Alternative(s) to fractional-diffusion equations in bedload transport models

Christophe Ancey, Valentin Métraux, François Mettra

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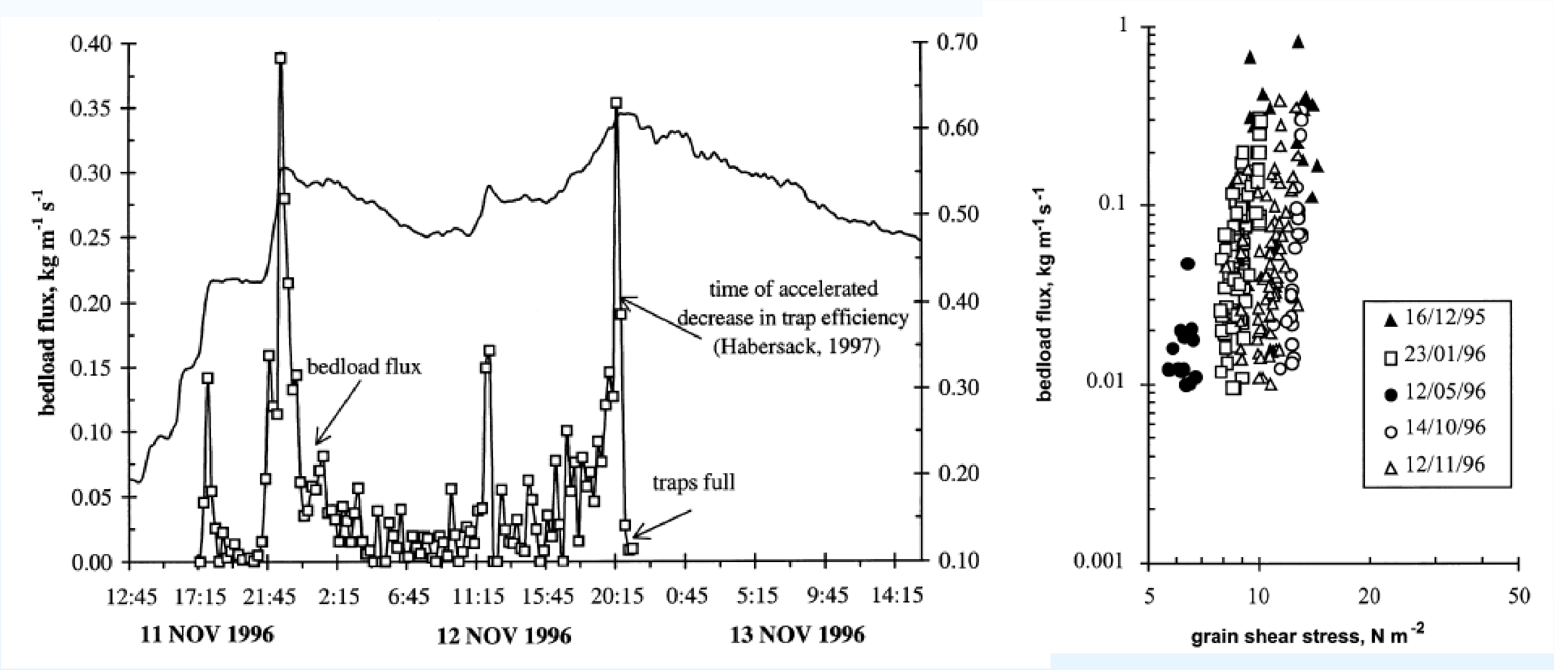
Outline

- Fluctuations in bedload transport rates
- Approaches to ‘stochastic’ bedload transport rate
- Extension of a birth-death Markov process
- Comparison with experiments

Fluctuations in sediment transport

Evidence that bedload transport is highly fluctuating, with pulses more or less correlated with the water discharge

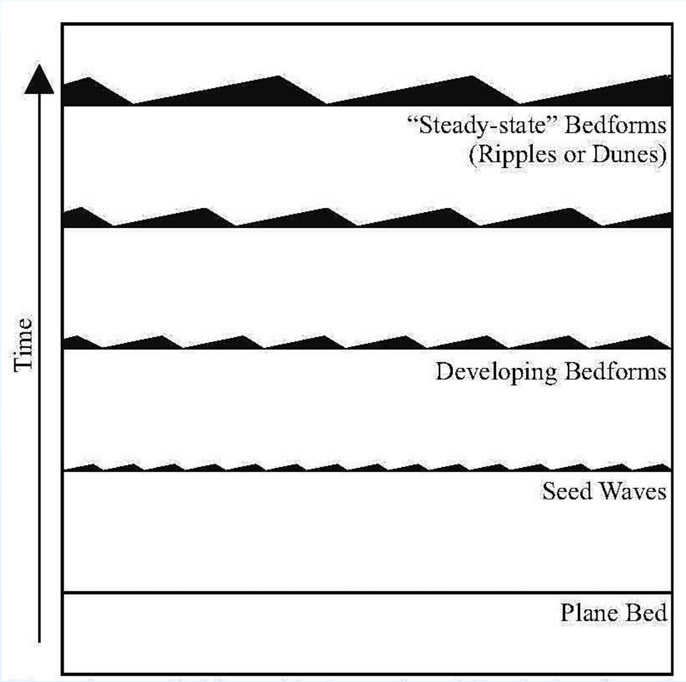
Tordera River (Spain) gravel-bed river (2% slope)

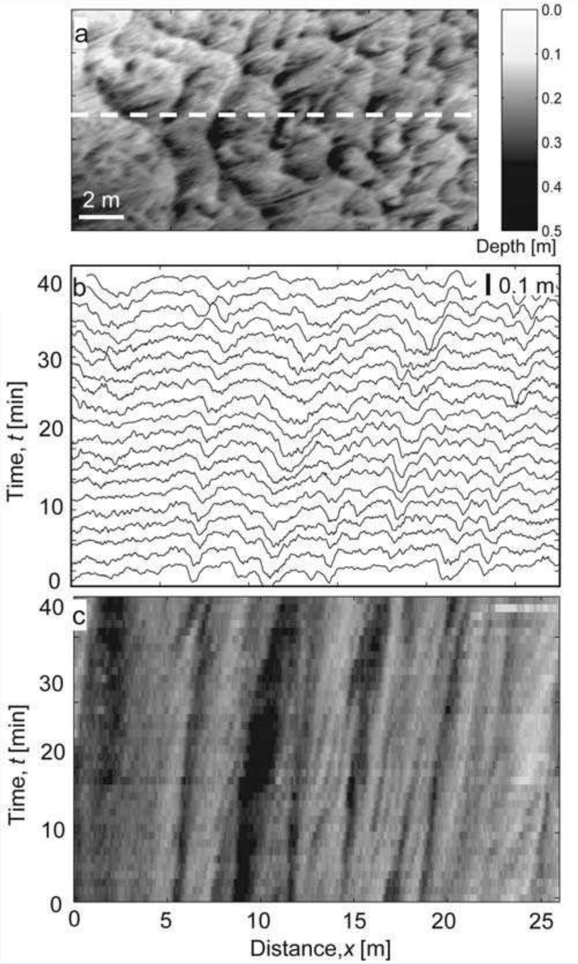


Garcia *et al.*, Geomorphology, **34**, 23–31, 2000.

Bedform

Bedforms: chaotic or regular forms?





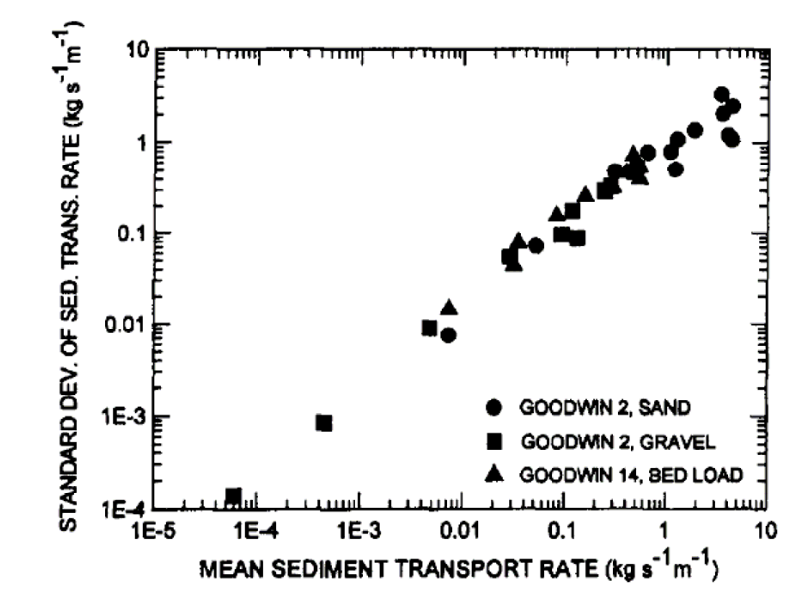
Coleman & Melville, *J. Hydr. Eng.*, **122** 301–310, 1996.
Jerolmack & Mohrig, *Water Resour. Res.*, **41** W12421, 2005.

Einstein and Hamamori's distributions

$$P(z) = -\frac{\sqrt{7}}{12} \ln\left(\frac{\sqrt{7}}{12} z + \frac{1}{4}\right)$$

with $z = (Q_s - \bar{Q}_s)/\sqrt{\operatorname{var} Q_s}$ and $\operatorname{var} Q_s = 7\bar{Q}_s^2/9$.

Goodwin Creek data



Kuhnle & Willis, *J. Hydr. Eng.* **124**, 1109–1114 1998.

Motivation

Approaches to bedload transport
- Einstein and Hamamori's distributions
- Advection diffusion equation
- Markov model
- Steady-state solution

Model extension

Bedform dynamics

Conclusions

Advection diffusion equation

The classic advection diffusion equation (ADE)

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x} + \frac{\partial}{\partial x}\left(K\frac{\partial c}{\partial x}\right)$$

can be generalized into a fractional ADE

$$\frac{\partial c}{\partial t} = u\frac{\partial c}{\partial x} + K\frac{\partial^\gamma c}{\partial x^\gamma}$$

with c particle concentration, K diffusivity, γ fractional exponent ($0 < \gamma < 2$, $\gamma = 2$ gives the classic ADE).
Few experimental evidence to date, but Nikora *et al.* found

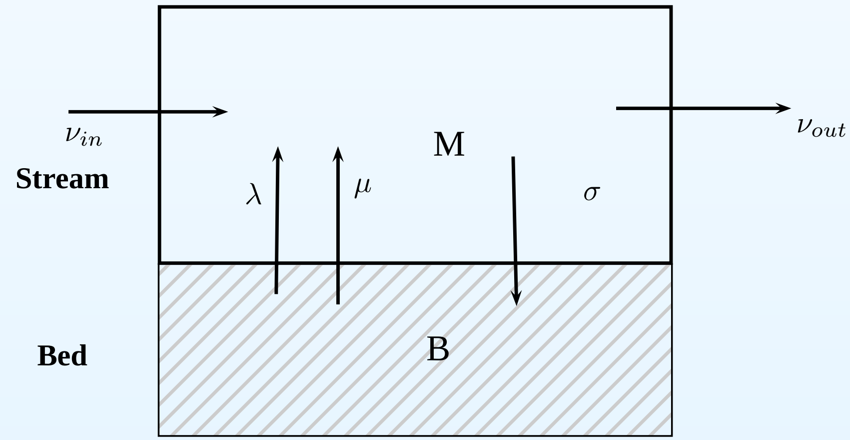
- $\operatorname{var} d(t) \propto t^\gamma$ with $\gamma \geq 2$ for short timescales (i.e., seconds)
- $\gamma < 2$ for long timescales (i.e., minutes)

Nikora *et al.*, *Water Resour. Res.* **38**, WR000513, 2002

Markov model

Emigration-immigration birth-death model

$$\frac{\partial}{\partial t}P(n;t) = (n+1)\alpha P(n+1;t) + (\beta + (n-1)\mu)\,P(n-1;t) - (\beta + n(\alpha+\mu))\,P(n;t),$$



Ancey et al., *J. Fluid. Mech.* **595**, 83–114, 2008;
Ancey, J. Geophys. Res., **115** F00A11, 2010.

Steady-state solution

Under steady state conditions, the solution is the negative binomial distribution

$$P(n) = \mathrm{NegBin}(n; r, p) = \frac{\Gamma(r+n)}{\Gamma(r)n!} p^r (1-p)^n, n = 0, 1, \ldots,$$

with $r = (\lambda + \nu_{in})/\mu$ and $p = 1 - \mu/(\sigma + \nu_{out})$, and where Γ denotes the gamma function.

$$\bar{N} = r\frac{1-p}{p} = (\lambda + \nu_{in})/(\sigma + \nu_{out} - \mu)$$

$$\mathrm{var}N = r\frac{1-p}{p^2} = \frac{(\lambda + \nu_{in})(\sigma + \nu_{out})}{(\sigma + \nu_{out} - \mu)^2}.$$

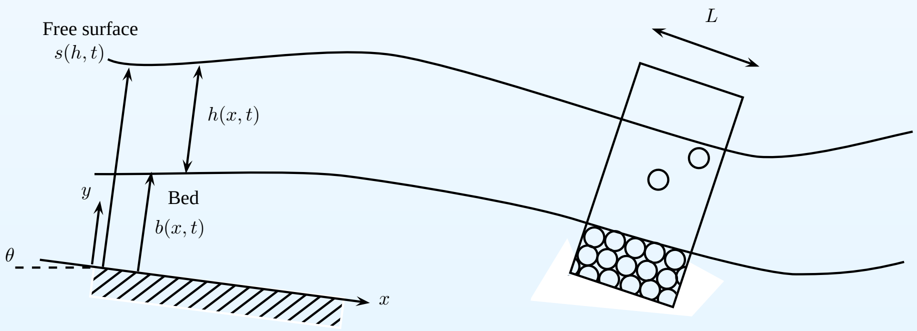
Note: (i) a tail falling off like $n^r(1-p)^n$. (ii) When $p \to 0$, $\sqrt{\mathrm{var}}/\bar{N} \to \sqrt{r}$

Exner equation

Bed mass conservation

$$(1-\phi_p)\frac{\partial b}{\partial t} = D - E = -\frac{\partial q_s}{\partial x},$$

ϕ_p bed porosity, D (E, resp.) volume rate of deposition (entrainment, resp.) per unit time and per unit bed surface onto the bed, and q_s solid discharge per unit width.



$$E = \frac{v_p}{L}(\lambda + \mu N) = v_p(\tilde{\lambda} + \mu\tilde{N}) \text{ and } D = \frac{v_p}{L}\sigma N = v_p\sigma\tilde{N}.$$

Large-number approximation

The master equation reads

$$\frac{\partial}{\partial t}P(n;t) = (n+1)\alpha P(n+1;t) + \{\beta + (n-1)\mu\}\, P(n-1;t) - \{\beta + n(\alpha+\mu)\}\, P(n;t),$$

In the $N \to \infty$ limit, $P(n+1;t) \approx P(N) + 1 \cdot P'(N)$; the master equation is approximated by a Fokker-Planck equation

$$\frac{\partial}{\partial t}P(n;t) = -\frac{\partial}{\partial n}(AP) + \frac{1}{2}\frac{\partial^2}{\partial n^2}(BP),$$

with $A = \beta + n(\mu - \alpha)$ (the *drift function*) and $B = (\alpha+\mu)n + \beta$ (the *diffusion function*). This PDE is equivalent to the SDE

$$dN = A(N)dt + B^{1/2}(N)d\mathcal{W}(t),$$

where $\mathcal{W}(t)$ is the Wiener process.

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Expansion

Introducing $\epsilon = L^{-1/2}$ and $\tilde{N}$ (number of moving particles per unit length)

$$\tilde{N} = \underbrace{\tilde{N}_*}_{\text{deterministic component}} + \underbrace{\epsilon\eta_1(t)}_{\text{first-order correction}} + \cdots .$$

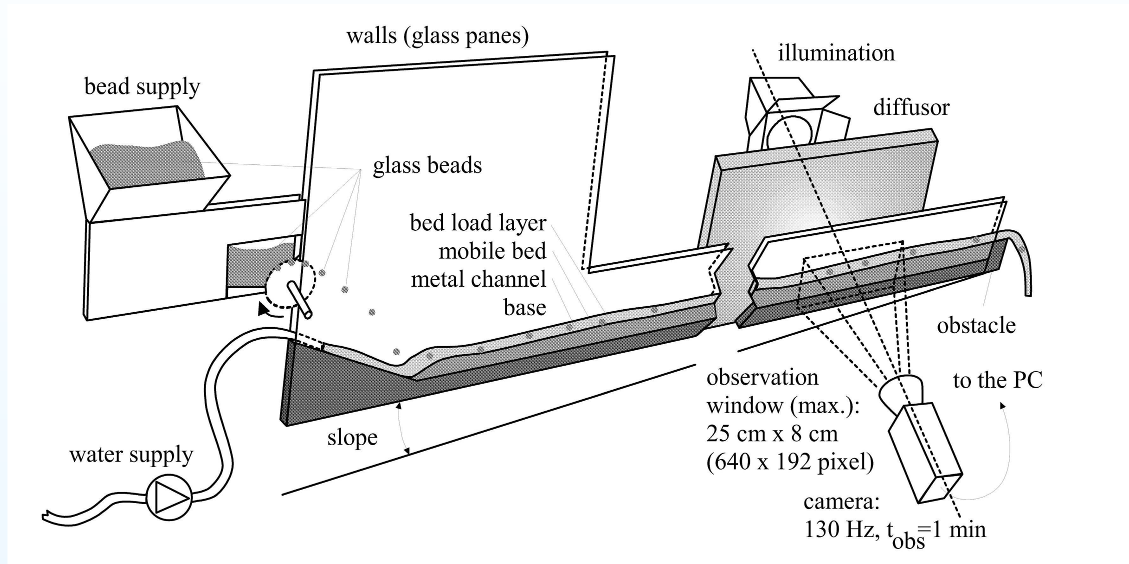
Deterministic component ("sure function")

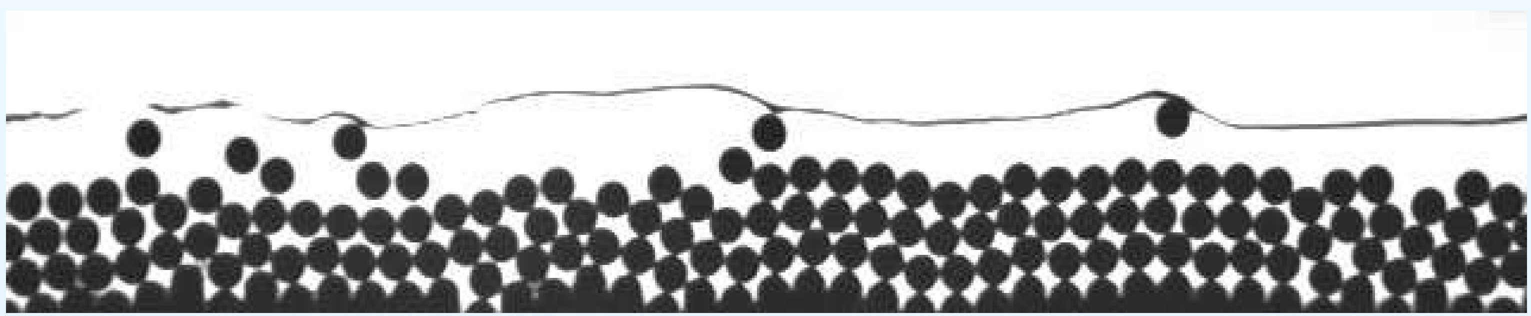
$$\tilde{N}_*(t) = \frac{\tilde{\lambda} - e^{-t(\sigma-\mu)}(\tilde{\lambda} + N_0(\mu - \sigma))}{\sigma - \mu}.$$

The long-term behavior can be approximated by an Ornstein-Uhlenbeck process

$$t \to \infty,\ \eta_1 \sim \mathcal{N}\left(0, \frac{D_o}{2k_o}\right) = \mathcal{N}\left(0, \frac{\tilde{\lambda}\sigma}{(\sigma-\mu)^2}\right).$$

Comparison with data





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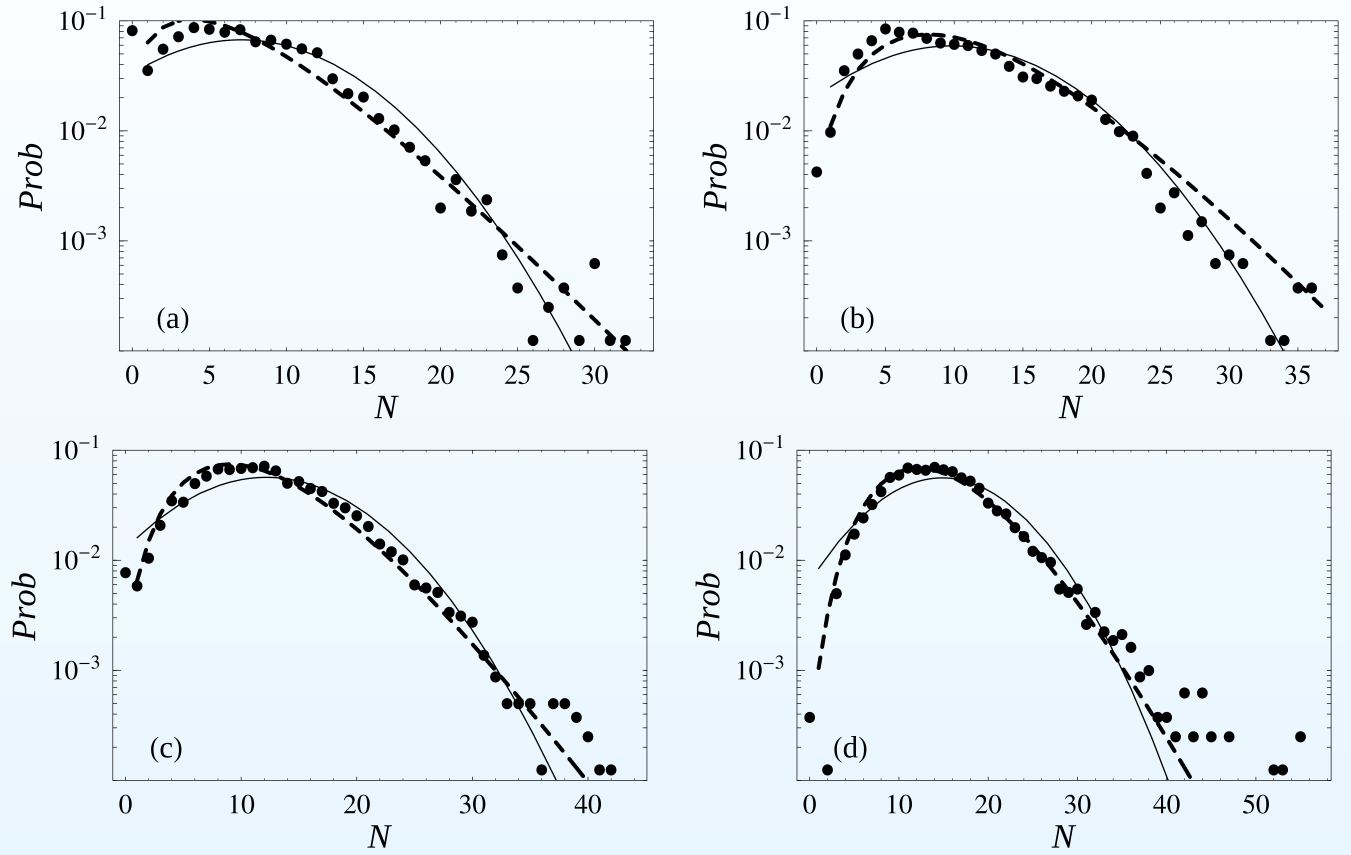
Comparison with data



dots: data. Dashed line: binomial distribution (steady-state solution).

Solid line: Gaussian approximation.

Implications for the Exner equation

Recall that in the large-system limit and at long times

$$\tilde{N} = \frac{\tilde{\lambda}}{\sigma - \mu} + \frac{\sqrt{\tilde{B}(n_*)}}{\sigma - \mu}\frac{d\mathcal{W}}{dt},$$

Substituting $b(x,t) = b_0(x,t) + \epsilon b_1(x,t) + \cdots$ into the Exner equation gives a hierarchy of equations

$$\psi\frac{\partial b_0}{\partial t} = (\sigma - \mu)\tilde{n}_* - \tilde{\lambda},$$

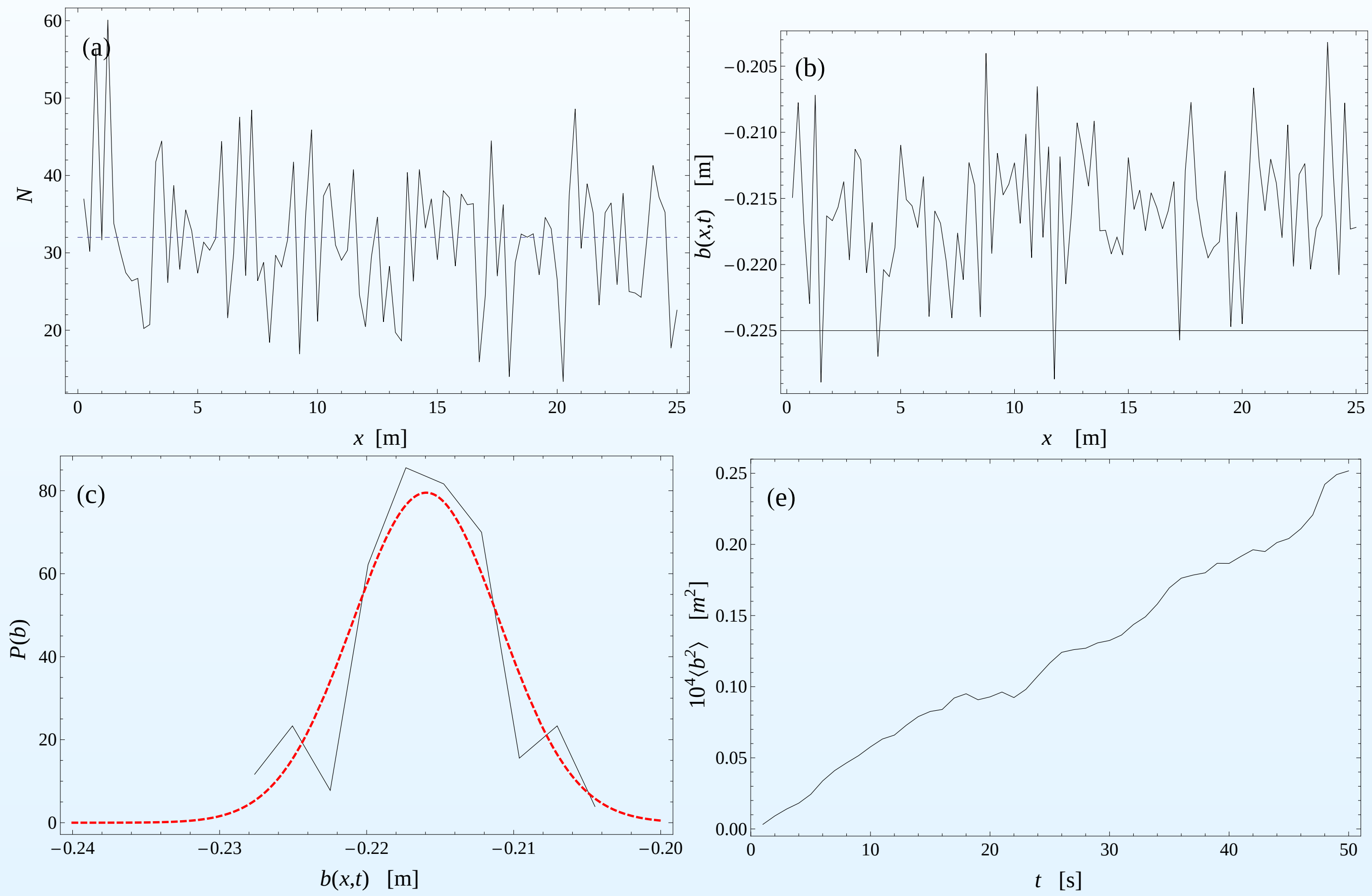
$$\psi\frac{\partial b_1}{\partial t} = (\sigma - \mu)\eta_1,$$

The first moments of b_1 are $\langle b_1 \rangle = 0$ and $\langle b_1^2 \rangle = \tilde{B}(N_\infty)t$.

$$\mathcal{S}(\omega) = \frac{\tilde{B}(N_\infty)}{2\pi}\frac{1}{\omega^2 + (\sigma - \mu)^2} \propto \omega^{-2} \text{ for } \omega \to \infty$$

Stochastic simulations

Bed evolution can be computed using SDE solvers (e.g., Euler-Maruyama method)



Issue: coupling with the stream

Conclusion

- reasonably simple framework for computing bedload transport in steep streams
- analytical and numerical results in agreement with experimental data
- explain and describe why wide fluctuations arise

but

- no fluid-bed interaction at the moment
- simplified physical picture (1D stream!)
- very slow numerical schemes