A homotopy-equivalence between Cacti and Configuration Spaces

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Master of Mathematics

by

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to

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for Maria, my love

Abstract

Let $k \geq 2$ be an integer. We show that the space $\mathcal{C}_k$ of ordered cacti with k lobes is homotopy-equivalent to the k-th ordered configuration space in the complex plane up to translation, denoted by $\mathcal{F}_k$. We then give an explicit homotopy-equivalence between $\mathcal{F}_k$ and $\mathcal{C}_k$.

Résumé

Soit $k \geq 2$ un entier. Nous montrons que l'espace $\mathcal{C}_k$ de cactus avec k lobes est homotopie-équivalente au k-ème espace de configurations dans le plan complexe à translation près, noté $\mathcal{F}_k$. Nous donnons ensuite une équivalence d'homotopie explicite entre $\mathcal{F}_k$ et $\mathcal{C}_k$.

Contents

Notations

We will use the following notations:

- 0: the trivial group
- I: the closed unit interval $[0, 1]$
- 1_X: the identity map on the space X
- S^n: the real n-sphere
- D^n: the real n-disk
- S_n: the symmetric group on n letters
- F_n: the free group on n generators
- $\mathbb{A}_2$: the affine group of the plane
- $\mathbb{Z}_n$: the cyclic group of order n
- Δ^n: the geometric n-simplex
- X_n : the n-skeleton of the CW-complex X
- X^+ : the one-point compactification of the space X

Chapter 1

Introduction

1.1 Motivation

In this paper, we examine a link between the spaces of configurations and cacti, both of which have ties to a wide variety of branches of mathematics.
The euclidean configuration spaces, which we will be studying in the particular case of dimension 2, that is to say, configurations in the plane, are fundamental in the theory of double loop spaces. They are also closely related to the world of algebraic geometry, by way of an identification with moduli spaces of curves, more specifically those of genus zero over the complex numbers with marked points. Kontsevich studied cell decompositions of these moduli spaces of curves, which provide insight into the study of configuration spaces, and vice versa. They are also related to the little discs operad of Boardman and Vogt, which was the first known non-trivial operad structure, instrumental in the formulation of the Deligne conjecture, a topological version of which was proved by McClure and Smith.
The spaces of cacti, first introduced by Voronov, are another very important example of operads: indeed, this operad was invented for its applications to string topology. The cacti can be related, like the configurations, to the little discs operad. In fact, Tarje Bargheer [Barg], in his own master's thesis, proved a result very similar to the central result of this paper: rather than linking together configuration spaces and cacti, his work linked the little discs operad with the cacti operad. These two approaches are two sides of the same coin: while Bargheer's approach concentrates on the operad-theory aspect, we focus here on the more geometric aspect. In addition, cacti play an important role in string topology according to Voronov, with applications to topological field theories in particular.

1.2 Outline

Throughout this paper, we will be dealing with "nice" spaces: aspherical CW-complexes. We will begin by talking about obstruction theory, and showing how this can facilitate the problem of showing that two aspherical CW-complexes have the same homotopy type just by considering their fundamental group. We will then apply this to two spaces that we wish to show as being homotopy-equivalent, namely, for any integer $k \geq 2$, the k-th ordered configuration space up to translation $\mathcal{F}_k$ and the space of ordered cacti with k lobes $\mathcal{C}_k$.

First, some quick definitions: an element of $\mathcal{F}_k$, or *configuration*, is simply an ordered k-tuple of distinct points of $\mathbb{C}$. We take it up to translation, so that the whole configuration can be shifted around at will, for example if we want to translate all the points so that their barycenter is located at the origin. An element of $\mathcal{C}_k$, or *cactus* is, roughly speaking, an arrangement in the plane of topological circles, each of which will be called a *lobe* of the cactus connected together without forming a "pearl necklace," along with a "root" on the edge of the entire structure.

We define a map

$$\Phi : \mathcal{F}_k \to \mathcal{C}_k$$

corresponding to a physical intuition of the resemblance between these spaces that was suggested to the author by Paolo Salvatore in a private communication. This deceptively simple idea is to imagine that each point in a configuration has a radial vector field that at each point in the plane has a norm inversely proportional to its distance from the first point. When these vector fields are added up over all the points in the configuration, it produces another vector field, whose flow lines we will consider. In the space of flow lines, by identifying each flow line with a point, it is not hard to see that we get circles around each of the points in the configuration. We will show that this arrangement of circles is in fact a cactus. With this definition, Φ will be our homotopy-equivalence.

Rather than trying to find a homotopy-inverse to Φ, which risks being unintuitive, we will consider the map $\widetilde{\Phi}$ defined on the orbit space of a free action of the cyclic group $\mathbb{Z}_k$ on $\mathcal{F}_k$. It will be the case that not only is

$$\widetilde{\Phi} : \mathcal{F}_k \,/\, \mathbb{Z}_k \to \mathcal{C}_k \,/\, \mathbb{Z}_k$$

well-defined, but the action of $\mathbb{Z}_k$ on both spaces, by rotation of the configuration or cactus by the angle $\frac{2\pi}{k}$, is free, so that the projection onto the orbit spaces is in fact a covering. Then by the unique lifting property of covering spaces, it will suffice to find a proof that $\widetilde{\Phi}$ is continuous, and find a

homotopy-inverse $\widetilde{s}$ of $\widetilde{\Phi}$, to uniquely lift these maps to Φ and s, which will then be homotopy-equivalences, and homotopy-inverses of each other.
However, we will not even really work with $\widetilde{\Phi}$. Rather, we will encode $\mathcal{F}_k \,/\, \mathbb{Z}_k$ and $\mathcal{C}_k \,/\, \mathbb{Z}_k$ combinatorially, that is to say, we will find spaces $\mathscr{F}_k$ and $\mathscr{C}_k \subset \mathscr{F}_k$ of data to come in the form of complex numbers with "summable" permutation labels. More precisely, an element of $\mathscr{C}_k$ or $\mathscr{F}_k$ is a tuple of complex numbers, labelled with permutations in such a way that when two numbers move around and collide, their corresponding labels sum. These complex numbers will be critical values of polynomials that come from configurations, where each point in the configuration is a root. These polynomials are seen as branched coverings of S^2 over S^2, while the corresponding label will represent the corresponding monodromy action on the set of roots of the polynomial. With this, we have homeomorphisms

$$\mathcal{F}_k \,/\, \mathbb{Z}_k \xrightarrow[\cong]{h_{\mathscr{F}}} \mathscr{F}_k,$$

$$\mathcal{C}_k \,/\, \mathbb{Z}_k \xrightarrow[\cong]{h_{\mathscr{C}}} \mathscr{C}_k\,.$$

We will then look at $\widetilde{\Phi}$ taken "downstairs" through these homeomorphisms, giving a map we will call $\overline{\Phi}$, giving us the diagram

$$\begin{array}{ccc}
\mathcal{F}_k & \underset{s}{\overset{\Phi}{\rightleftarrows}} & \mathcal{C}_k \\
\Big\downarrow{\scriptstyle \pi_{\mathcal{F}}} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{C}}} \\
\mathcal{F}_k \,/\, \mathbb{Z}_k & \underset{\widetilde{s}}{\overset{\widetilde{\Phi}}{\rightleftarrows}} & \mathcal{C}_k \,/\, \mathbb{Z}_k \\
{\scriptstyle h_{\mathscr{F}}}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle h_{\mathscr{C}}} \\
\mathscr{F}_k & \underset{\overline{s}}{\overset{\overline{\Phi}}{\rightleftarrows}} & \mathscr{C}_k
\end{array}$$

where

$$\mathcal{F}_k \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F}_k \,/\, \mathbb{Z}_k$$

$$\mathcal{C}_k \xrightarrow{\pi_{\mathcal{C}}} \mathcal{C}_k \,/\, \mathbb{Z}_k$$

are the canonical projections.
Let us give a quick explanation of how to do this. It is well-known that $\mathcal{F}_k$

can be homeomorphically identified with the space $\mathrm{o}\mathcal{P}_k$ of polynomials with no multiple roots and an ordering of the roots by sending a configuration to the polynomial admitting all the points in the configuration as roots. This homeomorphism carries over to a homeomorphic identification

$$\mathcal{F}_k \,/\, \mathbb{Z}_k \cong \mathrm{o}\mathcal{P}_k \,/\, \mathbb{A}_2,$$

where $\mathbb{A}_2$ is the group of transformations

$$z \mapsto az + b$$

for $a \in \mathbb{C}^*, b \in \mathbb{C}$, acting on the space of polynomials by right composition, so that

$$p(z) \sim p(az + b).$$

So we wish to homeomorphically identify polynomials up to right affine transformation with some appropriate space $\mathscr{F}_k$ such that we can homeomorphically identify $\mathscr{C}_k$, the subspace in which all the complex numbers have modulus one, with $\mathcal{C}_k \,/\, \mathbb{Z}_k$, and such that $\overline{\Phi}$ is particularly "nice," in that it is a "forgetful" map, it is easily shown to be a homotopy-equivalence, and also that it can be seen as an embedding. Then the maps from $\mathcal{F}_k$ to $\mathscr{F}_k$ and from $\mathcal{C}_k$ to $\mathscr{C}_k$, respectively, will be coverings composed with homeomorphisms, which are still coverings. So we can then lift the homotopy-equivalence

$$\overline{\Phi} : \mathscr{F}_k \to \mathscr{C}_k$$

in a unique manner to

$$\Phi : \mathcal{F}_k \to \mathcal{C}_k,$$

which we will have finally shown to be a homotopy-equivalence.
In analogy to what is true for the combinatorial data representing both configuration spaces and cacti, any configuration can be uniquely determined by a cactus, along with at most $k-1$ real numbers. We will use this to determine CW-decompositions of both spaces of cacti and configuration spaces, with respect to which Φ, like $\overline{\Phi}$ is simply a "forgetful" map. Although we only do this explicitly for the case $k = 3$, as for larger k, the combinatorial work required quickly gets tedious, the method we use for $k = 3$ is valid for all k.

Chapter 2

Proof of homotopy-equivalence

2.1 A bit of obstruction theory

Definition 2.1.1. A space X is called *aspherical* if all its higher homotopy groups are trivial.

Obstruction theory concerns itself, generally speaking, with finding homotopy obstructions to the existence of extensions and lifts of maps between CW-complexes. In this section, we will start with a constant map defined on the 0-skeleton of a CW-complex, and extend it to the 1-skeleton in a special way, so that in homotopy, this map represents a specified group homomorphism. We will then extend this map to the entire CW-complex, and show that it is a homotopy-equivalence.

Theorem. *Let X and Y be aspherical connected CW-complexes with isomorphic fundamental groups. Then X and Y are homotopy-equivalent.*

We concentrate on aspherical spaces, as this means that the obstructions we seek, which lie in cohomology groups with coefficients in homotopy groups, will often be trivial, as all higher homotopy groups are trivial. Such spaces are thus, in this regard, particularly "nice".
As a caveat, while this proof may seem at first glance to be constructive, it is not.
First, a technical lemma which will allow us, once we have constructed a map from X to Y and one from Y to X, to give a homotopic necessary and sufficient condition for these maps to be homotopy-inverses to one another.

Lemma 2.1.2. *Let $f, g : X \to Y$ be maps between CW-complexes X and Y with Y aspherical. Then f is homotopic to g if and only if $\pi_1 f = \pi_1 g$.*

Proof. The "only if" implication is clear, as it is well known that π_1 is a homotopy functor.

We will now assume that $\pi_1 f = \pi_1 g$ and construct a homotopy H from f to g. We will do this by first defining a candidate map on the 2-skeleton of $X \times I$, which is $X_2 \times \{0,1\} \amalg X_1 \times I$. Since it is obvious how to define H on the cells of type $e^2 \times \{0,1\}$, we only consider how to define H on cells of type $e^1 \times I$. Similarly, to define H on the $(n+1)$-skeleton of $X \times I$, it suffices to do so on $X_n \times I$.

We consider a cell in $X_1 \times I$, namely a prism whose base is a loop $\lambda : S^1 \to X$, the class $[\lambda] \in \pi_1 X$, and the image

$$\pi_1 f\left([\lambda]\right) = \pi_1 g\left([\lambda]\right) \in \pi_1 Y.$$

This last equality means that there is a homotopy

$$L : S^1 \times I \to Y$$

from $f \circ \lambda$ to $g \circ \lambda$, which gives us

$$H_2 : (X \times I)_2 \to Y.$$

We now try to extend H_2 to $(X \times I)_3$, by recursively attaching 3-cells e^3_α to the 2-skeleton of $X \times I$ via the attaching maps

$$\alpha : S^2 \to (X \times I)_2.$$

We thus have the pushout diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\alpha} & (X \times I)_2 \\ {\scriptstyle i}\downarrow & & \downarrow \\ D^3 & \longrightarrow & (X \times I)_2 \cup_\alpha e^3_\alpha \end{array}$$

Given that the disk D^3 is contractible, any map from S^2 to Y that factors through D^3 must be nullhomotopic. Thus, a necessary and sufficient condition for H_2 to be extensible to $(X \times I)_2 \cup_\alpha e^3_\alpha$, and by induction to the whole 3-skeleton, is for the composition

$$S^2 \xrightarrow{\alpha} (X \times I)_2 \xrightarrow{H_2} Y$$

to be nullhomotopic. This is always the case, because $\pi_2 Y = 0$. So we can extend H_2 to

$$H_3 : (X \times I)_3 \to Y.$$

By induction, using the very same arguments, we can extend

$$H_n : (X \times I)_n \to Y$$

to

$$H_{n+1} : (X \times I)_{n+1} \to Y.$$

Thus we get a homotopy between f and g. $\square$

Theorem 2.1.3. *Let X and Y be aspherical connected CW-complexes with isomorphic fundamental groups. Then X and Y are homotopy-equivalent.*

Proof. We will make the additional assumption that the 0-skeletons of both X and Y are reduced to a point, that is, that both have only one 0-cell. We can do this without loss of generality, as any CW-complex is homotopy-equivalent to one that has a decomposition with only one 0-cell.
In order to find such a homotopy-equivalence, we shall take an isomorphism

$$\varphi : \pi_1 X \to \pi_1 Y,$$

and construct a CW-map

$$f : X \to Y$$

such that

$$\varphi = \pi_1 f.$$

This f will be our homotopy-equivalence. Indeed, if such a map is found, its homotopy-inverse will simply be

$$g := \pi_1(\varphi^{-1})$$

by lemma 2.1.2. Let us now proceed to do construct f.
A consequence of Seifert-van Kampen's theorem (see [Hat, Proposition 1.26]) is that the fundamental group of a CW-complex X can be written

$$\pi_1 X = \pi_1(X_1)/N,$$

where N is the subgroup of $\pi_1(X_1)$ generated by the classes of the attaching maps $\alpha : S^1 \to X_1$ of 2-cells to X_1, also denoted by $\pi_2(X, X_1)$. The generators are classes of loops in X_1, where the only loops we consider are those to which we do not plan on attaching any higher-dimensional cells. This makes intuitive sense: if we attach a 2-cell to a loop, we should expect the resulting loop to be nullhomotopic, as we could then close the loop through the newly available cell. In fact, we can even restrict the generators further, taking only those loops whose trace is a 1-cell, as all others are simply concatenations of such

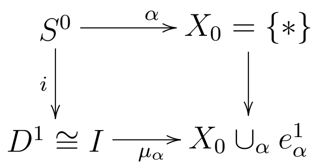
loops. This will simplify our work later.
We start with the trivial map $f_0 : X_0 \to Y_0$, which is trivial because we have taken CW-decompositions of X and Y containing only one 0-cell. Our task is now to give an appropriate extension

$$f_1 : X_1 \to Y_1$$

of f_0.
Consider the 1-cell e^1_α with attaching map

$$\alpha : S^0 \to X_0.$$

We have a pushout diagram



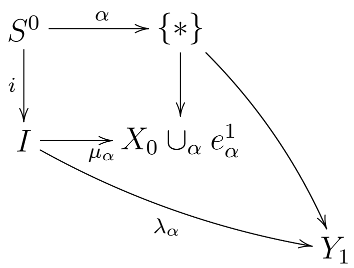
where both α and any map defined on X_0 are necessarily constant. So any map

$$\lambda_\alpha : I \to Y_1$$

satisfying

$$\lambda_\alpha(0) = \lambda_\alpha(1),$$

in other words, any loop $\lambda_\alpha : S^1 \to Y_1$, will make the diagram



commute. But which λ_α shall we pick?
The map μ_α is a loop

$$\mu_\alpha : S^1 \to X_0 \cup_\alpha e^1_\alpha \subset X,$$

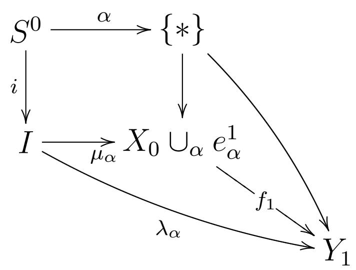
so we have

$$[\mu_\alpha] \in \pi_1 X \quad , \quad \varphi([\mu_\alpha]) \in \pi_1 Y.$$

We can take any λ_α in this class. Then by the pushout property, there exists a unique

$$f_1 : X_0 \cup_\alpha e^1_\alpha \to Y_1$$

making



commute. Of course, the same reasoning applies if we want to add an A-indexed number of 1-cells, replacing S^0 by $\coprod_{\alpha \in A} S^0$ and I by $\coprod_{\alpha \in A} I$. So we have an extension

$$f_1 : X_1 \to Y_1.$$

To further extend f_1 to X, we proceed in much the same manner as in the proof of lemma 2.1.2, using the fact that Y is aspherical. Thus, we have defined

$$f : X \to Y,$$

and it only remains to show that we indeed have

$$\pi_1 f = \varphi.$$

Let

$$\gamma : I \to X$$

be a loop. We begin by showing that

$$[f_1 \circ \gamma] = \varphi([\gamma]).$$

Since X is connected, we can assume without loss of generality that

$$\gamma(0) = \gamma(1) = * \in X_0.$$

As a reminder, we have

$$[\gamma] \in \pi_1 X = \pi_1(X_1)/\pi_2(X, X_1),$$

so γ is, up to homotopy, a loop in X_1, based in the 0-cell. Because $\pi_1(X_1)$ is generated by those loops that simply trace 1-cells, we can assume without loss of generality that γ is such a loop, "going around" a particular 1-cell, which we call e^1_β. So we have

$$\gamma = \mu_\beta,$$

and we know from above that the class $\varphi([\mu_\beta])$ contains a loop

$$\lambda_\beta : S^1 \to Y_1$$

such that

$$\lambda_\beta = f_1 \circ \mu_\beta = f_1 \circ \gamma,$$

so that

$$\varphi([\gamma]) = [\lambda_\beta] = \pi_1 f_1([\gamma]).$$

We have thus shown that φ and $\pi_1 f_1$ agree on the subgroup $\pi_1 X$ of $\pi_1(X_1)$. Furthermore, since f_1 is merely a restriction of f, $\pi_1 f$ and $\pi_1 f_1$ must also agree on $\pi_1 X$, which completes the proof. □

2.2 Configuration spaces and braid groups

Definition 2.2.1. Let $k \geq 2$ be an integer. The *k-th ordered configuration space up to translation* in $\mathbb{C}$ is the pointed space

$$\mathcal{F}_k := \{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_i \neq z_j \text{ when } i \neq j\} / \mathbb{C}.$$

If we map an element of the $k+1$-th configuration space to the element of the k-th configuration space consisting of its first k coordinates,

$$\begin{array}{rcl} \mathcal{F}_{k+1} & \twoheadrightarrow & \mathcal{F}_k \\ (z_1, \ldots, z_k, z_{k+1}) & \mapsto & (z_1, \ldots, z_k), \end{array}$$

we get a projection that defines a fiber bundle, whose fiber is $\mathbb{C}$ with k points deleted, which has the homotopy type of a wedge of k circles. So the bundle can be seen thusly:

$$\bigvee_{i=1}^{k} S^1 \to \mathcal{F}_{k+1} \twoheadrightarrow \mathcal{F}_k\,.$$

Let us consider the long exact sequence in homotopy of this fiber bundle. We recall that the fundamental group of $\bigvee_{i=1}^k S^1$ is F_k, the free group on k generators, and all higher homotopy groups are trivial. We show part of the long exact sequence in homotopy below:

$$\cdots \to 0 \to \pi_3\, \mathcal{F}_{k+1} \to \pi_3\, \mathcal{F}_k \to 0 \to \pi_2\, \mathcal{F}_k \to \pi_2\, \mathcal{F}_k \to F_k \to \cdots$$

Thus, the long exact sequence tells us that the third and higher homotopy groups of $\mathcal{F}_{k+1}$ are equal to those of $\mathcal{F}_k$. Let us now set about proving the following result:

Proposition 2.2.2. *For any $k \geq 2$, $\mathcal{F}_k$ is aspherical.*

Proof. We will start by showing that $\mathcal{F}_2$ has the homotopy type of a circle. We define

$$\begin{aligned} f : \mathcal{F}_2 &\rightarrow S^1 \\ (z_1, z_2) &\mapsto \arg(z_1 - z_2) \end{aligned}$$

and

$$\begin{aligned} g : S^1 &\rightarrow \mathcal{F}_2 \\ \theta &\mapsto (2e^{i\theta}, e^{i\theta}) \end{aligned}$$

With this, we already have

$$f \circ g = 1_{S^1}.$$

Furthermore, with

$$g \circ f(z_1, z_2) = \left(2\frac{z_1 - z_2}{|z_1 - z_2|}, \frac{z_1 - z_2}{|z_1 - z_2|}\right),$$

there is the intuitive homotopy

$$H\left((z_1, z_2), t\right) := \left(tz_1 + (1-t)2\frac{z_1 - z_2}{|z_1 - z_2|}, tz_2 + (1-t)\frac{z_1 - z_2}{|z_1 - z_2|}\right)$$

between $g \circ f$ and $1_{\mathcal{F}_2}$. To show that this is well-defined, it suffices to show that

$$tz_1 + (1-t)2\frac{z_1 - z_2}{|z_1 - z_2|} - tz_2 + (1-t)\frac{z_1 - z_2}{|z_1 - z_2|} = (z_1 - z_2)\left(t + \frac{1-t}{|z_1 - z_2|}\right) \neq 0$$

for any $z_1 \neq z_2$ and any $t \in I$. This is clearly the case.
Therefore, the fundamental group of $\mathcal{F}_2$ is $\mathbb{Z}$, and since we know that S^1 is aspherical, it follows that $\mathcal{F}_2$ is aspherical too.
Let us return for a moment to the long exact sequence in homotopy discussed above. As we saw, the third and all higher homotopy groups of $\mathcal{F}_{k+1}$ are equal to those of $\mathcal{F}_k$. So to find the third and higher homotopy groups of all configuration spaces, it suffices to find these groups for $\mathcal{F}_2$, which has the homotopy type of a circle. Thus the third and higher homotopy groups of the

k-th configuration space are trivial.
Furthermore, if we assume that $\pi_2\,\mathcal{F}_k = 0$, then we also have $\pi_2\,\mathcal{F}_{k+1} = 0$, and since $\pi_2\mathcal{F}_2 = 0$, by induction we get

$$\pi_i\,\mathcal{F}_k = 0 \text{ for all } i, k.$$

□

So we have seen that the configuration spaces are aspherical. They are also manifolds, as

$$\mathcal{F}_k := \{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_i \neq z_j \text{ when } i \neq j\} / \mathbb{C}$$

is an open set in the euclidean space $\mathbb{C}^k$, thus it is a CW-complex, for which we will not give a description here. We have thus proved that configuration spaces are aspherical CW-complexes.

Definition 2.2.3. The fundamental group of $\mathcal{F}_k$ will be denoted by

$$P\beta_k := \pi_1\,\mathcal{F}_k,$$

and referred to as the *pure braid group on k strands.* Similarly, we consider the k-th *unordered* configuration space

$$\mathcal{F}_k{}' := \mathcal{F}_k\,/S_k,$$

whose fundamental group will be denoted by

$$\beta_k := \pi_1\,\mathcal{F}_k{}',$$

and referred to as the *braid group on k strands.*

Remark. The unordered configuration spaces are also aspherical.

Example 2.2.4. As we will show in section 4.1, $\mathcal{F}_2$ and $\mathcal{U}_2$ have the homotopy type of a circle. It follows that

$$P\beta_2 = \mathbb{Z}, \beta_2 = \mathbb{Z}\,.$$

The braid group β_k is in some sense analogous to the symmetric group S_k. Indeed, an element of the braid group is simply the homotopy class of a loop in $\mathcal{F}_k{}'$, which is derived from the pointed topological space $\mathcal{F}_k$, let us say with base point $(1, \ldots, k)$. Now, a loop in $\mathcal{F}_k$ is simply a collection of k continuous, non-intersecting paths, one from 1 to 1, and so forth. On the other hand, a loop in $\mathcal{F}_k{}'$ will be a collection of k continuous non-intersecting paths, along with a permutation $\sigma \in S_k$ such that the path starting at 1 ends up not at 1, but at $\sigma(1)$, and so forth.

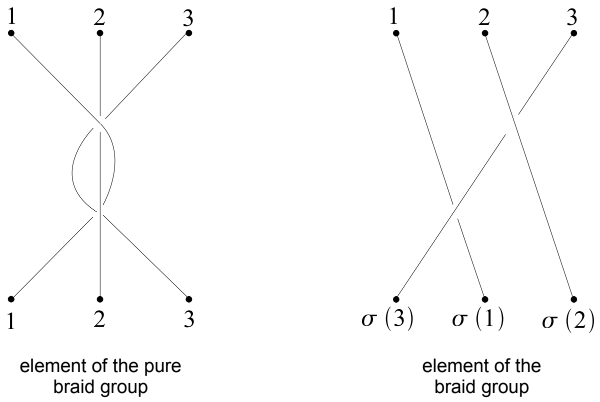


Figure 2.1: The braid group and the pure braid group

Hence, the homotopy class of a loop in $\mathcal{F}_k{}'$ can be seen as analogous to the permutation associated with the loop, as we have just explained.
This analogy gives us a presentation of the braid group β_k, generated by the elements σ_i associated with the permutations

$$\tau_i := (i\ i+1) \in S_k, i = 1, \ldots, k-1$$

and shown in the figure below:

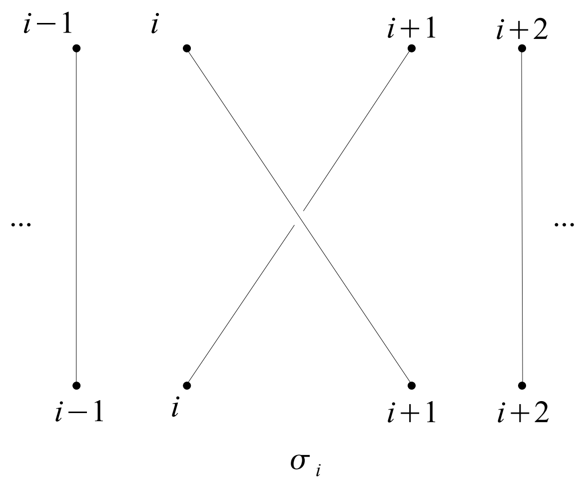


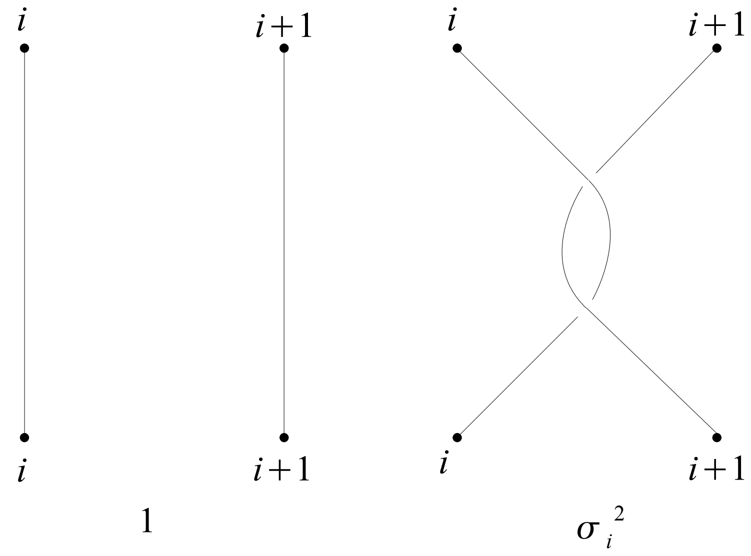
Figure 2.2: The generators σ_i of the braid group

Like the generators of the symmetric group, these σ_i satisfy the following identities:

1. $$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2$$

2. $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

However, unlike the generators of the symmetric group, these σ_i do not satisfy the idempotence condition $\sigma_i^2 = 1$. This can be easily seen in the figure below:



In fact, it is known that we have a presentation of β_k given by

$$\{\sigma_i, i = 1, \ldots, k-1 \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\},$$

which is sometimes used as a definition of β_k.
Returning to the analogy between β_k and S_k, we can project any element of the braid group onto the permutation which represents it. The kernel of this projection is none other than the pure braid group $P\beta_k$, in other words

$$S_k = \beta_k/P\beta_k.$$

With this, we can find a presentation of $P\beta_k$. Indeed, the only difference between our preferred presentation of β_k and that of S_k is that the symmetric group presentation has the additional relation of idempotence. Therefore, a presentation of $P\beta_k$ must be such that quotienting by these relations is the same as adding the idempotence relation to β_k. Thus, the generators of $P\beta_k$ are precisely the

$$\xi_{i,j} := \sigma_i \ldots \sigma_{j-1}\sigma_j^2\sigma_{j-1}^{-1} \ldots \sigma_i^{-1}, 1 \leq i < j \leq k,$$

where the σ_i are the generators of β_k. To see that quotienting these elements out is equivalent to adding the idempotence relation is straightforward.

2.3 Cacti

Following Salvatore [Sal], we will define a cactus as follows:

Definition 2.3.1. An *ordered cactus with k lobes* is the quotient of a partition of S^1 into k closed 1-manifolds I_j with pairwise disjoint interiors, by their boundaries, such that each I_j is of equal length and there is no cyclically ordered 4-tuple $(\theta_1, \theta_2, \theta_3, \theta_4)$ such that $\theta_1, \theta_3 \in \operatorname{int}(I_i), \theta_2, \theta_4 \in \operatorname{int}(I_j)$ and $i \neq j$. We denote the set of cacti with k lobes by $\mathcal{C}_k$.

In other words, chords of the circle are formed by linking together all the boundary points of the sets in the partition, and cacti are formed by collapsing those chords.

Terminology. The original unit circle will henceforth be referred to as the "outer circle", while the second condition in the definition will be called the "no-necklace condition".

As with configuration spaces, the symmetric group S_k acts freely on the space of cacti by permuting lobes, and we shall denote the space of unordered cacti with k lobes by $\mathcal{C}_k{}'$, giving us a principal covering

$$S_k \to \mathcal{C}_k \to \mathcal{C}_k{}' .$$

We would like to prove that the space of cacti with k lobes is an aspherical CW-complex. Just as earlier, we considered the map that "forgot" about the last point in the ordered configuration space, so will we now consider the map that will "forget" about the last ordered lobe in a cactus to give us Kaufmann's "forgetful" quasifibration[Kauf05]

$$\mathcal{C}_{k+1} \twoheadrightarrow \mathcal{C}_k .$$

To prove that the $\mathcal{C}_k$ are aspherical, we would like to proceed in a manner analog to the manner in which we proved that the k-th configuration space is aspherical, except that there is no convenient structure of a bundle or even a fibration [Kauf05, Remark 3.3.8] that would help us to do this. Fortunately, there is a quasi-fibration structure [Kauf05, Proposition 3.3.18], which for our purposes works in a similar way, in that there is still a long exact sequence in homotopy.
Kaufmann [Kauf05, Corollary 3.3.6] shows that the fiber over any cactus in $\mathcal{C}_{k+1}$ is homotopy-equivalent to a wedge of k circles. So once again the quasi-fibration can be seen as

$$\bigvee_{i=1}^{k} S^1 \to \mathcal{C}_{k+1} \twoheadrightarrow \mathcal{C}_k,$$

and part of the long exact sequence in homotopy is

$$\cdots \to 0 \to \pi_3\,\mathcal{C}_{k+1} \to \pi_3\,\mathcal{C}_k \to 0 \to \pi_2\,\mathcal{C}_{k+1} \to \pi_2\,\mathcal{C}_k \to F_k \to \cdots$$

Now let us consider the space $\mathcal{C}_2$ of cacti with two lobes. Using the same reasoning used to show that $\mathcal{F}_2$ has the homotopy type of a circle, we see that $\mathcal{C}_2$ also has the homotopy type of a circle. We can immediately deduce that all the $\mathcal{C}_k$ are aspherical, and that $\mathcal{C}_2$ has fundamental group $\mathbb{Z} = P\beta_2$.
As a reminder, our goal is to show that the spaces $\mathcal{C}_k$ and $\mathcal{F}_k$ are homotopy-equivalent. For us to be able to invoke theorem 2.1.3, it still remains to show that:

1. $\mathcal{C}_k$ is a CW-complex,
2. $\mathcal{C}_k$ has fundamental group $P\beta_k$.

Let us first give a CW-decomposition of $\mathcal{C}_k$. To do this, we will adopt a simpler notation, inspired by McClure and Smith [MS03], for the cell containing a given cactus. Starting at the zero point and moving counterclockwise around the outer circle, we write successively the indices of all the lobes we pass. For instance, the cactus below

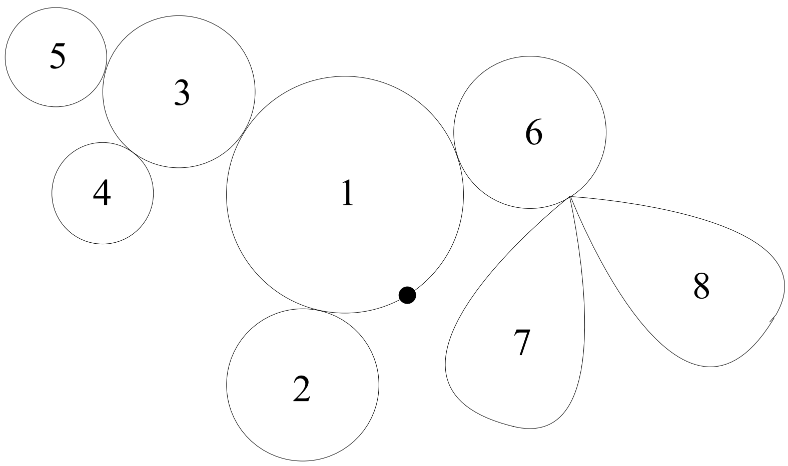


Figure 2.3: A cactus in the cell 16786135343121

lies in the cell 16786135343121.
Now, in $\mathcal{C}_k$, there are a finite number of such cells. We determine the dimension of each as follows. Let us say that in the description of a given cell, the index i appears n_i times, corresponding to the i-th lobe being seperated into n_i submanifolds on the outer circle. Since the length of lobe i is given by the

definition of a cactus, this corresponds to the geometric $(n_i - 1)$-simplex Δ^{n_i-1}. Thus, by doing this for all indices i, the cell is homeomorphic to the product

$$\prod_{i=1}^{k} \Delta^{n_i-1}, \tag{2.1}$$

which has dimension $\sum_{i=1}^{k}(n_i - 1)$. For example, we have

$$16786135343121 \cong \Delta^3 \times \Delta^1 \times \Delta^0 \times \Delta^0 \times \Delta^2 \times \Delta^0 \times \Delta^0 \times \Delta^0,$$

which is a 6-cell. We will use this simple mechanism later to give a CW-structure for $\mathcal{C}_3$.
We will resolve the second question following Kaufmann [Kauf05, Proposition 3.3.19]. Since we already know this to be true for the case $k = 2$, we can proceed by induction. Incidentally, we will also show that the fundamental group of $\mathcal{C}_k{}'$, the space of *unordered* cacti with k lobes, is the braid group β_k, also by induction. Therefore, assume that

$$\pi_1\,\mathcal{C}_k = P\beta_k, \pi_1\,\mathcal{C}_k{}' = \beta_k.$$

Firstly, we fix a base point c_k for the space $\mathcal{C}_k$, which we will take to be the *corolla cactus* $12\ldots k$, shown below:

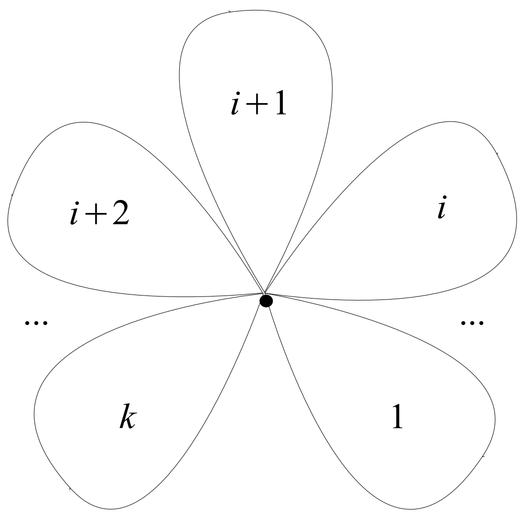


Figure 2.4: The corolla cactus c_k

Consider the path α_i, which can be seen as the one in which lobe $i+1$ "leapfrogs" over lobe i along the outer circle.

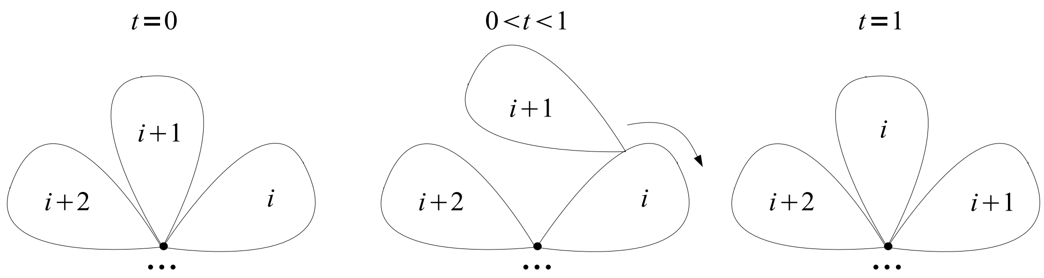


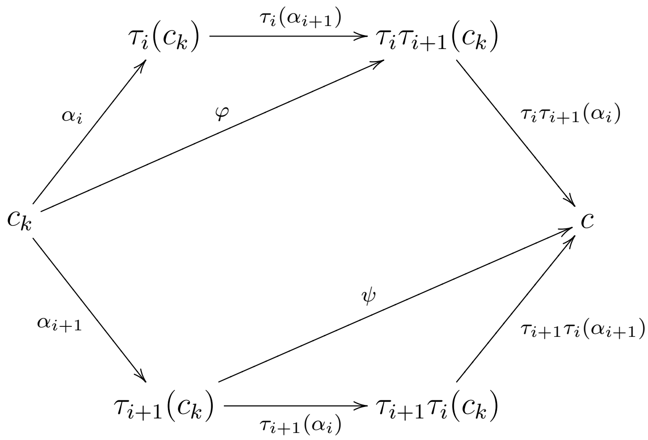
Figure 2.5: The path α_i

In the space of unordered cacti $\mathcal{C}_k{}'$, $f \circ \alpha_i$ is in fact a loop (we take $f : \mathcal{C}_k \to \mathcal{C}_k{}'$ as the canonical map that "forgets" the ordering). We wish to send the generator σ_i of the braid group to $[\alpha_i]$ to specify the isomorphism between $\pi_1 \, \mathcal{C}_k{}'$ and β_k. For this choice to be valid, we need to verify that

1. $$[\alpha_i][\alpha_j] = [\alpha_j][\alpha_i] \text{ if } |i-j| \geq 2,$$

2. $$[\alpha_i][\alpha_{i+1}][\alpha_i] = [\alpha_{i+1}][\alpha_i][\alpha_{i+1}].$$

That the first condition holds is fairly evident. To show that the second holds, we explicitly look at the action of the given paths on c_k:



Both of these paths end at $c := \tau_i\tau_{i+1}\tau_i(c_k) = \tau_{i+1}\tau_i\tau_{i+1}(c_k)$, which is a good sign. Another is that if φ is defined as the path that keeps lobes $i+1$ and $i+2$

stuck together while leapfrogging over lobe i, while ψ is defined as the path that keeps lobes $i+2$ and $i+1$ stuck together while leapfrogging over lobe i, both triangles and the central parallelogram each represent a path homotopy. Hence the application

$$\begin{array}{rcl} \pi_1 \mathcal{C}_k{}' & \to & \beta_k \\ [\alpha_i] & \mapsto & \sigma_i \end{array}$$

is a surjective homomorphism. It is in fact an isomorphism; it is known that the $[\alpha_i]$ generate $\pi_1 \mathcal{C}_k{}'$.
Now we recall the quasifibrations

$$\bigvee_{i=1}^{k} S^1 \to \mathcal{F}_{k+1} \twoheadrightarrow \mathcal{F}_k$$

and

$$\bigvee_{i=1}^{k} S^1 \to \mathcal{C}_{k+1} \twoheadrightarrow \mathcal{C}_k$$

seen earlier, and take the long exact sequences in homotopy of both of them. Since all spaces involved are aspherical, these long exact sequences are in fact short:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_k & \longrightarrow & P\beta_{k+1} & \longrightarrow & P\beta_k & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_k & \longrightarrow & \pi_1 \mathcal{C}_{k+1} & \longrightarrow & \pi_1 \mathcal{C}_k & \longrightarrow & 0 \end{array}$$

Thanks to our assumption that $\pi_1 \mathcal{C}_k$ and $P\beta_k$ are isomorphic, it suffices to find an isomorphism, and then by the five-lemma, any arrow from $\pi_1 \mathcal{C}_{k+1}$ to $P\beta_{k+1}$ that makes the resulting diagram commute will be an isomorphism, completing the proof.
To find an isomorphism $P\beta_k \to \pi_1 \mathcal{C}_k$, we recall the isomorphism $\beta_k \to \pi_1 \mathcal{C}_k{}'$ given earlier, and the fact that $S_k = \beta_k / P\beta_k$. We thus take the restriction of the isomorphism given earlier to the quotient group $P\beta_k$. Since $P\beta_k$ is generated by the $\xi_{i,j}$, we get the isomorphism

$$\begin{array}{rcl} P\beta_k & \to & \pi_1 \mathcal{C}_k \\ \xi_{i,j} & \mapsto & [\alpha_{i,j}] \, . \end{array}$$

Taking the further condition $\xi_{i,k+1} \mapsto [\alpha_{i,k+1}]$, we get a map $P\beta_{k+1} \to \pi_1 \mathcal{C}_{k+1}$ that makes the diagram commute, thus by the five-lemma, it is an isomorphism, proving that

$$\pi_1 \mathcal{C}_k = P\beta_k.$$

Now consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P\beta_{k+1} & \longrightarrow & \beta_{k+1} & \longrightarrow & S_{k+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_1\,\mathcal{C}_{k+1} & \longrightarrow & \pi_1\,{\mathcal{C}_{k+1}}' & \longrightarrow & S_{k+1} & \longrightarrow & 0
\end{array}$$

Again, the five-lemma, and the fact that $\pi_1\,\mathcal{C}_{k+1} = P\beta_{k+1}$ tell us that $\pi_1\,{\mathcal{C}_{k+1}}' = \beta_{k+1}$, proving that

$$\pi_1\,{\mathcal{C}_k}' = \beta_k.$$

Finally, we have shown that $\mathcal{F}_k$ and $\mathcal{C}_k$ are aspherical CW-complexes with the same fundamental group $P\beta_k$, and thus by theorem 2.1.3, we find that

$$\mathcal{F}_k \simeq \mathcal{C}_k\,.$$

One can also show, much in the same manner, that ${\mathcal{F}_k}'$ and ${\mathcal{C}_k}'$ are aspherical CW-complexes, and thus since they have the same fundamental group β_k, these spaces are also homotopy-equivalent.

Chapter 3

An explicit homotopy-equivalence

In section 2, we showed that the $\mathcal{C}_k$ was homotopy-equivalent to $\mathcal{F}_k$, without giving an explicit homotopy-equivalence, or even much of an intuition as to why this should be, other than the vague idea that it involved mapping a complex coordinate of a configuration to a lobe in a cactus.
In this section, we will first give an intuitive reason why the spaces of cacti and configuration spaces should be homotopy-equivalent, and provide an explicit map

$$\Phi : \mathcal{F}_k \to \mathcal{C}_k,$$

which we will show to be a homotopy-equivalence.

3.1 Definition of the map Φ

We return to configuration spaces, this time seen from a physicist's viewpoint. We will consider an element of the k-th configuration space as a physical configuration consisting of k charged particles lying in the plane, each creating a radial vector field whose norm is inversely proportional to the distance from the particle.
Consider a system with k particles lying in the plane, each with its own contribution to the total vector field. If the i-th charge is located at the complex point z_i, then its contribution to the total field at any point z is $\frac{1}{\overline{z-z_i}}$, and so the total field is

$$V(z) = \sum_{i=1}^{k} \frac{1}{\overline{z - z_i}} = \overline{\left(\frac{p'(z)}{p(z)}\right)},$$

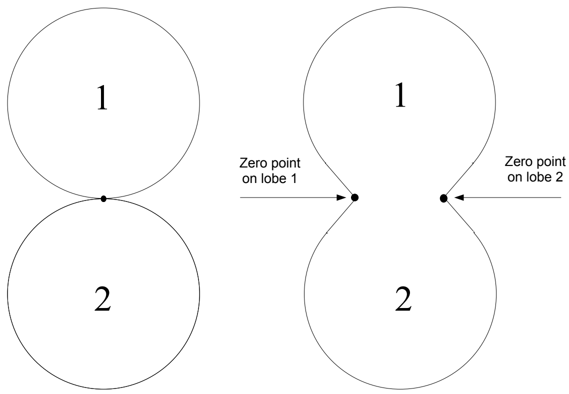
where

$$p(z) := (z - z_1) \dots (z - z_k).$$

One immediately notices that V, seen as a complex function, is antiholomorphic. Furthermore, the leading coefficient of p' is k, so the degree of p' is indeed $k-1$. Hence, there are a finite number of points, at least one and at most $k-1$ when allowing for multiple roots, where the total vector field vanishes.
Now, we consider the flow lines γ associated with the total vector field V, which satisfy

$$\gamma'(t) = V(\gamma(t)).$$

As before, we can represent this configuration as a graph in which the particles are represented by circles, but rather than drawing the circles as disjoint, we join two or more circles whenever there is a vanishing point of the vector field, from which flow lines lead from the particles represented by these circles.
These graphs - which indeed resemble cacti - make it easier for us to visualize configuration spaces. To see this, consider the case $k = 2$. We have two particles, and at the midpoint of their segment, the vector field vanishes, and there are two flow lines, each linking the vanishing point to one of the particles. Thus, up to choice of a circle and a point on the circle, the space $\mathcal{F}_2$ has but one element, which is the configuration we represent by two circles joined at one point, forming a figure eight. The zero point can be chosen anywhere along the perimeter of the figure eight, including at the join between the two circles. If the zero point is taken there, however, it must lie on only one of the two circles, as illustrated below:



Hence we reconfirm that the configuration space $\mathcal{F}_2$ is indeed homotopy-equivalent to S^1, as we showed in the proof of proposition 2.2.2.
Our first task will be to describe the flow lines of the vector field geometrically,

using the canonical identification of the complex plane with real euclidean 2-space. To get an intuitive sense of how to do this, we will begin with the simple case $k = 1$. In this case, we have one charge, located at the origin of the complex plane without loss of generality. The vector field is given by

$$V(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) = \bigtriangledown u(x, y),$$

where, for the complex variable $z = x + iy$, we have

$$u(x, y) = \frac{1}{2}\log(x^2 + y^2) = \log|z|.$$

It is a well-known fact (see [Lang, VIII,§1]) that if u and v are the real and imaginary parts of a holomorphic function $f = u + iv$, then the flow lines are level curves of v. Thus we need to find an appropriate v making f holomorphic. Consider the function

$$v(x, y) = \arctan\frac{y}{x} = \arg z$$

defined locally, which would make f the complex logarithm, which is known to be holomorphic. Therefore, each flow line can be seen as the locus of points with a given argument, as shown below:

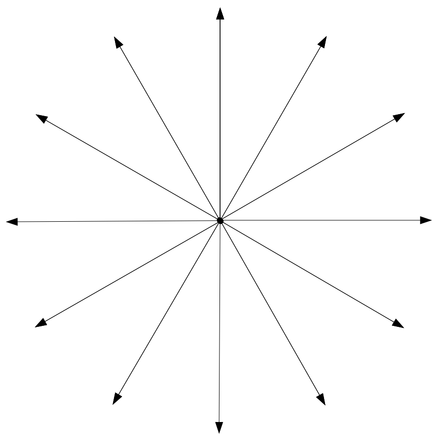


Figure 3.1: Flow lines for one charge

Since the vector field, the gradient operator, and holomorphic functions are additive, this is easily generalized to an arbitrary number of charges. Thus, for k charges $z_1, \ldots, z_k$, each flow line is a subspace of the locus

$$\left\{ z \in \mathbb{C} \mid \sum_{j=1}^{k} \arg(z - z_j) \equiv c \pmod{2\pi} \right\},$$

for some real number c. Furthermore, the total potential is

$$u(z) = \sum_{j=1}^{k} \log |z - z_j| = \mathrm{Re}\ \left(\log \prod_{j=1}^{k} (z - z_j) \right) = \mathrm{Re}\ \left(\log p(z) \right).$$

As we recall from earlier, the vector field in a point z in the complex plane is given by

$$V(z) = \overline{\left(\frac{p'(z)}{p(z)} \right)},$$

so that the vector field only vanishes in at most $k - 1$ points. For each of these points w, we would like to know how the flow lines associated with the vector field behave, specifically, whether or not they lead directly to or from w. Consider the multiplicity n of w as a root of the polynomial p'. Then in a small neighborhood of w, the vector field may be written

$$V(z) = \overline{z - w}^n f(z),$$

where $f(w) \neq 0$.

A simple way to describe flow lines leading to or from a vanishing point is to say that a flow line leading from, respectively towards, a vanishing point w can be characterized by the vector field being equal to the translated identity, respectively its opposite, in a small neighborhood of w. Formally, ignoring the nonzero factor $f(w)$, flow lines satisfy the equations

$$\begin{aligned} z - w &= e^{ix} \\ \overline{z - w}^n &= e^{-nix} \\ \overline{z - w}^n &= \pm z - w \end{aligned}$$

This equation has $n + 1$ solutions, corresponding to $n + 1$ incoming and outgoing flow lines near a vanishing point of multiplicity n. In particular, each vanishing point has at least two incoing flow lines. Given a vanishing point and an incoming flow line, any outgoing flow line can form a continuation of the incoming flow line.

We thus have a reason to consider larger subspaces of a locus

$$\left\{ z \in \mathbb{C} \mid \sum_{j=1}^{k} \arg(z - z_j) \equiv c \pmod{2\pi} \right\}$$

consisting of the union of flow lines that converge to, or from, a vanishing point not on the flow line.
This will give us a first step in the direction of finding a map

$$\mathcal{F}_k \to \mathcal{C}_k \, .$$

Given a configuration $(z_1, \ldots, z_k) \in \mathcal{F}_k$, we have the pattern of flow lines described earlier. We then consider the quotient space

$$\mathbb{C} - \{z_1 \ldots, z_k\} / \sim,$$

where

$$x \sim y \quad \Leftrightarrow \quad \begin{array}{l} x \text{ and } y \text{ lie on the same flow line} \\ \text{or} \\ x \text{ is at the limit of a flow line containing } y. \end{array} \tag{3.1}$$

At first glance, this quotient space should be a cactus, and we will show that in fact it is. To give an intuition as to why this should be, consider the case $k = 2$. In figure 3.2, we show on the right the configuration with flow lines, where the charges are represented by dots and the vanishing point by a cross, along with the "circle at infinity" in dotted lines. To fix things, we will say that the vanishing point is at the origin, and the charges are at points $\pm i$. Then all points in the upper half plane are linked by a flow line to a point on a circle surrounding the upper charge, and all points in the lower half plane are linked by a flow line to a point on a circle surrounding the lower charge. But these circles cannot be disjoint, because all points on the real line or on the open segment $(-i, i)$ are linked via flow lines both to points on circles surrounding both the upper and lower charges, so the two circles must share one point. So the quotient space described earlier looks like a cactus with two lobes.

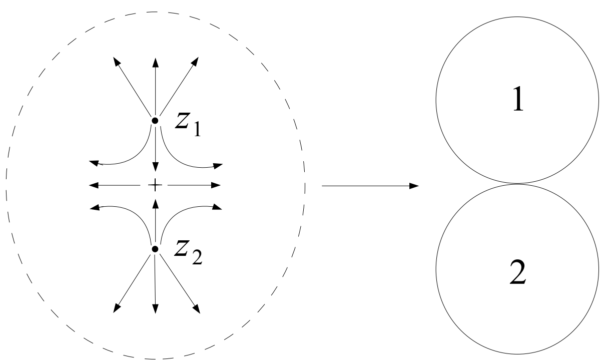


Figure 3.2: A configuration and its associated cactus

One can also see what we have defined as a graph. To form a graph from a configuration, we represent each charge by a labelled white vertex, and each vanishing point by a black vertex. We then form a connected graph by drawing in an edge between each pair of vertices such that there is a flow line between the two vanishing points or between the charge and the vanishing point represented by the pair of vertices, as shown in the example below:

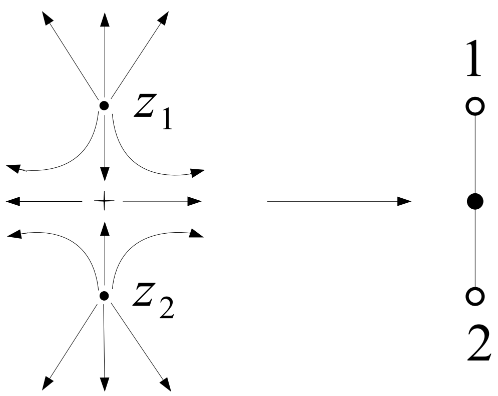


Figure 3.3: A configuration and its associated graph

From such a graph, one can in turn get a layout of circles in the plane, replacing each white vertex by a circle with the appropriate label, and each black vertex linked by an edge to several white vertices by an intersection point between the corresponding circles. The advantage of looking at such a graph is that the layout of circles forms a necklace if and only if the graph has a cycle.
We have set the grounds for believing that what we have done indeed yields a cactus, and so we must begin the technical work of showing that this is so. To do this, we recall the key points of the definition of a cactus, given on page 18:

1. Choice of a zero point and a lobe on which it lies
2. Constant speed parametrization of the outer circle
3. The no-necklace condition

For the first and second points, it is useful to remember that every flow line passes through exactly one point on the circle at infinity, and conversely, every point at infinity lies on exactly one flow line. Since we are identifying all points lying on one flow line, it follows that to understand the outer circle, it makes sense to take a closer look at the circle at infinity. We consider the standard parametrization of the circle at infinity consisting of fixing $r = \infty$ and taking

$$re^{i\theta}, \theta \in [0, 2\pi).$$

So, choosing a zero point on the outer circle is equivalent to choosing a point on the circle at infinity, which, in turn, is equivalent to choosing a value of $\theta \in [0, 2\pi)$. Now, fix such a value. If the flow line passing through this point at infinity leads directly from a charge, then this choice of θ leaves no ambiguity as to which lobe the point on the associated cactus lies on. If, however, the flow line leads to a vanishing point, then in the vicinity of this vanishing point, there may be more than one possible continuation of the line, each of which leads back to a different charge. Then to get one flow line, we must choose only one continuation of the flow line, which of course corresponds to a choice of one lobe on which the zero point lies.
The flow line containing the point at infinity corresponding to the angle θ, which we will denote by θ_∞, is a connected component of a locus

$$\left\{ z \in \mathbb{C} \mid \sum_{j=1}^{k} \arg(z - z_j) = \theta \right\}.$$

Going around the circle at infinity, the term $\sum_{j=1}^{k} \arg(z - z_j)$ becomes simply $k \arg z$, and so we have

$$\arg z = \frac{\theta}{k}, \theta \in [0, 2\pi).$$

This means that from the standard parametrization of the circle at infinity, we get a parametrization of the outer circle of the cactus at constant speed.
To verify the no-necklace condition, we need to show that the associated graph of any configuration must be a tree. This is not difficult to do. Let us denote the numbers of black vertices, white vertices, and edges by b, w and e, respectively. Recall that $w = k$, as there are k charges. Because flow lines lead away from charges and not towards them, all the edges in the graph represent flow lines leading towards a vanishing point, thus, all the edges are accounted for by taking all the incoming flow lines to a given vanishing point, and then repeating for all vanishing points. As we have seen, for a vanishing point of degree n, there are $n + 1$ incoming flow lines to this vanishing point. Also, the sum of the n over all the vanishing points equals $k - 1$, the degree of p'. Therefore we have

$$k - 1 = \sum_{i=1}^{b} e_i - 1,$$

where e_i is the number of edges connected to the black vertex i. Finally,

$$k - 1 = w - 1 = \sum_{i=1}^{b} e_i - 1 = e - b,$$

which implies

$$w + b - e = 1,$$

and so the graph is indeed a tree, and the configuration indeed yields a cactus. Hence, we have defined a map

$$\begin{array}{rcl} \Phi : \mathcal{F}_k & \to & \mathcal{C}_k \\ (z_1, \ldots, z_k) & \mapsto & \mathbb{C} - \{z_1, \ldots, z_k\} / \sim, \end{array}$$

where $\sim$ is the equivalence relation defined in (3.1). More formally, for a configuration $(z_1, \ldots, z_k) \in \mathcal{F}_k$, and an index $j = 1, \ldots, k$, we define

$$I_j\left(\Phi(z_1, \ldots, z_k)\right) := \{\theta \in [0, 2\pi) \mid z'_j \sim \theta_\infty\} \tag{3.2}$$

for some z'_j close to z_j.

By setting the zero point as the class of the point 0_∞ at infinity, this map becomes well-defined, and also gives a motivation for the name "zero point". In summary, we have found a surjective map

$$\Phi : \mathcal{F}_k \to \mathcal{C}_k$$

between homotopy-equivalent spaces, which we have claimed to be a homotopy-equivalence. As things stand, we have a long way to go before proving this, indeed, we have not even shown that Φ is continuous. We will bolster the claim that Φ is a homotopy-equivalence, by providing a map s in the other direction that should serve as a section and homotopy-inverse to Φ. In fact, we will show that s can even be seen as an embedding, whereby one can find a way of creating a configuration from a cactus, augmented by real numbers. Before we do this, however, let us take a step back. Indeed, there are some substantial difficulties in directly defining s. Fortunately, these difficulties can be circumvented by considering orbit spaces of free $\mathbb{Z}_k$-actions on both $\mathcal{F}_k$, by rotation of the whole configuration by an angle of $\frac{2\pi}{k}$, and $\mathcal{C}_k$, by rotating the zero point around the outer circle by the same angle. We must thus look at Φ "downstairs," at

$$\widetilde{\Phi} : \mathcal{F}_k \,/\, \mathbb{Z}_k \to \mathcal{C}_k \,/\, \mathbb{Z}_k,$$

which is well-defined. If we can find a homotopy-inverse $\widetilde{s}$ to $\widetilde{\Phi}$, then we can lift this using the unique lifting property of covering spaces, and find a homotopy-inverse s of Φ. Because the actions of $\mathbb{Z}_k$ on $\mathcal{F}_k$ and $\mathcal{C}_k$ are free, it follows that the canonical projections

$$\pi_{\mathcal{F}} : \mathcal{F}_k \to \mathcal{F}_k \,/\, \mathbb{Z}_k$$

and

$$\pi_{\mathcal{C}} : \mathcal{C}_k \to \mathcal{C}_k \,/\, \mathbb{Z}_k$$

are indeed $\mathbb{Z}_k$-coverings, so we are allowed to do this.

$$\begin{array}{ccc} \mathcal{F}_k & \underset{s}{\overset{\Phi}{\rightleftarrows}} & \mathcal{C}_k \\ {\scriptstyle \pi_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{C}}} \\ \mathcal{F}_k \,/\, \mathbb{Z}_k & \underset{\widetilde{s}}{\overset{\widetilde{\Phi}}{\rightleftarrows}} & \mathcal{C}_k \,/\, \mathbb{Z}_k \end{array}$$

Our strategy will be to make life easier by homeomorphically identifying $\mathcal{F}_k \,/\, \mathbb{Z}_k$ and $\mathcal{C}_k \,/\, \mathbb{Z}_k$ with spaces $\mathscr{F}_k$ and $\mathscr{C}_k$, respectively, that are easier to codify mathematically. If we call these homeomorphisms $h_{\mathscr{F}}$ and $h_{\mathscr{C}}$, respectively, we should then have the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_k & \underset{s}{\overset{\Phi}{\rightleftarrows}} & \mathcal{C}_k \\ {\scriptstyle \pi_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{C}}} \\ \mathcal{F}_k \,/\, \mathbb{Z}_k & \underset{\widetilde{s}}{\overset{\widetilde{\Phi}}{\rightleftarrows}} & \mathcal{C}_k \,/\, \mathbb{Z}_k \\ {\scriptstyle h_{\mathscr{F}}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle h_{\mathscr{C}}} \\ \mathscr{F}_k & \underset{\overline{s}}{\overset{\overline{\Phi}}{\rightleftarrows}} & \mathscr{C}_k\,. \end{array}$$

If $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{C}}$ are coverings and $h_{\mathscr{F}}$ and $h_{\mathscr{C}}$ are homeomorphisms, then it suffices to find a homotopy-inverse $\overline{s}$ to $\overline{\Phi}$, which we can then lift to Φ and s. This lifting is also unique, provided that basepoints in both spaces are fixed. For instance, we could fix the basepoint

$$(1, 2 \ldots, k) \in \mathcal{F}_k,$$

and the corresponding basepoint

$$\Phi(1, 2, \ldots, k) \in \mathcal{C}_k,$$

shown in the figure below:

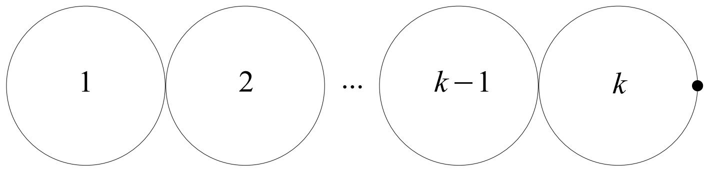


Figure 3.4: The cactus $\Phi(1, 2, \ldots, k)$

With this, the lift of $\overline{\Phi}$ is unique, and equal to the Φ we defined earlier.

3.2 An encoding of $\mathcal{F}_k \,/\, \mathbb{Z}_k$

Let us first talk about $\mathcal{F}_k{}'$, $\mathcal{F}_k{}' \,/\, \mathbb{Z}_k$, and a canonical identification between these spaces and certain spaces of polynomials. The elements of $\mathcal{F}_k{}' \times \mathbb{C}$ (the extra coordinate accounts for possible translations) are unordered k-tuples of distinct points in $\mathbb{C}$. Seeing these points as the roots of polynomials, if we define the space $\mathcal{P}_n$ of polynomials of degree n with no multiple roots, we can homeomorphically identify $\mathcal{F}_k{}' \times \mathbb{C}$ with $\mathcal{P}_k$ via

$$(z_1, \ldots, z_k) \mapsto (z - z_1) \ldots (z - z_k)$$

by the well-known result, which is a consequence of the fundamental theorem of algebra, that the symmetric product

$$SP^n(\mathbb{C}) := \mathbb{C}^n \,/ S_n$$

is homeomorphic to $\mathbb{C}^n$, which can also be interpreted as the bicontinuity of roots of polynomials with respect to their coefficients. For the rest of this paper, this result will be referred to as "root-coefficient bicontinuity."
It will be interesting, for reasons that will become apparent presently, to consider spaces of polynomials modulo *right affine equivalence*

$$p(z) \sim p(az + b), a \neq 0.$$

By quotienting out translations $z \mapsto z + b$, we get

$$\mathcal{P}_k \,/\, \mathbb{C} \cong \mathcal{F}_k{}'.$$

Now consider transformations $z \mapsto az$ for $a \neq 0$. Applying this transformation to the polynomial $(z - z_1) \ldots (z - z_k)$ gives

$$(az - z_1) \ldots (az - z_k) = a^k \left(z - \frac{z_1}{a}\right) \ldots \left(z - \frac{z_k}{a}\right).$$

Since we can take $a^k = 1$ without loss of generality, by quotienting these out, we get that
$$(z - z_1) \dots (z - z_k) \sim (z - \xi z_1) \dots (z - \xi z_k),$$
where ξ is a primitive k-th root of unity, and so a generator of $\mathbb{Z}_k$. So we have a canonical isomorphism
$$\mathcal{P}_k \,/\, \mathbb{A}_2 \cong \mathcal{F}_k' \,/\, \mathbb{Z}_k \,.$$
Extending this argument to the space $\mathrm{o}\mathcal{P}_k$ of polynomials in $\mathcal{P}_k$ with ordered roots (it is a principal S_k-covering of $\mathcal{P}_k$), we get a homeomorphic identification
$$\mathrm{o}\mathcal{P}_k \,/\, \mathbb{A}_2 \cong \mathcal{F}_k \,/\, \mathbb{Z}_k$$
via
$$[(z - z_1) \dots (z - z_k)] \leftrightarrow \pi_{\mathcal{F}}(z_1, \dots, z_k),$$
where we take the right affine equivalence class of the polynomial.
Let us in turn identify the space $\mathrm{o}\mathcal{P}_k \,/\, \mathbb{A}_2$ with something else. It is worth remembering that $S^2 = \mathbb{C} \cup \{\infty\}$, so every complex polynomial p can also be seen as a holomorphic self-map
$$p : S^2 \to S^2,$$
sending ∞ to ∞. We consider the *regular* and *critical* values and points of a polynomial.

Definition 3.2.1. Let p be a complex polynomial. Then $\zeta \in \mathbb{C}$ is

- a *critical point* of p if $p'(\zeta) = 0$
- a *regular point* of p if $p'(\zeta) \neq 0$

If ζ is a critical point, then $p(\zeta)$ is a *critical value* and all other values are *regular values.*

Remark. It is immediate that a polynomial of degree k can have at most $k-1$ critical points.

So if w is a critical value, the equation $p(z) = w$ has at most $k-1$ solutions, at least one of which will be a critical point. We denote these critical points by $\zeta_1, \dots \zeta_m$. We denote the multiplicity of the root ζ_i in the polynomial $p(z) - w$ by $e_p(\zeta_i)$, and we call this integer the *ramification index* of ζ_i.

Lemma 3.2.2. *Let p be a complex polynomial of degree k and $\zeta \in \mathbb{C}$. Then ζ is a regular point if and only if*
$$e_p(\zeta) = 1.$$

The space $o\mathcal{P}_k$ is the space of complex polynomials (with an ordering of the roots) of degree k such that 0 is a regular value. Given such a polynomial p, consider the finite subset B of S^2 consisting of $B = \{\infty, w_1, \dots, w_l\}$, where $w_1, \dots, w_l$ are the critical values of p. As p is a covering space, we consider the fiber over 0, which is a set of k points, indeed, they are the points $\{z_1, \dots, z_k\}$ in our configuration. Then, the group

$$\pi_1(S^2 - B, 0) = \pi_1(\mathbb{C} - \{w_1, \dots, w_l\}, 0) \cong F_l$$

acts on $\{z_1, \dots, z_k\}$ by monodromy, that is to say, we have a homomorphism $\mu_p : F_l \to S_k$, where F_l is the free group on l generators, each generator λ_j for $j = 1, \dots, l$ being a small loop based at 0 around w_j that is the boundary of a topological disk containing no other critical values.
Furthermore, this action is transitive. To see this, we return to the world of configurations, where $z_1, \dots, z_k$ represent charges, while $w_1, \dots, w_l$ represent vanishing points of the vector field given by

$$V(z) = \overline{\left(\frac{p'(z)}{p(z)}\right)}.$$

Any root is connected to a vanishing point by a flow line. If we isolate this vanishing point, and look at the monodromy generated by a loop around it in $\mathbb{C} - \{w_1, \dots, w_l\}$, it permutes all the roots connected to this vanishing point in a counterclockwise manner, preserving the cyclic ordering.
This correspondence is more explicitly given by a version of Riemann's existence theorem (see [App]) which we give below.

Theorem 3.2.3 (Riemann's existence theorem)**.** *Let $B \subset S^2$ be finite. Then there is a natural bijection between the following two sets:*

1. *Isomorphism classes of holomorphic mappings $f : \Sigma \to S^2$ of degree k of a compact connected Riemann surface Σ such that B is the set of critical values of f*
2. *Conjugacy classes of homomorphisms $\mu : \pi_1(S^2 - B) \to S_k$ such that the image of μ is transitive*

where

- $f : \Sigma \to S^2$ *and* $f' : \Sigma' \to S^2$ *are* isomorphic *if there exists a biholomorphic mapping $h : \Sigma \to \Sigma'$ such that $f' = f \circ h$*
- $\mu, \mu' : \pi_1(S^2 - B) \to S_k$ *are* conjugated *if there exists some $\tau \in S_k$ such that $\mu' = \tau\mu\tau^{-1}$*

Furthermore, if $b \in B$ and γ_b is a loop around b and $\mu([\gamma_b]) = (m_1)\dots(m_r)$, where the m_i are integers, and the (m_i) are pairwise disjoint m_i-cycles, then $p^{-1}(b) = \{u_1, \dots, u_r\}$, where

$$e_f(u_i) = m_i.$$

Consider the special case where $\Sigma = S^2$ and f is our polynomial p of degree k, with $B = \{\infty, w_1 \dots, w_l\}$, where $w_1, \dots w_l$ are the critical values of p. The first thing to notice is that biholomorphic self-maps of S^2 are precisely the Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}.$$

If we also require, as we must, since p is a polynomial, that ∞ be sent to ∞, then the biholomorphic self-maps of S^2 that preserve ∞ are precisely the affine transformations

$$z \mapsto az + b.$$

Theorem 3.2.4 (Riemann-Hurwitz formula)**.** *Let $f : \Sigma \to \Sigma'$ be a holomorphic mapping of degree k between Riemann surfaces Σ and Σ'. Then*

$$\chi(\Sigma) = k\chi(\Sigma') - \sum_{s \in \Sigma} (e_f(s) - 1).$$

By a generalization of lemma 3.2.2, only a finite number of terms in the sum are nonzero, so this is well-defined. Recall our special case, where we had a complex polynomial p, which extended to a holomorphic self-map of S^2 by sending ∞ to ∞. The critical points, as might be expected, are precisely the critical points of the polynomial p, and ∞. That ∞ is a critical point can be easily explained by the fact that close to infinity, the polynomial p behaves like its highest-degree monomial, which is z^k. Given an open chart U_∞ of S^2 containing ∞, we restrain p to $U_\infty - \{\infty\}$, which factors into a self-map of S^1, as $U_\infty - \{\infty\}$ contains a circle as a deformation retract. Locally, this self-map behaves like $z \mapsto z^k$, meaning that

$$e_p(\infty) = k.$$

So in this case, the Riemann-Hurwitz formula becomes

$$2 = 2k - (e_p(\infty) - 1) - \sum_{j=1}^{l} \sum_{i=1}^{r_j} (e_p(u_{j,i}) - 1),$$

where

$$p^{-1}(w_j) = \{u_{j,1}, \dots, u_{j,r_j}\}.$$

Since $e_p(\infty) = k$, we then get

$$k - 1 = \sum_{j=1}^{l} \sum_{i=1}^{r_j} (e_p(u_{j,i}) - 1) .$$

What this means for us is that one can identify $\mathrm{o}\mathcal{P}_k \,/\, \mathbb{A}_2$ with a set of group homomorphisms from a free group to S_k, with some data added (the values of the elements of B). This space has the advantage that it can be encoded fairly simply. A homomorphism $\mu : F_l \to S_k$ is given by a choice of l permutations $\sigma_1, \ldots, \sigma_l \in S_k$, for $\sigma_j = \mu(\lambda_j)$, where λ_j is the j-th generator of the free group, the homotopy class of a loop based in 0 that "runs around" w_j. The transitivity condition can also be easily identified to the requirement that

$$\sigma_l \ldots \sigma_1 \text{ be a } k - \text{cycle.} \tag{3.3}$$

The condition given by the Riemann-Hurwitz formula can be expressed as

$$\sum_{j=1}^{l} \sum_{i=1}^{r_j} (m_{j,i} - 1) = k - 1, \tag{3.4}$$

where σ_j is a pairwise disjoint product of an $m_{j,1}$-cycle, an $m_{j,2}$-cycle, and so forth up to an m_{j,r_j}-cycle.

In light of the implicit ordering of the tuples of permutations expressed in (3.3), it is also necessary, because S_k is not abelian, to give a canonical ordering of the $w_1, \ldots, w_l$, which will correspond to the ordering of the tuples of permutations. It is natural to order them by increasing argument, however this is not yet canonical. To make it so, we must also order by increasing modulus those numbers with the same argument. In other words, we need to order the coordinates of $(w_1, \ldots, w_l)$ so that

$$\arg(w_i) \leq \arg(w_{i+1}) \text{ and if } \arg(w_i) = \arg(w_{i+1}), \text{ then } |w_i| < |w_{i+1}|, \tag{3.5}$$

$$w_i \neq w_j \text{ for } i \neq j. \tag{3.6}$$

We will denote the space of l-tuples of complex numbers without repetition ordered in the way we just described by

$$\mathcal{O}rd_l(\mathbb{C}^*) := \left\{ (w_1, \ldots, w_l) \in (\mathbb{C}^*)^l \mid (3.5), (3.6) \right\},$$

with the euclidean topology.

Since each number w_j corresponds to a permutation σ_j, these must be ordered as well as satisfying (3.3) and (3.4). We will denote the space of ordered l-tuples of permutations satisfying (3.3) and (3.4) by

$$\mathcal{C}omp_l^k := \left\{(\sigma_1, \ldots, \sigma_l) \in (S_k)^l \mid (3.3), (3.4)\right\},$$

with the discrete topology.
Hence, let us consider the space

$${\mathscr{F}_k}^l := \mathcal{C}omp_l^k \times \overline{\mathcal{O}rd_l(\mathbb{C}^*)}, \mathscr{F}_k := \coprod_{l=1}^{k-1} {\mathscr{F}_k}^l \,/\, \sim_{\mathscr{F}},$$

with the product topology, the coproduct topology, and then the quotient topology.
We define the equivalence relation $\sim_{\mathscr{F}}$ by

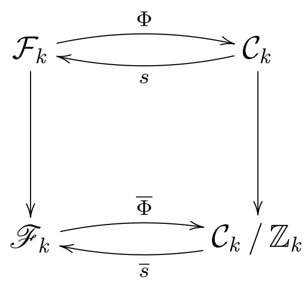
$$\begin{gathered}((\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_m, w_{m+2}, \ldots, w_l)) \\ \wr_{\mathscr{F}} \\ ((\sigma_1, \ldots, \sigma_{m+1}\sigma_m, \sigma_{m+2}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_{m+2} \ldots, w_l))\end{gathered} \tag{3.7}$$

and

$$\begin{gathered}((\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_{m+1}, \ldots, w_l)) \\ \wr_{\mathscr{F}} \\ \big((\sigma_1, \ldots, \sigma_{m+1}, \sigma_{m+1}\sigma_m\sigma_{m+1}^{-1}, \ldots, \sigma_l), (w_1, \ldots, w_{m+1}, w_m \ldots, w_l)\big)\end{gathered} \tag{3.8}$$

We will show in section 3.5 that this definition $\sim_{\mathscr{F}}$ will be defined such that we get a homeomorphism

$$\mathcal{F}_k \,/\, \mathbb{Z}_k \cong \mathscr{F}_k \,.$$



That we have decided to take the closure of $\mathcal{O}rd_l(\mathbb{C}^*)$ deserves an explanation. Indeed, it is necessary to do so for reasons that will be apparent soon.
The key concept here is that of the order of the complex numbers in the vector. As elements of this space are moved around slightly (we need to do this for reasons of continuity), their canonical ordering could change. There are two

ways this can happen. It could happen that a value with a larger argument than a collection of other values decreases in argument to end up on the same ray as this collection. If there is a value in this collection with a larger modulus than the value whose argument we have decreased, then a reordering would be necessary to remain in $\mathcal{O}rd_l(\mathbb{C}^*)$. We call this an "angular" reordering.
The second way values could be reordered is if of two values with same argument but different modulus, the one with the smaller modulus increases in modulus until it overtakes the other value. We call this a "radial" reordering.
Let us consider

$$(w_1, \ldots, w_l) \in \mathcal{O}rd_l(\mathbb{C}^*)$$

where $\arg(w_m) < \arg(w_{m+1})$, and take a sequence

$$\left\{w_{m+1}^r\right\}_{r\geq 1}$$

such that

$$w_{m+1}^1 = w_{m+1}, \arg\left(w_{m+1}^r\right) \downarrow \arg\left(w_m\right), |w_{m+1}^r| = |w_{m+1}|,$$

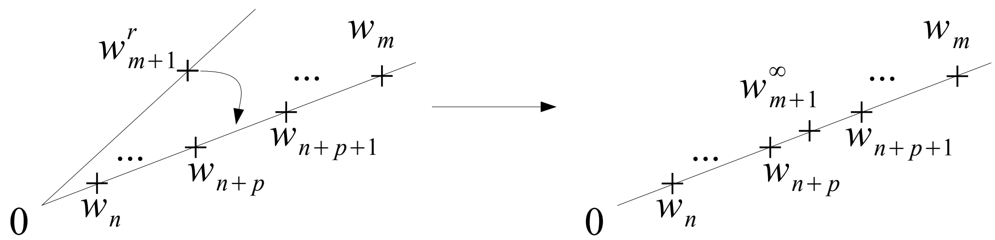
which converges to a limit w_{m+1}^∞. The induced sequence in $\mathcal{O}rd_l(\mathbb{C}^*)$,

$$\left\{\left(w_1, \ldots, w_m, w_{m+1}^r, w_{m+2}, \ldots, w_l\right)\right\}_{r\geq 1}$$

converges to

$$\left(w_1, \ldots, w_m, w_{m+1}^\infty, w_{m+2}, \ldots, w_l\right),$$

which lies not in $\mathcal{O}rd_l(\mathbb{C}^*)$, but in its closure $\overline{\mathcal{O}rd_l(\mathbb{C}^*)}$.



Thus, taking the closure eliminates the need to "angularly" reorder the coordinates in order to remain in $\mathcal{O}rd_l(\mathbb{C}^*)$, and in turn the need to account for the impact the reordering would have on the side of the data represented by elements of $\mathcal{C}omp_l^k$. We still need, however, to account for "radial" reordering, which we have done with the definition of $\sim_{\mathscr{F}}$.
We will, in an analogous way, define a space $\mathscr{C}_k$ to represent cacti in section 3.4.

3.3 A description of $\overline{\Phi}$

Consider

$$((\sigma_1, \ldots, \sigma_l), (w_1, \ldots, w_l)) \in \mathscr{F}_k .$$

We will denote by λ_j the loop whose monodromy is σ_j, as illustrated in figure 3.5.

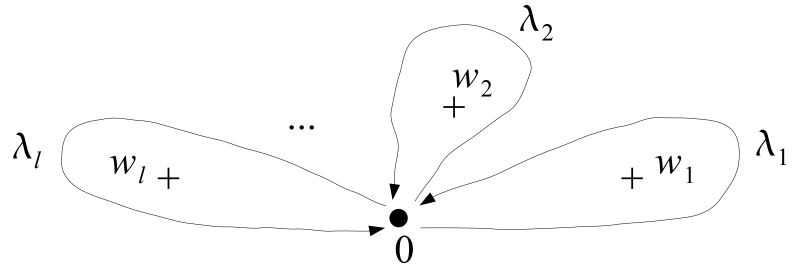


Figure 3.5: An illustration of our new configuration data

How can this data give us a cactus? Let us recall what Φ actually does. It takes a polynomial, and yields a cactus where the lobes correspond to roots, while the intersection points correspond to equivalence classes of critical points by the equivalence relation used to define Φ. Therefore, each of the w_j should represent one intersection point between lobes on a cactus, while its associated permutation σ_j, being the permutation of roots under monodromy, should list the lobes touching that intersection point, as well as giving a cyclic ordering for the lobes. The differences between the arguments of the w_j should give us angles on a lobe between intersection points.

Our first task will be to create a black-white tree that will be dual to our cactus. That the graph thus created is indeed a tree will follow from condition (3.4). As a reminder, each black vertex should represent one intersection point, while each white vertex should represent a lobe. Assume each σ_j has the decomposition

$$\sigma_j = (m_{j,1}) \ldots (m_{j,r_j}),$$

where the $m_{j,i}$ are integers and the $(m_{j,i})$ are pairwise disjoint $m_{j,i}$-cycles. Each of these $m_{j,i}$-cycles will represent a black vertex surrounded by $m_{j,i}$ vertices, each leading to a white vertex such that the cyclic ordering of the white vertices around the black vertex corresponds to that of the labels within the $m_{j,i}$-cycle. Since for a given j, there are r_j cycles, this means there are r_j black vertices, and so we have $\sum_{j=1}^{l} r_j$ black vertices. Naturally, there should be exactly k

white vertices. So we have a total of

$$V = k + \sum_{j=1}^{l} r_j$$

vertices in our graph. Now, for each index (j, i), we have $m_{j,i}$ edges, for a total of

$$E = \sum_{j=1}^{l} \sum_{i=1}^{r_j} m_{j,i}$$

edges. We can rewrite equation (3.4) to say

$$1 = k + \sum_{j=1}^{l} r_j - \sum_{j=1}^{l} \sum_{i=1}^{r_j} m_{j,i} = V - E,$$

and so the graph we created is indeed a tree, thus we will get an honest cactus. We still need to use the data $(w_1, \ldots, w_l)$ to assign $n-1$ angles to each white vertex that is surrounded by n black vertices. For now, if $w_j, \ldots, w_{j+m}$ have the same argument, we will consider the slightly different numbers $w'_j, \ldots, w'_{j+m}$, where for a small $\epsilon > 0$, we take

$$w'_{j+t} := w_{j+t} \cdot e^{i(t\epsilon)},$$

and will later set $\epsilon = 0$. Every black vertex corresponds to an $m_{j,i}$-cycle $(m_{j,i})$, which is a component of some σ_j. So if the black vertices corresponding to $(m_{j_1,i})$ and $(m_{j_2,i})$ are both connected to some white vertex, in other words, if the index of this white vertex appears in both cycles, then the angle we put on the graph, going from the black vertex associated with $(m_{j_1,i})$ to the one associated with $(m_{j_2,i})$, will be $\arg(w_{j_2}) - \arg(w_{j_1})$. We adopt the convention that if we draw a circle surrounding the graph, adding a positive angle is such that its projection of this new angle onto the circle moves in a counterclockwise direction.
We will have alongside the rays $w_1, \ldots, w_l$, the ray given by the positive real axis. We have by now established the correspondence between travelling counterclockwise around the origin and encountering rays, and travelling around the outer circle of a cactus and encountering intersection points. It would be appropriate to consider a similar correspondence between the positive real axis and the zero point. However, a full turn around $\mathbb{C}^*$ corresponds to a full turn on one lobe, in other words, a rotation of $\frac{2\pi}{k}$ around the outer circle. So starting at the positive real axis (the zero point on the cactus) and travelling around, we get back to the positive real axis "downstairs", but we arrive at

another zero point, distant from the first one by $\frac{2\pi}{k}$. So we have k candidates for the zero point, which indeed gives us an orbit in $\mathcal{C}_k \,/\, \mathbb{Z}_k$. One of these may be obtained by starting from the intersection point corresponding to w_1, and travelling in the clockwise direction around a lobe by $\arg(w_1)$.
It remains to show that we could have taken any ordering for those of the $w_1, \ldots, w_l$ that have the same argument. A "radial" reordering of the w_j would yield another ordering of the σ_j. As S_k is non-abelian, a different ordering of the σ_j might jeopardize the satisfaction of condition (3.3). Indeed, let us say we have ordered the w_j so that w_u comes "right before" w_v. Then $\sigma_l \ldots \sigma_v \sigma_u \ldots \sigma_1$ must be a k-cycle. Unfortunately, the fact that

$$\sigma_l \ldots \sigma_v \sigma_u \ldots \sigma_1$$

is a k-cycle does not imply that

$$\sigma_l \ldots \sigma_u \sigma_v \ldots \sigma_1$$

is a k-cycle too. Thus if we change the ordering so that w_v comes "right before" w_u, we must replace σ_u and σ_v with other permutations σ'_u and σ'_v such that

$$\sigma_l \ldots \sigma'_u \sigma'_v \ldots \sigma_1$$

is a k-cycle. If we take

$$\sigma'_v = \sigma_v, \sigma'_u = \sigma_v \sigma_u \sigma_v^{-1},$$

then we get that

$$\sigma_l \ldots \sigma'_u \sigma'_v \ldots \sigma_1 = \sigma_l \ldots \sigma_v \sigma_u \sigma_v^{-1} \sigma_v \ldots \sigma_1 = \sigma_l \ldots \sigma_v \sigma_u \ldots \sigma_1,$$

which is a k-cycle. So a new ordering will give us new data, but will it give us a new cactus? Fortunately for us, the answer is no, provided that the arguments of the complex numbers w_u and w_v that we want to switch around are identical. To prove this, consider the two sets of data

$$((\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_{m+1}, \ldots, w_l))$$

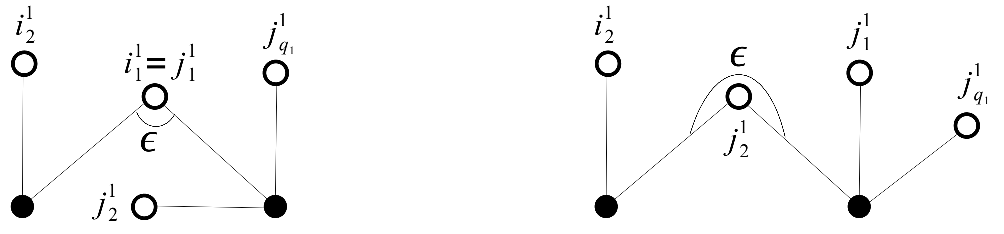
and

$$\left((\sigma_1, \ldots, \sigma_{m+1}, \sigma_{m+1} \sigma_m \sigma_{m+1}^{-1}, \ldots, \sigma_l), (w_1, \ldots, w_{m+1}, w_m, \ldots, w_l)\right).$$

It is immediate that if σ_m and σ_{m+1} are disjoint, then the cacti yielded by these data sets are identical. Let us write

$$\sigma_m = (i_1^1 \ldots i_{p_1}^1) \ldots (i_1^n \ldots i_{p_n}^n), \sigma_{m+1} = (j_1^1 \ldots j_{q_1}^1) \ldots (j_1^t \ldots j_{q_t}^t),$$

where we will assume without loss of generality that $i_1^1 = j_1^1$ and all other indices are not shared by σ_m and σ_{m+1}.
In the cacti generated by both these data sets, every black vertex not corresponding to $(i_1^1 \dots i_{p_1}^1)$ is surrounded by the same set of white vertices, with the same cyclic ordering. The set of white vertices in the second tree is merely the set of white vertices in the first, with the same cyclic ordering, except that we have permuted all the indices by σ_{m+1}, in this case, it just means we have replaced the index of the white vertex formerly known as i_1^1 by j_2^1. Since we have $\arg(w_m) = \arg(w_{m+1})$, the only part of the black-white tree that can change is the part in the following figure, where we collapse $\epsilon = 0$ in both cases.



old ordering *new ordering*

It is plain to see that in both cases, we get the same tree. Thus, the cactus really does not depend on the ordering we assign to those w_j that have the same argument.

Example 3.3.1. Consider the case $k = 8$ and the data given by the permutations

$$[(12), (23)(45), (36), (678), (58)],$$

and the nonzero complex numbers

$$\left(e^{i\alpha}, e^{i(\alpha+\beta)}, \frac{1}{2}e^{i(\alpha+\beta+\gamma)}, e^{i(\alpha+\beta+\gamma+\delta)}, 2e^{i(\alpha+\beta+\gamma+\delta)}\right),$$

as illustrated in figure 3.6.

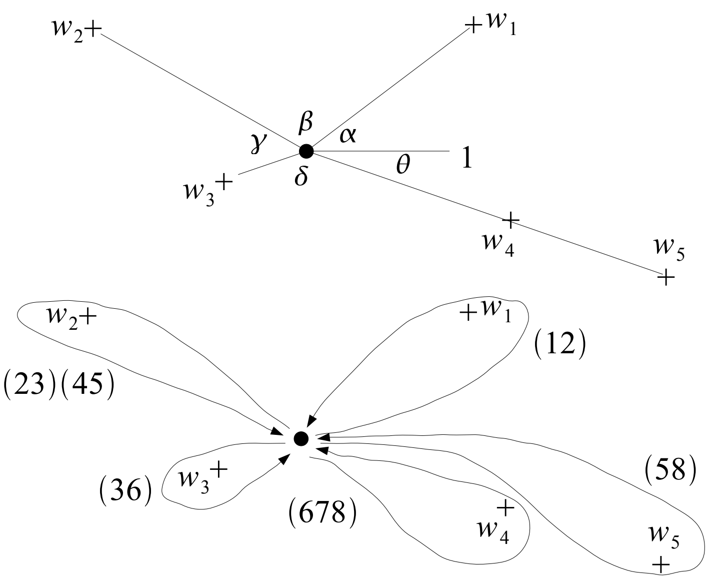


Figure 3.6: An example of configuration data

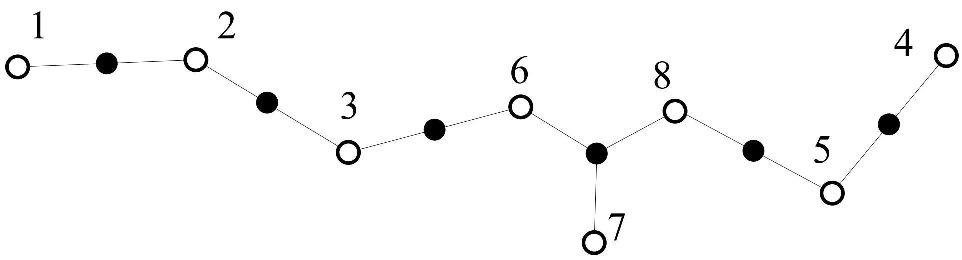
This data is acceptable, because

$$(58)(678)(36)(23)(45)(12) = (17548632),$$

which is an 8-cycle, thereby satisfying (3.3), and

$$\sum_{j=1}^{l}\sum_{i=1}^{r_j}(m_{j,i}-1) = (1)+(1+1)+(1)+(2)+(1) = 7 = 8-1 = k-1,$$

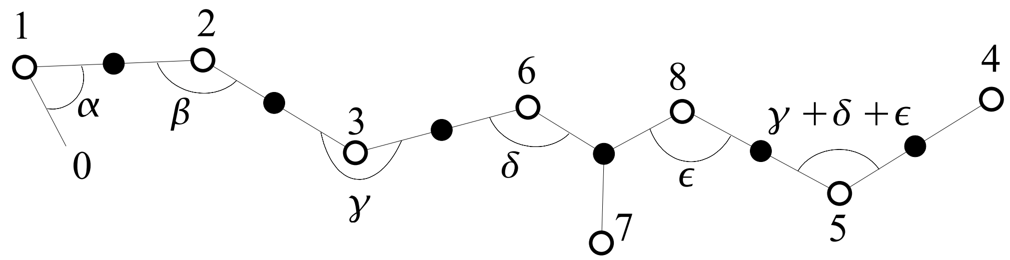
which satisfies (3.4). The black-white tree associated to the permutations is the following:



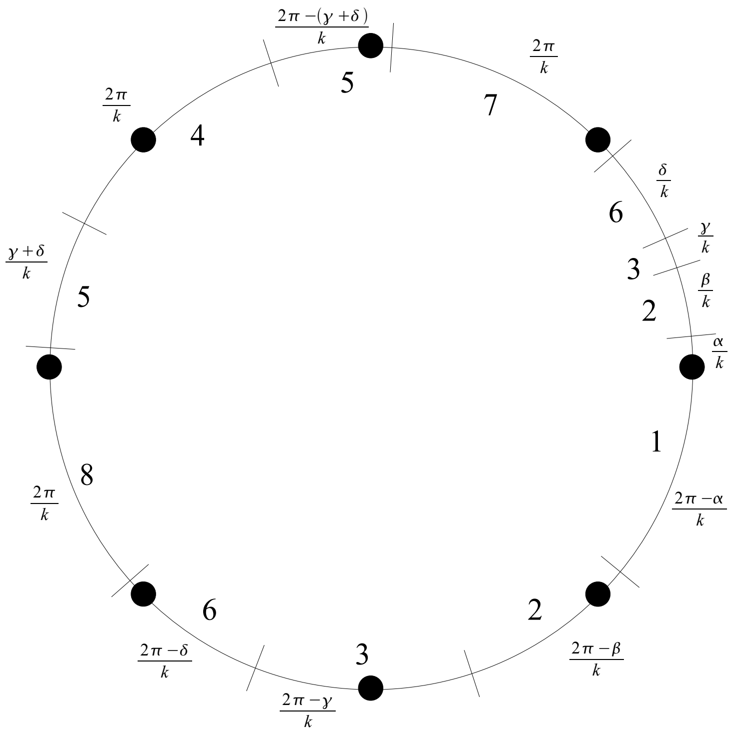
Now we add information about the angles. We take the numbers

$$\left(e^{i\alpha}, 2e^{i(\alpha+\beta)}, \frac{1}{2}e^{i(\alpha+\beta+\gamma)}, e^{i(\alpha+\beta+\gamma+\delta)}, 2e^{i(\alpha+\beta+\gamma+\delta+\epsilon)}\right).$$

With this, we get the black-white graph with angles,



which, when we set $\epsilon = 0$, yields the orbit



To confirm that reordering w_4 and w_5, which have the same argument, should yield the same orbit, consider the new data given by

$$[(12), (23)(45), (36), (58), (567)]$$

and

$$\left(e^{i\alpha}, 2e^{i(\alpha+\beta)}, \frac{1}{2}e^{i(\alpha+\beta+\gamma)}, 2e^{i(\alpha+\beta+\gamma+\delta)}, e^{i(\alpha+\beta+\gamma+\delta)}\right),$$

as illustrated in figure 3.7.

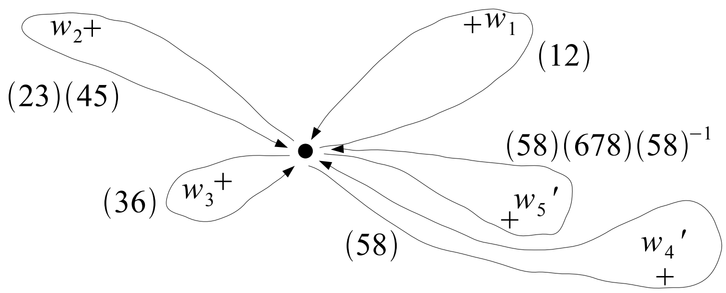
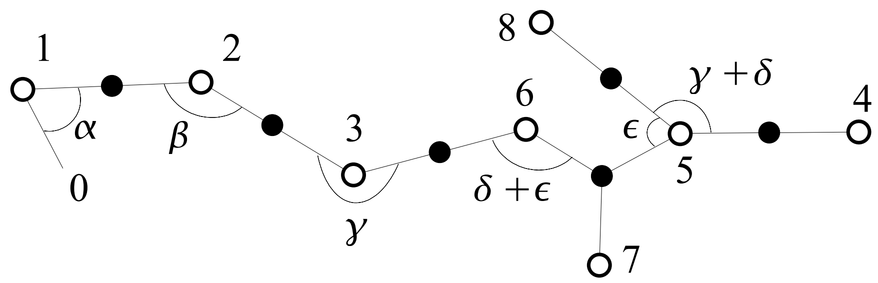


Figure 3.7: A different ordering

The data (where we have replaced $w_5' = e^{i(\alpha+\beta+\gamma+\delta)}$ with $e^{i(\alpha+\beta+\gamma+\delta+\epsilon)}$, to later set $\epsilon = 0$) give us the black-white graph



which, when we set $\epsilon = 0$, yields the same black-white tree and thus the same orbit as before.

It should be noted that in example 3.3.1, if we look at the lobe each of the potential zero points is located on, the cyclic ordering of these lobes corresponds to the 8-cycle (17548632). This is true in general. Indeed, turning once around 0 “downstairs” corresponds to going once around each loop in sequence, which corresponds to the monodromy action on the roots being precisely that of the k-cycle. It also represents a rotation of $\frac{2\pi}{k}$ around the outer circle. So if the k-cycle is $(i_1 i_2 \dots i_k)$, and if we consider one zero point on lobe i_1, then the next zero point, distant by exactly $\frac{2\pi}{k}$ on the outer circle, must be on lobe i_2. What we have done so far should give the reader a fairly good intuition as to what $\overline{\Phi}$ actually does. However, the encoding $\mathscr{F}_k$ is woefully dissimilar to $\mathcal{C}_k \,/\, \mathbb{Z}_k$. Recall that we wanted to express $\overline{\Phi}$ as a “forgetful” map, which we clearly cannot do until we find an encoding of $\mathcal{C}_k \,/\, \mathbb{Z}_k$ that is more compatible with $\mathscr{F}_k$, which we will now proceed to do.

3.4 An encoding of $\mathcal{C}_k \,/\, \mathbb{Z}_k$

As should be apparent from the work done in section 3.5, we only use the moduli of the coordinates of elements of $\overline{\mathcal{O}rd_l(\mathbb{C}^*)}$ to make the ordering canonical. However, as we have shown, had we chosen another ordering for coordinates with the same argument, we would still get the same result. In other words, as far as $\mathcal{C}_k \,/\, \mathbb{Z}_k$ is concerned, moduli are not that important. Can we then forget them altogether? The following definition,

$$\mathscr{C}_k{}^l := \mathcal{C}omp_l^k \times \Delta^l \amalg \mathcal{C}omp_l^k \times \Delta^{l-1}, \mathscr{C}_k := \coprod_{l=1}^{k-1} \mathscr{C}_k{}^l \,/ \sim_{\mathscr{C}},$$

is an encoding of $\mathcal{C}_k \,/\, \mathbb{Z}_k$ such that $\overline{\Phi}$ does precisely that. The coordinates of elements of the l-simplex Δ^l defined from the arguments of the w_j in (3.9) and (3.10).
Now, we also need to define things in such a way that $\overline{\Phi}$ will be continuous. It should be apparent by now that the continuity of $\overline{\Phi}$ will be a consequence of that of the argument function. Unfortunately, this is only continuous on $\mathbb{C}$ with one ray, say the positive real axis, deleted. Thus we will need to treat seperately the cases where $w_1 \in \mathbb{R}_+$ and $w_1 \notin \mathbb{R}_+$.
First, consider the case $w_1 \notin \mathbb{R}_+$. We consider the set of $l+1$ rays $\{1, w_1, \ldots, w_l\}$. Then $\overline{\Phi}$ should send $(w_1, \ldots, w_l)$ to

$$(\theta_0, \theta_1, \ldots, \theta_{l-1}, \theta_l) \in \Delta^l,$$

where

$$\begin{aligned} \theta_0 &= \arg(w_1) \\ \theta_1 &= \arg(w_2) - \arg(w_1) \\ &\vdots \\ \theta_{l-1} &= \arg(w_l) - \arg(w_{l-1}) \\ \theta_l &= 2\pi - \arg(w_l) \end{aligned} \tag{3.9}$$

Now, consider the case $w_1 \in \mathbb{R}_+$. In this case, the set $\{1, w_1, \ldots, w_l\}$ only contains l rays. Then $\overline{\Phi}$ should send $(w_1, \ldots, w_l)$ to

$$(\theta_1, \theta_2, \ldots, \theta_{l-1}, \theta_l) \in \Delta^{l-1},$$

where

$$\begin{aligned} \theta_1 &= \arg(w_2) \\ \theta_2 &= \arg(w_3) - \arg(w_2) \\ &\vdots \\ \theta_{l-1} &= \arg(w_l) - \arg(w_{l-1}) \\ \theta_l &= 2\pi - \arg(w_l) \end{aligned} \tag{3.10}$$

Now we define $\sim_{\mathscr{C}}$ as follows

$$((\sigma_1, \ldots, \sigma_l), (0, \theta_1, \ldots, \theta_l)) \quad \sim_{\mathscr{C}} \quad ((\sigma_1, \ldots, \sigma_l), (\theta_1, \ldots, \theta_l))$$

$$\text{and} \tag{3.11}$$

$$\begin{gathered} ((\sigma_1, \ldots, \sigma_{m-1}, \sigma_m, \ldots, \sigma_l), (\theta_0, \ldots, \theta_{m-2}, 0, \theta_m, \ldots, \theta_l)) \\ \wr_{\mathscr{C}} \\ ((\sigma_1, \ldots, \sigma_m\sigma_{m-1}, \ldots, \sigma_l), (\theta_0, \ldots, \theta_{m-2}, \theta_m, \ldots, \theta_l)) \end{gathered}$$

for $m \geq 2$.
This implies that there is a CW-decomposition of $\mathscr{C}_k$, with one n-cell and one $(n-1)$-cell for each n-tuple in $\mathcal{C}omp_n^k$, where elements of the $(n-1)$-cells correspond precisely to those $(w_1, \ldots, w_l)$ where $w_1 \in \mathbb{R}_+$. In other words, the $(n-1)$-cells are part of the boundary of the n-cells, with the obvious attaching maps.

3.5 $\overline{\Phi}$ is a homotopy-equivalence

In section 3.4, we discussed an encoding of the cacti that was very similar, at least at the point-set level, to that which we proposed in section 3.2 for configuration spaces. But we also gave our encoding of the cacti a topological structure, which we have yet to do for our encoding of configuration spaces. We will do this by giving an appropriate definition of $\sim_{\mathscr{F}}$. To do this, we return to our definition of $\sim_{\mathscr{C}}$, in (3.11). What this definition did was to adjust data to make sure that when points on the circle collide, things run smoothly. It is in this spirit that we will seek to define $\sim_{\mathscr{F}}$. In this space, points can collide in two "directions": radial and angular, while in the encoding of the cacti, we only had angular collisions. In other words, a point w_{m+1} may collide

with a point on the same ray, or with the ray containing points $w_n, \dots, w_m$ for $\arg(w_{n-1}) < \arg(w_n) = \dots = \arg(w_m) < \arg(w_{m+1})$. Fortunately, we have defined things intelligently enough that "angular" collisions require no reordering of data, and thus do not need to be included in the definition of $\sim_{\mathscr{F}}$.
It therefore suffices to take care of what happens when two points w_m and w_{m+1}, such that

$$\arg(w_m) = \arg(w_{m+1}), |w_m| < |w_{m+1}|$$

collide.

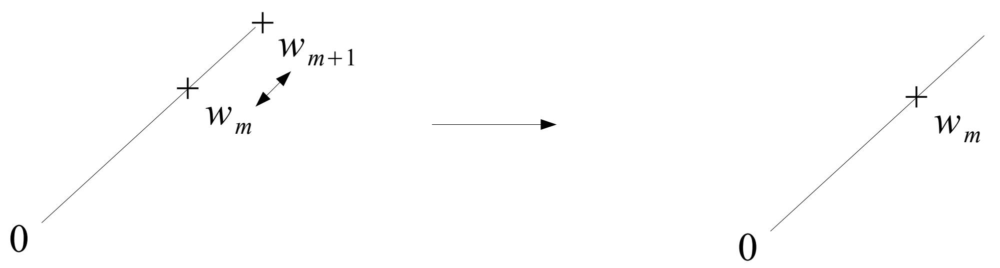


Figure 3.8: A collision

Since we do not want repetitions in the complex-number data, it is fairly straightforward that we need

$$\begin{gathered}((\sigma_1, \dots, \sigma_m, \sigma_{m+1}, \sigma_{m+2}, \dots, \sigma_l), (w_1, \dots, w_m, w_m, w_{m+2}, \dots, w_l)) \\ \wr_{\mathscr{F}} \\ ((\sigma_1, \dots, \sigma_{m+1}\sigma_m, \sigma_{m+2}, \dots, \sigma_l), (w_1, \dots, w_m, w_{m+2} \dots, w_l))\end{gathered}$$

for

$$\arg(w_m) = \arg(w_{m+1}), |w_m| < |w_{m+1}|.$$

Recall also our treatment, in section 3.5, of the "radial" reordering of complex values and their impact on the corresponding configuration data

$$\begin{gathered}((\sigma_1, \dots, \sigma_m, \sigma_{m+1}, \dots, \sigma_l), (w_1, \dots, w_m, w_{m+1}, \dots, w_l)) \\ \wr_{\mathscr{F}} \\ \big((\sigma_1, \dots, \sigma_{m+1}, \sigma_{m+1}\sigma_m\sigma_{m+1}^{-1}, \dots, \sigma_l), (w_1, \dots, w_{m+1}, w_m \dots, w_l)\big)\end{gathered}$$

These are of course precisely the definitions given in (3.7) and (3.8).
Our first indication that we have correctly defined $\sim_{\mathscr{C}}$ and $\sim_{\mathscr{F}}$ is that with them, we can take

$$\overline{\Phi} : \mathscr{F}_k \to \mathscr{C}_k$$

to be given by

$$\overline{\Phi}\left((\sigma_1, \ldots, \sigma_l), (w_1, \ldots, w_l)\right) = ((\sigma_1, \ldots, \sigma_l), (\theta_0, \theta_1, \ldots, \theta_{l-1}, \theta_l))$$

where the θ_i are defined from the arguments of the w_j, as in equation (3.9) and (3.10), and this will be well-defined.

Lemma 3.5.1. *The map $\overline{\Phi}$ is well-defined.*

Proof. To lighten the notation, we will write

$$\varphi_j := \arg(w_j).$$

We must show that $\overline{\Phi}$ sends elements equivalent under $\sim_{\mathscr{F}}$ to elements equivalent under $\sim_{\mathscr{C}}$.
Consider thus

$$\begin{gathered}((\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_{m+1}, w_{m+2}, \ldots, w_l)) \\ \wr_{\mathscr{F}} \\ ((\sigma_1, \ldots, \sigma_{m+1}\sigma_m, \sigma_{m+2}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_{m+2}, \ldots, w_l))\end{gathered}$$

where

$$\varphi_m = \varphi_{m+1}, |w_m| < |w_{m+1}|.$$

Then

$$\begin{gathered}\overline{\Phi}\left((\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_{m+1}, w_{m+2}, \ldots, w_l)\right) \\ \shortparallel \\ \left((\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_l), (\ldots, \varphi_m - \varphi_{m-1}, 0, \varphi_{m+1} - \varphi_m, \ldots)\right),\end{gathered}$$

$$\begin{gathered}\overline{\Phi}\left((\sigma_1, \ldots, \sigma_{m+1}\sigma_m, \sigma_{m+2}, \ldots, \sigma_l), (w_1, \ldots, w_m, w_{m+2} \ldots, w_l)\right) \\ \shortparallel \\ \left((\sigma_1, \ldots, \sigma_{m+1}\sigma_m, \sigma_{m+2}, \ldots, \sigma_l), (\ldots, \varphi_m - \varphi_{m-1}, \varphi_{m+1} - \varphi_m, \ldots)\right),\end{gathered}$$

and

$$\begin{gathered}((\sigma_1, \ldots, \sigma_m, \sigma_{m+1}, \sigma_{m+2}, \ldots, \sigma_l), (\ldots, \varphi_m - \varphi_{m-1}, 0, \varphi_{m+1} - \varphi_m, \ldots)) \\ \wr_{\mathscr{C}} \\ \left((\sigma_1, \ldots, \sigma_{m+1}\sigma_m, \sigma_{m+2}, \ldots, \sigma_l), (\ldots, \varphi_m - \varphi_{m-1}, \varphi_{m+1} - \varphi_m, \ldots)\right).\end{gathered}$$

Now, consider

$$\begin{gathered}((\sigma_1,\dots,\sigma_m,\sigma_{m+1},\dots,\sigma_l),(w_1,\dots,w_m,w_{m+1},\dots,w_l))\\ \wr_{\mathscr{F}}\\ \big((\sigma_1,\dots,\sigma_{m+1},\sigma_{m+1}\sigma_m\sigma_{m+1}^{-1},\dots,\sigma_l),(w_1,\dots,w_{m+1},w_m\dots,w_l)\big)\end{gathered}$$

for

$$\varphi_m=\varphi_{m+1}.$$

We have

$$\begin{gathered}\overline{\Phi}\left((\sigma_1,\dots,\sigma_m,\sigma_{m+1},\dots,\sigma_l),(w_1,\dots,w_m,w_{m+1},\dots,w_l)\right)\\ \shortparallel\\ ((\sigma_1,\dots,\sigma_m,\sigma_{m+1},\dots,\sigma_l),(\dots,\varphi_m-\varphi_{m-1},\varphi_{m+1}-\varphi_m,\varphi_{m+2}-\varphi_m,\dots))\\ \shortparallel\\ ((\sigma_1,\dots,\sigma_m,\sigma_{m+1},\dots,\sigma_l),(\dots,\varphi_{m+1}-\varphi_{m-1},0,\varphi_{m+2}-\varphi_{m+1},\dots))\,,\end{gathered}$$

$$\begin{gathered}\overline{\Phi}\left((\sigma_1,\dots,\sigma_{m+1},\sigma_{m+1}\sigma_m\sigma_{m+1}^{-1},\dots,\sigma_l),(w_1,\dots,w_{m+1},w_m\dots,w_l)\right)\\ \shortparallel\\ \big((\sigma_1,\dots,\sigma_{m+1},\sigma_{m+1}\sigma_m\sigma_{m+1}^{-1},\dots,\sigma_l),(\dots,\varphi_{m+1}-\varphi_{m-1},0,\varphi_{m+2}-\varphi_{m+1},\dots)\big)\,,\end{gathered}$$

and

$$\begin{gathered}((\sigma_1,\dots,\sigma_m,\sigma_{m+1},\dots,\sigma_l),(\dots,\varphi_{m+1}-\varphi_{m-1},0,\varphi_{m+2}-\varphi_{m+1},\dots))\\ \wr_{\mathscr{C}}\\ ((\sigma_1,\dots,\sigma_{m+1}\sigma_m,\dots,\sigma_l),(\dots,\varphi_{m+1}-\varphi_{m-1},\varphi_{m+2}-\varphi_{m+1},\dots))\\ \wr_{\mathscr{C}}\\ \big((\sigma_1,\dots,\sigma_{m+1},\sigma_{m+1}\sigma_m\sigma_{m+1}^{-1},\dots,\sigma_l),(\dots,\varphi_{m+1}-\varphi_{m-1},0,\varphi_{m+2}-\varphi_{m+1},\dots)\big)\,.\end{gathered}$$

Therefore $\overline{\Phi}$ is indeed well-defined. □

Let us now show that $\overline{\Phi}$ is continuous. As a reminder, we have

$$\overline{\Phi}\left((\sigma_1,\dots,\sigma_l),(w_1,\dots,w_l)\right)=((\sigma_1,\dots,\sigma_l),(\theta_0,\theta_1,\dots,\theta_{l-1},\theta_l))$$

for θ_i defined in (3.9) and (3.10).
We can take the argument function as being continuous on $\mathbb{C}-\mathbb{R}_+$, which, along with the equivalence relations $\sim_{\mathscr{C}}$ described in (3.11) and $\sim_{\mathscr{F}}$ described

in (3.7) and (3.8), means that $\overline{\Phi}$ is continuous.
Hence, we have found encodings of orbit spaces of a $\mathbb{Z}_k$-action on both the cacti and configuration spaces up to translation

$$
\begin{array}{ccc}
\mathcal{F}_k & \overset{\Phi}{\underset{s}{\rightleftarrows}} & \mathcal{C}_k \\
\downarrow{\scriptstyle \pi_{\mathcal{F}}} & & \downarrow{\scriptstyle \pi_{\mathcal{C}}} \\
\mathcal{F}_k \,/\, \mathbb{Z}_k & \overset{\widetilde{\Phi}}{\underset{\widetilde{s}}{\rightleftarrows}} & \mathcal{C}_k \,/\, \mathbb{Z}_k \\
{\scriptstyle h_{\mathscr{F}}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle h_{\mathscr{C}}} \\
\mathscr{F}_k & \overset{\overline{\Phi}}{\underset{\overline{s}}{\rightleftarrows}} & \mathscr{C}_k,
\end{array}
$$

where we now have the means to construct a map

$$\overline{s} : \mathscr{C}_k \to \mathscr{F}_k$$

defined by

$$\overline{s}\,((\sigma_1 \ldots, \sigma_l), (\theta_0, \theta_1, \ldots, \theta_{l-1}, \theta_l)) = ((\sigma_1 \ldots, \sigma_l), (w_1, \ldots, w_l))\,,$$

where, for positive real numbers $a_1, \ldots, a_l$, we have

$$w_j := \exp\left(i \sum_{m=0}^{j-1} \theta_m\right).$$

Of course, one could also require the input of positive real numbers $a_1, \ldots, a_l$, or equivalently real numbers $b_1, \ldots, b_l$ such that

$$a_j = e^{b_j},$$

and define

$$w_j := a_j \exp\left(i \sum_{m=0}^{j-1} \theta_m\right).$$

This permits us to view $\overline{\Phi}$ as a forgetful map, and $\overline{s}$ the map that "reminds," provided data in the form of real numbers is given (if none is given, we canonically take $b_1 = \ldots = b_l = 0$), hence embedding the cacti space into the configuration space.

As the title of this paper states, we seek a homotopy-equivalence "upstairs," and this starts by proving that $\overline{\Phi}$ and $\overline{s}$ are homotopy-inverses of one another. To this end, we will only look at the effect $\overline{\Phi}$ and $\overline{s}$ have on the complex-number data, as neither affects the permutation data. So we have that

$$\begin{array}{rcl} \overline{\Phi}\circ\overline{s}:\mathscr{C}_k & \to & \mathscr{C}_k \\ ((\sigma_1,\ldots,\sigma_l),(\theta_0,\theta_1,\ldots,\theta_{l-1},\theta_l)) & \mapsto & ((\sigma_1,\ldots,\sigma_l),(\psi_0,\psi_1,\ldots,\psi_{l-1},\psi_l)) \end{array}$$

where

$$\psi_0=\theta_0, \psi_j=\arg\left(\exp\left(i\sum_{m=0}^{j}\theta_m\right)\right)-\arg\left(\exp\left(i\sum_{m=0}^{j-1}\theta_m\right)\right)=\theta_j.$$

Therefore, we have

$$\overline{\Phi}\circ\overline{s}=1_{\mathscr{C}_k}.$$

The other composition, as might be expected, is not the identity. Indeed, we have

$$\begin{array}{rcl} \overline{s}\circ\overline{\Phi}:\mathscr{F}_k & \to & \mathscr{F}_k \\ ((\sigma_1,\ldots,\sigma_l),(w_1,\ldots,w_l)) & \mapsto & ((\sigma_1,\ldots,\sigma_l),(y_1,\ldots,y_l)) \end{array}$$

where

$$y_1=e^{i\varphi_1}, y_j=\exp\left(i\sum_{m=0}^{j-1}\varphi_{m+1}-\varphi_m\right)=e^{i\varphi_j},$$

for

$$\varphi_j:=\arg(w_j).$$

While this composition is not equal to the identity, it is certainly homotopic to the identity, via the homotopy

$$H:\mathscr{F}_k\times I\to\mathscr{F}_k$$

defined by

$$\begin{array}{c} H\left((\sigma_1,\ldots,\sigma_l),(w_1,\ldots,w_l),t\right) \\ \shortparallel \\ \left((\sigma_1,\ldots,\sigma_l),t(w_1,\ldots,w_l)+(1-t)(e^{i\varphi_1},\ldots,e^{i\varphi_l})\right). \end{array}$$

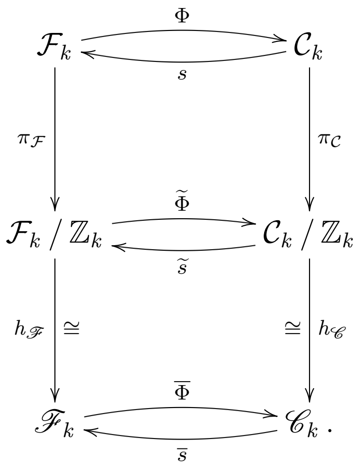
Therefore, we have

$$\overline{s}\circ\overline{\Phi}\simeq 1_{\mathscr{F}_k},$$

and $\overline{\Phi}$ is indeed a homotopy-equivalence, with homotopy-inverse $\overline{s}$.

3.6 Lifting $\overline{\Phi}$ to Φ

We recall the diagram



We must define $h_{\mathscr{F}}$ and $h_{\mathscr{C}}$ such that

1. $h_{\mathscr{F}}$ and $h_{\mathscr{C}}$ are homeomorphisms
2. $h_{\mathscr{C}} \circ \pi_{\mathcal{C}} \circ \Phi = \overline{\Phi} \circ h_{\mathscr{F}} \circ \pi_{\mathcal{F}}$

In order to prove that $\overline{\Phi}$ was a homotopy-equivalence, we gave $\mathscr{F}_k$ and $\mathscr{C}_k$ topologies that need not resemble those of $\mathcal{F}_k$ and $\mathcal{C}_k$. Indeed, if we hope to lift $\overline{\Phi}$ and $\overline{s}$ uniquely, it is necessary and sufficient that the vertical compositions be honest coverings, and since we know that the downward arrows on top are both coverings, it suffices to show that

$$\mathcal{F}_k \,/\, \mathbb{Z}_k \cong \mathscr{F}_k$$

and

$$\mathcal{C}_k \,/\, \mathbb{Z}_k \cong \mathscr{C}_k$$

and take these homeomorphisms.
First, we consider the configurations. As we have already shown in section, we know that

$$\mathrm{o}\mathcal{P}_k \,/\, \mathbb{A}_2 \cong \mathcal{F}_k \,/\, \mathbb{Z}_k \,.$$

We take the map

$$\mathrm{o}\mathcal{P}_k \,/\, \mathbb{A}_2 \to \mathscr{F}_k$$

defined as follows. Consider the class of the polynomial

$$p(z) = (z - z_1) \ldots (z - z_k)$$

by right affine equivalence. It has l critical values, which we call $w_1, \dots, w_l$, where $1 \le l \le k-1$. Without loss of generality, we can order these values according to 3.5.
By Riemann's existence theorem, there is a homomorphism

$$\mu_p : \pi_1 (\mathbb{C} - \{w_1, \dots, w_l\}, 0) \to S_k$$

with transitive image. Because we are working with polynomials with ordered roots, what we get is in fact such a homomorphism, rather than the conjugacy class thereof. Giving such a homomorphism is equivalent to giving an element $(\sigma_1, \dots, \sigma_l)$ of $\mathcal{C}omp_l^k$; as the fundamental group of $\mathbb{C} - \{w_1, \dots, w_l\}$ is the free group on l letters, each σ_j is merely $\mu_p(\lambda_j)$, where the λ_j is the j-th generator of the free group. Then $h_{\mathscr{F}}$ associates to the class of p, the class (by the equivalence $\sim_{\mathscr{F}}$) of

$$((\sigma_1, \dots, \sigma_l), (w_1, \dots, w_l)) .$$

Proposition 3.6.1. *The map $h_{\mathscr{F}}$ is well-defined, bijective and continuous.*

Proof. Let us take two right affine equivalent polynomials p and q, so that

$$q(z) = p(az + b).$$

Let w be a critical value of p, say for the critical point ζ_p. Then clearly $\zeta_q = \frac{\zeta_p - b}{a}$ is a critical point of q, and $q(\zeta_q) = w$. Hence w is also a critical value of q. By symmetry, it follows that right affine equivalent polynomials yield the same complex number data in $\mathscr{F}_k$. So the set of critical values of p and q is $\{w_1, \dots, w_l\}$, say. We take a loop λ_j around w_j and based at 0, and lift this by p and q, respectively. If we take

$$p(z) = (z - z_1) \dots (z - z_k),$$

then it follows that

$$q(z) = a^k (z - z_1') \dots (z - z_k'),$$

where

$$z_i' := \frac{z_i - b}{a}.$$

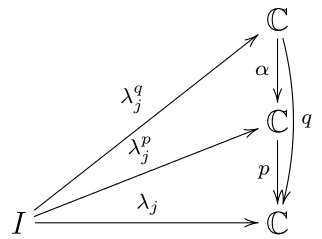
Let us assume that

$$z_u \cdot \lambda_j = z_v,$$

when lifted through p and show that

$$z_u' \cdot \lambda_j = z_v',$$

when lifted through $q = p \circ \alpha$, for $\alpha = (a, b) \in \mathbb{A}_2$.
We have



If $\lambda_j^p(0) = z_u$, then we have that $\lambda_j^p(1) = z_v$. Since we have $\lambda_j^q = \alpha^{-1} \circ \lambda_j^p$, it follows that

$$\lambda_j^q(0) = \alpha^{-1}\left(\lambda_j^p(0)\right) = \alpha^{-1}(z_u) = z'_u$$

and

$$\lambda_j^q(1) = \alpha^{-1}\left(\lambda_j^p(1)\right) = \alpha^{-1}(z_v) = z'_v.$$

Hence the correspondence is indeed well-defined.
To show bijectivity, we must show that if two polynomials p and q have the same data in $\mathscr{F}_k$, then they are right affine equivalent. To say p and q have the same data is to say that $\mu_p = \mu_q$, which in turn means that p and q are right affine equivalent, by Riemann's existence theorem. Hence the correspondence is also bijective.
Continuity is, roughly speaking, a consequence of root-coefficient bicontinuity, introduced at the beginning of section 3.2. Indeed, critical values are simply roots of the derivative plugged into the original polynomial. Now, the coefficients of the derivative depend continuously on those of the polynomial, upon which, in turn, the roots of the derivative depend continuously. Then, since every polynomial is continuous, the critical values of the polynomial depend continuously on its coefficients, and hence, as long as permutation data remain constant, continuity follows immediately from root-coefficient bicontinuity.
The only "special" cases are those where the permutation data do not remain constant, which only happens in the cases of collision and "radial" reordering. Fortunately, the quotient space has been defined precisely so that the correspondence remains continuous. □

To show that $h_{\mathscr{F}}$ is a homeomorphism, it remains to establish that it is proper. For this, it suffices to show that its extension to the one-point compactification of both spaces remains continuous. So what are $(\mathcal{F}_k \,/\, \mathbb{Z}_k)^+$ and $(\mathscr{F}_k)^+$?
To answer this question, it is necessary to explain what points in both spaces we would consider "close to infinity." The role of the point at infinity is, generally speaking, to "tie up loose ends" in the space, so that points close to infinity are those close to complementary of the set. Hence, in the space $\mathcal{F}_k$

infinity is reached where either two of the charges collide, or one of the charges gets very far from all the others. More formally, a point $(z_1, \dots, z_k) \in \mathcal{F}_k$ approaches infinity when

$$\begin{aligned} \min_{i,j} |z_i - z_j| &\to 0 \\ \max_{i,j} |z_i - z_j| &\to \infty \end{aligned}$$

In $\mathscr{F}_k$, infinity is reached where one critical value either vanishes or "explodes," more formally, a point $((\sigma_1, \dots, \sigma_l), (w_1, \dots, w_l)) \in \mathscr{F}_k$ approaches infinity when

$$\begin{aligned} \min_{j} |w_j| &\to 0 \\ \max_{j} |w_j| &\to \infty \end{aligned}$$

Let us first consider the case where z_i and z_j collide. Then the polynomial

$$p(z) = (z - z_1) \dots (z - z_k)$$

approaches one with a double root, hence this polynomial has a root that is also a critical point, hence it follows that zero is a critical value.
Now we consider the case of one of the z_i "exploding" with respect to another z_j.

Lemma 3.6.2. *Let $M > k$. If there exists a pair i, j such that $|z_i - z_j| > M$, then there exists a critical value w such that $|w| > \frac{M}{2}$.*

Proof. Given the polynomial $p(z) = (z - z_1) \dots (z - z_k)$, we consider the associated graph Γ described in section 3.1. We claim that there exists $x \in \Gamma$ such that

$$|x - z_i| > 1 \text{ for all } i = 1, \dots k.$$

This is proved by remarking that if for all $x \in \Gamma$, there existed an index u_x such that $|x - z_{u_x}| \leq 1$, then it would follow that

$$M < |z_i - z_j| \leq |x_1 - z_1| + \ldots + |x_k - z_k| \leq k < M,$$

where $u_{x_v} = v$ for all v.
We further claim that there exists an index u such that

$$|x - z_u| > \frac{M}{2}.$$

This is proved by remarking that if for all indices u, we had $|x - z_u| \leq \frac{M}{2}$, then it would follow that

$$M < |z_i - z_j| \leq |x - z_i| + |x - z_j| \leq M.$$

So far, we have shown that there exists $x \in \Gamma$ such that

$$|p(x)| > \frac{M}{2}.$$

If x is a critical point, we are done. If x is not a critical point, it lies on an edge, either between two black vertices, or between one black and one white vertex. If x lies between a white vertex and the black vertex corresponding to the critical point ζ, then we can take $w = p(\zeta)$, as

$$|w| = |p(\zeta)| \geq |p(x)| > \frac{M}{2}.$$

If x lies between the black vertices corresponding to the critical points ζ_1 and ζ_2, with $|p(\zeta_1)| \leq |p(x)| \leq |p(\zeta_2)|$, then we can take $w = p(\zeta_2)$, as

$$|w| = |p(\zeta_2)| \geq |p(x)| > \frac{M}{2},$$

completing the proof. □

Hence $h_{\mathscr{F}}$ remains continuous when extended to the one-point compactifications, and is thus proper, and a homeomorphism.
The requirement that

$$h_{\mathscr{C}} \circ \pi_{\mathcal{C}} \circ \Phi = \overline{\Phi} \circ h_{\mathscr{F}} \circ \pi_{\mathcal{F}},$$

along with the fact that $\overline{\Phi}$ is a "forgetful" map, leaves only one possible way to define $h_{\mathscr{C}}$. Indeed, $h_{\mathscr{F}}$ sends $\pi_{\mathcal{F}}(z_1, \ldots, z_k)$ to

$$((\sigma_1, \ldots, \sigma_l), (w_1, \ldots, w_l)),$$

where $w_1, \ldots, w_l$ are the critical values of $(z - z_1) \ldots (z - z_k)$, and $\sigma_1, \ldots, \sigma_l$ represent the monodromy. So we have

$$\overline{\Phi} \circ h_{\mathscr{F}} \circ \pi_{\mathcal{F}}(z_1, \ldots, z_k) = ((\sigma_1, \ldots, \sigma_l), (\theta_0, \theta_1, \ldots, \theta_{l-1}, \theta_l)),$$

where for $w_1, \ldots, w_l$ the critical values of $(z - z_1) \ldots (z - z_k)$, we have

$$\begin{aligned} \theta_0 &= \arg(w_1) \\ \theta_1 &= \arg(w_2) - \arg(w_1) \\ &\vdots \\ \theta_{l-1} &= \arg(w_l) - \arg(w_{l-1}) \\ \theta_l &= 2\pi - \arg(w_l) \end{aligned}$$

and for λ_j a small loop around w_j based at 0, we have

$$z_n \cdot \lambda_j = z_{\sigma_j(n)}.$$

It follows that we must define $h_{\mathscr{C}}$ in such a way that

$$h_{\mathscr{C}} \circ \pi_{\mathcal{C}} \circ \Phi(z_1, \ldots, z_k) = ((\sigma_1, \ldots, \sigma_l), (\theta_0, \theta_1, \ldots, \theta_{l-1}, \theta_l)).$$

As a reminder, in (3.2), we defined

$$I_j(\Phi(z_1, \ldots, z_k)) := \left\{ \theta \in [0, 2\pi) \mid \sum_{i=1}^{k} \arg(z_j' - z_i) = \sum_{i=1}^{k} \arg(\theta_\infty - z_i) \right\}$$

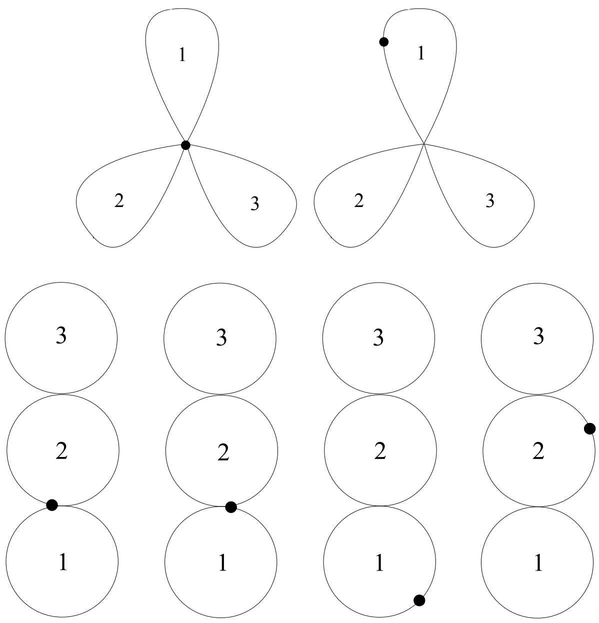
for some z_j' close to z_j, with the zero point being the class of 0_∞. Thus, $\pi_{\mathcal{C}} \circ \Phi(z_1, \ldots, z_k)$ is defined by the same partition of $[0, 2\pi)$, except that the zero points are the classes of $\left(\frac{2\pi n}{k}\right)_\infty$ for $n = 0, 1, \ldots, k-1$. Since the map Φ is surjective, this is sufficient to define $h_{\mathscr{C}}$.
One can use the same arguments we used to prove that $h_{\mathscr{F}}$ was bijective and continuous to show that $h_{\mathscr{C}}$ is also bijective and continuous. Furthermore, the domain and codomain of $h_{\mathscr{C}}$ are both compact spaces. Indeed, we saw earlier that $\mathcal{C}_k$ was decomposed into cells that are all products of geometric simplices, which are compact, and similarly, $\mathscr{C}_k$ is decomposed into cells that are products of geometric simplices and discrete spaces, which are also compact. Hence $h_{\mathscr{C}}$ is a bijective continuous map between compact spaces and thus it is a homeomorphism.

Chapter 4

Cell decompositions of $\mathcal{F}_k$ and $\mathcal{C}_k$

4.1 A CW-decomposition of $\mathcal{C}_k$

We will explicitly do the case $k = 3$, which can be generalized to any k.
We will give the decomposition we alluded to in section 2.3, which in the case $k = 3$ gives thirty-six different cells, though we only give the six of them from which all the other cells can be obtained by permuting the lobes by an element of S_3.



The first and second cacti from the left on the bottom row are distinct; in the first, the zero point is taken on lobe 1, whereas in the second, the zero point is taken on lobe 2. With this notation, the different cells become simply

$$\begin{array}{cccc} & 123 & 1231 & \\ 1232 & 2321 & 12321 & 23212 \end{array}$$

According to (2.1), we have

$$\begin{array}{cccc} & 123 \cong \Delta^0 & 1231 \cong \Delta^1 & \\ 1232 \cong \Delta^1 & 2321 \cong \Delta^1 & 12321 \cong \Delta^1 \times \Delta^1 & 23212 \cong \Delta^2 \end{array}$$

and so one CW-decomposition of $\mathcal{C}_3$ has

$$\begin{array}{rl} \text{six} & 0 - \text{cells} \\ \text{eighteen} & 1 - \text{cells} \\ \text{twelve} & 2 - \text{cells} \end{array}$$

This is a complete decomposition, as these are all the cells we can enumerate, combinatorially speaking.
For this decomposition, we will now describe the attaching maps. The only real work we have to do here is to find the boundaries of the 2-cells and see how cells of lower dimension attach to them. The 2-cells are, in cactus notation, 12321 and 23212. While they are both 2-cells, the former is homeomorphic to a rectangle, the latter to a triangle. Thus it stands to reason that the boundary of 12321 should have four disjoint components homeomorphic to 1-cells, while that of 23212 should have three.
Using our cacti notation, it is easy to find the components of the boundary of a cactus homeomorphic to a 2-cell. Indeed, a cactus is considered to be on the boundary of another if the former can be obtained by squeezing out one part of the outer circle, and so all components of the boundary of a cactus are found by deleting one index from the cactus notation, so long as this deletion does not mean that

- one type of index is not present anymore (we do not want to leave any lobes out)
- there are no repeated indices

Hence, the boundaries of the cells are

$$\begin{aligned} \partial(1231) = \partial(1232) = \partial(2321) &= 123 \\ \partial(12321) &= 2321 \amalg 1321 \amalg 1231 \amalg 1232 \\ \partial(23212) &= 3212 \amalg 2312 \amalg 2321 \end{aligned}$$

With this notation, it becomes easy to see how lower-dimensional cells are attached to the 2-cells. Indeed, all the attaching maps are inclusions, making $\mathcal{C}_3$ a *regular* CW-complex.
With the description of $\mathcal{C}_3$ as a 2-dimensional CW-complex, as well as a canonical embedding of $\mathcal{C}_3$ into $\mathcal{F}_3$, which is also a retraction of Φ, we can see the latter as a 4-dimensional open-cell complex.

4.2 An open-cell decomposition of $\mathcal{F}_k$

We first use the knowledge acquired in the last section about a CW-decomposition of $\mathcal{C}_3$, as well as the embedding of $\mathcal{C}_3 \times \mathbb{R} \times \mathbb{R}$ into $\mathcal{F}_3$ obtained in section 4.1, to find an open-cell decomposition of $\mathcal{F}_3$, and equivalently, a CW-decomposition of its one-point compactification $(\mathcal{F}_3)^+$. More formally, we will consider each of the six cells (up to permutation by an element of S_3) in the decomposition found in the last section, and associate each, one by one, with one or several cells in the new decomposition.
Let us start with the 0-cell $123 \cong \Delta^0$. We know that it represents either a configuration with one double critical value $w_1 = w_2$, or one with two distinct critical values w_1 and w_2, with $v(w_1) = v(w_2)$. In the first case, the only extra degree of freedom comes in the setting of $a = u(w_1) = u(w_2) = b$, and we get the 1-cell $123 \times \mathbb{R}$. In the second case, there are two free variables, a and b, where $a \leq b$ without loss of generality, as they are located on the same flow line, so we have the 2-cell $123 \times \mathbb{R} \times \mathbb{R}_+$. We can apply the same reasoning to the 1-cell $1231 \cong \Delta^1$, and so we get in this case the 2-cell $1231 \times \mathbb{R}$ and the 3-cell $1231 \times \mathbb{R} \times \mathbb{R}_+$.
Now consider the 1-cells $1232 \cong \Delta^1$ and $2321 \cong \Delta^1$. Both of these represent configurations with two distinct critical values w_1 and w_2, on different flow lines. There are two degrees of freedom in both cases a and b, meaning we get the 3-cells $1232 \times \mathbb{R} \times \mathbb{R}$ and $2321 \times \mathbb{R} \times \mathbb{R}$ in the open-cell decomposition of $\mathcal{F}_3$.
Finally, the 2-cells $12321 \cong \Delta^1 \times \Delta^1$ and $23212 \cong \Delta^2$ also represent configurations with two degrees of freedom, so we get the 4-cells $12321 \times \mathbb{R} \times \mathbb{R}$ and $23212 \times \mathbb{R} \times \mathbb{R}$, respectively, in our decomposition.

So we have found an open-cell decomposition of $\mathcal{F}_3$ with

$$\begin{array}{rl} \text{no} & 0-\text{cells} \\ \text{six} & 1-\text{cells} \\ \text{twelve} & 2-\text{cells} \\ \text{eighteen} & 3-\text{cells} \\ \text{twelve} & 4-\text{cells} \end{array}$$

From this open-cell decomposition of $\mathcal{F}_3$, we can find a CW-decomposition of $(\mathcal{F}_3)^+$ with

$$\begin{array}{rl} \text{one} & 0-\text{cell} \\ \text{twelve} & 1-\text{cells} \\ \text{eighteen} & 2-\text{cells} \\ \text{eighteen} & 3-\text{cells} \\ \text{twelve} & 4-\text{cells} \end{array}$$

Let us now explicitly give these cells, and their boundaries. But first we introduce some shorthand vocabulary. We say that an n-cell e^n "attaches to" an m-cell e^m for $m < n$ if the attaching map α from ∂e^n to the $(n-1)$-skeleton of the complex is such that

$$e^m \subset \mathrm{Im}(\alpha).$$

The 0-cell is, of course, the point at infinity. The point at infinity makes the space compact by, in essence, "tying up all the loose ends", and doing so on each open cell. So any cell in the open-cell decomposition of $\mathcal{F}_3$ that is homeomorphic to a Cartesian product $K \times \mathbb{R}$ with K compact produces a cell of the form $K \times \Delta^1$ in the CW-decomposition of $(\mathcal{F}_3)^+$, which attaches to the point at infinity. So the open cell $123 \times \mathbb{R}$ becomes $123 \times \Delta^1$ attached to the point at infinity.

It is well known that the one-point compactification of $\mathbb{R} \times \mathbb{R}$ is S^2, so any open cell homeomorphic to $K \times \mathbb{R} \times \mathbb{R}$ with K compact produces a cell of the form $K \times D^2$, with an attaching map sending the boundary of the disk to the point at infinity. So the open cells $1231 \times \mathbb{R} \times \mathbb{R}, 1232 \times \mathbb{R} \times \mathbb{R}, 2321 \times \mathbb{R} \times \mathbb{R}, 12321 \times \mathbb{R} \times \mathbb{R}$ and $23212 \times \mathbb{R} \times \mathbb{R}$, produce, respectively, cells of the form $1231 \times D^2, 1232 \times D^2, 2321 \times D^2, 12321 \times D^2$ and $23212 \times D^2$, attached in each case to the point at infinity.

Now, the one-point compactification of $\mathbb{R} \times \mathbb{R}_+$ is D^2, where the point at infinity is on the boundary of the disk. Hence any open cell homeomorphic to $K \times \mathbb{R} \times \mathbb{R}_+$ with K compact produce two cells of the form $K \times \Delta^1 \amalg K \times D^2$, where $K \times \Delta^1$ is attached to the point at infinity, and $K \times D^2$ is attached to

$K \times \Delta^1$ along the boundary of the disk. So the open cells $123 \times \mathbb{R} \times \mathbb{R}_+$ and $1231 \times \mathbb{R} \times \mathbb{R}_+$ yield, respectively, the cells $123 \times \Delta^1, 123 \times D^2, 1231 \times \Delta^1$ and $1231 \times D^2$, where $123 \times \Delta^1$ and $1231 \times \Delta^1$ are both attached to the point at infinity, and $123 \times D^2$ and $1231 \times D^2$ are attached to $123 \times \Delta^1$ and $1231 \times \Delta^1$, respectively, along the boundary of D^2.

The following table gives us the cells in our CW-decomposition of $(\mathcal{F}_3)^+$, up to permutation by an element of S_3:

$0-$ cells	$\{\infty\}$
$1-$ cells	$123 \times \Delta^1$
	$123 \times \Delta^1$
$2-$ cells	$1231 \times \Delta^1$
	$123 \times D^2$
	$1231 \times \Delta^1$
$3-$ cells	$1231 \times D^2$
	$1232 \times D^2$
	$2321 \times D^2$
$4-$ cells	$12321 \times D^2$
	$23212 \times D^2$

In the discussion of the boundaries, it must be noted that we will not be able, as we were with the CW-decomposition of $\mathcal{C}_3$, to write these in such a way as to make $(\mathcal{F}_3)^+$ a regular CW-complex, that is, the attaching maps will not be simple inclusions. Nonetheless, it is not diffcult to summarize the attaching maps of this CW-decomposition intuitively, leaving the technical details out.

Let us begin with the 2-cells. Both cells $1231 \times \Delta^1$ attach to the point at infinity, while the cell $123 \times D^2$ attaches to the 1-cell $123 \times \Delta^1$ along the boundary of D^2.

The 3-cell $1231 \times D^2$ attaches to the 2-cell $1231 \times \Delta^1$ along the boundary of D^2, while the other 3-cells $1232 \times D^2$ and $2321 \times D^2$ attach to the point at infinity.

Finally, both 4-cells attach to the point at infinity.

In addition to these attaching maps, there are also those inherited from the "cactus" part of $\mathcal{F}_3'$. Hence, $1231 \times \Delta^1$ attaches to $123 \times \Delta^1$, $1231 \times D^2, 2321 \times D^2, 1232 \times D^2$ all attach to $123 \times D^2$, while $12321 \times D^2$ and $23212 \times D^2$ attach to

$$2321 \times D^2 \amalg 1321 \times D^2 \amalg 1231 \times D^2 \amalg 1232 \times D^2$$

and

$$3212 \times D^2 \amalg 2312 \times D^2 \amalg 2321 \times D^2,$$

respectively.
Hence we get a CW-decomposition of $(\mathcal{F}_3)^+$with

$$\begin{aligned} \text{one} \quad & 0-\text{cell} \\ \text{twelve} \quad & 1-\text{cells} \\ \text{eighteen} \quad & 2-\text{cells} \\ \text{eighteen} \quad & 3-\text{cells} \\ \text{twelve} \quad & 4-\text{cells} \end{aligned}$$

which is equivalent to the open-cell decomposition of $\mathcal{F}_3$ found earlier in this section.
We will not pursue the matter further in this paper, but what we have done in sections 4.1 and 4.2 for the case $k = 3$ can be generalized to any k, in that $\mathcal{C}_k$ can be seen as a $(k-1)$-dimensional CW-complex, which is canonically embedded into the $(2k-2)$-dimensional open-cell complex $\mathcal{F}_k$, where the embedding is a retraction of Φ, by adjunction of data representable by at most $k-1$ real numbers. To see this, it is enough to notice that the main reasoning we used for the case $k = 3$ was of a combinatorial nature, which means that the same reasoning can be used for any k, though for even fairly low values of k, the work quickly gets tedious.

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