. For unitary scalar products, there are interesting relations between the structured condition numbers of these eigenvalue pairings.

THEOREM 5.1. *Let λ be a simple eigenvalue of $A \in \mathbb{G}$, where $\mathbb{G}$ is the automorphism group of a unitary scalar product on $\mathbb{K}^n$. For any unitarily invariant norm, the (absolute) unstructured eigenvalue condition number satisfies*

$$\kappa(A,\lambda) = \begin{cases} \kappa(A,1/\lambda) & \textit{for bilinear forms,} \\ \kappa(A,1/\bar{\lambda}) & \textit{for sesquilinear forms,} \end{cases}$$

whereas the (absolute) structured eigenvalue condition number satisfies

$$\kappa(A,\lambda;\mathbb{G}) = \begin{cases} |\lambda|^2\,\kappa(A,1/\lambda;\mathbb{G}) & \textit{for bilinear forms,} \\ |\lambda|^2\,\kappa(A,1/\bar{\lambda};\mathbb{G}) & \textit{for sesquilinear forms.} \end{cases}$$

Proof. We just prove the bilinear case, the proof for the sesquilinear case being similar. The scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ being unitary implies that αM is unitary for some $\alpha > 0$. If x and y are normalized right and left eigenvectors associated with λ, then $\widetilde{x} = \alpha\overline{My}$ and $\widetilde{y} = \alpha\overline{Mx}$ are right and left normalized eigenvectors belonging to the eigenvalue $1/\lambda$. It is easily checked that $|\widetilde{y}^*\widetilde{x}| = |y^*x|$, and since $\|\cdot\|$ is unitarily invariant, $\phi(\widetilde{x},\widetilde{y};\mathbb{K}^{n\times n}) = \phi(x,y;\mathbb{K}^{n\times n})$ so that $\kappa(A,\lambda) = \kappa(A,1/\lambda)$.

Let $E \in T_A\mathbb{G} = A\cdot\mathbb{L}$. Then $E = AH$ for some H in the Lie algebra $\mathbb{L}$ of $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ and

$$|y^*Ex| = |\lambda|\;|y^*Hx|. \tag{5.6}$$

Also, $A \in \mathbb{G} \Rightarrow M^TA = A^{-T}M^T$, αM unitary $\Rightarrow M^{-T} = \alpha^2\overline{M}$, and $H \in \mathbb{L} \Rightarrow \alpha^2M^TH\overline{M} = -H^T$. Hence,

$$\begin{aligned} |(\alpha\overline{Mx})^*E(\alpha\overline{My})| &= |\alpha^2x^TM^TAH\overline{My}| \\ &= |\alpha^2(x^TA^{-T})(M^TH\overline{M})\overline{y}| \\ &= \frac{1}{|\lambda|}|x^TH^T\overline{y}| \\ &= \frac{1}{|\lambda|}|y^*Hx| = \frac{1}{|\lambda|^2}|y^*Ex| \end{aligned}$$

so that from (2.10) and (2.11), $\kappa(A,\lambda;\mathbb{G}) = \kappa(A,1/\lambda;\mathbb{G})/|\lambda|^2$. ◻

Theorem 5.1 shows that the *relative structured* eigenvalue condition numbers for λ and $1/\lambda$ if the form is bilinear or λ and $1/\bar{\lambda}$ if the form is sesquilinear, are equal. On the other hand, the ratio between the *relative unstructured* eigenvalue condition numbers for λ and $1/\lambda$ (or λ and $1/\bar{\lambda}$) is $1/|\lambda|^2$. Hence, if we use a non–structure-preserving algorithm, we should compute the larger of λ and $1/\lambda$ (or $1/\bar{\lambda}$). In other words, we should compute whichever member of the pair $(\lambda, 1/\lambda)$ (or the pair $(\lambda, 1/\bar{\lambda})$) lies outside the unit circle and then obtain the other one by reciprocation.

5.3. Bounds for $\kappa(A,\lambda;\mathbb{G})$. Lower bounds for the eigenvalue structured condition number can be derived when $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is orthosymmetric and unitary.

THEOREM 5.2. *Let λ be a simple eigenvalue of $A \in \mathbb{G}$, where $\mathbb{G}$ is the automorphism group of an orthosymmetric and unitary scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ on $\mathbb{K}^n$. If $\mathbb{K}=\mathbb{C}$ or, if $\mathbb{K}=\mathbb{R}$ with λ real we have for both the Frobenius norm and the 2-norm $(\nu=2,F)$,*

- *for symmetric bilinear forms,*

$$\frac{|\lambda|}{\|A\|_2} \max_{\substack{b\in(\overline{Mx})^\perp\\ \|b\|_2=1}} |y^*b|\;\kappa_\nu(A,\lambda) \le \kappa_\nu(A,\lambda;\mathbb{G}) \le \max_{\substack{b\in(\overline{Mx})^\perp\\ \|b\|_2=1}} |y^*b|\;\kappa_\nu(A,\lambda),$$

- *for skew-symmetric bilinear or sesquilinear forms,*

$$\frac{|\lambda|}{\|A\|_2}\,\kappa_\nu(A,\lambda) \le \kappa_\nu(A,\lambda;\mathbb{G}) \le \kappa_\nu(A,\lambda).$$

For $\mathbb{K}=\mathbb{R}$ and λ complex, the lower bounds for the Frobenius norm need to be multiplied by $1/\sqrt{2}$.

Proof. Let x and y be right and left eigenvectors of A associated with λ normalized so that $\|x\|_2=\|y\|_2=1$. Let $\mathbb{L}$ be the Lie algebra of $\langle\cdot,\cdot\rangle_{\mathrm{M}}$. From (2.11) and (5.1) we have

$$\kappa(A,\lambda;\mathbb{G}) = \frac{1}{|y^*x|}\phi(x,y;A{\cdot}\mathbb{L}) = \frac{1}{|y^*x|}\max\left\{|y^*AHx| :\; H\in\mathbb{L},\; \|AH\|=1\right\}.$$

By definition of orthosymmetry and from Remark 3.3 we just need to prove the result for symmetric and skew-symmetric bilinear forms and for Hermitian sesquilinear forms.

Suppose first that $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is a symmetric bilinear form on $\mathbb{K}^n$. Let $H_\nu\in\mathbb{L}$ be such that $\|AH_\nu\|_\nu=1$ and $|y^*AH_\nu x|=\phi_\nu(x,y;A{\cdot}\mathbb{L})$, $\nu=2,F$. Let $b_\nu=AH_\nu x$. Theorem 4.1 implies that $(A^{-1}b_\nu)^TMx=0$. Since $M=M^T$ and $A\in\mathbb{G}$, that is, $A^{-1}=A^\star=M^{-1}A^TM$, we have

$$(A^{-1}b_\nu)^TMx=0 \Longleftrightarrow b_\nu^TMAM^{-1}Mx=\lambda\, b_\nu^TMx=0$$

so that $b_\nu\in(\overline{Mx})^\perp$. Also, $\|b_\nu\|_2=\|AH_\nu x\|_2\le 1$. Hence

$$\phi_\nu(x,y;A{\cdot}\mathbb{L}) = |y^*AH_\nu x| = |y^*b_\nu| \le \max\big\{|y^*b| :\; b\in(\overline{Mx})^\perp, \|b\|_2=1\big\},$$

which proves the upper bound. For the lower bound we take $v\in(\overline{Mx})^\perp$ of unit 2-norm and such that $|y^*v|=\max\big\{|y^*b| :\; b\in(\overline{Mx})^\perp, \|b\|_2=1\big\}$. From Lemma 4.2 there exists $S\in\mathbb{L}$ such that $Sx=v$ and $\|S\|_2=1$. Let $\widetilde{H}_\nu=\xi_\nu S$ with $\xi_\nu>0$ such that $\|A\widetilde{H}_\nu\|_\nu=1$, $\nu=2,F$. From $\|A\widetilde{H}_\nu\|_\nu\le\|A\|_\nu\|\widetilde{H}_\nu\|_2$ we have that $\xi_\nu\ge 1/\|A\|_\nu$. Hence

$$\begin{aligned}
\phi_\nu(x,y;A{\cdot}\mathbb{L}) &= |\lambda|\max\left\{|y^*Hx| :\; H\in\mathbb{L},\; \|AH\|_\nu=1\right\}\\
&\ge |\lambda||y^*\widetilde{H}_\nu x|\\
&\ge \frac{|\lambda|}{\|A\|_\nu}|y^*v|\\
&= \frac{|\lambda|}{\|A\|_\nu}\max\big\{|y^*b| :\; b\in(\overline{Mx})^\perp, \|b\|_2=1\big\}
\end{aligned}$$

proving the lower bound.

The lower bound for the skew-symmetric bilinear or Hermitian sesquilinear cases is derived in a similar way to that for the symmetric bilinear case. The only difference being that, from Lemma 4.2, there exists $S \in \mathbb{L}$ of unit 2-norm such that $Sx = y$ if the form is skew-symmetric bilinear and $Sx = \mu y$ for some $\mu \in \mathbb{C}$ such that $(\mu y)^* Mx \in \imath\mathbb{R}$, $|\mu| = 1$ when the form is Hermitian sesquilinear. $\square$

Note that $A \in \mathbb{G}$ implies

$$\lambda \langle x, x\rangle_{\mathrm{M}} = \langle Ax, x\rangle_{\mathrm{M}} = \langle x, A^{-1}x\rangle_{\mathrm{M}} = \frac{1}{\lambda}\langle x, x\rangle_{\mathrm{M}}. \tag{5.7}$$

Hence if $\lambda \neq \pm 1$ we have, for bilinear forms, $\langle x, x\rangle_{\mathrm{M}} = x^T Mx = 0$, that is, $x \in (\overline{Mx})^{\perp}$ so that

$$|y^* x| \le \max_{\substack{b \in (\overline{Mx})^{\perp} \\ \|b\|_2 = 1}} |y^* b| \le 1, \qquad \lambda \neq \pm 1. \tag{5.8}$$

If $\lambda = \pm 1$, then

$$Ax = \pm x \Leftrightarrow x = \pm A^{-1} x \Leftrightarrow x = \pm A^{\star} x \Leftrightarrow Mx = A^T\, Mx \Leftrightarrow (\overline{Mx})^* = \pm (\overline{Mx})^* A$$

so that $y = \overline{Mx}$ is a left eigenvector of A associated with λ. If $M = M^T$, then Theorem 5.2 implies that for both the 2-norm and Frobenius norm,

$$\kappa_\nu(A, \lambda; \mathbb{G}) = 0 \quad \text{ for } \lambda = \pm 1. \tag{5.9}$$

When $M = I$ and $\langle \cdot, \cdot \rangle$ is a sesquilinear form, $\mathbb{G}$ is the set of unitary matrices (see Table 3.1). But unitary matrices are normal and therefore $\kappa_\nu(A, \lambda) = 1$, $\nu = 2, F$. Thus we can expect $\kappa_\nu(A, \lambda; \mathbb{G}) \le 1$. Theorem 5.2 implies that the structured condition number is exactly 1. If $\langle \cdot, \cdot \rangle_M$ with $M = I$ is a real (symmetric) bilinear form, $\mathbb{G}$ is the set of orthogonal matrices. Theorem 5.2 combined with (5.8) and (5.9) says that $\kappa_\nu(A, \lambda; \mathbb{G}) = 0$ if $\lambda = \pm 1$ and $\kappa_\nu(A, \lambda; \mathbb{G}) = 1$ otherwise. We refer to [3] for a more general perturbation analysis of orthogonal and unitary eigenvalue problems, based on the Cayley transform.

Suppose $\mathbb{G}$ is the automorphism group of a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ ($M = -M^T$). For an eigenvalue λ of A with $|\lambda| \approx \|A\|_2$, the bounds in Theorem 5.2 imply

$$\kappa_\nu(A, \lambda; \mathbb{G}) \approx \kappa_\nu(A, \lambda), \quad \nu = 2, F.$$

From Theorem 5.1 we then have

$$|\lambda|^2 \kappa_\nu(A, 1/\lambda; \mathbb{G}) \approx \kappa_\nu(A, 1/\lambda), \quad \nu = 2, F$$

showing that if $|\lambda|$ is large, the unstructured eigenvalue condition number for $1/\lambda$ is much larger than the structured one. The lower bounds in Theorem 5.2 may not be tight when $\max(|\lambda|, 1/|\lambda|) \ll \|A\|_\nu$ as shown by the following example. Suppose that $M = J$ and that $\langle \cdot, \cdot \rangle_{\mathrm{J}}$ is a real bilinear form ($\mathbb{K} = \mathbb{R}$). Then $\mathbb{G}$ is the set of real symplectic matrices (see Table 3.1). Let us consider the symplectic matrix

$$A = \begin{bmatrix} D & D \\ 0 & D^{-1} \end{bmatrix}, \quad D = \mathrm{diag}(10^4, 10^2, 2). \tag{5.10}$$

Define the ratio

$$\rho = \kappa_F(A, \lambda; \mathbb{G}) / \kappa_F(A, \lambda) \le 1$$

TABLE 5.2

Condition numbers for the eigenvalues of the symplectic matrix A in (5.10), ratio ρ between the structured and unstructured condition number, and lower bound γ for this ratio.

λ	10^4	10^2	2	$1/2$	10^{-2}	10^{-4}
$\kappa_F(A,\lambda;\mathbb{G})$	1.2	1.2	1.5	0.4	1.2×10^{-4}	1.2×10^{-8}
ρ	0.87	0.87	0.89	0.22	8.7×10^{-5}	8.7×10^{-9}
γ	0.5	5×10^{-3}	1×10^{-4}	2.5×10^{-5}	5×10^{-7}	5×10^{-9}

between the structured and unstructured eigenvalue condition numbers. $\kappa_F(A,\lambda;\mathbb{G})$ is computed using (5.2) and its values and these of ρ are displayed in Table 5.2 together with the lower bound $\gamma = |\lambda|/(\sqrt{2}\|A\|_2)$ of Theorem 5.2. This example demonstrates the looseness of the bounds of Theorem 5.2 for eigenvalues in the interior of the spectrum. Hence for these eigenvalues the computable expressions in section 5.1 are of interest.

6. Conclusions.

We have derived directly computable expressions for structured eigenvalue condition numbers on a smooth manifold of structured matrices. Furthermore, we have obtained meaningful bounds on the ratios between the structured and unstructured eigenvalue condition numbers for a number of structures related to Jordan algebras, Lie algebras, and automorphism groups. We have identified classes of structured matrices for which this ratio is 1 or close to 1. Hence for these structures, the usual unstructured perturbation analysis is sufficient.

The important task of finding computable expressions for structured backward errors of nonlinearly structured eigenvalue problems is still largely open and remains to be addressed.

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REFERENCES

[1] E. Artin, *Geometric Algebra*, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers Inc., New York, 1957.

[2] P. Benner, V. Mehrmann, and H. Xu, *A numerically stable, structure preserving method for computing the eigenvalues of real Hamiltonian or symplectic pencils*, Numer. Math., 78 (1998), pp. 329–358.

[3] B. Bohnhorst, A. Bunse-Gerstner, and H. Fassbender, *On the perturbation theory for unitary eigenvalue problems*, SIAM J. Matrix Anal. Appl., 21 (2000), pp. 809–824.

[4] A. Bunse-Gerstner, R. Byers, and V. Mehrmann, *A chart of numerical methods for structured eigenvalue problems*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 419–453.

[5] R. Byers and D. Kressner, *On the condition of a complex eigenvalue under real perturbations*, BIT, 44 (2004), pp. 209–214.

[6] P. I. Davies, *Structured conditioning of matrix functions*, Electron. J. Linear Algebra, 11 (2004), pp. 132–161.

[7] H. Fassbender, D. S. Mackey, N. Mackey, and H. Xu, *Hamiltonian square roots of skew-Hamiltonian matrices*, Linear Algebra Appl., 287 (1999), pp. 125–159.

[8] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins University Press, Baltimore, MD, 1996.

[9] S. Graillat, *Structured Condition Number and Backward Error for Eigenvalue Problems*, Research Report RR2005-01, LP2A, University of Perpignan, Perpignan, France, 2005.

[10] D. J. Higham and N. J. Higham, *Backward error and condition of structured linear systems*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 162–175.

[11] D. J. Higham and N. J. Higham, *Structured backward error and condition of generalized eigenvalue problems*, SIAM J. Matrix Anal. Appl., 20 (1998), pp. 493–512.

[12] J. M. Lee, *Introduction to Smooth Manifolds*, Grad. Texts in Math. 218, Springer-Verlag, New York, 2003.

[13] D. S. Mackey, N. Mackey, and D. M. Dunlavy, *Structure preserving algorithms for perplectic eigenproblems*, Electron. J. Linear Algebra, 13 (2005), pp. 10–39.

[14] D. S. Mackey, N. Mackey, and F. Tisseur, *Structured factorizations in scalar product spaces*, SIAM J. Matrix Anal. Appl., 27 (2006), pp. 821–850.

[15] D. S. Mackey, N. Mackey, and F. Tisseur, *Structured Mapping Problems for Matrices Associated with Scalar Products, Part* I: *Lie and Jordan Algebras*, MIMS EPrint 2006.44, Manchester Institute for Mathematical Sciences, The University of Manchester, Manchester, UK, 2006.

[16] S. Noschese and L. Pasquini, *Eigenvalue condition numbers: Zero-structured versus traditional*, J. Comput. Appl. Math., 185 (2006), pp. 174–189.

[17] S. M. Rump, *Eigenvalues, pseudospectrum, and structured perturbations*, Linear Algebra Appl., 413 (2006), pp. 567–593.

[18] G. W. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, London, 1990.

[19] F. Tisseur, *A chart of backward errors and condition numbers for singly and doubly structured eigenvalue problems*, SIAM J. Matrix Anal. Appl., 24 (2003), pp. 877–897.

[20] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, Oxford, 1965.