Spread Spectrum for Universal Compressive Sampling

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***Abstract*—We propose a *universal* and *efficient* compressive sampling strategy based on the use of a spread spectrum technique. The method essentially consists in a random pre-modulation of the signal of interest followed by projections onto randomly selected vectors of an orthonormal basis. The effectiveness of the technique is induced by a decrease of coherence between the sparsity and the sensing bases. The sensing scheme is *universal* for a family of sensing bases in the sense that the number of measurements needed for accurate recovery is optimal and independent of the sparsity matrix. It is also *efficient* as sensing matrices with fast matrix multiplication algorithms can be used. These results are confirmed experimentally through analyses of the phase transition of the ℓ_1-minimization problem.**

I. Spread spectrum technique

Let $\boldsymbol{x} \in \mathbb{C}^N$ be an s-sparse digital signals in an orthonormal basis $\Psi = (\boldsymbol{\psi}_1, ..., \boldsymbol{\psi}_N) \in \mathbb{C}^{N\times N}$ and $\boldsymbol{\alpha} \in \mathbb{C}^N$ be its decomposition in this basis: $\boldsymbol{\alpha} = \Psi^* \boldsymbol{x}$. The spread spectrum technique consists in a pre-modulation of the original signal $\boldsymbol{x}$ by a wide-band signal $\boldsymbol{c} = (c_l)_{1\leqslant l \leqslant N} \in \mathbb{C}^N$, with $|c_l| = 1$ and random phases, and a projection onto m randomly selected vectors of another orthonormal basis $\Phi = (\boldsymbol{\phi}_1, ..., \boldsymbol{\phi}_N) \in \mathbb{C}^{N\times N}$ [2]. The indices $\Omega = \{l_1, \ldots, l_m\}$ of the selected vectors are chosen independently and uniformly at random from $\{1, \ldots, N\}$. We denote Φ^*_Ω the $m \times N$ matrix made of the selected rows of Φ^*. The measurement vector $\boldsymbol{y} \in \mathbb{C}^m$ thus reads as

$$\boldsymbol{y} = \mathsf{A}_\Omega \, \boldsymbol{\alpha} \text{ with } \mathsf{A}_\Omega = \Phi^*_\Omega \mathsf{C} \Psi \in \mathbb{C}^{m\times N}. \quad (1)$$

In the above equation, the matrix $\mathsf{C} \in \mathbb{C}^{N\times N}$ stands for the diagonal matrix associated to the sequence $\boldsymbol{c}$. Finally, we aim at recovering $\boldsymbol{\alpha}$ by solving the ℓ_1-minimization problem

$$\arg \min_{\bar{\boldsymbol{\alpha}} \in \mathbb{C}^N} \|\bar{\boldsymbol{\alpha}}\|_1 \text{ subject to } \boldsymbol{y} = \mathsf{A}_\Omega \bar{\boldsymbol{\alpha}}. \quad (2)$$

II. Reducing the mutual coherence by pre-modulation

In the absence of pre-modulation, i.e. when C is reduced to the identity matrix, the compressive sampling theory already demonstrates that a small number $m \ll N$ of random measurements is sufficient for an accurate and stable reconstruction of $\boldsymbol{\alpha}$ [1]. However, the recovery conditions depend on the mutual coherence $\mu = \max_{1\leqslant i,j\leqslant N} |\langle \boldsymbol{\phi}_i, \boldsymbol{\psi}_j \rangle|$ between Φ and Ψ. The performance is optimal when the bases are perfectly incoherent, i.e. $\mu = N^{-1/2}$, and unavoidably decreases when μ increases.

The spread spectrum technique proposed in this work significantly reduces the mutual coherence μ towards its optimal value [2]. In the presence of a digital pre-modulation by a random Rademacher or Steinhaus sequence $\boldsymbol{c} \in \mathbb{C}^N$, the mutual coherence $\mu = \max_{1\leqslant i,j\leqslant N} |\langle \boldsymbol{\phi}_i, \mathsf{C}\, \boldsymbol{\psi}_j \rangle|$ is essentially bounded by the *modulus-coherence* $\beta^2(\Phi, \Psi) = \max_{1\leqslant i,j\leqslant N} \sum_{k=1}^N |\phi^*_{ki} \psi_{kj}|^2$. Indeed, we can show that the mutual coherence μ satisfies

$$N^{-1/2} \leqslant \mu \leqslant \beta(\Phi, \Psi) \sqrt{2 \log(2N^2/\epsilon)}, \quad (3)$$

with probabilty at least $1 - \epsilon$.

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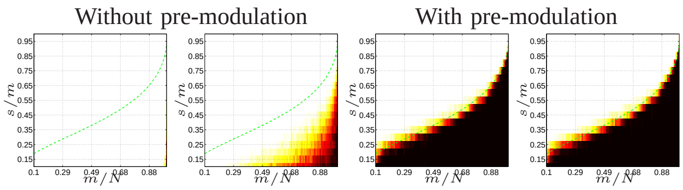

Fig. 1. Phase transition of the ℓ_1-minimization problem for sparse signals in the Fourier basis and random selection of **Fourier** (first and third panels) or **Hadamard** (second and fourth panels) measurements without and with random modulation. The dashed green line indicates the weak phase transition of Donoho-Tanner [3] and the color bar goes from white to black indicating a probability of recovery from 0 to 1. The domain of recovery becomes optimal with the spread spectrum technique.

Definition 1. *(Universal sensing basis) An orthonormal basis $\Phi \in \mathbb{C}^{N\times N}$ is called a universal sensing basis if all its entries ϕ_{ki}, $1 \leqslant k, i \leqslant N$, are of equal complex magnitude.*

For universal sensing bases Φ, e.g. the Fourier, Hadamard, or noiselet transform, we have $\beta(\Phi, \Psi) = N^{-1/2}$ whatever the sparsity matrix Ψ. The mutual coherence μ is thus equal to its optimal value, up to a logarithmic factor, whatever the sparsity matrix considered!

III. Spread spectrum universality

Theorem 1. *Let $\boldsymbol{c} \in \mathbb{C}^N$, with $N > 3$, be a random Rademacher or Steinhaus sequence and $\boldsymbol{y}$ satisfying equation (1). For universal sensing bases $\Phi \in \mathbb{C}^{N\times N}$ and for a universal constant $C > 0$, if $m \geqslant C\, s \log^8(N)$, then $\boldsymbol{\alpha}$ is the unique minimizer of the ℓ_1-minimization problem (2) with probability at least $1 - \mathcal{O}\left(N^{-\log^3(N)}\right)$.*

For universal sensing bases, the spread spectrum technique is thus *universal*: the recovery condition does not depend on the sparsity basis and the number of measurements needed to reconstruct sparse signals is optimal in the sense that it is reduced to the sparsity level s. The experimental study of the phase transitions of the ℓ_1-minimization problem confirms this result (see Figure 1). The spread spectrum technique is also *efficient* as the pre-modulation only requires a sample-by-sample multiplication between $\boldsymbol{x}$ and $\boldsymbol{c}$ and fast matrix multiplication algorithms are available for several universal sensing bases such as the Fourier, Hadamard, or noiselet bases.

IV. Conclusion

We presented a *universal* and *efficient* compressive sampling strategy based on spread spectrum. For applications such as radio interferometry and MRI, this technique is of great interest to optimize the number of measurements needed for an accurate recovery [4], [5].

References

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