Diameter Bounds for Planar Graphs

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Abstract

The *inverse degree* of a graph is the sum of the reciprocals of the degrees of its vertices. We prove that in any connected planar graph, the diameter is at most 5/2 times the inverse degree, and that this ratio is tight. To develop a crucial surgery method, we begin by proving the simpler related upper bounds $(4(|V|-1)-|E|)/3$ and $4|V|^2/3|E|$ on the diameter (for connected planar graphs), which are also tight.

1 Introduction

In this paper we examine the relation between "inverse degree" and diameter in connected planar simple graphs. The diameter $D(G)$ of a graph $G=(V,E)$ is the maximum distance between any pair of vertices, $D := \max_{u,v\in V} dist(u,v)$, where as usual the distance between two vertices is the minimum number of edges on any u-v path. The *inverse degree* $r(G) := \sum_{v\in V} d^{-1}(v)$, defined to be the sum of the inverses of the degrees, is a less well-studied quantity.

The history of the inverse degree stems from the conjecture-generating program Graffiti [2]. Let n denote $|V|$ and m denote $|E|$. Graffiti posited that the *mean distance* $\binom{n}{2}^{-1}\sum_{\{u,v\}\subset V} dist(u,v)$ is always at most the inverse degree $r(G)$. This was disproved by Erdős, Pach & Spencer [1], who also proved the tight bound $D = O\left(\frac{\log n}{\log\log n}\cdot r\right)$ in the process. Subsequently, Mukwembi [3] studied the diameter for various kinds of graphs in terms of inverse degrees. Among other things he conjectured that for any *planar* graph G, $D(G) \le \frac{9}{4}r(G) + O(1)$.

We disprove Mukwembi's conjecture and establish just how large D/r can be:

Theorem 1. *For any planar graph G, $D(G) < \frac{5}{2}r(G)$. There is an infinite family of planar graphs with $D(G) = \frac{5}{2}r(G) - O(1)$.*

The tight family we construct is very simple, but the bound $D(G) \le \frac{5}{2}r(G)$ turns out to be quite challenging. A natural approach is to use the arithmetic-harmonic mean inequality to bound r with the simpler quantity $r \ge \frac{n^2}{2m}$; to this end we prove the tight bound $D \le \frac{4n^2}{3m}$ using a simple "surgery argument."

However, the tight examples of graphs with $D = \frac{4n^2}{3m} - O(1)$ are non-regular (about 2/3 of vertices have degree 5, and 1/3 have degree 2) and so they are not tight for the ratio D/r (since our use of the arithmetic-harmonic mean is tight only for regular graphs). Indeed, the bounds $D \le \frac{4n^2}{3m}$ and $r \ge \frac{n^2}{2m}$ do not imply Theorem 1, but rather the weaker bound $D \le \frac{8}{3}r$. To actually prove Theorem 1 (in Section 3) we carefully engineer a more intricate version of the surgery argument.

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Figure 1: These planar graphs are depicted as if they were drawn on a cylindrical tube, with the dashed edges hidden on the back. Left: the graph L_8. Right: the graph T_{12}.

2 Initial Bounds from Surgery

In this section we focus on proving the less complex bound $D \leq \frac{4n^2}{3m}$ for connected planar graphs, and we prove that the ratio $\frac{4}{3}$ is best possible. We use the following sneaky attack on the problem:

Theorem 2. *For every connected planar graph,* $D \leq \frac{4(n-1)-m}{3}$.

We give the proof later in this section, introducing our surgery approach along the way. It gives the desired corollary:

Corollary 3. *For every connected planar graph,* $D \leq \frac{4n^2}{3m}$.

Proof. We know $(2(n-1)-m)^2 \geq 0$; rearranging yields $4(n-1)-m \leq 4\frac{(n-1)^2}{m}$, thus Theorem 2 yields $D(G) \leq \frac{4(n-1)-m}{3} \leq \frac{4(n-1)^2}{3m}$, which implies the corollary. □

We give some examples before proving Theorem 2. One example disproves Mukwembi's conjecture, and the others demonstrate the tightness of the above theorems. For any even integer $n \geq 4$, let L_n denote the graph with vertices v_j^i for $i \in \{1, 2\}, 1 \leq j \leq n/2$, such that distinct nodes $v_j^i, v_{j'}^{i'}$ are joined by an edge whenever $|j - j'| \leq 1$; the left side of Figure 1 illustrates L_8. Its diameter is $D(L_n) = n/2 - 1$, and its inverse degree is $r(L_n) = \frac{n-4}{5} + \frac{4}{3}$. Hence $D = \frac{5}{2}r - O(1)$ for this family of graphs and the second half of Theorem 1 is proven.

Here is the tight example for Corollary 3: for any n divisible by 3, take $L_{2n/3}$ and attach a path with $n/3$ additional nodes to v_1^1. The resulting graph has diameter $\frac{2n}{3} - 1$ and $m = 5\frac{n}{3} - 4 + \frac{n}{3}$ edges, so $\frac{4n^2}{3mD}$ tends to 1 as n tends to infinity.

Finally, Theorem 2 is best possible, up to an additive constant, for all possible values of m and n. *Euler's bound* says that in planar graphs having $n \geq 3$, we have $m \leq 3n - 6$; this maximum is achieved only for triangulations. For $n \geq 6$ divisible by 3, let T_n be obtained from gluing a sequence of $\frac{n}{3} - 1$ octahedra at opposite faces; we illustrate T_{12} in the right side of Figure 1. To demonstrate tightness of Theorem 2 we start with the extremal values of m. For $m = n - 1$ we have exact tightness: the path graph P_n has $D(P_n) = n - 1 = \frac{4(n-1)-m(P_n)}{3}$. For $m = 3n - 6$ when 3 divides n, the graph T_n has $D = \frac{n}{3} - 1$ and $3n - 6$ edges, which is tight for Theorem 2 up to an additive constant; other n are similar. More generally, for any n and any $n - 1 \leq m \leq 3n - 6$, taking $T_{3\lceil (m+2-n)/6 \rceil}$ and adding a path of $n - 3\lceil (m + 2 - n)/6 \rceil$ more vertices to one end gives an n-node, m-edge graph with $D = \frac{4(n-1)-m}{3} - O(1)$.

Now we give the proof of Theorem 2, which has some ingredients used later on: a *surgery* operation and decomposition into levels. In the proof, we will let st be a diameter of G, e.g. $dist_G(s,t) = D(G)$. We let V_i, the *ith level*, denote all vertices at distance i from s, hence $\biguplus_{i=0}^{D} V_i$ is a partition of V. We use the shorthand $V_{[i,j]}$ to mean $\bigcup_{i \le x \le j} V_x$ and $V_{\ge i}$ is analogous. Additionally, $G[X]$ denotes an induced subgraph and we will extend the subscript notation on V to mean induced subgraphs of G, for example $G_{\ge i} = G[V_{\ge i}]$.

Proof of Theorem 2. Assume for the sake of contradiction that G is a graph with $D(G) > \frac{4(n-1)-m}{3}$, and assume that n is minimal over all such graphs; we may clearly also assume that E is inclusion-wise *maximal*, i.e. for any $e \notin E$, either $G \cup \{e\}$ is non-planar or $D(G \cup \{e\}) < D(G)$.

Our first step is to show that G is 2-vertex-connected. Otherwise, pick a cut vertex v, then we can decompose G into graphs G_1, G_2 with $V(G_1) \cap V(G_2) = \{v\}, V(G_1) \cup V(G_2) = V(G), E(G_1) \cup E(G_2) = E(G)$, and $n(G_1), n(G_2) < n(G)$ (a *1-sum*). Since G was chosen such that n is minimal, both G_i's satisfy the conclusion of Theorem 2. Moreover it is easy to see $m(G) = m(G_1) + m(G_2)$ and $D(G) \le D(G_1) + D(G_2)$. Hence

$$D(G) \le D(G_1) + D(G_2) \le \tfrac{4(n(G_1)-1)-m(G_1)}{3} + \tfrac{4(n(G_2)-1)-m(G_2)}{3} = \tfrac{4(n(G)-1)-m(G)}{3},$$

contradicting the fact that G was chosen to be a counterexample. Thus G is indeed 2-vertex-connected.

We now consider the diameter st and the level decomposition mentioned previously. Note that there are no edges between any pair of vertices in V_i and V_j if $|i-j| > 1$. It is easy to see that if $|V_i| = 1$ for some $0 < i < D$ then V_i is a cut vertex, so we have (by 2-vertex-connectivity) that $|V_i| \ge 2$ for all $0 < i < D$.

To begin, suppose $|V_i| \le 2$ for all $i \ne 0$. Since each vertex in V_i can only connect to neighbours in $V_{i-1} \cup V_i \cup V_{i+1}$ the maximum degree is 5 (and 2 for s, 3 for t, 4 in V_1). Thus (assuming $n \ge 4$ which is easy to justify) we have $D = \lfloor \frac{n}{2} \rfloor$ and $m \le \lfloor \frac{5n-7}{2} \rfloor$, whence it is easy to verify $D \le (4(n-1)-m)/3$ as needed.

Hence, there exists a level of size ≥ 3. We need one well-known fact and a technical claim.

Fact 4. *Let G_1, G_2 be planar graphs with $V(G_1) \cap V(G_2) = \{u, v\}$ and $uv \in E(G_1), E(G_2)$. Define their* 2-sum *G by $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$. Then G is planar.*

Claim 5. *If $|V_i| = 2$, $0 < i < D$, then there is an edge joining the two vertices of V_i.*

Proof. Suppose otherwise. Let $V_i = \{u, v\}$. We will show that uv can be added to G without violating planarity, which will complete the proof, since G was chosen edge-maximal (and adding uv does not change D).

Since G is 2-vertex-connected, u is not a cut vertex, so $G[\{v\} \cup V_{>i}]$ is connected, and similarly for $G[\{u\} \cup V_{>i}]$. Thus there is a path P_R from u to v all of whose internal vertices lie in $V_{>i}$. Likewise there is a u-v path P_L all of whose internal vertices lie in $V_{<i}$ (e.g. concatenate shortest u-s and s-v paths).

Consider a plane drawing of G. The sub-drawing of $G_{\le i}$ must have u, v on a common face due to P_R, so $G_{\le i} \cup \{uv\}$ is planar. Likewise $G_{\ge i} \cup \{uv\}$ is planar and using Fact 4, $G \cup \{uv\}$ is planar as needed. ∎

Recall that there exists a level of size at least 3; let L be chosen minimal with $|V_{L+1}| \geq 3$. Let R be chosen maximal such that $R > L$, and all of the levels $V_{L+1}, V_{L+2}, \ldots, V_{R-1}$ have size at least 3. Thus either $R = D + 1$, or $R \leq D$ and $|V_R| < 3$. We break into several similar cases now.

Case $L > 0, R < D$. Thus $|V_L| = |V_R| = 2$. Consider the graph G' obtained by "surgery" from G by deleting all edges in $G_{[L,R]}$, deleting the isolated vertices $V_{[L+1,R-1]}$, then adding a clique on $V_L \cup V_R$. This is a planar graph by Fact 4 and Claim 5: it is obtained by two 2-sums from $G_{\leq L}$, K_4, and $G_{\geq R}$. We illustrate in Figure 2. Now G' is smaller than G; write $\Delta D = D(G) - D(G')$, $\Delta m = m(G) - m(G')$, and $\Delta n = n(G) - n(G')$. We have $\Delta n \geq 3\Delta D$ since all deleted levels had size at least 3. Moreover, since $G_{[L,R]}$ is a planar graph Euler's bound gives that we deleted at most $3(\Delta n + 4) - 6$ edges. Since we added 6 edges to the new clique, $\Delta m \leq 3\Delta n$. Thus $\frac{4(\Delta n) - \Delta m}{3} \geq \frac{\Delta n}{3} \geq \Delta D$ and from this it is easy to verify that G' is a smaller counterexample to Theorem 2, contradicting our choice of G.

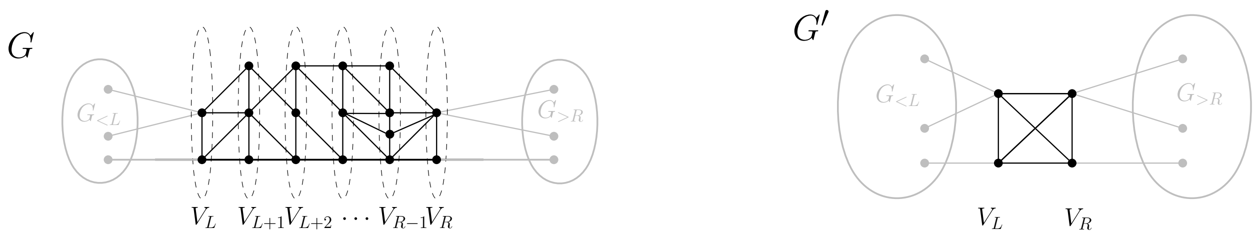

Figure 2: Depiction of how surgery changes a graph G (left) into G' (right). Note the V_i, G_i labels are with respect to the original graph. Gray parts are unaltered.

Case $L > 0, R \in \{D, D+1\}$. Let $X = V_{>L} \backslash \{t\}$. We delete all edges in $G_{\geq L}$, then the isolated vertices X, then we join the three vertices $V_L \cup \{t\}$ by a clique. Thus $\Delta m \leq 3(\Delta n + 3) - 6 - 3 = 3\Delta n$ and we proceed as before.

Case $L = 0, R < D$ is the mirror image of the previous case (e.g. the clique is added to $V_R \cup \{s\}$).

Case $L = 0, R \in \{D, D+1\}$. We have $n \geq 3D - 1$ since all levels in $V_{[1,D-1]}$ have size at least 3. Using Euler's bound, $4(n-1) - m \geq n + 2 > 3D$ and $D < \frac{4(n-1)-m}{3}$ as needed. □

3 Proof that $r(G) \geq \frac{2}{5}D(G)$ for Planar Graphs

The general idea in the proof of Theorem 1 is similar to what we did in the previous section, but the devil is in the details, because the terms $1/d(v)$ change in quite complex ways when we perform surgery on the graph. For example, it is no longer possible to easily argue that the selected counterexample G is 2-vertex-connected. Here is the sketch of how we prove $r(G) \geq \frac{2}{5}D(G)$.

- Define the *fitness* of a planar connected graph G to be $\mathcal{F}(G) := \frac{2}{5}D(G) - r(G)$. So we want to show no graph has positive fitness.

- Let n be minimal such that some n-vertex planar connected graph has positive fitness. Subject to this minimal n, take such a graph G having maximal fitness. If another graph G' exists such that $|V(G')| \leq |V(G)|$ and $\mathcal{F}(G') \geq \mathcal{F}(G)$ and at least one of the these two inequalities is strict, this contradicts our choice of G. Therefore, the proof strategy uses several parts, and in each part we either find such a G', or impose additional structure on G.

- Let st be any diameter of G. We show that except for s and t, every vertex has degree at least 3, and that s and t have degree at least 2.
- We lay out the graph G in levels, as in the previous proof: level V_i consists of all vertices at distance i from s, hence $\biguplus_{i=0}^{D} V_i$ is a partition of V.
- We arrive at a general "cornerstone" theorem (Theorem 20) showing that in many cases, a surgery like in Section 2 finds the desired G'.
- We clean up some additional cases, and thereby prove that G has at most 3 nodes per level, that no size-3 levels are adjacent, that for every size-2 level the contained nodes share an edge, and that the last level V_D has size 1.
- We use a computation (Section 3.7) to prove that this structured graph has $\mathcal{F}(G) < 0$, completing the proof.

3.1 Preliminaries

We reiterate the main tool in the proof.

Claim 6. *If G' is another graph obtained from G with $n(G') < n(G)$, and nonnegative reals $\Delta r, \Delta D$ satisfy $D(G') \geq D(G) - \Delta D$, $r(G') \leq r(G) - \Delta r$, and $\Delta r \geq \frac{2}{5}\Delta D$, then G' is smaller but at least as fit as G, contradicting our choice of G.*

Since adding an edge decreases r, we also have the following.

Claim 7 (Maximality)**.** *If $uv \notin E$ then either $G \cup \{uv\}$ is non-planar or $D(G \cup \{uv\}) < D(G)$. In particular, when u and v are in the same levels or adjacent levels, since adding uv would not change the diameter, we have that $G \cup \{uv\}$ is non-planar.*

We will repeatedly make use of the arithmetic-harmonic mean in the following way.

Proposition 8. *For any set S of vertices, $\sum_{v \in S} 1/d(v) \geq |S|^2/(\sum_{v \in S} d(v))$.*

Thus, the contribution to r by any set is at least as big as what it would give "on average" by counting all endpoints incident on S. Later, we will count $\sum_{v \in S} d(v)$ as twice the number of edges of $G[S]$, plus the number of edges with exactly one endpoint in S.

Suppose that every level of G, except possibly the first and last (V_0 and V_D) have size 3. Then $n \geq 3(D-1)+2$ and the following proposition shows that such graphs are not problematic.

Proposition 9. *If $n \geq 3(D-1)+2$, then $r(G) \geq \frac{2}{5}D$.*

Proof. The case that $n < 3$ is easy to verify, so assume $|E| \leq 3n-6$. Proposition 8 applied to $S = V$ implies that $r \geq n^2/(6n-12)$, and by hypothesis $D \leq (n+1)/3$. Therefore it is enough to prove $n^2/(6n-12) \geq \frac{2}{5}(n+1)/3$, which is easy to verify by cross-multiplying and solving the resulting quadratic. ∎

3.2 Small-Degree Vertices and cut Points

Proposition 10. *G does not have a degree-1 vertex.*

Proof. Let v be a degree-1 vertex with neighbour z. We may assume $|V| \geq 3$ so $d(z) \geq 2$. How do r and D change if we get another graph G' by deleting v? Clearly D decreases by at most 1; and $r(G') = r(G) - \frac{1}{1} - \frac{1}{d(z)} + \frac{1}{d(z)-1} \leq r(G) - 1/2$. We are done by Claim 6, taking $\Delta D = 1$ and $\Delta r = 1/2$. □

A repeated issue is that r is not monotonic, i.e. sometimes we can decrease r in a graph by adding extra vertices (e.g. consider the complete bipartite graphs, where $r(K_{2,10}) < r(K_{1,10})$. The following proposition is a first attack against this issue and shows that adding extra blocks (2-vertex-connected components) cannot decrease r.

Proposition 11. *If v is a cut vertex of G, then $G\backslash v$ has exactly two connected components, one containing s and one containing t.*

Proof. If the proposition is false, there is a cut vertex v such that a connected component H of $G\backslash\{v\}$ contains neither s nor t. Thus $G\backslash H$ contains s and t, moreover $D(G\backslash H) = D(G)$ since any simple s-t path goes through v at most once and hence does not use any vertex of H.

We want to argue that $r(G\backslash H) \leq r(G)$, which will complete the proof using Claim 6 with $\Delta D = \Delta r = 0$. It is enough to use very crude degree estimates. Let $|V(H)| = k$. Each vertex of H has degree at most k in G since each $u \in V(H)$ can only have neighbours in $V(H) \cup \{v\}\backslash\{u\}$. Moreover, the difference between $r(G\backslash H)$ and $r(G)$ is due only to vertices in $\{v\} \cup V(H)$. Clearly v has at least one neighbour not in H. Then

$$r(G) = r(G\backslash H) + \sum_{u \in H} \frac{1}{d_G(u)} + \frac{1}{d_G(v)} - \frac{1}{d_{G\backslash H}(v)} \geq r(G\backslash H) + \frac{k}{k} + 0 - 1 = r(G\backslash H),$$

as needed. □

Proposition 12. *Except possibly s and t, G does not have a degree-2 vertex.*

Proof. Let $v \notin \{s,t\}$ be a degree-2 vertex, with neighbours a, b. If a and b are non-adjacent, we can remove v and directly connect them, which decreases r by $1/2$ and decreases D by at most 1, which yields a contradiction by Claim 6.

Therefore assume a and b are adjacent. If both a and b have degree 2 then $G = K_3$ and $\mathcal{F}(G) < 0$, so we are done. If both a and b have degree at least 3, since $v \notin \{s,t\}$, $G\backslash\{v\}$ is a connected planar graph with diameter at least as large as that of G and $r(G\backslash\{v\}) \leq r(G) - 1/2 + 1/6 + 1/6 < r(G)$, so we are done by using Claim 6 with $\Delta D = \Delta r = 0$.

The final case is that a has degree 2 (w.l.o.g.) and b has degree at least 3. Then b is a cut vertex, implying by Proposition 11 that $a \in \{s,t\}$, say w.l.o.g. $a = s$, and $t \notin \{v,a,b\}$. But this contradicts edge-maximality in the following way: let by for $y \notin \{a,v\}$ be an edge on a common face with bv (see Figure 3(a)), then adding vy to G does not change the diameter. □

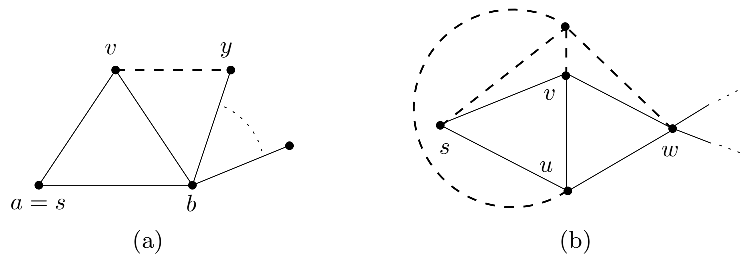

Figure 3: Dashed edges are added without violating planarity. (a) The edge vy contradicting the edge-maximality. (b) The distance 2 neighbourhood of s after ω-μ surgery and the added edges.

3.3 Basic Surgery: Case Analysis and Bonuses

The central idea for surgery comes from the first case of Theorem 2's proof.

Definition 13. *Given two levels V_L and V_R, to apply* surgery at V_L and V_R *means to delete all nodes in $V_{[L+1,R-1]}$ (and their incident edges) and then to connect each $u \in V_L$ to each $v \in V_R$ (we "add a biclique").*

We say a level of size 2 is *connected* if its vertices share an edge, and that a level of size 1 is always connected. Assuming the levels are connected and of size at most 2, Definition 13 is indeed the same surgery as in Section 2. As before we get:

Proposition 14. *Suppose $|V_L|, |V_R| \leq 2$ are connected levels with $L < R$. Surgery at V_L and V_R yields a connected planar graph G' with $D(G') = D(G) - (R - L - 1)$.*

We need a collection of *types* (cases) for our analysis. There are 7 types and V_L may satisfy one or none of them (i.e. the cases are not exhaustive; nonetheless they form the core of our arguments). Analogous cases for V_R are explained afterwards. Here are the 7 types for V_L:

- ω: $L = 0$, i.e. the level contains one end of the diameter st; for all other cases, $L > 0$.
- α: $|V_L| = 1$ and the node in V_L has 1 neighbour in V_{L-1}
- β: $|V_L| = 1$ and the node in V_L has 2 neighbours in V_{L-1}
- β': $|V_L| = 1$ and the node in V_L has ≥ 2 neighbours in V_{L-1} and ≥ 2 neighbours in V_{L+1}
- μ: $|V_L| = 2$, V_L is connected, and each node of V_L has 1 neighbour in V_{L-1}, in fact the same one
- ν: $|V_L| = 2$, V_L is connected, and each node of V_L has 2 neighbours in V_{L-1}
- ν': $|V_L| = 2$, V_L is connected, and each node of V_L has ≥ 2 neighbours in V_{L-1} and ≥ 2 neighbours in V_{L+1}

The analogous cases for the right-hand side are the same with $L = 0, L > 0$ replaced by $R = D, R < D$, V_L replaced by V_R, V_{L-1} replaced by V_{R+1}, and V_{L+1} replaced by V_{R-1} (note the sign changes).

Fix V_L, V_R each of size ≤ 2 with $L < R$, such that all levels in between have size at least 3. Our proof's cornerstone, which we complete at the end of Section 3.5, is to show that when L and R are each of one of the 7 types, provided there are at least 4 nodes between V_L and V_R, we can get a smaller G' which is at least as fit as G, by using surgery and some other "bonus" operations, contradicting our choice of G. After this cornerstone we deal with cases outside the 7 types.

First note that if both L and R are of type ω, Proposition 9 already ensures $r(G) \geq \frac{2}{5}D(G)$. If V_L is of type λ and V_R is of type ξ, we call the surgery type λ-ξ; we call ω-ω the *unneeded type* of surgery since we don't need to analyze it. It is essential to increase post-surgery fitness when possible. We now establish some values $bonus(\{\lambda, \xi\})$ (which are symmetric in λ and ξ) such that, after a λ-ξ surgery, we can increase the fitness by at least $bonus(\{\lambda, \xi\})$.

- We may take $bonus(\{\alpha, \beta\}) = bonus(\{\alpha, \beta'\}) = \frac{1}{10}$ because this surgery results in a degree-2 vertex, which may be shortcutted to decrease D by 1 and decrease r by 1/2, giving a $\frac{1}{2} - \frac{2}{5}$ increase in fitness.
- Similarly we may take $bonus(\{\alpha, \alpha\}) = \frac{2}{10}$.
- We may take $bonus(\{\omega, \beta\}) = bonus(\{\omega, \beta'\}) = \frac{13}{30}$ as follows. Consider a ω-β (or β') surgery, so V_R is a singleton $\{v\}$. After surgery s has only one neighbour, v, and v has degree at least 3. Then deleting s decreases the diameter by 1 and decreases r by at least $1 - 1/6$. Therefore there is a bonus of at least $1 - 1/6 - 2/5 = \frac{13}{30}$.
- Similarly we can get $bonus(\{\omega, \alpha\}) = 13/30 + 1/10 = 8/15$ because (w.l.o.g. in a ω-α surgery) the α vertex's right neighbour has degree at least 3 in the original and post-operation graphs, using Proposition 12.
- Finally we can get $bonus(\{\omega, \mu\}) = 1/12$ as follows. Consider a (w.l.o.g.) μ-ω surgery, where $V_L = \{u, v\}$ and the common neighbour of u, v in V_{L-1} is w. Post-surgery, the distance-2 neighbourhood of s is as shown in Figure 3(b). Add a new vertex and connect it to u, v, w, s; it is not hard to argue this preserves planarity. Not counting the increased degree at w, we decreased r by $\frac{1}{2} + \frac{2}{3} - \frac{1}{3} - \frac{3}{4} = \frac{1}{12}$ and preserved D. (Although this adds a vertex, the surgery theorems later on always delete at least 2 vertices, so overall the total number of vertices always decreases.)

3.4 First Analysis of Surgery

Now we give a lower bound on fitness increase due to surgery. It is convenient to assume when V_L is in cases β', ν' that each node in V_L has *exactly* two neighbours in V_{L-1} — call the rest *ghost neighbours*. Why is this ok? Keep in mind we want to prove a lower bound on the fitness increase from surgery. Due to the "≥ 2 neighbours in V_{L+1}" condition in these cases, surgery does not increase the degree of nodes in V_L. Further, by the convexity of $d(v) \mapsto \frac{1}{d(v)}$, the actual r increase including ghost neighbours will be no more than the "virtual r increase" ignoring ghost neighbours made by our analysis.

Here are the details. Let n_L denote $|V_L|$ and similarly for n_R. Let o_L denote, for each node in V_L, the number of "outside" neighbours such a node has in V_{L-1}; define o_R similarly with V_{R+1} in place of V_{L-1}. Thus n_L and o_L depend only on the type of L, and abusing notation, we write $n_\omega = n_\alpha = n_\beta = n_{\beta'} = 1, n_\mu = 2, n_\nu = n_{\nu'} = 2$ and $o_\omega = 0, o_\alpha = 1, o_\beta = o_{\beta'} = 2, o_\mu = 1, o_\nu = o_{\nu'} = 2$. Let $\overline{o}$ denote the number of neighbours each vertex of V_L has in $V_L \cup V_{L-1}$, so for any subscript X, $\overline{o}_X = o_X + (n_X - 1)$. Let $w = R - L - 1$ denote the number of levels in between, and recall that each of these w levels has at least 3 nodes. Let x denote the number of nodes in the deleted levels, hence we have $x \geq 3w$.

Before surgery, the sum of the degrees of the nodes in $V_{[L,R]}$ is at most $n_L o_L + 2(3(n_L + x + n_R) - 6) + n_R o_R$ — the terms count edges from V_{L-1} to V_L, in $G_{[L,R]}$, and from V_R to V_{R+1} respectively. We thereby use Proposition 8 to lower-bound the initial sum of the inverse degrees in $V_{[L,R]}$. Post-surgery, we know the degrees of the nodes in V_L are $\overline{o}_L + n_R$ and similarly for V_R.

Therefore, if G' indicates the result of applying surgery and bonus operations, and if we define the quantity $\mathcal{Q}$ as follows,

$$\mathcal{Q} := \underbrace{\frac{(n_L + x + n_R)^2}{n_L o_L + 2(3(n_L + x + n_R) - 6) + n_R o_R}}_{\text{lower bound on inverse degrees of } V_{[L,R]} \text{ in } G} - \Bigg(\underbrace{\frac{n_L}{\overline{o}_L + n_R} + \frac{n_R}{\overline{o}_R + n_L}}_{\text{inv. degrees of } V_L \cup V_R \text{ after surgery}} \Bigg) - \frac{2}{5} \underbrace{w}_{\Delta D} + bonus(L, R),$$

then using the definition of fitness, we have $\mathcal{F}(G') - \mathcal{F}(G) \geq \mathcal{Q}$.

Claim 15. *Let x, w be integers with $x \geq 3w$, $x \geq 2$, and $w \geq 0$. Except for $(w, x) \in \{(1, 3), (2, 6)\}$, the value $\mathcal{Q}$ is positive for all types of L, R (except the unneeded type $L = R = \omega$).*

That is to say, this analysis covers almost all needed cases. Note the case $w = 0$ does not fit the surgery-based motivation given above, but it will be useful in Section 3.6.

Proof of Claim 15. It is easy to verify that $\mathcal{Q} > \frac{x}{6} - 4 - \frac{2}{5}w \geq \frac{x}{6} - \frac{2x}{15} - 4$ so it is clearly positive for $x \geq 120$. We use a publicly posted `Sage` worksheet [5] to verify the remaining cases. (Note we've chosen things so that a λ-ξ surgery has the same analysis as a ξ-λ surgery, and such that the pairs $\{\beta, \beta'\}$ and $\{\nu, \nu'\}$ are analyzed in the same way. So our computation involves 14 surgery cases.) □

More generally, the exact same proof gives the following generalization, which is needed later.

Theorem 16. *Let $V'_R \subseteq V_R$, $L < R$, so that every s-t path intersects V'_R. Let X be the nodes not connected to s or t in $G \backslash V_L \backslash V'_R$ and let $x = |X|$. Let V_L be any of the 7 types. Let V'_R be of one of the 7 types, modified so that "in V_{R-1}" is replaced by "in X" and "in V_{R+1}" is replaced by "in $V_{R+1} \backslash X$." Assume that at least one of L, R is not of type ω. Let $w = R - L - 1$. If we delete X and connect V_L to V'_R by a biclique, then perform bonus operations, we get a smaller graph at least as fit as G, provided $x \geq 2$, $x \geq 3w$, $w \geq 0$ and $(w, x) \notin \{(1, 3), (2, 6)\}$.*

3.5 Completing the Cornerstone: The Case $w = 2, x = 6$

If $w = 2$ and $x = 6$ then $R = L + 3$ and $|V_{L+1}| = |V_{L+2}| = 3$, since all levels between V_L and V_R have size at least 3. We need:

Claim 17. *Let V_i be a level of size 2, whose vertices are connected by an edge, and let $j = i + 1$ or $j = i - 1$, with $|V_j| = 3$. Then it is not true that each vertex of V_i is adjacent to each vertex of V_j.*

Proof. The goal of the proof is similar to the result in Proposition 11: assume the opposite for the sake of contradiction, then show there is some part of the graph that can be deleted while decreasing r and leaving D unchanged. To do this, we need to establish some structure.

Let $V_i = \{u, v\}$; we consider the case $j = i+1$, the other case is analogous. Take a plane drawing of $G_{\geq i}$ with uv on the outer face; it follows that for some labelling $V_j = \{a, b, c\}$, the drawing of $G_{\geq i}$ has triangle uva containing vertex b and triangle uvb containing vertex c. By combining it with a drawing of $G_{\leq i}$ with uv on the outer face, we get the layout shown in Figure 4(a). Now by maximality ab is an edge of G: indeed, since u has no neighbours other than v, a, b, c in the drawing of $G_{\geq i}$, if ab is not present we can add it in a planar way by going next to the path aub. Similarly $bc \in E(G)$.

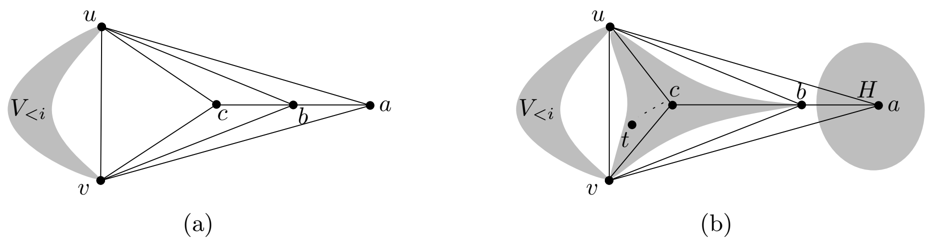

Figure 4: (a) A drawing of $G_{\leq j}$. If we delete uvb the remainder will have at least 3 connected components. (b) A drawing of G. One of the aforementioned connected components, H, does not contain s or t; we will delete it.

Now note that $G \backslash \{u, b, v\}$ has at least 3 components: one containing a, one containing c, and one containing $V_{<i}$. One of the first two does *not* contain t. Assume the first (the second case is analogous): denote the component containing a in $G \backslash \{u, b, v\}$ by H, so $H \not\ni t$ (see Figure 4(b)). It's not hard to see that any shortest s-t path avoids H, hence $D(G \backslash H) = D(G)$. Moreover we claim that $r(G \backslash H) < r(G)$, contradicting our choice of G. To see this, let k denote $|V(H)|$, note that each vertex in H has degree at most $k+2$, and that we drop the degrees of u, b, v by at most k, thus

$$\begin{aligned} r(G) - r(G \backslash H) &\geq k \frac{1}{k+2} + \sum_{i \in \{u,b,v\}} \frac{1}{\deg_G(i)} - \frac{1}{\deg_{G \backslash H}(i)} \\ &\geq \frac{k}{k+2} + \sum_{i \in \{u,b,v\}} \frac{1}{\deg_{G \backslash H}(i) + k} - \frac{1}{\deg_{G \backslash H}(i)} \\ &\geq \frac{k}{k+2} + 3(1/(k+3) - 1/3) = \frac{k}{(k+2)(k+3)} > 0 \end{aligned}$$

where in the second-to-last inequality we used the fact that $\deg_{G \backslash H}(i) \geq 3$ and $x \mapsto \frac{1}{x}$ is convex. ☐

Claim 17 allows us to bound the number of edges between a level $\{u, v\}$ with $uv \in E$ and an adjacent level of size 3: there are at most 5. It's also obvious that between a singleton level and an adjacent level of size 3, there are at most 3 edges. Accordingly, let z_L be 3 (resp. 5) when n_L is 1 (resp. 2) and similarly define z_R. In the situation that there are exactly two levels, each of size-3, between V_L and V_R, we can replace the quantity $\mathcal{Q}$ from the previous section by grouping the vertices in a different way; specifically we have $\mathcal{F}(G') - \mathcal{F}(G) \geq \mathcal{Q}'$ with $\mathcal{Q}'$ defined by

$$\mathcal{Q}' := \frac{n_L^2}{n_L \overline{o}_L + z_L} + \frac{x^2}{z_L + 2(3x-6) + z_R} + \frac{n_R^2}{n_R \overline{o}_R + z_R} - \frac{n_L}{\overline{o}_L + n_R} - \frac{n_R}{\overline{o}_R + n_L} + bonus(L, R) - \frac{2}{5} w.$$

Specifically, the first three terms lower-bound the contribution to $r(G)$ by vertices in V_L, in $V_{L+1} \cup V_{L+2}$, and $V_{L+3} = V_R$ respectively.

Claim 18. *The quantity $\mathcal{Q}'$ is positive when $w = 2, x = 6$ for all types of L, R (except the unneeded type $L = R = \omega$).*

Proof. This calculation is also done via computer at [5]. ☐

Corollary 19. *Let V_L and V_R be levels of one of the 7 types (except the unneeded type $L = R = \omega$), with $R = L + 3$ and $|V_{L+1}| = |V_{L+2}| = 3$. Applying surgery at V_L and V_R gives a graph which is smaller and more fit than G.*

Together with Theorem 16 this gives the heart of our proof:

Theorem 20 (Cornerstone)**.** *Let V_L, V_R be levels of size ≤ 2, with all levels between them of size ≥ 3. If V_L and V_R are each one of the 7 types, and there are at least 4 nodes between them, this contradicts our choice of G.*

3.6 Sufficiency of the 7 Cases

The structure we want to establish in G is that every level has size at most 3, and that two size-3 levels are never adjacent. We now show how to get from the cornerstone (Theorem 20) to this structure. We start with a general observation (which motivated our definition of the 7 cases).

Claim 21. *Suppose $V_i = \{u, v\}$ and $uv \in E$. Suppose $j = i \pm 1$, that u has 1 or fewer neighbours in V_j, and that v has at least one neighbour in V_j which is not a neighbour of u. Then this violates maximality.*

Proof. Take $j = i + 1$, the other case is analogous. Embed $G_{\geq i}$ with uv on the outer face. First if u has no neighbours in V_{i+1} then note u and a neighbour of v are on the outer face, hence we can add an edge between them without violating planarity in $G_{\geq i}$ (and hence without violating planarity in G, by Fact 4). Second, suppose u has exactly one neighbour x in V_{i+1}; at least one endpoint emanating from v adjacently to vu is of the form vy with $y \neq u, v, x$; then the path uvy lies on a face and the edge uy can be added without violating planarity. □

In the remainder of the section, we ensure all size-2 levels are connected, show that V_L always is in one of the 7 cases, deal with V_R's that fall outside the 7 cases, and then show the last level V_D has size 1.

Claim 22. *Any level of size 2 is connected, except possibly for the last level V_D.*

Proof. Let V_R be minimal, $R < D$, such that $V_R = \{u, v\}$ is of size 2 and uv is not an edge. If both u and v are connected to t in $G_{\geq R}$ then using the proof method of Claim 5, uv can be added without violating planarity, which contradicts maximality. Therefore assume only u has a path to t in $G_{\geq R}$. It now follows that v is an isolated vertex in $G_{\geq R}$, or else Proposition 11 is violated because of the cut point v.

Since v has degree at least 3 (by Proposition 12) and these neighbours are only in V_{R-1}, it follows that $|V_{R-1}| \geq 3$. Let L be maximal with $L < R$ such that $|V_L| \leq 2$. By our choice of R, we see V_L is connected if it has size 2. Moreover, each vertex in V_L has at least two neighbours in V_{L+1}, using $|V_{L+1}| \geq 3$ and Claim 21. So V_L is of one of the 7 cases.

Now look at u. If u has 2 or more neighbours in V_{R-1}, we can use surgery at V_L and u which is of type β' (Theorem 16: cutting out $R - L - 1 \geq 1$ levels of size 3, plus v). Otherwise, we can use surgery at V_L and the unique neighbour of u in V_{R-1}, which is a cut vertex of type α (Theorem 16: cutting out $R - L - 2 \geq 0$ levels of size 3, plus v and at least two nodes from V_{R-1}). □

The following corollary follows from the previous proof and induction:

Corollary 23. *Every level V_L such that $|V_L| \leq 2, |V_{L+1}| \geq 3$ falls in one of the 7 cases.*

Proposition 24. *Let V_R, $R < D$, be such that $|V_R| \leq 2$, and either $|V_{R-1}| \geq 4$, or both $|V_{R-2}|, |V_{R-1}| \geq 3$. Then we can perform surgery to increase the fitness of G.*

Proof. Let $L < R$ be maximal with $|V_L| \leq 2$. Using Corollary 23 (along with Corollary 19 or Theorem 16) we may assume V_R falls outside of the 7 types; using Claim 22 and Claim 21 this means that either $|V_R| = 1$ and it has one neighbour in V_{R-1} but ≥ 3 neighbours in V_{R+1}, or $|V_R| = 2$ and these vertices each have one neighbour (the same one) in V_{R-1} and one vertex of V_R has ≥ 3 neighbours in V_{R+1}.

In either case, only one vertex in V_{R-1}, call it v, is adjacent to V_R. Since v is a cut vertex we can do surgery on V_L and v — we apply Theorem 16 to levels L and $R' = R - 1$, on sets V_L and $V'_{R'} = \{v\}$ (here $V'_{R'}$ is of type α if $|V_R| = 1$ or β if $|V_R| = 2$). The set X is $V_{[L+1,R-1]}\backslash\{v\}$, and $w = R' - L - 1$ so $x = |X| \geq 3w + 2, w \geq 0$. This indeed satisfies the conditions of Theorem 16 so we are done. □

Proposition 25. *The size of the last level V_D is 1.*

Proof. Suppose $|V_D| > 1$ for the sake of contradiction. Let V_L be the rightmost level of size at most 2, which we know is one of the 7 types by Corollary 23. Let $v \in V_D\backslash\{t\}$. If $L = D - 1$ then it is not hard to see some face contains v and a vertex from V_{D-2}; adding an edge between this pair does not decrease the diameter, so contradicts edge-maximality. Otherwise ($L < D - 1$) apply surgery to V_L and t: we cut out 1 or more levels of size at least 3, plus the vertices of $V_D\backslash\{t\}$. Thus $x \geq 3w + 1, w \geq 1$ and Theorem 16 is satisfied. □

Combining the results just proven, we have the desired structure theorem: G is a graph where the first and last level have size 1, all levels have size at most 3, every level of size 2 is connected, and no two levels of size 3 are adjacent.

3.7 Computation

We finish by showing that our hypothetical G has $r \geq \frac{2}{5}D$.

Theorem 26. *Let G be a graph where the first and last level have size 1, all levels have size at most 3, every level of size 2 is connected, and no two levels of size 3 are adjacent. Then $r(G) \geq \frac{2}{5}D + \frac{37}{60}$.*

Proof. The most important fact about the structure is that, given the sizes of levels $i - 1, i, i + 1$, we can get a precise upper bound on the degrees of the nodes in level i, and thus also a lower bound on the sum of the inverse degrees for that level.

Given any two adjacent levels, we may upper bound the edges they share by a biclique. Furthermore, if a level of size 2 and a level of size 3 are adjacent, by Claim 17 we can upper bound their shared edges as being one edge short of a biclique. Hence let $\mathcal{S}(i, j) = i \cdot j$ unless $\{i, j\} = \{2, 3\}$ in which case $\mathcal{S}(i, j) = 5$. Thus:

- $\sum_{v \in V_0} 1/d(v) \geq 1/|V_1|$
- $\sum_{v \in V_D} 1/d(v) \geq 1/|V_{D-1}|$

- For $0 < i < D$ there are at most $\mathcal{E} := \mathcal{S}(|V_{i-1}|, |V_i|) + 2\binom{|V_i|}{2} + \mathcal{S}(|V_i|, |V_{i+1}|)$ endpoints incident on V_i; considering the degrees are integral and using convexity we see

$$\sum_{v \in V_i} 1/d(v) \geq \frac{\mathcal{E} \bmod |V_i|}{\lceil \mathcal{E}/|V_i| \rceil} + \frac{|V_i| - (\mathcal{E} \bmod |V_i|)}{\lfloor \mathcal{E}/|V_i| \rfloor} =: \mathcal{C}.$$

Since $\mathcal{C}$ is determined only by $|V_{i-1}|, |V_i|, |V_{i+1}|$, we write it as $\mathcal{C}(|V_{i-1}|, |V_i|, |V_{i+1}|)$. We therefore deduce for any sequence $(n_0, n_1, \dots, n_D)$ of level sizes of a graph G that

$$r(G) \geq \mathcal{R}(n_0, n_1, \dots, n_D) := 1/n_1 + 1/n_{D-1} + \sum_{i=1}^{D-1} \mathcal{C}(|V_{i-1}|, |V_i|, |V_{i+1}|).$$

Finally, we want to determine which valid sequence minimizes $\mathcal{R}(n_0, n_1, \dots, n_D) - \frac{2}{5}D$. Because $\mathcal{C}$ is a sum of local contributions, and because each level contributes 1 to the diameter, we can think of this last step as shortest path problem, as follows. Define a new digraph with vertex set

$$\{s, (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), t\},$$

where the (i,j)-vertices represent a pair of adjacent levels, s represents the start, and t the end. The intuition: we insert an arc from (i,j) to (k,ℓ) whenever $j = k$, representing three consecutive levels. The cost of such an edge should account for the r-contribution of the level corresponding to j, minus the contribution from lengthening the diameter.

Formally, we add an arc $(i,j) \to (j,k)$ for all i,j,k (with no consecutive 3s) having cost $\mathcal{C}(i,j,k) - \frac{2}{5}$; we add an arc $s \to (1,i)$ for all i having cost $1/i$; and we add an arc $(i,1) \to t$ for all i having cost $1/i - \frac{2}{5}$. Then it's easy to see that for any sequence of n_i's, $\mathcal{R} - \frac{2}{5}D$ is given by the cost of the $(D+1)$-edge path $s \to (n_0, n_1) \to (n_1, n_2) \to \cdots (n_{D-1}, n_D) \to t$ in the new digraph. Executing a shortest-path algorithm such as Bellman-Ford (see the worksheet at [5]) establishes that the shortest path from s to t has cost $\frac{37}{60}$, hence $r \geq \mathcal{R} \geq \frac{2}{5}D + \frac{37}{60}$ for these graphs (and that there are no negative dicycles). □

In fact $r \geq \frac{2}{5}D + \frac{37}{60}$ holds for all graphs, is best possible, and the unique graph with $r = \frac{2}{5}D + \frac{37}{60}$ is K_5^-. To establish this precise result, small adjustments to our proofs are necessary, as well as exhaustive searching on all planar graphs with up to 9 vertices.

4 Conclusion

The main techniques underlying our diameter bounds for planar graphs were the surgery operation (which preserves planarity), and the fact that every planar graph has at most a linear number of edges. One might try the same approach on the family of graphs excluding any fixed k-clique minor, since such graphs have $O(nk\sqrt{\log k})$ edges (e.g., see [4]). A perpendicular avenue for future research would be to find a tight relation in connected planar graphs between the mean distance and the diameter.

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