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Affine Lie–Poisson reduction, Yang–Mills magnetohydrodynamics, and superfluids

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Abstract

This paper develops the theory of affine Lie–Poisson reduction and applies this process to Yang–Mills and Hall magnetohydrodynamics for fluids and superfluids. As a consequence of this approach, the associated Poisson brackets are obtained by reduction of a canonical cotangent bundle. A Kelvin–Noether circulation theorem is presented and is applied to these examples.

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1. Introduction

The equations of motion of a non-relativistic *adiabatic compressible fluid* are given by

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}}\mathbf{u} = \dfrac{1}{\rho}\operatorname{grad} p, \\ \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0, \qquad \dfrac{\partial S}{\partial t} + \operatorname{div}(S\mathbf{u}) = 0, \end{cases} \tag{1.1}$$

where ρ is the mass density, S is the entropy density and p is the pressure. It was shown in Morrison and Greene (1980) that this system, as well as its magnetohydrodynamic extension, admits a noncanonical Poisson formulation, that is, equation (1.1) can be written as

$$\dot{f} = \{f, h\},$$

relative to a Hamiltonian function h. The study of the relativistic case was initiated in Białynicki-Birula *et al* (1984), Mayer (1984) and Holm and Kupershmidt (1984a). The present paper considers only non-relativistic fluids.

It is of great (mathematical and physical) interest to obtain these Poisson brackets by a reduction procedure from a canonical Hamiltonian formulation on a cotangent bundle.

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In Marsden *et al* (1984), the noncanonical Poisson bracket associated with (1.1) is obtained via Lie–Poisson reduction for a semidirect product group involving the diffeomorphisms group of the fluid container and the space of the advected quantities ρ and S. In the same spirit, the noncanonical Hamiltonian structure for *adiabatic Yang–Mills charged fluid* discovered in Gibbons *et al* (1983) is obtained by reduction from a canonical formulation in Gay-Balmaz and Ratiu (2008), by using a Kaluza–Klein point of view involving the automorphism group of the principal bundle of the theory.

Noncanonical Hamiltonian structures for a wide class of non-dissipative fluid models were derived in Holm and Kupershmidt (1984b), Holm (1987), Holm and Kupershmidt (1987), Holm and Kupershmidt (1988) and Holm (2001). These examples include *Yang–Mills magnetohydrodynamics*, *spin glasses* and various models of *superfluids*, and involve Lie–Poisson brackets *with cocycles*. Remarkably, from a mathematical point of view, the Hamiltonian structures of many of these models are identical. This Hamiltonian structure is studied in more detail, with an application to *liquid crystals*, in Holm (2002). We will refer to all these models as *complex fluids*.

In this paper we show the remarkable property that these Lie–Poisson brackets with cocycles can also be obtained by Poisson reduction from a canonical Hamiltonian structure. The cocycle in the Hamiltonian structure appears only after reduction and it is due to the presence of an affine term added to the cotangent lifted action. The associated reduction process is naturally called *affine Lie–Poisson reduction*.

An important example of such an affine action is given by the usual action of the automorphism group of a principal bundle on the connection forms. As a result, we obtain, in a natural way, covariant differentials and covariant divergences in the expression of the Poisson brackets and of the reduced equations. These gauge theory aspects in the case of complex fluids are mathematically and physically interesting since they represent a bridge to other possible gauge theories in physics.

We begin by recalling some needed facts about Lie–Poisson reduction for semidirect products (see Holm *et al* (1998)). Let $\rho : G \to \mathrm{Aut}(V)$ denote a *right* Lie group representation of G on the vector space V. As a set, the semidirect product $S = G \circledS V$ is the Cartesian product $S = G \times V$, whose group multiplication is given by

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + \rho_{g_2}(v_1)).$$

The Lie algebra of S is the semidirect product Lie algebra, $\mathfrak{s} = \mathfrak{g} \circledS V$, whose bracket has the expression

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) = [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1),$$

where $v\xi$ denotes the induced action of $\mathfrak{g}$ on V, that is,

$$v\xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho_{\exp(t\xi)}(v) \in V.$$

From the expression for the Lie bracket, it follows that for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi),$$

where $a\xi \in V^*$ and $v \diamond a \in \mathfrak{g}^*$ are given by

$$a\xi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho^*_{\exp(-t\xi)}(a) \qquad \text{and} \qquad \langle v \diamond a, \xi\rangle_{\mathfrak{g}} := -\langle a\xi, v\rangle_V,$$

and where $\langle\cdot,\cdot\rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\cdot,\cdot\rangle_V : V^* \times V \to \mathbb{R}$ are the duality parings.

Hamiltonian semidirect product theory. Let $S := G \circledS V$ be the semidirect product defined before. The lift of right translation of S on T^*S induces a right action on $T^*G \times V^*$. Consider a Hamiltonian function $H : T^*G \times V^* \to \mathbb{R}$ right invariant under the S-action on $T^*G \times V^*$. In particular, the function $H_{a_0} := H|_{T^*G\times\{a_0\}} : T^*G \to \mathbb{R}$ is invariant under the induced action of the isotropy subgroup $G_{a_0} := \{g \in G \mid \rho_g^* a_0 = a_0\}$ for any $a_0 \in V^*$. The following theorem is an easy consequence of the semidirect product reduction theorem (see Marsden *et al* (1984)) and the reduction by stages method (see Marsden *et al* (2007)).

Theorem 1.1. *For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^*R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent:*

(i) $\alpha(t)$ satisfies Hamilton's equations for H_{a_0} on T^*G.
(ii) The Lie–Poisson equation holds on $\mathfrak{s}^*$:

$$\frac{\partial}{\partial t}(\mu, a) = -\mathrm{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)}(\mu, a) = -\left(\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}}\mu + \frac{\delta h}{\delta a} \diamond a, a\frac{\delta h}{\delta \mu}\right), \qquad a(0) = a_0$$

where $\mathfrak{s}$ is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$. The associated Poisson bracket is the Lie–Poisson bracket on the semidirect product Lie algebra $\mathfrak{s}^*$, that is,

$$\{f, g\}(\mu, a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle.$$

The evolution of the advected quantities is given by $a(t) = \rho^*_{g(t)}(a_0)$.

2. Affine Lie–Poisson reduction

The goal of this section is to carry out a generalization of the standard process of Lie–Poisson reduction for Lie groups, which is motivated by the example of superfluids. The only modification lies in the fact that the Lie group G acts on its cotangent bundle by a cotangent lift *plus an affine term*. The principal result of this section states that, under some conditions, reducing a canonical symplectic form relative to a cotangent lift with an affine term is equivalent to reduce a magnetic symplectic form relative to the right-cotangent lift. At the reduced level, we obtain affine Lie–Poisson brackets and affine coadjoint orbits, whose affine terms depend on the affine term in the action.

Consider the cotangent lift $R_g^{T^*}$ of the right translation R_g on a Lie group G. Recall that $R_g^{T^*}$ is the right action of G on T^*G given by

$$R_g^{T^*}(\alpha_f) = T^*R_{g^{-1}}(\alpha_f).$$

Consider the map $\Psi_g : T^*G \to T^*G$ defined by

$$\Psi_g(\alpha_f) := R_g^{T^*}(\alpha_f) + C_g(f), \tag{2.1}$$

where $C : G \times G \to T^*G$ is a smooth map such that $C_g(f) \in T^*_{fg}G$, for all $f, g \in G$. The map Ψ_g is seen here as a modification of the cotangent lift by an affine term C. The following lemma gives the conditions guaranteeing that the map Ψ_g is a right action.

Lemma 2.1. *Consider the map Ψ_g defined in (2.1). The following are equivalent:*

(i) Ψ_g is a right action;
(ii) For all $f, g, h \in G$, the affine term C verifies the property

$$C_{gh}(f) = C_h(fg) + R_h^{T^*}(C_g(f)); \tag{2.2}$$

(iii) There exists a one-form $\alpha \in \Omega^1(G)$ such that $C_g(f) = \alpha(fg) - R_g^{T^*}(\alpha(f))$.

We denote by $\mathcal{C}(G)$ the space of all maps $C : G \times G \to T^*G$, $(g, f) \mapsto C_g(f) \in T^*_{fg}G$ verifying the property (2.2). Remark that given an affine term $C \in \mathcal{C}(G)$, the one-form α in item (iii) is only determined up to a right-invariant one-form. Denoting by $\Omega^1_R(G)$ the space of all right-invariant one-forms on G, we have an isomorphism between $\mathcal{C}(G)$ and $\Omega^1(G)\big/\Omega^1_R(G)$. This space is clearly isomorphic to the space $\Omega^1_0(G)$ of all one-forms α on G such that $\alpha(e) = 0$. We can now state the main result of this section.

Theorem 2.2. *Consider the symplectic manifold* $(T^*G, \Omega_{\mathrm{can}})$*, and the affine action*

$$\Psi_g(\beta_f) := R_g^{T^*}(\beta_f) + C_g(f),$$

where $C \in \mathcal{C}(G)$*. Let* $\alpha \in \Omega^1_0(G)$ *be the one-form associated with* Ψ_g*. Then the following hold:*

(i) The fiber translation $t_\alpha : (T^*G, \Omega_{\mathrm{can}}) \to (T^*G, \Omega_{\mathrm{can}} - \pi_G^* \mathbf{d}\alpha)$ *is a symplectic map. The action induced by* Ψ_g *on* $(T^*G, \Omega_{\mathrm{can}} - \pi_G^*\mathbf{d}\alpha)$ *through* t_α *is simply the cotangent lift* $R_g^{T^*}$*.*

(ii) Suppose that $\mathbf{d}\alpha$ *is* G*-invariant. Then the action* Ψ_g *is symplectic relative to the canonical symplectic form* Ω_{can}*.*

(iii) Suppose that there is a smooth map $\phi : G \to \mathfrak{g}^*$ *that satisfies*

$$\mathbf{i}_{\xi^L}\mathbf{d}\alpha = \mathbf{d}\langle\phi, \xi\rangle$$

for all $\xi \in \mathfrak{g}$*, where* ξ^L *is the left invariant extension of* ξ*. Then the map*

$$\mathbf{J}_\alpha = \mathbf{J}_R \circ t_\alpha - \phi \circ \pi_G,$$

where $\mathbf{J}_R(\alpha_f) = T_e^*L_f(\alpha_f)$ *is a momentum map for the action* Ψ_g *relative to the canonical symplectic form. We can always choose* ϕ *such that* $\phi(e) = 0$*. In this case, the nonequivariance cocycle of* $\mathbf{J}_\alpha$ *is* $\sigma = -\phi$*.*

(iv) The symplectic reduced space $\left(\mathbf{J}_\alpha^{-1}(\mu)\big/G_\mu^\sigma, \Omega_\mu\right)$ *is symplectically diffeomorphic to the affine coadjoint orbit*

$$\mathcal{O}_\mu^\sigma = \{\mathrm{Ad}_g^*\mu + \sigma(g) \mid g \in G\},$$

endowed with the affine orbit symplectic form

$$\omega_\sigma^+(\lambda)(\mathrm{ad}_\xi^*\lambda - \Sigma(\xi, \cdot), \mathrm{ad}_\eta^*\lambda - \Sigma(\eta, \cdot)) = \langle\lambda, [\xi, \eta]\rangle - \Sigma(\xi, \eta),$$

where $\Sigma(\xi, \cdot) := -T_e\sigma(\xi)$*. The symplectic diffeomorphism is induced by the* G_μ^σ*-invariant smooth map*

$$\psi : \mathbf{J}_\alpha^{-1}(\mu) \to \mathcal{O}_\mu^\sigma, \qquad \psi(\alpha_g) := \Psi_{g^{-1}}(\alpha_g).$$

The affine coadjoint orbits $\left(\mathcal{O}_\mu^\sigma, \omega_\sigma^+\right)$ are symplectic leaves in the affine Lie–Poisson space $\left(\mathfrak{g}^*, \{\,,\}_\sigma^+\right)$, where

$$\{f, g\}_\sigma^+(\mu) = \left\langle\mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right\rangle - \Sigma\left(\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right). \tag{2.3}$$

3. Affine Hamiltonian semidirect product theory

In this section we carry out the Poisson and symplectic reductions of a canonical cotangent bundle $(T^*S, \Omega_{\text{can}})$, where $S = G \circledS V$ is the semidirect product of a Lie group G and a vector space V and where S acts on its cotangent bundle by cotangent lift *plus an affine term*. We will see that this process is a particular case of the theory developed in the previous section.

Consider the semidirect product Lie group $S := G \circledS V$ associated with a right representation $\rho : G \to \text{Aut}(V)$. The cotangent lift of the right translation is given by

$$R^{T^*}_{(g,v)}(\alpha_f, (u, a)) = (R^{T^*}_g(\alpha_f), v + \rho_g(u), \rho^*_{g^{-1}}(a)) \in T^*_{(f,u)(g,v)}S.$$

We modify this cotangent lifted action by an affine term of the form

$$C_{(g,v)}(f, u) := (0_{fg}, v + \rho_g(u), c(g)), \tag{3.1}$$

for a group one-cocycle $c \in \mathcal{F}(G, V^*)$, that is, verifying the property $c(fg) = \rho^*_{g^{-1}}(c(f)) + c(g)$. The resulting affine right action on T^*S is therefore given by

$$\begin{aligned}\Psi_{(g,v)}(\alpha_f, (u, a)) &:= R^{T^*}_{(g,v)}(\alpha_f, (u, a)) + C_{(g,v)}(f, u) \\ &= \left(R^{T^*}_g(\alpha_f), v + \rho_g(u), \rho^*_{g^{-1}}(a) + c(g)\right).\end{aligned} \tag{3.2}$$

This action is clearly of the form (2.1), and it is readily verified that property (2.2) holds. This proves that $\Psi_{(g,v)}$ is a right action.

In the following lemma, we compute the one-form $\alpha \in \Omega^1_0(S)$ associated with C and we show that it verifies the hypotheses of theorem 2.2. Recall that α is defined by $\alpha(g, v) := C_{(g,v)}(e, 0)$.

Lemma 3.1. *The one-form $\alpha \in \Omega^1_0(S)$ associated with the affine term (3.1) is given by*

$$\alpha(g, v)(\xi_g, (v, u)) = \langle c(g), u\rangle, \tag{3.3}$$

for $(\xi_g, (v, u)) \in T_{(g,v)}S$. Moreover, $\mathbf{d}\alpha$ is S-invariant and its value at the identity is given by

$$\mathbf{d}\alpha(e, 0)((\xi, u), (\eta, w)) = \langle \mathbf{d}c(\xi), w\rangle - \langle \mathbf{d}c(\eta), u\rangle. \tag{3.4}$$

The map $\phi : S \to \mathfrak{s}^*$ defined by

$$\phi(g, v) = (\mathbf{d}c^T(v) - v \diamond c(g), -c(g)),$$

verifies the property

$$\mathbf{i}_{(\xi,u)^L}\mathbf{d}\alpha = \mathbf{d}\langle\phi, (\xi, u)\rangle,$$

where $(\xi, u)^L \in \mathfrak{X}(S)$ is the left-invariant vector field induced by $(\xi, u) \in \mathfrak{s}$.

Using the equality $\sigma = -\phi$, we obtain that the bilinear form Σ, appearing in the formula of the affine orbit symplectic form and the affine Lie–Poisson bracket, is given by

$$\begin{aligned}\Sigma((\xi, u), \cdot) &= -T_{(e,0)}\sigma(\xi, u) = -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (tu \diamond c(\exp(t\xi)) - \mathbf{d}c^T(tu), c(\exp(t\xi))) \\ &= (\mathbf{d}c^T(u), -\mathbf{d}c(\xi)),\end{aligned}$$

where $(\xi, u) \in \mathfrak{s}$.

The momentum map. By item (iii) of theorem 2.2 and using the one-form $\alpha \in \Omega^1_0(S)$ given by (3.3), we obtain that a momentum map for the right-action (3.2) is given by

$$\begin{aligned}\mathbf{J}_\alpha(\beta_f, (u, a)) &= \mathbf{J}_R(t_\alpha(\beta_f, (u, a))) - \phi(f, u) \\ &= (T^*L_f(\beta_f) + u \diamond a - \mathbf{d}c^T(u), a),\end{aligned} \tag{3.5}$$

with nonequivariance one-cocycle

$$\sigma(f,u) = -\phi(f,u) = (u \diamond c(f) - \mathbf{d}c^T(u), c(f)) \in \mathfrak{s}^*. \tag{3.6}$$

Poisson bracket and symplectic reduced spaces. Using formula (2.3) and the expression of Σ, we obtain that the reduced Poisson bracket on $\mathfrak{s}^*$ is given by

$$\{f,g\}^+_\sigma(\mu,a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle + \left\langle \mathbf{d}c\left(\frac{\delta f}{\delta \mu}\right), \frac{\delta g}{\delta a}\right\rangle - \left\langle \mathbf{d}c\left(\frac{\delta g}{\delta \mu}\right), \frac{\delta f}{\delta a}\right\rangle.$$

By item (iv) of theorem 2.2, the reduced space $\left(\mathbf{J}_\alpha^{-1}(\mu,a)\big/S^\sigma_{(\mu,a)}, \Omega_{(\mu,a)}\right)$ is symplectically diffeomorphic to the affine coadjoint orbit $\left(\mathcal{O}^\sigma_{(\mu,a)}, \omega^+_\sigma\right)$. More precisely, we have

$$\begin{aligned}\mathcal{O}^\sigma_{(\mu,a)} &= \{\mathrm{Ad}^*_{(g,u)}(\mu,a) + \sigma(g,u) | (g,u) \in S\}\\ &= \{(\mathrm{Ad}^*_g\mu + u \diamond (\rho^*_{g^{-1}}(a) + c(g)) - \mathbf{d}c^T(u), \rho^*_{g^{-1}}(a) + c(g)) | (g,u) \in S\}.\end{aligned} \tag{3.7}$$

The symplectic structure on $\mathcal{O}^\sigma_{(\mu,a)}$ is given by

$$\begin{aligned}&\omega^+_\sigma(\lambda,b)((\mathrm{ad}^*_\xi\lambda + u \diamond b - \mathbf{d}c^T(u), b\xi + \mathbf{d}c(\xi)), (\mathrm{ad}^*_\eta\lambda + w \diamond b - \mathbf{d}c^T(w), b\eta + \mathbf{d}c(\eta)))\\ &\qquad = \langle\lambda, [\xi,\eta]\rangle + \langle b, u\eta - w\xi\rangle + \langle \mathbf{d}c(\eta), u\rangle - \langle \mathbf{d}c(\xi), w\rangle.\end{aligned} \tag{3.8}$$

Affine Lie–Poisson Hamiltonian reduction for semidirect products. Consider a Hamiltonian function $H : T^*G \times V^* \to \mathbb{R}$ right-invariant under the G-action

$$(\alpha_h, a) \mapsto \left(R^{T^*}_g(\alpha_h), \theta_g(a)\right) = \left(R^{T^*}_g(\alpha_h), \rho^*_{g^{-1}}(a) + c(g)\right). \tag{3.9}$$

This G-action on $T^*G \times V^*$ is induced by the S-action (3.2) on T^*S. Note that we can think of this Hamiltonian $H : T^*G \times V^* \to \mathbb{R}$ as being the Poisson reduction of a S-invariant Hamiltonian $\overline{H} : T^*S \to \mathbb{R}$ by the normal subgroup $\{e\} \times V$ since $(T^*S)/(\{e\} \times V) \cong T^*G \times V^*$. In particular, the function $H_{a_0} := H|_{T^*G\times\{a_0\}} : T^*G \to \mathbb{R}$ is invariant under the induced action of the isotropy subgroup $G^c_{a_0}$ of a_0 relative to the affine action θ, for any $a_0 \in V^*$. The following theorem is a generalization of theorem 1.1 and is also a consequence of the reduction by stages method for nonequivariant momentum maps, together with the results obtained in section 2 and at the beginning of the present section.

Theorem 3.2. *For $\alpha(t) \in T^*_{g(t)}G$ and $\mu(t) := T^*R_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$, the following are equivalent:*

(i) $\alpha(t)$ satisfies Hamilton's equations for H_{a_0} on T^*G.

(ii) The following affine Lie–Poisson equation holds on $\mathfrak{s}^*$:

$$\frac{\partial}{\partial t}(\mu,a) = \left(-\mathrm{ad}^*_{\frac{\delta h}{\delta\mu}}\mu - \frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right), -a\frac{\delta h}{\delta \mu} - \mathbf{d}c\left(\frac{\delta h}{\delta \mu}\right)\right), \qquad a(0) = a_0$$

where $\mathfrak{s}$ is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g}\,\circledS\,V$. The associated Poisson bracket is the following affine Lie–Poisson bracket on the semidirect product Lie algebra $\mathfrak{s}^*$,

$$\{f,g\}^+_\sigma(\mu,a) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle a, \frac{\delta f}{\delta a}\frac{\delta g}{\delta \mu} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle + \left\langle \mathbf{d}c\left(\frac{\delta f}{\delta \mu}\right), \frac{\delta g}{\delta a}\right\rangle - \left\langle \mathbf{d}c\left(\frac{\delta g}{\delta \mu}\right), \frac{\delta f}{\delta a}\right\rangle.$$

The evolution of the advected quantities is given by $a(t) = \theta_{g(t)^{-1}}(a_0)$.

4. Hamiltonian approach to continuum theories of complex fluids

Recall that in the case of the motion of a fluid on an orientable manifold $\mathcal{D}$, the configuration space is the group $G = \mathrm{Diff}(\mathcal{D})$ of all diffeomorphisms of $\mathcal{D}$. In the case of incompressible fluids, one chooses the subgroup $\mathrm{Diff}_{\mathrm{vol}}(\mathcal{D})$ of all volume preserving diffeomorphisms, with respect to a fixed volume form on $\mathcal{D}$. Besides the diffeomorphism group, the other basic object is the vector space V^* of advected quantities on which G acts by representations. Typical advected quantities are, for example, the *mass density*, the *entropy density* or the *magnetic field*. One can obtain the fluid equations by choosing the appropriate Hamiltonian function and by applying the semidirect Lie–Poisson reduction process (theorem 1.1); see Marsden *et al* (1984) and Holm *et al* (1998).

The goal of this section is to extend these formulations to the case of complex fluids. At the reduced level, the affine Lie–Poisson equations for complex fluids are given in Holm (2002) (equation (3.44)). The two key observations we make regarding these equations are the following. First, they suggest that the configuration manifold $\mathrm{Diff}(\mathcal{D})$ has to be enlarged to a bigger group G in order to contain variables involving the Lie group $\mathcal{O}$ of order parameters. Second, they suggest that there is a new advected quantity on which the group G acts by affine representation. Making use of these two observations, we construct below the appropriate configuration space and the appropriate affine action for the dynamics of complex fluids. By using the general process of affine Lie–Poisson reduction developed before (theorem 3.2), we get (a generalization of) the equations given in Holm (2002).

Here and in all examples that follow, there are fields different from the velocity field for which we shall never specify the boundary conditions. We make the general assumption, valid throughout the paper, that all integrations by parts have vanishing boundary terms, or that the problem has periodic boundary conditions (in which case $\mathcal{D}$ is a boundaryless three-dimensional manifold). Of course if one would try to get an analytically rigorous result, the boundary conditions for all fields need to be carefully specified.

The configuration manifold. Consider a finite-dimensional Lie group $\mathcal{O}$. In applications $\mathcal{O}$ will be called the *order parameter Lie group*. Recall that in the case of the motion of a fluid on an orientable manifold $\mathcal{D}$, the configuration space is the group $G = \mathrm{Diff}(\mathcal{D})$ of all diffeomorphisms of $\mathcal{D}$. In the case of complex fluids, the basic idea is to enlarge this group to the semidirect product of groups $G = \mathrm{Diff}(\mathcal{D})\,Ⓢ\,\mathcal{F}(\mathcal{D}, \mathcal{O})$. Here $\mathcal{F}(\mathcal{D}, \mathcal{O})$ denotes the group of all mappings χ defined on $\mathcal{D}$ with values in the Lie group $\mathcal{O}$ of order parameters. The diffeomorphism group acts on $\mathcal{F}(\mathcal{D}, \mathcal{O})$ via the *right* action

$$(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathcal{O}) \mapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \mathcal{O}).$$

Therefore, the group multiplication is given by

$$(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi).$$

Recall that the tangent space to $\mathrm{Diff}(\mathcal{D})$ at η is

$$T_\eta \mathrm{Diff}(\mathcal{D}) = \{\mathbf{u}_\eta : \mathcal{D} \to T\mathcal{D} \mid \mathbf{u}_\eta(x) \in T_{\eta(x)}\mathcal{D}\},$$

the tangent space to $\mathcal{F}(\mathcal{D}, \mathcal{O})$ at χ is

$$T_\chi \mathcal{F}(\mathcal{D}, \mathcal{O}) = \{\nu_\chi : \mathcal{D} \to T\mathcal{O} \mid \nu_\chi(x) \in T_{\chi(x)}\mathcal{O}\}.$$

A direct computation shows that the tangent map of right translation is

$$TR_{(\varphi,\psi)}(\mathbf{u}_\eta, \nu_\chi) = (\mathbf{u}_\eta \circ \varphi, TR_\psi(\nu_\chi \circ \varphi)).$$

For simplicity we fix a volume form μ on $\mathcal{D}$. Therefore we can identify the cotangent space $T^*_\eta \mathrm{Diff}(\mathcal{D})$ with a space of one-forms over η, that is,

$$T^*_\eta \mathrm{Diff}(\mathcal{D}) = \{\mathbf{m}_\eta : \mathcal{D} \to T^*\mathcal{D} \mid \mathbf{m}_\eta(x) \in T^*_{\eta(x)}\mathcal{D}\}.$$

The cotangent space of $\mathcal{F}(\mathcal{D}, \mathcal{O})$ at χ is naturally given by

$$T^*_\chi \mathcal{F}(\mathcal{D}, \mathcal{O}) = \{\kappa_\chi : \mathcal{D} \to T^*\mathcal{O} \mid \kappa_\chi(x) \in T^*_{\chi(x)}\mathcal{O}\}.$$

Using these identifications, the cotangent lift of right translation is given by

$$R^{T^*}_{(\varphi,\psi)}(\mathbf{m}_\eta, \kappa_\chi) = J(\varphi)(\mathbf{m}_\eta \circ \varphi, T^*R_{\psi^{-1}}(\kappa_\chi \circ \varphi)),$$

where $J(\varphi)$ is the Jacobian determinant of the diffeomorphism φ. The Lie algebra $\mathfrak{g}$ of the semidirect product group is

$$\mathfrak{g} = \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o}),$$

and the Lie bracket is computed to be

$$\mathrm{ad}_{(\mathbf{u},\nu)}(\mathbf{v}, \zeta) = (\mathrm{ad}_\mathbf{u}\mathbf{v}, \mathrm{ad}_\nu\zeta + \mathbf{d}\nu \cdot \mathbf{v} - \mathbf{d}\zeta \cdot \mathbf{u}),$$

where $\mathrm{ad}_\mathbf{u}\mathbf{v} = -[\mathbf{u}, \mathbf{v}]$, $\mathrm{ad}_\nu\zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\mathrm{ad}_\nu\zeta(x) := \mathrm{ad}_{\nu(x)}\zeta(x)$ and $\mathbf{d}\nu \cdot \mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\mathbf{d}\nu \cdot \mathbf{v}(x) := \mathbf{d}\nu(x)(\mathbf{v}(x))$.

Using the previous identification of cotangent spaces, the dual Lie algebra $\mathfrak{g}^*$ can be identified with $\Omega^1(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ through the pairing

$$\langle(\mathbf{m}, \kappa), (\mathbf{u}, \nu)\rangle = \int_\mathcal{D} (\mathbf{m} \cdot \mathbf{u} + \kappa \cdot \nu)\mu.$$

The dual map to $\mathrm{ad}_{(\mathbf{u},\nu)}$ is

$$\mathrm{ad}^*_{(\mathbf{u},\nu)}(\mathbf{m}, \kappa) = (\pounds_\mathbf{u}\mathbf{m} + (\mathrm{div}\,\mathbf{u})\mathbf{m} + \kappa \cdot \mathbf{d}\nu, \mathrm{ad}^*_\nu\kappa + \mathrm{div}(\mathbf{u}\kappa)). \tag{4.1}$$

This formula needs some explanation. The symbol $\kappa \cdot \mathbf{d}\nu \in \Omega^1(\mathcal{D})$ denotes the one-form defined by

$$\kappa \cdot \mathbf{d}\nu(v_x) := \kappa(x)(\mathbf{d}\nu(v_x)).$$

The expression $\mathbf{u}\kappa$ denotes the 1-contravariant tensor field with values in $\mathfrak{o}^*$ defined by

$$\mathbf{u}\kappa(\alpha_x) := \alpha_x(\mathbf{u}(x))\kappa(x) \in \mathfrak{o}^*.$$

Since $\mathbf{u}\kappa$ is a generalization of the notion of a vector field, we denote by $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ the space of all 1-contravariant tensor fields with values in $\mathfrak{o}^*$. In (4.1), $\mathrm{div}(\mathbf{u})$ denotes the divergence of the vector field $\mathbf{u}$ with respect to the fixed volume form μ. Recall that it is defined by the condition

$$(\mathrm{div}\,\mathbf{u})\mu = \pounds_\mathbf{u}\mu.$$

This operator can be naturally extended to the space $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ as follows. For $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ we write $w = w_a\varepsilon^a$ where (ε^a) is a basis of $\mathfrak{o}^*$ and $w_a \in \mathfrak{X}(\mathcal{D})$. We define $\mathrm{div} : \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*) \to \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ by the equality

$$\mathrm{div} w := (\mathrm{div}\, w_a)\varepsilon^a.$$

Note that for $w = \mathbf{u}\kappa$ we have

$$\mathrm{div}(\mathbf{u}\kappa) = \mathbf{d}\kappa \cdot \mathbf{u} + (\mathrm{div}\,\mathbf{u})\kappa.$$

The space of advected quantities. In physical applications, the affine representation space V^* of $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$ is a direct product $V_1^* \oplus V_2^*$, where V_i^* are subspaces of the space of all tensor fields on $\mathcal{D}$ (possibly with values in a vector space). Moreover,

- V_1^* is only acted upon by the component $\mathrm{Diff}(\mathcal{D})$ of G;
- The action of G on V_2^* is affine, with the restriction that the affine term only depends on the second component $\mathcal{F}(\mathcal{D}, \mathcal{O})$ of G.

In this way, we obtain the affine representation

$$(a, \gamma) \in V_1^* \oplus V_2^* \mapsto (a\eta, \gamma(\eta, \chi) + C(\chi)) \in V_1^* \oplus V_2^*, \tag{4.2}$$

where $\gamma(\eta, \chi)$ denotes the representation of $(\eta, \chi) \in G$ on $\gamma \in V_2^*$, and $C \in \mathcal{F}(\mathcal{F}(\mathcal{D}, \mathcal{O}), V_2^*)$ satisfies the identity

$$C((\chi \circ \varphi)\psi) = C(\chi)(\varphi, \psi) + C(\psi). \tag{4.3}$$

Note that this is equivalent to say that the representation ρ and the affine term c of the previous section have the particular form

$$\rho^*_{(\eta,\chi)^{-1}}(a, \gamma) = (a\eta, \gamma(\eta, \chi)) \qquad \text{and} \qquad c(\eta, \chi) = (0, C(\chi)).$$

The infinitesimal action of $(\mathbf{u}, \nu) \in \mathfrak{g}$ on $\gamma \in V_2^*$ induced by the representation of G on V_2^* is

$$\begin{aligned}\gamma(\mathbf{u}, \nu) &:= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(\exp(t\mathbf{u}), \exp(t\nu)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(\exp(t\mathbf{u}), e)(e, \exp(t\nu)) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(\exp(t\mathbf{u}), e) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(e, \exp(t\nu)) =: \gamma\mathbf{u} + \gamma\nu.\end{aligned}$$

Therefore, for $(v, w) \in V_1 \oplus V_2$ we have

$$(v, w) \diamond (a, \gamma) = (v \diamond a + w \diamond_1 \gamma, w \diamond_2 \gamma),$$

where $\diamond_1$ and $\diamond_2$ are associated with the induced representations of the first and second components of G on V_2^*. On the right-hand side, the diamond operation $\diamond$ is associated with the representation of $\mathrm{Diff}(\mathcal{D})$ on V_1^*. The space V_1^* is naturally the dual of some space V_1 of tensor fields on $\mathcal{D}$. For example, the (p, q) tensor fields are naturally in duality with the (q, p) tensor fields. For $a \in V_1^*$ and $v \in V_1$, the duality pairing is given by

$$\langle a, v\rangle = \int_{\mathcal{D}} (a \cdot v)\mu,$$

where '·' denotes the contraction of tensor fields.

Since the affine cocycle has the particular form $c(\eta, \chi) = (0, C(\chi))$, we obtain that

$$\mathbf{d}c^T(v, w) = (0, \mathbf{d}C^T(w)).$$

By theorem 3.2, we obtain that the associated affine Lie–Poisson bracket is given by

$$\begin{aligned}\{f, g\}^+_\sigma(\mathbf{m}, \kappa, a, \gamma) = &\int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu + \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta\kappa}} \frac{\delta g}{\delta\kappa} + \mathbf{d}\frac{\delta f}{\delta\kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta\kappa} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ &+ \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right) + \int_{\mathcal{D}} \gamma \cdot \left(\frac{\delta f}{\delta\gamma}\frac{\delta g}{\delta \mathbf{m}} + \frac{\delta f}{\delta\gamma}\frac{\delta g}{\delta\kappa} - \frac{\delta g}{\delta\gamma}\frac{\delta f}{\delta \mathbf{m}} - \frac{\delta g}{\delta\gamma}\frac{\delta f}{\delta\kappa}\right)\mu \\ &+ \int_{\mathcal{D}} \left(\mathbf{d}C\left(\frac{\delta f}{\delta\kappa}\right) \cdot \frac{\delta g}{\delta\gamma} - \mathbf{d}C\left(\frac{\delta g}{\delta\kappa}\right) \cdot \frac{\delta f}{\delta\gamma}\right)\mu.\end{aligned}$$

The first four terms give the Lie–Poisson bracket on the dual Lie algebra

$$([\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})] \circledS [V_1 \oplus V_2])^* \cong \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \times V_1^* \times V_2^*.$$

The last term is due to the presence of the affine term C in the representation. Since C depends only on the group $\mathcal{F}(\mathcal{D}, \mathcal{O})$, this term does not involve the functional derivatives with respect to $\mathbf{m}$.

The symplectic leaves of this bracket are the affine coadjoint orbits in the dual Lie algebra $([\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o})] \circledS [V_1 \oplus V_2])^*$. The expression of the tangent spaces and of the affine orbit symplectic forms involves the bilinear form Σ which is defined in this case on $[\mathfrak{X}(\mathcal{D})\,Ⓢ\,\mathcal{F}(\mathcal{D},\mathfrak{o})]\,Ⓢ\,[V_1\oplus V_2]$ by

$$\Sigma((\mathbf{u}_1,\nu_1,v_1,w_1),(\mathbf{u}_2,\nu_2,v_2,w_2)) = \mathbf{d}C(\nu_2)\cdot w_1 - \mathbf{d}C(\nu_1)\cdot w_2.$$

For a Hamiltonian $h = h(\mathbf{m},\kappa,a,\gamma) : \Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathfrak{o}^*)\times V_1^*\times V_2^*\to\mathbb{R}$, the affine Lie–Poisson equations of theorem 3.2 become

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa\cdot\mathbf{d}\dfrac{\delta h}{\delta\kappa} - \dfrac{\delta h}{\delta a}\diamond a - \dfrac{\delta h}{\delta\gamma}\diamond_1\gamma \\ \dfrac{\partial}{\partial t}\kappa = -\operatorname{ad}^*_{\frac{\delta h}{\delta\kappa}}\kappa - \operatorname{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right) - \dfrac{\delta h}{\delta\gamma}\diamond_2\gamma + \mathbf{d}C^T\left(\dfrac{\delta h}{\delta\gamma}\right) \\ \dfrac{\partial}{\partial t}a = -a\dfrac{\delta h}{\delta \mathbf{m}} \\ \dfrac{\partial}{\partial t}\gamma = -\gamma\dfrac{\delta h}{\delta \mathbf{m}} - \gamma\dfrac{\delta h}{\delta\kappa} - \mathbf{d}C\left(\dfrac{\delta h}{\delta\kappa}\right). \end{cases} \tag{4.4}$$

Using formula (3.5), the momentum map of the affine right action of the semidirect product $[\operatorname{Diff}(\mathcal{D})\,Ⓢ\,\mathcal{F}(\mathcal{D},\mathcal{O})]\,Ⓢ\,[V_1\oplus V_2]$ on its cotangent bundle is computed to be

$$\begin{aligned}\mathbf{J}_\alpha(\mathbf{m}_\eta,\kappa_\chi,(v,w),(a,\gamma)) = (&T^*\eta\circ\mathbf{m}_\eta + T^*\chi\circ\kappa_\chi + v\diamond a\\ &+ w\diamond_1\gamma, T^*L_\chi\circ\kappa_\chi + w\diamond_2\gamma - \mathbf{d}C^T(w),(a,\gamma)).\end{aligned}$$

By the general theory, the nonequivariance cocycle of $\mathbf{J}_\alpha$ is given by $\sigma = -\phi$, where ϕ is computed to be

$$\begin{aligned}&\phi : [\operatorname{Diff}(\mathcal{D})\,Ⓢ\,\mathcal{F}(\mathcal{D},\mathcal{O})]\,Ⓢ\,[V_1\oplus V_2]\to\Omega^1(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathfrak{o}^*)\times V_1^*\times V_2^*,\\ &\phi(\eta,\chi,v,w) = (\mathbf{d}c^T(v,w) - (v,w)\diamond c(\eta,\chi), -c(\eta,\chi))\\ &\qquad = (-w\diamond_1 C(\chi), \mathbf{d}C^T(w) - w\diamond_2 C(\chi), 0, -C(\chi)).\end{aligned}$$

Basic example. Take $V_2^* := \Omega^1(\mathcal{D},\mathfrak{o})$, the space of all one-forms on $\mathcal{D}$ with values in $\mathfrak{o}$. This space is naturally the dual of the space $V_2 = \mathfrak{X}(\mathcal{D},\mathfrak{o}^*)$ of contravariant tensor fields with values in $\mathfrak{o}^*$, the duality pairing being given, for $\gamma\in\Omega^1(\mathcal{D},\mathfrak{o})$ and $w\in\mathfrak{X}(\mathcal{D},\mathfrak{o}^*)$, by

$$\langle\gamma,w\rangle := \int_{\mathcal{D}}(\gamma\cdot w)\mu,$$

where $\gamma\cdot w$ denotes the contraction of tensors.

We consider for (4.2) the affine representation defined by

$$(a,\gamma)\mapsto(a\eta, \operatorname{Ad}_{\chi^{-1}}\eta^*\gamma + \chi^{-1}T\chi). \tag{4.5}$$

One can check that $\gamma(\eta,\chi) := \operatorname{Ad}_{\chi^{-1}}\eta^*\gamma$ is a right representation of G on V_2^* and that $C(\chi) = \chi^{-1}T\chi$ verifies condition (4.3). The one-form $\gamma\in\Omega^1(\mathcal{D},\mathfrak{o})$ can be considered as a connection one-form on the trivial principal $\mathcal{O}$-bundle $\mathcal{O}\times\mathcal{D}\to\mathcal{D}$. The covariant differential associated with this principal connection will be denoted by $\mathbf{d}^\gamma$. Therefore, for a function $\nu\in\mathcal{F}(\mathcal{D},\mathfrak{o})$, we have

$$\mathbf{d}^\gamma\nu(\mathbf{v}) := \mathbf{d}\nu(\mathbf{v}) + [\gamma(\mathbf{v}),\nu].$$

The covariant divergence of $w\in\mathfrak{X}(\mathcal{D},\mathfrak{o}^*)$ is the function

$$\operatorname{div}^\gamma w := \operatorname{div} w - \operatorname{Tr}(\operatorname{ad}^*_\gamma w)\in\mathcal{F}(\mathcal{D},\mathfrak{o}^*),$$

defined as minus the adjoint of the covariant differential, that is,

$$\int_{\mathcal{D}}(\mathbf{d}^\gamma\nu\cdot w)\,\mu = -\int_{\mathcal{D}}(\nu\cdot\operatorname{div}^\gamma w)\,\mu \tag{4.6}$$

for all $\nu \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$. More generally, the space $\Omega^k(\mathcal{D}, \mathfrak{o})$ is, in a natural way, dual to the space $\Omega_k(\mathcal{D}, \mathfrak{o}^*)$ of k-contravariant skew symmetric tensor fields with values in $\mathfrak{o}^*$. The duality pairing is given by contraction and integration with respect to the fixed volume form μ. Therefore, we can define the divergence operators, div, $\mathrm{div}^\gamma : \Omega_k(\mathcal{D}, \mathfrak{o}^*) \to \Omega_{k-1}(\mathcal{D}, \mathfrak{o}^*)$, to be minus the adjoint of the exterior derivatives $\mathbf{d}$ and $\mathbf{d}^\gamma$, respectively. Note that we have $\Omega_1(\mathcal{D}, \mathfrak{o}^*) = \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ and $\Omega_0(\mathcal{D}, \mathfrak{o}^*) = \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$.

For the particular case of the affine action (4.5), we have

$$\gamma \mathbf{u} = \pounds_{\mathbf{u}} \gamma = \mathbf{d}^\gamma(\gamma(\mathbf{u})) + \mathbf{i}_{\mathbf{u}} \mathbf{d}^\gamma \gamma,$$
$$\gamma \nu = -\mathrm{ad}_\nu \gamma, \qquad \mathbf{d}C(\nu) = \mathbf{d}\nu \qquad \text{and} \qquad \mathbf{d}C^T(w) = -\mathrm{div}(w).$$

The diamond operations are given by

$$w \diamond_1 \gamma = (\mathrm{div}^\gamma w) \cdot \gamma - w \cdot \mathbf{i}\_\mathbf{d}^\gamma \gamma \quad \text{and} \quad w \diamond_2 \gamma = -\mathrm{Tr}(\mathrm{ad}^*_\gamma w).$$

The affine Lie–Poisson equations (4.4) become

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}} \mathbf{m} - \mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \kappa \cdot \mathbf{d}\dfrac{\delta h}{\delta \kappa} - \dfrac{\delta h}{\delta a} \diamond a \\ \quad - \left(\mathrm{div}^\gamma \dfrac{\delta h}{\delta \gamma}\right)\gamma + \dfrac{\delta h}{\delta \gamma} \cdot \mathbf{i}\_\mathbf{d}^\gamma \gamma \\ \dfrac{\partial}{\partial t}\kappa = -\mathrm{ad}^*_{\frac{\delta h}{\delta \kappa}} \kappa - \mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\kappa\right) - \mathrm{div}^\gamma \dfrac{\delta h}{\delta \gamma} \\ \dfrac{\partial}{\partial t} a = -a \dfrac{\delta h}{\delta \mathbf{m}} \\ \dfrac{\partial}{\partial t}\gamma = -\mathbf{d}^\gamma\left(\gamma\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\right) - \mathbf{i}_{\frac{\delta h}{\delta \mathbf{m}}} \mathbf{d}^\gamma \gamma - \mathbf{d}^\gamma \dfrac{\delta h}{\delta \kappa}. \end{cases} \tag{4.7}$$

So we recover, by a reduction from a canonical situation, equations (3.44) of Holm (2002), up to sign conventions, as well as their Hamiltonian structure. In matrix notation and with respect to local coordinates we have

$$\begin{bmatrix} \dot{m}_i \\ \dot{\kappa}_a \\ \dot{a} \\ \dot{\gamma}^a_i \end{bmatrix} = - \begin{bmatrix} m_k \partial_i + \partial_k m_i & \kappa_b \partial_i & (\Box \diamond a)_i & \partial_j \gamma^b_i - \gamma^b_{j,i} \\ \partial_k \kappa_a & \kappa_c C^c_{ba} & 0 & \delta^b_a \partial_j - C^b_{ca}\gamma^c_j \\ a \Box \partial_k & 0 & 0 & 0 \\ \gamma^a_k \partial_i + \gamma^a_{i,k} & \delta^a_b \partial_i + C^a_{cb}\gamma^c_i & 0 & 0 \end{bmatrix} \begin{bmatrix} (\delta h/\delta m)^k \\ (\delta h/\delta \kappa)^b \\ \delta h/\delta a \\ (\delta h/\delta \gamma)^j_b \end{bmatrix}. \tag{4.8}$$

This matrix appears in Holm and Kupershmidt (1988) (formula (2.26a)) and in Holm (2002) (formula (3.46)) as the common Hamiltonian structure for various hydrodynamical systems. See Cendra *et al* (2003) for another derivation of this Hamiltonian structure based on reduction.

The associated affine Lie–Poisson bracket is

$$\begin{aligned} \{f, g\}^+_\sigma(\mathbf{m}, \kappa, a, \gamma) = & \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right]\mu + \int_{\mathcal{D}} \kappa \cdot \left(\mathrm{ad}_{\frac{\delta f}{\delta \kappa}} \frac{\delta g}{\delta \kappa} + \mathbf{d}\frac{\delta f}{\delta \kappa} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \kappa} \cdot \frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ & + \int_{\mathcal{D}} a \cdot \left(\frac{\delta f}{\delta a}\frac{\delta g}{\delta \mathbf{m}} - \frac{\delta g}{\delta a}\frac{\delta f}{\delta \mathbf{m}}\right)\mu \\ & + \int_{\mathcal{D}} \left[\left(\mathbf{d}^\gamma \frac{\delta f}{\delta \kappa} + \pounds_{\frac{\delta f}{\delta \mathbf{m}}} \gamma\right) \cdot \frac{\delta g}{\delta \gamma} - \left(\mathbf{d}^\gamma \frac{\delta g}{\delta \kappa} + \pounds_{\frac{\delta g}{\delta \mathbf{m}}} \gamma\right) \cdot \frac{\delta f}{\delta \gamma}\right]\mu. \end{aligned}$$

The momentum map is computed to be

$$\begin{aligned} \mathbf{J}_\alpha(\mathbf{m}_\eta, \kappa_\chi, (v, w), (a, \gamma)) = & (T^*\eta \circ \mathbf{m}_\eta + T^*\chi \circ \kappa_\chi + v \diamond a \\ & + (\mathrm{div}^\gamma w) \cdot \gamma - w \cdot \mathbf{i}\_\mathbf{d}^\gamma \gamma, T^* L_\chi \circ \kappa_\chi + \mathrm{div}^\gamma w, (a, \gamma)). \end{aligned}$$

5. The circulation theorems

The Kelvin–Noether theorem is a version of the Noether theorem that holds for solutions of the Euler–Poincaré equations. For example, an application of this theorem to the compressible adiabatic fluid gives the Kelvin circulation theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t}\mathbf{u}^\flat = \oint_{\gamma_t} T\,\mathbf{d}s,$$

where $\gamma_t \subset \mathcal{D}$ is a closed curve which moves with the fluid velocity $\mathbf{u}$, T is the *temperature* and s denotes the *specific entropy*. The Kelvin–Noether theorem associated with Euler–Poincaré reduction for semidirect product is presented in Holm *et al* (1998). We now adapt this result to the case of affine Lie–Poisson reduction.

Kelvin–Noether theorem. Let $\mathcal{C}$ be a manifold on which G acts on the left and suppose we have an equivariant map $\mathcal{K} : \mathcal{C} \times V^* \to \mathfrak{g}^{**}$, that is, for all $g \in G$, $a \in V^*$, $c \in \mathcal{C}$, we have

$$\langle \mathcal{K}(gc, \theta_g(a)), \mu\rangle = \langle \mathcal{K}(c, a), \mathrm{Ad}^*_g \mu\rangle,$$

where gc denotes the action of G on $\mathcal{C}$, and θ_g is the affine action of G on V^*. Consider the map $J : \mathcal{C} \times \mathfrak{g}^* \times V^* \to \mathbb{R}$ defined by

$$J(c, \mu, a) := \langle \mathcal{K}(c, a), \mu\rangle.$$

We have the following result.

Theorem 5.1. *Fixing $c_0 \in \mathcal{C}$, let $\mu(t), a(t)$ satisfy the affine Lie–Poisson equations of theorem 3.2 and define $g(t)$ to be the solution of*

$$\dot{g}(t) = TR_{g(t)}\left(\frac{\delta h}{\delta \mu}\right), \qquad g(0) = e.$$

Let $c(t) = g(t)c_0$ and $J(t) := J(c(t), \mu(t), a(t))$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}J(t) = \left\langle \mathcal{K}(c(t), a(t)), -\frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right)\right\rangle.$$

In the case of dynamics on the group $G = \mathrm{Diff}(\mathcal{D})$, the standard choice for the equivariant map $\mathcal{K}$ is

$$\langle \mathcal{K}(c, a), \mathbf{m}\rangle := \oint_c \frac{1}{\rho}\mathbf{m}, \tag{5.1}$$

where $c \in \mathcal{C} = \mathrm{Emb}(S^1, \mathcal{D})$, the manifold of all embeddings of the circle S^1 in $\mathcal{D}$, $\mathbf{m} \in \Omega^1(\mathcal{D})$ and ρ is advected as $(J\eta)(\rho \circ \eta)$. There is a generalization of this map in the case of the group $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$; see section 7 in Gay-Balmaz and Ratiu (2008). Therefore, theorem 5.1 can be applied in the case of the affine Lie–Poisson equations (4.4). Nevertheless we shall not use this point of view here and we apply the Kelvin–Noether theorem only to the first component of the group G, namely the group $\mathrm{Diff}(\mathcal{D})$, and we obtain the following result.

Theorem 5.2. *Consider the affine Lie–Poisson equations for complex fluids (4.4). Suppose that one of the linear advected variables, say ρ, is advected as $(J\eta)(\rho \circ \eta)$. Then, using the map (5.1), we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t}\frac{1}{\rho}\mathbf{m} = \oint_{c_t}\frac{1}{\rho}\left(-\kappa\cdot\mathbf{d}\frac{\delta h}{\delta \kappa} - \frac{\delta h}{\delta a}\diamond a - \frac{\delta h}{\delta \gamma}\diamond_1 \gamma\right),$$

where c_t is a loop in $\mathcal{D}$ which moves with the fluid velocity $\mathbf{u}$, *defined by the equality*

$$\mathbf{u} := \frac{\delta h}{\delta \mathbf{m}}.$$

γ-circulation. The γ-circulation is associated with the equation

$$\frac{\partial}{\partial t}\gamma + \pounds_{\mathbf{u}}\gamma = -\mathbf{d}\nu + \mathrm{ad}_\nu \gamma.$$

Let η_t be the flow of the vector field $\mathbf{u}$, let c_0 be a loop in $\mathcal{D}$ and let $c_t := \eta_t \circ c_0$. Then, by change of variables, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t}\gamma = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_0}\eta_t^*\gamma = \oint_{c_0}\eta_t^*(\dot{\gamma} + \pounds_{\mathbf{u}}\gamma) = \oint_{c_0}\eta_t^*(-\mathbf{d}\nu + \mathrm{ad}_\nu\gamma) = \oint_{c_t}\mathrm{ad}_\nu\gamma \in \mathfrak{o}.$$

6. Applications

In this section, we apply the processes of affine Euler–Poincaré and Lie–Poisson reductions to Yang–Mills magnetohydrodynamics and superfluids. The theory developed in this paper applies to a wide range of fluids with internal degrees of freedom, including microfluids, spin glasses and liquid crystals. These examples are studied in detail in Gay-Balmaz and Ratiu (2008).

6.1. Yang–Mills magnetohydrodynamics

Magnetohydrodynamics models the motion of an electrically charged and perfectly conducting fluid. In the balance of momentum law, one must add the Lorentz force of the magnetic field created by the fluid in motion. In addition, the hypothesis of infinite conductivity leads one to the conclusion that magnetic lines are frozen in the fluid, i.e. that they are transported along the particle paths. This hypothesis leads to the equation

$$\frac{\partial}{\partial t}B + \pounds_{\mathbf{u}}B = 0.$$

This model can be extend to incorporate nonabelian Yang–Mills interactions and is known under the name of Yang–Mills magnetohydrodynamics; see Holm and Kupershmidt (1984b) for a derivation of this model. Recall that for Yang–Mills theory, the field B is seen as the curvature of a connection A on a principal bundle. Clearly the connection A represents the variable γ in the general theory developed previously, on which the automorphism group acts by affine transformations. This shows that the abstract formalism developed previously is very natural in the context of Yang–Mills theory. Note that there is a more general model of fluid motion with Yang–Mills charged particles, namely the Euler–Yang–Mills equations. The Hamiltonian structure of these equations is given in Gibbons *et al* (1983); see also Gay-Balmaz and Ratiu (2008) for the associated Lagrangian and Hamiltonian reductions.

As remarked in Holm and Kupershmidt (1988), at the reduced level, the Hamiltonian structure of Yang–Mills magnetohydrodynamics is given by the matrix (4.8). In this paragraph we carry out the corresponding affine Lie–Poisson reduction.

The group G is chosen to be the semidirect product of the diffeomorphism group with the group of $\mathcal{O}$-valued function on $\mathcal{D}$, that is, $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$. The order parameter Lie group $\mathcal{O}$ represents here the *symmetry group of the particles interaction*. For example, $\mathcal{O} = S^1$ corresponds to electromagnetism, $\mathcal{O} = SU(2)$ and $\mathcal{O} = SU(3)$ correspond to weak and strong interactions, respectively. The advected quantities are the *mass density* ρ, the *entropy density* S and the *potential of the Yang–Mills field* A. Therefore, we set

$$a = (\rho, S) \in V_1^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \qquad \text{and} \qquad A = \gamma \in \Omega^1(\mathcal{D}, \mathfrak{o}).$$

The action of $(\eta, \chi) \in G$ on $(\rho, S) \in V_1^*$ is the usual right representation of the fluid relabeling group on the mass density and entropy density. It is given by

$$(\rho, S)(\eta, \chi) = J\eta(\rho \circ \eta, S \circ \eta).$$

The right affine action of $(\eta, \chi) \in G$ on $A \in \Omega^1(M, \mathfrak{o})$ is given, as in the example (4.5), by

$$A \mapsto \mathrm{Ad}_{\chi^{-1}}\eta^* A + \chi^{-1} T\chi.$$

Since the variable $\kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ is interpreted as the *gauge-charge density*, we use the notations $Q \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) = T_0^*\mathcal{F}(\mathcal{D}, \mathcal{O})$ and $Q_\chi \in T_\chi^*\mathcal{F}(\mathcal{D}, \mathcal{O})$.

The reduced Hamiltonian $h : \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \times V_1^* \times V_2^* \to \mathbb{R}$ is given by

$$h(\mathbf{m}, Q, \rho, S, A) = \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\|\mathbf{m}\|^2\mu + \int_{\mathcal{D}} \varepsilon(\rho, S)\mu + \frac{1}{2}\int_{\mathcal{D}} \|\mathbf{d}^A A\|^2\mu,$$

where ε denotes the *internal energy density*, the norm in the first term is associated with a Riemannian metric g on $\mathcal{D}$, and the norm in the third term is associated with the metric (gk), on the vector bundle of $\mathfrak{o}$-valued k-forms on $\mathcal{D}$, induced by the metric g and by an Ad-invariant inner product k on $\mathfrak{o}$. The metric (gk) can be used to identify $\Omega_k(\mathcal{D}, \mathfrak{o}^*)$ and its dual $\Omega^k(\mathcal{D}, \mathfrak{o})$, by raising and lowering indices. Through this identification, the operators div and div^A act also on $\Omega^k(\mathcal{D}, \mathfrak{o})$.

The affine Lie–Poisson equations (4.7) associated with this Hamiltonian are

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}(\mathrm{grad}\, p + (gk)(\mathbf{i}\_B, \mathrm{div}^A B)^\sharp), \qquad B = \mathbf{d}^A A, \\ \dfrac{\partial}{\partial t} Q + \mathrm{div}(Q\mathbf{u}) = 0, \qquad \dfrac{\partial}{\partial t} A + \mathbf{d}^A(A(\mathbf{u})) + \mathbf{i}_{\mathbf{u}} B = 0, \\ \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \qquad \dfrac{\partial}{\partial t} S + \mathrm{div}(S\mathbf{u}) = 0, \end{cases} \tag{6.1}$$

where $\mathbf{m} = \rho\mathbf{u}^\flat$ and $p = \rho\mu_{\mathrm{chem}} + ST - \varepsilon$ is the *pressure*, given in terms of the *chemical potential* $\mu_{\mathrm{chem}} = \partial\varepsilon/\partial\rho$ and the *temperature* $T = \partial\varepsilon/\partial S$. The first equation admits the stress tensor formulation

$$\dot{\mathbf{m}} = -\mathrm{Div}\,\mathbf{T},$$

where $\mathbf{T}$ is the $(1, 1)$ stress tensor given by

$$\mathbf{T} = \mathbf{u} \otimes \rho\mathbf{u}^\flat + B \cdot B + q\delta, \qquad q = p - \tfrac{1}{2}\|B\|^2.$$

We now treat the particular case of magnetohydrodynamics, that is, the case $\mathcal{O} = S^1$. In order to recover the standard equations we suppose that $\mathcal{D}$ is three dimensional. In this case we can define the *magnetic potential* $\mathbf{A} := A^\sharp \in \mathfrak{X}(\mathcal{D})$ and the *magnetic field* $\mathbf{B} := (\star B)^\sharp \in \mathfrak{X}(\mathcal{D})$. Since the group is Abelian, covariant differentiation coincides with usual differentiation and the equality $\mathbf{d}^A A = B$ reads $\mathrm{curl}\,\mathbf{A} = \mathbf{B}$. Using the identities $(\mathrm{div} B)^\sharp = -\mathrm{curl}\,\mathbf{B}$ and $(\mathbf{i}_{\mathbf{u}} B)^\sharp = \mathbf{B} \times \mathbf{u}$ we obtain

$$g(\mathbf{i}\_B, \mathrm{div} B)^\sharp = -(\mathbf{i}_{(\mathrm{div} B)^\sharp} B)^\sharp = \mathbf{B} \times \mathrm{curl}\,\mathbf{B}.$$

Suppose that all particles have mass m. The electric charge q is such that $Q = \rho\frac{q}{m}$; therefore the equation for Q in (6.1) becomes

$$\frac{\partial}{\partial t} q + \mathbf{d}q(\mathbf{u}) = 0.$$

If we suppose that at time $t = 0$ all the particle have the same charge, then this charge remains constant for all time. Using these remarks and hypotheses, equations (6.1) become

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}\left(\operatorname{grad} p + \mathbf{B} \times \operatorname{curl}\mathbf{B}\right), \qquad \mathbf{B} = \operatorname{curl}\mathbf{A}, \\ \dfrac{\partial}{\partial t}\mathbf{A} + \operatorname{grad}[g(\mathbf{A}, \mathbf{u})] + \mathbf{B} \times \mathbf{u} = 0, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \qquad \dfrac{\partial}{\partial t}S + \operatorname{div}(S\mathbf{u}) = 0. \end{cases} \tag{6.2}$$

Thus, we have recovered the equations for magnetohydrodynamics.

Turning back to the general case and using theorem 3.2, we obtain the following result.

Hamiltonian reduction for Yang–Mills magnetohydrodynamics. Consider the unreduced right invariant Hamiltonian $H(\mathbf{m}_\eta, \rho_\chi, S, \mathbf{v}_s) = H_{(S,\mathbf{v}_s)}(\mathbf{m}_\eta, \rho_\chi)$,

$$H_{(S,\mathbf{v}_s)} : T^*(\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)) \to \mathbb{R},$$

whose value at the identity is given by h. A smooth path $(\mathbf{m}_\eta, Q_\chi) \in T^*[\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})]$ is a solution of Hamilton's equations associated with the Hamiltonian $H_{(\rho_0,S_0,A_0)}$ if and only if the curve

$$(\rho\mathbf{u}^\flat, Q) =: (\mathbf{m}, Q) := J(\eta^{-1})(\mathbf{m}_\eta \circ \eta^{-1}, T^*R_{\chi\circ\eta^{-1}}(Q_\chi \circ \eta^{-1}))$$

is a solution of the system (6.1) with initial conditions (ρ_0, S_0, A_0).

The evolution of the advected quantities is given by

$$\begin{aligned} \rho &= J(\eta^{-1})(\rho_0 \circ \eta^{-1}), \qquad S = J(\eta^{-1})(S_0 \circ \eta^{-1}), \\ A &= \operatorname{Ad}_{\chi\circ\eta^{-1}}\eta_* A_0 + (\chi \circ \eta^{-1})T(\chi^{-1} \circ \eta^{-1}) \\ &= \eta_*(\operatorname{Ad}_\chi A_0 + \chi T \chi^{-1}). \end{aligned}$$

This theorem is interesting from two points of view. Firstly, it allows us to recover the non-canonical Hamiltonian structure given in Holm and Kupershmidt (1988) by a reduction from a canonical cotangent bundle. Secondly, it generalizes to the nonabelian case the Hamiltonian reduction for magnetohydrodynamics given in Marsden *et al* (1984).

The associated affine Lie–Poisson bracket is that given in (4.9), where the third term takes the explicit form

$$\int_{\mathcal{D}} \rho \left(\mathbf{d}\left(\frac{\delta f}{\delta \rho}\right)\frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\left(\frac{\delta g}{\delta \rho}\right)\frac{\delta f}{\delta \mathbf{m}}\right)\mu + \int_{\mathcal{D}} S \left(\mathbf{d}\left(\frac{\delta f}{\delta S}\right)\frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\left(\frac{\delta g}{\delta S}\right)\frac{\delta f}{\delta \mathbf{m}}\right)\mu.$$

In the general case of Yang–Mills magnetohydrodynamics, the Kelvin–Noether theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t}\mathbf{u}^\flat = \oint_{c_t} T\,\mathbf{d}s - \oint_{c_t}\frac{1}{\rho}(gk)\left(\mathbf{i}_{\_}B, \operatorname{div}^A B\right),$$

where s is the *specific entropy*. The γ-circulation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t} A = 0,$$

where c_t is a loop which move with the fluid velocity $\mathbf{u}$, that is, $c_t = \eta_t \circ c_0$.

6.2. Hall magnetohydrodynamics

As we will see, Hall magnetohydrodynamics does not require the use of the affine Lie–Poisson reduction developed in this paper. However, in view of the next paragraph about superfluids, we quickly recall here from Holm (1987) the Hamiltonian formulation of these equations.

We will obtain this Hamiltonian structure by a Lie–Poisson reduction for semidirect products, associated with the direct product group $G := \mathrm{Diff}(\mathcal{D}) \times \mathrm{Diff}(\mathcal{D})$. The advected quantities are

$$(\rho, S; n) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}).$$

The variables ρ and S are, as before, the *mass density* and the *entropy density*, on which only the first diffeomorphism group acts as

$$(\rho, S) \mapsto (J\eta)(\rho \circ \eta, S \circ \eta).$$

The variable n is the *electron charge density*, on which only the second diffeomorphism group acts as

$$n \mapsto (J\xi)(n \circ \xi).$$

By Lie–Poisson reduction, for a Hamiltonian $h = h(\mathbf{m}, \rho, S; \mathbf{n}, n)$ defined on the dual Lie-algebra

$$([\mathfrak{X}(\mathcal{D}) \circledS (\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}))] \times [\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})])^* \cong [\Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D})] \times [\Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D})],$$

we obtain the coupled Lie–Poisson equations

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\pounds_{\frac{\delta h}{\delta \mathbf{m}}}\mathbf{m} - \mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\right)\mathbf{m} - \dfrac{\delta h}{\delta \rho} \diamond \rho - \dfrac{\delta h}{\delta S} \diamond S \\ \dfrac{\partial}{\partial t}\rho = -\mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}\rho\right) \\ \dfrac{\partial}{\partial t}S = -\mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{m}}S\right) \end{cases} \tag{6.3}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{n} = -\pounds_{\frac{\delta h}{\delta \mathbf{n}}}\mathbf{n} - \mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{n}}\right)\mathbf{n} - \dfrac{\delta h}{\delta n} \diamond n \\ \dfrac{\partial}{\partial t}n = -\mathrm{div}\left(\dfrac{\delta h}{\delta \mathbf{n}}n\right). \end{cases} \tag{6.4}$$

The Hamiltonian for Hall magnetohydrodynamics is

$$h(\mathbf{m}, \rho, S; \mathbf{n}, n) := \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho}\left\|\mathbf{m} - \frac{a\rho}{R}A\right\|^2 \mu + \int_{\mathcal{D}} \varepsilon(\rho, S)\mu + \frac{1}{2}\int_{\mathcal{D}} \|\mathbf{d}A\|^2 \mu, \tag{6.5}$$

where the one-from A, defined by

$$A := R\frac{\mathbf{n}}{n} \in \Omega^1(\mathcal{D}),$$

is the *magnetic vector potential*, the constants a, R are respectively the *ion charge-to-mass ratio* and the *Hall scaling parameter*, and the norms are taken with respect to a fixed Riemannian metric g on $\mathcal{D}$. Using the advection equations for ρ and n we obtain

$$\frac{\partial}{\partial t}(a\rho + n) = 0.$$

Thus, if we assume that $a\rho_0 + n_0 = 0$ for the initial conditions, we have $a\rho + n = 0$ for all time. Using the Hamiltonian (6.5), equations (6.3) and (6.4) are computed to be

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}(\mathrm{grad}\, p - (\mathbf{i}_{(\mathrm{div} B)^\sharp}B)^\sharp) \\ \dfrac{\partial}{\partial t}\rho = -\mathrm{div}\,(\rho\mathbf{u}), \qquad \dfrac{\partial}{\partial t}S = -\mathrm{div}\,(S\mathbf{u}) \\ \dfrac{\partial}{\partial t}A = -\mathbf{i}_{\mathbf{u}}B - \dfrac{R}{a\rho}\mathbf{i}_{(\mathrm{div} B)^\sharp}B, \end{cases} \tag{6.6}$$

where $\mathbf{m} = \rho\mathbf{u}^\flat + \frac{a\rho}{R}A$ and $p = \rho\mu_{\text{chem}} + ST - \varepsilon$ is the *pressure*. The first equation admits the stress tensor formulation

$$\dot{\mathbf{m}} + \dot{\mathbf{n}} = -\operatorname{Div}\mathbf{T},$$

where $\mathbf{T}$ is the (1, 1) stress tensor given by

$$\mathbf{T} = \mathbf{u} \otimes \rho\mathbf{u}^\flat + B \cdot B + q\delta, \qquad q = p - \tfrac{1}{2}\|B\|^2.$$

When $\mathcal{D}$ is three dimensional, we can define the *magnetic potential* $\mathbf{A} := A^\sharp$ and the *magnetic field* $\mathbf{B} := (\star B)^\sharp$. In this case the previous equations read

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\dfrac{1}{\rho}(\operatorname{grad} p + \mathbf{B} \times \operatorname{curl}\mathbf{B}) \\ \dfrac{\partial}{\partial t}\rho = -\operatorname{div}(\rho\mathbf{u}), \qquad \dfrac{\partial}{\partial t}S = -\operatorname{div}(S\mathbf{u}) \\ \dfrac{\partial}{\partial t}\mathbf{A} = \mathbf{u} \times \mathbf{B} + \dfrac{R}{a\rho}\mathbf{B} \times \operatorname{curl}\mathbf{B}. \end{cases} \tag{6.7}$$

These are the classical equations of Hall magnetohydrodynamics. Note that we can pass from the equations for magnetohydrodynamics to those for Hall magnetohydrodynamics by simply replacing the advection law for A by *Ohm's law*. In terms of the magnetic field B, one simply replace the advection law

$$\frac{\partial}{\partial t}B + \pounds_{\mathbf{u}}B = 0,$$

where $\mathbf{u}$ is the fluid velocity, by the equation

$$\frac{\partial}{\partial t}B + \pounds_{\mathbf{v}}B = 0,$$

where $\mathbf{v} = \mathbf{u} + \frac{R}{a\rho}(\operatorname{div}B)^\sharp$ is the *electron fluid velocity*.

Hamiltonian reduction for Hall magnetohydrodynamics. Consider the unreduced right invariant Hamiltonian $H(\mathbf{m}_\eta, \rho, S; \mathbf{n}_\xi, n) = H_{(\rho,S;n)}(\mathbf{m}_\eta, \mathbf{n}_\xi)$,

$$H_{(\rho,S;n)} : T^*(\operatorname{Diff}(\mathcal{D}) \times \operatorname{Diff}(\mathcal{D})) \to \mathbb{R},$$

whose value at the identity is given by h. Suppose that $a\rho_0 + n_0 = 0$. A smooth path $(\mathbf{m}_\eta, \mathbf{n}_\xi) \in T^*(\operatorname{Diff}(\mathcal{D}) \times \operatorname{Diff}(\mathcal{D}))$ is a solution of Hamilton's equations associated with $H_{(\rho_0,S_0;n_0)}$ if and only if the curve

$$(\mathbf{m}, \mathbf{n}) := (J(\eta^{-1})(\mathbf{m}_\eta \circ \eta^{-1}), J(\xi^{-1})(\mathbf{n}_\xi \circ \xi^{-1}))$$

is a solution of equations (6.6) where $A := \frac{R}{n}\mathbf{n} = -\frac{R}{a\rho}\mathbf{n}$, since $a\rho + n = 0$. Moreover, the evolution of the advected quantities is given by

$$\rho = J(\eta^{-1})(\rho_0 \circ \eta^{-1}), \qquad S = J(\eta^{-1})(S_0 \circ \eta^{-1}), \qquad n = J(\xi^{-1})(n_0 \circ \xi^{-1}).$$

Let us assume from now on that the initial conditions ρ_0 and n_0 are related by $a\rho_0 + n_0 = 0$. We have seen that this implies that $a\rho + n = 0$. From the relations above we conclude the interesting result that the action of $\xi^{-1} \circ \eta$ fixes n_0, that is, $J(\xi^{-1} \circ \eta)(n_0 \circ \xi^{-1} \circ \eta) = n_0$. Conversely, given this relation and the condition $a\rho_0 + n_0 = 0$, it is easily seen that $a\rho + n = 0$.

The Lie–Poisson bracket associated with these equations is clearly the sum of two Lie–Poisson brackets associated with the semidirect products $\operatorname{Diff}(\mathcal{D}) \circledS [\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D})]$ and $\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})$.

The Kelvin–Noether theorem associated with the variable $\mathbf{m}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t}\left(\mathbf{u}^\flat + \frac{a}{R}A\right) = \oint_{c_t} T\,\mathbf{d}s,$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t}\mathbf{u}^\flat = \oint_{c_t} T\,\mathbf{d}s + \oint_{c_t}\frac{1}{\rho}\mathbf{i}_{(\operatorname{div}B)^\sharp}B,$$

where c_t is a loop which moves with the *fluid velocity* $\mathbf{u}$, that is, $c_t = \eta_t \circ c_0$, and s is the *specific entropy*. The Kelvin–Noether theorem associated with the variable $\mathbf{n}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{d_t} A = 0,$$

where d_t is a loop which moves with the electron fluid velocity $\mathbf{v}$, that is, $d_t = \xi_t \circ d_0$.

6.3. Superfluids

Superfluidity is a rare state of matter encountered in few fluids at extremely low temperatures. Such materials exhibit strange behavior such as the lack of viscosity, the ability to flow through very small channels that are impermeable to ordinary fluids, and the fact that it can form a layer whose thickness is that of one atom on the walls of the container in which it is placed. In addition, the rotational speed of a superfluid is quantized, that is, the fluid can rotate only at certain values of the speed. Superfluidity is considered to be a manifestation of quantum mechanical effects at macroscopic level. Typical examples of superfluids are ^{3}He, whose atoms are fermions and the superfluid transition occurs by Cooper pairing between atoms rather than electrons, and ^{4}He, whose atoms are bosons and the superfluidity is a consequence of Bose–Einstein condensation in an interacting system.

For example, at temperatures close to absolute zero a solution of ^{3}He and ^{4}He has its hydrodynamics described by three velocities: two superfluid velocities $\mathbf{v}_s^1$, $\mathbf{v}_s^2$ and one normal fluid velocity $\mathbf{v}_n$. If other kinds of superfluid are present, one needs to introduce additional superfluid velocities. For a history of the equations considered below and the Hamiltonian structure for superfluids, see Holm and Kupershmidt (1987).

For simplicity we treat the *two-fluid model*, that is, the case of one superfluid velocity $\mathbf{v}_s$ and one normal-fluid velocity $\mathbf{v}_n$. Remarkably, this Hamiltonian structure can be obtained by affine Lie–Poisson reduction, with order parameter Lie group $\mathcal{O} = S^1$.

The linear advected quantity is the *entropy density* S on which a diffeomorphism η acts as

$$S \mapsto (J\eta)(S\circ\eta).$$

The affine advected quantity is the *superfluid velocity* $\mathbf{v}_s$, on which the element $(\eta,\chi) \in \operatorname{Diff}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D}, S^1)$ acts as

$$\mathbf{v}_s \mapsto \left(\eta^*\mathbf{v}_s^\flat + \mathbf{d}\chi\right)^\sharp.$$

This action is simply the affine representation (4.5) for the Lie group $\mathcal{O} = S^1$. Here the advected quantity $\mathbf{v}_s$ is a vector field and not a one-form, since it represents a velocity and hence formula (4.5) was changed accordingly. As will be seen, in this formalism, the *mass density* does not appear as an advected quantity in the representation space V^*; it is a momentum, that is, one of the variables in the dual Lie algebra $\mathfrak{g}^* = \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D})$.

The reduced Hamiltonian is

$$h(\mathbf{m},\rho,S,\mathbf{v}_s) = -\frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}}(\mathbf{m}\cdot\mathbf{v}_n)\mu + \int_{\mathcal{D}}\varepsilon(\rho,S,\mathbf{v}_s-\mathbf{v}_n)\mu, \tag{6.8}$$

where $\mathbf{v}_n = \mathbf{v}_n(\mathbf{m},\rho,S,\mathbf{v}_s)$ is the vector field defined by the implicit condition

$$\mathbf{m} - \rho\mathbf{v}_n^\flat = \frac{\partial\varepsilon}{\partial\mathbf{r}}(\rho,S,\mathbf{v}_s-\mathbf{v}_n). \tag{6.9}$$

By the implicit function theorem, the above relation defines a unique vector field $\mathbf{v}_n$, provided the function ε verifies the condition that

$$u_x \mapsto \frac{\partial^2 \varepsilon}{\partial \mathbf{r}^2}(r, s, v_x) \cdot u_x - r u_x$$

is a bijective linear map.

The vector field $\mathbf{v}_n$ is interpreted as the *velocity of the normal flow*. The *internal energy density* ε is seen here as a function of three variables $\varepsilon = \varepsilon(\rho, S, \mathbf{r}) : \mathbb{R} \times \mathbb{R} \times T\mathcal{D} \to \mathbb{R}$. We make the following definitions:

$$\mu_{\text{chem}} := \frac{\partial \varepsilon}{\partial \rho}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \mathcal{F}(\mathcal{D}), \qquad T := \frac{\partial \varepsilon}{\partial S}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \mathcal{F}(\mathcal{D}),$$

$$\mathbf{p} := \frac{\partial \varepsilon}{\partial \mathbf{r}}(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \in \Omega^1(\mathcal{D}).$$

The interpretation of the quantities μ_{chem}, T, and $\mathbf{p}$ is obtained from the following thermodynamic derivative identity for the internal energy (superfluid first law):

$$\mathbf{d}(\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n)) = \mu_{\text{chem}} \mathbf{d}\rho + T\mathbf{d}S + \mathbf{p} \cdot \nabla_{\_}(\mathbf{v}_s - \mathbf{v}_n) \in \Omega^1(\mathcal{D}),$$

where ∇ denotes the Levi-Civita covariant derivative associated with the metric g. The function μ_{chem} is the *chemical potential*, T is the *temperature* and $\mathbf{p}$ is the *relative momentum density*.

The affine Lie–Poisson equations (4.7) associated with the Hamiltonian (6.8) are computed to be

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{m} = -\operatorname{Div} \mathbf{T}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{v}_n + \mathbf{p}^\sharp) = 0, \qquad \dfrac{\partial}{\partial t}S + \operatorname{div}(S\mathbf{v}_n) = 0, \\ \dfrac{\partial}{\partial t}\mathbf{v}_s + \operatorname{grad}\left(g(\mathbf{v}_s, \mathbf{v}_n) + \mu_{\text{chem}} - \dfrac{1}{2}\|\mathbf{v}_n\|^2\right) + \left(\mathbf{i}_{\mathbf{v}_n}\mathbf{d}\mathbf{v}_s^\flat\right)^\sharp = 0, \end{cases} \tag{6.10}$$

where $\mathbf{m} = \rho \mathbf{v}_n^\flat + \mathbf{p}$ and $\mathbf{T}$ is the superfluid stress tensor given by

$$\mathbf{T} = \mathbf{v}_n \otimes \mathbf{m} + \mathbf{p}^\sharp \otimes \mathbf{v}_s^\flat + p\delta, \qquad p := -\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n) + \mu_{\text{chem}}\rho + ST.$$

The last equation can be rewritten as

$$\frac{\partial}{\partial t}\mathbf{v}_s + \nabla_{\mathbf{v}_s}\mathbf{v}_s = -\operatorname{grad}\left(\mu_{\text{chem}} - \frac{1}{2}\|\mathbf{v}_s - \mathbf{v}_n\|^2\right) + \left(\mathbf{i}_{\mathbf{v}_s - \mathbf{v}_n}\mathbf{d}\mathbf{v}_s^\flat\right)^\sharp. \tag{6.11}$$

When $\mathcal{D}$ is three dimensional, the last term reads

$$\left(\mathbf{i}_{\mathbf{v}_s - \mathbf{v}_n}\mathbf{d}\mathbf{v}_s^\flat\right)^\sharp = \left(\star \mathbf{d}\mathbf{v}_s^\flat\right)^\sharp \times (\mathbf{v}_s - \mathbf{v}_n) = \operatorname{curl} \mathbf{v}_s \times (\mathbf{v}_s - \mathbf{v}_n) = (\mathbf{v}_n - \mathbf{v}_s) \times \operatorname{curl} \mathbf{v}_s.$$

Thus we have recovered equations (1a)–(1d) in Holm and Kupershmidt (1987), in the particular case of the two-fluids model.

Hamiltonian reduction for superfluids. Consider the right-invariant Hamiltonian function $H(\mathbf{m}_\eta, \rho_\chi, S, \mathbf{v}_s) = H_{(S,\mathbf{v}_s)}(\mathbf{m}_\eta, \rho_\chi)$ induced by h. A curve

$$(\mathbf{m}_\eta, \rho_\chi) \in T^*[\operatorname{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)]$$

is a solution of Hamilton's equations associated with the superfluid Hamiltonian $H_{(S_0, \mathbf{v}_{s0})}$ if and only if the curve

$$(\mathbf{m}, \rho) := J(\eta^{-1})(\mathbf{m}_\eta \circ \eta^{-1}, \rho_\chi \circ \eta^{-1})$$

is a solution of the system (6.10) with initial conditions $(S_0, \mathbf{v}_{s0})$.

The evolution of the advected quantities is given by

$$S = J(\eta^{-1})(S_0 \circ \eta^{-1}) \qquad \text{and} \qquad \mathbf{v}_s = \left(\eta_*\left(\mathbf{v}_{s0}^\flat + \mathbf{d}\chi^{-1}\right)\right)^\sharp.$$

Note that the evolution of the *superfluid vorticity* is given by

$$\mathbf{d}\mathbf{v}_s^\flat = \eta_* \mathbf{d}\mathbf{v}_{s0}^\flat;$$

therefore, the irrotationality condition $\mathbf{d}\mathbf{v}_s = 0$ (curl $\mathbf{v}_s = 0$ for the three-dimensional case) is preserved.

The associated Poisson bracket for superfluids is

$$\begin{aligned}\{f, g\}(\mathbf{m}, \rho, S, \mathbf{v}_s) = {} & \int_{\mathcal{D}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}\right] \mu + \int_{\mathcal{D}} \rho \cdot \left(\mathbf{d}\frac{\delta f}{\delta \rho} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta \rho} \cdot \frac{\delta f}{\delta \mathbf{m}}\right) \mu \\ & + \int_{\mathcal{D}} S \cdot \left(\mathbf{d}\frac{\delta f}{\delta S} \cdot \frac{\delta g}{\delta \mathbf{m}} - \mathbf{d}\frac{\delta g}{\delta S} \cdot \frac{\delta f}{\delta \mathbf{m}}\right) \mu \\ & + \int_{\mathcal{D}} \left[\left(\mathbf{d}\frac{\delta f}{\delta \rho} + \pounds_{\frac{\delta f}{\delta \mathbf{m}}} \mathbf{v}_s^\flat\right) \cdot \frac{\delta g}{\delta \mathbf{v}_s}^\sharp - \left(\mathbf{d}\frac{\delta g}{\delta \rho} + \pounds_{\frac{\delta g}{\delta \mathbf{m}}} \mathbf{v}_s^\flat\right) \cdot \frac{\delta f}{\delta \mathbf{v}_s}^\sharp\right] \mu. \end{aligned} \tag{6.12}$$

The γ-circulation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{c_t} \mathbf{v}_s^\flat = 0,$$

where c_t is a loop which moves with the *normal fluid velocity* $\mathbf{v}_n$.

6.4. *Superfluid Yang–Mills magnetohydrodynamics*

In this paragraph we combine the Hamiltonian structures of Yang–Mills magnetohydrodynamics and superfluid dynamics to obtain a new physical model for the theory of superfluids Yang–Mills magnetohydrodynamics as well as the corresponding Hamiltonian structure. In the Abelian case we recover the theory and the Hamiltonian structure derived in Holm and Kupershmidt (1987). We need a slight generalization of the geometric framework developed in section 4, namely we consider the group semidirect product

$$\mathrm{Diff}(\mathcal{D}) \circledS (\mathcal{F}(\mathcal{D}, \mathcal{O}) \times \mathcal{F}(\mathcal{D}, S^1)),$$

where $\mathcal{F}(\mathcal{D}, \mathcal{O}) \times \mathcal{F}(\mathcal{D}, S^1)$ is a direct product of groups on which $\mathrm{Diff}(\mathcal{D})$ acts as

$$(\chi_1, \chi_2) \mapsto (\chi_1 \circ \eta, \chi_2 \circ \eta).$$

The affine advected quantities are the *potential of the Yang–Mills fluid* A and the *superfluid velocity* $\mathbf{v}_s$, on which (η, χ_1, χ_2) acts as

$$A \mapsto \mathrm{Ad}_{\chi_1^{-1}} \eta^* A + \chi_1^{-1} T \chi_1 \qquad \text{and} \qquad \mathbf{v}_s \mapsto \left(\eta^* \mathbf{v}_s^\flat + \mathbf{d}\chi_2\right)^\sharp.$$

The reduced Hamiltonian is defined on the dual of the Lie algebra

$$[\mathfrak{X}(\mathcal{D}) \circledS (\mathcal{F}(\mathcal{D}, \mathfrak{o}) \times \mathcal{F}(\mathcal{D}))] \circledS (\mathcal{F}(\mathcal{D}) \times \Omega^1(\mathcal{D}, \mathfrak{o}) \times \mathfrak{X}(\mathcal{D}))$$

and is given by

$$\begin{aligned} h(\mathbf{m}, Q, \rho, S, A, \mathbf{v}_s) = {} & -\frac{1}{2} \int_{\mathcal{D}} \rho \|\mathbf{v}_n\|^2 \mu + \int_{\mathcal{D}} (\mathbf{m} \cdot \mathbf{v}_n) \mu \\ & + \int_{\mathcal{D}} \varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n) \mu + \frac{1}{2} \int_{\mathcal{D}} \|\mathbf{d}^A A\|^2 \mu, \end{aligned}$$

where the *normal fluid velocity* $\mathbf{v}_n$ is defined as in (6.9). This is simply the Hamiltonian (6.8) plus the energy of the Yang–Mills field. The norms are respectively associated with the metrics

g and (gk), where g is a Riemannian metric on $\mathcal{D}$ and k is an Ad-invariant inner product on $\mathfrak{o}$. The affine Lie–Poisson equations associated with this Hamiltonian are computed to be

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{m} = -\mathrm{Div}\,\mathbf{T}, \qquad \frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\mathbf{v}_n + \mathbf{p}^\sharp) = 0, \\ \frac{\partial}{\partial t}Q + \mathrm{div}(Q\mathbf{v}_n) = 0, \qquad \frac{\partial}{\partial t}S + \mathrm{div}(S\mathbf{v}_n) = 0, \\ \frac{\partial}{\partial t}\mathbf{v}_s + \mathrm{grad}\left(g(\mathbf{v}_s,\mathbf{v}_n) + \mu_{\mathrm{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2\right) + \left(\mathbf{i}_{\mathbf{v}_n}\mathbf{d}\mathbf{v}_s^\flat\right)^\sharp = 0, \\ \frac{\partial}{\partial t}A + \mathbf{d}^A(A(\mathbf{v}_n)) + \mathbf{i}_{\mathbf{v}_n}B = 0, \qquad B := \mathbf{d}^A A, \end{cases} \tag{6.13}$$

where $\mathbf{m} = \rho\mathbf{v}_n^\flat + \mathbf{p}$ and $\mathbf{T}$ is the $(1,1)$ stress tensor given by

$$\mathbf{T} := \mathbf{v}_n \otimes \mathbf{m} + \mathbf{p}^\sharp \otimes \mathbf{v}_s^\flat + B \cdot B + q\delta, \qquad q = -\varepsilon(\rho, S, \mathbf{v}_s - \mathbf{v}_n) + \mu_{\mathrm{chem}}\rho + ST - \tfrac{1}{2}\|B\|^2.$$

The corresponding Hamiltonian reduction and affine Lie–Poisson bracket can be found as before and the evolutions of the advected quantities are given by

$$S = J(\eta^{-1})(S_0 \circ \eta^{-1}),\ A = \eta_*\left(\mathrm{Ad}_\chi A_0 + \chi_1 T\chi_1^{-1}\right) \qquad \text{and} \qquad \mathbf{v}_s = \left(\eta_*\left(\mathbf{v}_{s0}^\flat + \mathbf{d}\chi_2^{-1}\right)\right)^\sharp.$$

The γ-circulation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t}\mathbf{v}_s^\flat = 0, \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t} A = 0,$$

where c_t is a loop which moves with the *normal fluid velocity* $\mathbf{v}_n$.

6.5. *Superfluid Hall magnetohydrodynamics*

The Hamiltonian formulation of superfluid Hall magnetohydrodynamics is given in Holm and Kupershmidt (1987). As one can guess, the Hamiltonian structure of these equations combines the Hamiltonian structures of Hall magnetohydrodynamics and of superfluids. This is still true at the group level and we will obtain the equations by affine Lie–Poisson reduction associated with the group

$$G := [\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)] \times \mathrm{Diff}(\mathcal{D}).$$

In this expression, the symbol '$\times$' denotes the *direct product* of the two groups. The advected quantities are

$$(S, \mathbf{u}; n) \in \mathcal{F}(\mathcal{D}) \times \mathfrak{X}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}).$$

The variable S is the *entropy density* of the normal flow, the other variables will be interpreted later. The action of $(\eta, \chi; \xi) \in G$ is given by

$$(S, \mathbf{u}; n) \mapsto (J\eta(S \circ \eta), (\eta^*\mathbf{u}^\flat + \mathbf{d}\chi)^\sharp; J\xi(n \circ \xi)).$$

The resulting affine Lie–Poisson equations consist of two systems, the affine Lie–Poisson equations associated with the variables $(\mathbf{m}, \rho, S, \mathbf{u})$ and the Lie–Poisson equations associated with the variables $(\mathbf{n}, n)$.

The Hamiltonian of superfluid Hall magnetohydrodynamics is defined on the dual Lie algebra

$$\begin{aligned}&([(\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})) \circledS (\mathcal{F}(\mathcal{D}) \oplus \mathfrak{X}(\mathcal{D}))] \times [\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D})])^* \\ &\qquad \cong \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \Omega^1(\mathcal{D}) \times \mathcal{F}(\mathcal{D})\end{aligned}$$

and is given by

$$h(\mathbf{m},\rho,S,\mathbf{u};\mathbf{n},n) := -\frac{1}{2}\int_{\mathcal{D}} \rho\|\mathbf{v}_n\|^2\mu + \int_{\mathcal{D}}\left(\left(\mathbf{m}-\frac{a\rho}{R}A\right)\cdot\mathbf{v}_n\right)\mu + \int_{\mathcal{D}} \varepsilon(\rho,S,\mathbf{v}_s-\mathbf{v}_n)\mu + \frac{1}{2}\int_{\mathcal{D}}\|\mathbf{d}A\|^2\mu, \tag{6.14}$$

where $\mathbf{v}_n$ is the *velocity of the normal flow*, $\mathbf{v}_s := \mathbf{u} - \frac{a}{R}A^\sharp$ is the *superfluid velocity* and ε is the *internal energy density*. The one-form A is defined by

$$A := R\frac{\mathbf{n}}{n}.$$

The norm in the first term is taken with respect to a fixed Riemannian metric g on $\mathcal{D}$. The velocity $\mathbf{v}_n$ is the function $\mathbf{v}_n = \mathbf{v}_n(\mathbf{m},\rho,S,\mathbf{u};\mathbf{n},n)$ defined by the implicit condition

$$\mathbf{m} - \rho\mathbf{v}_n^\flat - \frac{a\rho}{R}A = \frac{\partial\varepsilon}{\partial\mathbf{r}}(\rho,S,\mathbf{v}_s-\mathbf{v}_n).$$

By the implicit function theorem, the above relation defines a unique function $\mathbf{v}_n$, provided the function ε verifies the condition that the linear map

$$u_x \mapsto \frac{\partial^2\varepsilon}{\partial\mathbf{r}^2}(r,s,v_x,w_x)\cdot u_x - ru_x$$

is bijective for all $(r,s,v_x,w_x) \in \mathbb{R}\times\mathbb{R}\times T\mathcal{D}\times T\mathcal{D}$. Using the equations for ρ and n we obtain

$$\frac{\partial}{\partial t}(a\rho+n) = 0.$$

Thus, if we assume that the initial conditions verify $a\rho_0+n_0 = 0$, then we have $a\rho+n = 0$ for all time. In this case, the affine Lie–Poisson equations associated with the Hamiltonian (6.14) are given by

$$\begin{cases} \dfrac{\partial}{\partial t}(\mathbf{m}+\mathbf{n}) = -\operatorname{Div}\mathbf{T}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{v}_n + \mathbf{p}^\sharp) = 0, \qquad \dfrac{\partial}{\partial t}S + \operatorname{div}(S\mathbf{v}_n) = 0, \\ \dfrac{\partial}{\partial t}A = -\mathbf{i}_{\mathbf{v}_n}B - \dfrac{1}{\rho}\mathbf{i}_{\mathbf{p}^\sharp}B - \dfrac{R}{a\rho}\mathbf{i}_{(\operatorname{div}B)^\sharp}B, \\ \dfrac{\partial}{\partial t}\mathbf{v}_s = -\operatorname{grad}\left(g(\mathbf{v}_s,\mathbf{v}_n) + \mu_{\mathrm{chem}} - \dfrac{1}{2}\|\mathbf{v}_n\|^2\right) \\ \quad + \left(\dfrac{a}{R\rho}\mathbf{i}_{\mathbf{p}^\sharp}B + \dfrac{1}{\rho}\mathbf{i}_{(\operatorname{div}B)^\sharp}B - \mathbf{i}_{\mathbf{v}_n}\mathbf{d}\mathbf{v}_s^\flat\right)^\sharp, \end{cases} \tag{6.15}$$

where $\mathbf{m} = \rho\mathbf{v}_n^\flat + \frac{a\rho}{R}A + \mathbf{p}$ and $\mathbf{T}$ is $(1,1)$ stress tensor given by

$$\mathbf{T} = \mathbf{v}_n\otimes\left(\rho\mathbf{v}_n^\flat + \mathbf{p}\right) + \mathbf{p}^\sharp + B\cdot B + q\delta, \qquad q = -\varepsilon(\rho,S,\mathbf{v}_s-\mathbf{v}_n) + \rho\mu_{\mathrm{chem}} + ST - \tfrac{1}{2}\|B\|^2.$$

These are the equations for superfluid Hall magnetohydrodynamics as given in Holm and Kupershmidt (1987) equations (35a)–(35e). When $\mathcal{D}$ is three dimensional, the two last equations read

$$\frac{\partial}{\partial t}A^\sharp = \left(\mathbf{v}_n + \frac{1}{\rho}\mathbf{p}^\sharp - \frac{R}{a\rho}\operatorname{curl}\mathbf{B}\right)\times\mathbf{B} \text{ and } \frac{\partial}{\partial t}\mathbf{v}_s = -\operatorname{grad}\left(g(\mathbf{v}_s,\mathbf{v}_n) + \mu_{\mathrm{chem}} - \frac{1}{2}\|\mathbf{v}_n\|^2\right) + \mathbf{v}_n\times\operatorname{curl}\mathbf{v}_s + \frac{1}{\rho}\left(\operatorname{curl}\mathbf{B} - \frac{a}{R}\mathbf{p}^\sharp\right)\times\mathbf{B}.$$

Hamiltonian reduction for superfluid Hall magnetohydrodynamics. Consider the right-invariant Hamiltonian function $H(\mathbf{m}_\eta, \rho_\chi, S, \mathbf{u}; \mathbf{n}_\xi, n) = H_{(S,\mathbf{u};n)}(\mathbf{m}_\eta, \rho_\chi; \mathbf{n}_\xi)$ induced by h and suppose that we have $a\rho_0 + n_0 = 0$. A smooth curve

$$(\mathbf{m}_\eta, \rho_\chi; \mathbf{n}_\xi) \in T^*[(\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, S^1)) \times \mathrm{Diff}(\mathcal{D})]$$

is a solution of Hamilton's equations associated with $H_{(S_0,\mathbf{u}_0;n_0)}$ and with the initial condition ρ_0 if and only if the curve

$$(\mathbf{m}, \rho; \mathbf{n}) := (J(\eta^{-1})(\mathbf{m}_\eta \circ \eta^{-1}), J(\eta^{-1})(\rho_\chi \circ \eta^{-1}); J(\xi^{-1})(\mathbf{n} \circ \xi^{-1}))$$

is a solution of the equations (6.15), where $\mathbf{v}_s = \mathbf{u} - aA^\sharp/R = \mathbf{u} - a\mathbf{n}/n$.

The Poisson bracket for superfluid Hall magnetohydrodynamics is the sum of the affine Lie–Poisson bracket associated with the variables $(\mathbf{m}, \rho, S, \mathbf{u})$ and the Lie–Poisson bracket associated with the variables $(\mathbf{n}, n)$.

The γ-circulation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t} \mathbf{u}^\flat = 0.$$

Using the definition $\mathbf{v}_s := \mathbf{u} - \frac{a}{R}A^\sharp$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c_t} \left(\mathbf{v}_s^\flat + \frac{a}{R}A^\sharp\right) = 0,$$

where c_t is a loop which moves with the *normal fluid velocity* $\mathbf{v}_n$. The Kelvin–Noether theorem associated with the variable $\mathbf{n}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{d_t} A = 0,$$

where d_t is a loop which moves with the *electron fluid velocity* $\mathbf{v}$.

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