Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 255 (2008) 1225–1272
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Induced and coinduced Banach Lie–Poisson spaces and integrability

Anatol Odzijewicz [a], Tudor S. Ratiu [b,*]

[a] *Institute of Mathematics, University of Bialystok, Lipowa 41, PL-15424 Bialystok, Poland*
[b] *Section de Mathématiques and Bernoulli Center, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland*

Received 22 December 2007; accepted 2 June 2008
Available online 9 July 2008
Communicated by D. Voiculescu

Abstract

The Poisson induction and coinduction procedures are used to construct Banach Lie–Poisson spaces as well as related systems of integrals in involution. This general method applied to the Banach Lie–Poisson space of trace class operators leads to infinite Hamiltonian systems of k-diagonal trace class operators which have infinitely many integrals. The bidiagonal case is investigated in detail.

Keywords: Banach Lie–Poisson space; Induction; Coinduction; Momentum map; Coadjoint orbit; Integrable Hamiltonian system

Contents

* Corresponding author.
E-mail addresses: aodzijew@uwb.edu.pl (A. Odzijewicz), tudor.ratiu@epfl.ch (T.S. Ratiu).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.06.001

1. Introduction

This paper continues the investigation of Banach Lie–Poisson spaces introduced in [12] and also studied in [4,13]. The theory of Banach Lie–Poisson spaces gives a natural generalization in the functional analytical context of the Poisson geometry of finite-dimensional integrable Hamiltonian systems. It gives also a solid mathematical foundation for the theory of Hamiltonian systems with infinitely many degrees of freedom. The interest in these system was initiated in [3] and [8] and from then on they have played an important role in mathematics and physics. With few notable exceptions, for infinite-dimensional systems, the Lie–Poisson bracket formulation is mostly formal. It is our belief that these formal approaches can be given a solid functional analytic underpinning. The present paper formulates such an approach for various families of integrable systems which arise in a natural way when one investigates Banach Lie–Poisson spaces of trace class operators.

The paper is organized as follows. Section 2 presents the general theory of induced and coinduced Banach Lie–Poisson structures (Propositions 2.1–2.4) and gives a method of construction of systems of integrals in involution. The associated involution theorem (Proposition 2.2 and Corollary 2.3) is an analogue of the classical R-matrix method for Banach Lie–Poisson spaces.

Section 3 investigates the Banach Lie–Poisson geometry of several classes of spaces of trace class operators. The general constructions of Section 2 are implemented explicitly to these spaces. The multi-diagonal Banach Lie group, its Lie algebra, and its dual are introduced and studied (Propositions 3.1 and 3.2). The naturally induced and coinduced Poisson structures on the preduals of their Banach Lie algebras are presented.

Section 4 formulates the equations of motion induced by the Casimir functions of the Banach Lie–Poisson space of trace class operators relative to the various induced and coinduced Poisson brackets discussed previously. These systems represent a k-diagonal version of the semi-infinite Toda system which is obtained from this point of view if $k = 2$. The solution of the systems associated to two different splittings of the space of trace class operators in terms of group decompositions are also presented.

Section 5 emphasizes the important particular case of bidiagonal operators. The Banach Lie group of upper bidiagonal bounded operators is studied in detail and the topological and symplectic structure of the generic coadjoint orbit is presented (Proposition 5.1). The Banach space analogue of the Flaschka map (defined for the first time in [7]) is analyzed and its relationship to the coadjoint orbits is pointed out (Propositions 5.2 and 5.3). There are new, typical infinite-dimensional, phenomena that appear in this context. For example, as opposed to the finite-dimensional case, the Banach space of lower bidiagonal trace class operators does not form a single coadjoint orbit and there are non-algebraic invariants for the coadjoint orbits. As an example of the theory, the semi-infinite Toda lattice is rigorously investigated using the method of orthogonal polynomials first introduced, to our knowledge, in [5]. The explicit solution of this system is obtained, both in action–angle as well as in the original variables, thereby extending the formulas in [10] from the finite to the semi-infinite Toda lattice.

Conventions. In this paper all Banach manifolds and Lie groups are real. The definition of the notion of a Banach Lie subgroup follows Bourbaki [6], that is, a subgroup H of a Banach Lie group G is necessarily a submanifold (and not just injectively immersed). In particular, Banach Lie subgroups are necessarily closed.

2. Induced and coinduced Banach Lie–Poisson spaces

In this section we shortly review some material from [12] and present constructions that are necessary for the development of the ideas in the rest of the paper.

Preliminaries. Let us recall how a given Banach Lie–Poisson structure induces and coinduces similar structures on other Banach spaces. All the proofs of the statements below can be found in [12]. Throughout this paper, unless specified otherwise, all objects are over $\mathbb{R}$.

A *Banach Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ is a Banach space $\mathfrak{g}$ that is also a Lie algebra such that the Lie bracket is a bilinear continuous map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Thus the adjoint and coadjoint maps $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_x\, y := [x, y]$, and $\mathrm{ad}^*_x : \mathfrak{g}^* \to \mathfrak{g}^*$ are also continuous for each $x \in \mathfrak{g}$. Here $\mathfrak{g}^*$ denotes the dual of $\mathfrak{g}$, that is, the Banach space of all linear continuous functionals on $\mathfrak{g}$.

A *Banach Lie–Poisson space* $(\mathfrak{b}, \{\cdot,\cdot\})$ is defined to be a real Poisson manifold such that $\mathfrak{b}$ is a Banach space and the dual $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation. We need to explain what does it mean for $\mathfrak{b}$ to be a Banach Poisson manifold. The Poisson bracket induces the derivation $h \mapsto \{\cdot, h\}$ on $C^\infty(\mathfrak{b})$ which defines a map $X_h : \mathfrak{b} \to \mathfrak{b}^{**}$ by $\langle X_h(b), Df(b)\rangle = \{f, h\}(b)$ for any $b \in \mathfrak{b}$ and f a smooth real-valued function defined in an open subset of $\mathfrak{b}$ containing b. Thus, $X_h(b) \in \mathfrak{b}^{**} \cong T^{**}_b\mathfrak{b}$ and therefore $X_h(b)$ is not a tangent vector to $\mathfrak{b}$ at b. The requirement that $\mathfrak{b}$ be a Banach Poisson manifold is that $X_h(b) \in \mathfrak{b} \cong T_b\mathfrak{b}$ for all $b \in \mathfrak{b}$.

Denote by $[\cdot,\cdot]$ the restriction of the Poisson bracket $\{\cdot,\cdot\}$ from $C^\infty(\mathfrak{b})$ to the Lie subalgebra $\mathfrak{b}^*$. The following criterion characterizes the Banach Lie–Poisson structure. The Banach space $\mathfrak{b}$ is a Banach Lie–Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ if and only if its dual $\mathfrak{b}^*$ is a Banach Lie algebra $(\mathfrak{b}^*, [\cdot,\cdot])$ satisfying $\mathrm{ad}^*_x\, \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**}$ for all $x \in \mathfrak{b}^*$. Moreover, the Poisson bracket of $f, h \in C^\infty(\mathfrak{b})$ is given by

$$\{f, h\}(b) = \big\langle \big[Df(b), Dh(b)\big], b\big\rangle, \tag{2.1}$$

where $b \in \mathfrak{b}$ and $Df(b) \in \mathfrak{b}^*$ denotes the Fréchet derivative of f at the point b. If h is a smooth function on $\mathfrak{b}$, the associated *Hamiltonian vector field* is given by

$$X_h(b) = -\,\mathrm{ad}^*_{Dh(b)}\, b \in \mathfrak{b} \tag{2.2}$$

for any $b \in \mathfrak{b}$. Therefore Hamilton's equations are

$$\frac{d}{dt} b(t) = -\,\mathrm{ad}^*_{Dh(b(t))}\, b(t). \tag{2.3}$$

In particular, $h \in C^\infty(\mathfrak{b})$ is a Casimir function, that is, $\{f, h\} = 0$ for all $f \in C^\infty(\mathfrak{b})$ if and only if

$$\mathrm{ad}^*_{Dh(b)}\, b = 0 \quad \text{for all } b \in \mathfrak{b}. \tag{2.4}$$

Given two Banach Lie–Poisson spaces $(\mathfrak{b}_1, \{,\}_1)$ and $(\mathfrak{b}_2, \{,\}_2)$, a smooth map $\varphi : \mathfrak{b}_1 \to \mathfrak{b}_2$ is said to be *canonical* or a *Poisson map* if

$$\{f, h\}_2 \circ \varphi = \{f \circ \varphi, h \circ \varphi\}_1 \tag{2.5}$$

for any two smooth locally defined functions f and h on $\mathfrak{b}_2$. Like in the finite-dimensional case, (2.5) is equivalent to

$$X_h^2 \circ \varphi = T\varphi \circ X_{h\circ\varphi}^1 \tag{2.6}$$

for any smooth locally defined function h on $\mathfrak{b}_2$. Therefore, the flow of a Hamiltonian vector field is a Poisson map and Hamilton's equations $\dot{f} = \{f, h\}$ in Poisson bracket formulation are valid. If the Poisson map φ is, in addition, linear, then it is called a *linear Poisson map*.

Given the Banach Lie–Poisson spaces $(\mathfrak{b}_1, \{,\}_1)$ and $(\mathfrak{b}_2, \{,\}_2)$ there is a unique Banach Poisson structure $\{,\}$ on the product space $\mathfrak{b}_1 \times \mathfrak{b}_2$ such that:

(i) the canonical projections $\pi_1 : \mathfrak{b}_1 \times \mathfrak{b}_2 \to \mathfrak{b}_1$ and $\pi_2 : \mathfrak{b}_1 \times \mathfrak{b}_2 \to \mathfrak{b}_2$ are Poisson maps;
(ii) $\pi_1^*(C^\infty(\mathfrak{b}_1))$ and $\pi_2^*(C^\infty(\mathfrak{b}_2))$ are Poisson commuting subalgebras of $C^\infty(\mathfrak{b}_1 \times \mathfrak{b}_2)$.

This unique Poisson structure on $\mathfrak{b}_1 \times \mathfrak{b}_2$ is called the *product Poisson structure* and its bracket is given by the formula

$$\{f, g\}(b_1, b_2) = \{f_{b_2}, g_{b_2}\}_1(b_1) + \{f_{b_1}, g_{b_1}\}_2(b_2), \tag{2.7}$$

where $f_{b_1}, g_{b_1} \in C^\infty(\mathfrak{b}_2)$ and $f_{b_2}, g_{b_2} \in C^\infty(\mathfrak{b}_1)$ are the partial functions given by $f_{b_1}(b_2) := f_{b_2}(b_1) := f(b_1, b_2)$ and $g_{b_1}(b_2) := g_{b_2}(b_1) := g(b_1, b_2)$. In addition, this formula shows that this unique Banach Poisson structure is Lie–Poisson and that the inclusions $\iota_1 : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}_1 \times \mathfrak{b}_2$, $\iota_2 : \mathfrak{b}_2 \hookrightarrow \mathfrak{b}_1 \times \mathfrak{b}_2$ given by $\iota_1(b_1) := (b_1, 0)$ and $\iota_2(b_2) := (0, b_2)$, respectively, are also linear Poisson maps.

Induced structures. Let $\mathfrak{b}_1$ be a Banach space, $(\mathfrak{b}, \{\cdot,\cdot\})$ a Banach Lie–Poisson space, and $\iota : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$ an injective continuous linear map with closed range. Then $\ker \iota^*$ is an ideal in the Banach Lie algebra $(\mathfrak{b}^*, [\cdot,\cdot])$ if and only if $\mathfrak{b}_1$ carries a unique Banach Lie–Poisson bracket $\{\cdot,\cdot\}_1^{\mathrm{ind}}$ such that

$$\{F \circ \iota, G \circ \iota\}_1^{\mathrm{ind}} = \{F, G\} \circ \iota \tag{2.8}$$

for any $F, G \in C^\infty(\mathfrak{b})$; see [12, Proposition 4.10]. This Poisson structure on $\mathfrak{b}_1$ is said to be *induced* by the mapping ι and it is given by

$$\{f, g\}_1^{\mathrm{ind}}(b_1) = \left\langle [\iota^*]\left(\left[\left[\iota^*\right]^{-1}\left(Df(b_1)\right), \left[\iota^*\right]^{-1}\left(Dg(b_1)\right)\right]_1\right), b_1\right\rangle \tag{2.9}$$

for any $f, g \in C^\infty(\mathfrak{b}_1)$ and $b_1 \in \mathfrak{b}_1$, where $[\iota^*] : \mathfrak{b}^*/\ker\iota^* \to \mathfrak{b}_1^*$ is the Banach space isomorphism induced by $\iota^* : \mathfrak{b}^* \to \mathfrak{b}_1^*$ and $[\cdot,\cdot]_1$ denotes the Lie bracket on the quotient Lie algebra $\mathfrak{b}^*/\ker\iota^*$.

Let us assume now that the range $\iota(\mathfrak{b}_1)$ is a closed split subspace of $\mathfrak{b}$, that is, there exists a projector $R = R^2 : \mathfrak{b} \to \mathfrak{b}$ such that $\iota(\mathfrak{b}_1) = R(\mathfrak{b})$. Taking in (2.8) $F := f \circ \iota^{-1} \circ R$, $G := g \circ \iota^{-1} \circ R \in C^\infty(\mathfrak{b})$ for $f, g \in C^\infty(\mathfrak{b}_1)$ and noting that $\iota^{-1} \circ R \circ \iota = \mathrm{id}_{\mathfrak{b}_1}$, we get

$$\begin{aligned} \{f, g\}_1^{\mathrm{ind}}(b_1) &= \left\{f \circ \iota^{-1} \circ R, g \circ \iota^{-1} \circ R\right\}\left(\iota(b_1)\right) \\ &= \left\langle\left[D\left(f \circ \iota^{-1} \circ R\right)\left(\iota(b_1)\right), D\left(g \circ \iota^{-1} \circ R\right)\left(\iota(b_1)\right)\right], \iota(b_1)\right\rangle. \end{aligned} \tag{2.10}$$

We shall make use of this formula in Section 3.

We return now to the general case, that is, we consider an arbitrary quasi-immersion $\iota : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$ of Banach spaces which means that the range $\iota(\mathfrak{b}_1)$ is closed but does not necessarily possess a closed complement.

Proposition 2.1. *Let $\iota : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$ be a quasi-immersion of Banach Lie–Poisson spaces* (*so* range ι *is a closed subspace of* $\mathfrak{b}$ *and* $\ker \iota^*$ *is an ideal in the Banach Lie algebra* $\mathfrak{b}^*$). *Assume that there is a connected Banach Lie group G with Banach Lie algebra $\mathfrak{g} := \mathfrak{b}^*$. Then the G-coadjoint orbit $\mathcal{O}_{\iota(b_1)} := \mathrm{Ad}^*_G \, \iota(b_1)$ is contained in $\iota(\mathfrak{b}_1)$ for any $b_1 \in \mathfrak{b}_1$. In addition, if $N \subset G$ is a closed connected normal Lie subgroup of G whose Lie algebra is* $\ker \iota^*$*, then the N-coadjoint action restricted to $\iota(\mathfrak{b}_1)$ is trivial. Therefore the Banach Lie group $G/N := \{[g] := gN \mid g \in G\}$ naturally acts on $\iota(\mathfrak{b}_1)$ and the orbit of $\iota(b_1)$ under this action coincides with $\mathcal{O}_{\iota(b_1)}$ for any $b_1 \in \mathfrak{b}_1$.*

Proof. Since $\ker \iota^*$ is an ideal in $\mathfrak{g} = \mathfrak{b}^*$, it follows that $[x, y] \in \ker \iota^*$ for all $x \in \mathfrak{g}$ and $y \in \ker \iota^*$. Therefore, since $\ker \iota^*$ is closed in $\mathfrak{g}$, it follows that $\mathrm{Ad}_{\exp x}\, y = e^{\mathrm{ad}_x} y \in \ker \iota^*$ for any $x \in \mathfrak{g}$ and $y \in \ker \iota^*$. This shows that for any $g \in G$ in an open neighborhood of the identity element of G we have $\mathrm{Ad}_g \ker \iota^* \subset \ker \iota^*$. Since G is connected, it is generated by a neighborhood of the identity and we conclude that $\mathrm{Ad}_g \ker \iota^* \subset \ker \iota^*$ for any $g \in G$.

The upper index $^\circ$ on a set denotes the annihilator of that set relative to a duality pairing; the annihilator of a set is always a vector subspace. Let $b_1 \in \mathfrak{b}_1$ and $g \in G$. Since $\ker \iota^* = \iota(\mathfrak{b}_1)^\circ$, closedness of $\iota(\mathfrak{b}_1)$ in $\mathfrak{b}$ implies that $(\ker \iota^*)^\circ = \iota(\mathfrak{b}_1)$. Thus, for any $g \in G$ and $x \in \ker \iota^*$, we have

$$\left\langle \mathrm{Ad}^*_g \, \iota(b_1), x \right\rangle = \left\langle \iota(b_1), \mathrm{Ad}_g \, x \right\rangle = 0$$

which proves that $\mathrm{Ad}^*_G \, \iota(b_1) \subset \iota(\mathfrak{b}_1)$.

Now let $N \subset G$ be a closed connected normal Lie subgroup of G with Banach Lie algebra $\ker \iota^* \subset \mathfrak{g}$. For any $b_1 \in \mathfrak{b}_1$, $x \in \mathfrak{g} = \mathfrak{b}^*$, $y \in \ker \iota^*$, we have

$$\left\langle \mathrm{ad}^*_y \, \iota(b_1), x \right\rangle = \left\langle \iota(b_1), [y, x] \right\rangle = 0$$

since $\ker \iota^*$ is an ideal in $\mathfrak{g}$ and $\ker \iota^* = \iota(\mathfrak{b}_1)^\circ$. Since this is valid for all $x \in \mathfrak{g}$, it follows that $\mathrm{ad}^*_y \, \iota(b_1) = 0$ for all $y \in \ker \iota^*$ and $b_1 \in \mathfrak{b}_1$. Using the exponential map, this shows that $\mathrm{Ad}^*_n \, \iota(b_1) = \iota(b_1)$ for any n in a neighborhood of the identity in N. Since N is connected, it is generated by a neighborhood of the identity and we conclude that $\mathrm{Ad}^*_n \, \iota(b_1) = \iota(b_1)$ for all $n \in N$.

The quotient $G/N := \{[g] := gN \mid g \in G\}$ is a Banach Lie group and the projection $G \to G/N$ is a smooth surjective submersive Banach Lie group homomorphism [6, Chapter III, §1.6]. Since the coadjoint action of N on $\iota(\mathfrak{b}_1)$ is trivial, the Banach Lie group G/N acts smoothly on $\iota(\mathfrak{b}_1)$ by $[g] \cdot \iota(b_1) := \mathrm{Ad}^*_{g^{-1}} \, \iota(b_1)$. The orbit of a fixed element $\iota(b_1) \in \iota(\mathfrak{b}_1)$ by this group action is obviously equal to the G-orbit $\mathcal{O}_{\iota(b_1)}$. □

Coinduced structures. Let $(\mathfrak{b}, \{,\})$ be a Banach Lie–Poisson space and $\pi : \mathfrak{b} \to \mathfrak{b}_1$ a continuous linear surjective map onto the Banach space $\mathfrak{b}_1$. Then $\mathfrak{b}_1$ carries a unique Banach Lie–Poisson bracket $\{,\}_1^{\mathrm{coind}}$ making π into a linear Poisson map, that is,

$$\{f \circ \pi, g \circ \pi\} = \{f, g\}_1^{\mathrm{coind}} \circ \pi \tag{2.11}$$

for any $f, g \in C^\infty(\mathfrak{b}_1)$ if and only if $\pi^*(\mathfrak{b}_1^*) \subset \mathfrak{b}^*$ is closed under the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{b}^*$; see [12, Proposition 4.8]. This unique Poisson structure on $\mathfrak{b}_1$ is said to be *coinduced* by the Banach Lie–Poisson structure on $\mathfrak{b}$ and the linear continuous map π. It should be noted that $\operatorname{im}\pi^*$ is a closed subspace of $\mathfrak{b}^*$ since $\operatorname{im}\pi^* = (\ker\pi)^\circ$. To determine the coinduced bracket on $\mathfrak{b}_1$ note that $\pi^* : \mathfrak{b}_1^* \to \mathfrak{b}^*$ is an injective linear continuous map whose closed range is a Banach Lie subalgebra of $\mathfrak{b}^*$. Thus, on $\operatorname{im}\pi^*$ we can invert π^*. The coinduced bracket on $\mathfrak{b}_1$ has then the form

$$\{f, g\}_1^{\mathrm{coind}}(b_1) = \left\langle \left(\pi^*\right)^{-1}\left[\pi^*\left(Df(b_1)\right), \pi^*\left(Dg(b_1)\right)\right], b_1\right\rangle \tag{2.12}$$

for any $f, g \in C^\infty(\mathfrak{b}_1)$ and $b_1 \in \mathfrak{b}_1$.

Let us assume that $\ker\pi$ admits a closed complement. This is equivalent to the existence of a linear continuous injective map $\iota : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$ with closed range such that $\pi \circ \iota = \mathrm{id}_{\mathfrak{b}_1}$. Thus (2.11) implies that

$$\{f, g\}_1^{\mathrm{coind}} = \{f \circ \pi, g \circ \pi\} \circ \iota \tag{2.13}$$

for any $f, g \in C^\infty(\mathfrak{b}_1)$.

Assume now that the Banach Lie–Poisson space $\mathfrak{b}$ splits into a direct sum $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ of closed Banach subspaces. Denote by $R_j : \mathfrak{b} \to \mathfrak{b}$ the projection onto $\mathfrak{b}_j$, for $j = 1, 2$. So we have the following relations: $R_1 + R_2 = \mathrm{id}_{\mathfrak{b}}$, $R_1^2 = R_1$, $R_2^2 = R_2$, $R_2 R_1 = R_1 R_2 = 0$, $\mathfrak{b}_1 := \operatorname{im} R_1$, and $\mathfrak{b}_2 := \operatorname{im} R_2$. Dualizing we get the projectors $R_1^*, R_2^* : \mathfrak{b}^* \to \mathfrak{b}^*$ satisfying $R_1^* + R_2^* = \mathrm{id}_{\mathfrak{b}^*}$, $(R_1^*)^2 = R_1^*$, $(R_2^*)^2 = R_2^*$, $R_2^* R_1^* = R_1^* R_2^* = 0$. The relationship between these spaces is given by

$$\ker R_1 = \operatorname{im} R_2 = \mathfrak{b}_2 \quad \text{and} \quad \ker R_2 = \operatorname{im} R_1 = \mathfrak{b}_1, \tag{2.14}$$

$$\ker R_1^* = \operatorname{im} R_2^* = (\operatorname{im} R_1)^\circ \cong \mathfrak{b}_2^* \quad \text{and} \quad \ker R_2^* = \operatorname{im} R_1^* = (\operatorname{im} R_2)^\circ \cong \mathfrak{b}_1^*, \tag{2.15}$$

$$\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \quad \text{and} \quad \mathfrak{b}^* = \mathfrak{b}_2^\circ \oplus \mathfrak{b}_1^\circ \cong \mathfrak{b}_1^* \oplus \mathfrak{b}_2^*. \tag{2.16}$$

Let $\iota_j : \mathfrak{b}_j \hookrightarrow \mathfrak{b}$ be the inclusion determined by the splitting $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ for $j = 1, 2$. Denote by $\pi_j : \mathfrak{b} \to \mathfrak{b}_j$ the projection determined by the projector $R_j : \mathfrak{b} \to \mathfrak{b}$, that is, $\iota_j \circ \pi_j = R_j$ and note that $\pi_j \circ \iota_j = \mathrm{id}_{\mathfrak{b}_j}$. We summarize these notations in the following diagram.

$$\begin{tikzcd}
 & \mathfrak{b} \arrow[dl, shift right, "\pi_1"'] \arrow[dr, shift left, "\pi_2"] & \\
\mathfrak{b}_1 \arrow[ur, shift right, "\iota_1"'] & & \mathfrak{b}_2 \arrow[ul, shift left, "\iota_2"]
\end{tikzcd}$$

From (2.13) we get

$$\{f, g\}_j^{\mathrm{coind}} = \{f \circ \pi_j, g \circ \pi_j\} \circ \iota_j \tag{2.17}$$

or, explicitly

$$\{f,g\}_j^{\mathrm{coind}}(b_j) = \big\langle \big[D(f\circ\pi_j)\big(\iota_j(b_j)\big), D(g\circ\pi_j)\big(\iota_j(b_j)\big)\big], \iota_j(b_j)\big\rangle, \quad \text{where } b_j\in\mathfrak{b}_j. \tag{2.18}$$

The following proposition presents some properties of the induced and coinduced structures on $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

Proposition 2.2. *Assume that* $\operatorname{im} R_1^*$ *and* $\operatorname{im} R_2^*$ *are Banach Lie subalgebras of* $\mathfrak{b}^*$. *Then*:

(i) $\mathfrak{b}_j$ *has a Banach Lie–Poisson structure coinduced by* π_j *and the expression of the coinduced bracket* $\{,\}_j^{\mathrm{coind}}$ *on* $\mathfrak{b}_j$ *is given by* (2.17). *The Hamiltonian vector field of* $h\in C^\infty(\mathfrak{b}_j)$ *at* $b_j\in\mathfrak{b}_j$ *is given by*

$$X_h(b_j) = -\pi_j\big(\mathrm{ad}^*_{\pi_j^* Dh(b_j)}\,\iota_j(b_j)\big), \quad j=1,2, \tag{2.19}$$

where $Dh(b_j)\in\mathfrak{b}_j^*$ *and* ad_x *is the adjoint action of* $x\in\mathfrak{b}^*$ *on* $\mathfrak{b}^*$.

(ii) *The Banach space isomorphism* $R := \frac{1}{2}(R_1 - R_2) : \mathfrak{b}\to\mathfrak{b}$ *defines a new Banach Lie–Poisson structure*

$$\{f,g\}_R(b) := \big\langle \big[R^* Df(b), Dg(b)\big] + \big[Df(b), R^* Dg(b)\big], b\big\rangle \tag{2.20}$$

on $\mathfrak{b}$, $f,g\in C^\infty(\mathfrak{b})$, *that coincides with the product structure on* $\mathfrak{b}_1\times\bar{\mathfrak{b}}_2$, *where* $\mathfrak{b}_1$ *carries the coinduced bracket* $\{,\}_1^{\mathrm{coind}}$ *and* $\bar{\mathfrak{b}}_2$ *denotes* $\mathfrak{b}_2$ *endowed with the Lie–Poisson bracket* $-\{,\}_2^{\mathrm{coind}}$.

(iii) *The inclusion maps* $\iota_1 : (\mathfrak{b}_1, \{,\}_1^{\mathrm{coind}}) \hookrightarrow (\mathfrak{b}, \{,\}_R)$ *and* $\iota_2 : (\bar{\mathfrak{b}}_2, \{,\}_2^{\mathrm{coind}}) \hookrightarrow (\mathfrak{b}, \{,\}_R)$ *are linear injective Poisson maps with closed range.*

(iv) *The map* ι_j *induces from* $(\mathfrak{b}, \{,\}_R)$ *a Banach Lie–Poisson structure on* $\mathfrak{b}_j$ *which coincides with the coinduced structure described in* (i), *for* $j=1,2$.

Proof. (i) By hypothesis, the range $\operatorname{im} R_j^*$ of the map $R_j^* : \mathfrak{b}^*\to\mathfrak{b}^*$ is a Banach Lie subalgebra of $\mathfrak{b}^*$. Thus π_j coinduces a Banach Lie–Poisson structure on $\mathfrak{b}_j^*$. Let $h\in C^\infty(\mathfrak{b}_j)$ and note that for any function $f\in C^\infty(\mathfrak{b}_j)$ and $b_j\in\mathfrak{b}_j$ we have

$$\begin{aligned}\big\langle Df(b_j), X_h(b_j)\big\rangle &= \{f,h\}_j^{\mathrm{coind}}(b_j) = \big\langle \big[D(f\circ\pi_j)\big(\iota_j(b_j)\big), D(h\circ\pi_j)\big(\iota_j(b_j)\big)\big], \iota_j(b_j)\big\rangle \\ &= \big\langle \big[\pi_j^* Df(b_j), \pi_j^* Dh(b_j)\big], \iota_j(b_j)\big\rangle \\ &= \big\langle \pi_j^* Df(b_j), -\mathrm{ad}^*_{\pi_j^* Dh(b_j)}\,\iota_j(b_j)\big\rangle \\ &= \big\langle Df(b_j), -\pi_j\,\mathrm{ad}^*_{\pi_j^* Dh(b_j)}\,\iota_j(b_j)\big\rangle,\end{aligned}$$

which proves formula (2.19).

(ii) Let $b = b_1 + b_2 \in \mathfrak{b}_1\oplus\mathfrak{b}_2$. Then $R_j(b) = b_j$, for $j=1,2$. A direct verification shows then that

$$\begin{aligned}\{f,g\}_R(b) &= \big\langle \big[R^* Df(b), Dg(b)\big] + \big[D(b), R^* Dg(b)\big], b\big\rangle \\ &= \frac{1}{2}\big\langle \big[R_1^* Df(b) - R_2^* Df(b), R_1^* Dg(b) + R_2^* Dg(b)\big], b\big\rangle\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\big\langle\big[R_1^*Df(b)+R_2^*Df(b), R_1^*Dg(b)-R_2^*Dg(b)\big], b\big\rangle \\
&=\big\langle\big[R_1^*Df(b), R_1^*Dg(b)\big], R_1b+R_2b\big\rangle-\big\langle\big[R_2^*Df(b), R_2^*Dg(b)\big], R_1b+R_2b\big\rangle \\
&=\big\langle\big[R_1^*Df(b), R_1^*Dg(b)\big], R_1b\big\rangle-\big\langle\big[R_2^*Df(b), R_2^*Dg(b)\big], R_2b\big\rangle \\
&=\{f_{b_2}, g_{b_2}\}_1^{\mathrm{coind}}(b_1)-\{f_{b_1}, g_{b_1}\}_2^{\mathrm{coind}}(b_2),
\end{aligned}
$$

where in the third equality we have used the fact that $[R_1^*Df(b), R_1^*Dg(b)] \in \operatorname{im} R_1^* = (\operatorname{im} R_2)^\circ$ and $[R_2^*Df(b), R_2^*Dg(b)] \in \operatorname{im} R_2^* = (\operatorname{im} R_1)^\circ$ and $b = b_1 + b_2$ with $b_j \in \mathfrak{b}_j$. To prove the last equality above it suffices to note that

$$
\begin{aligned}
&D_1 f_{b_2}(b_1)\cdot\delta b_1 = Df(b)\cdot\delta b_1 = Df(b)\cdot R_1\delta b_1 \quad \text{and} \\
&D_2 f_{b_1}(b_2)\cdot\delta b_2 = Df(b)\cdot\delta b_2 = Df(b)\cdot R_2\delta b_2
\end{aligned}
$$

for any $\delta b_j \in \mathfrak{b}_j$, where D_j is the Fréchet derivative on $\mathfrak{b}_j$, for $j = 1, 2$. The last expression is that of the product Banach Lie–Poisson structure on $\mathfrak{b}_1 \times \overline{\mathfrak{b}}_2$ (see (2.7)).

(iii) This is an immediate consequence of (ii) and the general fact, recalled earlier for products of Banach Lie–Poisson spaces, that these inclusions are Poisson maps with closed range.

(iv) Let $\{,\}_j^{\mathrm{ind}}$ and $\{,\}_j^{\mathrm{coind}}$ be the induced and coinduced brackets on $\mathfrak{b}_j$ from $(\mathfrak{b}, \{\cdot,\cdot\}_R)$ and $(\mathfrak{b}, \{\cdot,\cdot\})$, respectively. Therefore,

$$
\{F, G\}_R \circ \iota_j = \{F \circ \iota_j, G \circ \iota_j\}_j^{\mathrm{ind}} \tag{2.21}
$$

for any $F, G \in C^\infty(\mathfrak{b})$ and, by (2.17),

$$
\{f, g\}_j^{\mathrm{coind}} = (-1)^{j-1}\{f \circ \pi_j, g \circ \pi_j\} \circ \iota_j \tag{2.22}
$$

for any $f, g \in C^\infty(\mathfrak{b}_j)$. Apply relation (2.21) to the functions $F := f \circ \pi_j$, $G := g \circ \pi_j$ and use $\pi_j \circ \iota_j = \mathrm{id}_{\mathfrak{b}_j}$, $\pi_j \circ R = \frac{1}{2}(-1)^{j-1}\pi_j$, and (2.22) to get for any $b_j \in \mathfrak{b}_j$

$$
\begin{aligned}
\{f, g\}_j^{\mathrm{ind}}(b_j) &= \{f \circ \pi_j, g \circ \pi_j\}_R\big(\iota_j(b_j)\big) \\
&= \big\langle\big[R^*D(f \circ \pi_j)\big(\iota_j(b_j)\big), D(g \circ \pi_j)\big(\iota_j(b_j)\big)\big], \iota_j(b_j)\big\rangle \\
&\quad + \big\langle\big[D(f \circ \pi_j)\big(\iota_j(b_j)\big), R^*D(g \circ \pi_j)\big(\iota_j(b_j)\big)\big], \iota_j(b_j)\big\rangle \\
&= \big\langle\big[R^*\pi_j^*Df(b_j), \pi_j^*Dg(b_j)\big], \iota_j(b_j)\big\rangle \\
&\quad + \big\langle\big[\pi_j^*Df(b_j), R^*\pi_j^*Dg(b_j)\big], \iota_j(b_j)\big\rangle \\
&= (-1)^{j-1}\big\langle\big[\pi_j^*Df(b_j), \pi_j^*Dg(b_j)\big], \iota_j(b_j)\big\rangle \\
&= (-1)^{j-1}\big\langle\big[D(f \circ \pi_j)\big(\iota_j(b_j)\big), D(g \circ \pi_j)\big(\iota_j(b_j)\big)\big], \iota_j(b_j)\big\rangle \\
&= (-1)^{j-1}\{f \circ \pi_j, g \circ \pi_j\}\big(\iota_j(b_j)\big) \\
&= \{f, g\}_j^{\mathrm{coind}}(b_j). \qquad \square
\end{aligned}
$$

This proposition implies the following involution theorem.

Corollary 2.3. *In the notations and hypotheses of Proposition* 2.2 *we have*:

(i) *The Casimir functions on* $(\mathfrak{b}, \{\cdot,\cdot\})$ *are in involution on* $(\mathfrak{b}, \{\cdot,\cdot\}_R)$ *and restrict to functions in involution on* $\mathfrak{b}_j$, $j = 1, 2$.
(ii) *If* H *is a Casimir function on* $\mathfrak{b}$, *then its restriction* $H \circ \iota_j$ *to* $\mathfrak{b}_j$ *has the Hamiltonian vector field*

$$\begin{aligned} X_{H\circ\iota_1}(b_1) &= \pi_1\big(\operatorname{ad}^*_{R_2^* DH(\iota_1(b_1))}\iota_1(b_1)\big), \\ X_{H\circ\iota_2}(b_2) &= \pi_2\big(\operatorname{ad}^*_{R_1^* DH(\iota_2(b_2))}\iota_2(b_2)\big) \end{aligned} \tag{2.23}$$

for any $b_1 \in \mathfrak{b}_1$ *and* $b_2 \in \mathfrak{b}_2$, *where* $\iota_j : \mathfrak{b}_j \hookrightarrow \mathfrak{b}$ *is the inclusion,* $j = 1, 2$.

Proof. (i) Let $F, H \in C^\infty(\mathfrak{b})$ be Casimir functions for the Lie–Poisson bracket $\{,\}$, that is, $\operatorname{ad}^*_{DF(b)} b = 0$ and $\operatorname{ad}^*_{DH(b)} b = 0$ for any $b \in \mathfrak{b}$ (see (2.4)). Therefore

$$\begin{aligned} \{F, H\}_R(b) &= \big\langle \big[R^* DF(b), DH(b)\big] + \big[DF(b), R^* DH(b)\big], b\big\rangle \\ &= -\big\langle R^* DF(b), \operatorname{ad}^*_{DH(b)} b\big\rangle + \big\langle R^* DH(b), \operatorname{ad}^*_{DF(b)} b\big\rangle = 0 \end{aligned}$$

which shows that F and H are in involution relative to $\{,\}_R$. Then statements (iii) and (iv) of Proposition 2.2 show that $F \circ \iota_j$, $H \circ \iota_j$ are also in involution on $\mathfrak{b}_j$, $j = 1, 2$.

(ii) Since H is a Casimir function on $\mathfrak{b}$, we have $\operatorname{ad}^*_{DH(b)} b = 0$ for any $b \in \mathfrak{b}$ (see (2.4)). Therefore, since $R_1^* + R_2^* = \mathrm{id}_{\mathfrak{b}^*}$, we get for any $b_1 \in \mathfrak{b}_1$

$$0 = \operatorname{ad}^*_{DH(\iota_1(b_1))}\iota_1(b_1) = \operatorname{ad}^*_{R_1^* DH(\iota_1(b_1))}\iota_1(b_1) + \operatorname{ad}^*_{R_2^* DH(\iota_1(b_1))}\iota_1(b_1).$$

A similar relation holds for any $b_2 \in \mathfrak{b}_2$. So, we have

$$-\operatorname{ad}^*_{R_j^* DH(\iota_j(b_j))} = \operatorname{ad}^*_{R_{j+1}^* DH(\iota_j(b_j))}, \tag{2.24}$$

where j is taken modulo 2.

Since $\iota_j \circ \pi_j = R_j$ we get

$$\begin{aligned} \pi_j^* D(H \circ \iota_j)(b_j) &= D(H \circ \iota_j)(b_j) \circ \pi_j = DH\big(\iota_j(b_j)\big) \circ \iota_j \circ \pi_j \\ &= DH\big(\iota_j(b_j)\big) \circ R_j = R_j^* DH\big(\iota_j(b_j)\big), \end{aligned}$$

so (2.19) and (2.24) yield

$$\begin{aligned} \iota_j\big(X_{H\circ\iota_j}(b_j)\big) &= -(\iota_j \circ \pi_j)\big(\operatorname{ad}^*_{\pi_j^* D(H\circ\iota_j)(b_j)}\iota_j(b_j)\big) = -R_j\big(\operatorname{ad}^*_{R_j^* DH(\iota_j(b_j))}\iota_j(b_j)\big) \\ &= R_j\big(\operatorname{ad}^*_{R_{j+1}^* DH(\iota_j(b_j))}\iota_j(b_j)\big) = \operatorname{ad}^*_{R_{j+1}^* DH(\iota_j(b_j))}\iota_j(b_j). \end{aligned} \tag{2.25}$$

The last equality follows from the fact that $\operatorname{ad}^*_{R_{j+1}^* x}\iota_j(b_j) \in \operatorname{im} R_j = \operatorname{im}\iota_j$ for any $x \in \mathfrak{b}^*$ and $b_j \in \mathfrak{b}_j$. Indeed, for any $y \in (\operatorname{im} R_j)^\circ = \operatorname{im} R_{j+1}^*$ we have

$$\big\langle \operatorname{ad}^*_{R_{j+1}^* x}\iota_j(b_j), y\big\rangle = \big\langle \iota_j(b_j), \big[R_{j+1}^* x, y\big]\big\rangle = 0$$

because $[R_{j+1}^* x, y] \in \operatorname{im} R_{j+1}^* = (\operatorname{im} R_j)^\circ$ by hypothesis (the image of R_{j+1}^* is a Banach Lie subalgebra of $\mathfrak{b}^*$) and $\iota_j(b_j) \in \operatorname{im} R_j$. Therefore, $\operatorname{ad}^*_{R_{j+1}^* x} \iota_j(b_j) \in (\operatorname{im} R_j)^{\circ\circ} = \overline{\operatorname{im} R_j} = \operatorname{im} R_j$.

Finally, applying π_j to (2.25) yields (2.23). □

Taken together, Proposition 2.2 and Corollary 2.3 give a construction of integrals in involution. This construction is an infinite-dimensional version of the classical R-matrix method. However, as was seen, the extension of this construction to the Banach Lie–Poisson space context is not direct and needs additional functional analytic hypotheses.

Proposition 2.4. *Let $(\mathfrak{b}, \{,\})$ be a Banach Lie–Poisson space and let $R_1, R_3 : \mathfrak{b} \to \mathfrak{b}$ be projectors. Assume that $\operatorname{im} R_{21} = \operatorname{im} R_{23} =: \mathfrak{b}_2$, where $R_{21} := \mathrm{id}_{\mathfrak{b}} - R_1$, $R_{23} := \mathrm{id}_{\mathfrak{b}} - R_3$, and denote $\mathfrak{b}_1 := \operatorname{im} R_1$, $\mathfrak{b}_3 := \operatorname{im} R_3$. We summarize this situation in the diagram*

$$\begin{tikzcd}
 & \mathfrak{b} \arrow[dl, shift left, "\pi_1"'] \arrow[dr, "\pi_{21}"] & & \mathfrak{b} \arrow[dl, "\pi_{23}"'] \arrow[dr, shift left, "\pi_3"] & \\
\mathfrak{b}_1 \arrow[ur, shift left, "\iota_1"'] & & \mathfrak{b}_2 & & \mathfrak{b}_3 \arrow[ul, shift left, "\iota_3"]
\end{tikzcd}$$

where $\pi_1, \pi_{21}, \pi_{23}, \pi_3$ are the projections onto the ranges of R_1, R_{21}, R_{23}, and R_3, respectively, according to the splittings $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 = \mathfrak{b}_2 \oplus \mathfrak{b}_3$, and $\iota_1 : \mathfrak{b}_1 \hookrightarrow \mathfrak{b}$, $\iota_3 : \mathfrak{b}_3 \hookrightarrow \mathfrak{b}$ are the inclusions.

Then one has:

(i) *If $\mathfrak{b}_2^\circ$ is a Banach Lie subalgebra of $\mathfrak{b}^*$, then $\Phi_{31} := \pi_3 \circ \iota_1 : (\mathfrak{b}_1, \{,\}_1^{\mathrm{coind}}) \to (\mathfrak{b}_3, \{,\}_3^{\mathrm{coind}})$ and $\Phi_{13} := \pi_1 \circ \iota_3 : (\mathfrak{b}_3, \{,\}_3^{\mathrm{coind}}) \to (\mathfrak{b}_1, \{,\}_1^{\mathrm{coind}})$ are mutually inverse linear Poisson isomorphisms.*

(ii) *If $\mathfrak{b}_1^\circ$ and $\mathfrak{b}_3^\circ$ are Banach Lie subalgebras of $\mathfrak{b}^*$, then $\mathfrak{b}_2$ has two coinduced Banach Lie–Poisson brackets $\{,\}_{21}^{\mathrm{coind}}$ and $\{,\}_{23}^{\mathrm{coind}}$ which are not isomorphic in general.*

Proof. (i) Since $\mathfrak{b}_2^\circ = (\operatorname{im} R_{21})^\circ = \operatorname{im} R_1^*$ (see (2.15)) is a Banach Lie subalgebra of $\mathfrak{b}^*$ it follows that R_1 coinduces a Banach Lie–Poisson bracket $\{,\}_1^{\mathrm{coind}}$ on $\mathfrak{b}_1$. Similarly, the relation $\mathfrak{b}_2^\circ = (\operatorname{im} R_{23})^\circ = \operatorname{im} R_3^*$ implies that R_3 coinduces a Banach Lie–Poisson bracket $\{,\}_3^{\mathrm{coind}}$ on $\mathfrak{b}_3$.

Let us notice that $\Phi_{31} \circ \Phi_{13} = \pi_3 \circ \iota_1 \circ \pi_1 \circ \iota_3 = \pi_3 \circ R_1 \circ \iota_3 = \pi_3 \circ (\mathrm{id}_{\mathfrak{b}} - R_{21}) \circ \iota_3 = \pi_3 \circ \iota_3 - \pi_3 \circ R_{21} \circ \iota_3 = \mathrm{id}_{\mathfrak{b}_3}$ since $\pi_3 \circ R_{21} = 0$. One proves similarly that $\Phi_{13} \circ \Phi_{31} = \mathrm{id}_{\mathfrak{b}_1}$.

From $\ker \pi_1 = \ker \pi_3 = \mathfrak{b}_2$ and $b - (\iota_3 \circ \pi_3)(b) \in \ker \pi_3$ for any $b \in \mathfrak{b}$, it follows that $\pi_1 \circ \iota_3 \circ \pi_3 = \pi_1$. Therefore, if $f, g \in C^\infty(\mathfrak{b}_1)$ we get from (2.17) and the fact that $\pi_1 : \mathfrak{b} \to \mathfrak{b}_1$ is a Poisson map

$$\begin{aligned}
\{f \circ \Phi_{13}, g \circ \Phi_{13}\}_3^{\mathrm{coind}} &= \{f \circ \pi_1 \circ \iota_3, g \circ \pi_1 \circ \iota_3\}_3^{\mathrm{coind}} \\
&= \{f \circ \pi_1 \circ \iota_3 \circ \pi_3, g \circ \pi_1 \circ \iota_3 \circ \pi_3\} \circ \iota_3 \\
&= \{f \circ \pi_1, g \circ \pi_1\} \circ \iota_3 = \{f, g\}_1^{\mathrm{coind}} \circ \pi_1 \circ \iota_3 = \{f, g\}_1^{\mathrm{coind}} \circ \Phi_{13}.
\end{aligned}$$

It is shown in a similar way that $\Phi_{31} : \mathfrak{b}_1 \to \mathfrak{b}_3$ is a Poisson map.

(ii) By (2.14) we have $\mathfrak{b}_1^\circ = \operatorname{im} R_{21}^*$ and $\mathfrak{b}_3^\circ = \operatorname{im} R_{23}^*$ which, by hypothesis, are Banach Lie subalgebras of $\mathfrak{b}^*$. Therefore, π_{21} and π_{23} coinduce Poisson brackets $\{,\}_{21}^{\mathrm{coind}}$ and $\{,\}_{23}^{\mathrm{coind}}$ on $\mathfrak{b}_2$. □

Remark. A concrete counterexample where the two coinduced Poisson structures are not isomorphic is obtained for $\mathfrak{b}_2 = I_{+,1}^1$ which is the predual of two different Banach Lie algebras $I_{-,1}^\infty$ and L_A^∞. For the definition of these Banach spaces see Section 3. The two different coinduced brackets are given by (3.45) and (3.46).

3. Induction and coinduction from $L^1(\mathcal{H})$

In this section we introduce several Banach spaces that will be used later on and implement the constructions presented in Section 2 in these concrete cases.

The Banach Lie–Poisson space $L^1(\mathcal{H})$. The Banach space of trace class operators $(L^1(\mathcal{H}), \|\cdot\|_1)$ on a separable Hilbert space $\mathcal{H}$ has a canonical Banach Lie–Poisson bracket defined by

$$\{f, g\}(\rho) = \operatorname{Tr}\big(\rho\big[Df(\rho), Dg(\rho)\big]\big), \tag{3.1}$$

where $f, g \in C^\infty(L^1(\mathcal{H}))$ and the Fréchet derivatives $Df(\rho)$, $Dg(\rho)$ are regarded as elements of the Banach Lie algebra $(L^\infty(\mathcal{H}), \|\cdot\|_\infty)$ of bounded operators on $\mathcal{H}$, identified with the dual of $L^1(\mathcal{H})$ by the strongly nondegenerate pairing

$$\langle \rho, x\rangle = \operatorname{Tr}(\rho x), \quad \text{for } \rho \in L^1(\mathcal{H}),\ x \in L^\infty(\mathcal{H}). \tag{3.2}$$

Hamilton's equations defined by the Poisson bracket (3.1) are easily verified to be given in Lax form (see [12] for details)

$$\frac{d\rho}{dt} = \big[Dh(\rho), \rho\big]. \tag{3.3}$$

The orthonormal basis $\{|n\rangle\}_{n=0}^\infty$ of $\mathcal{H}$, that is, $\langle n|m\rangle = \delta_{nm}$ for $n, m \in \mathbb{N} \cup \{0\}$, induces the Schauder basis $\{|n\rangle\langle m|\}_{n,m=0}^\infty$ of $L^1(\mathcal{H})$ since it is orthonormal in the Hilbert space $L^2(\mathcal{H})$ of Hilbert–Schmidt operators and $L^1(\mathcal{H}) \subset L^2(\mathcal{H})$. Thus, every trace class operator $\rho \in L^1(\mathcal{H})$ can be uniquely expressed as

$$\rho = \sum_{n,m=0}^\infty \rho_{nm}|n\rangle\langle m|, \tag{3.4}$$

where the series is convergent in the $\|\cdot\|_1$ topology. The coordinates $\rho_{nm} \in \mathbb{R}$ are given by $\rho_{nm} = \operatorname{Tr}(\rho|m\rangle\langle n|)$. The rank one projectors $|l\rangle\langle k|$ thought of as elements of $L^\infty(\mathcal{H})$, by giving their values on the Schauder basis of $L^1(\mathcal{H})$ as $\operatorname{Tr}(|l\rangle\langle k|\,|n\rangle\langle m|) = \delta_{kn}\delta_{lm}$, form a biorthogonal family of functionals (see [9]) in $L^\infty(\mathcal{H})$ associated to the given Schauder basis $\{|n\rangle\langle m|\}_{n,m=0}^\infty$ of $L^1(\mathcal{H})$. Therefore, each bounded operator $x \in L^\infty(\mathcal{H})$ can be uniquely expressed as

$$x = \sum_{l,k=0}^\infty x_{lk}|l\rangle\langle k|, \tag{3.5}$$

where the series is convergent in the w*-topology. The coordinates $x_{lk} \in \mathbb{R}$ are also given by $x_{lk} = \operatorname{Tr}(x|k\rangle\langle l|)$. Recall that w*-convergence of the series (3.5) means that the numerical series

$$\sum_{l,k=0}^{\infty} x_{lk} \operatorname{Tr}\bigl(\rho|l\rangle\langle k|\bigr) = \sum_{l,k=0}^{\infty} x_{lk}\rho_{kl} = \operatorname{Tr}(x\rho)$$

is convergent for any $\rho \in L^1(\mathcal{H})$.

Since the separable Hilbert space $\mathcal{H}$ is fixed throughout this paper we shall simplify the notation by writing $L^1 := L^1(\mathcal{H})$ and $L^\infty := L^\infty(\mathcal{H})$.

Shift operator notation. The *shift operator*

$$S := \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \tag{3.6}$$

and its adjoint

$$S^T := \sum_{n=0}^{\infty} |n+1\rangle\langle n|, \tag{3.7}$$

turn out to give a very convenient coordinate description of various objects that we shall study in this paper. Note that the matrix of S has all entries of the upper diagonal equal to one and all other entries equal to zero whereas the matrix of S^T has all entries of the lower diagonal equal to one and all other entries equal to zero. To facilitate various subsequent computations, we note that

$$S^k\bigl(S^T\bigr)^k = \mathbb{I}, \qquad \bigl(S^T\bigr)^k S^k = \mathbb{I} - \sum_{i=0}^{k-1} p_i, \quad \text{for } k = 1, 2, \ldots, \tag{3.8}$$

where $p_i = |i\rangle\langle i| : \mathcal{H} \to \mathcal{H}$ are the orthogonal projectors on $\mathbb{R}|i\rangle \subset \mathcal{H}$ for any $i \in \mathbb{N} \cup \{0\}$. Let $L_0^\infty \subset L^\infty$ and $L_0^1 \subset L^1$ denote the closed subspaces of diagonal operators and define the bounded linear operators $s, \tilde{s}$ on both L_0^∞ and L_0^1 by

$$\begin{aligned} &Sx = s(x)S \qquad \text{or} \quad xS^T = S^T s(x), \quad \text{and} \\ &S^T x = \tilde{s}(x)S^T \quad \text{or} \quad xS = S\tilde{s}(x) \end{aligned} \tag{3.9}$$

for $x \in L_0^\infty$ or $x \in L_0^1$. The effect of the map s is that the ith coordinate of $s(x)$ equals the $(i+1)$st coordinate of x, that is, $s(x_0, x_1, x_2, \ldots, x_n, \ldots) := (x_1, x_2, \ldots, x_n, \ldots)$ for any $(x_0, x_1, x_2, \ldots, x_n, \ldots) \in \ell^\infty \cong L_0^\infty$. Similarly, the effect of the map $\tilde{s}$ is that the ith coordinate of $\tilde{s}(x)$ equals the $(i-1)$st coordinate of x and the zero coordinate of $\tilde{s}(x)$ is zero, that is, $\tilde{s}(x_0, x_1, x_2, \ldots, x_n, \ldots) := (0, x_0, x_1, x_2, \ldots, x_n, \ldots)$. Thus

$$s^k \circ \tilde{s}^k = \mathrm{id} \quad \text{and} \quad \tilde{s}^k \circ s^k = M_{\mathbb{I} - \sum_{i=0}^{k-1} p_i}, \quad k = 1, 2, \ldots, \tag{3.10}$$

where $M_y : L_0^\infty \to L_0^\infty$ is defined by $M_y(x) := yx$ for any $y \in L_0^\infty$. The following identities are useful in several computations later on:

$$\operatorname{Tr}\bigl(\rho s(x)\bigr) = \operatorname{Tr}\bigl(\tilde{s}(\rho)x\bigr) \quad \text{and} \quad \operatorname{Tr}\bigl(s(\rho)x\bigr) = \operatorname{Tr}\bigl(\rho\tilde{s}(x)\bigr) \tag{3.11}$$

for any $\rho \in L_0^1$ and $x \in L_0^\infty$, which means that s and $\tilde{s}$ are mutually adjoint operators.

Any $x \in L^\infty$ and $\rho \in L^1$ can be written as

$$x = \sum_{j=1}^{\infty} \bigl(S^T\bigr)^j x_{-j} + x_0 + \sum_{i=1}^{\infty} x_i S^i, \tag{3.12}$$

$$\rho = \sum_{j=1}^{\infty} \bigl(S^T\bigr)^j \rho_j + \rho_0 + \sum_{i=1}^{\infty} \rho_{-i} S^i, \tag{3.13}$$

where $x_i, x_0, x_{-j} \in L_0^\infty$ and $\rho_j, \rho_0, \rho_{-i} \in L_0^1$. *Note the different conventions*: the indices of the lower diagonals for the bounded operators are negative whereas for the trace class operators they are positive. This convention simplifies many formulas later on.

The expressions (3.12) and (3.13) suggest the introduction, for every $k \in \mathbb{Z}$, of the Banach subspaces

$$L_k^\infty := \bigl\{\rho \in L^\infty \bigm| \rho_{nm} = 0 \text{ for } m \neq n+k\bigr\} \subset L^\infty, \tag{3.14}$$

$$L_k^1 := \bigl\{\rho \in L^1 \bigm| \rho_{nm} = 0 \text{ for } m \neq n+k\bigr\} \subset L^1 \tag{3.15}$$

consisting of operators whose only non-zero elements lie on the kth diagonal. We have the following Schauder decompositions

$$L^\infty = \bigoplus_{k \in \mathbb{Z}} L_k^\infty \quad \text{and} \quad L^1 = \bigoplus_{k \in \mathbb{Z}} L_k^1. \tag{3.16}$$

See [14, Chapter III, §15], namely Definition 15.1 (p. 485), Definition 15.3 (p. 487), and Theorem 15.1 (p. 489) for a detailed discussion of this concept and generalizations. The duality relations between the various spaces L_n^∞ and L_k^1 are given by

$$\operatorname{Tr}(\rho_k x_n) = \delta_{kn} \operatorname{Tr}(\rho_k x_k) \quad \text{if } \rho_k \in L_k^1 \text{ and } x_n \in L_{-n}^\infty. \tag{3.17}$$

Finally, note that if $k \geqslant 0$ then $S^k \in L_k^\infty$, $(S^T)^k \in L_{-k}^\infty$, and

$$S^l\bigl(S^T\bigr)^j = \begin{cases} S^{l-j}, & \text{if } l \geqslant j, \\ (S^T)^{j-l}, & \text{if } l \leqslant j \end{cases} \tag{3.18}$$

which implies

$$\langle \rho, x\rangle = \sum_{k \in \mathbb{Z}} \operatorname{Tr} \rho_i x_i \tag{3.19}$$

if ρ and x are expressed in the form (3.13) and (3.12).

Banach subspaces of $L^1(\mathcal{H})$ and $L^\infty(\mathcal{H})$. Given the Schauder basis $\{|n\rangle\langle m|\}_{n,m=0}^\infty$ of L^1 (or biorthogonal family of L^∞) inducing the direct sum splitting (3.16), define the *transposition operator* $T : L^1 \to L^1$ (or $T : L^\infty \to L^\infty$) by $(\rho^T)_{ij} := \rho_{ji}$ for any $i, j \in \mathbb{N} \cup \{0\}$. We construct the following Banach subspaces of L^1:

- $L^1_- := \bigoplus_{k=-\infty}^0 L^1_k$ and $L^1_+ := \bigoplus_{k=0}^\infty L^1_k$;
- $L^1_S := \{\rho \in L^1 \mid \rho = \rho^T\}$ and $L^1_A := \{\rho \in L^1 \mid \rho = -\rho^T\}$;
- $L^1_{-,k} := \bigoplus_{i=-k+1}^0 L^1_i$ and $L^1_{+,k} := \bigoplus_{i=0}^{k-1} L^1_i$, for $k \geqslant 1$;
- $I^1_{-,k} := \bigoplus_{i=-\infty}^{-k} L^1_i$ and $I^1_{+,k} := \bigoplus_{i=k}^\infty L^1_i$, for $k \geqslant 1$;
- $L^1_{S,k} := L^1_S \cap (L^1_{+,k} + L^1_{-,k})$ and $L^1_{A,k} := L^1_A \cap (L^1_{+,k} + L^1_{-,k})$, for $k \geqslant 1$.

Relative to operator multiplication, $I^1_{-,k}$ is an ideal in L^1_-, $I^1_{+,k}$ is an ideal in L^1_+, but neither is an ideal in L^1. Therefore, relative to the commutator bracket, the same is true in the associated Banach Lie algebras.

Similarly, using the biorthogonal family of functionals $\{|l\rangle\langle k|\}_{l,k=0}^\infty$ in L^∞ inducing the direct sum splitting (3.16), we construct the following Banach subspaces of L^∞:

- $L^\infty_- := \bigoplus_{k=-\infty}^0 L^\infty_k$ and $L^\infty_+ := \bigoplus_{k=0}^\infty L^\infty_k$;
- $L^\infty_S := \{x \in L^\infty \mid x^T = x\}$ and $L^\infty_A := \{x \in L^\infty \mid x^T = -x\}$;
- $L^\infty_{-,k} := \bigoplus_{i=-k+1}^0 L^\infty_i$ and $L^\infty_{+,k} := \bigoplus_{i=0}^{k-1} L^\infty_i$, for $k \geqslant 1$;
- $I^\infty_{-,k} := \bigoplus_{i=-\infty}^{-k} L^\infty_i$ and $I^\infty_{+,k} := \bigoplus_{i=k}^\infty L^\infty_i$, for $k \geqslant 1$;
- $L^\infty_{S,k} := L^\infty_S \cap (L^\infty_{+,k} + L^\infty_{-,k})$ and $L^\infty_{A,k} := L^\infty_A \cap (L^\infty_{+,k} + L^\infty_{-,k})$, for $k \geqslant 1$.

The following splittings of Banach spaces of trace class operators

$$L^1 = L^1_- \oplus I^1_{+,1}, \qquad L^1 = L^1_S \oplus I^1_{+,1}, \qquad L^1_- = L^1_{-,k} \oplus I^1_{-,k} \tag{3.20}$$

and of bounded operators

$$L^\infty = L^\infty_+ \oplus I^\infty_{-,1}, \qquad L^\infty = L^\infty_+ \oplus L^\infty_A, \qquad L^\infty_+ = L^\infty_{+,k} \oplus I^\infty_{+,k} \tag{3.21}$$

will be used below. The strongly nondegenerate pairing (3.2) relates the splittings (3.20) and (3.21) by

$$\begin{aligned}
&\big(L^1_-\big)^* \cong \big(I^1_{+,1}\big)^\circ = L^\infty_+, \qquad \big(L^1_S\big)^* \cong \big(I^1_{+,1}\big)^\circ = L^\infty_+, \qquad \big(L^1_{-,k}\big)^* \cong \big(I^1_{-,k}\big)^\circ = L^\infty_{+,k}, \\
&\big(I^1_{+,1}\big)^* \cong \big(L^1_-\big)^\circ = I^\infty_{-,1}, \qquad \big(I^1_{+,1}\big)^* \cong \big(L^1_S\big)^\circ = L^\infty_A, \qquad \big(I^1_{-,k}\big)^* \cong \big(L^1_{-,k}\big)^\circ = I^\infty_{+,k}
\end{aligned} \tag{3.22}$$

where, as usual, $\circ$ denotes the annihilator of the Banach subspace in the dual of the ambient space.

The splittings (3.20) and (3.21) define six projectors of L^1 and L^∞, respectively. Let $P^1_-, P^1_0, P^1_+ : L^1 \to L^1$ be the projectors whose ranges are $I^1_{-,1}$, L^1_0, and $I^1_{+,1}$ defined by the splitting $L^1 = I^1_{-,1} \oplus L^1_0 \oplus I^1_{+,1}$. In particular $P^1_- + P^1_0 + P^1_+ = \mathbb{I}$. Let $P^1_{-,k} : L^1_- \to L^1_-$ be the projector whose range is $L^1_{-,k}$ defined by the splitting $L^1_- = L^1_{-,k} \oplus I^1_{-,k}$. Define the six projectors

$$R_- := P^1_- + P^1_0, \qquad R_S := P^1_- + P^1_0 + T \circ P^1_-, \qquad R_{-,k} := P^1_{-,k},$$
$$R_+ := P^1_+, \qquad R_{S,+} := P^1_+ - T \circ P^1_-, \qquad R_{ik} := R_-|_{L^1_-} - R_{-,k} \tag{3.23}$$

associated to the splittings (3.20). The order of presentation of these projectors corresponds to the order of the splittings in (3.20).

Similarly, the six projectors associated to the dual splittings (3.21) are given by

$$R^*_- := P^\infty_+ + P^\infty_0, \qquad R^*_S := P^\infty_+ + P^\infty_0 + T \circ P^\infty_-, \qquad R^*_{-,k} := P^\infty_{+,k},$$
$$R^*_+ := P^\infty_-, \qquad R^*_{S,+} := P^\infty_- - T \circ P^\infty_-, \qquad R^*_{ik} := R^*_-|_{L^\infty_+} - P^\infty_{+,k}, \tag{3.24}$$

where $P^\infty_-, P^\infty_0, P^\infty_+ : L^\infty \to L^\infty$ are the projectors whose ranges are $I^\infty_{-,1}, L^\infty_0, I^\infty_{+,1}$ defined by the splitting $L^\infty = I^\infty_{-,1} \oplus L^\infty_0 \oplus I^\infty_{+,1}$ and $P^\infty_{+,k} : L^\infty_+ \to L^\infty_+$ is the projector with range $L^\infty_{+,k}$ defined by the splitting $L^\infty_+ = L^\infty_{+,k} \oplus I^\infty_{+,k}$.

All Banach spaces appearing in (3.21), with the exception of L^∞_k and $L^\infty_{+,k}$, are Banach subalgebras of L^∞ or L^∞_+ whereas $I^\infty_{+,k}$, for $k \in \mathbb{N}$, are ideals of the Banach algebra L^∞_+ (but not of L^∞). Therefore, $I^\infty_{+,k}$ define a filtration of L^∞_+ and hence $L^\infty_{+,k} \cong L^\infty_+/I^\infty_{+,k}$ inherits the structure of an associative Banach algebra. Thus all these associative Banach algebras are naturally Banach Lie algebras. The same considerations apply to the Banach ideals $I^\infty_{-,k} \subset L^\infty_-$.

It will be useful in our subsequent development to distinguish between the projectors defined in (3.23) and (3.24) and the corresponding maps onto their ranges. We shall denote by π_-, π_+, π_S, and $\pi_{S,+}$ the maps on L^1 equal to R_-, R_+, R_S, and $R_{S,+}$ but viewed as taking values in $\operatorname{im} R_- = L^1_-$, $\operatorname{im} R_+ = I^1_{+,1}$, $\operatorname{im} R_S = L^1_S$, and $\operatorname{im} R_{S,+} = I^1_{+,1}$, respectively. Similarly, denote by $\pi_{-,k}$ and π_{ik} the maps on L^1_- equal to $R_{-,k}$ and R_{ik}, but viewed as having values in $\operatorname{im} R_{-,k} = L^1_{-,k}$ and $\operatorname{im} R_{ik} = I^1_{-,k}$, respectively. For the projectors on L^∞ we shall denote by $\pi^\infty_+, \pi^\infty_-, \pi^\infty_S$, and π^∞_A the maps equal to R^*_-, R^*_+, R^*_S, and $R^*_{S,+}$ viewed as having values in $\operatorname{im} R^*_- = L^\infty_+$, $\operatorname{im} R^*_+ = I^\infty_{-,1}$, $\operatorname{im} R^*_S = L^\infty_S$, and $\operatorname{im} R^*_{S,+} = L^\infty_A$, respectively. Finally, let $\pi^\infty_{+,k}$ and π^∞_{ik} denote the maps on L^∞_+ equal to $R^*_{-,k}$ and R^*_{ik} viewed as having values in $\operatorname{im} R^*_{-,k} = L^\infty_{+,k}$ and $\operatorname{im} R^*_{ik} = I^\infty_{+,k}$, respectively.

Associated Banach Lie groups. Note that the Banach Lie group

$$GL^\infty := \{x \in L^\infty \mid x \text{ is invertible}\} \tag{3.25}$$

has Banach Lie algebra L^∞ and is open in L^∞. Define the closed Banach Lie subgroup of upper triangular operators in GL^∞ by

$$GL^\infty_+ := GL^\infty \cap L^\infty_+. \tag{3.26}$$

Since GL^∞_+ is open in L^∞_+, we can conclude that its Banach Lie algebra is L^∞_+. Define the closed Banach Lie subgroup of orthogonal operators in GL^∞ by

$$O^\infty := \{x \in L^\infty \mid xx^T = x^T x = \mathbb{I}\}. \tag{3.27}$$

The Banach Lie algebra L_A^∞ of O^∞ consists of all bounded skew-symmetric operators.

Denote by

$$GI_{+,k}^\infty := \left(\mathbb{I} + I_{+,k}^\infty\right) \cap GL_+^\infty = \left\{\mathbb{I} + \varphi \;\middle|\; \varphi \in I_{+,k}^\infty,\ \mathbb{I} + \varphi \text{ is invertible in } GL_+^\infty\right\} \tag{3.28}$$

the open subset of $\mathbb{I} + I_{+,k}^\infty$ formed by the group of all bounded invertible upper triangular operators whose strictly upper $(k-1)$-diagonals are identically zero and whose diagonal is the identity. This is a closed normal Banach Lie subgroup of GL_+^∞ whose Lie algebra is the closed ideal $I_{+,k}^\infty$.

Remark. Unlike the situation encountered in finite-dimensions, the set $\mathbb{I} + I_{+,k}^\infty$ does not consist only of invertible bounded linear isomorphisms. An example of an operator in $\mathbb{I} + I_{+,2}^\infty$ that is not onto is given by $\mathbb{I} - S^2$, where S is the shift operator defined in (3.6), since $\sum_{n=0}^\infty \frac{1}{n+1}|n\rangle \notin \operatorname{im}(\mathbb{I} - S^2)$.

Returning to the general case, define the product

$$x \circ_k y := \sum_{l=0}^{k-1}\left(\sum_{i=0}^{l} x_i s^i(y_{l-i})\right)S^l \tag{3.29}$$

of the elements $x = \sum_{i=0}^{k-1} x_i S^i$ and $y = \sum_{i=0}^{k-1} y_i S^i \in L_{+,k}^\infty$, where x_i, y_i are diagonal operators. Relative to $\circ_k$, the Banach space $L_{+,k}^\infty$ is an associative Banach algebra with unity. It is easy to see that the projection map $\pi_{+,k}^\infty : L_+^\infty \to (L_{+,k}^\infty, \circ_k)$ is an associative Banach algebra homomorphism whose kernel is $I_{+,k}^\infty$. So, it defines a Banach algebra isomorphism $[\pi_{+,k}^\infty] : L_+^\infty / I_{+,k}^\infty \to (L_{+,k}^\infty, \circ_k)$ of the factor Banach algebra $L_+^\infty / I_{+,k}^\infty$ with $(L_{+,k}^\infty, \circ_k)$.

The associative algebra $L_{+,k}^\infty$ with the commutator bracket

$$[x, y]_k := x \circ_k y - y \circ_k x = \sum_{l=0}^{k-1}\sum_{i=0}^{l}\left(x_i s^i(y_{l-i}) - y_i s^i(x_{l-i})\right)S^l \tag{3.30}$$

is the Banach Lie algebra of the group

$$GL_{+,k}^\infty = \left\{ g = \sum_{i=0}^{k-1} g_i S^i \;\middle|\; g_i \in L_0^\infty,\ |g_0| \geqslant \varepsilon(g_0)\mathbb{I} \text{ for some } \varepsilon(g_0) > 0 \right\} \tag{3.31}$$

of invertible elements in $(L_{+,k}^\infty, \circ_k)$. In (3.31), the inequality $|g_0| \geqslant \varepsilon(g_0)\mathbb{I}$ means the component wise inequalities for the diagonal operators, that is, $|g_{0p}| \geqslant \varepsilon(g_0)$ for all $p \in \mathbb{N} \cup \{0\}$, where $g_0 := \operatorname{diag}(g_{00}, g_{01}, \ldots, g_{0p}, \ldots)$.

Remark. It is important to note that invertibility in the Banach algebra $(L_{+,k}^\infty, \circ_k)$ does not mean invertibility of the operator on $\mathcal{H}$. For example, $\mathbb{I} - S^2 \in GL_{+,3}^\infty$, that is, $\mathbb{I} - S^2$ is an invertible element in $(L_{+,3}^\infty, \circ_3)$, but $\mathbb{I} - S^2$ is not an invertible operator, as noted in the previous remark.

Note that $(L^{\infty}_{+,k}, [\cdot,\cdot]_k)$ is not a Banach Lie subalgebra of L^{∞}_{+}. Since $\pi^{\infty}_{+,k} : L^{\infty}_{+} \to L^{\infty}_{+,k}$ is also a Banach Lie algebra homomorphism one has

$$[x, y]_k = \pi^{\infty}_{+,k}\big([x, y]\big) \quad \text{for } x, y \in L^{\infty}_{+,k}. \tag{3.32}$$

Note that $\pi^{\infty}_{+,k}(GL^{\infty}_{+}) \subset GL^{\infty}_{+,k}$, since every invertible operator in L^{∞}_{+} is mapped by the homomorphism $\pi^{\infty}_{+,k}$ to an invertible element of $L^{\infty}_{+,k}$. Moreover, if $x \in \pi^{\infty}_{+,k}(GL^{\infty}_{+}) \subset GL^{\infty}_{+,k}$, then

$$\big(\pi^{\infty}_{+,k}|_{GL^{\infty}_{+}}\big)^{-1}(x) = \big\{g(\mathbb{I} + \psi) \;\big|\; \mathbb{I} + \psi \in GI^{\infty}_{+,k}\big\} \quad \text{for some } g \in \big(\pi^{\infty}_{+,k}|_{GL^{\infty}_{+}}\big)^{-1}(x).$$

Indeed, if $g' \in (\pi^{\infty}_{+,k}|_{GL^{\infty}_{+}})^{-1}(x)$, then there exists some $g\psi \in I^{\infty}_{+,k}$, since g is invertible, such that $g^{-1}g' = \mathbb{I} + \psi \in GI^{\infty}_{+,k}$. The next proposition shows that the restriction of $\pi^{\infty}_{+,k}$ to GL^{∞}_{+} has range equal to $GL^{\infty}_{+,k}$.

Proposition 3.1. *The Banach Lie group homomorphism $\pi^{\infty}_{+,k}|_{GL^{\infty}_{+}} : GL^{\infty}_{+} \to GL^{\infty}_{+,k}$ is surjective and induces a Banach Lie group isomorphism $\widetilde{\pi^{\infty}_{+,k}} : GL^{\infty}_{+}/GI^{\infty}_{+,k} \to GL^{\infty}_{+,k}$ for any $k = 1, 2, \ldots$.*

Proof. To show that $\pi^{\infty}_{+,k} : GL^{\infty}_{+} \to GL^{\infty}_{+,k}$ is surjective is equivalent to proving that for any $g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} \in GL^{\infty}_{+,k}$ there exists $\varphi_k \in I^{\infty}_{+,k}$ such that

$$g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + \varphi_k \in GL^{\infty}_{+}. \tag{3.33}$$

Assume for the moment that (3.33) holds. We shall draw a consequence from it. By (3.31), $g_0 + g_1 S + \cdots + g_{k-1}S^{k-1}$ is in $GL^{\infty}_{+,k}$ if and only if g_0 is invertible. Decompose $\varphi_k = \alpha_k S^k g_0 + \alpha_{k+1}$, where $\alpha_{k+1} \in I^{\infty}_{+,k+1}$. Choosing $N \in \mathbb{N}$ large enough so that $\mathbb{I} - \frac{1}{N}\alpha_k S^k \in GL^{\infty}_{+}$, we obtain

$$\begin{aligned} GL^{\infty}_{+} \ni {} & \left(\mathbb{I} - \frac{1}{N}\alpha_k S^k\right)^N \big(g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + \alpha_k S^k g_0 + \alpha_{k+1}\big) \\ & = g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + \varphi_{k+1}, \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} \varphi_{k+1} = {} & \left(\sum_{j=2}^{N} \binom{N}{j} (-1)^j \frac{1}{N^j}\big(\alpha_k S^k\big)^j\right)\big(g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + \alpha_k S^k g_0 + \alpha_{k+1}\big) \\ & + \alpha_{k+1} - \alpha_k S^k\big(g_1 S + \cdots + g_{k-1}S^{k-1} + \alpha_k S^k g_0 + \alpha_{k+1}\big) \in I^{\infty}_{+,k+1}. \end{aligned} \tag{3.35}$$

Therefore, if $g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + \varphi_k \in GL^{\infty}_{+}$ for some $\varphi_k \in I^{\infty}_{+,k}$, then there exists some $\varphi_{k+1} \in I^{\infty}_{+,k+1}$ such that $g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + \varphi_{k+1} \in GL^{\infty}_{+}$.

Now we prove the proposition by induction on k.

If $k = 1$, then $g_0 \in GL^{\infty}_{+}$ by definition. Next, let us assume that (3.33) holds. As we just saw, it follows that (3.34) holds. Consider then $g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + g_k S^k \in GL^{\infty}_{+,k}$ and decompose it in the group $GL^{\infty}_{+,k}$ as $g_0 + g_1 S + \cdots + g_{k-1}S^{k-1} + g_k S^k = (\mathbb{I} + g_k S^k g_0^{-1}) \circ_k (g_0 + g_1 S + \cdots + g_{k-1}S^{k-1})$. Let us assume, that $\|g_k\| < \min(1, \|g_0\|)$ which implies that

$\|g_k S g_0^{-1}\| < 1$ and hence that $\mathbb{I} + g_k S^k g_0^{-1} \in GL_+^\infty$. By (3.34) there exists $\varphi_{k+1} \in I_{+,k+1}^\infty$ such that $g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi_{k+1} \in GL_+^\infty$. Thus we get

$$\left(\mathbb{I} + g_k S^k g_0^{-1}\right)\left(g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} + \varphi_{k+1}\right) = g_0 + g_1 S + \cdots + g_k S^k + \psi_{k+1} \in GL_+^\infty$$

for

$$\psi_{k+1} = \left(\mathbb{I} + g_k S^k g_0^{-1}\right)\varphi_{k+1} + g_k S^k g_0^{-1}\left(g_1 S + \cdots + g_{k-1} S^{k-1}\right) \in I_{+,k+1}^\infty$$

which proves the assertion (3.33) for any element in the connected component of $GL_{+,k}^\infty$. Since $\{\mathbb{I} + g_1 S + \cdots + g_k S^k \mid g_1, \ldots, g_k \text{ diagonal operators in } L^\infty\}$ is a connected Banach Lie subgroup of the connected component of the identity in $GL_{+,k}^\infty$ and any element of $GL_{+,k}^\infty$ can be written as a product of an element of this group and the Banach Lie subgroup $GL_{+,1}^\infty$ of diagonal operators, it follows that (3.33) holds for any element in $GL_{+,k}^\infty$. $\square$

Remark. There is a shorter proof of this proposition based on the following general remark, pointed out by the referee. If $\varphi : G \to H$ is a homomorphism of Banach Lie groups whose tangent map $T_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is surjective, then $\varphi(G)$ contains the identity component of H because it is generated (as a group) by the image of the exponential map. In the concrete situation of the proposition, the surjectivity of the induced Banach Lie algebra homomorphism is very easy to show.

In the Banach Lie group $(GL_{+,k}^\infty, \circ_k)$, the inverse $g^{-1} = g_0^{-1} + h_1 S + \cdots + h_{k-1} S^{k-1}$ of $g = g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \in GL_{+,k}^\infty$ is given for all $p = 1, \ldots, k-1$ by

$$\begin{aligned} h_p = -g_0^{-1}\Bigg[\sum_{r=1}^{p}(-1)^{r-1} \sum_{i_1+\cdots+i_r=p,\ i_1,\ldots,i_q\geqslant 1} g_{i_1} s^{i_1}\left(g_0^{-1} g_{i_2}\right) s^{i_1+i_2}\left(g_0 g_{i_3}\right) \\ \cdots s^{i_1+\cdots+i_{q-1}}\left(g_0^{-1} g_{i_q}\right) \cdots s^{i_1+\cdots+i_{r-1}}\left(g_0^{-1} g_{i_r}\right)\Bigg] s^p\left(g_0^{-1}\right). \end{aligned} \tag{3.36}$$

For example, here are the first elements:

$$\begin{aligned} h_1 &= -g_0^{-1} g_1 s\left(g_0^{-1}\right), \\ h_2 &= -g_0^{-1}\left[g_2 - g_1 s\left(g_0^{-1} g_1\right)\right] s^2\left(g_0^{-1}\right), \\ h_3 &= -g_0^{-1}\left[g_3 - g_2 s^2\left(g_0^{-1} g_1\right) - g_1 s\left(g_0^{-1} g_2\right) + g_1 s\left(g_0^{-1} g_1\right) s^2\left(g_0^{-1} g_1\right)\right] s^3\left(g_0^{-1}\right), \\ h_4 &= -g_0^{-1}\left[g_4 - g_3 s^3\left(g_0^{-1} g_1\right) - g_2 s^2\left(g_0^{-1} g_2\right) - g_1 s\left(g_0^{-1} g_3\right) + g_2 s^2\left(g_0^{-1} g_1\right) s^3\left(g_0^{-1} g_1\right)\right. \\ &\quad + g_1 s\left(g_0^{-1} g_1\right) s^2\left(g_0^{-1} g_2\right) + g_1 s\left(g_0^{-1} g_2\right) s^3\left(g_0^{-1} g_1\right) \\ &\quad \left.- g_1 s\left(g_0^{-1} g_1\right) s^2\left(g_0^{-1} g_1\right) s^3\left(g_0^{-1} g_1\right)\right] s^4\left(g_0^{-1}\right), \\ h_5 &= -g_0^{-1}\left[g_5 - g_4 s^4\left(g_0^{-1} g_1\right) - g_3 s^3\left(g_0^{-1} g_2\right) - g_2 s^2\left(g_0^{-1} g_3\right) - g_1 s\left(g_0^{-1} g_4\right)\right. \\ &\quad + g_3 s^3\left(g_0^{-1} g_1\right) s^4\left(g_0^{-1} g_1\right) + g_2 s^2\left(g_0^{-1} g_2\right) s^4\left(g_0^{-1} g_1\right) + g_2 s^2\left(g_0^{-1} g_1\right) s^3\left(g_0^{-1} g_2\right) \end{aligned}$$

$$
\begin{aligned}
&+ g_1 s(g_0^{-1} g_3) s^4(g_0^{-1} g_1) + g_1 s(g_0^{-1} g_2) s^3(g_0^{-1} g_2) + g_1 s(g_0^{-1} g_1) s^2(g_0^{-1} g_3) \\
&- g_2 s^2(g_0^{-1} g_1) s^3(g_0^{-1} g_1) s^4(g_0^{-1} g_1) - g_1 s(g_0^{-1} g_2) s^3(g_0^{-1} g_1) s^4(g_0^{-1} g_1) \\
&- g_1 s(g_0^{-1} g_1) s^2(g_0^{-1} g_1) s^3(g_0^{-1} g_2) - g_1 s(g_0^{-1} g_1) s^2(g_0^{-1} g_2) s^4(g_0^{-1} g_1) \\
&+ g_1 s(g_0^{-1} g_1) s^2(g_0^{-1} g_1) s^3(g_0^{-1} g_1) s^4(g_0^{-1} g_1)\big] s^5(g_0^{-1}).
\end{aligned}
$$

Coinduced Banach Lie–Poisson structures. After these preliminary remarks and notations let us apply the results of the previous section to the Banach Lie–Poisson space L^1. We shall drop the upper indices "ind" and "coind" on the Poisson brackets because it will be clear from the context which brackets are induced and coinduced on various subspaces.

We start with points (i) of Proposition 2.2 and Proposition 2.4. So let us consider the diagram

$$
\begin{tikzcd}
& L^1 \arrow[dl, shift right, "\pi_S"'] \arrow[dr, shift left, "\pi_{S,+}"] & & L^1 \arrow[dl, shift right, "\pi_+"'] \arrow[dr, shift left, "\pi_-"] & \\
L^1_S \arrow[ur, shift right, "\iota_S"'] & & I^1_{+,1} \arrow[ul, shift left, "\iota_{S,+}"] \arrow[ur, shift right, "\iota_+"'] & & L^1_- \arrow[ul, shift left, "\iota_-"]
\end{tikzcd}
$$

where we recall that $\pi_S, \pi_{S,+}, \pi_+$ and π_- are the projections onto the ranges of R_S, R_{S+}, R_+, and R_-, respectively, and $\iota_S, \iota_{S,+}, \iota_+$, and ι_- are inclusions. We see from the above that the assumptions in part (i) of Proposition 2.4 are satisfied because $(I^1_{+,1})^\circ = L^\infty_+$ is a Banach Lie subalgebra of $(L^1)^* = L^\infty$. Thus we can conclude the following facts.

(i) By Proposition 2.4(i) it follows that L^1_S and L^1_- are isomorphic Banach Lie–Poisson spaces with the Poisson brackets defined by formula (2.18). They are given, respectively, by

$$
\{f, g\}_S(\sigma) = \operatorname{Tr}\big(\iota_S(\sigma)\big[D(f \circ \pi_S)\big(\iota_S(\sigma)\big), D(g \circ \pi_S)\big(\iota_S(\sigma)\big)\big]\big) \tag{3.37}
$$

for $\sigma \in L^1_S$ and $f, g \in C^\infty(L^1_S)$ and

$$
\{f, g\}_-(\rho) = \operatorname{Tr}\big(\iota_-(\rho)\big[D(f \circ \pi_-)\big(\iota_-(\rho)\big), D(g \circ \pi_-)\big(\iota_-(\rho)\big)\big]\big) \tag{3.38}
$$

for $\rho \in L^1_-$ and $f, g \in C^\infty(L^1_-)$.

The linear continuous maps $\Phi_{-,S} := \pi_- \circ \iota_S : L^1_S \to L^1_-$ and $\Phi_{S,-} := \pi_S \circ \iota_- : L^1_- \to L^1_S$ are mutually inverse isomorphisms of the Banach Lie–Poisson spaces $(L^1_S, \{,\}_S)$ and $(L^1_-, \{,\}_-)$. The coadjoint actions of the Banach Lie group GL^∞_+ on L^1_- and L^1_S are given by

$$
\big(\mathrm{Ad}^+\big)^*_{g^{-1}} \rho = \pi_-\big(g \iota_-(\rho) g^{-1}\big) \quad \text{for } \rho \in L^1_-, \tag{3.39}
$$

$$
\big(\mathrm{Ad}^S\big)^*_{g^{-1}} \sigma = \pi_S\big(g \iota_S(\sigma) g^{-1}\big) \quad \text{for } \sigma \in L^1_S \tag{3.40}
$$

and $g \in GL^\infty_+$. Differentiating these formulas relative to g at the identity, we get

$$\left(\mathrm{ad}^+\right)^*_x \rho = -\pi_-\left(\left[x, \iota_-(\rho)\right]\right) \quad \text{for } \rho \in L^1_-, \tag{3.41}$$

$$\left(\mathrm{ad}^S\right)^*_x \sigma = -\pi_S\left(\left[x, \iota_S(\sigma)\right]\right) \quad \text{for } \sigma \in L^1_S \tag{3.42}$$

for $x \in L^\infty_+$. The isomorphisms $\Phi_{-,S} : L^1_S \to L^1_-$ and $\Phi_{S,-} : L^1_- \to L^1_S$ are equivariant relative to these coadjoint actions, that is,

$$\left(\mathrm{Ad}^S\right)^*_{g^{-1}} \circ \Phi_{S,-} = \Phi_{S,-} \circ \left(\mathrm{Ad}^+\right)^*_{g^{-1}}, \tag{3.43}$$

$$\left(\mathrm{Ad}^+\right)^*_{g^{-1}} \circ \Phi_{-,S} = \Phi_{-,S} \circ \left(\mathrm{Ad}^S\right)^*_{g^{-1}} \tag{3.44}$$

for any $g \in GL^\infty_+$.

(ii) By (3.22), $I^1_{+,1}$ is the predual of the two Banach Lie algebras $I^\infty_{-,1}$ and L^∞_A. Thus (3.20)–(3.24) and point (ii) of Proposition 2.4 imply that $I_{+,1}$ carries two different Lie–Poisson brackets, namely by (2.18) we have

$$\{f, g\}_+(\rho) = \mathrm{Tr}\big(\iota_+(\rho)\big[D(f \circ \pi_+)\big(\iota_+(\rho)\big), D(g \circ \pi_+)\big(\iota_+(\rho)\big)\big]\big) \tag{3.45}$$

and

$$\{f, g\}_{S,+}(\rho) = \mathrm{Tr}\big(\iota_{S+}(\rho)\big[D(f \circ \pi_{S,+})\big(\iota_{S,+}(\rho)\big), D(g \circ \pi_{S,+})\big(\iota_{S,+}(\rho)\big)\big]\big), \tag{3.46}$$

where $\rho \in I^1_{+,1}$, $f, g \in C^\infty(I^1_{+,1})$.

The coadjoint actions $(\mathrm{Ad}^-)^*$ and $(\mathrm{Ad}^A)^*$ of the groups $GI^\infty_{-,1}$ and O^∞, respectively on $I^1_{+,1}$, are given by

$$\left(\mathrm{Ad}^-\right)^*_{h^{-1}} \rho = \pi_+\big(h \iota_+(\rho) h^{-1}\big) \quad \text{for } h \in GI^\infty_{-,1} \tag{3.47}$$

and

$$\left(\mathrm{Ad}^A\right)^*_{g^{-1}} \rho = \pi_{S+}\big(g \iota_{S,+}(\rho) g^{-1}\big) \quad \text{for } g \in O^\infty, \tag{3.48}$$

where $\rho \in I^1_{+,1}$. We shall not pursue the investigation of this interesting case in this paper.

Induced Banach Lie–Poisson structures. We begin with the study of the lower triangular case. Denote by $\iota_{-,k} : L^1_{-,k} \hookrightarrow L^1_-$ the inclusion and let $\iota^{-1}_{-,k} : \iota_{-,k}(L^1_{-,k}) \to L^1_{-,k}$ be its inverse (defined, of course, only on the range of $\iota_{-,k}$). Then $\iota^*_{-,k} : L^\infty_+ \to L^\infty_{+,k}$. Since $\ker \iota^*_{-,k} = I^\infty_{+,k}$ is an ideal in L^∞_+, by Proposition 2.1 we have $(\mathrm{Ad}^+)^*_{g^{-1}} \iota_{-,k}(L^1_{-,k}) \subset \iota_{-,k}(L^1_{-,k})$ for any $g \in GL^\infty_+$. Therefore there are GL^∞_+ and L^∞_+ coadjoint actions on $L^1_{-,k}$ defined by

$$\left(\mathrm{Ad}^{+,k}\right)^*_{g^{-1}} \rho := \iota^{-1}_{-,k}\big(\pi_-\big(g(\iota_- \circ \iota_{-,k})(\rho) g^{-1}\big)\big) \quad \text{for } \rho \in L^1_{-,k} \text{ and } g \in GL^\infty_+, \tag{3.49}$$

$$\left(\mathrm{ad}^{+,k}\right)^*_x \rho := \iota^{-1}_{-,k}\big(\pi_-\big[x, (\iota_- \circ \iota_{-,k})(\rho)\big]\big) \quad \text{for } \rho \in L^1_{-,k} \text{ and } x \in L^\infty_+. \tag{3.50}$$

Since the action (3.49) is trivial for all elements of the closed normal Lie subgroup $GI^\infty_{+,k}$, it induces the coadjoint action of the group $GL^\infty_{+,k} \cong GL^\infty_+ / GI^\infty_{+,k}$ given by (3.49) that will be also denoted by $(\mathrm{Ad}^{+,k})^*$. Similarly, the Lie algebra action (3.50) is trivial for all elements in the closed ideal $I^\infty_{+,k}$ so it induces the coadjoint action of the Lie algebra $L^\infty_{+,k} \cong L^\infty / I^\infty_{+,k}$ on $L^1_{-,k}$ denoted also by $(\mathrm{ad}^{+,k})^*$.

One can express (3.49) and (3.50) in terms of the expansions $\rho = \rho_0 + S^T \rho_1 + \cdots + (S^T)^{k-1}\rho_{k-1} \in L^1_{-,k}$, $x = x_0 + x_1 S + \cdots + x_{k-1} S^{k-1} \in L^\infty_{+,k}$, and $g = g_0 + g_1 S + \cdots + g_{k-1} S^{k-1} \in GL^\infty_{+,k}$ in the following way:

$$\left(\mathrm{Ad}^{+,k}\right)^*_{g^{-1}} \rho = \sum_{i,j,l=0,\ j \geqslant i+l}^{k-1} \left(S^T\right)^{j-i-l} \tilde{s}^l \left[s^j \left(\tilde{s}^i (g_i)\right) \rho_j h_l \right], \tag{3.51}$$

where the diagonal operators h_l are expressed in terms of the g_i in (3.36), and (using (3.18))

$$\left(\mathrm{ad}^{+,k}\right)^*_x \rho = \sum_{j=0}^{k-1} \left(S^T\right)^j \sum_{i=j}^{k-1} \left(\tilde{s}^{i-j}(\rho_i x_{i-j}) - \rho_i s^j (x_{i-j})\right). \tag{3.52}$$

By (3.30) and (3.19), the Lie–Poisson bracket on $L^1_{-,k}$ is given by

$$\begin{aligned} \{f,g\}_k(\rho) &= \mathrm{Tr}\left(\rho\left[Df(\rho), Dg(\rho)\right]_k\right) \\ &= \sum_{l=0}^{k-1} \sum_{i=0}^{l} \mathrm{Tr}\left[\rho_l \left(\frac{\delta f}{\delta \rho_i}(\rho) s^i \left(\frac{\delta g}{\delta \rho_{l-i}}(\rho)\right) - \frac{\delta g}{\delta \rho_i}(\rho) s^i \left(\frac{\delta f}{\delta \rho_{l-i}}(\rho)\right)\right)\right] \end{aligned} \tag{3.53}$$

for $f, g \in C^\infty(L^1_{-,k})$, where $\frac{\delta f}{\delta \rho_i}(\rho)$ denotes the partial functional derivative of f relative to ρ_i defined by $Df(\rho) = \frac{\delta f}{\delta \rho_0}(\rho) + \frac{\delta f}{\delta \rho_1}(\rho) S + \cdots + \frac{\delta f}{\delta \rho_{k-1}}(\rho) S^{k-1}$.

If in the previous formulas we let $k = \infty$ one obtains the Lie–Poisson bracket on L^1_-. Indeed, the Lie–Poisson bracket $\{f,g\}_-$ on L^1_- given by (3.38) expressed in the coordinates $\{\rho_i\}_{i=0}^\infty$ equals (3.53) for $k = \infty$.

Proposition 3.2. *The Lie–Poisson bracket* (3.53) *on* $L^1_{-,k}$ *coincides with the induced bracket* (2.10) *determined by the inclusion* $\iota_{-,k} : L^1_{-,k} \hookrightarrow L^1_-$ *and the Lie–Poisson bracket* (3.38) *on* L^1_-.

Proof. We need to prove that the induced bracket (2.10) evaluated on two linear functionals $x, y \in L^\infty_{+,k} \cong (L^1_{-,k})^* \subset C^\infty(L^1_{-,k})$ coincides with $[x,y]_k$. To see this we note that $D(x \circ \iota^{-1}_{-,k} \circ R_{-,k})(\iota_{-,k}(\rho)) = \iota_{+,k} x \in L^\infty_+$, where $\iota_{+,k} : L^\infty_{+,k} \hookrightarrow L^\infty_+$ is the inclusion. Then, a direct verification shows that for any $\rho \in L^1_{-,k}$ we have

$$\{x,y\}^{\mathrm{ind}}(\rho) = \left\langle [\iota_{+,k} x, \iota_{+,k} y], \iota_{-,k} \rho \right\rangle = \mathrm{Tr}\left([x,y]\rho\right) = \mathrm{Tr}\left([x,y]_k \rho\right)$$

by (3.30). □

Let us study now the symmetric representation of $(L^1_{-,k}, \{\cdot,\cdot\}_{-,k})$ for $k \in \mathbb{N} \cup \{\infty\}$. This will be done by using the Banach Lie–Poisson space isomorphism $\Phi_{S,-} := \pi_S \circ \iota_- : L^1_- \to L^1_S$. Let $\pi_{-,k} : L^1_- \to L^1_{-,k}$ and $\pi_{S,k} : L^1_S \to L^1_{S,k}$ be the projections with the indicated ranges and

$\iota_{S,k}: L^1_{S,k} \to L^1_S$ the inclusion. Define $\Phi_{S,-,k} := \pi_{S,k} \circ \Phi_{S,-} \circ \iota_{-,k} : L^1_{-,k} \to L^1_{S,k}$. The following commutative diagram illustrates these maps:

$$\begin{array}{ccc} L^1_- & \xrightarrow{\Phi_{S,-}} & L^1_S \\ \iota_{-,k}\Big\uparrow\Big\downarrow\pi_{-,k} & & \iota_{S,k}\Big\uparrow\Big\downarrow\pi_{S,k} \\ L^1_{-,k} & \xrightarrow{\Phi_{S,-,k}} & L^1_{S,k}. \end{array}$$

Pushing forward the Poisson bracket $\{\cdot,\cdot\}_k$ on $L^1_{-,k}$ by the Banach space isomorphism $\Phi_{S,-,k}$ endows $L^1_{S,k}$ with an isomorphic Poisson structure denoted by $\{\cdot,\cdot\}_{S,k}$. From Propositions 2.4 and 3.2, all the maps in the diagram above are linear Poisson maps, with the exception of $\pi_{-,k}$ and $\pi_{S,k}$ which are not Poisson. Recall that GL^∞_+ acts on L^1_- and L^1_S by (3.39) and (3.40), respectively, and that GL^∞_+ (and hence $GL^\infty_{+,k}$) acts on $L^1_{-,k}$ by (3.49). Using the isomorphisms $\Phi_{S,-}$ and $\Phi_{S,-,k}$ to push forward these actions to L^1_S and $L^1_{S,k}$, respectively, all the maps in the diagram above are also GL^∞_+-equivariant. Consequently, one has the GL^∞_+-invariant filtrations

$$\iota_{-,1}(L^1_{-,1}) \hookrightarrow \iota_{-,2}(L^1_{-,2}) \hookrightarrow \cdots \hookrightarrow \iota_{-,k}(L^1_{-,k}) \hookrightarrow \iota_{-,k+1}(L^1_{-,k+1}) \hookrightarrow \cdots \hookrightarrow L^1_-, \tag{3.54}$$

$$\iota_{S,1}(L^1_{S,1}) \hookrightarrow \iota_{S,2}(L^1_{S,2}) \hookrightarrow \cdots \hookrightarrow \iota_{S,k}(L^1_{S,k}) \hookrightarrow \iota_{S,k+1}(L^1_{S,k+1}) \hookrightarrow \cdots \hookrightarrow L^1_S \tag{3.55}$$

of Banach Lie–Poisson spaces predual to the sequence

$$L^\infty_+ \longrightarrow \cdots \longrightarrow L^\infty_{+,k} \longrightarrow L^\infty_{+,k-1} \longrightarrow \cdots \longrightarrow L^\infty_{+,2} \longrightarrow L^\infty_{+,1} \tag{3.56}$$

of Banach Lie algebras in which each arrow is the surjective projector $\pi^\infty_{+,k,k-1}: L^\infty_{+,k} \to L^\infty_{+,k-1}$ that maps k-diagonal upper triangular operators to $(k-1)$-diagonal upper triangular operators by eliminating the kth diagonal. We have $\pi^\infty_{+,k,k-1} \circ \pi^\infty_{+,k} = \pi^\infty_{+,k-1}$.

4. Dynamics generated by Casimirs of $L^1(\mathcal{H})$

We begin by presenting Hamilton's equations on L^1_- and L^1_S given by arbitrary smooth functions h and f defined on the relevant Banach Lie–Poisson spaces. Using formula (2.19) of Proposition 2.2, one obtains Hamilton's equations

$$\frac{d}{dt}\rho = \pi_-\big(\big[D(h\circ\pi_-)\big(\iota_-(\rho)\big), \iota_-(\rho)\big]\big) \quad \text{for } \rho \in L^1_- \text{ and } h \in C^\infty\big(L^1_-\big), \tag{4.1}$$

$$\frac{d}{dt}\sigma = \pi_S\big(\big[D(f\circ\pi_S)\big(\iota_S(\sigma)\big), \iota_S(\sigma)\big]\big) \quad \text{for } \sigma \in L^1_S \text{ and } f \in C^\infty\big(L^1_S\big), \tag{4.2}$$

on the isomorphic Banach Lie–Poisson spaces $(L^1_-, \{\cdot,\cdot\}_-)$ and $(L^1_S, \{\cdot,\cdot\}_S)$; from Section 3 we know that this isomorphism is $\Phi_{S,-} := \pi_S \circ \iota_- : (L^1_-, \{\cdot,\cdot\}_-) \xrightarrow{\sim} (L^1_S, \{\cdot,\cdot\}_S)$. Therefore, if $f \circ \Phi_{S,-} = h$ then Eqs. (4.1) and (4.2) give the same dynamics. Recall that $\pi_-: L^1 \to L^1_-$ and $\pi_S: L^1 \to L^1_S$ are, by definition, the projectors $P^1_- + P_0: L^1 \to L^1$ and $\pi_S := P^1_- + P_0 + T \circ P^1_-: L^1 \to L^1$ considered as maps on their ranges (see (3.23) and the subsequent comments) and $\iota_-: L^1_- \hookrightarrow L^1$, $\iota_S: L^1_S \hookrightarrow L^1$ are the inclusions.

Dynamics associated to the restriction of Casimirs. Now let us observe that the family of functions $I_l \in C^\infty(L^1)$ defined by

$$I_l(\rho) := \frac{1}{l}\operatorname{Tr}\rho^l \quad \text{for } l \in \mathbb{N}, \tag{4.3}$$

are Casimir functions on the Banach Lie–Poisson space $(L^1, \{\cdot,\cdot\})$. This follows from (3.1) since one has

$$DI_l(\rho) = \rho^{l-1} \in L^1 \subset L^\infty \cong \left(L^1\right)^*. \tag{4.4}$$

Restricting I_l to $\iota_- : L^1_- \hookrightarrow L^1$ and $\iota_S : L^1_S \hookrightarrow L^1$ we obtain for all $l \in \mathbb{N}$

$$I_l^-(\rho) := I_l\big(\iota_-(\rho)\big) \quad \text{for } \rho \in L^1_-, \tag{4.5}$$

$$I_l^S(\sigma) := I_l\big(\iota_S(\sigma)\big) \quad \text{for } \sigma \in L^1_S. \tag{4.6}$$

According to Corollary 2.3(i), (4.5) and (4.6) form two infinite families of integrals in involution

$$\left\{I_l^-, I_m^-\right\}_- = 0 \quad \text{and} \quad \left\{I_l^S, I_m^S\right\}_S = 0 \quad \text{for } l, m \in \mathbb{N}. \tag{4.7}$$

Since $I_l^S \circ \Phi_{S,-} \neq I_l^-$, the Hamiltonians I_l^- and I_l^S define on $(L^1_-, \{\cdot,\cdot\}_-)$ (or $(L^1_S, \{\cdot,\cdot\}_S)$) different families of dynamical systems.

Firstly, we shall investigate the systems associated to the Hamiltonians I_l^- given by (4.5). As we shall see, the framework of the Banach Lie–Poisson space $(L^1_-, \{\cdot,\cdot\}_-)$ is more natural in this case. Hence, taking into account Corollary 2.3(ii), substituting I_l^- into (4.1), then applying ι_- to (4.1), and using (4.4), yields the family of Hamilton equations on L^1_-

$$\frac{\partial \iota_-(\rho)}{\partial t_l} = \left(P^1_- + P^1_0\right)\left[\left(P^\infty_+ + P^\infty_0\right)\left(\left[\iota_-(\rho)\right]^{l-1}\right), \iota_-(\rho)\right] \tag{4.8}$$

or, equivalently, in Lax form

$$\frac{\partial \iota_-(\rho)}{\partial t_l} = -\left[P^\infty_-\left(\left[\iota_-(\rho)\right]^{l-1}\right), \iota_-(\rho)\right] = \left[P^\infty_0\left(\left[\iota_-(\rho)\right]^{l-1}\right), \iota_-(\rho)\right], \tag{4.9}$$

where t_l denotes the time parameter for the lth flow.

Eq. (4.8) implies that its solution is given by the coadjoint action of the group GL^∞_+ on the dual L^1_- of its Lie algebra. Hence, there is some smooth curve $\mathbb{R} \ni t_l \mapsto h_+(t_l) \in GL^\infty_+$ satisfying $(\mathrm{Ad}^+)^*_{h_+(t_l)^{-1}} \circ (\mathrm{Ad}^+)^*_{h_+(s_l)^{-1}} = (\mathrm{Ad}^+)^*_{h_+(t_l+s_l)^{-1}}$ such that

$$\iota_-\big(\rho(t_l)\big) = \left(\mathrm{Ad}^+\right)^*_{h_+(t_l)^{-1}}\rho(0) = \left(P^1_- + P^1_0\right)\left(h_+(t_l)\iota_-\big(\rho(0)\big)h_+(t_l)^{-1}\right) \tag{4.10}$$

is the solution of (4.8) with initial condition $\rho(0)$ for $t_l = 0$.

On the other hand, the solution of (4.9) is given by

$$\iota_-\big(\rho(t_l)\big) = h_-(t_l)^{-1}\iota_-\big(\rho(0)\big)h_-(t_l), \tag{4.11}$$

for a smooth one-parameter subgroup $\mathbb{R} \ni t_l \mapsto h_-(t_l) \in GL_-^\infty$ that can be explicitly determined. We shall do this by using the decomposition $\iota_-(\rho) = \rho_0 + \rho_-$, where $\rho_- = \sum_{i=1}^\infty (S^T)^i \rho_i$ and $\rho_i \in L_0^1$ if $i \in \mathbb{N} \cup \{0\}$. Since $P_0^\infty([\iota_-(\rho)]^{l-1}) = \rho_0^{l-1}$, Eq. (4.9) becomes

$$\frac{\partial}{\partial t_l}\iota_-(\rho) = \left[\rho_0^{l-1}, \rho_0 + \rho_-\right] = \left[\rho_0^{l-1}, \rho_-\right]$$

which is equivalent to

$$\frac{\partial}{\partial t_l}\rho_- = \left[\rho_0^{l-1}, \rho_-\right] \quad \text{and} \quad \frac{\partial}{\partial t_l}\rho_0 = 0. \tag{4.12}$$

It immediately follows that its solution is given by (4.11) with

$$h_-(t_l) = e^{-t_l \rho_0(0)^{l-1}}, \tag{4.13}$$

where $\rho(0) = \rho_0(0) + \rho_-(0)$ is the initial value of ρ at time $t_l = 0$.

Note that $h_-(t_l) \in GL_-^\infty$ is in fact a diagonal operator which can also be obtained from the decomposition

$$e^{t_l[\iota_-(\rho(0))]^{l-1}} = k_-(t_l)h_-(t_l)^{-1}, \tag{4.14}$$

where $k_-(t_l) \in GI_{-,1}^\infty$. It follows that we can write the solution also in the form

$$\iota_-\big(\rho(t_l)\big) = k_-(t_l)^{-1}\big[\iota_-\big(\rho(0)\big)\big]k_-(t_l). \tag{4.15}$$

Finally, note that in (4.10) we can choose $h_+(t_l) = h_-(t_l)$ since also $h_-(t_l) \in GL_+^\infty$.

Let us analyze the system (4.9) in more detail. We begin by noting that there is an isometry between ℓ^∞ and the diagonal bounded linear operators $L_0^\infty \subset L^\infty$ and between ℓ^1 and the diagonal trace class operators $L_0^1 \subset L^1$. Fix a strictly lower triangular element

$$\nu_- = \sum_{i=1}^{k-1}\left(S^T\right)^i \nu_i \in L_{-,k}^1 \quad \text{where } k \in \mathbb{N} \cup \{\infty\}, \tag{4.16}$$

and define the map $\mathcal{J}_{\nu_-} : \ell^\infty \times \ell^1 \to L_{-,k}^1$ by

$$\mathcal{J}_{\nu_-}(\mathbf{q}, \mathbf{p}) := \mathbf{p} + e^{\mathbf{q}}\nu_- e^{-\mathbf{q}}, \tag{4.17}$$

where, on the right-hand side, we identify $\mathbf{p}$ and $\mathbf{q}$ with diagonal operators and $e^{\mathbf{q}}$ is the exponential of $\mathbf{q}$. It is easy to see that this map is smooth and that $\mathcal{J}_{\nu_-}(\mathbf{q}, \mathbf{p}) = \mathcal{J}_{\nu_-}(\mathbf{q} + \alpha\mathbb{I}, \mathbf{p})$, for any $\alpha \in \mathbb{R}$.

The weak symplectic manifold ($\boldsymbol{\ell^\infty \times \ell^1}, \boldsymbol{\omega}$). In this paper we shall often work with the weak symplectic manifold $(\ell^\infty \times \ell^1, \omega)$, where ℓ^∞ and ℓ^1 are the Banach spaces of bounded real sequences and absolutely convergent real series, respectively, whose norms are given by

$$\|\mathbf{q}\|_\infty := \sup_{k=0,1,\ldots} |q_k|, \qquad \mathbf{q} := \{q_k\}_{k=0}^\infty \in \ell^\infty,$$

and

$$\|\mathbf{p}\|_1 := \sum_{k=0}^{\infty} |p_k|, \qquad \mathbf{p} := \{p_k\}_{k=0}^{\infty} \in \ell^1.$$

The strongly nondegenerate duality pairing

$$\langle \mathbf{q}, \mathbf{p} \rangle = \sum_{k=0}^{\infty} q_k p_k, \quad \text{for } \mathbf{q} \in \ell^{\infty},\ \mathbf{p} \in \ell^1, \tag{4.18}$$

establishes the Banach space isomorphism $(\ell^1)^* = \ell^{\infty}$ and the weak symplectic form ω is given by

$$\omega\big((\mathbf{q}, \mathbf{p}), (\mathbf{q}', \mathbf{p}')\big) := \langle \mathbf{q}, \mathbf{p}' \rangle - \langle \mathbf{q}', \mathbf{p} \rangle, \quad \text{for } \mathbf{q}, \mathbf{q}' \in \ell^{\infty},\ \mathbf{p}, \mathbf{p}' \in \ell^1. \tag{4.19}$$

The differential form ω is conveniently written as

$$\omega = \sum_{k=0}^{\infty} \mathbf{d} q_k \wedge \mathbf{d} p_k \tag{4.20}$$

in the coordinates q_k, p_k. Let us elaborate on the notation used in (4.20). If $\mathbf{p} = \{p_k\}_{k=0}^{\infty} \in \ell^1$, denote by $\{\partial/\partial p_k\}_{k=0}^{\infty}$ the basis of the tangent space $T_{\mathbf{p}}\ell^1$ corresponding to the standard Schauder basis $\{|k\rangle\}_{k=0}^{\infty}$ of ℓ^1. The same basis in ℓ^{∞} has a different meaning: every element $\mathbf{a} := \{a_k\}_{k=0}^{\infty} \in \ell^{\infty}$ can be uniquely written as a *weakly* convergent series $\mathbf{a} = \sum_{k=0}^{\infty} a_k |k\rangle$. With this notion of basis in ℓ^{∞}, given $\mathbf{q} \in \ell^{\infty}$, the sequence $\{\partial/\partial q_k\}_{k=0}^{\infty}$ denotes the basis of the tangent space $T_{\mathbf{q}}\ell^{\infty}$ corresponding to $\{|k\rangle\}_{k=0}^{\infty}$. Thus, any smooth vector field X on $\ell^{\infty} \times \ell^1$ is written as

$$X(\mathbf{q}, \mathbf{p}) = \sum_{k=0}^{\infty} \left(A_k(\mathbf{q}, \mathbf{p}) \frac{\partial}{\partial q_k} + B_k(\mathbf{q}, \mathbf{p}) \frac{\partial}{\partial p_k} \right),$$

where $\{A_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^{\infty}$ and $\{B_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^1$.

If Y is another vector field whose coefficients are $\{C_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^{\infty}$, $\{D_k(\mathbf{q}, \mathbf{p})\}_{k=0}^{\infty} \in \ell^1$, employing the usual conventions for the exterior derivatives of coordinate functions to represent elements in the corresponding dual spaces, formula (4.20) gives

$$\left(\sum_{k=0}^{\infty} \mathbf{d} q_k \wedge \mathbf{d} p_k \right)(X, Y)(\mathbf{q}, \mathbf{p}) = \sum_{k=0}^{\infty} \big(A_k(\mathbf{q}, \mathbf{p}) D_k(\mathbf{q}, \mathbf{p}) - C_k(\mathbf{q}, \mathbf{p}) B_k(\mathbf{q}, \mathbf{p}) \big)$$

which equals (4.19). It is in this sense that the writing in (4.20) represents the weak symplectic form (4.19).

Denote by $B^1_{-,k}$ the space of bidiagonal trace class operators having non-zero entries only on the main and the lower $(k-1)$st diagonal. For $\nu_- = (S^T)^{k-1} \nu_{k-1} \in L^1_{-k+1} \subset L^1_{-,k}$, we shall prove in Proposition 5.2 that the map $\mathcal{J}_{\nu_-} : \ell^{\infty} \times \ell^1 \to B^1_{-,k}$ is a momentum map in the following sense: for any two locally defined smooth functions $\varphi, \psi \in C^{\infty}(B^1_{-,k})$, we have $\{\varphi \circ \mathcal{J}_{\nu_-},$

$\psi \circ \mathcal{J}_{\nu_-}\}_\omega = \{\varphi, \psi\} \circ \mathcal{J}_{\nu_-}$. See [12] for the general definition and properties of momentum maps on Banach Poisson and weak symplectic manifolds.

The solution of the system (4.9). We shall argue below, in analogy with the finite-dimensional case, that $(\mathbf{q}, \mathbf{p})$ can be considered as angle–action coordinates for the Hamiltonian system (4.9). We begin by recalling that the solution of (4.9) is given by $\iota_-(\rho(t_l)) = h_-(t_l)^{-1}\iota_-(\rho(0))h_-(t_l)$, where $h_-(t_l) = e^{-t_l\rho_0(0)^{l-1}}$, $\rho(0) = \rho_0(0) + \rho_-(0) \in L^1_-$ is the initial value of the variable ρ at $t_l = 0$, $\rho_0 \in L^1_0$ a diagonal operator, and ρ_- a strictly lower triangular operator. Therefore, $h_-(t_l)h_-(t_m) = h_-(t_m)h_-(t_l)$ for any $l, m \in \mathbb{N}$, and hence the product

$$h_-(t) := h_-(t_1, t_2, \ldots) := \prod_{l=1}^{\infty} h_-(t_l) \tag{4.21}$$

is independent on the order of the factors and it exists as an invertible bounded operator if we assume that $t := (t_1, t_2, \ldots) \in \ell^\infty_0$ which means that t has only finitely many non-zero elements.

One also has

$$h_-(t)^{-1}\mathcal{J}_{\nu_-}(\mathbf{q}, \mathbf{p})h_-(t) = \mathcal{J}_{\nu_-}\left(\mathbf{q} + \sum_{l=1}^{\infty} t_l\rho_0(0)^{l-1}, \mathbf{p}\right) \quad \text{for } t \in \ell^\infty_0, \tag{4.22}$$

which shows that the flow in the coordinates $(\mathbf{q}, \mathbf{p})$ is described by a straight line motion in $\mathbf{q}$ with $\mathbf{p}$ conserved. If this would be a finite-dimensional system, since $(\mathbf{q}, \mathbf{p})$ are also Darboux coordinates (see (4.19) or (4.20)), we would say that they are action–angle coordinates on $\mathcal{J}_{\nu_-}(\ell^\infty \times \ell^1)$.

In infinite dimensions, even the definition of action–angle coordinates presents problems. First, if the symplectic form is strong, the Darboux theorem (that is, the symplectic form is locally constant) is valid; see the proof of Theorem 3.2.2 in [1]. Second, if the symplectic form is weak, which is our case, the Darboux theorem fails in general, even if the manifold is a Hilbert space; Marsden's classical counterexample can be found and discussed in Exercise 3.2H of [1]. Third, even if one could show in a particular case that the Darboux theorem holds, there still is the problem of coordinates. In the case presented above, the action–angle coordinates were constructed explicitly. In general, on Banach weak symplectic manifolds this may well be impossible.

The solution of the system (4.6). We return now to the systems described by the family of integrals in involution I^S_l given by (4.6). By Corollary 2.3(ii), substituting I^S_l into (4.2), applying ι_S to (4.2), and using (4.4), yields the family of Hamilton equations on L^1_S

$$\frac{\partial \iota_S(\sigma)}{\partial t_l} = \left(P^1_- + P^1_0 + T \circ P^1_-\right)\left[\left(P^\infty_+ + P^\infty_0 + T \circ P^\infty_-\right)\left(\left[\iota_S(\sigma)\right]^{l-1}\right), \iota_S(\sigma)\right] \tag{4.23}$$

or, equivalently, in Lax form

$$\frac{\partial \iota_S(\sigma)}{\partial t_l} = -\left[\left(P^\infty_- - T \circ P^\infty_-\right)\left(\left[\iota_S(\sigma)\right]^{l-1}\right), \iota_S(\sigma)\right], \tag{4.24}$$

where t_l denotes the time parameter for the lth flow.

From (4.23) it follows that the solution of this equation can be written in terms of the coadjoint action of the Banach Lie group GL_+^∞ on the dual L_S^1 of its Lie algebra. More precisely, the solution is necessarily of the form

$$\iota_S(\sigma(t_l)) = (\mathrm{Ad}^S)^*_{\bar{g}_+(t_l)^{-1}}\sigma(0) = (P_-^1 + P_0^1 + T \circ P_-^1)(\bar{g}_+(t_l)\iota_S(\sigma(0))\bar{g}_+(t_l)^{-1}) \quad (4.25)$$

for some smooth curve $\mathbb{R} \ni t_l \mapsto \bar{g}_+(t_l) \in GL_+^\infty$ and $\sigma(0)$ the initial condition for $t_l = 0$.

On the other hand, the solution of (4.24) is

$$\iota_S(\sigma(t_l)) = g_S(t_l)^T \iota_S(\sigma(0)) g_S(t_l), \quad (4.26)$$

where $\mathbb{R} \ni t_l \mapsto g_S(t_l) \in O^\infty$ is a smooth curve that will be determined in the next proposition by the same method as in the finite-dimensional case.

Proposition 4.1. *Assume that we have the decomposition* (*we set here* $t = t_l$)

$$e^{t[\iota_S(\sigma(0))]^{l-1}} = g_S(t)g_+(t) \quad (4.27)$$

for $g_S(t) \in O^\infty$ *and* $g_+(t) \in GL_+^\infty$. *Then*

$$\iota_S(\sigma(t)) := g_S(t)^T[\iota_S(\sigma(0))]g_S(t) = g_+(t)[\iota_S(\sigma(0))]g_+(t)^{-1} \quad (4.28)$$

is the solution of (4.24) *with initial condition* $\iota_S(\sigma(0))$.

Proof. To prove the first equality in (4.28), use (4.27) to get

$$g_S(t) = e^{t[\iota_S(\sigma(0))]^{l-1}} g_+(t)^{-1}$$

and hence

$$g_+(t)e^{-t[\iota_S(\sigma(0))]^{l-1}}[\iota_S(\sigma(0))]e^{t[\iota_S(\sigma(0))]^{l-1}} g_+(t)^{-1} = g_+(t)[\iota_S(\sigma(0))]g_+(t)^{-1}$$

since $\iota_S(\sigma(0))$ commutes with $e^{t[\iota_S(\sigma(0))]^{l-1}}$.

Let $\iota_S(\sigma(t)) := g_S(t)^{-1}[\iota_S(\sigma(0))]g_S(t)$. Taking the time derivative of (4.27) and multiplying on the right by $g_S(t)^{-1}$ and on the left by $g_+(t)^{-1}$ we get

$$[\iota_S(\sigma(t))]^{l-1} = g_S(t)^{-1}\dot{g}_S(t) + \dot{g}_+(t)g_+(t)^{-1}$$

which is equivalent to the equations

$$g_S(t)^{-1}\dot{g}_S(t) = (P_-^\infty - T \circ P_-^\infty)([\iota_S(\sigma(t))]^{l-1}), \quad (4.29)$$

$$\dot{g}_+(t)g_+(t)^{-1} = (P_+^\infty + P_0^\infty + T \circ P_-^\infty)([\iota_S(\sigma(t))]^{l-1}). \quad (4.30)$$

Therefore

$$\begin{aligned}
&\frac{d}{dt}\iota_S\big(\sigma(t)\big)\\
&\quad= -g_S(t)^{-1}\dot{g}_S(t)g_S(t)^{-1}\big[\iota_S\big(\sigma(0)\big)\big]g_S(t) + g_S(t)^{-1}\big[\iota_S\big(\sigma(0)\big)\big]\dot{g}_S(t)\\
&\quad= -\big(P_-^\infty - T\circ P_-^\infty\big)\big(\big[\iota_S\big(\sigma(t)\big)\big]^{l-1}\big)\iota_S\big(\sigma(t)\big) + \iota_S\big(\sigma(t)\big)\big(P_-^\infty - T\circ P_-^\infty\big)\big(\big[\iota_S\big(\sigma(t)\big)\big]^{l-1}\big)\\
&\quad= -\big[\big(P_-^\infty - T\circ P_-^\infty\big)\big(\big[\iota_S(\sigma)\big]^{l-1}\big), \iota_S(\sigma)\big]
\end{aligned}$$

which is (4.24). □

This proposition shows that the solution (4.28) of the system (4.24) can be expressed using the Iwasawa decomposition for the operator $e^{t[\iota_S(\sigma(0))]^{l-1}}$. Note that since the Gram–Schmidt method is valid in separable Hilbert spaces for a Hilbert basis indexed by $\mathbb{N}$, each operator has an Iwasawa decomposition. However, an Iwasawa decomposition $GL^\infty = O^\infty \cdot GL_0^\infty \cdot GI_{+,1}^\infty$ for the Banach Lie group GL^∞ is *not* valid. The reason is that the closed Banach Lie subalgebras corresponding to these three factors do not sum up to the whole space L^∞ which implies that the map intervening in the Iwasawa decomposition is not smooth. This is in sharp contrast with the polar decomposition theorem; see the appendix in [11].

Note also that (4.28) produces a smooth curve $g_+(t) \in GL_+^\infty$ satisfying (4.25) even without the projection operator in that formula. This follows also directly from (4.28) and (3.40).

The previous general considerations involving Proposition 2.1, imply that the families of flows given by (4.1) or (4.2) and, in particular by (4.9) or (4.24), not only preserve the symplectic leaves of L_-^1 and L_S^1, but also the filtrations (3.54) and (3.55), respectively. This remark has some important consequences which we discussed below.

We turn now to the study of Hamiltonian systems induced on the filtrations (3.54) and (3.55). A *k-diagonal Hamiltonian system* is, by definition, a Hamiltonian system on $(L_{-,k}^1, \{\cdot,\cdot\}_k)$. Since the map $\Phi_{S,-,k} : (L_{-,k}^1, \{\cdot,\cdot\}_k) \to (L_{S,k}^1, \{\cdot,\cdot\}_{S,k})$ introduced at the end of Section 3 is a Banach Lie–Poisson space isomorphism, we can regard k-diagonal Hamiltonian systems as being defined also on $(L_{S,k}^1, \{\cdot,\cdot\}_{S,k})$. From (3.52), Hamilton's equations on $(L_{-,k}^1, \{\cdot,\cdot\}_k)$ defined by an arbitrary function $h_k \in C^\infty(L_{-,k}^1)$ are given by

$$\frac{d}{dt}\rho_j = -\sum_{l=j}^{k-1}\left(\tilde{s}^{l-j}\left(\rho_l\frac{\delta h_k}{\delta\rho_{l-j}}\right) - \rho_l s^j\left(\frac{\delta h_k}{\delta\rho_{l-j}}\right)\right) \quad \text{for } j = 0, 1, 2, \ldots, k-1. \tag{4.31}$$

Note that for all $n > k$ (including $n = \infty$), any $h_k \in C^\infty(L_{-,k}^1)$ can be smoothly extended to $h_n := h_k \circ \pi_{kn} \in C^\infty(L_{-,n}^1)$, where $\pi_{kn} : L_{-,n}^1 \to L_{-,k}^1$ is the projection that eliminates the last lower $n-k$ diagonals of an operator in $L_{-,n}^1 := \bigoplus_{i=-n+1}^0 L_i^1$. Conversely, any $h_n \in C^\infty(L_{-,n}^1)$ gives rise to a smooth function $h_k := h_n \circ \iota_{nk} \in C^\infty(L_{-,k}^1)$, where $\iota_{nk} : L_{-,k}^1 \hookrightarrow L_{-,n}^1$ is the natural inclusion. Since the flow defined by $h \in C^\infty(L_-^1)$ preserves the filtration (3.54) (see Proposition 2.1) it follows that if the initial condition $\rho(0) \in L_{-,k}^1$ its trajectory is necessarily contained in $L_{-,k}^1$. This means that in order to solve the system (4.31) for a given $k \in \mathbb{N}$, it suffices to solve the Hamiltonian system given by the extension of h_k to $(L_-^1, \{\cdot,\cdot\}_-)$ for initial conditions in $L_{-,k}^1$.

Let us now specialize the functions $h_k \in C^\infty(L_{-,k}^1)$ and $f_k \in C^\infty(L_{S,k}^1)$ to

$$I_l^{-,k}(\rho) := I_l^-\big(\iota_{-,k}(\rho)\big) = I_l\big((\iota_- \circ \iota_{-,k})(\rho)\big) \quad \text{for } \rho \in L^1_{-,k}, \tag{4.32}$$

$$I_l^{S,k}(\sigma) := I_l^S\big(\iota_{S,k}(\sigma)\big) = I_l\big((\iota_S \circ \iota_{S,k})(\sigma)\big) \quad \text{for } \sigma \in L^1_{S,k}, \tag{4.33}$$

respectively, where $\iota_{-,k} : L^1_{-,k} \hookrightarrow L^1_-$ and $\iota_{S,k} : L^1_{S,k} \hookrightarrow L^1_S$ are the inclusions. Note that since $I_l^{S,k} \circ \Phi_{S,-,k} \neq I_l^{-,k}$, the dynamics induced by the functions $I_l^{-,k}$ and $I_l^{S,k}$ are different in spite of the fact that the Poisson structures on $L^1_{-,k}$ and $L^1_{S,k}$ are isomorphic. Therefore, we see that on has the family of Hamiltonian systems indexed by $k \in \mathbb{N}$ which have an infinite number of integrals in involution indexed by $l \in \mathbb{N}$. For $k = 2$ the system is the semi-infinite Toda lattice. Therefore, the *k-diagonal semi-infinite Toda systems* are defined to be the Hamiltonian systems on $L^1_{S,k}$ associated to the functions $I_l^{S,k}$, $l \in \mathbb{N}$.

An important consequence of the fact that the Poisson brackets on $L^1_{-,k}$ and $L^1_{S,k}$ are induced is that the method of solution of the corresponding Hamilton equations for $I_l^{-,k}$ and $I_l^{S,k}$, respectively, can be obtained by solving these equations on L^1_- and L^1_S, respectively. Namely, it suffices to work with the equations of motion (4.9) and (4.24) with initial conditions $\rho(0) \in L^1_{-,k}$ and $\sigma(0) \in L^1_{S,k}$, respectively, and use Proposition 4.1. We shall do this in the rest of the paper for a special case related to the semi-infinite Toda system.

5. The bidiagonal case

In this section we shall study in great detail the bidiagonal case consisting of operators that have only two non-zero diagonals: the main one and the lower $k-1$ diagonal. As an application of the obtained results we give a rigorous functional analytic formulation of the integrability of the semi-infinite Toda lattice.

The coordinate description of the bidiagonal subcase. Due to their usefulness in the study of the Toda lattice, we shall express in coordinates several formulas from Section 3 adapted to the subalgebra $B^\infty_{+,k} \subset L^\infty_{+,k}$, $k \geqslant 2$, consisting of bidiagonal elements

$$x := x_0 + x_{k-1}S^{k-1} = \sum_{i=0}^{\infty}\big(x_{0,ii}|i\rangle\langle i| + x_{k-1,ii}|i\rangle\langle i+k-1|\big), \tag{5.1}$$

where x_0, x_{k-1} are diagonal operators whose entries are given by the sequences $\{x_{0,ii}\}_{i=0}^\infty$, $\{x_{k-1,ii}\}_{i=0}^\infty \in \ell^\infty$, respectively. The subalgebra $B^\infty_{+,k}$ of $L^\infty_{+,k}$ is hence formed by upper triangular bounded operators that have only two non-zero diagonals, namely the main diagonal and the strictly upper $k-1$ diagonal.

The predual of $B^\infty_{+,k}$ is $B^1_{-,k}$ which consists of lower triangular trace class operators having only two non-vanishing diagonals, namely the main one and the strictly lower $k-1$ diagonal ($k \geqslant 2$), that is, they are of the type

$$\rho = \rho_0 + (S^{k-1})^T\rho_{k-1} = \sum_{i=0}^{\infty}\big(\rho_{0,ii}|i\rangle\langle i| + \rho_{k-1,ii}|i+k-1\rangle\langle i|\big), \tag{5.2}$$

where ρ_0 and ρ_{k-1} are diagonal operators whose entries are given by the sequences $\{\rho_{0,ii}\}_{i=0}^{\infty}$, $\{\rho_{k-1,ii}\}_{i=0}^{\infty} \in \ell^1$, respectively. The Banach Lie subgroup $GB_{+,k}^{\infty}$ of $GL_{+,k}^{\infty}$ whose Banach Lie algebra is $B_{+,k}^{\infty}$ has elements given by

$$g = g_0 + g_{k-1}S^{k-1} = \sum_{i=0}^{\infty}\bigl(g_{0,ii}|i\rangle\langle i| + g_{k-1,ii}|i\rangle\langle i+k-1|\bigr), \tag{5.3}$$

where g_0 and g_{k-1} are diagonal operators whose entries are given by the sequences $\{g_{0,ii}\}_{i=0}^{\infty}$, $\{g_{k-1,ii}\}_{i=0}^{\infty} \in \ell^{\infty}$, respectively, and the sequence $\{g_{0,ii}\}_{i=0}^{\infty}$ is bounded below by a strictly positive number (that depends on g_0).

The product of $g, h \in GB_{+,k}^{\infty}$ in $GL_{+,k}^{\infty}$ is given by

$$\begin{aligned} g \circ_k h &= g_0 h_0 + \bigl(g_0 h_{k-1} + g_{k-1} s^{k-1}(h_0)\bigr)S^{k-1} \\ &= \sum_{i=0}^{\infty} g_{0,ii}h_{0,ii}|i\rangle\langle i| + \sum_{i=0}^{\infty}(g_{ii}h_{k-1,ii} + g_{k-1,ii}h_{0,i+k-1,i+k-1})|i\rangle\langle i+k-1| \end{aligned} \tag{5.4}$$

and the inverse of g in $GB_{+,k}^{\infty}$ is given by

$$\begin{aligned} g^{-1} &= g_0^{-1} - g_0^{-1} g_{k-1} s^{k-1}\bigl(g_0^{-1}\bigr)S^{k-1} \\ &= \sum_{i=0}^{\infty}\frac{1}{g_{0,ii}}|i\rangle\langle i| - \sum_{i=0}^{\infty}\frac{g_{k-1,ii}}{g_{0,ii}g_{0,i+k-1,i+k-1}}|i\rangle\langle i+k-1|. \end{aligned} \tag{5.5}$$

The Lie bracket of $x, y \in B_{+,k}^{\infty}$ has the expression

$$\begin{aligned} [x,y]_k &= \bigl(x_{k-1}\bigl(s^{k-1}(y_0) - y_0\bigr) - y_{k-1}\bigl(s^{k-1}(x_0) - x_0\bigr)\bigr)S^{k-1} \\ &= \sum_{i=0}^{\infty}\bigl(x_{k-1,ii}(y_{0,i+k-1,i+k-1} - y_{0,ii}) - y_{k-1,ii}(x_{0,i+k-1,i+k-1} - x_{0,ii})\bigr)|i\rangle\langle i+k-1|. \end{aligned} \tag{5.6}$$

The group coadjoint action $(\mathrm{Ad}^{+,k})_{g^{-1}}^{*} : B_{-,k}^{1} \to B_{-,k}^{1}$ for $g := g_0 + g_{k-1}S^{k-1} \in GB_{+,k}^{\infty} \subset GL_{+,k}^{\infty}$ and Lie algebra coadjoint action $(\mathrm{ad}^{+,k})_x^{*} : B_{-,k}^{1} \to B_{-,k}^{1}$, for $x := x_0 + x_{k-1}S^{k-1} \in B_{+,k}^{\infty} \subset L_{+,k}^{\infty}$ are given by

$$\begin{aligned} \bigl(\mathrm{Ad}^{+,k}\bigr)_{g^{-1}}^{*}\rho &= \rho_0 + g_0^{-1}g_{k-1}\rho_{k-1} - \tilde{s}^{k-1}\bigl(g_0^{-1}g_{k-1}\rho_{k-1}\bigr)\left(\mathbb{I} - \sum_{j=0}^{k-2}p_j\right) \\ &\quad + \bigl(S^T\bigr)^{k-1}s^{k-1}(g_0)g_0^{-1}\rho_{k-1} \\ &= \sum_{i=0}^{\infty}\left(\rho_{0,ii} + \rho_{k-1,ii}\frac{g_{k-1,ii}}{g_{0,ii}} - \rho_{k-1,ii}\frac{g_{k-1,ii}}{g_{0,i-k+1,i-k+1}}\right)|i\rangle\langle i| \\ &\quad + \sum_{i=0}^{\infty}\rho_{k-1,ii}\frac{g_{0,i+k-1,i+k-1}}{g_{0,ii}}|i+k-1\rangle\langle i| \end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\left(\mathrm{ad}^{+,k}\right)_x^*\rho &= \tilde{s}^{k-1}(\rho_{k-1}x_{k-1}) - \rho_{k-1}x_{k-1} + \left(S^T\right)^{k-1}\rho_{k-1}\left(x_0 - s^{k-1}(x_0)\right) \\ &= \sum_{i=0}^{\infty}(\rho_{k-1,ii}x_{k-1,ii} - \rho_{k-1,ii}x_{k-1,ii})|i\rangle\langle i| \\ &\quad + \sum_{i=0}^{\infty}\rho_{k-1,ii}(x_{0,ii} - x_{0,i+k-1,i+k-1})|i+k-1\rangle\langle i|,\end{aligned} \tag{5.8}$$

where $\rho := \rho_0 + (S^T)^{k-1}\rho_{k-1} \in B^1_{-,k}$.

Since $(B^1_{-,k})^* = B^\infty_{+,k}$ and the duality pairing is given by the trace of the product, it follows that the Lie–Poisson bracket and its associated Hamiltonian vector field on $B^1_{-,k}$ are given by

$$\begin{aligned}&\{f,h\}_{0,k-1}(\rho) \\ &= \mathrm{Tr}\left[\rho_{k-1}\left(\frac{\partial f}{\partial\rho_{k-1}}\left(s^{k-1}\left(\frac{\partial h}{\partial\rho_0}\right) - \frac{\partial h}{\partial\rho_0}\right) - \frac{\partial h}{\partial\rho_{k-1}}\left(s^{k-1}\left(\frac{\partial f}{\partial\rho_0}\right) - \frac{\partial f}{\partial\rho_0}\right)\right)\right] \\ &= \sum_{i=0}^{\infty}\rho_{k-1,ii}\left[\frac{\partial f}{\partial\rho_{k-1,ii}}\left(\frac{\partial h}{\partial\rho_{0,i+k-1,i+k-1}} - \frac{\partial h}{\partial\rho_{0,ii}}\right)\right. \\ &\quad \left. - \frac{\partial h}{\partial\rho_{k-1,ii}}\left(\frac{\partial f}{\partial\rho_{0,i+k-1,i+k-1}} - \frac{\partial f}{\partial\rho_{0,ii}}\right)\right]\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}X_h^{0,k-1}(\rho) &= \mathrm{Tr}\left[\rho_{k-1}\left(s^{k-1}\left(\frac{\partial h}{\partial\rho_0}\right) - \frac{\partial h}{\partial\rho_0}\right)\frac{\partial}{\partial\rho_{k-1}} - \frac{\partial h}{\partial\rho_{k-1}}\left(s^{k-1}\left(\frac{\partial}{\partial\rho_0}\right) - \frac{\partial}{\partial\rho_0}\right)\right] \\ &= \sum_{i=0}^{\infty}\rho_{k-1,ii}\left[\left(\frac{\partial h}{\partial\rho_{0,i+k-1,i+k-1}} - \frac{\partial h}{\partial\rho_{0,ii}}\right)\frac{\partial}{\partial\rho_{i+k-1,i}}\right. \\ &\quad \left. - \frac{\partial h}{\partial\rho_{k-1,ii}}\left(\frac{\partial}{\partial\rho_{0,i+k-1,i+k-1}} - \frac{\partial}{\partial\rho_{0,ii}}\right)\right]\end{aligned} \tag{5.10}$$

for $f,h \in C^\infty(B^1_{-,k})$. Like in Section 4, in (5.10) we have used the standard coordinate conventions from finite-dimensions to write a vector field. The precise meaning of the symbols $\partial/\partial\rho_{k-1} = \{\partial/\partial\rho_{i+k-1,i}\}_{i=0}^\infty$ and $\partial/\partial\rho_0 = \{\partial/\partial_{0,ii}\}_{i=0}^\infty$ is that they form the Schauder basis of the tangent space $T_\rho B^1_{-,k}$ corresponding to the Schauder basis $\{|i+k-1\rangle\langle i|, |i\rangle\langle i|\}_{i=0}^\infty$ of $B^1_{-,k}$. Thus Hamilton's equations in terms of diagonal operators are

$$\frac{d}{dt}\rho_0 = \rho_{k-1}\frac{\partial h}{\partial\rho_{k-1}} - \tilde{s}^{k-1}\left(\rho_{k-1}\frac{\partial h}{\partial\rho_{k-1}}\right), \tag{5.11}$$

$$\frac{d}{dt}\rho_{k-1} = \rho_{k-1}\left(s^{k-1}\left(\frac{\partial h}{\partial\rho_0}\right) - \frac{\partial h}{\partial\rho_0}\right) \tag{5.12}$$

or, in coordinates, for $i \in \mathbb{N} \cup \{0\}$, $k \geqslant 2$,

$$\frac{d}{dt}\rho_{0,ii} = \rho_{k-1,ii}\frac{\partial h}{\partial \rho_{k-1,ii}} - \rho_{k-1,i-k+1,i-k+1}\frac{\partial h}{\partial \rho_{k-1,i-k+1,i-k+1}}, \tag{5.13}$$

$$\frac{d}{dt}\rho_{k-1,ii} = \rho_{k-1,ii}\left(\frac{\partial h}{\partial \rho_{0,i+k-1,i+k-1}} - \frac{\partial h}{\partial \rho_{0,ii}}\right). \tag{5.14}$$

Structure of the generic coadjoint orbit. A coadjoint orbit

$$\mathcal{O}_\nu := \left\{ \left(\mathrm{Ad}^{+,k}\right)^*_{g^{-1}} \nu \;\middle|\; g \in GB^\infty_{+,k} \right\},$$

through the element $\nu = \nu_0 + (S^T)^{k-1}\nu_{k-1} \in B^1_{-,k}$ is said to be *generic* if $\nu_{k-1,ii} \neq 0$ for $i = 0, 1, 2, \ldots$.

Let us denote by $GL_0^{\infty,k-1}$ the Banach Lie subgroup of $(k-1)$-periodic elements of GL_0^∞, that is, $g_0 \in GL_0^{\infty,k-1}$ if and only if $s^{k-1}(g_0) = g_0$. Denote by $L_0^{\infty,k-1}$ the Banach Lie algebra of $GL_0^{\infty,k-1}$.

Proposition 5.1.

(i) *One has the following equalities*

$$Z\left(GB^\infty_{+,k}\right) = \left(GB^\infty_{+,k}\right)_\nu = GL_0^{\infty,k-1}, \tag{5.15}$$

where $Z(GB^\infty_{+,k})$ is the center of $GB^\infty_{+,k}$ and $(GB^\infty_{+,k})_\nu$ is the stabilizer of the generic element $\nu \in B^1_{-,k}$.

(ii) *The generic orbit*

$$\mathcal{O}_\nu \cong GB^\infty_{+,k}/GL_0^{\infty,k-1} \tag{5.16}$$

is a Banach Lie group.

(iii) *One has the relation*

$$\mathcal{O}_\nu = \nu_0 + \mathcal{O}_{(S^T)^{k-1}\nu_{k-1}} \tag{5.17}$$

between the coadjoint orbits through $\nu = \nu_0 + (S^T)^{k-1}\nu_{k-1}$ and through $(S^T)^{k-1}\nu_{k-1}$.

Proof. Part (i) follows from a direct verification. Since $GL_0^{\infty,k-1}$ is a normal Banach Lie group of $GB^\infty_{+,k}$ the quotient $GB^\infty_{+,k}/GL_0^{\infty,k-1}$ is also a Banach Lie group (see [6]). This proves (ii). Part (iii) follows from (5.7). □

We conclude from (5.17) that to describe any $\mathcal{O}_\nu$ it suffices to study coadjoint orbits through the $(k-1)$-lower diagonal elements, $k \geqslant 2$.

Since the Banach Lie group $GB^\infty_{+,k}$ and the generic element $\nu \in B^1_{-,k}$ satisfy all the hypotheses of Theorems 7.3 and 7.4 in [12] we conclude:

- The map $\iota_\nu : GB^\infty_{+,k}/GL_0^{\infty,k-1} \to B^1_{-,k}$ given by $\iota_\nu([g]) := (\mathrm{Ad}^{+,k})^*_{g^{-1}}\nu$ is a weak injective immersion. This means that its derivative is injective but no conditions on the closedness of its range or the fact that it splits are imposed. The map ι_ν is not an immersion as we now show by using Theorem 7.5 in [12].
  Since the coadjoint stabilizer Lie algebra $(B^\infty_{+,k})_\nu$ is equal to the center

$$Z\big(B^\infty_{+,k}\big) = \big\{x = x_0 + x_{k-1}S^{k-1} \in B^\infty_{+,k} \bigm| s^{k-1}(x_0) = x_0,\ x_{k-1} = 0\big\}$$

  it follows that its annihilator is

$$\begin{aligned}\big(\big(B^\infty_{+,k}\big)_\nu\big)^\circ = \big\{\rho = \rho_0 + \big(S^T\big)^{k-1}\rho_{k-1} \in B^1_{-,k} \bigm| \mathrm{Tr}(x_0\rho_0) = 0,\\ \text{for all } x_0 \in L^\infty_0 \text{ such that } s^{k-1}(x_0) = x_0\big\}.\end{aligned}$$

  Because

$$\mathrm{Tr}\big(x_0\big(\big(\mathrm{ad}^{+,k}\big)^*_x\nu\big)_0\big) = \mathrm{Tr}\big(x_0\big(\mathrm{ad}^{+,k}\big)^*_x\nu\big) = \mathrm{Tr}\big([x_0, x]_k\nu\big) = 0$$

  for any $x_0 \in Z(B^\infty_{+,k})$ and any $x \in B^\infty_{+,k}$, we have $S_\nu \subset ((B^\infty_{+,k})_\nu)^\circ$, where

$$S_\nu := \big\{\big(\mathrm{ad}^{+,k}\big)^*_x\nu \bigm| x \in B^\infty_{+k}\big\}$$

  is the characteristic subspace of the Banach Lie–Poisson structure of $B^1_{-,k}$ at ν. Moreover, the bounded operator $K_\nu : x \in B^\infty_{+,k} \mapsto (\mathrm{ad}^{+,k})^*_x\nu \in B^1_{-,k}$ has non-closed range $\mathrm{im}\, K_\nu = S_\nu$ and thus the inclusion $S_\nu \subset ((B^\infty_{+,k})_\nu)^\circ$ is strict. To see that the range of K_ν is not closed, one uses the Banach space isomorphisms $B^1_{-,k} \cong \ell^1 \times \ell^1$ and $B^\infty_{+,k} \cong \ell^\infty \times \ell^\infty$ and shows that the two components of K_ν are both bounded linear operators with non-closed range. Therefore, since Theorem 7.5 in [12] states that ι_ν is an immersion if and only if $S_\nu = ((B^\infty_{+,k})_\nu)^\circ$, this argument shows that ι_ν is only a weak immersion.
- The quotient space $GB^\infty_{+,k}/GL_0^{\infty,k-1}$ is a weak symplectic Banach manifold relative to the closed two-form

$$\begin{aligned}&\omega_\nu\big([g]\big)\big(T_g\pi(g \circ_k x), T_g\pi(g \circ_k y)\big) = \mathrm{Tr}\big(\nu[x, y]_k\big)\\ &\quad = \sum_{i=0}^\infty \nu_{k-1,ii}\big(x_{k-1,ii}(y_{0,i+k-1,i+k-1} - y_{ii}) - y_{k-1,ii}(x_{0,i+k-1,i+k-1} - x_{0,ii})\big),\end{aligned} \tag{5.18}$$

  where $x, y \in B^\infty_{+,k}$, $g \in GB^\infty_{+,k}$, $[g] := \pi(g)$, $\pi : GB^\infty_{+,k} \to GB^\infty_{+,k}/GL_0^{\infty,k-1}$ is the canonical projection, and $T_g\pi : T_gGB^\infty_{+,k} \to T_{[g]}(GB^\infty_{+,k}/GL_0^{\infty,k-1})$ is its derivative at g. In this formula we have used the fact that the value at g of the left invariant vector field ξ_x on $GB^\infty_{+,k}$ generated by x is $g \circ_k x$.
- Relative to the Banach manifold structure on $\mathcal{O}_\nu$ making $\iota_\nu : GB^\infty_{+,k}/GL_0^{\infty,k-1} \longrightarrow \mathcal{O}_\nu$ into a diffeomorphism, the push forward of the weak symplectic form (5.18) has the expression

$$\omega_{\mathcal{O}_\nu}(\rho)\left(\left(\mathrm{ad}^{+,k}\right)^*_x\rho, \left(\mathrm{ad}^{+,k}\right)^*_y\rho\right) = \mathrm{Tr}\left(\rho[x,y]_k\right)$$
$$= \sum_{i=0}^{\infty} \rho_{k-1,ii}\left(x_{k-1,ii}(y_{0,i+k-1,i+k-1} - y_{0,ii}) - y_{k-1,ii}(x_{0,i+k-1,i+k-1} - x_{0,ii})\right), \quad (5.19)$$

where $x, y \in B^\infty_{+,k}$ and $\rho \in \mathcal{O}_\nu$.

We shall express the pull back $\pi^*\omega_\nu$ of the weak symplectic form ω_ν in terms of the diagonal operators represented by $\{g_{0,ii}\}_{i=0}^\infty \in \ell^\infty$ and $\{g_{k-1,ii}\}_{i=0}^\infty \in \ell^\infty$ defining the element $g \in GB^\infty_{+,k}$. If $x = x_0 + x_{k-1}S^{k-1}$, $y = y_0 + y_{k-1}S^{k-1} \in B^\infty_{+,k}$, and $\nu = \nu_0 + (S^T)^{k-1}\nu_{k-1} \in B^1_{-,k}$, (5.18) yields

$$\left(\pi^*\omega_\nu\right)(g)(g \circ_k x, g \circ_k y) = \omega_\nu\left([g]\right)\left(T_g\pi(g \circ_k x), T_g\pi(g \circ_k y)\right) = \mathrm{Tr}\left(\nu[x,y]_k\right)$$
$$= \sum_{i=0}^{\infty} \nu_{k-1,ii}\left(x_{k-1,ii}(y_{0,i+k-1,i+k-1} - y_{0,ii}) - y_{k-1,ii}(x_{0,i+k-1,i+k-1} - x_{0,ii})\right), \quad (5.20)$$

where ν_{k-1} has the diagonal entries $\{\nu_{k-1,ii}\}_{i=0}^\infty$. The left-invariant vector field ξ_x on $GB^\infty_{+,k}$ generated by x has the expression

$$\xi_x = \sum_{i=0}^{\infty} g_{0,ii}x_{0,ii}\frac{\partial}{\partial g_{0,ii}} + \sum_{i=0}^{\infty}(g_{0,ii}x_{k-1,ii} + g_{k-1,ii}x_{0,i+k-1,i+k-1})\frac{\partial}{\partial g_{k-1,ii}}.$$

The symbols $\{\partial/\partial g_{0,ii}, \partial/\partial g_{k-1,ii}\}_{i=0}^\infty$ denote the biorthogonal family in the tangent space $T_g B^\infty_{+,k}$ corresponding to the standard biorthogonal family $\{|i\rangle\langle i|, |i\rangle\langle i+k-1|\}_{i=0}^\infty$ in $B^\infty_{+,k}$. We shall use, as in finite-dimensions, the exterior derivative on real-valued smooth functions, in particular coordinates, to represent elements in the dual space. With this convention, we have

$$\pi^*\omega_\nu = \sum_{i=0}^{\infty} \mathbf{d}\log g_{0,ii} \wedge \mathbf{d}\left(\nu_{k-1,ii}\frac{g_{k-1,ii}}{g_{0,ii}} - \nu_{k-1,ii}\frac{g_{k-1,ii}}{g_{0,i-k+1,i-k+1}}\right), \quad (5.21)$$

where, as usual, any element with negative index is set equal to zero. To show this, we evaluate the right-hand side of (5.21) on ξ_x and ξ_y and observe that it equals the right-hand side of (5.20). Note that the computations make sense since $\nu_{k-1} \in \ell^1$.

The action of the coadjoint isotropy subgroup $(GB^\infty_{+,k})_\nu = GL_0^{\infty,k-1}$ on $GB^\infty_{+,k}$ is given by $g_{0,ii} \mapsto h_{0,ii}g_{0,ii}$, $g_{k-1,ii} \mapsto h_{0,ii}g_{k-1,ii}$, where $h_{0,ii} = h_{0,i+k-1,i+k-1}$. As expected, the right-hand side of (5.21) is invariant under this transformation and its interior product with any tangent vector to the orbit of the normal subgroup $GL_0^{\infty,k-1}$ is zero. This shows, once again, that (5.21) naturally descends to the quotient group $GB^\infty_{+,k}/GL_0^{\infty,k-1}$.

In order to understand the structure of $\mathcal{O}_\nu$, define the action $\alpha^k : GB^\infty_{+,k} \times L^1_{-k+1} \to L^1_{-k+1}$ by

$$\alpha^k_g\left(\left(S^T\right)^{k-1}\nu_{k-1}\right) := \left(S^T\right)^{k-1}s^{k-1}(g_0)g_0^{-1}\nu_{k-1}. \quad (5.22)$$

The projector $\delta^k : B^1_{-,k} \to L^1_{-k+1}$ defined by the splitting $B^1_{-,k} = L^1_{-k+1} \oplus L^1_0$ is a $GB^\infty_{+,k}$- equivariant map relative to the coadjoint and the α^k-actions of $GB^\infty_{+,k}$, that is, the diagram

$$\begin{tikzcd} B^1_{-,k} \arrow[r, "(\mathrm{Ad}^{-,k})^*_{g^{-1}}"] \arrow[d, "\delta^k"'] & B^1_{-,k} \arrow[d, "\delta^k"] \\ L^1_{-k+1} \arrow[r, "\alpha^k_g"] & L^1_{-k+1} \end{tikzcd}$$

commutes for any $g \in GB^\infty_{+,k}$. We observe that the stabilizer $GL_0^{\infty,k-1}$ of the α^k-action does not depend on the choice of the generic element $(S^T)^{k-1}\nu_{k-1} \in L^1_{k-1}$. The orbits of the coadjoint action of the subgroup $GL_0^{\infty,k-1}$ on $(\delta^k)^{-1}((S^T)^{k-1}\nu_{k-1})$ are of the form

$$\Delta_{\nu_0,\nu_{k-1}} + \left(S^T\right)^{k-1}\nu_{k-1} \subset \left(\delta^k\right)^{-1}\left(\left(S^T\right)^{k-1}\nu_{k-1}\right) \subset B^1_{-,k},$$

where

$$\Delta_{\nu_0,\nu_{k-1}} := \nu_0 + \operatorname{im}\mathcal{N}_{\nu_{k-1}} \subset L^1_0 \tag{5.23}$$

are affine spaces for each $\nu_0 \in L^1_0$ and the linear operator $\mathcal{N}_{\nu_{k-1}} : L^\infty_0 \to L^1_0$ is defined by

$$\mathcal{N}_{\nu_{k-1}}(g_{k-1}) := \nu_{k-1}g_{k-1} + \tilde{s}(\nu_{k-1}g_{k-1})\left(\mathbb{I} - \sum_{j=0}^{k-2} p_j\right).$$

The orbits of the α^k-action of $GB^\infty_{+,k}$ on L^1_{-k+1} are

$$GB^\infty_{+,k} \cdot \left(\left(S^T\right)^{k-1}\nu_{k-1}\right) = \left\{\left(S^T\right)^{k-1} s^{k-1}(g_0)g_0^{-1}\nu_{k-1} \;\middle|\; g_0 \in GL^\infty_0\right\} =: \Delta_{\nu_{k-1}}. \tag{5.24}$$

Note that if $\Delta_{\nu_{k-1}} = \Delta_{\nu'_{k-1}}$ then $\operatorname{im}\mathcal{N}_{\nu_{k-1}} = \operatorname{im}\mathcal{N}_{\nu'_{k-1}}$ and so $\Delta_{\nu_0,\nu_{k-1}} = \Delta_{\nu_0,\nu'_{k-1}}$. These remarks show that the coadjoint orbit $\mathcal{O}_\nu$ is diffeomorphic to the product $(\nu_0 + \operatorname{im}\mathcal{N}_{\nu_{k-1}}) \times \Delta_{\nu_{k-1}}$ of the affine space $\Delta_{\nu_0,\nu_{k-1}}$ with the α^k-orbit $\Delta_{\nu_{k-1}}$. This diffeomorphism does not depend on the choice of $(S^T)^{k-1}\nu'_{k-1} \in \Delta_{\nu_{k-1}}$. Additionally, one identifies the set of generic coadjoint orbits with the total space $\mathbb{L}_k$ of the vector bundle $\mathbb{L}_k \to L^\infty_0/\alpha^k(GL^\infty_0)$, whose fiber at $[\nu_{k-1}]$ is $L^1_0/\operatorname{im}\mathcal{N}_{\nu_{k-1}}$. The vector space $L^1_0/\operatorname{im}\mathcal{N}_{\nu_{k-1}}$ is not Banach since $\operatorname{im}\mathcal{N}_{\nu_{k-1}}$ is not closed in L^1_0 because the operator $\mathcal{N}_{\nu_{k-1}} : L^\infty_0 \to L^1_0$ is compact. Consequently, the bundle $\mathbb{L}_k \to L^\infty_0/\alpha^k(GL^\infty_0)$ does not have the structure of a Banach vector bundle and does not have fixed typical fiber.

The momentum map. Let us now study an important particular case of the map $\mathcal{J}_{\nu_-}$ by taking in (4.17) the element $\nu_- = (S^T)^{k-1}\nu_{k-1} \in L^1_{-k+1} \subset L^1_{-,k}$. The map (4.17), denoted in this case $\mathcal{J}_{\nu_{k-1}} : \ell^\infty \times \ell^1 \to B^1_{-,k}$, becomes

$$\mathcal{J}_{\nu_{k-1}}(\mathbf{q}, \mathbf{p}) = \mathbf{p} + \left(S^T\right)^{k-1}\nu_{k-1}e^{s^{k-1}(\mathbf{q})-\mathbf{q}}. \tag{5.25}$$

Recall that we identify ℓ^1 with L_0^1 and ℓ^∞ with L_0^∞. Having fixed $(S^T)^{k-1}\nu_{k-1} \in L^1_{-k+1}$, define the action of $GB^\infty_{+,k}$ on $\ell^\infty \times \ell^1$ by

$$\begin{aligned}&\boldsymbol{\sigma}_g^{\nu_{k-1}}(\mathbf{q},\mathbf{p})\\ &\quad := \left(\mathbf{q} + \log g_0, \mathbf{p} + g_{k-1}g_0^{-1}\nu_{k-1}e^{s^{k-1}(\mathbf{q})-\mathbf{q}} - \tilde{s}^{k-1}\left(g_{k-1}g_0^{-1}\nu_{k-1}e^{s^{k-1}(\mathbf{q})-\mathbf{q}}\right)\right), \quad (5.26)\end{aligned}$$

where $g := g_0 + g_{k-1}S^{k-1} \in GB^\infty_{+,k}$ and $(\mathbf{q},\mathbf{p}) \in \ell^\infty \times \ell^1$. The coordinate form of the action (5.26) is

$$q_i' = q_i + \log g_{0,ii}, \tag{5.27}$$

$$p_i' = p_i + \frac{g_{k-1,ii}}{g_{0,ii}}\nu_{k-1,ii}e^{q_{k+1}-q_k} - \frac{g_{k-1,ii}}{g_{0,k-1,k-1}}\nu_{k-1,ii}e^{q_k-q_{k-1}} \tag{5.28}$$

for $i \in \mathbb{N} \cup \{0\}$. Using (5.27) and (5.28) one shows that

$$\sum_{i=0}^\infty p_i'\mathbf{d}q_i' = \sum_{i=0}^\infty p_i\mathbf{d}q_i - \mathbf{d}Q,$$

where the function $Q : \ell^\infty \to \mathbb{R}$ is given by

$$Q(\mathbf{q}) := \operatorname{Tr}\left(g_0^{-1}g_{k-1}\nu_{k-1}e^{s^{k-1}(\mathbf{q})-\mathbf{q}}\right) = \sum_{i=0}^\infty \frac{g_{k-1,ii}}{g_{0,ii}}\nu_{k-1,ii}e^{q_{k+1}-q_k}. \tag{5.29}$$

Thus we see that ω is invariant relative to the $\boldsymbol{\sigma}^{\nu_{k-1}}$-action, that is, $(\boldsymbol{\sigma}_g^{\nu_{k-1}})^*\omega = \omega$ for any $g \in GB^\infty_{+,k}$.

Proposition 5.2. *The smooth map $\mathcal{J}_{\nu_{k-1}} : \ell^\infty \times \ell^1 \to B^1_{-,k}$ given by* (5.25) *is constant on the $\boldsymbol{\sigma}^{\nu_{k-1}}$-orbits of the subgroup $GL_0^{\infty,k-1}$. In addition*:

- (i) *$\mathcal{J}_{\nu_{k-1}}$ is a momentum map. More precisely, $\{f \circ \mathcal{J}_{\nu_{k-1}}, g \circ \mathcal{J}_{\nu_{k-1}}\}_\omega = \{f, g\}_{0,k-1} \circ \mathcal{J}_{\nu_{k-1}}$, for all $f, g \in C^\infty(B^1_{-,k})$, where $\{\cdot,\cdot\}_\omega$ is the canonical Poisson bracket of the weak symplectic Banach space $(\ell^\infty \times \ell^1, \omega)$ given by the weak symplectic form* (4.20) *and $\{,\}_{0,k-1}$ is the Lie–Poisson bracket on $B^1_{-,k}$ given by* (5.9).
- (ii) *$\mathcal{J}_{\nu_{k-1}}$ is $GB^\infty_{+,k}$-equivariant, that is, $\mathcal{J}_{\nu_{k-1}} \circ \boldsymbol{\sigma}_g^{\nu_{k-1}} = (\mathrm{Ad}^{-,k})^*_{g^{-1}} \circ \mathcal{J}_{\nu_{k-1}}$ for any $g \in GB^\infty_{+,k}$.*
- (iii) *$\mathcal{J}_{\nu_{k-1}}(\ell^\infty \times \ell^1) = (\delta^k)^{-1}(\Delta_{\nu_{k-1}})$, $\mathcal{J}_{\nu_{k-1}}(\ell^\infty \times \{0\}) = \Delta_{\nu_{k-1}}$, and hence $(\ell^\infty \times \ell^1)/\boldsymbol{\sigma}^{\nu_{k-1}}(GL_0^{\infty,k-1}) \cong \mathcal{J}_{\nu_{k-1}}(\ell^\infty \times \ell^1)$ consists of those coadjoint orbits which are projected by δ^k to the α^k-orbit $\Delta_{\nu_{k-1}}$.*

Proof. To prove (i), let $f, g \in C^\infty(B^1_{-,k})$ and notice that

$$\frac{\partial(f \circ \mathcal{J}_{\nu_{k-1}})}{\partial\mathbf{q}} \in \left(L^\infty\right)^* \quad \text{and} \quad \frac{\partial(f \circ \mathcal{J}_{\nu_{k-1}})}{\partial\mathbf{p}} \in \left(L^1\right)^* = L^\infty$$

because $\mathbf{q} \in L^\infty$ and $\mathbf{p} \in L^1$. However, by (5.25),

$$\frac{\partial(f \circ \mathcal{J}_{\nu_{k-1}})}{\partial \mathbf{q}} = \left(\frac{\partial f}{\partial \rho_{k-1}} \circ \mathcal{J}_{\nu_{k-1}}\right)(\mathbf{q}, \mathbf{p})(\rho_{k-1} \circ \mathcal{J}_{\nu_{k-1}})(\mathbf{q}, \mathbf{p})\big(S^{k-1} - \mathbb{I}\big) \in L^1 \tag{5.30}$$

since $(\rho_{k-1} \circ \mathcal{J}_{\nu_{k-1}})(\mathbf{q}, \mathbf{p}) \in L^1$ and

$$\frac{\partial(f \circ \mathcal{J}_{\nu_{k-1}})}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial \rho_0} \circ \mathcal{J}_{\nu_{k-1}}\right)(\mathbf{q}, \mathbf{p}) \in L^\infty. \tag{5.31}$$

Note that (5.30) and (5.31) imply that $\{f \circ \mathcal{J}_{\nu_{k-1}}, h \circ \mathcal{J}_{\nu_{k-1}}\}_\omega$ makes sense for any $f, h \in C^\infty(B^1_{-,k})$.

Thus, using the formula for the canonical bracket on the weak symplectic Banach space $(\ell^\infty \times \ell^1, \omega)$ and the fact that the duality pairing $(L^\infty)^* \times L^\infty \to \mathbb{R}$ restricted to $L^1 \times L^\infty$ equals to the trace of the product of operators, we get

$$\begin{aligned}
&\{f \circ \mathcal{J}_{\nu_{k-1}}, g \circ \mathcal{J}_{\nu_{k-1}}\}_\omega(\mathbf{q}, \mathbf{p}) \\
&\quad = \left\langle \frac{\partial(f \circ \mathcal{J}_{\nu_{k-1}})}{\partial \mathbf{q}}, \frac{\partial(g \circ \mathcal{J}_{\nu_{k-1}})}{\partial \mathbf{p}} \right\rangle - \left\langle \frac{\partial(g \circ \mathcal{J}_{\nu_{k-1}})}{\partial \mathbf{q}}, \frac{\partial(f \circ \mathcal{J}_{\nu_{k-1}})}{\partial \mathbf{p}} \right\rangle \\
&\quad = \operatorname{Tr}\Bigg[(\rho_{k-1} \circ \mathcal{J}_{\nu_{k-1}})(\mathbf{q}, \mathbf{p})\Bigg(\big(S^{k-1} - \mathbb{I}\big)\left(\frac{\partial g}{\partial \rho_0} \circ \mathcal{J}_{\nu_{k-1}}\right)(\mathbf{q}, \mathbf{p})\left(\frac{\partial f}{\partial \rho_{k-1}} \circ \mathcal{J}_{\nu_{k-1}}\right)(\mathbf{q}, \mathbf{p}) \\
&\qquad - \big(S^{k-1} - \mathbb{I}\big)\left(\frac{\partial f}{\partial \rho_0} \circ \mathcal{J}_{\nu_{k-1}}\right)(\mathbf{q}, \mathbf{p})\left(\frac{\partial g}{\partial \rho_{k-1}} \circ \mathcal{J}_{\nu_{k-1}}\right)(\mathbf{q}, \mathbf{p})\Bigg)\Bigg] \\
&\quad = \big(\{f, g\}_{0,k-1} \circ \mathcal{J}_{\nu_{k-1}}\big)(\mathbf{q}, \mathbf{p})
\end{aligned}$$

by (5.9).

Parts (ii) and (iii) are proved by direct verifications. □

Let us define the map $\Phi^{\nu_{k-1}}(g) : GB^\infty_{+,k} \to \ell^\infty \times \ell^1$ by

$$\Phi^{\nu_{k-1}}(g) := \boldsymbol{\sigma}_g^{\nu_{k-1}}(\mathbf{0}, \mathbf{0}), \tag{5.32}$$

or, in coordinates,

$$\Phi^{\nu_{k-1}}(g_0, g_{k-1}) = \big(\log g_0, g_{k-1} g_0^{-1} \nu_{k-1} - \tilde{s}^{k-1}\big(g_{k-1} g_0^{-1} \nu_{k-1}\big)\big), \tag{5.33}$$

which shows that $\Phi^{\nu_{k-1}}$ is smooth and injective.

Proposition 5.3. *The following diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & GL_0^{\infty,k-1} & \longrightarrow & GB^\infty_{+,k} & \xrightarrow{\pi} & GB^\infty_{+,k}/GL_0^{\infty,k-1} & \longrightarrow & 1 \\
 & & & & \Big\downarrow{\scriptstyle \Phi^{\nu_{k-1}}} & & \Big\downarrow{\scriptstyle \iota_{\nu_{k-1}}} & & \\
0 \times \{\mathbf{p}\} & \longrightarrow & L_0^{\infty,k-1} \times \{\mathbf{p}\} & \longrightarrow & \ell^\infty \times \ell^1 & \xrightarrow{\mathcal{J}_{\nu_{k-1}}} & (\delta^k)^{-1}(\Delta_{\nu_{k-1}}) & \longrightarrow & 0
\end{array}$$

commutes. The first row is an exact sequence of Banach Lie groups. The second row is also exact in the following sense: the map $\mathcal{J}_{\nu_{k-1}}$ is onto and its level sets are all of the form $L_0^{\infty,k-1} \times \{\mathbf{p}\}$, where $\mathbf{p} \in L_0^1$. In addition,

$$\left(\Phi^{\nu_{k-1}}\right)^* \omega = \pi^* \omega_{\nu_{k-1}}, \tag{5.34}$$

where ω and $\omega_{\nu_{k-1}}$ are the weak symplectic forms (4.19) *and* (5.21) *on $\ell^\infty \times \ell^1$ and $GB^\infty_{+,k}/GL_0^{\infty,k-1}$, respectively. We also have*

$$\Phi^{\nu_{k-1}}\left(\pi^{-1}\left([g]\right)\right) = \mathcal{J}^{-1}_{\nu_{k-1}}\left(\iota_{\nu_{k-1}}\left([g]\right)\right) \tag{5.35}$$

for any $g \in GI^\infty_{+,0,k-1}$.

Proof. Commutativity is verified using (5.7), (5.25), and (5.32). The identities (5.34) and (5.35) are obtained by direct verifications. □

Remarks. (i) The analysis of the coadjoint orbit $\mathcal{O}_\nu \cong GB^\infty_{+,k}/GL_0^{\infty,k-1}$ through the generic element $\nu \in B^1_{-,k}$ carried out in this section shows that it is diffeomorphic to $\Delta_{\nu_0,\nu_{k-1}} \times \Delta_{\nu_{k-1}}$. For an arbitrary $(\nu_0', \nu_{k-1}') \in \Delta_{\nu_0,\nu_{k-1}} \times \Delta_{\nu_{k-1}}$, the manifolds $\iota^{-1}_{\nu_{k-1}}(\{\nu_0'\} \times \Delta_{\nu_{k-1}})$ and $\iota^{-1}_{\nu_{k-1}}(\Delta_{\nu_0,\nu_{k-1}} \times \{\nu_{k-1}'\})$ are Lagrangian submanifolds in the sense that their tangent spaces are maximal isotropic.

(ii) If $k = 2$ we have $B^1_{-,2} = L^1_{-,2}$ and $GB^\infty_{+,2} = GL^\infty_{+,2}$. If, in addition, we consider the finite-dimensional case, that is, instead of $L^1_{-,2}$ we work with the traceless $n \times n$ matrices having non-zero entries only on the main and the first lower diagonals, then $\mathcal{J}_{\nu_1}$ is a symplectic diffeomorphism of $\mathbb{R}^{2(n-1)}$, endowed with the canonical symplectic structure, with a single coadjoint orbit of the upper bidiagonal group through a strictly lower diagonal element all of whose entries are non-zero.

(iii) If $k = 2$ and we consider the generic infinite-dimensional case, that is, ν_1 has all entries different from zero, then the map $\mathcal{J}_{\nu_1}$ does not provide a morphism of weak symplectic manifolds between $\ell^\infty \times \ell^1$ and a single coadjoint orbit of $GL^\infty_{+,2}$. The relation between these spaces is more complicated and is explained in the diagram of Proposition 5.3. Each $GL^\infty_{+,2}$-coadjoint orbit through a generic element $S^T\nu_1$ is only weakly symplectic and Poisson injectively weakly immersed in $L^1_{-,2}$ but not equal to it.

(iv) If $k = 2$ and we consider the infinite-dimensional case with ν_1 having also some vanishing entries, the structure of the $GL^\infty_{+,2}$-coadjoint orbit through $S^T\nu_1$ reduces to the two previous cases as we shall explain below. Let i_0 be the first index for which the entry $\nu_{1,i_0i_0} = 0$. Formula (5.7) shows that the first $i_0 \times i_0$ block of $\mathcal{O}_{S^T\nu_1}$ is that of a finite-dimensional orbit of the upper bidiagonal group of matrices of size $i_0 \times i_0$ and that the coadjoint action preserves this block. Let i_1 be the next index for which $\nu_{1,i_1i_1} = 0$. Again by (5.7) it follows that there is an $i_1 \times i_1$ block of $\mathcal{O}_{S^T\nu_1}$ that is preserved by the coadjoint action and that is equal to a finite-dimensional orbit of the upper bidiagonal group of matrices of size $i_1 \times i_1$. Continuing in this fashion we arrive either at an infinite sequence of orbits of finite-dimensional upper bidiagonal groups (in the case that there is an infinity of indices i_s such that $\nu_{1,i_si_s} = 0$, $s \in \mathbb{N} \cup \{0\}$) or to a generic infinite dimensional orbit of $GL^\infty_{+,2}$ (if there are only finitely many indices i_s, $s = 0, 1, \ldots, r$, such that $\nu_{1,i_si_s} = 0$). In the latter case, the last infinite block is preserved by the coadjoint action and we are in the generic case of an orbit of $GL^\infty_{+,2}$ but on the space complementary to the $r + 1$ finite-dimensional blocks of sizes $i_0 \times i_0, \ldots, i_r \times i_r$. Thus, decomposing the orbit as described, the problem of classification of the general $GL^{\infty}_{+,2}$-coadjoint orbit is reduced to the finite-dimensional case and to the generic infinite-dimensional case.

(v) One can restrict the Hamiltonians $I_l^{S,k}$ given by (4.33) to $B^1_{-,k}$ but these functions are not in involution because the inclusion of $B^1_{-,k}$ in $L^1_{-,k}$ is not Poisson. Indeed, as recalled in Section 2, the inclusion would be Poisson if and only if the kernel of its dual map is an ideal in $L^{\infty}_{+,k}$ which is easily seen to be false unless $k=2$, in which case we have

$$\left(I_1^{S,2}\circ\mathcal{J}_{\nu_1}\right)(\mathbf{q},\mathbf{p})=\sum_{i=0}^{\infty}p_i \tag{5.36}$$

and

$$H_2(\mathbf{q},\mathbf{p}):=\left(I_2^{S,2}\circ\mathcal{J}_{\nu_1}\right)(\mathbf{q},\mathbf{p})=\frac{1}{2}\sum_{i=0}^{\infty}p_i^2+\sum_{i=0}^{\infty}\nu_{1,ii}^2e^{2(q_{i+1}-q_i)}. \tag{5.37}$$

The function H_2 is, up to a renormalization of constants, the Hamiltonian of the semi-infinite Toda lattice. The first integral $I_1^{S,2}\circ\mathcal{J}_{\nu_1}$ is the total momentum of the system which generates the translation action given by the subgroup $\mathbb{R}_+\mathbb{I}$. All integrals $I_l^{S,2}\circ\mathcal{J}_{\nu_1}$, $l\in\mathbb{N}$, give the full Toda lattice hierarchy on $\ell^{\infty}\times\ell^1$ as we shall see in the next paragraph.

These considerations show that the momentum map $\mathcal{J}_{\nu_1}:\ell^{\infty}\times\ell^1\to I^1_{-,0,1}=L^1_{-,2}$ is an infinite-dimensional analogue of the Flaschka map (see [7]). In fact, as we shall see, $\mathcal{J}_{\nu_1}$ plays the role of the Flaschka map for the system of integrals in involution (4.33) for $k\geqslant 2$.

The semi-infinite Toda lattice. In this example we illustrate the theory of the k-diagonal Hamiltonian systems by the detailed investigation of the semi-infinite Toda lattice which is an example of a bidiagonal system (see Remark (v) at the end of Section 5). We shall follow the method of orthogonal polynomials first proposed in [5], as far as we know. We shall extend below the results in [10] for the finite Toda lattice by explicitly solving the semi-infinite Toda lattice including the construction of action–angle variables.

The family of Hamiltonians $I_l^{S,2}\in C^{\infty}(L^1_{S,2})$, $l\in\mathbb{N}$, leads to the chain of Hamilton equations

$$\frac{\partial}{\partial t_l}\boldsymbol{\rho}=[\boldsymbol{\rho},B_l],\quad \text{where } B_l:=P_-^{\infty}\left(\boldsymbol{\rho}^l\right)-\left(P_-^{\infty}\left(\boldsymbol{\rho}^l\right)\right)^T, \tag{5.38}$$

on the Banach Lie–Poisson space $(L^1_{S,2},\{\cdot,\cdot\}_{S,2})$ (or on the space $(L^1_{-,2},\{\cdot,\cdot\}_2)$ isomorphic to it) induced from (4.24) by the inclusion $\iota_{S,2}:L^1_{S,2}\hookrightarrow L^1_S$.

The selfadjoint trace class operator $\boldsymbol{\rho}\in L^1_{S,2}$ acts on the orthonormal basis $\{|k\rangle\}_{k=0}^{\infty}$ of $\mathcal{H}$ as follows:

$$\boldsymbol{\rho}|k\rangle=\rho_{k-1,k}|k-1\rangle+\rho_{kk}|k\rangle+\rho_{k,k+1}|k+1\rangle, \tag{5.39}$$

where $k\in\mathbb{N}\cup\{0\}$ and we set $\rho_{-1,0}=0$.

Note that if $\boldsymbol{\rho}$ is replaced by $\boldsymbol{\rho}':=c\boldsymbol{\rho}+b\mathbb{I}$, where $b,c\in\mathbb{R}$, $c\neq 0$, then the equations (5.38) remain unchanged by rescaling the time $t_l':=c^{-l}t_l$. As will be explained later, the norm $\|\boldsymbol{\rho}\|_{\infty}$

and the positivity $\boldsymbol{\rho} \geqslant 0$ are preserved by the evolution defined by (5.38). Taking into account the above facts, we can assume, without loss of generality, that $\|\boldsymbol{\rho}\|_\infty < 1$ and $\boldsymbol{\rho} \geqslant 0$. Consequently, from now on we shall work with generic initial conditions $\boldsymbol{\rho}(0)$ for the Hamiltonian system (5.38), i.e.,

$$\lambda_m(0) \neq \lambda_n(0), \quad \text{for } n \neq m, \tag{5.40}$$

$$\lambda_m(0) > 0 \quad \text{and} \quad \sup_{m\in\mathbb{N}\cup\{0\}}\big\{\lambda_m(0)\big\} < 1, \tag{5.41}$$

where $\lambda_m(0)$ are the eigenvalues of $\boldsymbol{\rho}(0)$. This means that $\boldsymbol{\rho}(0)$ has simple spectrum, $\boldsymbol{\rho}(0) \geqslant 0$, and $\|\boldsymbol{\rho}(0)\|_\infty < 1$. These hypotheses imply that $\rho_{k,k+1}(0) > 0$ for all $k \in \mathbb{N}\cup\{0\}$ and are consistent with the physical interpretation of the semi-infinite Toda system. Let us denote by $\Omega^1_{-,2} \subset L^1_{S,2}$ the set consisting of operators satisfying (5.40) and (5.41).

From (5.39), it follows that

$$|k\rangle = P_k(\boldsymbol{\rho})|0\rangle, \tag{5.42}$$

where the polynomials $P_k(\lambda) \in \mathbb{R}[\lambda]$, $k \in \mathbb{N}\cup\{0\}$, are obtained by solving the three-term recurrence equation

$$\lambda P_k(\lambda) = \rho_{k-1,k}P_{k-1}(\lambda) + \rho_{kk}P_k(\lambda) + \rho_{k,k+1}P_{k+1}(\lambda) \tag{5.43}$$

with initial condition $P_0(\lambda) \equiv 1$. Note that the degree of $P_k(\lambda)$ is k.

We show now that the operator $\boldsymbol{\rho} \in L^1_{S,2}$ evolving according to (5.38) also has simple spectrum independent of all times t_l. To do this, we write the spectral resolution

$$\boldsymbol{\rho} = \sum_{m=0}^{\infty} \lambda_m \mathbb{P}_m, \qquad \mathbb{P}_m\mathbb{P}_n = \delta_{mn}\mathbb{P}_n, \qquad \sum_{m=0}^{\infty}\mathbb{P}_m = \mathbb{I}, \tag{5.44}$$

where

$$\mathbb{P}_m := \frac{|\lambda_m\rangle\langle\lambda_m|}{\langle\lambda_m|\lambda_m\rangle} \tag{5.45}$$

are the projectors on the one-dimensional eigenspaces spanned by the eigenvector $|\lambda_m\rangle$. From (5.38) one obtains

$$\left(\frac{\partial}{\partial t_l}\lambda_k\right)\mathbb{P}_n\mathbb{P}_k + (\lambda_n - \lambda_k)\left[\left(\frac{\partial}{\partial t_l}\mathbb{P}_n\right)\mathbb{P}_k - \mathbb{P}_n B_l \mathbb{P}_k\right] = 0 \tag{5.46}$$

for any $n, k \in \mathbb{N}\cup\{0\}$ and $l \in \mathbb{N}$. Putting $n = k$ in (5.46) one finds

$$\frac{\partial}{\partial t_l}\lambda_n = 0 \tag{5.47}$$

for any $n \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$. Thus $\lambda_m = \lambda_m(0) \neq \lambda_n$ for $n \neq m$ and we can conclude that the coefficients in

$$|\lambda_m\rangle = \sum_{l=0}^{\infty} P_l(\lambda_m)|l\rangle \tag{5.48}$$

are the values $P_l(\lambda_m)$ at the eigenvalue λ_m of the polynomials $P_l(\lambda)$ which are orthogonal relative to the L^2-scalar product given by the measure σ in (5.52).

Taking $n \neq k$ in (5.46) and using properties of orthogonal projectors one obtains

$$\frac{\partial}{\partial t_l}\mathbb{P}_n = [\mathbb{P}_n, B_l] \quad \text{for any } n \in \mathbb{N} \cup \{0\} \text{ and } l \in \mathbb{N}. \tag{5.49}$$

Similarly, for the resolvent

$$R_\lambda := (\boldsymbol{\rho} - \lambda\mathbb{I})^{-1} = \sum_{m=0}^{\infty} \frac{1}{\lambda_m - \lambda}\mathbb{P}_m \tag{5.50}$$

by (5.49) one has

$$\frac{\partial}{\partial t_l}R_\lambda = \sum_{m=0}^{\infty} \frac{1}{\lambda_m - \lambda}[\mathbb{P}_m, B_l] = [R_\lambda, B_l]. \tag{5.51}$$

Note that (5.42) implies that the vector $|0\rangle$ is cyclic for $\boldsymbol{\rho}$. Thus, one has a unitary isomorphism of $\mathcal{H}$ with $L^2(\mathbb{R}, d\sigma)$, where the measure

$$d\sigma(\lambda) := d\langle 0|\mathbb{P}_\lambda 0\rangle = \sum_{m=0}^{\infty} \mu_m \delta(\lambda - \lambda_m)\,d\lambda, \tag{5.52}$$

is given by the orthogonal resolution of the unity $\mathbb{P} : \mathbb{R} \ni \lambda \mapsto \mathbb{P}_\lambda \in L^\infty(\mathcal{H})$ for

$$\rho = \int_{\mathbb{R}} \lambda\, d\mathbb{P}_\lambda.$$

The *masses* μ_m in (5.52) are given by

$$\mu_m^{-1} = \langle \lambda_m|\lambda_m\rangle = \sum_{l=0}^{\infty}\big(P_l(\lambda_m)\big)^2. \tag{5.53}$$

Using $\mathbb{P}_m|0\rangle = \mu_m|\lambda_m\rangle$ and $\mu_m = \langle 0|\mathbb{P}_m 0\rangle$, one obtains from (5.49) the differential equation

$$\frac{\partial}{\partial t_l}\mu_m = 2\langle \lambda_m|B_l 0\rangle \mu_m = 2\big(\lambda_m^l - \langle 0|\boldsymbol{\rho}^l 0\rangle\big)\mu_m \tag{5.54}$$

for any $l \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$. In order to prove the second equality in (5.54) we notice that

$$B_l = \boldsymbol{\rho}^l - P_0^\infty\big(\boldsymbol{\rho}^l\big) - 2\big[P_-^\infty\big(\boldsymbol{\rho}^l\big)\big]^T, \tag{5.55}$$

$$\big[P_-^\infty\big(\boldsymbol{\rho}^l\big)\big]^T|0\rangle = 0, \tag{5.56}$$

$$P_0^\infty\big(\boldsymbol{\rho}^l\big)|0\rangle = \big\langle 0\big|P_0^\infty\big(\boldsymbol{\rho}^l\big)0\big\rangle|0\rangle = \big\langle 0\big|\boldsymbol{\rho}^l 0\big\rangle|0\rangle, \tag{5.57}$$

which implies

$$\langle \lambda_m | B_l 0\rangle = \lambda_m^l - \big\langle 0\big|\boldsymbol{\rho}^l 0\big\rangle. \tag{5.58}$$

Using (5.54) and noticing that

$$\sigma_k = \big\langle 0\big|\boldsymbol{\rho}^k 0\big\rangle \tag{5.59}$$

one obtains the system of equations

$$\frac{\partial}{\partial t_l}\sigma_k = 2(\sigma_{k+l} - \sigma_l\sigma_k), \tag{5.60}$$

where $\sigma_0 = 1$, $k \in \mathbb{N} \cup \{0\}$, $l \in \mathbb{N}$, for the moments

$$\sigma_k = \int_{\mathbb{R}} \lambda^k\, d\sigma(\lambda) = \sum_{m=0}^{\infty} \lambda_m^k \mu_m \tag{5.61}$$

of the measure (5.52). Let us remark here that in the considered case the moment problem is determined, i.e., the moments σ_k determine the measure (5.52) in a unique way (see, e.g. [2, Chapter 2, Sections 1 and 2]).

Let us comment on the formulas obtained above. Introduce the diagonal trace class operators $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in L_0^1$ by defining their mth components to be the eigenvalues λ_m, the masses μ_m, and the moments σ_m, $m \in \mathbb{N} \cup \{0\}$, respectively. On the subset $\Omega_{-,2}^1$ one has three naturally defined smooth coordinate systems:

(i) $\boldsymbol{\rho} \in \Omega_{-,2}^1$,
(ii) $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in L_0^1 \times L_0^1$, where $\operatorname{Tr}\boldsymbol{\mu} = 1$ and $\boldsymbol{\mu} > 0$,
(iii) $\boldsymbol{\sigma} \in L_0^1$ with first component $\sigma_0 = 1$, $\boldsymbol{\sigma} > 0$, and $\mathbf{d}_0 > 0$,

where $\mathbf{d}_0 := \sum_{k=0}^\infty d_{0k}|k\rangle\langle k|$, and

$$d_{0k} := \det\begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \dots & \sigma_k \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \dots & \sigma_{k+1} \\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \dots & \sigma_{k+2} \\ \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \dots & \sigma_{k+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_k & \sigma_{k+1} & \sigma_{k+2} & \sigma_{k+3} & \dots & \sigma_{2k} \end{bmatrix} > 0, \tag{5.62}$$

with the convention that $d_{0,-1} = 1$. In order to see that $\boldsymbol{\sigma} \in L^1_0$ we notice that

$$\sum_{k=0}^{\infty} \sigma_k = \sum_{k=0}^{\infty} \langle 0 | \boldsymbol{\rho}^k 0 \rangle \leqslant \sum_{k=0}^{\infty} \| \boldsymbol{\rho}^k \| \leqslant \sum_{k=0}^{\infty} \| \boldsymbol{\rho} \|^k = \frac{1}{1 - \|\boldsymbol{\rho}\|_\infty} < +\infty.$$

We also define $\mathbf{d}_1 := \sum_{k=0}^{\infty} d_{1k} |k\rangle\langle k|$, where

$$d_{1k} := \det \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \dots & \sigma_{k-1} & \sigma_{k+1} \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \dots & \sigma_k & \sigma_{k+2} \\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \dots & \sigma_{k+1} & \sigma_{k+3} \\ \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \dots & \sigma_{k+2} & \sigma_{k+4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \sigma_k & \sigma_{k+1} & \sigma_{k+2} & \sigma_{k+3} & \dots & \sigma_{2k-1} & \sigma_{2k+1} \end{bmatrix} \tag{5.63}$$

for $n \in \mathbb{N} \cup \{0\}$.

The transformation from $\boldsymbol{\rho}$-coordinates to $\boldsymbol{\sigma}$-coordinates is given by formula (5.59). The inverse transformation to (5.59) has the form

$$\begin{aligned} \boldsymbol{\rho} &= S^T \rho_1 + \rho_0 + \rho_1 S \\ &= S^T \big[\tilde{s}(\mathbf{d}_0) s(\mathbf{d}_0)\big]^{1/2} \mathbf{d}_0^{-1} + \mathbf{d}_0^{-1}\mathbf{d}_1 - \tilde{s}\big(\mathbf{d}_0^{-1}\mathbf{d}_1\big) + \big[\tilde{s}(\mathbf{d}_0) s(\mathbf{d}_0)\big]^{1/2} \mathbf{d}_0^{-1} S, \end{aligned} \tag{5.64}$$

or, in components (see, e.g., [2]),

$$\rho_{kk} = d_{0k}^{-1} d_{1k} - d_{0,k-1}^{-1} d_{1,k-1} \quad \text{and} \quad \rho_{k,k+1} = (d_{0,k-1} d_{0,k+1})^{1/2} d_{0k}^{-1} > 0. \tag{5.65}$$

Formula (5.61) gives the transformation from $(\boldsymbol{\lambda}, \boldsymbol{\mu})$-coordinates to $\boldsymbol{\sigma}$-coordinates. The inverse transformation to (5.61) is obtained by expanding the so-called Weyl function $\langle 0 | R_\lambda 0 \rangle$ in a Laurent series

$$\langle 0 | R_\lambda 0 \rangle = \sum_{m=0}^{\infty} \frac{\mu_m}{\lambda_m - \lambda} = - \sum_{k=0}^{\infty} \frac{\sigma_k}{\lambda^{k+1}} \tag{5.66}$$

for $|\lambda| > \sup_{m \in \mathbb{N} \cup \{0\}} \{|\lambda_m|\} = \|\boldsymbol{\rho}\|_\infty$. So, one finds $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ by computing the Mittag-Leffler decomposition of the left-hand side of (5.66).

The passage from $\boldsymbol{\rho}$-coordinates to $(\boldsymbol{\lambda}, \boldsymbol{\mu})$-coordinates is obtained by composing the previously described transformations. This can also be done directly constructing the spectral resolution for $\boldsymbol{\rho}$.

After these remarks we present Hamilton's equations (5.38) in the coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$

$$\frac{\partial}{\partial t_l} \boldsymbol{\lambda} = \big\{\boldsymbol{\lambda}, I_l^{S,2}\big\}_{S,2} = 0, \tag{5.67}$$

$$\frac{\partial}{\partial t_l} \boldsymbol{\mu} = \big\{\boldsymbol{\mu}, I_l^{S,2}\big\}_{S,2} = 2\big(\boldsymbol{\lambda}^l - \operatorname{Tr}\big(\boldsymbol{\lambda}^l \boldsymbol{\mu}\big)\big)\boldsymbol{\mu} \tag{5.68}$$

or, in components,

$$\frac{\partial}{\partial t_l}\lambda_m = 0 \quad \text{and} \quad \frac{\partial}{\partial t_l}\mu_m = 2\left(\lambda_m^l - \sum_{n=0}^{\infty}\lambda_n^l\mu_n\right)\mu_m \tag{5.69}$$

and in the coordinates $\boldsymbol{\sigma}$

$$\frac{\partial}{\partial t_l}\boldsymbol{\sigma} = \{\boldsymbol{\sigma}, I_l^{S,2}\} = 2\big(s^l(\boldsymbol{\sigma}) - \sigma_l\boldsymbol{\sigma}\big) \tag{5.70}$$

whose coordinate expression was already given in (5.60). In deducing equations (5.67), (5.68), and (5.70) we used (5.59) and (5.61).

Let us observe now that (5.60) implies that

$$\frac{\partial\sigma_k}{\partial t_l} = \frac{\partial\sigma_l}{\partial t_k} \tag{5.71}$$

for $k, l \in \mathbb{N}$. Thus there exists a function $\tau(t_1, t_2, \ldots)$ of infinitely many variables $(t_1, t_2, \ldots) =: \mathbf{t} \in \ell^\infty$ such that

$$\sigma_k = \frac{1}{2}\frac{\partial}{\partial t_k}\log\tau, \quad k \in \mathbb{N}. \tag{5.72}$$

In order to be consistent with the notation assumed in the theory of integrable systems, we have called this function τ-function.

Substituting (5.72) into (5.60) we obtain the system of linear partial differential equations

$$\frac{\partial^2\tau}{\partial t_l\partial t_k} = 2\frac{\partial\tau}{\partial t_{k+l}}, \quad k, l \in \mathbb{N}, \tag{5.73}$$

on the τ-function.

In order to find the explicit form of the τ-function, use (5.61), substitute (5.72) into (5.69), and integrate both sides of the resulting equation to get

$$\begin{aligned} &\mu_m(t_1, t_2, \ldots, t_{l-1}, t_l, t_{l+1}, \ldots) \\ &\quad = \mu_m(t_1, t_2, \ldots, t_{l-1}, 0, t_{l+1}, \ldots)\frac{\tau(t_1, t_2, \ldots, t_{l-1}, 0, t_{l+1}, \ldots)}{\tau(t_1, t_2, \ldots, t_{l-1}, t_l, t_{l+1}, \ldots)}e^{2\lambda_m^l t_l}. \end{aligned} \tag{5.74}$$

Iterating (5.74) relative to $l \in \mathbb{N}$ yields the final formula for $\mu_m(t_1, t_2, \ldots)$, namely

$$\mu_m(t_1, t_2, \ldots) = \mu_m(0, 0, \ldots)\frac{\tau(0, 0, \ldots)}{\tau(t_1, t_2, \ldots)}e^{2\sum_{l=1}^{\infty}\lambda_m^l t_l}. \tag{5.75}$$

Since $\sum_{m=0}^{\infty}\mu_m(t_1, t_2, \ldots) = 1$, we get the following expression for the τ-function

$$\tau(t_1, t_2, \ldots) = \tau(0, 0, \ldots)\sum_{m=0}^{\infty}\mu_m(0, 0, \ldots)e^{2\sum_{l=1}^{\infty}\lambda_m^l t_l}. \tag{5.76}$$

Let us show that the series in (5.76) is convergent if $\boldsymbol{\mu}(0) \in L^1_0 \cong \ell^1$ and $\mathbf{t} \in \ell^\infty$. In order to do this we prove that the linear operator defined by

$$(\Lambda \mathbf{t})_m := \sum_{l=1}^{\infty} \lambda_m^l t_l$$

is bounded on ℓ^∞. This follows from

$$\begin{aligned} \|\Lambda \mathbf{t}\|_\infty &= \sup_{m\in\mathbb{N}} \left| \sum_{l=1}^{\infty} \lambda_m^l t_l \right| \leqslant \|\mathbf{t}\|_\infty \sup_{m\in\mathbb{N}} \left| \sum_{l=1}^{\infty} \lambda_m^l \right| \\ &= \|\mathbf{t}\|_\infty \sup_{m\in\mathbb{N}} \frac{\lambda_m}{1-\lambda_m} = \|\mathbf{t}\|_\infty \frac{\|\boldsymbol{\rho}\|_\infty}{1-\|\boldsymbol{\rho}\|_\infty}. \end{aligned}$$

Thus the sequence $\{e^{2\sum_{l=1}^{\infty} \lambda_m^l t_l}\}_{m\in\mathbb{N}} \in \ell^\infty$. Since $\{\mu_m(0,0,\ldots)\}_{m\in\mathbb{N}} \in \ell^1$, the series in (5.76) converges.

Summarizing, we see that the substitution of (5.76) into (5.72) and (5.74) gives the $\mathbf{t} := (t_1, t_2, \ldots)$-dependence of the moments $\sigma_k(\mathbf{t})$ and the masses $\mu_m(\mathbf{t})$, respectively. The dependence of $\rho_{kk}(\mathbf{t})$ and $\rho_{k,k+1}(\mathbf{t})$ on $\mathbf{t}$ is given by (5.64), (5.62), and (5.63) which express these quantities in terms of $\sigma_m(\mathbf{t})$. From the discussion above we see that the conditions (5.40), (5.41) are preserved by the $\mathbf{t}$-evolution.

Next, using (5.72), (5.75), and the formula

$$P_n(\lambda_m) = \frac{1}{\sqrt{d_{0,n-1} d_{0,n}}} \det \begin{bmatrix} \sigma_0 & \sigma_1 & \sigma_2 & \ldots & \sigma_n \\ \sigma_1 & \sigma_2 & \sigma_3 & \ldots & \sigma_{n+1} \\ \sigma_2 & \sigma_3 & \sigma_4 & \ldots & \sigma_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{n-1} & \sigma_n & \sigma_{n+1} & \ldots & \sigma_{2n-1} \\ 1 & \lambda_m & \lambda_m^2 & \ldots & \lambda_m^n \end{bmatrix} \tag{5.77}$$

obtained by orthonormalizing the monomials λ^n, $n \in \mathbb{N} \cup \{0\}$, with respect to the measure σ (see, e.g., [2]), we obtain from (5.48) the $\mathbf{t}$-dependence of the eigenvectors $|\lambda_m(\mathbf{t})\rangle$ and the corresponding projectors $\mathbb{P}_m(\mathbf{t})$, $m \in \mathbb{N} \cup \{0\}$.

Formula (5.48) defines the operator $O : \mathcal{H} \to \mathcal{H}$ whose matrix in the basis $\{|k\rangle\}_{k=0}^{\infty}$ is given by $O_{kl}(\mathbf{t}) := P_l(\mathbf{t})(\lambda_k)$. One has the following identities:

$$\boldsymbol{\rho}(\mathbf{t}) O(\mathbf{t}) = O(\mathbf{t}) \boldsymbol{\lambda}(\mathbf{t}), \tag{5.78}$$

$$O(\mathbf{t}) \boldsymbol{\mu}(\mathbf{t}) O(\mathbf{t})^T = \mathbb{I} \tag{5.79}$$

relating the operators $\boldsymbol{\rho}(\mathbf{t})$, $\boldsymbol{\lambda}(\mathbf{t})$, $\boldsymbol{\mu}(\mathbf{t})$, and $O(\mathbf{t})$ for any $\mathbf{t}$. Since $\boldsymbol{\lambda}(\mathbf{t}) = \boldsymbol{\lambda}(\mathbf{0})$, where $\mathbf{0} := (0, 0, \ldots)$, we obtain from (5.78) and (5.79)

$$\boldsymbol{\rho}(\mathbf{t}) = O(\mathbf{t}) O(\mathbf{0})^{-1} \boldsymbol{\rho}(\mathbf{0}) \big(O(\mathbf{t}) O(\mathbf{0})^{-1}\big)^{-1} = Z(\mathbf{t})^T \boldsymbol{\rho}(\mathbf{0}) Z(\mathbf{t}), \tag{5.80}$$

where $Z(\mathbf{t}) := O(\mathbf{0}) \boldsymbol{\mu}(\mathbf{0})^{1/2} \big(O(\mathbf{t}) \boldsymbol{\mu}(\mathbf{t})^{1/2}\big)^T$ is an orthonormal operator, i.e., $Z(\mathbf{t})^T Z(\mathbf{t}) = \mathbb{I}$. As shown in Sections 3 and 4, the flows $\mathbf{t} \mapsto \boldsymbol{\rho}(\mathbf{t})$ can be expressed in terms of the coadjoint action

$(\mathrm{Ad}^{S,2})^*: GL^\infty_{+,2} \to \mathrm{Aut}(L^1_{S,2})$ of the bidiagonal group $GL^\infty_{+,2}$ on the Banach Lie–Poisson space $L^1_{S,2} \cong L^1_{-,2}$, namely,

$$\begin{aligned}
\boldsymbol{\rho}(\mathbf{t}) &= \left(\mathrm{Ad}^{S,2}\right)^*_{g(\mathbf{t})^{-1}} \boldsymbol{\rho}(\mathbf{0}) \\
&= S^T s\left(g_0(\mathbf{t})\right) g_0(\mathbf{t})^{-1} \rho_1(\mathbf{0}) + \rho_0(\mathbf{0}) + g_0(\mathbf{t})^{-1} g_1(\mathbf{t}) \rho_1(\mathbf{0}) - \tilde{s}\left(g_0(\mathbf{t})^{-1} g_1(\mathbf{t}) \rho_1(\mathbf{0})\right) \\
&\quad + s\left(g_0(\mathbf{t})\right) g_0(\mathbf{t})^{-1} \rho_1(\mathbf{0}) S \\
&= \sum_{i=0}^{\infty} \rho_{i,i+1}(\mathbf{0}) \frac{g_{i+1,i+1}(\mathbf{t})}{g_{ii}(\mathbf{t})} |i+1\rangle\langle i| \\
&\quad + \sum_{i=0}^{\infty} \left(\rho_{ii}(\mathbf{0}) + \rho_{i,i+1}(\mathbf{0}) \frac{g_{i+1,i}(\mathbf{t})}{g_{ii}(\mathbf{t})} - \rho_{i,i+1}(\mathbf{0}) \frac{g_{i+1,i}(\mathbf{t})}{g_{i+1,i+1}(\mathbf{t})} \right) |i\rangle\langle i| \\
&\quad + \sum_{i=0}^{\infty} \rho_{i,i+1}(\mathbf{0}) \frac{g_{i+1,i+1}(\mathbf{t})}{g_{ii}(\mathbf{t})} |i\rangle\langle i+1|
\end{aligned} \tag{5.81}$$

(the symmetric version of (5.7)), where $\rho_0 := \mathrm{diag}(\rho_{00}, \rho_{11}, \ldots)$, $\rho_1 := \mathrm{diag}(\rho_{01}, \rho_{12}, \ldots)$, $g_0 := (g_{00}, g_{11}, \ldots)$, and $g_1 := (g_{10}, g_{21}, \ldots) \in L^1_0$.

In order to find the time dependence $\mathbf{t} \mapsto g(\mathbf{t}) = g_0(\mathbf{t}) + g_1(\mathbf{t})S$ for $g(\mathbf{t}) \in GL^\infty_{+,2}$ let us note that from (5.7) and the three-term recurrence relation (5.43) it follows that

$$\begin{aligned}
g_{kk}(\mathbf{t}) &= g_{00}(\mathbf{t}) \frac{\rho_{00}(\mathbf{t}) \cdots \rho_{k-1,k-1}(\mathbf{t})}{\rho_{00}(\mathbf{0}) \cdots \rho_{k-1,k-1}(\mathbf{0})} \\
&= g_{00}(\mathbf{t}) \frac{P_{kk}(\mathbf{0})}{P_{kk}(\mathbf{t})} = g_{00}(\mathbf{t}) \sqrt{\frac{d_{0,k-1}(\mathbf{0}) d_{0k}(\mathbf{t})}{d_{0k}(\mathbf{0}) d_{0,k-1}(\mathbf{t})}}
\end{aligned} \tag{5.82}$$

and

$$\begin{aligned}
g_{k+1,k}(\mathbf{t}) &= g_{00}(\mathbf{t}) \left(\frac{\rho_{00}(\mathbf{t}) \cdots \rho_{k-1,k-1}(\mathbf{t})}{\rho_{00}(\mathbf{0}) \cdots \rho_{k-1,k-1}(\mathbf{0})} \right) \left(\frac{\rho_{00}(\mathbf{t}) + \cdots + \rho_{kk}(\mathbf{t}) - \rho_{00}(\mathbf{0}) - \cdots - \rho_{kk}(\mathbf{0})}{\rho_{kk}(\mathbf{0})} \right) \\
&= g_{00}(\mathbf{t}) \frac{P_{k+1,k}(\mathbf{0}) P_{k+1,k+1}(\mathbf{t}) - P_{k+1,k}(\mathbf{t}) P_{k+1,k+1}(\mathbf{0})}{P_{kk}(\mathbf{t}) P_{k+1,k+1}(\mathbf{t})} \\
&= g_{00}(\mathbf{t}) \frac{d_{1k}(\mathbf{t}) \sqrt{d_{0,k+1}(\mathbf{0})} - d_{1k}(\mathbf{0}) \sqrt{d_{0,k+1}(\mathbf{t})}}{\sqrt{d_{0k}(\mathbf{0}) d_{0k}(\mathbf{t}) d_{0,k-1}(\mathbf{t}) d_{0,k+1}(\mathbf{0})}},
\end{aligned} \tag{5.83}$$

where $P_{kl}(\mathbf{t})$ are the coefficients of the polynomial $P_n(\mathbf{t})(\lambda) = P_{nn}(\mathbf{t})\lambda^n + P_{n,n-1}(\mathbf{t})\lambda^{n-1} + \cdots + P_{n1}(\mathbf{t})\lambda + P_{n0}(\mathbf{t})$. The last equalities in (5.82) and (5.83) are obtained using (5.77), (5.62), and (5.63) to get the expressions

$$P_{kk}(\mathbf{t}) = \sqrt{\frac{d_{0,k-1}(\mathbf{t})}{d_{0k}(\mathbf{t})}} \quad \text{and} \quad P_{k+1,k}(\mathbf{t}) = \frac{-d_{1k}(\mathbf{t})}{\sqrt{d_{0k}(\mathbf{t}) d_{0,k+1}(\mathbf{t})}}.$$

Recall that $d_0(\mathbf{t})$ and $d_1(\mathbf{t})$ are given by (5.62) and (5.63), respectively.

Finally, taking in (5.26) (for $k = 2$) $g_0(\mathbf{t})$ and $g_1(\mathbf{t})$ given by (5.82) and (5.83), we obtain the explicit expression for the time evolution of the position $\mathbf{q}(\mathbf{t})$ and the momentum $\mathbf{p}(\mathbf{t})$ for all flows in the Toda hierarchy described by the Hamiltonians

$$H_l(\mathbf{q},\mathbf{p}) := \left(I_l^{S,2} \circ \mathcal{J}_{\nu_1}\right)(\mathbf{q},\mathbf{p}),$$

where $\mathcal{J}_{\nu_1} : \ell^1 \times \ell^\infty \to L^1_{-,2} \cong L^1_{S,2}$ is the Flaschka map given by (5.25) for $k = 2$ and $I_l^{S,2} = I_l^S \circ \iota_{S,2} = I_l \circ \iota_S \circ \iota_{S,2}$ are the restrictions to $L^1_{S,2}$ of the Casimir functions I_l of L^1 (see (4.33)).

Note that the formulas giving the group element $g(\mathbf{t})$ depend on $g_{00}(\mathbf{t})$. This first component cannot be determined but it does not matter because $g_{00}(\mathbf{t})\mathbb{I}$ is in the center of $GL^\infty_{+,2}$ and hence the coadjoint action defined by it is trivial. Also, in terms of the variables $\mathbf{q}$ and $\mathbf{p}$, the action of this group element is a translation in $\mathbf{q}$ and has no effect on $\mathbf{p}$. This corresponds to the flow of $I_1^{S,2}$.

To solve the semi-infinite Toda system one takes an initial condition $\boldsymbol{\rho}(\mathbf{0})$ which determines a coadjoint orbit of $GL^\infty_{+,2}$ in $L^1_{S,2}$. In the generic case, when all entries on the upper (hence also lower) diagonal of $\boldsymbol{\rho}(\mathbf{0})$ are strictly positive, the solution of the semi-infinite Toda lattice was given above. If some upper diagonal entries of $\boldsymbol{\rho}(\mathbf{0})$ vanish, see Remark (iv). Then the Toda lattice equations decouple and we get a smaller Toda system for each block. On the infinite block, the solution is as above. On each finite block one obtains a finite dimensional Toda lattice whose solution is known (see, e.g., [10]). The method we used above for the semi-infinite case can be also used in the finite case; one works then with measures σ having finite support and uses finite orthogonal polynomials. If one implements the solution method described in this section to this finite-dimensional case the results in [10] are reproduced.

Acknowledgments

This work was begun while both authors were at the Erwin Schrödinger International Institute for Mathematical Physics in the Fall of 2003 during the program *The Geometry of the Moment Map* and hereby thank ESI for its hospitality. Some of the work on this paper was done during the program *Geometric Mechanics* at the Bernoulli Center of the EPFL in the Fall of 2004. A.O. thanks the Bernoulli Center for its hospitality and excellent working conditions during his extended stay there. We are grateful to D. Beltiţă, H. Flaschka, and the referee for several useful discussions and remarks that influenced our presentation. The authors thank the Polish and Swiss National Science Foundations (Polish State Grant P03A 0001 29 and Swiss NSF Grant 200021-109111/1) for partial support.

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