EPFL, Section de Mathématiques

Semester Project

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Topics in Homological Algebra

Lev Kiwi

Advisors:

Dr. Scherer

Others [topological invariants] were discovered by Poincaré. They are all tied up with his homology theory which is perhaps the most profound and far reaching creation in all topology.

- S. Lefschetz

Abstract

In this project, we start by giving some basic results about (co)homology of modules and groups, most of them taken from either [**Wei94**],[**HS97**] or [**Rot79**]. We then focus on the topological aspects and give a nice interpretation of the (co)homology of a group via topology; more precisely, the (co)homology of a group G is equal to the (co)homology of any Eilenberg-MacLane space $K(G, 1)$. This will bring us to the conjecture of Eilenberg-Ganea which states that the cohomological dimension of any group G is equal to the geometric dimension of G.

Contents

CHAPTER 1

Generalities

This chapter serves to provide all the necessary definitions and basic results in group (co)homology.

Let R be a commutative ring.

1. Review of chain complexes and (co)homology

We recall a few facts about chain complexes.

Definition. A *chain complex* C (over R) is a sequence of R-modules $(C_n)_{n\in\mathbb{Z}}$ and morphisms of R-modules $(d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$, called *differentiations*,

$$C: \quad \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

A *morphism of chain complexes* $f : C \to C'$ is a collection of morphisms of R-modules $(f_n : C_n \to C'_n)_{n\in\mathbb{Z}}$ such that the diagram

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "d'_n"] & C'_{n-1} \end{tikzcd}$$

commutes for all n.

For a chain complex C, we define its *n-th homology module* by

$$H_n(C) := \ker d_n / \operatorname{im} d_{n+1}.$$

If $f : C \to C'$ is a morphism of chain complexes, we define

$$\begin{aligned} H_n(f) : H_n(C) &\to H_n(C') \\ [z] &\mapsto [f_n(z)] \end{aligned}$$

for all $[z] \in H_n(C)$.

Similarly, we define cochain complexes and cohomolgy.

Definition. A *cochain complex* D (over R) is a sequence of R-modules $(D^n)_{n\in\mathbb{Z}}$ and morphisms of R-modules $(\delta^n : D^n \to D^{n+1})_{n\in\mathbb{Z}}$,

$$D: \quad \cdots \longrightarrow D^{n-1} \xrightarrow{\delta^{n-1}} D^n \xrightarrow{\delta^n} D^{n+1} \longrightarrow \cdots$$

such that $\delta^{n+1}\delta^n = 0$ for all $n \in \mathbb{Z}$.

We define its *n-th cohomology module* by

$$H^n(D) := \ker \delta^n / \operatorname{im} \delta^{n-1}.$$

Remark. There are several ways to obtain a cochain complex from a chain complex C. The most simple way is to set $C^n := C_{-n}$ and $\delta^n := d_{-n}$. Since we will work with positive chain complexes (i.e. $C_n = 0$ for all $n < 0$) it is more interesting to define $C^n := C_n^* = \mathrm{Hom}(C_n, R)$ and $\delta^n := d_{n+1}^* = \mathrm{Hom}(d_{n+1}, R)$. Let $\mathrm{Hom}(C, R)$ denote this cochain complex.

Definition. If C is a chain complex, we define its *n-th cohomology module* by

$$H^n(C) := H^n(\mathrm{Hom}(C, R)).$$

Definition. Let $f, g : C \to C'$ be morphisms of chain complexes. We say that f is *homotopic* to g if there exists a collection $(h_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ of morphisms of R-modules such that

$$f_n - g_n = d'_{n+1}h_n + h_{n-1}d_n$$

for all $n \in \mathbb{Z}$. The collection $(h_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ is called *homotopy*.

One can see that a homotopy defines an equivalence relation on $\mathrm{Hom}(C, C')$. We omit the elementary proof that homology is homotopy invariant.

Proposition 1.1. *Let $f, g : C \to C'$ be homotopic morphisms of chain complexes, then $H_n(f) = H_n(g)$.*

The following theorem is also a classical result.

Theorem 1.2. *Let $0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$ be an exact sequence of chain complexes. Then there is a long exact sequence of modules*

$$\cdots \longrightarrow H_n(C') \xrightarrow{H_n(i)} H_n(C) \xrightarrow{H_n(p)} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \longrightarrow \cdots .$$

Moreover, the connection $\partial_n : H_n(C'') \to H_{n-1}(C')$ *is defined by*

$$\partial_n([z]) := [i_{n-1}^{-1}(d_n(y))], \qquad \text{for any } y \in p_n^{-1}(z).$$

And similarly for cochain complexes and cohomology.

Theorem 1.3. *Let $0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$ be an exact sequence of cochain complexes. Then there is a long exact sequence of modules*

$$\cdots \longrightarrow H^n(C') \xrightarrow{H^n(i)} H^n(C) \xrightarrow{H^n(p)} H^n(C'') \xrightarrow{\partial^n} H^{n+1}(C') \longrightarrow \cdots .$$

Moreover, the connection $\partial^n : H^n(C'') \to H^{n+1}(C')$ *is defined by*

$$\partial^n([z]) := [i_{n+1}^{-1}(\delta^n(y))], \qquad \text{for any } y \in p_n^{-1}(z).$$

2. Projective and injective modules

Here we define two dual concepts: projective and injective modules.

Definition. A R-module M is said to be *free* if there exists a set $(m_i)_{i\in I}$, called a *basis* of M, such that each $m \in M$ has a unique expression

$$m = \sum_i r_i m_i$$

for some $r_i \in R$ with almost all $r_i = 0$.

Proposition 1.4. *Let M be a free R-module and let $B = (m_i)_{i\in I}$ be a basis. For any map $f : B \to N$ and any R-module N, there exists a unique morphism of R-modules $\bar{f} : M \to N$ such that the diagram*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, hook, "i"'] & N \\ M \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$

commutes.

COROLLARY 1.5. *Let $\beta : M \to N$ be an epimorphism of R-modules. If P is free, then for any morphism $\alpha : P \to N$ there exists a morphism $\gamma : P \to M$ such that $\beta\gamma = \alpha$ as in the following commutative diagram*

$$\begin{tikzcd} & P \arrow[d, "\alpha"] \arrow[dl, dotted, "\gamma"'] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0. \end{tikzcd}$$

We can slightly weaken the hypothesis and generalize to projective modules.

DEFINITION. A R-module P is said to be *projective* if for any epimorphism of R-modules $\beta : M \to N$ and any morphism $\alpha : P \to N$ there exists a morphism $\gamma : P \to M$ such that $\beta\gamma = \alpha$.

THEOREM 1.6. *Let P be a R-module. Then the following statements are equivalent:*

1. *P is projective,*
2. *$\mathrm{Hom}(P, -)$ is exact,*
3. *every exact sequence of R-modules $0 \to M' \to M \to P \to 0$ splits,*
4. *P is a summand of a free module, i.e. there exists a R-module M such that $P \oplus M$ is free,*
5. *there exists $(p_i)_{i\in I} \subset P$ and $(\varphi_i : P \to R)_{i\in I}$ such that for all $x \in P$, $\varphi_i(x) = 0$ for almost all $i \in I$, and $x = \sum_i \varphi_i(x)p_i$.*

PROOF. We know P is projective if for any exact sequence $M \xrightarrow{\beta} N \to 0$ and any $\alpha : P \to N$ there exists a morphism $\gamma : P \to M$ such that $\beta\gamma = \alpha$. This is exactly saying that for any $\alpha \in \mathrm{Hom}(P, N)$ there exists a morphism $\gamma \in \mathrm{Hom}(P, N)$ such that $\beta\gamma = \beta_*(\gamma) = \alpha$. But this is equivalent to the statement that $\beta_* : \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N)$ is an epimorphism. Therefore the functor $\mathrm{Hom}(P, -)$ is right exact. Since $\mathrm{Hom}(P, -)$ is clearly left exact, (1) is equivalent to (2).

Suppose (1) and let $0 \to M' \to M \to P \to 0$ be an exact sequence of R-modules. Because P is projective there is a morphism γ that makes the following diagram commute

$$\begin{tikzcd} & P \arrow[d, "\mathrm{Id}_P"] \arrow[dl, dotted, "\gamma"'] & \\ M \arrow[r] & P \arrow[r] & 0. \end{tikzcd}$$

Thus γ provides the desired splitting of the sequence. Hence (1) implies (3).

Suppose now (3). Consider the exact sequence $0 \to \ker\alpha \to F \xrightarrow{\alpha} P \to 0$ where F is the free R-module generated by all the elements of P, i.e. $F = \bigoplus_{p\in P} Rp$ and the morphism α is given by universal property of free modules. By hypothesis, this sequence splits. Consequently there exists $r : P \to F$ such that $\alpha r = \mathrm{Id}_P$. One can prove easily that

$$\begin{aligned} \ker\alpha \oplus P &\to F \\ (x, p) &\mapsto x + r(p) \end{aligned}$$

defines an isomorphism of R-modules. Hence (3) implies (4).

Assume (4) and let F be such a free R-module where P is the summand. We then construct the diagram

$$\begin{tikzcd} F \arrow[r, bend left, "p"] \arrow[d, dotted, "\gamma"] & P \arrow[l, bend left, "i"] \arrow[d, "\alpha"] & \\ M \arrow[r, "\beta"] & N \arrow[r] & 0, \end{tikzcd}$$

with $pi = \mathrm{Id}_P$. The universal property of free modules yields $\gamma : F \to M$ for any morphism $\alpha : P \to N$ and any epimorphism $\beta : M \to N$. Thus (4) implies (1).

To conclude the proof, observe that (5) is equivalent to (4). The proof may also be found in [**Rot79**, Chapter 3, Theorem 3.15]. $\square$

The dual notion of projective modules is injective modules.

Definition. A R-module E is said to be *injective* if for any morphism $\alpha : M \to E$ and any monomorphism of R-modules $\beta : M \to N$ there exists a morphism $\gamma : N \to E$ such that $\gamma\beta = \alpha$ as in the following commutative diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "\alpha"] \arrow[r, "\beta"] & N. \arrow[ul, dotted, "\gamma"'] \end{tikzcd}$$

Theorem 1.7. *Let E be a R-module. Then the following statements are equivalent:*

(1) *E is injective,*
(2) $\mathrm{Hom}(-, E)$ *sends monomorphisms to epimorphisms,*
(3) *every exact sequence of R-modules* $0 \to E \to M \to M' \to 0$ *splits.*

Proof. Let $0 \to M \xrightarrow{\beta} N$ be an exact sequence of R-modules. Apply $\mathrm{Hom}(-, E)$ to this sequence to obtain $\mathrm{Hom}(N, E) \xrightarrow{\beta^*} \mathrm{Hom}(M, E) \to 0$. Suppose that E is injective. Let $\alpha \in \mathrm{Hom}(M, E)$, then there exists $\gamma : N \to E$ such that $\alpha = \gamma\beta = \beta^*(\gamma)$ and thus β^* is an epimorphism. Hence (1) is equivalent to (2).

Let us prove that (1) is equivalent to (3). If E is injective, we construct the diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & E \arrow[u, "\mathrm{Id}_E"] \arrow[r] & M \arrow[ul, dotted, "\gamma"'] \end{tikzcd}$$

on any exact sequence $0 \to E \to M \to M' \to 0$ of R-modules. One can see that γ provides the desired splitting of the sequence.

Conversely, consider the diagram

$$\begin{tikzcd} & E \arrow[r, "\beta'"] & P \\ 0 \arrow[r] & M \arrow[u, "\alpha"] \arrow[r, "\beta"] & N, \arrow[u, "\alpha'"'] \end{tikzcd}$$

where P is the pushout of α and β, these being the same morphsims as in the first part of the proof. Since β is a monomorphism, β' is also a monomorphism. We can consider the exact sequence $0 \to E \xrightarrow{\beta'} P \to \operatorname{coker}\beta' \to 0$. By hypothesis, there exists $s : P \to E$ such that $s\beta' = \mathrm{Id}_E$. Defining $\gamma := s\alpha'$ gives $\gamma\beta = s\alpha'\beta = s\beta'\alpha = \mathrm{Id}_E\,\alpha = \alpha$. Therefore E is injective. $\square$

3. Resolutions and extensions

We introduce here the concept of resolutions of modules. They provide the chain complexes that we use to define (co)homology for modules in general.

DEFINITION. Let M be a R-module. A *projective* (resp. *free*) *resolution* of M (over R) is an exact sequence of R-modules

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0.$$

where each P_n is a projective (resp. free).

An *injective resolution* of M (over R) is an exact sequence of R-modules

$$0 \to M \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} E^1 \to \cdots \to E^n \xrightarrow{\delta^n} E^{n+1} \to \cdots$$

where each E^n is an injective.

THEOREM 1.8. *Every R-module M has a free resolution.*

PROOF. For any R-module M, there exists a free module F_0 and an exact sequence

$$0 \to S_0 \to F_0 \xrightarrow{\varepsilon} M \to 0.$$

We apply the same argument to the R-module S_0 to obtain an exact sequence

$$0 \to S_1 \xrightarrow{\sigma_1} F_1 \xrightarrow{\varphi_1} S_0 \to 0.$$

By induction we have exact sequences

$$0 \to S_n \xrightarrow{\sigma_n} F_n \xrightarrow{\varphi_n} S_{n-1} \to 0$$

for $n > 0$.

We assemble all those sequences into the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[rr, "d_3"] \arrow[dr] & & F_2 \arrow[rr, "d_2"] \arrow[dr] & & F_1 \arrow[rr, "d_1"] \arrow[dr] & & F_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
\cdots & & S_2 \arrow[ur] \arrow[dr] & & S_1 \arrow[ur] \arrow[dr] & & S_0 \arrow[ur] \arrow[dr] & & & \\
\cdots & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

with $d_n := \sigma_{n-1}\varphi_n$. Let us now prove the exactness of the top row. Since σ_n is a monomorphism, φ_n is an epimorphism and $\ker \varphi_n = \operatorname{im} \sigma_n$ we have:

$$\ker d_n = \ker(\sigma_{n-1}\varphi_n) = \ker \varphi_n = \operatorname{im} \sigma_n = \operatorname{im}(\sigma_n \varphi_{n+1}) = \operatorname{im} d_{n+1}.$$

Therefore the sequence above is exact. □

COROLLARY 1.9. *Every R-module M has a projective resolution.*

THEOREM 1.10. *Every R-module M has an injective resolution.*

PROOF. The proof is dual to the preceding theorem if we assume that every R-module can be imbedded in an injective R-module. The reader can find the proof of this useful fact in [**Rot79**, Chapter 3, Theorem 3.27]. □

DEFINITION. Let M be a R-module. If

$$X := \quad \cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_0 \to M \to 0$$

is a chain complex, we define the *deleted complex* of X to be the chain complex

$$X_M := \quad \cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_0 \to 0.$$

We define similarly Y_N for any cochain complex of the form

$$Y := \quad 0 \to N \to Y^0 \xrightarrow{\delta^0} Y^1 \to \cdots \to Y^n \xrightarrow{\delta^n} Y^{n+1} \to \cdots$$

Theorem 1.11. *Consider the solid commutative diagram of R-modules*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dotted, "\exists f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dotted, "\exists f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dotted, "\exists f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & X_2 \arrow[r, "\partial_2"] & X_1 \arrow[r, "\partial_1"] & X_0 \arrow[r, "\alpha"] & N \arrow[r] & 0.
\end{tikzcd}$$

If each P_n is projective and if the bottom row is exact, then there exits a morphism of chain complexes $f_* : P_M \to X_N$. Moreover, this morphism is unique up to chain homotopy.

We say that such a morphism f_* is a chain morphism over f.

Proof. We first show the existence of such a morphism of chain complexes by induction on n. If $n = 0$ we draw the diagram

$$\begin{tikzcd}
 & P_0 \arrow[d, "f\varepsilon"] & \\
X_0 \arrow[r, "\alpha"] & N \arrow[r] & 0
\end{tikzcd}$$

and since α is an epimorphism and P_0 is projective, there exists a morphism of R-modules $f_0 : P_0 \to X_0$ such that $\alpha f_0 = f\varepsilon$.

Suppose $f_k : P_k \to X_k$ constructed for all $k \leq n$, and consider the diagram

$$\begin{tikzcd}
P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & P_{n-1} \arrow[d, "f_{n-1}"] \\
X_{n+1} \arrow[r, "\partial_{n+1}"] & X_n \arrow[r, "\partial_n"] & X_{n-1}.
\end{tikzcd}$$

Since $\partial_n f_n d_{n+1} = f_{n-1} d_n d_{n+1} = 0$ and $\ker \partial_n = \operatorname{im} \partial_{n+1}$ we have $\operatorname{im}(f_n d_{n+1}) \subset \operatorname{im}(\partial_{n+1})$. Therefore, we may consider the diagram

$$\begin{tikzcd}
 & P_{n+1} \arrow[dl, dotted, "f_n"'] \arrow[d, "f_n d_{n+1}"] & \\
X_{n+1} \arrow[r, "\partial_{n+1}"] & \operatorname{im} \partial_{n+1} \arrow[r] & 0
\end{tikzcd}$$

where f_n is given by the projectivity of P_{n+1}.

We prove now that f_* is unique up to a chain homotopy. Let $f'_* : P_M \to X_N$ be a second morphism of chain complexes such that the above diagram commutes. By induction, we construct a chain homotopy from f_* to f'_*. Let $h_n : P_n \to X_{n+1}$ be the zero map for every $n < 0$. Suppose $h_k : P_k \to X_{k+1}$ constructed for all $k < n$, such that

$$f_k - f'_k = \partial_{k+1} h_k + h_{k-1} d_k.$$

We have that $\operatorname{im}(f_n - f'_n - h_{n-1} d_n) \subset \operatorname{im} \partial_{n+1} = \ker \partial_n$, since

$$\begin{aligned}
&\partial_n (f_n - f'_n - h_{n-1} d_n) \\
&= \partial_n f_n - \partial_n f'_n - \underbrace{\partial_n h_{n-1} d_n}_{=(f_{n-1} - f'_{n-1} - h_{n-2} d_{n-1}) d_n} \\
&= \underbrace{\partial_n f_n - f_{n-1} d_n}_{=0} - \underbrace{(\partial_n f'_n - f'_{n-1} d_n)}_{=0} + h_{n-2} \underbrace{d_{n-1} d_n}_{=0} \\
&= 0.
\end{aligned}$$

Hence $\operatorname{im}(f_n - f'_n - h_n d_{n+1}) \subset \operatorname{im} \partial_{n+1}$. Therefore, we may consider the diagram

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dotted, "h_{n+1}"'] \arrow[d, "f_n - f'_n - h_n d_{n+1}"] & \\ X_{n+2} \arrow[r, "\partial_{n+1}"] & \operatorname{im} \partial_{n+1} \arrow[r] & 0, \end{tikzcd}$$

where h_{n+1} is given by the projectivity of P_{n+1}. $\square$

4. Homology and Cohomology of R-modules

This section defines the functors Tor and Ext. For convenience, let $\otimes$ denote $\otimes_R$, the tensor product over R.

Definition. Let M, N be R-modules. Let P_M be the deleted complex of a projective resolution of M. Form the complex $P_M \otimes N$ by applying the functor $(-) \otimes N$ on each degree, i.e.

$$P_M \otimes N : \qquad \cdots \to P_n \otimes N \stackrel{d_n \otimes \mathrm{Id}_N}{\longrightarrow} P_{n-1} \otimes N \to \cdots \to P_0 \otimes N \to 0.$$

We define $\operatorname{Tor}_n^R(M, N)$ as the n-th homology of the chain complex $P_M \otimes N$, i.e.

$$\operatorname{Tor}_n^R(M, N) := H_n(P_M \otimes N) = \ker(d_n \otimes \mathrm{Id}_N)/\operatorname{im}(d_{n+1} \otimes \mathrm{Id}_N).$$

Remark. We sometimes write $H_n(M, N)$ for $\operatorname{Tor}_n^R(M, N)$.

We have to show that $\operatorname{Tor}_n^R(M, N)$ is independent of the choice of a projective resolution for M.

Theorem 1.12. *Let M, N be R-modules. If $\operatorname{Tor}_n^R(M, N)$ and $\overline{\operatorname{Tor}_n^R(M, N)}$ are the homology modules associated respectively to the projective resolutions P_M and $\overline{P_M}$ of M, then there exists a natural isomorphism $\operatorname{Tor}_n^R(M, N) \cong \overline{\operatorname{Tor}_n^R(M, N)}$.*

Proof. Consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{Id}_M"] & 0 \\ \cdots \arrow[r] & \overline{P_2} \arrow[r] & \overline{P_1} \arrow[r] & \overline{P_0} \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

By the theorem 1.11 there exists a morphism of chain complexes $i : P_M \to \overline{P_M}$ over Id_M. If we turn the above diagram upside down, the same theorem gives us a morphism of chain complexes $j : \overline{P_M} \to P_M$ over Id_M. By composition, we have that $ji : P_M \to P_M$ and $ij : \overline{P_M} \to \overline{P_M}$ are morphisms of chain complexes over Id_M. Moreover, they are homotopic to the trivial morphisms of chain complexes. If we now apply the functor $(-) \otimes N$, they remain homotopic, hence their homology modules are isomorphic. Consequently, $\operatorname{Tor}_n^R(M, N) \cong \overline{\operatorname{Tor}_n^R(M, N)}$ and the isomorphism is given by $i_* := H_n(i \otimes N) : H_n(P_M \otimes N) \to H_n(\overline{P_M} \otimes N)$.

The proof of the naturality of i_* may be found in [**Rot79**, Chapter 6, Theorem 6.11]. $\square$

Definition. Let M, N be R-modules. Let P_M be the deleted complex of a projective resolution of M. Form the complex $\operatorname{Hom}(P_M, N)$ by applying the functor $\operatorname{Hom}(-, N)$ to each degree, i.e.

$$\operatorname{Hom}(P_M, N) : \quad 0 \to \operatorname{Hom}(P_0, N) \stackrel{d_1^*}{\to} \cdots \to \operatorname{Hom}(P_n, N) \stackrel{d_{n+1}^*}{\to} \operatorname{Hom}(P_{n+1}, N) \to \cdots$$

We define $\operatorname{Ext}_R^n(M, N)$ by taking the n-th cohomology of the cochain complex $\operatorname{Hom}(P_M, N)$, i.e.

$$\operatorname{Ext}_R^n(M, N) := H^n(\operatorname{Hom}(P_M, N)) = \ker d_{n+1}^*/\operatorname{im} d_n^*.$$

Remark. We sometimes write $H^n(M,N)$ for $\operatorname{Ext}^n_R(M,N)$.

Theorem 1.13. *Let M, N be R-modules. If $\operatorname{Ext}^n_R(N,M)$ and $\overline{\operatorname{Ext}^n_R(N,M)}$ are the cohomology modules associated respectively to the projective resolutions P_M and $\overline{P_M}$ of M, then there exists a natural isomorphism $\operatorname{Ext}^n_R(N,M) \cong \overline{\operatorname{Ext}^n_R(N,M)}$.*

Proof. The proof is similar to the proof of 1.12. □

Remark. In fact, the cohomology of M with coefficients in N can be defined using injective resolutions. Let E_N be the deleted complex of an injective resolution of N. Form the complex $\operatorname{Hom}(M,E_N)$ by applying the functor $\operatorname{Hom}(M,-)$ to each degree, i.e.

$$\operatorname{Hom}(M,E_N): \quad 0 \to \operatorname{Hom}(M,E^0) \xrightarrow{\delta^0_*} \cdots \to \operatorname{Hom}(M,E^n) \xrightarrow{\delta^n_*} \operatorname{Hom}(M,E^{n+1}) \to \cdots$$

The *n-th cohomology* of M with coefficients in N is the n-th cohomology module of the cochain complex $\operatorname{Hom}(N,E_M)$, i.e.

$$H^n(M,N) = H^n(\operatorname{Hom}(M,E_N)) = \ker \delta^n_* / \operatorname{im} \delta^{n-1}_*.$$

The reader may find the proof of

$$H^n(\operatorname{Hom}(M,E_N)) = H^n(\operatorname{Hom}(P_M,N)) = \operatorname{Ext}^n_R(M,N)$$

in [**Rot79**, Chapter 7, Theorem 7.8].

Lemma 1.14. *Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be an exact sequence of R-modules. Let P' and P'' be projective resolutions of M' and M''. Then there exists a projective resolution P of M such that the sequence of chain complexes*

$$0 \to P'_{M'} \xrightarrow{i_*} P_M \xrightarrow{p_*} P''_{M''} \to 0$$

is exact.

Proof. First, we consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K'_0 \arrow[d] & & K''_0 \arrow[d] & \\
 & P'_0 \arrow[d, "\varepsilon'"] & & P''_0 \arrow[d, "\varepsilon''"] & \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d] & M \arrow[r, "p"] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

where K'_0, K''_0 are the kernels of ε' and ε''. Since P''_0 is projective, there exists $\sigma : P''_0 \to M$ such that $p\sigma = \varepsilon''$. If we set $P_0 := P'_0 \oplus P''_0$ and define $i_0 : P'_0 \to P_0$ by $x' \mapsto (x',0)$ as well as $p_0 : P_0 \to P''_0$ by $(x',x'') \mapsto x''$, then the sequence

$$0 \to P'_0 \xrightarrow{i_0} P_0 \xrightarrow{p_0} P''_0 \to 0$$

is exact. Now define $\varepsilon : P_0 \to M$ by $(x',x'') \mapsto i\varepsilon' x' + \sigma x''$. We show that ε is an epimorphism. Let $m \in M$. Since ε'' is an epimorphism, there exists $x'' \in P''_0$ such that $\varepsilon'' x'' = pm$. Therefore, $p(m - \sigma x'') = 0$ and since the sequence is exact, we can find an element $m' \in M'$ such that $im' = m - \sigma x''$. By surjectivity of ε' there exists $x' \in P_0'$ such that $\varepsilon' x' = m'$.

$$
\begin{tikzcd}
x' \in P_0' \arrow[r, mapsto] \arrow[d, dotted, mapsto, "\exists"] & (x', 0) \arrow[d, mapsto] \\
m' \in M' \arrow[r, dotted, mapsto, "\exists"] & m - \sigma x''
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & x'' \in P_0'' x' \arrow[d, dotted, mapsto, "\exists"] \\
m \in M \arrow[r, mapsto] & pm
\end{tikzcd}
$$

We then have $m = m - \sigma x'' + \sigma x'' = \varepsilon(x', x'')$. This proves the surjectivity of ε.

Set $K_0 := \ker \varepsilon$, then one easily completes the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, dotted] & K_0' \arrow[r, dotted] \arrow[d] & K_0 \arrow[r, dotted] \arrow[d] & K_0'' \arrow[r, dotted] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "i_0"] \arrow[d, "\varepsilon'"] & P_0 \arrow[r, "p_0"] \arrow[d, "\varepsilon"] & P_0'' \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d] & M \arrow[r, "p"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

into a commutative diagram with exact rows and columns.

By induction, we can iterate the process to

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K_{n+1}' \arrow[d] & & K_{n+1}'' \arrow[d] & \\
 & P_{n+1}' \arrow[d, "d_{n+1}'"] & & P_{n+1}'' \arrow[d, "d_{n+1}''"] & \\
0 \arrow[r] & K_n' \arrow[r] \arrow[d] & K_n \arrow[r] & K_n'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}
$$

where K_{n+1}' and K_{n+1}'' are the kernels of d_{n+1}' and d_{n+1}''. In the end we obtain a projective resolution since the direct sum of projective modules is projective. $\square$

Theorem 1.15. *Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be an exact sequence of R-modules and let N be a R-module. Then there exists a long exact sequence in homology*

$$
\begin{aligned}
\cdots \to \operatorname{Tor}_n^R(M', N) &\xrightarrow{\operatorname{Tor}_n^R(i,N)} \operatorname{Tor}_n^R(M, N) \xrightarrow{\operatorname{Tor}_n^R(p,N)} \operatorname{Tor}_n^R(M'', N) \\
&\xrightarrow{\partial_n} \operatorname{Tor}_{n-1}^R(M', N) \longrightarrow \cdots .
\end{aligned}
$$

Proof. By the preceding lemma, we can create an exact sequence of morphisms of chain complexes

$$0 \to P'_{M'} \to P_M \to P''_{M''} \to 0.$$

If we now apply the functor $(-) \otimes N$ we still have an exact sequence of morphisms of chain complexes

$$0 \to P'_{M'} \otimes N \to P_M \otimes N \to P''_{M''} \otimes N \to 0$$

since it is a split exact sequence. Apply theorem 1.2 to obtain the desired long exact sequence. □

Theorem 1.16. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules and let N be a R-module. Then there exists a long exact sequence in cohomology*

$$\begin{aligned} \cdots \to \operatorname{Ext}^n_R(M'', N) &\xrightarrow{\operatorname{Ext}^n_R(p,N)} \operatorname{Ext}^n_R(M, N) \xrightarrow{\operatorname{Ext}^n_R(i,N)} \operatorname{Ext}^n_R(M', N) \\ &\xrightarrow{\partial^n} \operatorname{Ext}^{n+1}_R(M'', N) \longrightarrow \cdots \end{aligned}$$

Proof. Apply the functor $\operatorname{Hom}(-, N)$ to the (split) exact sequence of morphisms of chain complexes

$$0 \to P'_{M'} \to P_M \to P''_{M''} \to 0$$

given by the lemma above. We obtain an exact sequence of cochain complexes

$$0 \to \operatorname{Hom}(P''_{M''}, N) \to \operatorname{Hom}(P_M, N) \to \operatorname{Hom}(P'_{M'}, N) \to 0.$$

Apply theorem 1.3. □

5. Homology and Cohomology of groups

We can now talk about the (co)homology of groups specifically.

Definition. Let G be a group (written multiplicatively), the *integral group ring* $\mathbb{Z}G$ is the free abelian group with basis G whose elements are

$$\{\sum_{x\in G} m_x x \quad | \quad m_x \in \mathbb{Z} \text{ and almost all } m_x = 0\}$$

and whose multiplication is induced by the multiplication of G.

For convenience, we say *G-module* instead of $\mathbb{Z}G$-module and $\otimes_G$ denotes $\otimes_{\mathbb{Z}G}$, the tensor product over $\mathbb{Z}G$.

Definition. Let G be a group, and N a G-module. Consider the integers $\mathbb{Z}$ as a trivial G-module and define the *n-th homology* group and the *n-th cohomology* group of G with coefficients in N to be

$$H_n(G, N) := \operatorname{Tor}^{\mathbb{Z}G}_n(\mathbb{Z}, N) \qquad H^n(G, N) := \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, N).$$

From now on, let G denote a group.

Definition. A G-module M is called trivial if every element of G acts as the identity on M. Any abelian group can be regarded as a trivial G-module for any group G.

Theorem 1.17. *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of G-modules, then there exists a long exact sequence in homology*

$$\cdots \to H_n(G, M') \xrightarrow{\overline{\alpha}} H_n(G, M) \xrightarrow{\overline{\beta}} H_n(G, M'') \xrightarrow{\partial_n} H_{n-1}(G, M') \longrightarrow \cdots .$$

PROOF. Let

$$\cdots P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z}$$

be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ ($\mathbb{Z}$ is seen as a trivial G-module).

Consider the exact sequence of chain complexes

$$0 \to M' \otimes_G P_{\mathbb{Z}} \to M \otimes_G P_{\mathbb{Z}} \to M'' \otimes_G P_{\mathbb{Z}} \to 0.$$

This sequence is exact because a projective module is flat. Apply theorem 1.2 to obtain the wished long exact sequence. □

THEOREM 1.18. *Let $0 \to M' \overset{\alpha}{\to} M \overset{\beta}{\to} M'' \to 0$ be an exact sequence of G-modules, then there exists a long exact sequence in cohomology*

$$\cdots \to H^n(G, M') \xrightarrow{\overline{\alpha}} H^n(G, M) \xrightarrow{\overline{\beta}} H^n(G, M'') \xrightarrow{\partial_n} H^{n+1}(G, M') \longrightarrow \cdots .$$

PROOF. The proof is similar to the one given above, except that we instead apply the functor $\mathrm{Hom}(P_{\mathbb{Z}}, -)$ to our exact sequence and use theorem 1.3. □

The following theorem characterises (co)homology with coefficients in some arbitrary module.

THEOREM 1.19 (Universal coefficient theorem)**.** *Let G be a group and M a trivial G-module. Then for $n \geq 0$ there exist split exact sequences*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G, \mathbb{Z}), M) \to H^n(G, M) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(G, \mathbb{Z}), M) \to 0,$$

and

$$0 \to H_n(G, \mathbb{Z}) \otimes_{\mathbb{Z}} M \to H_n(G, M) \to \mathrm{Tor}^{\mathbb{Z}}_1(H_{n-1}(G, \mathbb{Z}), M) \to 0.$$

PROOF. Subsequently, we will give a topological proof of this theorem using the Eilenberg-MacLane space. □

6. Homology groups

In this section, we give a few techniques to compute low homology groups.

DEFINITION. The *augmentation map* of G is the ring homomorphism $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ given by $\sum m_x x \mapsto \sum m_x$. The kernel of ε is called the *augmentation ideal* and is written $\mathfrak{g}$.

LEMMA 1.20. *Let M be a G-module, then $\mathbb{Z} \otimes_G M \cong M/\mathfrak{g}M$.*

PROOF. Consider the exact sequence $0 \to \mathfrak{g} \to \mathbb{Z}G \overset{\varepsilon}{\to} \mathbb{Z} \to 0$, and apply the functor $(-) \otimes_G M$. This yields the exact sequence

$$\mathfrak{g} \otimes_G M \xrightarrow{i \otimes_G \mathrm{Id}_M} \mathbb{Z}G \otimes_G M \to \mathbb{Z} \otimes_G M \to 0.$$

Since $\mathbb{Z}G \otimes_G M \cong M$ and the image of $i \otimes_G \mathrm{Id}_M$ up to this isomorphism is $\mathfrak{g}M$, that implies $\mathbb{Z} \otimes_G M \cong M/\mathfrak{g}M$. □

LEMMA 1.21. *The abelian group $\mathfrak{g}$ is free with basis $\{x - 1 \mid x \in G - \{1\}\}$.*

PROOF. If $\varepsilon(\sum m_x x) = 0$ then $\sum m_x = 0$. Therefore $\sum m_x x = \sum m_x x - \sum m_x 1 = \sum m_x(x - 1)$. So $\mathfrak{g}$ is generated by the $x - 1$, $x \in G$. Suppose now that $\sum m_x(x - 1) = 0$, hence $\sum m_x x = \sum m_x 1$ and since it is an equality in our free group $\mathbb{Z}G$, we have $m_x = 0$ for all $x \in G - \{1\}$. □

DEFINITION. For a G-module M, we define M_G to be the maximal quotient of M that is G-trivial, i.e that G acts on trivially. Hence $M_G = M/S$ where S is the submodule generated by $\{xm - m : x \in G, m \in M\}$.

LEMMA 1.22. *For a G-module M, $M_G \cong M/\mathfrak{g}M \cong \mathbb{Z} \otimes_G M$.*

PROOF. For all $x \in G$ and $m \in \mathbb{Z}$, $xm - m = (x-1)m$. Hence $S = \mathfrak{g}M$. □

THEOREM 1.23.

$$H_0(G, M) \cong M_G.$$

In particular, if M is G-trivial, then $H_0(G, M) = M$.

PROOF. If

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is an exact sequence, then the right exactness of $(-) \otimes_G M$ gives us the exact sequence

$$P_1 \otimes_G M \xrightarrow{d_1 \otimes \mathrm{Id}_M} P_0 \otimes_G M \xrightarrow{\varepsilon \otimes \mathrm{Id}_M} \mathbb{Z} \otimes_G M \to 0.$$

Hence, $H_0(G, M) = \operatorname{coker}(d_1 \otimes \mathrm{Id}_M) \cong \mathbb{Z} \otimes_G M$. The preceding lemma implies the result. □

REMARK. There exists a similar formula for the 0-th cohomology group. Define $M^G := \{m \in M | gm = m, \forall g \in G\}$ as the subset of *fixed points* of M. One can prove (with the bar resolution) that

$$H^0(G, M) \cong M^G.$$

In particular, if M is G-trivial, then $H^0(G, M) = M$. The proof can be found in [**Rot79**, Chapter 5, theorem 5.15].

THEOREM 1.24.

$$H_1(G, \mathbb{Z}) \cong \mathfrak{g}/\mathfrak{g}^2$$

where $\mathbb{Z}$ is G-trivial.

PROOF. Consider the exact sequence of G-modules

$$0 \to \mathfrak{g} \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

This gives us an exact sequence in homology

$$H_1(G, \mathbb{Z}G) \to H_1(G, \mathbb{Z}) \xrightarrow{\partial} H_0(G, \mathfrak{g}) \to H_0(G, \mathbb{Z}G) \xrightarrow{\bar{\varepsilon}} H_0(G, \mathbb{Z}) \to 0.$$

By the preceding theorem, $H_0(G, \mathbb{Z}) \cong \mathbb{Z}$ (since $\mathbb{Z}$ is G-trivial) and $H_0(G, \mathbb{Z}G) \cong \mathbb{Z} \otimes_G \mathbb{Z}G \cong \mathbb{Z}$ by the above lemma. Furthermore, an endomorphism of $\mathbb{Z}$ is either 0 or a monomorphism, but $\bar{\varepsilon}$ is an epimorphism by exactness, hence $\bar{\varepsilon}$ is not 0 and is an isomorphism.

Since $\mathbb{Z}G$ is projective, $\mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$ and we have $H_1(G, \mathbb{Z}G) = 0$. By exactness of the sequence, $\partial : H_1(G, \mathbb{Z}) \to H_0(G, \mathfrak{g})$ is an isomorphism. Finally, by the theorem above $H_0(G, \mathfrak{g}) \cong \mathfrak{g}/\mathfrak{g}\mathfrak{g} = \mathfrak{g}/\mathfrak{g}^2$. □

THEOREM 1.25.

$$H_1(G, \mathbb{Z}) \cong G/[G, G]$$

where $[G, G]$ denotes the commutator subgroup.

PROOF. We shall map out an isomorphism from the multiplicative group $G/[G, G]$ to the additive group $\mathfrak{g}/\mathfrak{g}^2$. Define

$$\begin{aligned} \theta : G &\to \mathfrak{g}/\mathfrak{g}^2 \\ x &\mapsto x - 1 + \mathfrak{g}^2. \end{aligned}$$

It is a group homomorphism, since

$$\begin{aligned}\theta(xy) &= xy - 1 + \mathfrak{g}^2 \\ &= (xy-1) - (x-1)(y-1) + \mathfrak{g}^2 \\ &= (xy-1) - (xy - x - y + 1) + \mathfrak{g}^2 \\ &= (x-1) + (y-1) + \mathfrak{g}^2 \\ &= \theta(x) + \theta(y).\end{aligned}$$

Since $\mathfrak{g}$ is an abelian group, $[G, G] \subset \ker\theta$. Therefore, the induced map

$$\begin{aligned}\overline{\theta} : G/[G,G] &\to \mathfrak{g}/\mathfrak{g}^2 \\ [x] &\mapsto x - 1 + \mathfrak{g}^2\end{aligned}$$

is well defined.

To prove that $\overline{\theta}$ is an isomorphism, we exhibit $\overline{\theta}^{-1}$. Define

$$\begin{aligned}\varphi : \mathfrak{g} &\to G/[G,G] \\ (x-1) &\mapsto x[G,G].\end{aligned}$$

It is straightforward to show that $\mathfrak{g}^2 \subset \ker\varphi$ and the induced map

$$\begin{aligned}\overline{\varphi} : \mathfrak{g}/\mathfrak{g}^2 &\to G/[G,G] \\ (x-1) + \mathfrak{g}^2 &\mapsto x[G,G]\end{aligned}$$

is $\overline{\theta}^{-1}$. □

Remark. This theorem is known in topology as the Hurewicz theorem. We shall give later a topological proof of this theorem using Eilenberg-MacLane spaces.

7. Homology and Cohomology of cyclic groups

In this section, we give some computations for cyclic groups. The following obvious Lemma is pretty useful.

Lemma 1.26. *Let $C_\infty = \{\dots, t^{-2}, t^{-1}, 1, t, t^2, \dots\}$ be the infinite cyclic group and $C_k = \{1, t, \dots, t^{k-1}\}$ the cyclic group of order $k > 0$, $k \in \mathbb{N}$. Then*

$$\mathbb{Z}C_\infty = \mathbb{Z}[t, t^{-1}] \qquad \mathbb{Z}C_k = \mathbb{Z}[t]/(t^k - 1).$$

Proposition 1.27.

$$H_n(C_\infty, \mathbb{Z}) = H^n(C_\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n > 1 \end{cases}$$

Proof. Consider the following projective $\mathbb{Z}C_\infty$-resolution of $\mathbb{Z}$

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}C_\infty \arrow[r] \arrow[d, equal] & \mathbb{Z}C_\infty \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}[t,t^{-1}] \arrow[r, "d_1"] & \mathbb{Z}[t,t^{-1}] \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

where

$$\begin{aligned}d_1 : \mathbb{Z}[t,t^{-1}] &\to \mathbb{Z}[t,t^{-1}] \\ t^k &\mapsto t^k(t-1)\end{aligned}$$

and

$$\begin{aligned} \varepsilon : \mathbb{Z}[t,t^{-1}] &\to \mathbb{Z} \\ \sum_k m_k t^k &\mapsto \sum_k m_k \end{aligned}$$

One can easily see that d_1 is a monomorphism, $\ker \varepsilon = \operatorname{im} d_1$ and ε is an epimorphism. Recall that for any $\mathbb{Z}C_\infty$-module M, $M \otimes_{\mathbb{Z}C_\infty} \mathbb{Z}C_\infty \cong M$ via $m \otimes x \mapsto xm$. If we now apply $(-) \otimes_{\mathbb{Z}G} \mathbb{Z}$ to the deleted complex, we obtain the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}] \arrow[r, "d_1"] \arrow[d, "\cong"] & \mathbb{Z} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}[t,t^{-1}] \arrow[r] \arrow[d, "\cong"] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\partial_1"] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

One can see that $\partial_1 = 0$ (use the fact that $\mathbb{Z}$ is $\mathbb{Z}[t,t^{-1}]$-trivial). Hence the homology groups are

$$H_n(C_\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n > 1. \end{cases}$$

To compute the cohomology groups, we apply the functor $\operatorname{Hom}_{\mathbb{Z}C_\infty}(-, \mathbb{Z})$ to the deleted complex above. Recall that $\operatorname{Hom}_{\mathbb{Z}C_\infty}(\mathbb{Z}C_\infty, M) \cong M$ via $f \mapsto f(1)$ for any $\mathbb{Z}C_\infty$-module M. We obtain

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}C_\infty}(\mathbb{Z}C_\infty, \mathbb{Z}) \arrow[r, "d_1^*"] \arrow[d, "\cong"] & \operatorname{Hom}_{\mathbb{Z}C_\infty}(\mathbb{Z}C_\infty, \mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] & \cdots \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\delta^0"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots. \end{tikzcd}$$

One can see that $\delta^0 = 0$ (once again this is due to the triviality of the action), therefore

$$H^n(C_\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n > 1. \end{cases} \qquad \square$$

Proposition 1.28.

$$H_n(C_k, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ C_k & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

and

$$H^n(C_k, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \text{ odd} \\ C_k & n \text{ even.} \end{cases}$$

Proof. Consider the following $\mathbb{Z}C_k$-resolution of $\mathbb{Z}$

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}C_k \arrow[r, "d_2"] \arrow[d, equal] & \mathbb{Z}C_k \arrow[r, "d_1"] \arrow[d, equal] & \mathbb{Z}C_k \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathbb{Z} \arrow[r] & 0 \\ \cdots \arrow[r] & \mathbb{Z}[t]/(t^k-1) \arrow[r, "d_2"] & \mathbb{Z}[t]/(t^k-1) \arrow[r, "d_1"] & \mathbb{Z}[t]/(t^k-1) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

where $\varepsilon : \mathbb{Z}[t]/(t^k - 1) \to \mathbb{Z} : t^k \mapsto 1$ and the d_n are defined by

$$\begin{aligned} d_n : \mathbb{Z}[t]/(t^k - 1) &\to \mathbb{Z}[t]/(t^k - 1) \\ t^l &\mapsto t^l(t - 1) \end{aligned}$$

if n is odd, and

$$\begin{aligned} d_n : \mathbb{Z}[t]/(t^k - 1) &\to \mathbb{Z}[t]/(t^k - 1) \\ t^l &\mapsto t^l(1 + t + \cdots + t^{k-1}) \end{aligned}$$

if n is even.

One can easily see that the sequence is exact. If we apply the functor $(-)\otimes_{\mathbb{Z}C_k}\mathbb{Z}$, we obtain via the isomorphism $t^l \otimes 1 \mapsto t^l 1 = 1$ the sequence

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0.$$

Thus when n is even $\partial_n : \mathbb{Z} \to \mathbb{Z} : m \mapsto km$ is the multiplication by k and when n is odd $\partial_n = 0$. Take homology to obtain the result.

To compute the cohomology groups, we apply the functor $\mathrm{Hom}_{\mathbb{Z}C_k}(-, \mathbb{Z})$ to our resolution. Via the isomorphism $\mathrm{Hom}_{\mathbb{Z}C_k}(\mathbb{Z}C_k, \mathbb{Z}) \cong \mathbb{Z}$, we obtain the complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta^0} \mathbb{Z} \xrightarrow{\delta^1} \mathbb{Z} \xrightarrow{\delta^2} \mathbb{Z} \xrightarrow{\delta^3} \cdots$$

If n is odd then we can identify δ^n as the multiplication by k and if n is even then $\delta^n = 0$. Take cohomology of this cochain complex to obtain the result. □

For a finite cyclic group C_k, we now want to compute the (co)homology with coefficients in an arbitrary C_k-module.

Proposition 1.29. *Let M be a C_k-module and $n > 2$. Then*

$$H^n(C_k, M) \cong H^{n-2}(C_k, M).$$

Proof. This follows from the fact that the resolution is periodic of period 2. □

Therefore it suffices to compute $H^1(C_k, M)$ and $H^2(C_k, M)$. Remark that there is an analogous result for homology, which is also of period 2.

Proposition 1.30. *Let t be a generator of C_k and consider the C_k - homomorphisms $\phi, \psi : M \to M$ defined by*

$$\phi m = (t - 1)m, \text{ and } \psi m = (t^{k-1} + t^{k-2} + \cdots + t + 1)m \ \forall m \in M.$$

Then

$$H^1(C_k, M) = \ker\psi/\operatorname{im}\phi \text{ and } H^2(C_k, M) = \ker\phi/\operatorname{im}\psi$$

as well as

$$H_1(C_k, M) = \ker\phi/\operatorname{im}\psi \text{ and } H_2(C_k, M) = \ker\psi/\operatorname{im}\phi.$$

Furthermore, $H^0(C_k, M) = \ker\phi$ and $H_0(C_k, M) = \operatorname{coker}\phi$.

Proof. Again, we will just prove these results for cohomology. Recall the 2-periodic $\mathbb{Z}C_k$-resolution of $\mathbb{Z}$ defined by

$$\begin{aligned} d_n : \mathbb{Z}[t]/(t^k - 1) &\to \mathbb{Z}[t]/(t^k - 1) \\ t^l &\mapsto t^l(t - 1) \end{aligned}$$

if n is odd, and

$$\begin{aligned} d_n : \mathbb{Z}[t]/(t^k - 1) &\to \mathbb{Z}[t]/(t^k - 1) \\ t^l &\mapsto t^l(1 + t + \cdots + t^{k-1}) \end{aligned}$$

if n is even. We apply the functor $\mathrm{Hom}_{\mathbb{Z}C_k}(-,M)$ to the resolution. The result is just a consequence of the commutativity of the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}C_k}(\mathbb{Z}C_k, M) \arrow[r, "d_1^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbb{Z}C_k}(\mathbb{Z}C_k, M) \arrow[r, "d_2^*"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbb{Z}C_k}(\mathbb{Z}C_k, M) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & M \arrow[r, "\phi"] & M \arrow[r, "\psi"] & M \arrow[r] & \cdots ,
\end{tikzcd}$$

where the isomorphism $\mathrm{Hom}_{\mathbb{Z}C_k}(\mathbb{Z}C_k, M) \to M$ is just evaluation on the identity. □

The two previous propositions completely describe (co)homology of finite cyclic groups. We conclude this chapter with a classical example.

Example. We consider the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong C_2$ on $\mathbb{C}$ and want to compute $H^2(C_2, \mathbb{C}^*)$. Now $\mathbb{C}^*$ is a $\mathbb{Z}[\sigma]/(\sigma^2-1)$-module with

$$\sigma(x+iy) = x - iy.$$

This means that our morphisms ϕ and ψ are defined as follows on $\mathbb{C}^*$:

$$\phi(z) = (\sigma - 1)z = \bar{z}z^{-1} = \bar{z}^2/\mathscr{N}(z)$$

and

$$\psi(z) = (\sigma+1)z = z\bar{z} = \mathscr{N}(z).$$

So

$$H^2(C_2, \mathbb{C}^*) \cong \ker\phi/\operatorname{im}\psi \cong \mathbb{R}^*/\mathscr{N}\mathbb{C}^* \cong \{\pm 1\} \cong C_2$$

as $\mathbb{Z}\,\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-modules, using the fact that the norm sends complex numbers to positive real numbers.

8. The bar resolution

In this section, we present an explicit resolution of $\mathbb{Z}$ over a given group G.

We first describe the *non-normalized homogeneous* bar resolution. Let $\bar{B}_n$, $n \geq 0$, be the free abelian group on the set of all $(n+1)$-tuples $(y_0, y_1, \ldots, y_n)$ of elements of G. Define a left G-module structure on $\bar{B}_n$ by

$$y(y_0, y_1, \ldots, y_n) = (yy_0, yy_1, \ldots, yy_n),\ y \in G.$$

Also define the differential in the sequence

$$\bar{B} : \cdots \to \bar{B}_n \xrightarrow{\partial_n} \bar{B}_{n-1} \to \cdots \to \bar{B}_1 \xrightarrow{\partial_1} \bar{B}_0$$

by the simplicial boundary formula

$$\partial_n(y_0, y_1, \ldots, y_n) = \sum_{i=0}^{n} (-1)^i (y_0, \ldots, \hat{y}_i, \ldots, y_n),$$

where $\hat{y}_i$ means that we omit the i-th element. Finally define the augmentation $\varepsilon : \bar{B}_0 \to \mathbb{Z}$ by

$$\varepsilon(y) = 1.$$

Proposition 1.31. *The chain complex $\bar{B}$ is a free G-resolution of $\mathbb{Z}$.*

Proof. We have that ∂_n and ε are G-module homomorphisms. Moreover, a trivial calculation yields

$$\partial_{n-1}\partial_n = 0 \text{ for } n \geq 2 \text{ and } \varepsilon\partial_1 = 0.$$

Clearly, $\bar{B}_n$ is a free G-module, for example $(1, y_1, \ldots, y_n)$ is a basis. It remains to show that the chain complex is acyclic. We show that it admits a contracting homotopy $\bar{\Delta}$. Define

$$\bar{\Delta}_{-1}(1) = 1, \text{ and } \bar{\Delta}_n(y_0, \ldots, y_n) = (1, y_0, \ldots, y_n).$$

Again one can verify that that $\bar{\Delta}$ is indeed a contracting homotopy, i.e that

$$\varepsilon\bar{\Delta}_{-1} = 1,\ \partial_1\bar{\Delta}_0 + \bar{\Delta}_{-1}\varepsilon = 1,$$

and

$$\partial_{n+1}\bar{\Delta}_n + \bar{\Delta}_{n-1}\partial_n = 1,\ n \geq 1. \qquad \square$$

The complex $\bar{B}$ is the non-normalized bar resolution in homogeneous form. Let now $D_n \subseteq \bar{B}_n$ be the subgroup generated by the $(n+1)$-tuples $(y_0, y_1, \ldots, y_n)$ where $y_i = y_{i+1}$ for at least one value of i in $\{0, \ldots, n-1\}$. Such a D_n will be called *degenerate*. Thus D_n is a submodule of $\bar{B}_n$ generated by the degenerated $(n+1)$-tuples with $y_0 = 1$.

Lemma 1.32. *We have that $\partial D_n \subseteq D_{n-1}$.*

Proof. Take $(y_0, y_1, \ldots, y_n)$ degenerate with $y_j = y_{j+1}$. Then $\partial_n(y_0, y_1, \ldots, y_n)$ is a linear combination of degenerated n-tuples together with the term

$$(-1)^j(y_0, \ldots, y_{j-1}, y, y_{j+2}, \ldots, y_n) + (-1)^{j+1}(y_0, \ldots, y_{j-1}, y, y_{j+2}, \ldots, y_n),$$

where $y = y_j = y_{j+1}$. $\square$

Therefore the submodules D_n yield a subcomplex D called the *degenerate* subcomplex of $\bar{B}$.

Proposition 1.33. *The quotient complex $B := \bar{B}/D$ is a G-free resolution of $\mathbb{Z}$.*

Proof. It suffices to see that $\bar{\Delta}_n D_n \subseteq D_{n+1}$ and thus we can take the contracting homotopy Δ induced by $\bar{\Delta}$. $\square$

This complex B is the *normalized* resolution in homogeneous form.

We now want a resolution that is inhomogeneous. Let $\bar{B}'_n, n \geq 0$ be the free left G-module on the set of all n-tuples $[x_1|x_2|\ldots|x_n]$ of elements of G. The differential in the sequence

$$\bar{B}' : \cdots \to \bar{B}'_n \xrightarrow{\partial_n} \bar{B}'_{n-1} \to \cdots \to \bar{B}'_1 \xrightarrow{\partial_1} \bar{B}'_0$$

is defined by

$$\begin{aligned}\partial_n[x_1|x_2|\ldots|x_n] &= x_1[x_2|\ldots|x_n] \\ &+ \sum_{i=1}^{n-1}(-1)^i[x_1|x_2|\ldots|x_ix_{i+1}|\ldots|x_n] \\ &+ (-1)^n[x_1|x_2|\ldots|x_{n-1}].\end{aligned}$$

The augmentation $\varepsilon : \bar{B}'_0 \to \mathbb{Z}$ is defined by

$$\varepsilon[\,] = 1.$$

This already makes $\bar{B}'$ into a chain complex since again it is easy to verify that $\partial_n\partial_{n+1} = 0$ for all $n \geq 0$. In particular,

$$\begin{aligned}\partial_1[x_1] &= x_1[\,] - [\,], \\ \partial_2[x_1, x_2] &= x_1[x_2] - [x_1x_2] + [x_2], \\ \partial_3[x_1, x_2, x_3] &= x_1[x_2, x_3] - [x_1x_2, x_3] + [x_1, x_2x_3] - [x_1, x_2].\end{aligned}$$

Proposition 1.34. *There is an isomorphism of chain complexes between $\bar{B}$ and $\bar{B}'$.*

PROOF. Define $\phi_n : \bar{B}_n \to \bar{B'}_n$ by

$$\phi_n(1, y_1, \dots, y_n) = [y_1|y_1^{-1}y_2|\dots|y_{n-1}^{-1}y_n]$$

and $\psi_n : \bar{B'}_n \to \bar{B}_n$ by

$$\psi_n[x_1|\dots|x_n] = (1, x_1, x_1x_2, \dots, x_1x_2\dots x_n).$$

One checks easily that ϕ_n, ψ_n are mutual inverses and that ϕ, ψ defined in this way are inverses and morphism of chain complexes, i.e. commute with differentials in every degree. □

Furthermore, if we define $D'_n := \phi_n D_n$, then D'_n is the submodule of $\bar{B'}_n$ generated by the n-tuples $[x_1|x_2|\dots|x_n]$ with at least one x_i equal to 1. The modules D'_n define a subcomplex D' of $\bar{B}'$ that we call *degenerate*. The quotient complex

$$B' := \bar{B}'/D'$$

is a G-free resolution of $\mathbb{Z}$ isomorphic to B called the *normalized resolution in inhomogeneous form.*

CHAPTER 2

Homological algebra and topology

In this chapter we will try to understand group (co)homology via topology. First we will recall some general facts about (co)homology of topological spaces. Then we introduce some very useful topological spaces (CW-complexes) and give some tools to compute their (co)homology. Finally we define and construct some *Eilenberg-MacLane spaces* $K(G,n)$. We use them to give us the topological interpretation of the (co)homology of a group.

1. Review of the (co)homology of a topological space

In this section, we recall some basic facts about (co)homology. We assume that the reader already followed a course in algebraic topology and knows almost everything in this section.

DEFINITION. Let $n \in \mathbb{N}$, define the *topological n-simplex* to be the topological space

$$\Delta^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1, t_i \geq 0, \forall i\}.$$

A *singular n-simplex* in a topological space X is a continuous map $\sigma : \Delta^n \to X$. Define $S_n(X)$ to be the free abelian group with basis all singular n-simplices. More formally, $S_n(X) := F_{\mathcal{A}b}(\mathrm{Hom}_{\mathcal{T}op}(\Delta^n, X))$.

We now define for $n \in \mathbb{N}$ and $0 \leq i \leq n$, the i-th *face map* to be the continuous map

$$\begin{aligned} \delta^i : \Delta^{n-1} &\to \Delta^n \\ (t_0, \dots, t_{n-1}) &\mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}). \end{aligned}$$

We define the *boundary* on every element of the basis of $S_n(X)$ as

$$d_n \alpha = \sum_{i=0}^{n} (-1)^i \alpha \circ \delta^i \in S_{n-1}(X) \text{ if } n > 0$$

for all $\alpha \in S_n(X)$. Let $d_0 := 0$ be the zero morphism.

THEOREM 2.1. *The graded abelian group $S(X) := (S_n(X))_{n\geq 0}$ equipped with differentials $(d_n : S_n(X) \to S_{n-1}(X))_{n\geq 0}$;*

$$\cdots \to S_{n+1}(X) \xrightarrow{d_{n+1}} S_n(X) \xrightarrow{d_n} S_{n-1}(X) \to \cdots \to S_1(X) \xrightarrow{d_1} S_0(X) \to 0$$

is a chain complex.

PROOF. Use the fact that $\delta^j \delta^k = \delta^k \delta^{j-1}$ if $k < j$ to show that $d_n d_{n+1} = 0$. □

REMARK. We can say more about $S_n(X)$. If we define for $0 \leq j \leq n$ the j-th *degeneracy maps* to be the continuous map

$$\begin{aligned} \sigma^j : \Delta^{n+1} &\to \Delta^n \\ (t_0, \dots, t_{n+1}) &\mapsto (t_0, \dots, t_{j-1}, t_j + t_{j+1}, \dots, t_{n+1}). \end{aligned}$$

Then define $\tilde{d}_i : S_n(X) \to S_{n-1}(X)$ to be the function $\alpha \mapsto \alpha \circ \delta^i$ and $s_j : S_{n+1}(X) \to S_n(X)$ as the function $\alpha \mapsto \alpha \circ \sigma^j$ for $1 \leq i, j, \leq n$. Then $S(X)$ equipped with the $(\tilde{d}_i)_i$ and $(s_j)_j$ is a simplicial set.

Definition. Let X be a topological space. The n-th *(singular) homology group* is

$$H_n(X) = H_n(S(X)) = \ker d_n / \operatorname{im} d_{n+1}.$$

If $f : X \to Y$ is a continuous map, then we can define a homomorphism $f_\# : S_n(X) \to S_n(Y)$ by $f_\#(\sum a_\alpha \alpha) := \sum a_\alpha f \circ \alpha$, where $a_\alpha \in \mathbb{Z}$ almost all equal to zero.

Lemma 2.2. *If $f : X \to Y$ is a continuous map, then the diagram*

$$\begin{tikzcd} S_n(X) \arrow[r, "d_n"] \arrow[d, "f_\#"'] & S_{n-1}(X) \arrow[d, "f_\#"] \\ S_n(Y) \arrow[r, "d_n"] & S_{n-1}(Y) \end{tikzcd}$$

commutes.

Proof. A short calculation gives us the result. □

Therefore, we can define $H_n(f) : H_n(X) \to H_n(Y)$ by $H_n(f)([z]) := [f_\#(z)]$. To see that its well defined, we need to show that

$$f_\#(\ker d_n) \subset \ker d_n, \quad \text{and} \quad f_\#(\operatorname{im} d_n) \subset \operatorname{im} d_n.$$

All of this is done carefully in [**Rot88**, Chapter 4, lemma 4.9].

Theorem 2.3.

$$H_n : \mathcal{T}op \to \mathcal{A}b$$

is a functor for all $n \geq 0$.

Definition. Let A be a subspace of a topological space X. We define the *relative singular chain complex* to be the quotient $S(X, A) := S(X)/S(A)$. Define now the n-th *relative homology group* as

$$H_n(X, A) := H_n(S(X, A)).$$

If G is an abelian group, define the n-th *relative homology group with coefficients in G* to be

$$H_n(X, A; G) := H_n(S(X, A) \otimes G).$$

Similarly, define the n-th *relative cohomology group with coefficients in G* as the cohomology of the dual complex associated to the relative singular chain complex, i.e.

$$H^n(X, A; G) := H^n(\operatorname{Hom}(S(X, A), G)).$$

Remark. One can see that the relative (co)homology group (with coefficient in an abelian group G) is a well defined functor from the category of topological spaces to abelian groups.

1.1. Important theorems from algebraic topology. We recall here theorems given by the algebraic topology. The interested reader will find the missing proofs in any good algebraic topology book, such as [**Hat02**], [**Rot88**], [**Bre93**], etc.

Theorem 2.4 (Long exact sequence for a pair in homology)**.** *If A is a subspace of X, there exists a long exact sequence*

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

In addition, if $f:(X,A)\to(Y,B)$ is a continuous map of pairs (i.e. $f:X\to Y$ is continuous and $f(A)\subset B$), then we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f|_{A*}"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "\overline{f}_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f|_{A*}"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. The proof is given by applying theorem 1.2 to the exact sequence

$$0 \to S(A) \to S(X) \to S(X)/S(A) \to 0.$$

The naturality of ∂ gives us the second part of the theorem. □

We have a similar exact sequence for cohomology.

Theorem 2.5 (Long exact sequence for a pair in cohomology)**.** *If A is a subspace of X and G an abelian group, then there exists a long exact sequence*

$$\cdots \to H^n(X,A;G) \xrightarrow{j^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\partial} H^{n+1}(X,A;G) \to \cdots$$

Theorem 2.6. *Let $A'\subset A\subset X$ be topological spaces, then there exists a long exact sequence*

$$\cdots \to H_n(A,A') \to H_n(X,A') \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,A') \to \cdots$$

and for a given commutative diagram of pairs

$$\begin{tikzcd}
(A,A') \arrow[r] \arrow[d] & (X,A') \arrow[r] \arrow[d] & (X,A) \arrow[d] \\
(B,B') \arrow[r] & (Y,B') \arrow[r] & (Y,B),
\end{tikzcd}$$

we have the commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A,A') \arrow[r] \arrow[d] & H_n(X,A') \arrow[r] \arrow[d] & H_n(X,A) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(A,A') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_n(B,B') \arrow[r] & H_n(Y,B') \arrow[r] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B,B') \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Apply theorem 1.2 to the exact sequence

$$0 \to S(A)/S(A') \to S(X)/S(A') \to S(X)/S(A) \to 0.$$ □

Remark. Apply the last theorem to the commutative diagram of pairs

$$\begin{tikzcd}
(A,\emptyset) \arrow[r] \arrow[d] & (X,\emptyset) \arrow[r] \arrow[d] & (X,A) \arrow[d] \\
(A,B) \arrow[r] & (X,B) \arrow[r] & (X,A)
\end{tikzcd}$$

to obtain the commutative diagram

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\partial"] \arrow[d, "\partial'"'] & H_{n-1}(A) \arrow[dl, "i_*"] \\ H_{n-1}(A,B) & \end{tikzcd}$$

where ∂ is the connecting morphism given by the pair (X, A) and ∂' is given by the triple (X, A, B).

Theorem 2.7 (Universal coefficients theorem for homology). *Let X be a topological space and G an abelian group, then there exists for all $n \geq 0$ a split exact sequence*

$$0 \to H_n(X) \otimes G \xrightarrow{\alpha} H_n(X;G) \to \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G) \to 0,$$

where $\alpha([x] \otimes g) = [x \otimes g]$.

Proof. The reader shall find the proof in [**Rot88**, Chapter 9, Theorem 9.32]. □

The cohomological statement is the following.

Theorem 2.8 (Universal coefficients theorem for cohomology). *Let X be a topological space and G an abelian group, then there exists for all $n \geq 0$ a split exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), G) \to H^n(X;G) \xrightarrow{\beta} \mathrm{Hom}(H_n(X), G) \to 0,$$

where $\beta([f]) : H_n(X) \to G : [x] \mapsto f(x)$.

Proof. The reader shall find the proof in [**Bre93**, Chapter 5, Theorem 7.1]. □

Theorem 2.9 (Excision for homology). *Let $U \subset A \subset X$ subspaces such that $\overline{U} \subset A^\circ$ (where $\overline{U}$ denote the closure of U and A° is the interior of A). Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces for each n an isomorphism*

$$i_* : H_n(X - U, A - U) \to H_n(X, A).$$

Proof. The reader shall find the proof in [**Hat02**, Chapter 2, Theorem 2.20]. □

We have the same theorem also for cohomology groups.

Theorem 2.10 (Excision for cohomology). *Let $U \subset A \subset X$ subspaces such that $\overline{U} \subset A^\circ$ and let G be an abelian group. Then the inclusion $i : (X-U, A-U) \hookrightarrow (X, A)$ induces for each n an isomorphism*

$$i^* : H^n(X - U, A - U; G) \to H^n(X, A; G).$$

Proof. The reader shall find a sketch of the proof in [**Hat02**, Chapter 3, Section 3.1]. □

As for the cohomology, theorems 2.4, 2.6, 2.7 and 2.9 are true with coefficient in any group G.

1.2. Reduced Homology. Since the homology of a point is 0 in non-zero degree and $\mathbb{Z}$ in degree zero, one could be interested to have another homology theory with 0 everywhere. That is the idea of reduced homology.

DEFINITION. Let X be a topological space, we define the *augmented chain complex*, denoted by $\tilde{S}(X)$, the chain complex

$$\cdots \to S_n(X) \overset{d_n}{\to} S_{n-1}(X) \to \cdots \to S_1(X) \overset{d_1}{\to} S_0(X) \overset{d_0}{\to} \mathbb{Z} \to 0$$

where d_0 is defined to be $\sum n_\alpha \alpha \mapsto \sum n_\alpha$. One can see easily that $\tilde{S}(X)$ is a chain complex.

The n-th *reduced homology group* of X is the homology of the augmented chain complex, i.e.

$$\tilde{H}_n(X) := H_n(\tilde{S}(X)).$$

REMARK. One can see easily that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$, and $H_n(X) = \tilde{H}_n(X)$ for $n > 0$. Hence the reduced homology of a point is zero everywhere.

DEFINITION. Let X be a topological space and $A \subset X$ a subspace. We say that A is a *deformation retract* of X if there exists a continuous map $F : X \times I \to X$ such that $F(-, 0) = \mathrm{Id}_X$, $F(X, 1) = A$ and $F|_{A\times I}(-, t) = \mathrm{Id}_A$ for all $t \in I$.

In fact, a deformation retract is a special case of homotopy.

THEOREM 2.11. *Let X be a space and A a subspace such that A is a nonempty closed subspace that is a deformation retract of some neighborhood in X, then there exists a long exact sequence in reduced homology*

$$\cdots \to \tilde{H}_n(A) \overset{i_*}{\to} \tilde{H}_n(X) \overset{j_*}{\to} \tilde{H}_n(X/A) \overset{\partial}{\to} \tilde{H}_{n-1}(A) \to \cdots$$

where $i : A \hookrightarrow X$ is the inclusion and $j : X \to X/A$ the quotient map.

We will not proof this theorem, in fact we will give a more general result using relative homology.

PROPOSITION 2.12. *Let (X, A) be a pair such that A is a nonempty closed subspace which is a deformation retract of some neighborhood in X, and let $q : (X, A) \to (X/A, A/A)$ be the quotient map. Then we have an induced isomorphism*

$$q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A).$$

for all n.

PROOF. Let $A \subset V \subset X$ such that A is a deformation retract of V. This induces a homotopy equivalence $(V, A) \simeq (A, A)$. Therefore $H_n(V, A) = H_n(A, A) = 0$. The same fact appears on the quotient $(V/A, A/A) \simeq (A/A, A/A)$, hence $H_n(V/A, A/A) = 0$. That implies the existence of a commutative square

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\cong"] \arrow[d, "q_*"'] & H_n(X,V) \arrow[d, "q_*"] \\ H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, V/A) \end{tikzcd}$$

where the horizontal arrows are the isomorphisms given by the long exact sequences of the triple (X, V, A) and $(X/A, V/A, A/A)$. The excision theorem gives us another commutative square

$$\begin{tikzcd} H_n(X,V) \arrow[d, "q_*"'] & H_n(X-A, V-A) \arrow[l, "\cong"'] \arrow[d, "q_*"] \\ H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A). \arrow[l, "\cong"'] \end{tikzcd}$$

One sees that the right vertical arrow is an isomorphism because $X - A$ and $V - A$ are homeomorphic respectively to $X/A - A/A$ and $V/A - A/A$. To finish the proof consider the commutative diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong"] \arrow[d, "q_*"'] & H_n(X,V) \arrow[d, "q_*"'] & H_n(X-A,V-A) \arrow[l, "\cong"'] \arrow[d, "q_*"] \\
H_n(X/A,A/A) \arrow[r, "\cong"] & H_n(X/A,V/A) & H_n(X/A-A/A,V/A-A/A). \arrow[l, "\cong"']
\end{tikzcd}$$

The preceding arguments implies that all the arrows are isomorphisms. □

2. CW-Complexes

We define in this section the notion of CW-complexes. This class of spaces forms a very interesting category with a lot of nice properties. The reader will see that most of topological spaces that he has in mind are naturally CW-complexes.

Definition. Let $X^{(0)}$ be a discrete set of points. Suppose $X^{(n-1)}$ has been defined, let $(f_{\partial e} : S^{n-1} \to X^{(n-1)})_{e \in E_n}$ be a collection of maps indexed by a set E_n, called *attaching maps*. Define a space Y,

$$Y := \coprod_e D^n_e$$

where $D^n_e = D^n$ for each e, and another space B,

$$B := \coprod_e S^{n-1}_e$$

where $S^{n-1}_e = S^{n-1}$ for each e.

The maps $f_{\partial e}$ give us a continuous map

$$f : B \to X^{(n-1)} \quad : \quad x \mapsto f_{\partial e}(x), \forall x \in S^{n-1}_e.$$

Then define the space

$$X^{(n)} := X^{(n-1)} \cup_f Y = (X^{(n-1)} \amalg Y)/x \sim f(x), \forall x \in B.$$

This space can also be describe as the pushout

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, hook, "i"] & X^{(n-1)} \arrow[d] \\
Y \arrow[r] & X^{(n)}
\end{tikzcd}$$

and is called the *n-th skeleton*. Finally, define the space X to be the union $\bigcup_n X^{(n)}$ embedded with the *weak topology*, i.e. U is an open subset of X if $U \cap X^{(n)}$ is open in $X^{(n)}$ for all $n \geq 0$. Such spaces are called *CW-complexes* (the C stands for "closure finite" and the W for "weak topology").

Moreover, for each e we define the *characteristic map* of the cell e to be the canonical map $f_e : D^n_e \to X$. The image of f_e is denoted by X_e and is called a *closed cell* and the image of the open disk $D^n_e - S^{n-1}_e$, denoted by U_e, is called an *open cell.* However, open cells are generally not open in X, they are open in $X^{(n)}$.

We say that Y is a *subcomplex* of X if it is a union of some of the closed cells which is a CW-complex with the same attachment maps. For example, the n-th skeleton are subcomplexes.

Remark. One can see that U is an open subset of X if and only if $f_e^{-1}(U)$ is open for each e. And a function $g : X \to Y$ is continuous if and only if $gf_e : D_e^n \to Y$ is continuous for all e.

For reasons of convenience we identify e to the open cell U_e. Consequently, we can write that $X = \bigcup\{e : e \in E_n,\ \text{for some } n\}$. We can naturally define the *dimension* of any cell e, such that the n-th skeleton $X^{(n)} = \bigcup\{e : \dim(e) \leq n\}$. Sometimes, a cell of dimension n shall be denoted by e^n.

Proposition 2.13. *Let X be a CW-complex, then*

(1) *for any subset A of X that has no two points in the same open cell (i.e for all $x, y \in A$ and for all e, $x \notin e$ or $y \notin e$), then A is closed and discrete,*
(2) *for any compact $C \subset X$, C is contained in a finite union of open cells,*
(3) *for each closed cell $\overline{e}$ of X, there exists a finite subcomplex of X containing it.*

Proof. The reader shall find the proof in [**Bre93**, Chapter 4, Proposition 8.1] □

Corollary 2.14. *Let X be a CW-complex and C a compact subset of X, then there exists a finite subcomplex containing C.*

To finish this section, we state a very useful theorem involving CW-complexes. We have to say that this theorem is not trivial and requires a technical proof.

Theorem 2.15 (Cellular approximation)**.** *Let $f : X \to Y$ be a continuous map between CW-complexes. Then there exists a map $h : X \to Y$ such that f is homotopic to h, and $h(X^n) \subset Y^n$ for some $n \geq 0$.*

Proof. The reader shall find the proof in [**Hat02**, Chapter 4, Theorem 4.8]. □

There is many interesting theorems about CW-complexes. For example the Whitehead theorem says that if we have a weak equivalence between CW-complexes then it is a homotopy equivalence.

3. Cellular Homology

Singular homology is not always easy to compute, but for CW-complexes we can use the information of its cell structure to calculate it. In this section our goal is to define the cellular homology. This homology theory can also be used to compute the homology of any cellular space.

Definition. Let X be a topological space. A *filtration* is a sequence of subspaces $(X^n)_{n\in\mathbb{Z}}$ such that $X^n \subset X^{n+1}$ for all $n \in \mathbb{Z}$. A filtration is *cellular* if it satisfies in addition:

(1) $H_k(X^n, X^{n-1}) = 0$ for all $k \neq n$,
(2) for all $m \geq 0$ and $\sigma : \Delta^m \to X$, there exists an integer n such that $\operatorname{im}\sigma \subset X^n$.

A topological space is a *cellular space* if it has a cellular filtration. For two cellular spaces X and Y, a *cellular map* is a continuous map $f : X \to Y$ such that $f(X^n) \subset Y^n$ for all $n \in \mathbb{Z}$.

Remark. One can see that CW-complexes are cellular spaces. Indeed its skeletons form a cellular filtration.

Definition. Let X be a cellular space and $k \geq 0$, we define

$$W_k(X) := H_k(X^k, X^{k-1})$$

and $d_k : W_k(X) \to W_{k-1}(X)$ to be the composition

$$H_k(X^k, X^{k-1}) \xrightarrow{\partial} H_{k-1}(X^{k-1}) \xrightarrow{i_*} H_{k-1}(X^{k-1}, X^{k-2})$$

where $i : (X^k, \emptyset) \to (X^k, X^{k-1})$ is the inclusion and ∂ is given by theorem 2.4.

Lemma 2.16. *For any cellular space X, $W_*(X) = (W_k(X))_{k\geq 0}$ is a chain complex with differentials $(d_k : W_k(X) \to W_{k-1}(X))_{k\geq 0}$*

Proof. We have to show that $d_k d_{k+1} = 0$. Indeed, $d_k d_{k+1}$ is the composition

$$H_{k+1}(X^{k+1}, X^k) \xrightarrow{\partial} H_k(X^k) \xrightarrow{i_*} H_k(X^k, X^{k-1}) \xrightarrow{\partial} H_{k-1}(X^{k-1})$$
$$\xrightarrow{i_*} H_{k-1}(X^{k-1}, X^{k-2}),$$

and by theorem 2.4, we have that this sequence is zero. □

$(W_*(X), (d_k)_{k\geq 0})$ is commonly called the *cellular chain complex* of X.

Lemma 2.17. *Let X be a cellular space and let $p \geq q$. Then*

(1) $H_n(X^p, X^q) = 0$ *if $n \leq q$ or if $n > p$,*
(2) $H_n(X, X^q) = 0$ *if $q \geq n$,*
(3) $H_n(X, X^q) \cong H_n(X^{n+1}, X^q)$ *if $q < n$,*
(4) $H_n(X, X^{-1}) \cong H_n(X, X^{-k})$ *for all n and $k \geq 1$.*

Proof. The reader shall find the proof in [**Rot88**, Chapter 8, lemma 8.35]. □

Theorem 2.18. *Let X be a cellular space and $n \geq 0$, then*

$$H_n(W_*(X)) \cong H_n(X, X^{-1}).$$

Proof. By theorem 2.6 (the long exact sequence for the triple (X^{n+1}, X^n, X^{-1})) we have the commutative diagram

$$\begin{tikzcd} H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, "\partial'"'] & H_n(X^n) \arrow[dl, "\lambda_*"] \\ H_n(X^n, X^{-1}) & \end{tikzcd}$$

where λ_* is induced by the inclusion $(X^n, \emptyset) \xrightarrow{\lambda} (X^n, X^{-1})$. Now take homology of the sequence of pairs

$$(X^n, \emptyset) \xrightarrow{\lambda} (X^n, X^{-1}) \xrightarrow{j} (X^n, X^{n-1}).$$

This gives us the commutative diagram

$$\begin{tikzcd} & H_n(X^n) \arrow[dl, "\lambda_*"'] \arrow[d, "u_*"] \\ H_n(X^n, X^{-1}) \arrow[r, "j_*"'] & H_n(X^n, X^{n-1}). \end{tikzcd}$$

Therefore we have

$$\begin{tikzcd} H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, "\partial'"'] & H_n(X^n) \arrow[dl, "\lambda_*"'] \arrow[d, "u_*"] \\ H_n(X^n, X^{-1}) \arrow[r, "j_*"'] & H_n(X^n, X^{n-1}) \end{tikzcd}$$

which is commutative.

Recall that the differential on the cellular chain complex appears here as $d_{n+1} = u_*\partial$. Hence, we have the commutative diagram

$$\begin{tikzcd} H_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial'"'] \arrow[dr, "d_{n+1}"] & \\ H_n(X^n, X^{-1}) \arrow[r, "j_*"'] & H_n(X^n, X^{n-1}). \end{tikzcd}$$

Argue in a similar way to obtain the commutative diagram

$$\begin{tikzcd} H_n(X^n, X^{n-1}) \arrow[r, "\partial''"] \arrow[dr, "d_n"'] & H_{n-1}(X^{n-1}, X^{-1}) \arrow[d, "i_*"] \\ & H_{n-1}(X^{n-1}, X^{n-2}). \end{tikzcd}$$

If we now put all together, we have a commutative diagram with exact columns and an exact row

$$\begin{tikzcd} & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial'"'] \arrow[dr, "d_{n+1}"] & & 0 \arrow[d] \\ 0 \arrow[r] & H_n(X^n, X^{-1}) \arrow[r, "j_*"'] \arrow[d] & H_n(X^n, X^{n-1}) \arrow[r, "\partial''"] \arrow[dr, "d_n"'] & H_{n-1}(X^{n-1}, X^{-1}) \arrow[d, "i_*"] \\ & H_n(X^{n+1}, X^{-1}) \arrow[d] & & H_{n-1}(X^{n-1}, X^{n-2}) \\ & 0. & & \end{tikzcd}$$

Indeed, the row is exact since it is a piece of the long exact sequence of the triple (X^n, X^{n-1}, X^{-1}) and the zero on the left is $H_n(X^{n-1}, X^{-1})$ which is trivial by lemma 2.17 part (1). The exactness of the columns is obtained by the long exact sequence of the triples (X^{n+1}, X^n, X^{-1}) and $(X^{n-1}, X^{n-2}, X^{-1})$. Use again lemma 2.17 part (1) to have all the zeros.

We now have a usual diagram chase (for the first isomorphism, use lemma 2.17 part (3)):

$$\begin{aligned} H_n(X, X^{-1}) &\cong H_n(X^{n+1}, X^{-1}) \\ &\cong H_n(X^n, X^{-1})/\operatorname{im}\partial' \\ &\cong \operatorname{im} j_*/\operatorname{im}(j_*\partial') \\ &= \operatorname{im} j_*/\operatorname{im} d_{n+1} \\ &= \ker \partial''/\operatorname{im} d_{n+1} \\ &= \ker i_*\partial''/\operatorname{im} d_{n+1} \\ &= \ker d_n/\operatorname{im} d_{n+1} = H_n(W_*(X)). \end{aligned}$$

□

Corollary 2.19. *If X is a cellular space with $X^{-1} = \emptyset$, then*

$$H_n(W_*(X)) \cong H_n(X)$$

for all $n \geq 0$.

If X is now a CW-complex, one would like to describe the relative homology groups using somehow its cell structure.

Definition. Let X be a CW-complex and Y a subcomplex of X, define

$$X_Y^k := X^{(k)} \cup Y.$$

Proposition 2.20. *The sequence $(X_Y^k)_k$ is a cellular filtration of X.*

Proof. Let E and E' be the cell decompositions of X and Y, hence $E' \subset E$. Let S_k be the set composed by choosing exactly one point of each cell of dimension k in $E - E'$. To reduce notation let X^k denote X_Y^k. Remark that $H_n(X^k - S_k, X^{k-1}) = 0$ (this is true since X^{k-1} is a strong deformation retract of $X^k - S_k$, the unconvinced reader can find the proof in [**Rot88**, Chapter 8, lemma 8.28]). Consider now the long exact sequence of the triple $(X^k, X^k - S_k, X^{k-1})$ and look at the piece

$$H_n(X^k-S_k, X^{k-1}) \to H_n(X^k, X^{k-1}) \to H_n(X^k, X^k-S_k) \xrightarrow{\partial} H_{n-1}(X^k-S_k, X^{k-1}).$$

Therefore, we have an isomorphism (given by the inclusion $(X^k, X^{k-1}) \hookrightarrow (X^k, X^k - S_k)$)

$$H_n(X^k, X^{k-1}) \cong H_n(X^k, X^k - S_k).$$

Moreover, one can see that $\overline{X^{k-1}} \subset (X^k - S_k)^\circ$. By the excision theorem, we have that

$$H_n(X^k, X^k - S_k) \cong H_n(X^k - X^{k-1}, (X^k - X^{k-1}) - S_k).$$

Remark that $X^k - X^{k-1} = \coprod\{e \in E - E' : \dim e = k\}$ and the homology group of a disjoint union is the sum of the homology groups, therefore

$$H_n(X^k - X^{k-1}, (X^k - X^{k-1}) - S_k) \cong \bigoplus_{\{e:\dim e = k\}} H_n(e, e - S_k).$$

Let $s_e \in S_k$ denote the point chosen in the cell e. Thus we have homeomorphisms $e \cong \mathbb{R}^k$ and $e - s_e \cong \mathbb{R}^k - \{0\} \simeq S^{k-1}$. Consider the long exact sequence in reduced homology of the pair $(\mathbb{R}^k, \mathbb{R}^k - \{0\})$

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_{n+1}(\mathbb{R}^k) \arrow[r] \arrow[d, equal] & \tilde{H}_{n+1}(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \arrow[r] \arrow[d, equal] & \tilde{H}_n(\mathbb{R}^k - \{0\}) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_n(\mathbb{R}^k) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \tilde{H}_{n+1}(\mathbb{R}^k, \mathbb{R}^k - \{0\}) \arrow[r, "\cong"] & \tilde{H}_n(S^{k-1}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Suppose $n > 0$, then we have

$$H_n(X^k, X^{k-1}) \cong \bigoplus_e H_{n-1}(S^{k-1})$$

and $\bigoplus_e H_{n-1}(S^{k-1}) = 0$ if $k \neq n$.

If $n = 0$, we can show by hands that $H_0(X^k, X^{k-1}) = \bigoplus_e H_0(\mathbb{R}^k, \mathbb{R}^k - \{0\}) = 0$ for $k \neq 0$.

To finish the proof, we need to prove that for any $\sigma : \Delta^m \to X$ there exists an n such that $\operatorname{im}\sigma \subset X^n$. Indeed, $\operatorname{im}\sigma$ is compact and lies in a finite subcomplex of X, take n big enough to obtain the result. $\square$

Definition. Let X be a CW-complex and Y a subcomplex of X, define the *relative cellular chain complex* $W_*(X,Y) = (W_k(X,Y))_{k\in\mathbb{Z}}$ to be the cellular chain complex associated to the filtration $(X_Y^k)_{k\in\mathbb{Z}}$ of X, i.e.

$$W_k(X,Y) = H_k(X_Y^k, X_Y^{k-1}),$$

and the differentials are defined by the composition

$$H_k(X_Y^k, X_Y^{k-1}) \xrightarrow{\partial} H_{k-1}(X_Y^{k-1}) \xrightarrow{i_*} H_{k-1}(X_Y^{k-1}, X_Y^{k-2}).$$

Corollary 2.21. *Let X be a CW-complex and Y a subcomplex, then we have an isomorphism*
$$H_n(W_*(X,Y)) \cong H_n(X,Y).$$

Proof. Clear by theorem 2.18 since $X_Y^{-1} = Y$. □

Corollary 2.22. *Let X be a CW-complex and Y a subcomplex, then $W_k(X,Y)$ is a free abelian group of rank equal to the number of k-cells e in $E - E'$.*

Proof. Suppose $k > 0$, then we have seen in the proof of proposition 2.20 that
$$W_k(X,Y) = H_k(X_Y^k, X_Y^{k-1}) \cong \bigoplus_e \tilde{H}_{k-1}(S^{k-1}) \cong \bigoplus_e \mathbb{Z}[e].$$

For $k = 0$, the reader shall find the proof in [**Rot88**, Chapter 8, Theorem 8.39] □

We now have a nice way to compute the chain complex of any CW-complex. We now need to understand our cellular differentials.

Definition. Let $f : S^n \to S^n$ be continuous and $n > 0$. The *degree* of f, is the integer $d = \deg(f)$ such that $f_*(\sigma) = d\sigma$ where $f_* : H_n(S^n) \to H_n(S^n)$ is the induced homomorphism and σ is any element of the cyclic group $H_n(S^n)$.

This definition has a sense since any homomorphism $\mathbb{Z} \to \mathbb{Z}$ is of the form $x \mapsto dx$.

Remark. If $f \simeq g$ then $\deg(f) = \deg(g)$. And if f is a homotopy equivalence, then $\deg(f) = \pm 1$. Indeed, suppose $fg \simeq \mathrm{Id}$, then $\deg(fg) = \deg(f)\deg(g) = \deg(\mathrm{Id}) = 1$.

Theorem 2.23. *Let X be a CW-complex and $e_\alpha \in W_n(X)$ for $n > 1$ (where we identify generators of $W_n(X)$ with the n-cells via the isomorphism above). Then*
$$d_n(e_\alpha) = \sum_\beta \deg(p_\beta f_{\partial e_\alpha}) e_\beta$$
where e_β are the cells of dimension $n-1$ corresponding to generators of $W_{n-1}(X)$, $f_{\partial e_\alpha} : S^{n-1}_{e_\alpha} \to X^{n-1}$ is the attaching map of the cell e_α and $p_\beta : X^{n-1} \to S^{n-1}_{e_\beta}$ is the quotient map identifying $X^{n-1} - e_\beta$ to a point.

Proof. Let e_α be a cell of dimension n in X. For reason of convenience, write D_α instead of $D^n_{e_\alpha}$. Moreover, let $f_\alpha : D^n_\alpha :\to X^n$ be the characteristic map of the cell e_α, $f_{\partial\alpha} : \partial D^n_\alpha \to X^{n-1}$ the attaching map, $q : X^{n-1} \to X^{n-1}/X^{n-2}$ the quotient map. We can see that the space X^{n-1}/X^{n-2} is homotopic to the bouquet of spheres $\bigvee S^{n-1}_\beta$, hence we can define $q_\beta : X^{n-1}/X^{n-2} \to S^{n-1}_\beta$ to be the quotient map identifying to a point the subspace $X^{n-1} - e_\beta$.

Now construct the diagram

$$\begin{tikzcd}
H_n(D^n_\alpha, \partial D^n_\alpha) \arrow[r, "\partial", "\cong"'] \arrow[d, "f_{\alpha *}"'] & \tilde{H}_{n-1}(\partial D^n_\alpha) \arrow[r, "q_{\beta *} q_* f_{\partial\alpha *}"] \arrow[d, "f_{\partial\alpha *}"] & \tilde{H}_{n-1}(S^{n-1}_\beta) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \tilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \tilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}).
\end{tikzcd}$$

The morphism ∂ is given by the long exact sequence of the pair $(D^n_\alpha, \partial D^n_\alpha)$ and is hence an isomorphism. The morphism ∂_n is given by the long exact sequence of the pair (X^n, X^{n-1}). The naturality of the connecting morphism ensure the commutativity of the upper left square. The commutativity of the upper right square and the triangle is definitional. The commutativity of the last square is given by proposition 2.12.

We now study the image by d_n of our cell e_α.

$$d_n(e_\alpha) = j_{n-1} f_{\partial\alpha *}\partial[D^n_\alpha] = \sum_\beta d_{\alpha\beta} e_\beta$$

where $[D^n_\alpha]$ is a generator of $H_n(D^n_\alpha, \partial D^n_\alpha)$ which is sent on e_α via $f_{\alpha *}$ and the $d_{\alpha\beta}$ are some coefficients in $\mathbb{Z}$. To characterize them, project on $\tilde{H}_{n-1}(S^{n-1}_\beta)$ (via the isomorphisms that are below in our diagram) to isolate $d_{\alpha\beta}$. Hence, $d_{\alpha\beta}$ will be the degree of the homomorphism $q_\beta q f_{\partial\alpha}$. □

Examples. We shall give two examples how one uses this formula to compute the homology of some closed surfaces.

(1) Let M_g be the closed orientable surface of genus g. One can construct it as a CW-complex by taking X^0 to be a point, $X^1 = S^1_{a_1} \vee S^1_{b_1} \vee \cdots \vee S^1_{a_g} \vee S^1_{b_g}$ a wedge of $2g$ circles. One sees that $\pi_1(X^1, x_0) = *_{a_i, b_j}\mathbb{Z}$ is the free group on generators a_i, b_j, and $X^2 = X^1 \cup_{f_\partial} D^2$ where we attach the disk D^2 by a continuous map $f_\partial : S^1 \to X^1$ which represent $[a_1, b_1] \ldots [a_g, b_g] \in \pi_1(X^1, x_0)$ (here the brackets $[x, y]$ denote the commutator $xyx^{-1}y^{-1}$).

We are now interested to use the formula to compute d_2. Since we have only one 2-cell, d_2 is fully determined by the image of e^2. We know $d_2(e^2) = \sum_\beta \deg(q_\beta q f_{\partial e^2}) e_\beta$. Remark that $q_\beta q_* f_{\partial e^2}$ is homotopic to the trivial map for any $\beta = a_i$ or b_i. Indeed, after passing to the quotient, our attaching map becomes a representative of the word $a_i a_i^{-1}$ in $\mathbb{Z}a_i$ or $b_i b_i^{-1}$ in $\mathbb{Z}b_i$ which is trivial. Therefore, the degree of such map is 0 and d_2 reveals to be the zero map. Moreover, d_1 is the zero map because the rank of the 0-th homology group is the number of path components.

We obtain the homology of this complex by taking the homology of the chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_2 = 0} \bigoplus_{i=1}^{2g} \mathbb{Z} \xrightarrow{d_1 = 0} \mathbb{Z} \to 0,$$

i.e.

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n = 0 \\ \bigoplus_{i=1}^{2g} \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n > 2. \end{cases}$$

(2) Let N_g be the closed nonorientable surface of genus g. As in the preceding example, it is a CW-complex by attaching this time a cell of dimension 2, denoted e^2, on the wedge of circles $S^1_{a_1} \vee \cdots \vee S^1_{a_g}$ by a representative f_∂ of the word $a_1^2 \ldots a_g^2$. As above, we use the cellular boundary formula to compute $d_2(e^2)$. One see that $q_\beta q f_\partial$ is homotopic to the map $z \to z^2$ for each β. Taking the homology gives us a map of degree 2. Thus we have $d_2(e^2) = 2e^1_{a_1} + \cdots + 2e^1_{a_g}$. We now want to take homology of the chain complex

$$0 \to \mathbb{Z} \xrightarrow{d_2} \bigoplus_{i=1}^{g} \mathbb{Z} \xrightarrow{d_1 = 0} \mathbb{Z} \to 0.$$

We may modify our natural basis of $\bigoplus_{i=1}^{g}\mathbb{Z}$ to the one of the form $\{e^1_{a_1},\dots,e^1_{a_{g-1}},e^1_{a_1}+\dots+e^1_{a_g}\}$ to obtain the homology

$$H_n(N_g)=\begin{cases}\mathbb{Z} & n=0\\ (\bigoplus_{i=1}^{g-1}\mathbb{Z})\oplus\mathbb{Z}_2 & n=1\\ 0 & n>1.\end{cases}$$

We also can talk about cellular cohomology. We give here some results without proof.

Definition. Let X be a cellular topological space and G an abelian group. The *cellular cochain complex* (with coefficient in G) is the cochain complex

$$W^k(X;G):=H^k(X^k,X^{k-1};G),$$

with differentials δ^k defined as the composition

$$H^k(X^k,X^{k-1};G)\overset{j_k}{\to}H^k(X^k;G)\overset{\partial_k}{\to}H^{k+1}(X^{k+1},X^k;G).$$

where j_* is induced by the inclusion of pairs $j:(X^k,\emptyset)\to(X^k,X^{k-1})$ and ∂_k is the connecting morphism of the long exact of a pair for cohomology (see theorem 2.5).

One can see that for a cellular space X, $(W^*(X;G),(\delta^n)_n)$ is a cochain complex. To argue that, one can use similar arguments that are in the proof of lemma 2.16.

Theorem 2.24. *Let X be a CW-complex, and G an abelian group. Then*

$$H^n(X;G)\cong H^n(W^*(X;G)),$$

and the cellular cochain complex is isomorphic to the dual of the cellular chain complex, more precisely

$$W^*(X;G)\cong \mathrm{Hom}(W_*(X),G).$$

Proof. The reader shall find the proof in [**Hat02**, Chapter 3, Theorem 3.5]. □

4. $K(G,n)$

In this section we construct the *Eilenberg-MacLane space*. They will play an important role in the last section since they give a topological interpretation of the (co)homology of groups.

Definition. Let (X,x_0) be a path connected topological space. We say that (X,x_0) is of *type* (G,n) if $\pi_n(X,x_0)\cong G$ and $\pi_k(X,x_0)=0$ for all $k\neq n$, $k>0$. Moreover, if X has the homotopy type of a CW-complex, then we say that it is an *Eilenberg-MacLane space* $K(G,n)$.

Theorem 2.25. *let G be a group and $n\geq 1$. If $n\geq 2$ suppose G is commutative. Then there exists an Eilenberg-MacLane space $K(G,n)$.*

Proof. We prove first the case where $n=1$. Let

$$1\to R\to F\to G\to 1$$

be a presentation of G where F is a free group and R a normal subgroup of F. We shall build a CW-complex with this data.

Take X^1 to be a wedge sum of circles $\bigvee_\alpha S^1_\alpha$ where α are the generators of F as a free group. Hence $\pi_1(X^1)\cong F$ via the Seifert van-Kampen theorem. Let β be generators of R as a normal subgroup of F. Each β corresponds to a unique element in $\pi_1(X^1,x_0)$ which can be represented by a map $\tilde\beta:S^1\to X^1$. This defines for all β an attachment map

$$\tilde\beta:\partial D^2_\beta\to X^1.$$

Put them together and obtain a map

$$f : \coprod_{\beta} \partial D^2_{\beta} \to X^1.$$

The 2-skeleton is defined naturally by

$$X^2 := X^1 \cup_f (\coprod_{\beta} D^2_{\beta}).$$

Via the Seifert van-Kampen theorem one can see that $\pi_1(X^2, x_0) \cong F/R \cong G$.

Suppose by induction on $m \geq 2$ that we constructed the m-skeleton X^m, such that

$$\pi_r(X^m, x_0) = \begin{cases} G & r = 1 \\ 0 & 1 < r < m. \end{cases}$$

Choose generators $\gamma \in \pi_m(X^m, x_0)$, and look at some representatives

$$\tilde{\gamma} : (S^m, s_0) \to (X^m, x_0).$$

As before, this defines an attachment map for each γ. Put them together to obtain

$$f : \coprod_{\gamma} \partial D^{m+1}_{\gamma} \to X^m$$

and define

$$X^{m+1} := X^m \cup_f (\coprod_{\gamma} D^{m+1}_{\gamma}).$$

Draw the long exact sequence in homotopy of the based pair (X^{m+1}, X^m, x_0)

$$\begin{aligned} \cdots &\longrightarrow \pi_{m+1}(X^{m+1}, X^m, x_0) \xrightarrow{\partial_{m+1}} \\ &\longrightarrow \pi_m(X^m, x_0) \longrightarrow \pi_m(X^{m+1}, x_0) \longrightarrow \pi_m(X^{m+1}, X^m, x_0) \xrightarrow{\partial_m} \\ &\longrightarrow \pi_{m-1}(X^m, x_0) \longrightarrow \pi_{m-1}(X^{m+1}, x_0) \longrightarrow \pi_{m-1}(X^{m+1}, X^m, x_0). \end{aligned}$$

Remark that $\pi_r(X^{m+1}, X^m, x_0) = 0$ for all $r \leq m$. Indeed, let $[\alpha] \in \pi_r(X^{m+1}, X^m, x_0)$ i.e. $\alpha : (S^r, \partial S^r) \to (X^{m+1}, X^m)$, by the cellular approximation theorem our map α is homotopic to a cellular map. Hence $\alpha(S^r) \subset X^r \subset X^m$, i.e. $[\alpha] = 0$. Therefore, we have an induced isomorphism $\pi_r(X^m, x_0) \cong \pi_r(X^{m+1}, x_0)$ for all $r = 1, \ldots, m-1$. We want to show that $\pi_m(X^{m+1}, x_0) = 0$ i.e. ∂_{m+1} is an epimorphism. We show that every generator in $\pi_m(X^m, x_0)$ is the image of an element in $\pi_{m+1}(X^{m+1}, X^m, x_0)$. By construction the homotopy classes of our attaching maps $[\tilde{\gamma}]$ generate $\pi_m(X^m, x_0)$. By the very definition of the boundary map ∂_{m+1}, we see that the class of the characteristic map of $\tilde{\gamma}$ is sent on the homotopy class of the attachment map $\tilde{\gamma}$. Hence ∂_{m+1} is an epimorphism.

Therefore, we constructed a topological space X^{m+1} whose homotopy groups until the degree m are

$$\pi_r(X^{m+1}, x_0) = \begin{cases} G & r = 1 \\ 0 & 1 < r < m+1. \end{cases}$$

Define $X := \bigcup_m X^m$, it is a CW-complex and of type $(G, 1)$.

Suppose now $n > 1$ and G commutative. As before, take a free abelian resolution of G

$$0 \to N \to L \to G \to 0$$

and define X^n to be a wedge of n-dimensional spheres $\bigvee S^n_{\alpha}$ where α are the generators of L. The n-th homotopy group of X^n is L. Hence for the generators β

of N there exist continuous functions $\tilde{\beta} : S^n \to X^n$ whose homotopy classes in $\pi_n(X^n, x_0)$ corresponds to β. This defines for each β an attachment map

$$\tilde{\beta} : \partial D_\beta^{n+1} \to X^n.$$

As before, use this data to construct X^{n+1} by glueing $\coprod_\beta D_\beta^{n+1}$ on X^n. It remains to show that $\pi_n(X^{n+1}, x_0) \cong L/N \cong G$. First we show $H_n(X^{n+1}) = L/N$. For this consider the Mayer-Vitoris sequence associated to the push-out

$$\begin{tikzcd} \bigvee S_\beta^n \arrow[r, "\bigvee \tilde{\beta}"] \arrow[d, hook] & X^n \arrow[d] \\ \bigvee D_\beta^{n+1} \arrow[r] & X^{n+1}. \end{tikzcd}$$

The exact sequence is

$$\underbrace{H_n(\bigvee S_\beta^n)}_{\cong N} \to \underbrace{H_n(\bigvee D^{n+1})}_{=0} \oplus \underbrace{H_n(X^n)}_{\cong L} \to H_n(X^{n+1}) \xrightarrow{\partial} \underbrace{H_{n-1}(\bigvee S_\beta^n)}_{=0}.$$

This implies $H_n(X^{n+1}) \cong L/N$ and by the Hurewicz theorem (see below) we have $\pi_n(X^{n+1}, x_0) \cong L/N$. By construction, we have also that $\pi_k(X^{n+1}, x_0) = 0$ for all $k < n$ (this is given once more by the cellular approximation theorem). The higher homotopy groups are killed by induction as before. □

Theorem 2.26 (Hurewicz)**.** *Let X be a path connected topological space, then there exists a homomorphism of groups*

$$h_n : \pi_n(X, x_0) \to H_n(X, x_0)$$

defined as $h_n([S^n \xrightarrow{\alpha} X]) := \alpha_*(1)$, where $\alpha_* : H_n(S^n) \to H_n(X)$ is the induced homomorphism and 1 is the generator of $H_n(S^n)$ identified with the infinite cyclic group.

If in addition $\pi_k(X, x_0) = 0$ for all $k = 1, \dots n-1$ and $n > 1$, then h_n is an isomorphism of groups. If $n = 1$, then h_n induces an isomorphism

$$\pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \xrightarrow{\cong} H_1(X).$$

We have to say here that this theorem is totally non-trivial and a lot of work must be done to show it. See for example [**Hat02**, Chapter 4, Theorem 4.32] and [**Spa66**, Chapter 7, Sections 4 & 5].

We will see in the future that we can construct $K(G,n)$ by a more conceptual way using a group actions. For that we need to recall some facts about covering spaces.

Definition. Let X be a topological space which is path connected and locally path connected. A simply-connected covering space of X is a called a *universal cover* of X.

Theorem 2.27. *Let X be a path connected and locally path connected topological space. Then X admits a universal cover if and only if X is locally relatively simply connected (i.e. each point $x \in X$ has a neighborhood U such that $\pi_1(U, u) \xrightarrow{i_*} \pi_1(X, u)$ is trivial for all $u \in U$).*

Proof. The reader shall find the proof in [**Bre93**, Chapter 3, Theorem 8.4]. □

Corollary 2.28. *Any CW-complex admits a universal cover.*

DEFINITION. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and $p' : (\tilde{X}', \tilde{x}'_0) \to (X, x_0)$ be two covering space. We say that p is *equivalent* to p' if there exists a homeomorphism $f : (\tilde{X}, \tilde{x}_0) \to (\tilde{X}', \tilde{x}'_0)$ such that $p = p' \circ f$.

THEOREM 2.29 (Classification of covering spaces)**.** *Let X be a path connected, locally path connected and locally relatively simply connected. Then we have a one-to-one correspondence between equivalence classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. The correspondence is given by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to any covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$.*

PROOF. The reader shall find the proof in [**Hat02**, Chapter 1, Theorem 1.38]. □

REMARK. In view of the classification theorem, one sees that a universal cover of a space X (path connected, locally path connected and locally relatively simply connected) is unique up to homeomorphism. Indeed, it corresponds to the trivial subgroup of $\pi_1(X, x_0)$. Therefore, it has sense to say "the" universal cover of X.

We recall now some facts about group actions.

DEFINITION. Let G be a group acting on a topological space $\tilde{X}$, i.e. there exists a group homomorphism $G \to \text{Homeo}(\tilde{X}, \tilde{X})$. We say then that $\tilde{X}$ is a *G-space*. We define the *orbit space* to be the space $\tilde{X}/G := \tilde{X}/\sim$ where $gx \sim x$ for all $g \in G$ and $x \in \tilde{X}$. The action of G on $\tilde{X}$ is said to be a *covering space action* if for all $x \in \tilde{X}$ there exists an open neighborhood U such that if $gU \cap U \neq \emptyset$, then $g = e$.

There is a natural way to obtain such action for connected, locally path connected spaces.

DEFINITION. Let $p : \tilde{X} \to X$ be a covering space. The group

$$G(\tilde{X}) := \{g : \tilde{X} \to \tilde{X} \mid g \text{ is a homeomorphism, and } pg = p\}$$

is called the group of *deck transformations* of $p : \tilde{X} \to X$. Moreover, we say that the covering space $p : \tilde{X} \to X$ is *regular* if $G(\tilde{X})$ acts transitively on the fiber $p^{-1}(x)$ for all $x \in \tilde{X}$.

By the unique lifting property, one can see that the group of deck transformations acts freely on $\tilde{X}$, since only the identity deck transformation can fix a point.

PROPOSITION 2.30. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space of path connected spaces, with X locally path connected. Then:*

(1) *The covering space is regular if and only if $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$ is a normal subgroup.*
(2) *If p is regular, then $G(\tilde{X}) \cong \pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

PROOF. The reader shall find the proof in [**Hat02**, Chapter 1, Proposition 1.39]. □

REMARK. With the same notation as in the proposition, if $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is the universal cover and is in addition regular, then $\pi_1(X, x_0) \cong G(\tilde{X})$.

The explicit isomorphism in (2) is given by the following. Take $[\lambda] \in \pi_1(X, x_0)$ and let $\hat{\lambda}$ be the unique lift of λ with $\hat{\lambda}(0) = \tilde{x}_0$. Then we know that $\hat{\lambda}(1) \in p^{-1}(x_0)$. Since the action of the group of deck transformations is free, there exists a unique $g_\lambda \in G(\tilde{X})$ such that $g_\lambda(\tilde{x}_0) = \hat{\lambda}(1)$. Define

$$\begin{aligned} \varphi : \pi_1(X, x_0) &\to G(\tilde{X}) \\ [\lambda] &\to g_\lambda. \end{aligned}$$

This function is a group homomorphism, surjective, with kernel equal to $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. We then have an induced isomorphism of groups $\pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \xrightarrow{\sim} G(\tilde{X})$.

We state now a very useful theorem about G-spaces, it is somehow the generalization of proposition 2.30.

THEOREM 2.31. *Let $(\tilde{X}, \tilde{x}_0)$ be a G-space with a covering space action. Then:*

(1) *The quotient map $p : \tilde{X} \to \tilde{X}/G$ is a regular covering space.*
(2) *If $\tilde{X}$ is path connected, then G is isomorphic to the group of deck transformations of p.*
(3) *If $\tilde{X}$ is path connected and locally path connected, then*

$$G \cong \pi_1(\tilde{X}/G, \tilde{x}_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

PROOF. The reader shall find the proof in [**Hat02**, Chapter 1, Proposition 1.40] □

In order to begin the "conceptual" construction of $K(G, 1)$, we need an additional definition.

DEFINITION. Let X be a CW-complex and let G act on X. We say that the action is *cellular* if for each cell e, the image by g, ge is again a cell of X. This action is *free* if our action freely permutes the (open) cells of X. Such a space X is sometimes called a G-complex.

Let G be a group endowed with the discrete topology. We define $E_n(G)$ and $B_n(G)$ to be the sets

$$E_nG = G^{n+1} \qquad B_nG = G^n.$$

Moreover, define for $i, j = 0, \ldots, n$

$$\begin{aligned} d_i : E_nG &\to E_{n-1}G \\ (g_1, \ldots, g_{n+1}) &\mapsto \begin{cases} (g_2, \ldots, g_{n+1}) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) & i = 1, \ldots, n \end{cases} \end{aligned}$$

$$\begin{aligned} s_j : E_nG &\to E_{n+1}G \\ (g_1, \ldots, g_{n+1}) &\mapsto (g_1, \ldots, g_{i-1}, e, g_i, \ldots, g_{n+1}) \end{aligned}$$

and similarly

$$\begin{aligned} d_i : B_nG &\to B_{n-1}G \\ (g_1, \ldots, g_n) &\mapsto \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & i = 1, \ldots, n-1 \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases} \end{aligned}$$

$$\begin{aligned} s_j : B_nG &\to B_{n+1}G \\ (g_1, \ldots, g_n) &\mapsto (g_1, \ldots, g_{i-1}, e, g_i, \ldots, g_n). \end{aligned}$$

We now define $p_n : E_nG \to B_nG$ to be the projection on the first components.

THEOREM 2.32. *Let G be a group with its discrete topology, E_nG, B_nG and p_n as above for $n \geq 0$. Then:*

(1) *$E_*G = (E_nG)_{n\geq 0}$ and $B_*G = (B_nG)_{n\geq 0}$ are simplicial spaces.*
(2) *E_*G is a G-simplicial space whose action by G is free, and $B_*G \cong E_*G/G$ as simplicial space.*
(3) *E_*G is contractible as a simplicial set.*

(4) *The realization of E_*G, denoted by EG, is a CW-complex and inherits a free cellular G-action.*
(5) *The realization of B_*G, denoted by BG, is a $K(G,1)$.*

PROOF. (1) One checks easily the simplicial identities for B_*G and E_*G

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } i < j \\
s_i s_j &= s_{j+1} s_i && \text{if } i \leq j \\
d_i s_j &= s_{j-1} d_i && \text{if } i < j \\
d_j s_j &= \mathrm{Id} = d_{j+1} s_j && \\
d_i s_j &= s_j d_{i-1} && \text{if } i > j+1.
\end{aligned}$$

(2) We define on E_nG a right G-action as follow

$$E_nG \times G \to E_nG : ((g_1, \dots, g_{n+1}), g) \mapsto (g_1, \dots, g_{n+1}g).$$

Clearly, this action respects the faces and the degeneracies maps. This action is without doubt a free action and it is easy to see that $B_*G \cong E_*G/G$ as simplicial spaces.

(3) To show that E_*G is contractible (as simplicial set), we have to define a morphism $h_* : E_*G \times \Delta[1] \to E_*G$ of simplicial sets such that the following diagram commutes for all $n \geq 0$

$$\begin{tikzcd}
E_nG \arrow[d, hook, "\iota_0"'] \arrow[drr, bend left, "c_e"] & & \\
E_nG \times \Delta[1] \arrow[rr, "h_n"] & & E_nG. \\
E_nG \arrow[u, hook', "\iota_1"] \arrow[urr, bend right, "\mathrm{Id}_{E_nG}"'] & &
\end{tikzcd}$$

Define

$$h_n((g_1, \dots, g_{n+1}), (\underbrace{0 \dots 0}_{k \text{ times}} 1 \dots 1)) := (\underbrace{e, \dots, e}_{k \text{ times}}, g_{k+1}, \dots, g_{n+1}).$$

Clearly, the diagram above is commutative and h_n commutes with the faces and degeneracies maps.

(4) We first show that

$$EG = (\coprod_k E_kG \times \Delta^k)/\sim$$

(where the equivalences relations are $(d_i x, t) \sim (x, \delta^i t)$ and $(s_j x, t) \sim (x, \sigma^j t)$) is a CW-complex. Let $EG^{(0)}$ be $G \times \Delta^0$ seen as a set of points to which we glued for all of them a 0-cell. Suppose constructed $EG^{(k-1)}$ for $k \geq 1$. Consider $(G^{k+1} \backslash \cup_j \operatorname{im} s_j) \times \Delta^k$, i.e. to each non degenerate point of $E_kG = G^{k+1}$ we associated a k-cell. Since G is discrete, it also can be seen as

$$\coprod_{x \in G^{k+1} \backslash \cup \operatorname{im} s_j} \Delta^k_x.$$

We define the attachment maps as

$$\begin{aligned}
f : \partial((G^{k+1} \backslash \cup_j \operatorname{im} s_j) \times \Delta^k) &\longrightarrow EG^{(k-1)} \\
(x, \delta^i t) &\longmapsto (d_i x, t).
\end{aligned}$$

In fact, every element of the boundary is of this form so. Consequently, the attachment map above is well defined. Hence, define

$$EG^{(k)} := EG^{(k-1)} \cup_f ((G^{k+1} \setminus \cup_j \operatorname{im} s_j) \times \Delta^k).$$

Moreover one sees that

$$\begin{aligned} EG^{(k)} &= EG^{(k-1)} \cup_f ((G^{k+1} \setminus \cup_j \operatorname{im} s_j) \times \Delta^k) \\ &\cong (\coprod_{0 \le n \le k} E_n G \times \Delta^n)/\sim \end{aligned}$$

because when we associate a cell to a degenerate element $x \in G^n$ we have that this cell collapses into an existing cell of lower dimension via the equivalence relations $(s_j x, t) \sim (x, \sigma^j t)$.

Therefore, $EG = \cup_k EG^{(k)}$ and so is a CW-complex. Look now at the action of G

$$\begin{aligned} EG \times G &\to EG \\ ([x,t], g) &\mapsto [xg, t]. \end{aligned}$$

Our action is well defined since G respects the faces and degeneracies. It is easy to see that the action is cellular. We now show that the action freely permutes (open) cells. Let $[x,t]$ be in EG, since EG is a CW-complex there exists an n such that $[x,t] \in EG^{(n)}$, so we can suppose that x is not degenerated. Since we are looking at how the action acts on open cells, we may suppose $t \notin \partial\Delta^n$. Consequently, if $[x,t] = [xg,t]$, then $g = e$ because we cannot reduce $[x,t]$ via our equivalence relation and the relation $[x,t] = [xg,t]$ becomes $(x,t) = (xg,t)$.

(5) Remark that the realization of $p_* : E_*G \to B_*G$ is exactly the quotient map $p : EG \to EG/G$ and since the action on EG is cellular, we have that it is a covering space action. By theorem 2.31, $p : EG \to BG$ is a normal covering space and $\pi_1(BG) \cong G$ since any CW-complex is locally path connected. Draw the long exact sequence of a fibration, to see that $\pi_n(BG) = 0$ for $n > 1$. Indeed, EG is contractible since E_*G is contractible as a simplicial set, so $\pi_n(EG) = 0$. Since the fiber of our covering space is G, we then have

$$\pi_{q+1}(BG, x_0) = \pi_q(G, e) \tag{$*$}$$

for all $q \ge 1$. Since G is discrete we have $\pi_q(G,e) = 0$ for all $q \ge 1$. Therefore BG is a $K(G,1)$. $\square$

Remark. Let G, H be two topological groups, then we have an obvious isomorphism

$$(G \times H)^n \cong G^n \times H^n.$$

This induces isomorphisms of simplicial spaces

$$E_*(G \times H) \cong E_*G \times E_*H \qquad B_*(G \times H) \cong B_*G \times B_*H. \tag{$**$}$$

We also know that the geometric realization commutes with products. So if we suppose now that G is a commutative topological group which has the homotopy type of a CW-complex, then we can define a structure of commutative topological group on EG and BG which has a homotopy type of a CW-complex. It is not so easy to see that EG and BG are CW-complex because we cannot give a trivial cellular decomposition as in the point (4) above. Moreover, the multiplication on BG is defined by the following

$$BG \times BG \cong B(G \times G) \to BG.$$

We remark that associativity of this multiplication is deduced by the fact that the isomorphisms in $(**)$ commutes with projections.

To obtain a $K(G,n)$ in a similar conceptual way (for a commutative discrete group G), we have to iterate our previous construction. Set $B^0G = G$ and $B^nG := BB^{n-1}G$, therefore by $(*)$

$$\pi_q(B^nG, x_0) \cong \pi_{q-1}(B^{n-1}G, x_0) \cong \ldots \cong \pi_{q-(n-1)}(BG, x_0) = \begin{cases} G & q = n \\ 0 & q \neq n. \end{cases}$$

This construction is inspired by the one given in [**May99**, Chapter 16, Section 5] and [**Dwy09**].

We gave until now two different constructions of an Eilenberg-MacLane space $K(G,n)$ and it is legitimate to ask ourself if there is a link between them. That is exactly the goal of the following theorem.

Theorem 2.33. *Let $n \in \mathbb{N}$, $n > 0$ and G be a group, commutative if $n > 1$. Then all Eilenberg-MacLane spaces $K(G,n)$ are homotopic.*

Proof. First suppose $n \geq 2$ and G commutative. Let $[X, K(G,n)]$ denote the set of homotopy classes of maps from X to $K(G,n)$. Recall that we have a natural isomorphism of groups given by the Hurewicz theorem

$$G \cong \pi_n(K(G,n), x_0) \xrightarrow{h_n} H_n(K(G,n)).$$

Let i_n denote its inverse map. Remark that we have an isomorphism of abelian groups between $H^n(K(G,n);G)$ and $\mathrm{Hom}_{\mathbb{Z}}(H_n(K(G,n)),G)$ given by the universal coefficient theorem. Indeed, the direct summand $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(K(G,n)),G) = 0$ because $K(G,n)$ is $(n-1)$-connected (the Hurewicz theorem implies $H_{n-1}(K(G,n)) = 0$). Therefore $i_n : H_n(K(G,n)) \to G$ can be seen in $H^n(K(G,n);G)$, let $\hat{i}_n$ denote the corresponding element. We can now define a map

$$\begin{aligned} [X, K(G,n)] &\longrightarrow H^n(X;G) \\ [X \xrightarrow{f} K(G,n)] &\longmapsto f^*(\hat{i}_n) \end{aligned}$$

where $f^* : H^n(K(G,n);G) \to H^n(X;G)$ is the induced map in cohomology with coefficient in G by f. If $g \simeq f$ then the induced homomorphisms are equal. Therefore, the map above is well defined.

Suppose now $n = 1$ and G not necessarily commutative. Define the map

$$\begin{aligned} [X, K(G,1)] &\longrightarrow \mathrm{Hom}_{\mathcal{G}rp}(\pi_1(X,x_0), G) \\ [X \xrightarrow{f} K(G,1)] &\longmapsto f_* \end{aligned}$$

where $f_* : \pi_1(X,x_0) \to \pi_1(K(G,1),x_0)$ is the induced map. This map is clearly well defined.

We assume those two maps are bijection of sets. I have to say here that this statement is not a trivial result and is proved in [**MT68**, Chapter 1, Theorem 1] using obstruction theory. One can find another proof of the claim in the case where X is a CW-complex (which is sufficient in our case) in [**Bre93**, Chapter 7, Section 12].

Suppose first $n > 1$, if we replace X by a $K(G',n)$ Eilenberg-MacLane space, via the universal coefficient theorem we have an isomorphism $H^n(K(G',n);G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(K(G',n)),G)$ (the direct summand $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(K(G',n)),G) = 0$ by the same reason as before). Moreover, the Hurewicz theorem states $H_n(K(G',n)) \cong G'$. We then have a one-to-one correspondence

$$[K(G',n), K(G,n)] \overset{\sim}{\leftrightarrow} \mathrm{Hom}_{\mathbb{Z}}(G',G)$$

where G, G' are seen as $\mathbb{Z}$-modules. If $G' = G$ let $K'(G,n)$ denote $K(G',n)$. Then we can find continuous maps $f : K'(G,n) \to K(G,n)$ and $g : K(G,n) \to K'(G,n)$ whose homotopy classes corresponds to $\mathrm{Id}_G \in \mathrm{Hom}_{\mathbb{Z}}(G,G)$. Therefore $f \circ g \simeq \mathrm{Id}_{K(G,n)}$ and $g \circ f \simeq \mathrm{Id}_{K'(G,n)}$ since Id_G corresponds also to $\mathrm{Id}_{K(G,n)}$ and $\mathrm{Id}_{K'(G,n)}$. Indeed, lets draw the sequence of bijections

$$\begin{array}{ccccccc} [K(G,n),K(G,n)] & \longrightarrow & H^n(K(G,n);G) & \xrightarrow{\beta} & \mathrm{Hom}(H_n(K(G,n)),G) & \xrightarrow{h_n^*} & \mathrm{Hom}_{\mathbb{Z}}(G,G) \\ \mathrm{Id}_{K(G,n)} & \longmapsto & \mathrm{Id}^*_{K(G,n)}(\hat{i}_n) = \hat{i}_n & \longmapsto & i_n & \longmapsto & h_n^*(i_n) = i_n \circ h_n = \mathrm{Id}_G \end{array}$$

Where β is the isomorphism given by the universal coefficient theorem and h_n^* is the pullback of the Hurewicz map. If $n = 1$, one can argue in a similar way to obtain the result. □

We have seen that EG has an action of G which is free and cellular. One may ask if the Eilenberg-MacLane $K(G,1)$ constructed in theorem 2.25 has also a cover with an action of G which is free and cellular. To understand totally what happens, we need a characterization of universal covers of CW-complexes.

Theorem 2.34. *Let X be a connected CW-complex and $p : \tilde{X} \to X$ a covering space. Then $\tilde{X}$ can be equipped with a structure of CW-complex such that p is a cellular map and the dimension of $\tilde{X}$ is equal to the dimension of X.*

Proof. Let be $x \in X$. Consider the fiber $p^{-1}(x) = \{\tilde{x}_i : i \in I, p(\tilde{x}_i) = x\}$ for some index set I. Now, take $e \in E$ a cell and let $x_e = f_e(0) \in e$, where $f_e : D^n \to X$ is the characteristic map of e. Consider the diagram

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_i) \arrow[d, "p"] \\ (D^n, 0) \arrow[ur, dotted, "\tilde{f}_{ei}"] \arrow[r, "f_e"] & (X, x_e). \end{tikzcd}$$

where $\tilde{f}_e$ is given by the lifting criterion. Indeed,

$$f_{e*}(\pi_1(D^n, 0)) = 0 = p_*(\pi_1(\tilde{X}, \tilde{x}_i)).$$

Then let $\tilde{e}_i$ denote $\tilde{f}_{ei}$. The reader shall find in [**Rot88**, Chapter 10, Theorem 10.43] the detailed proof that $\tilde{X}$ is a CW-complex with cells $\tilde{E} = \{\tilde{e}_i : e \in E, i \in I\}$ and characteristic maps $(\tilde{f}_{ei})_{e \in E, i \in I}$. □

Let now X be an Eilenberg-MacLane space $K(G,1)$ and consider $\tilde{X}$ its universal cover. Since X is CW-complex, $\tilde{X}$ is also a CW-complex. One can remark that the fiber of any point $x \in X$ is in bijection with G, i.e. $p^{-1}(x) \overset{\sim}{\leftrightarrow} G$. Then the index set I in the proof above is exactly G. By proposition 2.30, we have that $\pi_1(X, x_0) \cong G$ is isomorphic to the group of deck transformations of $\tilde{X}$, $G(\tilde{X})$. And it is a matter of fact that this group acts freely and cellularly on $\tilde{X}$.

5. Homology and cohomology of groups via topology

In this section we give the main result linking (co)homology of groups and the Eilenberg-MacLane spaces $K(G,1)$.

Theorem 2.35. *Let G be a group, M a trivial G-module, and $K(G,1)$ an Eilenberg-MacLane space for G, then*

$$H_n(G,M) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, M) \cong H_n(K(G,1); M),$$
$$H^n(G,M) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}, M) \cong H^n(K(G,1); M).$$

PROOF. Take BG as model for our Eilenberg-MacLane space. Then one can see that, $\tilde{W}_*(EG)$, the augmented cellular chain complex of EG is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Indeed, EG is a contractible CW-complex equipped with a free cellular G-action, therefore $W_n(EG)$ is a free $\mathbb{Z}G$-module. Take now the deleted complex, i.e. $W_*(EG)$ and apply the functor $(-) \otimes_G M$. Since $\mathbb{Z}G \otimes_G M \cong M$, one sees that

$$W_*(EG) \otimes_G M = (\bigoplus_\alpha \mathbb{Z}Ge^n_\alpha) \otimes_G M \cong \bigoplus_\alpha Me^n_\alpha = W_*(BG) \otimes_\mathbb{Z} M.$$

Hence

$$\begin{aligned} H_n(G, M) &= Tor^{\mathbb{Z}G}_n(\mathbb{Z}, M) \\ &\cong H_n(W_*(EG) \otimes_G M) \\ &\cong H_n(W_*(BG) \otimes_\mathbb{Z} M) \\ &\cong H_n(BG; M). \end{aligned}$$

For cohomology, apply the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$. Since $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \cong M$, we have

$$\mathrm{Hom}_{\mathbb{Z}G}(W_*(EG), M) = \mathrm{Hom}_{\mathbb{Z}G}(\bigoplus_\alpha \mathbb{Z}Ge^n_\alpha, M) \cong \mathrm{Hom}_\mathbb{Z}(W_*(BG), M).$$

Hence

$$\begin{aligned} H^n(G, M) &= Ext^n_{\mathbb{Z}G}(\mathbb{Z}, M) \\ &\cong H^n(\mathrm{Hom}_{\mathbb{Z}G}(W_*(EG), M)) \\ &\cong H^n(\mathrm{Hom}_\mathbb{Z}(W_*(BG), M)) \\ &\cong H^n(BG; M). \end{aligned}$$

To finish the proof, use theorem 2.33. □

REMARK. We can now use this theorem to transpose results from algebraic topology to (co)homology of groups. Indeed, a topological proof of the theorem 1.25 in chapter 1 is given by the Hurewicz theorem in algebraic topology and the theorem 2.35 above.

As a second example, we will prove the Universal Coefficient Theorem for homology of groups.

THEOREM 2.36 (Universal Coefficient Theorem)**.** *Let G be a group and M a trivial G-module. Then for $n \geq 0$ there exist split exact sequences*

$$0 \to \mathrm{Ext}^1_\mathbb{Z}(H_{n-1}(G, \mathbb{Z}), M) \to H^n(G, M) \to \mathrm{Hom}_\mathbb{Z}(H_n(G, \mathbb{Z}), M) \to 0,$$

and

$$0 \to H_n(G, \mathbb{Z}) \otimes_\mathbb{Z} M \to H_n(G, M) \to \mathrm{Tor}^\mathbb{Z}_1(H_{n-1}(G, \mathbb{Z}), M) \to 0.$$

PROOF. Let X be a $K(G, 1)$ Eilenberg-MacLane space. By the Universal Coefficient Theorem for Homology and Cohomology, we have the exact sequences

$$0 \to \mathrm{Ext}^1_\mathbb{Z}(H_{n-1}(X), M) \to H^n(X; M) \to \mathrm{Hom}_\mathbb{Z}(H_n(X), M) \to 0,$$

and

$$0 \to H_n(X) \otimes_\mathbb{Z} M \to H_n(X; M) \to \mathrm{Tor}^\mathbb{Z}_1(H_{n-1}(X), M) \to 0.$$

for all $n \geq 0$ (we look here M as an abelian group)

By theorem 2.35 we have $H_n(G, M) = H_n(X; M)$ and $H^n(G, M) = H^n(X; M)$. Replace them in the exact sequences above to obtain the result. □

EXAMPLE. Let $\mathbb{Z}_2$ be the cyclic group of order 2, we will compute its homology using theorem 2.35. First consider the space $S^\infty := \bigcup_{n\geq 0} S^n$ with its $\mathbb{Z}_2$-action

$$\begin{aligned}\mathbb{Z}_2 \times S^\infty &\to S^\infty\\ (k,x) &\mapsto (-1)^k x.\end{aligned}$$

One can see that this action is a nice covering space action and since S^∞ is contractible, we have that $S^\infty/\mathbb{Z}_2$ is a $K(\mathbb{Z}_2, 1)$ space. The space $S^\infty/\mathbb{Z}_2$ is commonly called the *real infinite projective space* and is denoted $\mathbb{R}P^\infty$. By theorem 2.35 we have that $H_n(\mathbb{Z}_2, M) \cong H_n(\mathbb{R}P^\infty; M)$ for any trivial $\mathbb{Z}_2$-module M.

We know that $\mathbb{R}P^\infty$ has a CW-structure with one n-cell for each $n \geq 0$. We obtain it as a quotient from S^∞, seen as a CW-complex with *two* n-cells (upper and lower hemispheres) for each $n \geq 0$, where we glue the points of the upper hemisphere to those of the lower hemisphere via the antipodal map $x \mapsto -x$.

Let e^n_+ and e^n_- denote the upper and lower hemispheres of $S^n \subset S^\infty$. We then have the relations

$$S^n = e^n_+ \cup e^n_-, \qquad S^{n-1} = e^n_+ \cap e^n_-.$$

Define $\pi_\pm : S^n \to S^n/e^n_\mp \cong e^n_\pm/S^{n-1}$ the quotient map which identifies the lower or upper hemisphere to a point, and write a_n for the antipodal map from $S^n \to S^n$. Moreover, we have a homeomorphisms $h : D^n/S^{n-1} \to S^n$ and $p_\pm : e^n_\pm/S^{n-1} \to D^n/S^{n-1}$ where $p_\pm$ are induced by the projections of $e^n_\pm$ on D^n (seen to as the equatorial plan). We then have a commutative diagram

$$\begin{tikzcd}
S^n \arrow[r, "\pi_+"] \arrow[d, "a_n"'] & e^n_+/S^{n-1} \arrow[r, "p_+"] \arrow[d, "a_n"] & D^n/S^{n-1} \arrow[r, "h"] \arrow[d, equal] & S^n \arrow[d, equal] \\
S^n \arrow[r, "\pi_-"] & e^n_-/S^{n-1} \arrow[r, "p_-"] & D^n/S^{n-1} \arrow[r, "h"] & S^n.
\end{tikzcd}$$

By theorem 2.23, we have that for $n \geq 1$

$$\begin{aligned}d_{n+1}([e^{n+1}_+]) &= \underbrace{\deg(h \circ p_+ \circ \pi_+)}_{=1}[e^n_+] + \underbrace{\deg(h \circ p_- \circ \pi_- \circ a_n)}_{=\deg(a_n)=(-1)^{n+1}}[e^n_-]\\ &= [e^n_+] + (-1)^{n+1}[e^n_-].\end{aligned}$$

We used the fact that the degree of the antipodal map $S^n \to S^n$ is $(-1)^{n+1}$. Indeed, it is composed of $n+1$ reflections each of degree -1.

By construction of $\mathbb{R}P^\infty$, we have that $e^n = e^n_+ = e^n_-$. This implies for $n > 1$

$$d_{n+1}([e^{n+1}]) = (1 + (-1)^{n+1})[e^n].$$

Hence, $d_n = 0$ if n is even and $d_n = 2$ (the multiplication by 2) if n is odd. Therefore, the cellular chain complex of $\mathbb{R}P^\infty$ is

$$\cdots \to \mathbb{Z}[e^{2k+1}] \xrightarrow{0} \mathbb{Z}[e^{2k}] \xrightarrow{2} \mathbb{Z}[e^{2k-1}] \xrightarrow{0} \cdots \to \mathbb{Z}[e^2] \xrightarrow{2} \mathbb{Z}[e^1] \xrightarrow{0} \mathbb{Z}[e^0] \to 0.$$

Take homology to have

$$H_n(\mathbb{Z}_2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0\\ \mathbb{Z}_2 & n \text{ odd}\\ 0 & n \text{ even.}\end{cases}$$

We have seen that the cohomology of a CW-complex can be computed by taking the cohomology of the dual of the cellular chain complex. Hence we have

$$H^n(\mathbb{Z}_2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0\\ 0 & n \text{ odd}\\ \mathbb{Z}_2 & n \text{ even.}\end{cases}$$

In fact, we can compute the (co)homology of any cyclic group via *infinite lens spaces*; those spaces are $K(\mathbb{Z}_m, 1)$ (cf. [**Hat02**, Chapter 2, Example 2.43]).

6. Cohomological dimension

In the preceding section we have seen that Eilenberg-MacLane spaces might be useful to compute (co)homology of groups. In particular, when the group is "very big" it gives us sometimes (when it has no torsion) "relatively small" projective resolutions. In this section, our goal would be to clarify this notion of groups which has "relatively small" projective resolutions and what happens with their Eilenberg-MacLane spaces.

At the very end of this section, we will proof the theorem of Eilenberg-Ganea-Wall and state the Eilenberg-Ganea conjecture. We will also mention some recent work around this conjecture.

Definition. Let R be a commutative ring and M any R-module. We say that $\operatorname{proj\,dim}_R M \leq n$ if M admits a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

of length n.

Lemma 2.37. *Let R be a commutative ring and M any R-module. Then the following conditions are equivalent:*

(1) $\operatorname{proj\,dim}_R M \leq n$,
(2) $\operatorname{Ext}^i_R(M, -) = 0$ *for all* $i > n$,
(3) $\operatorname{Ext}^{n+1}_R(M, -)$,
(4) *let* $0 \to K \to P_{n-1} \to \cdots P_0 \to M \to 0$ *be an exact sequence of R-modules such that each P_i is projective, then K is also a projective module.*

Proof. It is very easy to see that (4) implies (1) implies (2) implies (3). We show that (3) implies (4). Let

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to R \to 0$$

be an exact sequence as in the hypothesis. Complete this sequence to an R-projective resolution

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dr, "q"] & & P_n \arrow[rr, "d_n"] \arrow[dr, "p"] & & P_{n-1} \arrow[r] & \cdots \\ & & L \arrow[ur, hook, "i"] & & K \arrow[ur, hook, "j"] & & \end{tikzcd}$$

with p, q the cokernel of d_{n+2} and d_{n+1} respectively.

Apply now the functor $\operatorname{Hom}(-, N)$ to our resolution and remark that $\ker d^*_{n+2} = \operatorname{im} d^*_{n+1}$ by hypothesis. Therefore, if we have $\varphi : P_{n+1} \to N$ such that $\varphi \circ d_{n+2} = 0$ (i.e. $\varphi \in \ker d^*_{n+2}$) then there exists $\psi : P_n \to N$ such that $\varphi = \psi \circ d_{n+1}$. Apply the argument for $N = L$ and $\varphi = q$, we then have that the exact sequence

$$0 \to L \overset{i}{\hookrightarrow} P_n \overset{q}{\to} K \to 0$$

splits. Let $s : P_n \to L$ denote the induced morphism, we have to show that $s \circ i = \mathrm{Id}_L$. Indeed,

$$s \circ i \circ q = s \circ d_{n+1} = q = \mathrm{Id}_L \circ q,$$

since q is an epimorphism, $s \circ i = \mathrm{Id}_L$. Therefore $P_n \cong L \oplus K$ and that shows us K is projective because P_n is projective. $\square$

Consider now a group Γ and set $R = \mathbb{Z}\Gamma$, $M = \mathbb{Z}$ seen as a trivial $\mathbb{Z}\Gamma$-module.

DEFINITION. The *cohomological dimension* of Γ is the integer

$$\operatorname{cd}\Gamma := \min\{n : \operatorname{proj}\dim_{\mathbb{Z}\Gamma}\mathbb{Z} \leq n\}.$$

If $\operatorname{proj}\dim_{\mathbb{Z}\Gamma}\mathbb{Z} > n$ for all n, then we say that $\operatorname{cd}\Gamma = \infty$.

By the lemma above, we have also

$$\begin{aligned}\operatorname{cd}\Gamma &= \inf\{n : \mathbb{Z} \text{ admits a projective resolution of length } n\}\\ &= \inf\{n : H^i(\Gamma, -) = 0, \text{ for all } i > n\}\\ &= \sup\{n : H^n(\Gamma, M) \neq 0 \text{ for some } \Gamma\text{-module } M\}.\end{aligned}$$

PROPOSITION 2.38. *If* $\operatorname{cd}\Gamma < \infty$*, then*

$$\operatorname{cd}\Gamma = \sup\{n : H^n(\Gamma, F) \neq 0 \textit{ for some free } \Gamma\textit{-module } F\}.$$

PROOF. Let $n = \operatorname{cd}\Gamma$. The very definition of the cohomological dimension says that $H^n(\Gamma, M) \neq 0$ for some Γ-module M. Therefore, we can find a free module F which maps onto M. Consider then the long exact sequence in cohomology of the exact sequence

$$0 \to \ker f \to F \xrightarrow{f} M \to 0$$

to show that $H^n(\Gamma, F) \neq 0$. □

PROPOSITION 2.39. $\operatorname{cd}\Gamma = 0$ *if and only if* Γ *is the trivial group.*

PROOF. Suppose first that $\Gamma = \{1\}$. Then $\mathbb{Z}\Gamma \cong \mathbb{Z}$. Therefore, we have a projective $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$

$$\cdots \to 0 \to \mathbb{Z}\Gamma \xrightarrow{\sim} \mathbb{Z} \to 0.$$

We then have $H^1(\Gamma, -) = 0$, and that implies $\operatorname{cd}\Gamma = 0$.

Conversely, suppose $\operatorname{cd}\Gamma = 0$ and consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \to 0.$$

By the very definition of the cohomological dimension we have that $\mathbb{Z}$ is a projective Γ-module. Consider now the augmentation morphism

$$\mathbb{Z}\Gamma \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$$

and extend it onto an exact sequence

$$0 \to K \hookrightarrow \mathbb{Z}\Gamma \twoheadrightarrow \mathbb{Z} \to 0$$

where K is the kernel of ε. Since $\mathbb{Z}$ is projective, the sequence splits. Therefore, there exists a retract $s : \mathbb{Z} \to \mathbb{Z}\Gamma$ such that $\varepsilon \circ s = \mathrm{Id}_{\mathbb{Z}}$. Let $\sum_{\text{finite}} m_x x = s(1)$, since it is a morphism of Γ-modules,

$$\sum m_x gx = g\sum m_x x = s(g1) = s(1) = \sum m_x x.$$

since $\mathbb{Z}\Gamma$ is a free module, we have that $m_{g^{-1}x} = m_x$ for all $g \in \Gamma$. Hence $m_x = m$ for all $x \in \Gamma$. Therefore,

$$1 = \varepsilon(s(1)) = \varepsilon(m\sum_{\text{finite}} x) = mn$$

and since $n \in \mathbb{N}$, that implies $m = n = 1$, so $s(1) = x$ for some $x \in \Gamma$. Since it is a morphism of Γ-modules, we have $gx = x$ for all $g \in \Gamma$. That shows Γ is trivial. □

As we have seen in the last section, any Eilenberg-MacLane space $K(\Gamma, 1)$, gives us a projective $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$ (induced by the free action on the cells). So we can consider a similar notion of dimension.

Definition. The *geometric dimension*, denoted $\operatorname{geom}\dim\Gamma$, is defined to be the minimal dimension of an Eilenberg-MacLane space $K(\Gamma, 1)$ (for its structure of CW-complex).

It is totally clear that for any group Γ, we have

$$\operatorname{cd}\Gamma \leq \operatorname{geom}\dim\Gamma.$$

We are now interested to know when the equality holds, i.e. $\operatorname{cd}\Gamma = \operatorname{geom}\dim\Gamma$.

Corollary 2.40. *If* $\operatorname{cd}\Gamma = 0$*, then* $\operatorname{geom}\dim\Gamma = 0$.

Proof. It is clear by the previous proposition and if we take $K(\Gamma, 1)$ to be a point. □

Theorem 2.41 (Stallings - Swan)**.** $\operatorname{cd}\Gamma = 1$ *if and only if* Γ *is free and non-trivial.*

Proof. Suppose Γ is free and non-trivial, then there exists a set of generators $S \subset \Gamma$. Then consider the free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$

$$0 \to \mathbb{Z}\Gamma[S] \xrightarrow{d_1} \mathbb{Z}\Gamma \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where ε is the augmentation map, $\mathbb{Z}\Gamma[S]$ is the free $\mathbb{Z}\Gamma$-module generated by S and $d_1(s) := s - 1$ for all $s \in S$. Clearly $H^2(\Gamma, -) = 0$. That implies $\operatorname{cd}\Gamma \leq 1$, but $\operatorname{cd}\Gamma \neq 0$ since Γ is non-trivial. Hence $\operatorname{cd}\Gamma = 1$.

The converse is proved for finitely generated groups by Stalling in [**Sta68**] and by Swan in [**Swa69**] for the general case. □

Corollary 2.42. *If* $\operatorname{cd}\Gamma = 1$*, then* $\operatorname{geom}\dim\Gamma = 1$.

Proof. By the previous theorem, we know that Γ is free and non-trivial. Construct the wedge of circles composed with one circle for each generator of Γ. That gives us an Eilenberg-MacLane $K(\Gamma, 1)$ of dimension 1. □

We give now some group theoretic results about the cohomological dimension. We will come back later to our problem.

Proposition 2.43. *If* $\Gamma' \subset \Gamma$*, then* $\operatorname{cd}\Gamma' \leq \operatorname{cd}\Gamma$*. Moreover, if* $\operatorname{cd}\Gamma < \infty$ *and* $|\Gamma : \Gamma'| < \infty$*, then* $\operatorname{cd}\Gamma' = \operatorname{cd}\Gamma$.

Proof. The first inequality is clear since a $\mathbb{Z}\Gamma$-projective resolution can be seen as a $\mathbb{Z}\Gamma'$-projective resolution. For the second part, the reader shall find the proof in [**Bro82**, Chapter 8, Proposition 2.4 (a)]. □

Corollary 2.44. *If* $\operatorname{cd}\Gamma < \infty$*, then* Γ *is torsion free.*

Proof. Suppose Γ contains a nontrivial finite cyclic subgroup Γ', then $\operatorname{cd}\Gamma' = \infty$ (cf. proposition 1.28) and by the preceding proposition that implies $\operatorname{cd}\Gamma = \infty$. □

Remark. This corollary justifies the change of notion in this section for our group. We had in mind that the group Γ must be infinite (if $1 \leq \operatorname{cd}\Gamma < \infty$) and is torsion free.

Proposition 2.45. *For any group* Γ*, there exists a free* $\mathbb{Z}\Gamma$*-resolution of* $\mathbb{Z}$ *of length equal to* $\operatorname{cd}\Gamma$.

To proof this proposition we need before a lemma.

Lemma 2.46 (Eilenberg trick)**.** *Let* P *be a projective module over a ring* R*, then there exists a free module* F *such that* $P \oplus F \cong F$.

PROOF. Since P is projective there exists a R-module Q such that $P \oplus Q$ is free. Define the module

$$F := \bigoplus_{n\in\mathbb{N}} P \oplus Q.$$

F is free and if we add a copy of P it does not change anything because we can write F as $(\bigoplus_n P) \oplus (\bigoplus_n Q)$, hence $F \oplus P \cong F$. ☐

PROOF OF THE PROPOSITION. Take a partial free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$

$$F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \to \cdots \to F_0 \to \mathbb{Z} \to 0.$$

Let $P = \ker d_{n-1}$. Since $\operatorname{cd}\Gamma = n$, P is projective and by the Eilenberg trick there exists F such that $P \oplus F \cong F$. Replace F_{n-1} by $F_{n-1} \oplus F$ and set $d_{n-1}|_F = 0$ we then have a free $\mathbb{Z}\Gamma$-resolution. If we complete it with the kernel of our new d_{n-1}. This kernel is free since $P \oplus F$ is free. ☐

We might now be interested in another type of finiteness, where each P_i in our projective resolution are finitely generated. That leads us to the following considerations.

PROPOSITION 2.47. *Let R be a ring and M a R-module. Then the following conditions are equivalent:*

(1) *there exists an exact sequence $R^m \to R^n \to M \to 0$ for some $m, n \in \mathbb{N}$,*
(2) *there exists an exact sequence $P_1 \to P_0 \to M \to 0$ where P_1, P_0 are finitely generated projective R-modules,*
(3) *M is finitely generated and given any exact sequence $0 \to K \to P \to M \to 0$ with P finitely generated and projective, implies that K is also finitely generated.*

If the conditions holds, we then say that M is finitely presented *and an exact sequence as in (1) is said a* finite presentation *of M with n generators and m relations.*

To show the proposition, we need a lemma.

LEMMA 2.48 (Schanuel)**.** *Let*

$$0 \to K \xrightarrow{i} P \xrightarrow{q} M \to 0$$

and

$$0 \to K' \xrightarrow{i'} P' \xrightarrow{q'} M \to 0$$

be two exact sequences with P and P' projective R-modules. Then $P \oplus K' \cong P' \oplus K$.

PROOF. Let Q be the pullback of the diagram

$$\begin{tikzcd} & P' \arrow[d, "q'"] \\ P \arrow[r, "q"] & M. \end{tikzcd}$$

We then have an induced map $j' : K' \to Q$ given by the commutative solid diagram

$$\begin{tikzcd} K' \arrow[dr, dotted, "\exists! j'"] \arrow[drr, bend left, "i'"] \arrow[ddr, bend right, "0"'] & & \\ & Q \arrow[r] \arrow[d] & P' \arrow[d, "q'"] \\ & P \arrow[r, "q"] & M. \end{tikzcd}$$

Same for $j : K \to Q$. Therefore, we have a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K' \arrow[r, equal] \arrow[d, "j'"] & K' \arrow[d, "i'"] & \\
0 \arrow[r] & K \arrow[r, "j"] \arrow[d, equal] & Q \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d, "q'"] & 0 \\
0 \arrow[r] & K \arrow[r, "i"] & P \arrow[r, "q"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns, and since P and P' are projective modules, the sequences in our diagram containing Q splits. Hence $P \oplus K' \cong Q \cong P' \oplus K$. $\square$

Proof of the proposition. We have clearly that (3) implies (1) implies (2). We show that (2) implies (3). Suppose (2), then looking at the sequence $P_1 \to P_0 \to M \to 0$ gives us clearly that M is finitely generated. Now, consider the exact sequences

$$0 \to K' \to P_0 \to M \to 0$$

and

$$0 \to K \to P \to M \to 0$$

where K' is the kernel of $P_0 \to M$, and apply Schanuel's lemma to obtain $P \oplus K' \cong P_0 \oplus K$. Since the lefthand side is finitely generated, it implies that $P_0 \oplus K$ is also finitely generated, and so is K. $\square$

The Schanuel's lemma can be generalize to the following lemma.

Lemma 2.49. *Let*

$$0 \to P_n \to \cdots \to P_0 \xrightarrow{q} M \to 0$$

and

$$0 \to P'_n \to \cdots \to P'_0 \xrightarrow{q} M \to 0$$

be two exact sequences such that P_i and P'_j are projective R-modules for all $i, j = 1, \ldots, n-1$. Then

$$P_n \oplus P'_{n-1} \oplus P_{n-2} \oplus \cdots \cong P'_n \oplus P_{n-1} \oplus P'_{n-2} \oplus \cdots$$

We can the deduce as before the following proposition.

Proposition 2.50. *Let M be an R-module and $n \geq 0$ an integer. Then the following conditions are equivalent:*

(1) *there exists a partial resolution $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ with each F_i free and finitely generated,*
(2) *there exists a partial resolution $P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ with each P_i projective and finitely generated,*
(3) *M is finitely generated and given any exact sequence $0 \to K \to P_k \to \cdots \to P_0 \to M \to 0$ with $k < n$ and each P_i finitely generated and projective, implies that K is also finitely generated.*

If the condition holds, then we say that M is of type FP_n. *An exact sequence as in (2) is said a to be of* finite type.

Proof. The reader shall find the proof in [**Bro82**, Chapter 8, Proposition 4.3]. □

Remark. With this definition, a module is finitely presented if and only if it is of type FP_1. The condition FP_0 says just that the module is finitely generated.

Proposition 2.51. *Let M be an R-module. Then the following conditions are equivalent:*

(1) *M admits a free resolution of finite type,*
(2) *M admits a projective resolution of finite type,*
(3) *M is of type FP_n for all $n \geq 0$.*

If the condition holds, we say that M is of type FP_∞.

Proof. It is clear that (1) implies (2) implies (3). Suppose we have (3) and suppose in addition that we constructed a partial free resolution of finite type

$$F_n \xrightarrow{d_n} \cdots \to F_1 \to F_0 \to M \to 0.$$

Extend the exact sequence with the kernel of d_n, i.e.

$$0 \to K \to F_n \xrightarrow{d_n} \cdots \to F_1 \to F_0 \to M \to 0.$$

Since M is of type FP_{n+1}, K is finitely generated, that means that K is of type FP_0. Therefore, there exists a free finitely generated module F_{n+1} such that $F_{n+1} \xrightarrow{p} K \to 0$. Then one can build the exact sequence (of finite type)

$$\begin{tikzcd} F_{n+1} \arrow[r] \arrow[dr, "p"'] & F_n \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0. \\ & K \arrow[u, hook, "i"'] & & & & \end{tikzcd}$$

By induction, we obtain a free resolution of M of finite type. □

Definition. Let Γ be a group. We say now that Γ is of *type FP_n* (resp of *type FP_∞*) if $\mathbb{Z}$ is of type FP_n (resp. FP_∞) as a Γ-module.

We say that Γ is of *type FP* (resp. of *type FL*) if $\mathbb{Z}$ admits a projective (resp. free) $\mathbb{Z}\Gamma$-resolution of finite type and of finite length.

Proposition 2.52. *Γ is of type FP if and only if* $\operatorname{cd}\Gamma < \infty$ *and Γ is of type FP_∞.*

Proof. First suppose Γ is of type FP, then it admits a projective $\mathbb{Z}\Gamma$-resolution of finite type and of finite length. Hence $\operatorname{cd}\Gamma < \infty$ and clearly Γ is of type FP_∞.

Conversely, let $n = \operatorname{cd}\Gamma$. Take a partial projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$

$$P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z} \to 0,$$

and extend it with the kernel into an exact sequence of the form

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0.$$

By the very definition of the cohomological dimension, we have that K is projective and since Γ is of type FP_∞, K is projective. That implies Γ is of type FP. □

We see here that a group which is of type FP is not necessarily of type FL. To understand better what is a group of type FL, we need to introduce before some tools.

Definition. Let P be a finitely generated projective R-module. We say that P is *stably free* if there exists a free R-module F of finite rank such that $P \oplus F$ is free.

Suppose Γ to be of type FP and let $n = \operatorname{cd}\Gamma$. Take a partial resolution of $\mathbb{Z}$, $F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ where each F_i is free and finitely generated. Extend it to the resolution

$$0 \to P \xrightarrow{i} F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0.$$

Since the cohomological dimension is n we have that P is projective and since Γ is of type FP_∞, P is in addition finitely generated.

Proposition 2.53. *With the above notation, Γ is of type FL if and only if P is stably free.*

Proof. If P is stably free, there exists a free module F of finite rank such that $P \oplus F$ is free. Since P is finitely generated, $P \oplus F$ is also finitely generated. Set the differential $d_n = i \oplus 0 : P \oplus F \to F_{n-1}$ to obtain a free resolution of finite type and finite length.

Conversely, let

$$0 \to F'_n \to F'_{n-1} \to \cdots \to F'_0 \to \mathbb{Z} \to 0$$

be a free $\mathbb{Z}\Gamma$-resolution of finite type (given by the fact that Γ is of type FL and $\operatorname{cd}\Gamma = n$). Then

$$P \oplus F'_{n-1} \oplus F_{n-2} \oplus \cdots \cong F'_n \oplus F_{n-1} \oplus F'_{n-2} \oplus \cdots$$

by the generalization of the Schanuel's lemma. □

Proposition 2.54. *For a group Γ, if there exists an Eilenberg-MacLane space $K(\Gamma, 1)$ which is a finite complex, then Γ is of type FL.*

Proof. The proof is obvious. □

We now return to our first problem; when does the cohomological dimension and the geometric dimension coincide?

Theorem 2.55 (Eilenberg-Ganea-Wall)**.** *Let Γ be an arbitrary group, and $n = \max\{\operatorname{cd}\Gamma, 3\}$. Then there exists an Eilenberg-MacLane space $K(\Gamma, 1)$ of dimension n. Moreover, if Γ is finitely presented and of type FL then $K(\Gamma, 1)$ can be taken finite.*

To obtain such Eilenberg-MacLane space $K(\Gamma, 1)$ we will need a slightly different construction of the one given in theorem 2.25. We will have to use the universal cover. That is not surprising since we will need to deal at the end with some finite cellular Γ-complex (which will give us our desired free resolution of $\mathbb{Z}$). We do this construction in the following lemma.

Lemma 2.56. *(Alternative construction) Let Γ be a group, then there exists an Eilenberg-MacLane space $K(\Gamma, 1)$.*

Proof. We construct it inductively. As in theorem 2.25 we construct a 2-skeleton, associated to our favorite presentation of Γ, to obtain a path connected space X^2 with $\pi_1(X^2, x) \cong \Gamma$. Note that its universal cover, denoted by $\tilde{X}^2$ has a trivial first homology group by Hurewicz theorem.

Suppose now that we constructed a space X^{k-1} with

$$\pi_1(X^{k-1}, x) = \Gamma, \quad \text{and} \quad \pi_i(X^{k-1}, x) = 0$$

for all $1 < i < k-1$ and its universal cover, denoted by $\tilde{X}^{k-1}$, has $H_i(\tilde{X}^{k-1}) = 0$ for $0 < i < k-1$. By the Hurewicz theorem, we have

$$\pi_{k-1}(\tilde{X}^{k-1}, \tilde{x}) \cong H_{k-1}(\tilde{X}^{k-1}).$$

Let $(z_\alpha)_\alpha$ be generators of $H_{k-1}(\tilde{X}^{k-1})$ seen as a Γ-module, then there exist continuous maps

$$g_{\partial\alpha} : S^{k-1} \to \tilde{X}^{k-1}$$

which corresponds via the Hurewicz morphism to the generators z_α. Define the attaching maps as the composition

$$\begin{tikzcd}
\partial D^k \arrow[r, "g_{\partial\alpha}"'] \arrow[rr, bend left, "f_{\partial\alpha}"] & \tilde{X}^{k-1} \arrow[r, "p"'] & X^{k-1}.
\end{tikzcd}$$

The k-skeleton is then the space

$$X^k := X^{k-1} \cup_f \bigvee_\alpha D^k_\alpha.$$

Let the universal cover of X^k be denoted by $\tilde{X}^k$. We must show that $H_i(\tilde{X}^k) = 0$ for all $1 < i < k$. Remark first that $\tilde{X}^{k-1} \subset \tilde{X}^k$. Indeed, if we look carefully at the proof of theorem 2.34, one can see that $\tilde{X}^k$ is obtained by attaching k-cells to $\tilde{X}^{k-1}$ by the attaching maps $g_{\partial\alpha,\gamma} := \gamma \circ g_{\partial\alpha}$ where γ is a deck transformation of $\tilde{X}^{k-1}$ and hence corresponds to an element in Γ. Draw the long exact sequence of the pair $(\tilde{X}^k, \tilde{X}^{k-1})$

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H_k(\tilde{X}^k, \tilde{X}^{k-1}) \arrow[dll, "\partial_k"] \\
H_{k-1}(\tilde{X}^{k-1}) \arrow[r] & H_{k-1}(\tilde{X}^k) \arrow[r] & H_{k-1}(\tilde{X}^k, \tilde{X}^{k-1}) \arrow[dll, "\partial_{k-1}"] & \\
H_{k-2}(\tilde{X}^{k-1}) \arrow[r] & H_{k-2}(\tilde{X}^k) \arrow[r] & \cdots &
\end{tikzcd}$$

Since $H_i(\tilde{X}^k, \tilde{X}^{k-1}) = 0$ for $0 < i < k$, we then have $H_i(\tilde{X}^k) \cong H_i(\tilde{X}^{k-1}) = 0$ for all $0 < i < k-1$.

We must now verify that $H_{k-1}(\tilde{X}^k) = 0$. We will show that ∂_k in the following exact sequence is surjective

$$\begin{tikzcd}
 & & \cdots \arrow[r] & H_k(\tilde{X}^k, \tilde{X}^{k-1}) \arrow[dlll, "\partial_k"] \\
H_{k-1}(\tilde{X}^{k-1}) \arrow[r] & H_{k-1}(\tilde{X}^k) \arrow[r] & 0. &
\end{tikzcd}$$

Remark that $H_k(\tilde{X}^k, \tilde{X}^{k-1}) = W_k(\tilde{X}^k)$ is the cellular chain group which is a free Γ-module with basis one element for each k-cell in X^k (it is in fact a *free abelian group* with basis one element for each k-cell in $\tilde{X}^k$, but we prefer to look at it as a Γ-module with a different basis). Let $(v_\alpha)_\alpha$ be such a basis of $H_k(\tilde{X}^k, \tilde{X}^{k-1})$. To be precise, we define v_α to be the image by the induced map of the characteristic map g_α. Explicitly, we have for all α

$$\begin{aligned} g_{\alpha *} : H_k(D^k_\alpha, \partial D^k_\alpha) &\to H_k(\tilde{X}^k, \tilde{X}^{k-1}) \\ [D^k_\alpha] &\mapsto v_\alpha \end{aligned}$$

where $[D_\alpha^k]$ is a generator of $H_k(D_\alpha^k, \partial D_\alpha^k) \cong \mathbb{Z}$. We then have the commutative diagram

$$\begin{tikzcd} H_k(D_\alpha^k, \partial D_\alpha^k) \arrow[r, "\partial_k", "\sim"'] \arrow[d, "g_{\alpha *}"'] & H_{k-1}(\partial D_\alpha^k) \arrow[d, "g_{\partial\alpha *}"] \\ H_k(\tilde{X}^k, \tilde{X}^{k-1}) \arrow[r, "\partial_k"] & H_{k-1}(\tilde{X}^{k-1}). \end{tikzcd}$$

We then have $z_\alpha = \partial_k v_\alpha$ for any generators z_α of $H_{k-1}(\tilde{X}^{k-1})$. Indeed, we defined above z_α to be exactly the image by $g_{\partial\alpha *}$ of the generator of $H_{k-1}(\partial D_\alpha^k)$. Hence ∂_k is surjective and a posteriori $H_{k-1}(\tilde{X}^k) = 0$. (Remark that we have already seen the diagram above in the proof of theorem 2.23.)

Remark that this implies by the Hurewicz theorem that $\pi_i(\tilde{X}^k, \tilde{x}) = 0$ for all $0 < i < k$. By the long exact sequence of a fibration, we deduce that $\pi_i(X^k, x) = 0$ for all $1 < i < k$, and we then finished the inductive step. Now define $X = \cup_k X^k$, it is an Eilenberg-MacLane space $K(\Gamma, 1)$. □

Remark. One can see that by construction $H_i(\tilde{X}^k) = 0$ for all $0 < i < k$ and it is easy to see that we always have $H_i(\tilde{X}^k) = 0$ for all $i > k$ (use the long exact sequence for the pair drawn above). So the only nontrivial homology group would be in degree k. Now suppose $H_{k-1}(\tilde{X}^{k-1})$ is a free $\mathbb{Z}\Gamma$-module with basis (z_α), then ∂_k would be an isomorphism and that would imply $H_k(\tilde{X}^k) = 0$.

So if this happens, we will have a contractible covering space of X^k, consequently X^k is a $K(\Gamma, 1)$.

Proof of the theorem. We consider first the case where $n = \infty$. Then use the preceding lemma to obtain an infinite $K(\Gamma, 1)$-complex.

Let us now consider the case where $3 \leq n < \infty$. We use the construction in the preceding lemma to obtain $\tilde{X}^{n-1}$ (this makes sense by hypothesis on n). Consider its (reduced) cellular chain complex

$$W_{n-1}(\tilde{X}^{n-1}) \to W_{n-2}(\tilde{X}^{n-1}) \to \cdots \to W_0(\tilde{X}^{n-1}) \to \mathbb{Z} \to 0.$$

It is a partial free Γ-resolution of $\mathbb{Z}$ of length $n-1$. Since $\operatorname{cd}\Gamma \leq n$, we have that

$$H_{n-1}(\tilde{X}^{n-1}) = \ker\{W_{n-1}(\tilde{X}^{n-1}) \to W_{n-2}(\tilde{X}^{n-1})\}$$

is a projective Γ-module. Use the Eilenberg trick to obtain a free Γ-module F such that $H_{n-1}(\tilde{X}^{n-1}) \oplus F \cong F$. Choose a basis $(f_\omega)_{\omega\in\Omega}$ of F and replace X^{n-1} by

$$\overline{X}^{n-1} := X^{n-1} \vee (\bigvee_{\omega\in\Omega} S_\omega^{n-1}).$$

Let $\tilde{\overline{X}}^{n-1}$ be the universal cover of $\overline{X}^{n-1}$. One can see that via the proof of theorem 2.34 we can describe the universal cover of $\overline{X}^{n-1}$ as the universal cover of X^{n-1} to which we added $\#\Omega \cdot \#\Gamma$ $(n-1)$-spheres, i.e.

$$\tilde{\overline{X}}^{n-1} = \tilde{X}^{n-1} \vee (\bigvee_{\substack{\omega\in\Omega \\ \gamma\in\Gamma}} S_{\omega,\gamma}^{n-1}).$$

Therefore, its cellular chain complex will be the same in degrees smaller than $n-1$ and in degree $n-1$ it will be $W_{n-1}(\tilde{X}^{n-1})$ to which we added F. Moreover $d_{n-1}|_F = 0$ since the degree of the attachment maps followed by the collapse is 0 for all $(n-1)$-sphere that we attached. Remark that the homotopy groups of $\overline{X}^{n-1}$ and X^{n-1} are the same in degrees smaller than $n-1$, in particular $\pi_1(\overline{X}^{n-1}, x_0) = \pi_1(X^{n-1}, x_0)$ since we added spheres of dimension 2 or greater (we use again strongly our hypothesis that $n \geq 3$).

We now have that $H_{n-1}(\tilde{\overline{X}}^{n-1})$ is free (as a Γ-module) with a certain basis (z_α). As in the preceding lemma, we attach to it α cells of dimension n to obtain a complex $\overline{X}^n = \overline{X}^{n-1} \cup_f \bigcup_\alpha D^n_\alpha$. Since $H_{n-1}(\tilde{\overline{X}}^{n-1})$ is free, $\overline{X}^n$ is a $K(\Gamma,1)$, and so we can stop our inductive construction and enjoy this CW-complex of $\operatorname{geom\,dim} = n$.

We now prove the case where Γ is finitely presented and of type FL. We have to show that we can finish with a finite CW-complex $\overline{X}^n$. Since Γ is finitely presented, we can construct a finite X^2 with the same technique as in the lemma above. For the inductive step, suppose that we constructed a finite X^{k-1}. We now have to build a finite X^k. By the very definition of FP, $H_{k-1}(\tilde{X}^{k-1})$ is a finitely generated projective Γ-module since it is the kernel of the partial free Γ-resolution

$$W_{k-1}(\tilde{X}^{k-1}) \to W_{k-2}(\tilde{X}^{k-1}) \to \cdots \to W_0(\tilde{X}^{k-1}) \to \mathbb{Z} \to 0.$$

If $k = n$, then $H_{n-1}(\tilde{X}^{n-1})$ is in addition stably free (using the fact that the group is of type FL, cf. proposition 2.53). Hence there exists a finitely generated free Γ-module F such that $H_{n-1}(\tilde{X}^{n-1}) \oplus F$ is free and finitely generated. Define as above $\overline{X}^{n-1}$ and $\overline{X}^n$ to obtain the desired finite complex. $\square$

Corollary 2.57 (Eilenberg-Ganea)**.** *If* $\operatorname{cd}\Gamma \geq 3$*, then* $\operatorname{cd}\Gamma = \operatorname{geom\,dim}\Gamma$.

We now know that $\operatorname{cd}\Gamma = \operatorname{geom\,dim}\Gamma$ almost every time except when $\operatorname{cd}\Gamma = 2$. It might happen that $\operatorname{cd}\Gamma = 2$ and $\operatorname{geom\,dim}\Gamma = 3$ but we don't have at this day any proof of it nor any counter-example.

Conjecture 2.58 (Eilenberg-Ganea)**.** *Let* Γ *be a group, then*

$$\operatorname{cd}\Gamma = \operatorname{geom\,dim}\Gamma.$$

Some recent work made by M.Bestvina and N.Brady in **[BB97]** have shown that this conjecture is related to the following conjecture.

Conjecture 2.59 (Whitehead)**.** *Any connected subcomplex of an aspherical 2-complex is aspherical.*

Recall that an aspherical 2-complex X is a CW-complex of dimension 2 which has $\pi_k(X, x_0) = 0$ for all $k > 1$. It is equivalent to say that X is a $K(\Gamma,1)$ of dimension 2 for some group Γ.

They constructed a group H_L (using right angled Artin group associated to a finite flag complex L) which is a potential counter-example of the Eilenberg-Ganea conjecture, if not there exists a counter-example of the Whitehead conjecture.

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