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The composition factors of the functor of permutation modules

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ABSTRACT

Let k be a field, let $\Pi_k(G)$ be the Grothendieck group of permutation kG -modules, where G is a finite group, and let $\mathbb{C}\Pi_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \Pi_k(G)$. In this article, we find all the composition factors of the biset functor $\mathbb{C}\Pi_k$.

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1. Introduction

Let k be a field of characteristic p , where p is a prime number or 0.

Let G be a finite group, let $B(G)$ be the Grothendieck group of G -sets (Definition 11) and let $\Pi_k(G)$ be the Grothendieck group of permutation kG -modules (Definition 13). We can define a map $\theta_G : B(G) \rightarrow \Pi_k(G)$ which is natural and surjective by definition. Now if we tensor everything with \mathbb{C} and if G varies, $\mathbb{C}B$ and $\mathbb{C}\Pi_k$ become \mathbb{C} -linear biset functors and $\theta : \mathbb{C}B \rightarrow \mathbb{C}\Pi_k$ is a natural transformation.

Recall that the simple biset functors $S_{H,V}$ are parametrized by pairs (H, V) , where H is a finite group and V a simple $\mathbb{C}\text{Out}(H)$ -module. If $k = \mathbb{Q}$ then we have that $\mathbb{C}\Pi_{\mathbb{Q}} = \mathbb{C}R_{\mathbb{Q}}$, where $R_{\mathbb{Q}}(G)$ is the ordinary Grothendieck group of $\mathbb{Q}G$ -modules, and Serge Bouc proves that $\mathbb{C}R_{\mathbb{Q}} = S_{1,\mathbb{C}}$ is a simple biset functor [1, Proposition 4.4.8].

We want to generalize this to an arbitrary field k . More precisely, we want to find the composition factors of $\mathbb{C}\Pi_k$. In order to do this, we need the composition factors of $\mathbb{C}B$. They were determined by Serge Bouc and they are the simple functors $S_{H,\mathbb{C}}$, where H is a B-group. A B-group is defined by a technical condition (see Definition 16).

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Recall that a p -hypo-elementary group is the semi-direct product of a p -group with a cyclic p' -group. If $p = 0$, this means that the group is cyclic.

In this article, we will prove the following two theorems:

Theorem 1. *The composition factors of the functor $\mathbb{C}\Pi_k$ are the simple functors $S_{H,\mathbb{C}}$, where H is a p -hypo-elementary B -group and where \mathbb{C} is the trivial $\mathbb{C}\text{Out}(H)$ -module.*

Theorem 2. *Let $G \cong P \rtimes C_n$ be a p -hypo-elementary group (P is a p -group and C_n a cyclic p' -group). Then G is a B -group if and only if:*

- (i) P is elementary abelian;
- (ii) *In a decomposition of P as a direct sum of simple $\mathbb{F}_p C_n$ -modules, every simple $\mathbb{F}_p C_n$ -module appears at most one time, except the trivial module, which appears 0 or 2 times;*
- (iii) *The action of C_n on P is faithful.*

Serge Bouc proved that any finite group G has a largest quotient $\beta(G) = G/N$ which is a B -group, uniquely determined by G . In the course of the proof of Theorem 1, we also prove the following result:

Theorem 3. *Let G be a finite group. Then $\beta(G)$ is p -hypo-elementary if and only if G itself is p -hypo-elementary.*

This article begins with some background results. Then we define a natural transformation θ between the Burnside functor $\mathbb{C}B$ and the functor of permutation modules $\mathbb{C}\Pi_k$. Using this map and the classification of the composition factors of $\mathbb{C}B$ obtained by Serge Bouc [2] (see also Chapter 5 of [1]), we will find the composition factors of the functor of permutation modules $\mathbb{C}\Pi_k$. So we will have the proof of Theorem 1. Then we make a classification of the p -hypo-elementary groups which are B -groups (Theorem 2).

All groups are supposed finite, all vector spaces are finite dimensional and all modules are finitely generated left modules. Let G and H be finite groups. We write $G \gg H$ if H is isomorphic to a quotient of G . By $H \leq_G G$, we denote a subgroup H of G , up to conjugacy (in G). All G -sets and all (H, G) -bisets are finite. We denote by $[U]$ the isomorphism class of U (where U can be a group, a vector space, a module, a G -set, an (H, G) -biset, ...).

2. Background on biset functors

2.1. The category of biset functors

Definition 4. Let G and H be finite groups. Then $B(H, G)$ is the Grothendieck group of the isomorphism classes of finite (H, G) -bisets (for the disjoint union).

Notation 5. Let G be a group. We denote by Id_G the (G, G) -biset G where the two actions are defined by left and right multiplication in G . We also denote by Id_G the image of Id_G in $B(G, G)$.

Definition 6. (See [1], Definition 3.1.6.) We define the category \mathcal{C} as follows:

- The objects of \mathcal{C} are all finite groups;
- If G and H are finite groups, then

$$\text{Hom}_{\mathcal{C}}(G, H) = \mathbb{C} \otimes_{\mathbb{Z}} B(H, G);$$

- The composition of morphisms in \mathcal{C} is the \mathbb{C} -linear extension of the composition in $B(H, G)$, defined by $v \circ u = v \times_H u$ for all finite groups G, H and K , for all morphisms $u \in B(H, G)$ and for all morphisms $v \in B(K, H)$;
- For any finite group G , the identity morphism of G in \mathcal{C} is equal to $\text{Id}_{\mathbb{C}} \otimes_{\mathbb{Z}} \text{Id}_G$.

Definition 7. (See [1], Definition 3.2.2.) A *biset functor defined on \mathcal{C}* with values in $\mathbb{C}\text{-mod}$ is a \mathbb{C} -linear functor from \mathcal{C} to the category $\mathbb{C}\text{-mod}$ of all finite dimensional \mathbb{C} -vector spaces.

Biset functors over \mathcal{C} , with values in $\mathbb{C}\text{-mod}$, are the objects of a category, denoted by \mathcal{F} , where morphisms are natural transformations of functors, and composition of morphisms is composition of natural transformations.

Remark 8. For convenience, we have tensored everything with \mathbb{C} . But actually we only need a field of characteristic 0 which contains all roots of unity.

Proposition 9. (See [1], Proposition 3.2.8.) The category \mathcal{F} is a \mathbb{C} -linear abelian category. In particular, if θ is a morphism of biset functors and G is a finite group, then

$$(\text{Ker } \theta)(G) = \text{Ker } \theta_G, \quad (\text{Coker } \theta)(G) = \text{Coker } \theta_G.$$

The simple objects of the category \mathcal{F} are labeled by pairs (G, V) , where G is a finite group and V a simple $\mathbb{C}\text{Out}(G)$ -module. We denote by $S_{G,V}$ the simple functor associated to (G, V) . If $F \in \mathcal{F}$ is a simple functor, then $F \cong S_{G,V}$ where G is the smallest group (unique up to isomorphism) such that $F(G) \neq \{0\}$ and $V = F(G)$. We can define a notion of isomorphism on those pairs such that two simple functors are isomorphic if and only if the corresponding pairs are isomorphic [1, Theorem 4.3.10].

Proposition 10. (See [1], Lemma 4.3.9.) Let G be a finite group and V a simple $\mathbb{C}\text{Out}(G)$ -module. If H is a finite group such that $S_{G,V}(H) \neq \{0\}$, then G is isomorphic to a subquotient of H .

2.2. Three biset functors

In this section, we want to define three biset functors.

Definition 11. Let G be a finite group. Then $B(G)$ is the Grothendieck group of the set of isomorphism classes of finite G -sets (for disjoint union). Then $B(G)$ is a ring (called the Burnside ring of G), where the multiplication is defined by

$$[U] \cdot [V] = [U \times V]$$

for all G -sets U and V (extended to $B(G)$ by bilinearity).

Let G et H be two finite groups. For every (finite) (H, G) -biset U , we can define the following map:

$$\begin{aligned} B([U]): B(G) &\rightarrow B(H), \\ [V] &\mapsto [U \times_G V] \end{aligned}$$

for every (finite) G -set V . This extends by \mathbb{C} -linearity to a map $\mathbb{C}B([U]): \mathbb{C}B(G) \rightarrow \mathbb{C}B(H)$, where $\mathbb{C}B(G) = \mathbb{C} \otimes_{\mathbb{Z}} B(G)$.

Now we can define $\mathbb{C}B(u)$ for every $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$. Let $u = \sum_{i=1}^n \lambda_i [U_i]$ where $\lambda_i \in \mathbb{C}$ and U_i is an (H, G) -biset, for every $i = 1, \dots, n$. Then $B(u) = \sum_{i=1}^n \lambda_i B([U_i])$. This defines a structure of biset functor $\mathbb{C}B$.

If G is a finite group, then $\{[G/K] \mid K \leqslant_G G\}$ is a \mathbb{C} -basis of $\mathbb{C}B(G)$. But in the rest of this article, we will use another \mathbb{C} -basis, which is the following:

Theorem 12. (See Gluck [3], Yoshida [4].) Let G be a finite group. If H is a subgroup of G , denote by e_H^G the element of $\mathbb{C}B(G)$ defined by

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leqslant H} |K| \mu(K, H) [G/K],$$

where μ is the Möbius function of the poset of subgroups of G .

Then $e_H^G = e_K^G$ if the subgroups H and K are conjugate in G , and the elements e_H^G , for $H \leqslant_G G$, are the primitive idempotents of the \mathbb{C} -algebra $\mathbb{C}B(G)$.

In particular $\{e_H^G \mid H \leqslant_G G\}$ is a \mathbb{C} -basis of $\mathbb{C}B(G)$.

In the next two definitions, the Grothendieck groups are taken with respect to direct sums, that is, with respect to the relations $[M \oplus N] = [M] + [N]$.

Definition 13. We define $\Pi_k(G)$ as the Grothendieck group of the set of isomorphism classes of permutation kG -modules with respect to direct sums. For every (H, G) -biset U we define:

$$\begin{aligned} \Pi_k([U]): \quad \Pi_k(G) &\rightarrow \Pi_k(H), \\ [kP] &\mapsto [kU \otimes_{kG} kP] = [k(U \times_G P)] \end{aligned}$$

for every permutation kG -module kP . As for B , we can extend scalars to \mathbb{C} and define $\mathbb{C}\Pi_k(u) : \mathbb{C}\Pi_k(G) \rightarrow \mathbb{C}\Pi_k(H)$ for $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$, where $\mathbb{C}\Pi_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \Pi_k(G)$. This defines a structure of biset functor $\mathbb{C}\Pi_k$.

Definition 14. We define $\text{pp}_k(G)$ as the Grothendieck group of the set of isomorphism classes of p -permutation kG -modules (i.e. direct sums of indecomposable trivial source kG -modules) with respect to direct sums. For every (H, G) -biset U we define:

$$\begin{aligned} \text{pp}_k([U]): \quad \text{pp}_k(G) &\rightarrow \text{pp}_k(H), \\ [M] &\mapsto [kU \otimes_{kG} M] \end{aligned}$$

for every trivial source kG -module M . As for B and Π_k , we can extend scalars to \mathbb{C} and define $\mathbb{C}\text{pp}_k(u) : \mathbb{C}\text{pp}_k(G) \rightarrow \mathbb{C}\text{pp}_k(H)$ for $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$, where $\mathbb{C}\text{pp}_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \text{pp}_k(G)$. This defines a structure of biset functor $\mathbb{C}\text{pp}_k$.

Moreover, the functor $\mathbb{C}\Pi_k$ is a subfunctor of $\mathbb{C}\text{pp}_k$.

Remark 15. In the case $p = 0$, every kG -module is a p -permutation kG -module and $\mathbb{C}\text{pp}_k(G)$ is the ordinary Grothendieck group of kG -modules.

2.3. The Burnside biset functor

We describe in this section the composition factors of the functor $\mathbb{C}B$.

Definition 16. (See [1], Notation 5.2.2 and Definition 5.4.6.) If N is a normal subgroup of G , we define the number $m_{G,N}$ by

$$m_{G,N} = \frac{1}{|G|} \sum_{XN=G} |X| \mu(X, G) \in \mathbb{Q},$$

where μ is the Möbius function of the poset of subgroups of G .

A finite group G is a *B-group* (over \mathbb{C}) if for every non-trivial normal subgroup N of G , we have $m_{G,N} = 0$.

We denote by $\text{B-gr}(\mathcal{C})$ the class of all finite *B-groups* and by $[\text{B-gr}(\mathcal{C})]$ a set of representatives of isomorphism classes of finite *B-groups.*

A subset \mathcal{A} of $[\text{B-gr}(\mathcal{C})]$ is *closed* if for $G \in \mathcal{A}$ and $H \in [\text{B-gr}(\mathcal{C})]$ with $H \gg G$, we have $H \in \mathcal{A}$.

Remark 17. (See [1], Example 5.2.3.) We have that $m_{G,N} = m_{G,N\Phi(G)}$, for all normal subgroups N of G . In particular, $m_{G,\Phi(G)} = m_{G,1} = 1$. This implies that if G is a *B-group*, then the Frattini subgroup $\Phi(G)$ of G is trivial.

Definition 18. (See [1], Theorem 5.4.11.) Let G be a finite group. Then $\beta(G)$ is defined to be the quotient G/N of G , where N is a normal subgroup of G such that $m_{G,N} \neq 0$ and G/N is a *B-group*.

Remark 19. In the above definition, $\beta(G)$ is well defined, up to group isomorphism. But the normal subgroup N is in general not unique.

Notation 20. (See [1], Notation 5.4.3.) Let G be a finite group. Then \mathbf{e}_G denote the subfunctor of $\mathbb{C}B$ generated by $e_G^G \in \mathbb{C}B(G)$, where e_G^G is the idempotent defined in Theorem 12.

Theorem 21. (See [1], Proposition 5.5.1.)

1. Let G be a *B-group*. Then the subfunctor \mathbf{e}_G of $\mathbb{C}B$ has a unique maximal subfunctor, equal to

$$\mathbf{j}_G = \sum_{\substack{H \in [\text{B-gr}(\mathcal{C})] \\ H \gg G, H \neq G}} \mathbf{e}_H,$$

and the quotient $\mathbf{e}_G/\mathbf{j}_G$ is isomorphic to the simple functor $S_{G,\mathbb{C}}$.

2. If $F \subseteq F'$ are subfunctors of $\mathbb{C}B$ such that F'/F is simple, then there exists a unique $G \in [\text{B-gr}(\mathcal{C})]$ such that $\mathbf{e}_G \subseteq F'$ and $\mathbf{e}_G \not\subseteq F$. In particular, $\mathbf{e}_G + F = F'$, $\mathbf{e}_G \cap F = \mathbf{j}_G$, and $F'/F \cong S_{G,\mathbb{C}}$.

Remark 22. (See [1], Remark 5.5.2.) The “composition factors” (i.e. the simple subquotients) of the Burnside functor $\mathbb{C}B$ on \mathcal{C} are exactly the functors $S_{G,\mathbb{C}}$, where G is an object of \mathcal{C} (i.e. a finite group) which is a *B-group*.

Theorem 23. There is an isomorphism of lattices between the poset of subfunctors of $\mathbb{C}B$ and the poset of closed subsets of $[\text{B-gr}(\mathcal{C})]$.

Proof. This bijection is a consequence of Theorem 5.4.14 and Proposition 5.5.3 of [1]. \square

To be more precise, here is a description of this bijection: Let \mathcal{A} be a closed subset of $[\text{B-gr}(\mathcal{C})]$. We want to define the subfunctor $F_{\mathcal{A}}$ of $\mathbb{C}B$ associated to the set \mathcal{A} . We set $\mathcal{B} = \mathcal{B}_{\mathcal{A}}$ by

$$\mathcal{B} = \{G \in \mathcal{C} \mid \beta(G) \in \mathcal{A}\} = \{G \in \mathcal{C} \mid \exists H \in \mathcal{A}, G \gg H\}.$$

Then, for every group G , we have

$$F_{\mathcal{A}}(G) = \bigoplus_{\substack{H \leqslant_G G \\ H \in \mathcal{B}}} \mathbb{C}e_H^G = \bigoplus_{\substack{H \leqslant_G G \\ \beta(H) \in \mathcal{A}}} \mathbb{C}e_H^G.$$

More precisely, the set $\{e_H^G \mid H \leqslant_G G, H \in \mathcal{B}\} = \{e_H^G \mid H \leqslant_G G, \beta(H) \in \mathcal{A}\}$ is a \mathbb{C} -basis of $F_{\mathcal{A}}(G)$.

Conversely, if F is a subfunctor of $\mathbb{C}B$, we define the associated closed subset \mathcal{A} of $[\text{B-gr}(\mathcal{C})]$ by

$$\mathcal{A} = \{H \in [\text{B-gr}(\mathcal{C})] \mid e_H^H \in F(H)\}.$$

Remark that the functor $\mathbb{C}B$ corresponds to the set $[\text{B-gr}(\mathcal{C})]$.

3. The biset functor of permutation modules

We want to construct a morphism between $\mathbb{C}B$ and $\mathbb{C}\Pi_k$ and then use this morphism and the composition factors of $\mathbb{C}B$ to find those of $\mathbb{C}\Pi_k$.

Proposition 24. *There is a morphism of biset functor (i.e. a \mathbb{C} -linear natural transformation) θ between $\mathbb{C}B$ and $\mathbb{C}\Pi_k$ such that*

$$\theta_G([G/L]) = [k(G/L)]$$

for all finite group G and every subgroup L of G , where we denote by θ_G the map $\theta(G) : \mathbb{C}B(G) \rightarrow \mathbb{C}\Pi_k(G)$ and $[G/L]$, $[k(G/L)]$ are the isomorphism classes of the G -set G/L and the kG -module $k(G/L)$, respectively.

Proof. We extend by \mathbb{C} -linearity the definition of θ_G to $\mathbb{C}B(G)$ and then it is easy to check that θ is well defined and a \mathbb{C} -linear natural transformation. \square

Remark 25. We know that $(\text{Coker } \theta)(G) = \text{Coker } \theta_G$ for all finite group G (Proposition 9), consequently the image of the natural transformation θ is $\mathbb{C}\Pi_k$.

Definition 26. A group H is said to be p -hypo-elementary (or cyclic modulo p) if the quotient $H/O_p(H)$ is cyclic ($O_p(H)$ is the largest normal p -subgroup of G); in other words, H has a normal p -subgroup for which the quotient is a cyclic p' -group.

If $p = 0$, a 0-hypo-elementary group is a cyclic group.

We denote by \mathcal{H} the set of all finite p -hypo-elementary groups.

The aim now is to determine the kernel of θ . We will use the fact that $(\text{Ker } \theta)(G) = \text{Ker } \theta_G$ (Proposition 9) and that the set $\{e_H^G \mid H \leqslant_G G\}$ is a basis of $\mathbb{C}B(G)$ for all finite groups G . To do this, we need the following lemma, due to Conlon. We denote by \bar{k} the algebraic closure of k .

Lemma 27. *Let G be a finite group and E be the set of conjugacy classes of pairs (H, g) , where H is a p -hypo-elementary subgroup of G and g a generator of $H/O_p(H)$. Then we have an isomorphism*

$$\mathbb{C} \text{pp}_{\bar{k}}(G) \cong \bigoplus_{(H, g) \in E} \mathbb{C}.$$

Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}B(G) & \xrightarrow{\cong} & \bigoplus_{H \leqslant G} \mathbb{C} \\ \downarrow \theta_G & & \downarrow \lambda \\ \mathbb{C}pp_{\bar{k}}(G) & \xrightarrow{\cong} & \bigoplus_{(H,g) \in E} \mathbb{C} \end{array}$$

We still write e_H^G for the primitive idempotent in $\bigoplus_{H \leqslant G} \mathbb{C}$ which is the image of the primitive idempotent $e_H^G \in \mathbb{C}B(G)$. We write $\varepsilon_{H,g}$ for the primitive idempotents in $\bigoplus_{(H,g) \in E} \mathbb{C}$. The map λ sends e_H^G to $\sum_{(H,g) \in E} \varepsilon_{H,g}$ if H is p -hypo-elementary, and zero otherwise.

Remark 28. The map θ_G is defined between $\mathbb{C}B(G)$ and $\mathbb{C}\Pi_{\bar{k}}(G)$, but as $\mathbb{C}\Pi_{\bar{k}}$ is a subfunctor of $\mathbb{C}pp_{\bar{k}}$, we can extend it to $\mathbb{C}pp_{\bar{k}}$ by composing with the inclusion.

Proof. A proof of this result in characteristic $p \neq 0$ can be found in [5], p. 188. The case $p = 0$ is completely straightforward. \square

Proposition 29. Let G be a finite group. The set

$$B_{\text{Ker}} = \{e_H^G \mid H \leqslant G, H \notin \mathcal{H}\}$$

is a basis of $\text{Ker } \theta_G$. Moreover, the set

$$B_{\text{Im}} = \{\theta(e_H^G) \mid H \leqslant G, H \in \mathcal{H}\}$$

is a basis of $\text{Im } \theta_G$.

Proof. We considered the following composition of applications

$$\mathbb{C}B(G) \rightarrow \mathbb{C}\Pi_k(G) \xrightarrow{f} \mathbb{C}\Pi_{\bar{k}}(G) \xrightarrow{\iota} \mathbb{C}pp_{\bar{k}}(G),$$

where f is the scalar extension from k to \bar{k} and ι is the inclusion. Clearly, the map ι is injective. Let M and N be two permutation kG -modules such that $f([M]) = f([N])$, that is $\bar{k} \otimes_k M \cong \bar{k} \otimes_k N$. As M and N are finitely generated, this implies that $L \otimes_k M \cong L \otimes_k N$, for some finite dimensional field extension L of k . But, by Exercise 2, p. 138 of [6], this implies that $M \cong N$. So, we have that f is also injective.

Now the composition map $g = \iota \circ f \circ \theta_G$ is exactly the map θ_G of Lemma 27, and so B_{Ker} is a subset of $\text{Ker } g$ and the set $\{g(e_H^G) \mid H \leqslant G, H \in \mathcal{H}\}$ is linearly independent. As $\iota \circ f$ is injective, this implies that B_{Ker} is a subset of $\text{Ker } \theta_G$ and the set B_{Im} is linearly independent. We already know that B_{Ker} is a linearly independent set. If n is the number of conjugacy classes of subgroups of G , we have that:

$$n = \dim_{\mathbb{C}} \mathbb{C}B(G) = \dim_{\mathbb{C}} \text{Ker } \theta_G + \dim_{\mathbb{C}} \text{Im } \theta_G \geqslant |B_{\text{Ker}}| + |B_{\text{Im}}| = n$$

and so we must have equality, which proves the result. \square

We now have a basis of $\text{Ker } \theta_G$ and $\text{Im } \theta_G$ for every finite group G and we will use this to study the image of the composition factors of $\mathbb{C}B$.

Let G be a finite B -group. If G is not a p -hypo-elementary group, then $e_G^G \in \text{Ker } \theta_G$, hence $\mathbf{e}_G \subseteq \text{Ker } \theta$. So we can assume that G is a p -hypo-elementary group. Now the functor $\theta(\mathbf{e}_G)/\theta(\mathbf{j}_G)$ is a

quotient of the simple functor $\mathbf{e}_G/\mathbf{j}_G \cong S_{G,\mathbb{C}}$. Hence it is either 0 or isomorphic to $S_{G,\mathbb{C}}$. It is zero if and only if $\theta(\mathbf{e}_G) = \theta(\mathbf{j}_G)$, i.e. if $\mathbf{e}_G \subseteq \mathbf{j}_G + \text{Ker } \theta$. But $e_G^G \in \mathbf{e}_G(G)$, and $e_G^G \notin \mathbf{j}_G(G) + \text{Ker } \theta_G$, as $G \in \mathcal{H}$. Hence $\theta(\mathbf{e}_G) \neq \theta(\mathbf{j}_G)$ and $\theta(\mathbf{e}_G)/\theta(\mathbf{j}_G) \cong S_{G,\mathbb{C}}$.

Now we found the image of every composition functor of $\mathbb{C}B$ in $\mathbb{C}\Pi_k$ and as every composition factor has a preimage in $\mathbb{C}B$, this gives us the complete list of composition factors of $\mathbb{C}\Pi_k$. So we have proved the following theorem.

Theorem 30. *The composition factors of the functor $\mathbb{C}\Pi_k$ are the simple functors $S_{H,\mathbb{C}}$, where H is a p -hypo-elementary B -group and where \mathbb{C} is the trivial $\mathbb{C}\text{Out}(H)$ -module.*

Remarks 31.

1. With the same method, we can find an infinite sequence of subfunctors of $\mathbb{C}\Pi_k$ such that every successive quotient is simple. But to do this we need to make a choice. This sequence is finite if we evaluate it in a finite group.
2. With the same method, we can find the description of all the subfunctors of $\mathbb{C}\Pi_k$. We use the description of the subfunctors of $\mathbb{C}B$ and we obtain a bijection between the subfunctor of $\mathbb{C}\Pi_k$ and the closed subset of $[\text{B-gr}(\mathcal{C})] \cap \mathcal{H}$.
3. If $p = 0$, the unique B -group which is cyclic is $\mathbf{1}$ so we obtain that $\mathbb{C}\Pi_k \cong S_{\mathbf{1},\mathbb{C}}$, which is Proposition 4.4.8 of [1].

Moreover, the above proof implies the next theorem:

Theorem 32. *Let G be a finite group. Then $\beta(G)$ is p -hypo-elementary if and only if G itself is p -hypo-elementary.*

Proof. Let G be a finite group and H be a subgroup of G . By Corollary 29, we know that e_H^G is an element of $\ker \theta_G$ if and only if $H \notin \mathcal{H}$.

On the other hand, using the bijection between the subfunctors of $\mathbb{C}B(G)$ and the closed subsets of $[\text{B-gr}(\mathcal{C})]$, we find that the kernel $\text{Ker } \theta$ corresponds to the set $\mathcal{N} = [\text{B-gr}(\mathcal{C})] \cap \{G \mid G \notin \mathcal{H}\}$. But this implies that e_H^G is in $\ker \theta_G$ if and only if there exists $L \in \mathcal{N}$ such that $H \gg L$, which is equivalent to $\beta(H) \gg L$. But this implies that $e_H^G \in \ker \theta_G$ if and only if $\beta(H) \in \mathcal{N}$ (because \mathcal{N} is closed), that is, if and only if $\beta(H) \notin \mathcal{H}$.

If we put together those two results, we obtain that $H \notin \mathcal{H}$ if and only if $\beta(H) \notin \mathcal{H}$, which proves that $H \in \mathcal{H}$ if and only if $\beta(H) \in \mathcal{H}$. \square

4. B-groups and p -hypo-elementary groups

Now, to make precise Theorem 30, we want to find which p -hypo-elementary groups are also B -groups. Recall that the rational number $m_{G,N}$ is defined in Definition 16.

Proposition 33. (See [1], Proposition 5.6.4.) *If N is a minimal normal abelian subgroup of G , then*

$$m_{G,N} = 1 - \frac{|K_G(N)|}{|N|}$$

where $K_G(N)$ is the set of complements of N in G .

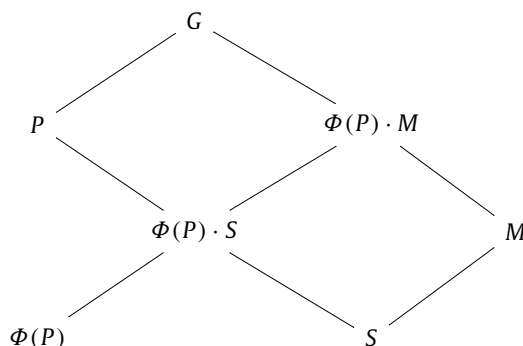
In particular, if the group G is solvable, then G is a B -group if and only if $|K_G(N)| = |N|$ for all minimal normal subgroups N of G .

Remark 34. As all p -hypo-elementary groups are solvable, we will use the second part of this proposition to determine which p -hypo-elementary groups are B -groups.

The aim is to find necessary and sufficient conditions for a p -hypo-elementary group to be a B -group. In order to do this, we need some results that can be stated in a more general case. We suppose that $G = P \rtimes H$ where P is a p -group of order p^m and H a p' -group of order n . This notation is kept throughout this section.

Proposition 35. *The Frattini subgroup $\Phi(G)$ contains $\Phi(P)$.*

Proof. Let M be a maximal subgroup of G . We have to show that $\Phi(P)$ is contained in M . Clearly, P is the unique p -Sylow subgroup of G , consequently P is the set of p -elements of G . So if we set $S = M \cap P$, then S is a normal p -Sylow subgroup of M . If $S = P$, then it is clear that $\Phi(P) \subseteq P \subseteq M$, so we can suppose that $S \neq P$. As $\Phi(P)$ is a characteristic subgroup of P and P is a normal subgroup of G , G normalizes $\Phi(P)$. As a consequence, $\Phi(P) \cdot S$ and $\Phi(P) \cdot M$ are subgroups. We now have the following diagram:



As M is maximal we have that $\Phi(P) \cdot M$ is either M or G . But

$$(\Phi(P) \cdot M) \cap P = \Phi(P) \cdot S \neq P$$

(because $S \neq P$) so $\Phi(P) \cdot M$ must be M , which implies that $\Phi(P) \subseteq M$. \square

Corollary 36. *If G is a B -group, then P is an elementary abelian group.*

Proof. This is a consequence of Proposition 35 and Remark 17. \square

So we suppose now that P is an elementary abelian group. So P is an \mathbb{F}_p -vector space on which H acts, namely an $\mathbb{F}_p H$ -module. As H is a p' -group, P is a semi-simple $\mathbb{F}_p H$ -module.

Proposition 37. *Suppose that H is a cyclic group of order n . If G is a B -group, then the group H acts faithfully on P .*

Proof. The case $n = 1$ is clear, so we can suppose that $n \geq 2$. Let d be the divisor of n such that $\text{Ker } \varphi = C_d$, where $\varphi : H \rightarrow \text{Aut}(P)$ is the action of H on P . To show that the action is faithful, we have to show that $d = 1$.

So we suppose that $d > 1$. As C_d acts trivially on P and H is abelian, C_d is a central subgroup of G , and so in particular a normal subgroup. As $d > 1$, there exists a minimal normal subgroup N of C_d (it is even central). But then N has at most one complement in G : If C is a complement of N in G , then C contains the unique Sylow p -subgroup P of G . Consequently, we have that $C = P \rtimes L$, where L is a subgroup of H . But then L is a complement of N in H , which is cyclic, so there is at most one possibility for L .

But N should have $|N| > 1$ complements in G because G is a B -group (Proposition 33). So we must have that $d = 1$, that is, the action is faithful. \square

Lemma 38. *Let G be a finite group and let H, K and L be subgroups of G such that $H \leq K \leq H \cdot L$. Then $K = H(K \cap L)$.*

Proof. Clear. \square

Proposition 39. *If H acts faithfully on P , then a minimal normal subgroup N of G is always contained in P .*

Proof. Let N be a minimal normal (non-trivial) subgroup of G . If $N \cap P \neq 1$ then by minimality of N , we have that $N \subseteq P$. We can now suppose that $N \cap P = 1$. Then, as N and P are normal in G , $[N, P] = 1$, which implies that $N \subseteq C_G(P)$.

We have that $P \leq C_G(P) \leq P \cdot H$ so by Lemma 38 we have that $C_G(P) = P \cdot (C_G(P) \cap H) = P \cdot C_H(P)$. But as H acts faithfully on P , $C_H(P) = 1$ so that $N \subseteq C_G(P) = P$. This is impossible because this implies that $N = 1$, which contradicts the assumption on N . \square

Proposition 40. *Let N be a normal subgroup of G contained in P . Then every complement of N is of the form $S \rtimes Q$, where S is a normal subgroup of G which is a complement of N in P and Q is a subgroup of G conjugate to H .*

Proof. Let C be a complement of N in G (i.e. $N \cap C = 1$ and $N \cdot C = G$). We define $S = C \cap P$, which is a normal p -Sylow subgroup of C . By the Schur–Zassenhaus theorem [7, Theorem 7.41], there exists a subgroup Q of C such that $C = S \rtimes Q$. Notice that the order of Q is n so that Q is conjugate to H (by the second part of the Schur–Zassenhaus theorem [7], Theorem 7.42). Now S is normal in C and in P (which is abelian) so also in $G = P \rtimes Q$.

Conversely, let $C = S \rtimes Q$ such that S is a normal subgroup of G which is a complement of N in P and Q is a subgroup of G of order n (hence conjugate to H). Clearly, we have that $N \cap C = N \cap S = 1$ and $N \cdot C = N \cdot (S \rtimes Q) = (N \cdot S) \rtimes Q = P \rtimes Q = G$, which proves that C is a complement of N in G . \square

Remark 41. We will need those results in the following case: If G is a p -hypo-elementary group (i.e. H is cyclic) and a B -group, then, by Proposition 37, the action of H on P is faithful. Let N be a minimal normal subgroup of G . Then, by Proposition 39, N is contained in P and Proposition 40 applies.

For the rest of this part, $G = P \rtimes H$ is a p -hypo-elementary group. This means that H is a cyclic group. We suppose that it is a B -group, so we know that P is elementary abelian, P is a semi-simple $\mathbb{F}_p H$ -module and H acts faithfully on P . We now decompose P into his isotypic components (see [6], pp. 46–47 for a definition):

$$P \cong \bigoplus_{i=1}^t P_i$$

and for each isotypic component, there exists a simple $\mathbb{F}_p H$ -module S_i and an integer m_i such that

$$P_i \cong \bigoplus_{j=1}^{m_i} S_i.$$

We can suppose that S_1 is the trivial $\mathbb{F}_p H$ -module (if necessary, we add $P_1 = \{0\}$). For each $1 \leq i \leq t$, there exists an integer s_i such that $|S_i| = p^{s_i}$.

We are now able to find the number of complements of a minimal normal subgroup N of G and find the necessary and sufficient condition such that this number is $|N|$.

Let N be a minimal normal subgroup of G . By Proposition 39, N is a subgroup of P on which H acts, i.e. an $\mathbb{F}_p H$ -submodule of P . Furthermore, the minimality of N implies that N is simple. So there exists an integer $1 \leq l \leq t$ such that $N \cong S_l$. Then N is a submodule of P_l . We know by Proposition 40 that a complement of N in G is of the form $C \rtimes Q$ where C is a normal subgroup of G which is a complement of N in P and Q is a cyclic subgroup of G of order n . So C is a complement of N in P not only as a group but also as a module. Our first step will be to determine the number of possibilities for C . In order to do this, we use the fact that the isotypic components are unique up to isomorphism [6, pp. 46–47] which implies that a complement of N in P (as a module) is of the form

$$H_l \oplus \bigoplus_{\substack{i=1 \\ i \neq l}}^t P_i$$

where H_l is a complement (as a module) of N in P_l .

By using the isomorphism $\mathbb{F}_p H \cong \mathbb{F}_p[t]/\langle t^n \rangle$ and the Chinese remainder theorem, we can see that the simple $\mathbb{F}_p H$ -modules can be seen as fields and this field structure contains the structure of $\mathbb{F}_p H$ -module. So we can see $N \cong S_l$ as the field $\mathbb{F}_{p^{s_l}}$ and the module P_l as a vector space over this field. Moreover, in order to find the number of complements of N in P_l , it is sufficient to count the number of complements as $\mathbb{F}_{p^{s_l}}$ -vector space. This is an easy exercise: N has

$$p^{s_l(m_l-1)}$$

complements in P_l and so also in P .

As every complement of N is of the form $C \rtimes Q$ and we know the number of possibilities for C , we only need to find the number of possibilities for Q (when C is fixed). In fact, this is equal to the number of complements of P in G divided by the number of complements of C in $C \rtimes Q$, where Q is an arbitrary subgroup of G conjugate to H . In order to make this calculation, we need the next lemma:

Lemma 42. *The number of complements of P in G is equal to p^{m-m_1} .*

Proof. We have to find the number of conjugates of H . Let E be the set of conjugates of H . By the Schur–Zassenhaus theorem [7, Theorem 7.41], we know that P acts transitively on E . Let S be the stabilizer of H in P . Then the number of conjugates of H is equal to $|P|/|S|$. So, in order to conclude, we have to determine the order of S . After some easy calculation, we can find that $S = P_1$ and consequently the number of conjugates of H is p^{m-m_1} . \square

This gives us the two numbers we have to divide: The number of complements of P in G is p^{m-m_1} and the number of complements of C in $C \rtimes Q$ for a fixed Q is:

$$\begin{cases} p^{m-s_1-(m_1-1)} = p^{m-m_1} & \text{if } l = 1, \\ p^{m-s_l-m_1} & \text{if } l \neq 1. \end{cases}$$

So the number of possibilities for Q is

$$\begin{cases} 1 & \text{if } l = 1, \\ p^{s_l} & \text{if } l \neq 1. \end{cases}$$

To conclude, the number of complements of N in G is:

$$\begin{cases} p^{m_1-1} & \text{if } l = 1, \\ p^{s_l m_l} & \text{if } l \neq 1. \end{cases}$$

But for G to be a B -group, we must have that the number of complements of N is $|N| = p^{s_l}$. So if $l = 1$, $m_1 = 2$ (or $m_1 = 0$ if $P_1 = 0$) and if $l \neq 1$ then $m_l = 1$, which concludes the proof of the following theorem:

Theorem 43. *Let $G \cong P \rtimes C_n$ be a p -hypo-elementary group (P is a p -group and C_n a cyclic p' -group). Then G is a B -group if and only if:*

- (i) P is elementary abelian;
- (ii) In a decomposition of P as a direct sum of simple $\mathbb{F}_p C_n$ -modules, every simple $\mathbb{F}_p C_n$ -module appears at most one time, except the trivial module, which appears 0 or 2 times;
- (iii) The action of C_n on P is faithful.

Remarks 44.

1. The above theorem can be generalized to other groups ($P \rtimes H$, where P is a p -group and H a solvable p' -group). For more details, see [8].
2. If $p = 0$, we have proved that the only cyclic B -group is **1**. So we recover a known result given in [2], Example 7.2.5.

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References

- [1] S. Bouc, Biset Functors for Finite Groups, Springer-Verlag, Berlin, 2010.
- [2] S. Bouc, Foncteurs d'ensembles munis d'une double action, J. Algebra 183 (1996) 664–736.
- [3] D. Gluck, Idempotent formula for the Burnside ring with applications to the p -subgroup simplicial complex, Illinois J. Math. 25 (1981) 63–67.
- [4] T. Yoshida, Idempotents in Burnside rings and Dress induction theorem, J. Algebra 80 (1983) 90–105.
- [5] D. Benson, Representations and Cohomology. Basic Representation Theory of Finite Groups and Associative Algebras, vol. 1, Cambridge University Press, Cambridge, 2004.
- [6] C.W. Curtis, I. Reiner, Methods of Representation Theory: With Applications to Finite Groups and Orders, vol. 1, Wiley, New York, 1981.
- [7] J.J. Rotman, An Introduction to the Theory of Groups, Springer-Verlag, New York, 1995.
- [8] M. Baumann, PhD thesis in preparation, Ecole Polytechnique Fédérale de Lausanne.