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OPENNESS AND CONVEXITY FOR MOMENTUM MAPS

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Abstract. The purpose of this paper is finding the essential attributes underlying the convexity theorems for momentum maps. It is shown that they are of a topological nature; more specifically, we show that convexity follows if the map is open onto its image and has the so-called local convexity data property. These conditions are satisfied in all the classical convexity theorems and hence they can, in principle, be obtained as corollaries of a more general theorem that has only these two hypotheses. We also prove a generalization of the so-called Local-to-Global Principle that only requires the map to be closed and to have a normal topological space as domain, instead of using a properness condition. This allows us to generalize the Flaschka-Ratiu convexity theorem to noncompact manifolds.

1. Introduction

The problem of describing the image of a momentum map defined on a symplectic manifold has generated a large amount of research in the past twenty years and it remains to this day one of the most active areas in symplectic geometry and its applications to Hamiltonian dynamics, especially bifurcation theory. In 1982 Atiyah [3] and, independently, Guillemin and Sternberg [11] proved the following result about the convexity of the image of the momentum map associated to the action of a torus T on a compact symplectic manifold.

Theorem (Atiyah-Guillemin-Sternberg)**.** *Let M be a compact connected symplectic manifold on which a torus T acts in a Hamiltonian fashion with associated invariant momentum map $J : M \to \mathfrak{t}^*$. Here $\mathfrak{t}$ denotes the Lie algebra of T and $\mathfrak{t}^*$ is its dual; both are isomorphic as Abelian Lie algebras to $\mathbb{R}^{\dim T}$. Then the image $J(M)$ of J is a compact convex polytope in $\mathbb{R}^{\dim T}$, called the T-momentum polytope. Moreover, it is equal to the convex hull of the image of the fixed point set of the T-action. The fibers of J are connected.*

The motivation for this result was the *Kostant linear convexity theorem* [25] which states that the projection of an adjoint orbit of a compact connected Lie group onto the Lie algebra of any of its maximal tori is the convex hull of the corresponding Weyl group orbit. This in turn is a generalization of a classical result of Schur [34] and Horn [16] (which in the case of the unitary group $U(n)$ is Kostant's theorem) which states that the set of diagonals of an isospectral set of Hermitian matrices equals the convex hull of the $n!$ points obtained by permuting all the eigenvalues.

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Another problem with important mathematical and physical consequences is to describe all possible eigenvalues of the sum $A + B$ of two Hermitian matrices A and B as each one of them ranges over an isospectral set. The isospectral sets of Hermitian matrices are precisely the coadjoint orbits $\mathcal{O}_\mu$ of $U(n)$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and μ_i are the eigenvalues. If one requires, in addition, that the eigenvalues of the sum $A + B$ be sorted in decreasing order, this problem amounts to describing the set $J(\mathcal{O}_\mu \times \mathcal{O}_\lambda) \cap \mathfrak{t}^*_+$ where $J(A, B) = A + B$ and $\mathfrak{t}^*_+$ is a positive Weyl chamber in $\mathfrak{t}^* = \{\text{real diagonal matrices}\}$. This problem is a particular case of the following more general situation: a compact Lie group G acts on a compact symplectic manifold M with associated equivariant momentum map $J : M \to \mathfrak{g}^*$. Equivariance is not an assumption, since any momentum map of a compact group can be averaged to give an equivariant momentum map for the same canonical G-action.

Guillemin and Sternberg [12] proved that $J(M) \cap \mathfrak{t}^*$ is a union of compact convex polytopes and Kirwan [20] showed that this set is connected, thereby concluding that $J(M) \cap \mathfrak{t}^*_+$ is a compact convex polytope. We will refer to this as the G-momentum polytope.

Theorem (Guillemin-Kirwan-Sternberg)**.** *Let M be a compact connected symplectic manifold on which the compact connected Lie group G acts in a Hamiltonian fashion with associated equivariant momentum map $J : M \to \mathfrak{g}^*$. Here $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Let T be a maximal torus of G, $\mathfrak{t}$ its Lie algebra, $\mathfrak{t}^*$ its dual, and $\mathfrak{t}^*_+$ the positive Weyl chamber relative to a fixed ordering of the roots. Then $J(M) \cap \mathfrak{t}^*_+$ is a compact convex polytope, called the G-momentum polytope. The fibers of J are connected.*

Important results about the description of the $U(n)$-momentum polytope were obtained by Knutson and Tao [24].

Sjamaar [35] has given another proof of the convexity theorems, based on ideas coming from Kähler and algebraic geometry. His proof, even though it gives the most complete information on these polytopes, uses strongly the symplectic form and it is not known how to generalize his technique to other manifolds and other types of actions, such as Poisson actions of Poisson-Lie groups.

The case of compact symplectic manifolds is rich but quite particular. For non-compact manifolds, the previous results no longer hold and a counterexample was given by Prato [33]. Conditions under which the T or G-momentum polytopes are convex were given by Condevaux, Dazord, and Molino [7] and later by Hilgert, Neeb, and Plank [14]. These papers show that the proof of the convexity of the image of the momentum map rests on the following result which we give here in the formulation due to the second group of authors:

Theorem (Local-to-Global Principle)**.** *Let $\Psi : X \to V$ be a locally fiber connected map from a connected locally connected Hausdorff topological space X to a finite dimensional vector space V, with local convexity data $(C_x)_{x \in X}$ such that all convex cones C_x are closed in V. Suppose that Ψ is a proper map. Then $\Psi(X)$ is a closed locally polyhedral convex subset of V, the fibers $\Psi^{-1}(v)$ are all connected, and $\Psi : X \to \Psi(X)$ is an open mapping.*

We elaborate now on the hypotheses of this theorem. A map $\Psi : X \to V$ is said to have ***local convexity data*** if for each $x \in X$ there exists an arbitrarily small open neighborhood U_x of x and a convex cone C_x with vertex $\Psi(x)$ in V such that

$\Psi(U_x)$ is a neighborhood of the vertex $\Psi(x)$ in C_x and such that $\Psi|_{U_x} : U_x \to C_x$ is an open map, where C_x is endowed with the subspace topology inherited from V. A map $\Psi : X \to V$ is said to be ***locally fiber connected*** if for each $x \in X$ there is an open neighborhood U_x of x such that $\Psi^{-1}(\Psi(u)) \cap U_x$ is connected for all $u \in U_x$. In Definitions 2.7 and 2.15 we will elaborate on these conditions. A continuous map between two topological spaces with Hausdorff domain is said to be ***proper*** if it is closed and all its fibers are compact subsets of its domain.

In spite of its generality this theorem cannot be applied to situations where the fibers $\Psi^{-1}(v)$ are either not compact or the map Ψ is not closed because both conditions are necessary for Ψ to be a proper map. This is, for example, one of the difficulties in the (direct) proof of the convexity theorem due to Flaschka and Ratiu [10] that contains as an important particular instance the case of Poisson-Lie group actions on compact symplectic manifolds. Alekseev [2] reproved the Poisson-Lie group action result mentioned above by other means, reducing it to the case covered by the Guillemin-Sternberg-Kirwan convexity result. His method strongly uses the structure of Poisson-Lie groups and it is not known how to extend it to other types of actions.

The main purpose of this paper is finding the essential features underlying all the results mentioned above that ensure convexity. As we will see these properties are of a topological nature; more specifically, we will show that convexity is rooted on the map being open onto its image and having local convexity data. These conditions happen to be satisfied in all the classical convexity theorems that we discussed previously and hence they can in principle be obtained as corollaries of the following general theorem that can be found in Section 2:

Theorem. *Let $f : X \to V$ be a continuous map from a connected Hausdorff topological space X to a Banach space V that is open onto its image and has local convexity data. Then the image $f(X)$ is locally convex. If, in addition, $f(X)$ is closed in V, then it is convex.*

Note first that V is allowed to be infinite dimensional. Second, unlike the local convexity data condition that can be found in [7, 14], we do not assume the cones C_x to be closed since it is not a reasonable assumption in infinite dimensions.

In the light of the result above, the convexity problem reduces to giving necessary and sufficient conditions for a map that has local convexity data to be open onto its image. In the paper (Section 3) we will provide those characterizations for the momentum maps associated to compact Lie group actions on symplectic manifolds. We will split the problem into two cases: when the map has connected fibers and when it has only the locally fiber connectedness property. We shall also show that the openness of the momentum map can be determined just by looking at its image and we will illustrate this with two examples.

It is worth noting at this stage that Montaldi and Tokieda [30] proved that the openness of the momentum map (relative to its image endowed with the subspace topology) implies persistence of extremal relative equilibria under every perturbation of the value of the momentum map, provided the isotropy subgroup of this value is compact. So the openness property of the momentum map onto its image has interesting dynamical consequences.

Section 2 contains a generalization of the Local-to-Global Principle in [7, 14] that only requires the map to be closed and to have a normal topological space as domain, instead of using the properness condition. This degree of generality is

needed to obtain convexity directly from the Local-to-Global Principle in some of the examples that we present and that generalize various results in the literature. For instance, in Theorem 3.5 we extend a result of Prato [33] where we only require the properness of a single component of the momentum map to conclude convexity. Additionally, using our generalization of the Local-to-Global Principle we are able to drop in Section 4 the compactness hypothesis on the manifold in the Flaschka-Ratiu convexity result [10].

It is worth mentioning that the generalization of the Local-to-Global Principle presented in the paper includes infinite dimensional situations. This suggests that one could, in principle, use this tool in dealing with convexity problems such as those in the papers of Bloch, Flaschka, and Ratiu [4] or of Neumann [31]. The implementation of this idea is not free of difficulties and remains an open problem. This is due to the lack of a Marle-Guillemin-Sternberg normal form in the infinite dimensional setting which makes the local convexity data property very difficult to check.

2. Openness and local convexity

In this section we will study topological properties of maps that have *local convexity data*, a notion that we will introduce below. This property holds typically for momentum maps and, under certain supplementary topological conditions on the map and on its domain of definition, it implies that the map is open onto its image. We will also show that local convexity of the image of a map that has local convexity data is implied by openness of that map.

The Klee Theorem. The passage from local convexity to convexity is given by a classical result of Klee [21] which we now present.

Definition 2.1. Let V be a topological vector space. If $x, y \in V$, the ***straight line segment***, or simply ***segment***, $[x, y]$ is defined by $[x, y] := \{(1-\lambda)x + \lambda y \mid 0 \leq \lambda \leq 1\}$. A subset $X \subset V$ is said to be ***convex*** if for any $x, y \in X$ we have $[x, y] \subset X$. A subset $Y \subset V$ is called ***locally convex*** if each point $y \in Y$ has a neighborhood V_y whose intersection with Y is convex. A ***polygonal path*** is a continuous path that is the union of segments. A subset $X \subset V$ is said to be ***polygonally connected*** if any two points can be joined by a polygonal path lying entirely in X.

The following lemma is due to Kakutani and Tukey and will be used in the proof of Klee's Theorem that insures the passage from local to global convexity.

Lemma 2.2. *In a topological vector space V, a connected locally convex set X is polygonally connected.*

Proof. Let p be an arbitrary point of X and let X_p be the set of all points of X which can be joined to p by a polygonal path. The strategy is to prove that X_p is both an open and closed subset of X. The result then follows by connectivity of X.

(i) X_p is open in X: Take an arbitrary point $x \in X_p$ and a neighborhood U_x of x in X chosen such that $U_x \cap X$ is convex. This is possible since X is locally convex. Each $y \in U_x \cap X$ can be joined to p by the polygonal path obtained by adding to the polygonal path joining p to x the straight line segment from x to y that is guaranteed to lie entirely in $U_x \cap X$. Thus p can be joined to $y \in U_x$ by a polygonal path that lies entirely in X, which proves that $y \in X_p$. Since $y \in U_x \cap X$

was arbitrary, this shows that the open set $U_x \cap X$ in the relative topology of X lies in X_p, that is, that X_p is open in X.

(ii) X_p is closed in X: We will show that $\overline{X}_p \cap X = X_p$. To prove the non-trivial inclusion $\overline{X}_p \cap X \subset X_p$, take an element $y \in \overline{X}_p \cap X$ and recall that for any neighborhood U_y of y in the topological vector space V we have $U_y \cap X_p \neq \varnothing$. We can choose, by hypothesis, U_y such that $U_y \cap X$ is convex. Thus there is an element $z \in U_y \cap X_p$ such that the straight line segment joining z to y lies entirely in $U_y \cap X$. However, $z \in X_p$, so there is a polygonal path that lies entirely in X joining p to z. Adding to this path the segment joining z to y yields a polygonal path lying entirely in X that joins p to y, which proves that $y \in X_p$. □

The next theorem is due to Klee [21] and gives the connection between local convexity and convexity.

Theorem 2.3 (Klee)**.** *Each closed connected and locally convex subset of a topological vector space is convex.*

Remark 2.4. A quick inspection of the proof of this result shows that one can relax the closedness condition by requiring that the subset is closed just in a convex subset of the vector space endowed with the relative topology. This remark will be used in Theorem 2.13.

Maps with local convexity data. Let $f : X \to V$ be a continuous map defined on a connected Hausdorff topological space X with values in a locally convex vector space V. On the topological space X define the following equivalence relation: declare two points $x, y \in X$ to be equivalent if and only if $f(x) = f(y) = v$ and they belong to the same connected component of $f^{-1}(v)$. The topological quotient space will be denoted by $X_f := X/R$, the projection map by $\pi_f : X \to X_f$, and the induced map on X_f by $\widetilde{f} : X_f \to V$. Thus, $\widetilde{f} \circ \pi_f = f$ uniquely characterizes $\widetilde{f}$. The map $\widetilde{f}$ is continuous and if the fibers of f are connected, then it is also injective.

The following elementary topological fact will be used several times later on.

Lemma 2.5. *Let $f : X \to Y$ be a continuous map between two topological spaces. Assume that f has connected fibers and is open or closed. Then for every connected subset C of Y the inverse image $f^{-1}(C)$ is connected.*

Proof. Suppose that f is an open map, $C \subset Y$ is connected, and $f^{-1}(C)$ is not connected. Then there exist two open sets U_1, U_2 in X such that $f^{-1}(C) = (U_1 \cap f^{-1}(C)) \cup (U_2 \cap f^{-1}(C))$, $U_1 \cap f^{-1}(C) \neq \varnothing$, $U_2 \cap f^{-1}(C) \neq \varnothing$, and $U_1 \cap U_2 \cap f^{-1}(C) = \varnothing$. Note that $C = f(f^{-1}(C)) = f((U_1 \cap f^{-1}(C)) \cup (U_2 \cap f^{-1}(C))) = f(U_1 \cap f^{-1}(C)) \cup f(U_2 \cap f^{-1}(C)) \subset (f(U_1) \cap C) \cup (f(U_2) \cap C)$. Conversely, since $f(U_1) \cap C \subset C$ and $f(U_2) \cap C \subset C$ it follows that $(f(U_1) \cap C) \cup (f(U_2) \cap C) \subset C$, which proves that $(f(U_1) \cap C) \cup (f(U_2) \cap C) = C$. Also, $f(U_1) \cap C \supset f(U_1 \cap f^{-1}(C)) \neq \varnothing$ and $f(U_2) \cap C \supset f(U_2 \cap f^{-1}(C)) \neq \varnothing$. By openness of f, the sets $f(U_1)$ and $f(U_2)$ are open in Y so that connectedness of C implies that $f(U_1) \cap f(U_2) \cap C \neq \varnothing$.

If $c \in f(U_1) \cap f(U_2) \cap C$, then $f^{-1}(c) = (U_1 \cap f^{-1}(c)) \cup (U_2 \cap f^{-1}(c))$. The inclusion $\supset$ is obvious. To prove the reverse inclusion $\subset$, let $x \in f^{-1}(c) \subset f^{-1}(C)$. Thus $x \in U_1 \cap f^{-1}(C)$ or $x \in U_2 \cap f^{-1}(C)$. Since $x \in f^{-1}(c)$ by hypothesis, this implies that $x \in U_1 \cap f^{-1}(c)$ or $x \in U_2 \cap f^{-1}(c)$, which proves the inclusion $\subset$. Note also that $U_1 \cap f^{-1}(c) \neq \varnothing$ since $c \in f(U_1)$. Similarly, $U_2 \cap f^{-1}(c) \neq \varnothing$. Finally,

$U_1 \cap U_2 \cap f^{-1}(c) \subset U_1 \cap U_2 \cap f^{-1}(C) = \varnothing$. Thus the fiber $f^{-1}(c)$ can be written as the disjoint union of the two open nonempty sets $U_1 \cap f^{-1}(c)$ and $U_2 \cap f^{-1}(c)$, which contradicts the connectedness hypothesis of the fibers of f.

The proof for f a closed map is identical to the one above by repeating the same argument for U_1 and U_2 closed subsets of X. □

Definition 2.6. Let V be a topological vector space. A set $C \subset V$ is called a ***cone*** with ***vertex*** v_0 if for each $\lambda \geq 0$ and for each $v \in C$, $v \neq v_0$, we have $(1-\lambda)v_0 + \lambda v \in C$. If the set C is, in addition, convex, then C is called a ***convex cone***. Note that, by definition, the vertex $v_0 \in C$.

Definition 2.7. The continuous map $f : X \to V$ defined on a connected locally connected Hausdorff topological space X with values in a locally convex topological vector space V is said to have **local convexity data** if for each $x \in X$ and every sufficiently small neighborhood U_x of x there exists a convex cone $C_{x,f(x),U_x}$ in V with vertex at $f(x)$ such that

- **(VN):** $f(U_x) \subset C_{x,f(x),U_x}$ is a neighborhood of the vertex $f(x)$ in the cone $C_{x,f(x),U_x}$ and
- **(SLO):** $f|_{U_x} : U_x \to C_{x,f(x),U_x}$ is an open map and for any neighborhood $U_x^{'}$ of x, $U_x^{'} \subset U_x$, the set $f(U_x^{'})$ is a neighborhood of the vertex $f(x)$ in the cone $C_{x,f(x),U_x}$,

where the cone $C_{x,f(x),U_x}$ is endowed with the subspace topology induced from V. If the associated cones $C_{x,f(x),U_x}$ are such that $C_{x,f(x),U_x} \cap f(X)$ is closed in $f(X)$, then we say that f has ***local convexity data with closed cones***.

Remark 2.8. We are using the (VN) condition as an abbreviation for "vertex neighborhood condition" and the (SLO) condition as an abbreviation for "strong local openness condition". Note that in the case when the associated cones $C_{x,f(x),U_x}$ are closed in $f(X)$ the second condition in (SLO) is automatically implied by the openness of $f|_{U_x} : U_x \to C_{x,f(x),U_x}$.

Remark 2.9. Let V be a locally convex topological vector space, C_1 and C_2 two cones in V with vertex at zero, and V_0 a neighborhood of zero in V. Suppose that $C_1 \cap C_2 \cap V_0$ is a neighborhood of zero in C_2. Then $C_2 \subset C_1$.

Indeed, since scalar multiplication is a continuous operation with respect to the subspace topology on C_2 induced by the topology of V and since V is a locally convex topological space, we obtain that for every $x \in C_2$ there exists some $\lambda > 0$ such that $\lambda x \in C_1 \cap C_2 \cap V_0 \subset C_1$. Since C_1 is a cone, it follows that $t\lambda x \in C_1$ for every $t > 0$ and hence $x \in C_1$.

By translation, the same property holds if the common vertices of the cones C_1 and C_2 are at some other point of V. We shall use this observation several times in the remarks that follow.

Remark 2.10. $C_{x,f(x),U_x}$ does not depend on the neighborhood U_x in the sense that if $U_x^{'} \subset U_x$ is another neighborhood of x, then $C_{x,f(x),U_x^{'}} = C_{x,f(x),U_x}$.

Indeed, we have $f(U_x^{'}) \subset C_{x,f(x),U_x^{'}}$ is a neighborhood of the vertex $f(x)$ in $C_{x,f(x),U_x^{'}}$, $f(U_x) \subset C_{x,f(x),U_x}$ is a neighborhood of the vertex $f(x)$ in $C_{x,f(x),U_x}$, and $f(U_x^{'}) = V_{f(x)}^{'} \cap C_{x,f(x),U_x^{'}} \subset f(U_x) = V_{f(x)} \cap C_{x,f(x),U_x}$, where $V_{f(x)}^{'}$ and $V_{f(x)}$ are two open neighborhoods of $f(x)$ in V. By the argument used in Remark 2.9

it follows that $C_{x,f(x),U_x'} \subset C_{x,f(x),U_x}$. The (SLO) condition shows that $f(U_x^{'}) = f(U_x^{'}) \cap C_{x,f(x),U_x'} \cap C_{x,f(x),U_x}$ is a neighborhood of the vertex $f(x)$ in $C_{x,f(x),U_x}$. Again Remark 2.9 implies that $C_{x,f(x),U_x} \subset C_{x,f(x),U_x'}$.

Thus, since $C_{x,f(x),U_x'}$ is independent of the neighborhood U_x, we shall write $C_{x,f(x)}$ for the cone in Definition 2.7.

Remark 2.11. In Remark 2.12 we shall need the following statement (see e.g. [6] or [15]): a topological space is connected if and only if every open covering $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ has the property that for each pair of points x, y there exists a finite sequence $\{\alpha_1, ..., \alpha_k\} \subset \mathcal{A}$ such that $x \in U_{\alpha_1}$, $y \in U_{\alpha_k}$, and $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \varnothing$ for all $i = 1, \ldots, k-1$. This finite family of open sets $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is called a ***finite chain*** linking x to y. A topological space that has the property that for each open cover and arbitrary points x, y there is a finite chain formed by elements of this open cover linking x to y is also called ***well-chained***. The statement above asserts hence that a topological space is connected if and only if it is well-chained.

Remark 2.12. $C_{x,f(x)}$ depends only on the connected components of $f^{-1}(f(x))$, that is, if y is in the same connected component of $f^{-1}(f(x))$, then $C_{x,f(x)} = C_{y,f(y)}$.

Let U_x be the open neighborhood of $x \in X$ guaranteed by Definition 2.7. Let $y \in f^{-1}(f(x))$ be in the same connected component of $f^{-1}(f(x))$ as x and $y \in U_x$. For this y, let U_y be the open neighborhood of y in Definition 2.7, which we can choose such that $U_y \subset U_x$. The fact that $f(U_y) \subset f(U_x)$ and $f(U_y)$ is a neighborhood of $f(x)$ in $C_{y,f(x)}$ and $f(U_x)$ is a neighborhood of $f(x)$ in $C_{x,f(x)}$ implies that $C_{y,f(x)} \subset C_{x,f(x)}$. For the reverse inclusion observe that $f(U_y) \subset C_{y,f(x)}$ is a neighborhood of $f(x)$ and by the (SLO) condition we also have that $f(U_y)$ is open in $C_{x,f(x)}$, which shows that $f(U_y) \cap C_{x,f(x)} = f(U_y) \cap C_{x,f(x)} \cap C_{y,f(x)}$ is an open neighborhood of $f(x)$ in $C_{x,f(x)}$. By Remark 2.9, we obtain $C_{x,f(x)} \subset C_{y,f(x)}$. Thus for any $y \in U_x$ that also lies in the same connected component of $f^{-1}(f(x))$ as x, we have $C_{x,f(x)} = C_{y,f(x)}$.

So, on the connected component E_x of the fiber $f^{-1}(f(x))$ containing x, using Remark 2.11, we obtain $C_{x,f(x)} = C_{y,f(x)}$ for any $y \in E_x$. Thus, if the fibers $f^{-1}(f(x))$ are connected we can erase x from $C_{x,f(x)}$.

Our strategy to prove local convexity for the image of a map that has local convexity data is to prove that it is open onto its image.

Theorem 2.13. *Let X be a connected Hausdorff topological space, V a locally convex topological vector space, and $f : X \to V$ a continuous map that has local convexity data. If f is open onto its image, then $f(X)$ is a locally convex subset of V. Moreover, if $f(X)$ is closed in a convex subset of V, then it is convex.*

Proof. Let $v \in f(X)$ be arbitrary and take $x \in f^{-1}(v)$. By the condition (VN), there exists a neighborhood $U_x \subset X$ of x such that $f(U_x) \subset C_{x,f(x)}$ is open in $f(X)$; $C_{x,f(x)}$ is the convex cone with vertex at $v = f(x)$ given in Definition 2.7. Thus, shrinking U_x if necessary, using condition (SLO), and the local convexity of the topological vector space V, we can find a convex neighborhood V_v of v in V such that $f(U_x) = V_v \cap C_{x,f(x)}$. Since f is open onto its image, the neighborhood V_v can be shrunk further to a convex neighborhood of v, also denoted by V_v, such that $f(U_x) = V_v \cap f(X)$. Taking this as the neighborhood of v and shrinking U_x if necessary, we get $V_v \cap C_{x,f(x)} = f(U_x) = V_v \cap f(X)$. Since the intersection of two convex sets is convex, it follows that $V_v \cap C_{x,f(x)}$ is also convex. Thus the point

$v \in f(X)$ has a neighborhood V_v in V such that $V_v \cap f(X)$ is convex; that is, $f(X)$ is locally convex (see Definition 2.1).

Now assume, in addition, that $f(X)$ is closed in a convex subset C of V. Connectedness of X and continuity of f imply that $f(X)$ is connected. Therefore, $f(X)$ is a closed, connected, and locally convex subset of C (by what was just proved), so Klee's theorem (see Theorem 2.3) ensures that $f(X)$ is convex in C. Since C is convex in V, this implies that $f(X)$ is convex in V. □

Remark 2.14. Note that in Theorem 2.13 it was not assumed that the cones that give the local convexity data are closed. This could be useful for convexity theorems with infinite dimensional range. It is also worth noting that connectivity of X was not really used. In other words, the theorem above holds for each connected component of X.

It is not obvious when a map that has local convexity data is open onto its image. In order to give sufficient conditions under which this happens we need a few more preliminary results. First we need the following concept.

Definition 2.15. Let X and Y be two topological spaces and $f : X \to Y$ a continuous map. The subset $A \subset X$ satisfies the ***locally fiber connected condition (LFC)*** if A does not intersect two different connected components of the fiber $f^{-1}(f(x))$, for any $x \in A$.

Let X be a connected, locally connected, Hausdorff topological space and V a locally convex topological vector space. The continuous map $f : X \to V$ is said to be **locally fiber connected** if for each $x \in X$, any open neighborhood of x contains a neighborhood U_x of x such that U_x satisfies the **(LFC)** condition.

Recall that X_f denotes the quotient topological space of X whose points are the connected components of the fibers of the continuous map $f : X \to V$, and $\widetilde{f} : X_f \to V$ is the map such that $\widetilde{f} \circ \pi_f = f$. Note that a subset $A \subset X$ satisfies (LFC) if and only if $\widetilde{f}|_{\pi_f(A)}$ is injective. Similarly, f is locally fiber connected if and only if for any $x \in X$, any open neighborhood of x contains an open neighborhood U_x of x such that the restriction of $\widetilde{f}$ to $\pi_f(U_x)$ is injective.

Lemma 2.16. *Assume that the continuous map $f : X \to V$ has local convexity data and is also locally fiber connected.*

(i) *The quotient projection $\pi_f : X \to X_f$ is an open map.*
(ii) *If f has connected fibers, then X_f is Hausdorff.*

Proof. **(i)** We begin by noting that it suffices to prove that for each $x \in X$ and for each open connected neighborhood U_x that satisfies (VN), (SLO), and (LFC), its image $\pi_f(U_x)$ is open in X_f. This is so because any open neighborhood of x contained in U_x also satisfies the same three properties.

Take x to be an arbitrary point in X and U_x a connected neighborhood of x that satisfies (VN), (SLO), and (LFC). We will prove that $\pi_f^{-1}(\pi_f(U_x))$ is open in X. To show this, let y be arbitrary in $\pi_f^{-1}(\pi_f(U_x))$. Then the connected component E_y of $f^{-1}(f(y))$ that contains y intersects U_x. Let $y' \in E_y \cap U_x$. Choose a neighborhood $U_{y'} \subset U_x$ that also satisfies (VN), (SLO) and (LFC). Since E_y is a connected space with respect to the induced topology from X, it is well-chained with respect to this topology; hence we can find a finite chain of open sets in X satisfying (VN), (SLO) and (LFC) such that $U_1 = U_{y'}$ as above, $y \in U_n$ and $U_i \cap U_{i+1} \cap E_y \neq \varnothing$. The set

$W_{yy'} := \bigcap_{i=1}^{n-1} f(U_i \cap U_{i+1})$ is an open subset of $C_{y,f(y)}$ because of (SLO) and due to the fact that the associated cones depend only on the connected components of the fibers. Additionally, $W_{yy'}$ is nonempty since by construction it contains the point $f(E_y)$, because $U_i \cap U_{i+1} \cap E_y \neq \varnothing$, for any $i \in \{1, \ldots, n-1\}$. We now show that $O_y := U_n \cap f^{-1}(W_{yy'})$ is an open subset of $\pi_f^{-1}(\pi_f(U_x))$ that contains y. Indeed, O_y is open in X because it is an open subset of U_n. By the construction of O_y, y clearly belongs to O_y. It remains to be shown that $O_y \subset \pi_f^{-1}(\pi_f(U_x))$.

Let $z \in O_y$ be arbitrary. The connected component E_z of $f^{-1}(f(z))$ that contains z intersects U_n. By the construction of O_y, the fiber $f^{-1}(f(z))$ intersects every open set $U_i \cap U_{i+1}$ of the finite chain that links y and y'. The (LFC) property guarantees that the connected component of $f^{-1}(f(z))$ that intersects $U_n \cap U_{n-1}$ has to be E_z. Repeating the argument we obtain that $E_z \cap U_{y'} \neq \emptyset$ and $E_z \cap U_{y'} \subset U_x$. Then $E_z \subset \pi_f^{-1}(\pi_f(U_x))$ for every $z \in O_y$, which proves that $O_y \subset \pi_f^{-1}(\pi_f(U_x))$ and, consequently, $\pi_f^{-1}(\pi_f(U_x))$ is an open subset of X.

(ii) Since all the fibers of f are connected and f is continuous, it follows that the graph of the relation determined by f is closed. As π_f is open we obtain that X_f is a Hausdorff space. □

Remark 2.17. Notice that neither local convexity data nor the locally fiber connected condition alone would imply openness of π_f. Indeed, if we consider the example where X is the square in $\mathbb{R}^2$ with vertices $(2,2)$, $(-2,2)$, $(-2,-2)$, $(2,-2)$ minus the interior of the square with vertices $(0,1)$, $(-1,0)$, $(0,-1)$, $(1,0)$, and $f(x,y) = y$, then π_f is not open. However, f has local convexity data but it does not satisfy the locally fiber connected condition. If we consider the example where X is the square in $\mathbb{R}^2$ with vertices $(2,2)$, $(-2,2)$, $(-2,-2)$, $(2,-2)$ minus the interior of the rotated square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$, and $f(x,y) = y$, then again π_f is not open and f satisfies the locally fiber connected condition but does not have local convexity data, precisely because it does not satisfy the (SLO) condition.

An immediate consequence of Lemma 2.16 and Theorem 2.13 is the following corollary. It states that for a map that has local convexity data and connected fibers the condition to be open onto its image, and consequently to have a locally convex image, is implied by the condition to be closed onto its image or by the stronger condition to be a proper map.

Corollary 2.18. *Let X be a connected, locally connected, Hausdorff topological space, V a locally convex topological vector space, and $f : X \to V$ a continuous map that has local convexity data. Assume that f is a closed map onto its image $f(X)$ and that it has connected fibers. Then f is open onto its image $f(X)$ and $f(X)$ is locally convex. Moreover, if $f(X)$ is closed, then it is convex.*

Proof. The hypothesis implies that the induced map $\widetilde{f} : X_f \to f(X)$, uniquely determined by the equality $\widetilde{f} \circ \pi_f = f$, is a homeomorphism. Indeed, closedness of $\widetilde{f}$ follows from the identity $\widetilde{f}(A) = f(\pi_f^{-1}(A))$ for any subset A of X_f. Since $\widetilde{f}$ is open onto $f(X)$ it follows that $f = \widetilde{f} \circ \pi_f$ is also open onto its image. The rest is a consequence of Theorem 2.13. □

Note that by Lemma 2.5, a necessary condition for the map f that has connected fibers to be open onto its image is that the inverse image of any connected set in

$f(X)$ is connected in X. The next proposition states that if f has local convexity data with closed cones, this condition is also sufficient.

Proposition 2.19. *Let $f : X \to V$ be a continuous map that has local convexity data with closed cones. If the fibers of f are connected and for every point $v \in f(X)$ and for all small neighborhoods V_v of v the set $f^{-1}(V_v)$ is connected, then f is open onto its image.*

Proof. Suppose that f is not open onto its image, has local convexity data with closed cones, and has connected fibers. So there exists a point $x \in X$ and an open neighborhood U_x (included in a neighborhood of x from the definition of local convexity data) such that $V_{f(x)} \cap C_{x,f(x)} = f(U_x) \subsetneq f(X) \cap V_{f(x)}$ for some open neighborhood $V_{f(x)}$ of $f(x)$ in V. Consequently, $(V_{f(x)} \cap f(X)) \backslash f(U_x) \neq \varnothing$ is open in $f(X)$ since $f(U_x) = V_{f(x)} \cap C_{x,f(x)} \cap f(X)$ is closed in the topology of $V_{f(x)} \cap f(X)$ due to the fact that $C_{x,f(x)} \cap f(X)$ is closed in $f(X)$ (by the closed cone hypothesis). We can also choose $V_{f(x)}$ small enough so that the hypothesis holds for it; that is, $f^{-1}(V_{f(x)} \cap f(X))$ is connected.

Note that connectedness of the fibers, and thus bijectivity of $\widetilde{f}$, implies that $\widetilde{f}^{-1}(f(A)) = \pi_f(A)$ for any subset A of X.

The sets that enter in the equality

$$\widetilde{f}^{-1}(V_{f(x)} \cap f(X)) = \widetilde{f}^{-1}((V_{f(x)} \cap f(X)) \backslash f(U_x)) \cup \widetilde{f}^{-1}(f(U_x))$$

or equivalently

$$\pi_f(f^{-1}(V_{f(x)} \cap f(X))) = \widetilde{f}^{-1}((V_{f(x)} \cap f(X)) \backslash f(U_x)) \cup \pi_f(U_x)$$

are all open because π_f is open and we also have that $\widetilde{f}^{-1}((V_{f(x)} \cap f(X)) \backslash f(U_x)) \cap \widetilde{f}^{-1}(f(U_x)) = \varnothing$. But this contradicts the connectivity of $\pi_f(f^{-1}(V_{f(x)} \cap f(X)))$. □

The Local-to-Global Principle. The rest of this section is dedicated to the generalization of the so-called Local-to-Global Principle used in the convexity proof of Condevaux, Dazord and Molino [7] and of Hilgert, Neeb, and Plank [14]. On the one hand we will prove that one can drop the condition on the compactness of the fibers necessary in the classical proof and still have the same conclusions, and on the other hand we will extend this result to maps that have an infinite dimensional target. In order to avoid the compactness condition (we will call a topological space compact if it is Hausdorff and every open cover has a finite subcover) on the fibers we need to further investigate the topology of the problem. We start with a short account of some topological results needed in the sequel.

Theorem 2.20 ([9])**.** *Let $f : X \to Y$ be a continuous mapping.*

- **(i)** *f is closed if and only if for every $B \subset Y$ and every open set $A \subset X$ which contains $f^{-1}(B)$, there exists an open set $C \subset Y$ containing B and such that $f^{-1}(C) \subset A$.*
- **(ii)** *f is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{-1}(y)$, there exists in Y a neighborhood V_y of the point y such that $f^{-1}(V_y) \subset U$.*

We now prove a crucial lemma needed for the generalization of the Local-to-Global Principle.

Lemma 2.21. *Let X be a normal first countable topological space and V a locally convex topological vector space. Let $f : X \to V$ be a continuous map that has local convexity data and satisfies the locally fiber connected condition. Suppose f is a closed map. Then the following hold:*

(i) *The projection $\pi_f : X \to X_f$ is also a closed map.*
(ii) *The quotient X_f is a Hausdorff topological space.*

Proof. **(i)** Let $[x]$ be an arbitrary point in X_f and $U \subset X$ an arbitrary open set that contains ${\pi_f}^{-1}([x]) = E_x$, the connected component of $f^{-1}(f(x))$ that contains x. Denote by $F := f^{-1}(f(x)) \setminus E_x$, which is the union of all (closed) connected components of $f^{-1}(f(x))$ minus E_x. We claim that F is a closed subset of X. Indeed, if $z \in \overline{F}$, then there exist a sequence $\{z_n\}_{n\in\mathbb{N}}$ in F which converges to z. Continuity of f implies that $z \in f^{-1}(f(x))$. If $z \in E_x$, then any neighborhood of z intersects at least one other connected component of the fiber $f^{-1}(f(x))$, which contradicts the locally fiber connected condition. Since $z \in f^{-1}(f(x))$ it follows that $z \in F$ and hence F is closed. The same argument as above shows that (LFC) implies that E_x is also closed in X.

Due to the normality of X, there exist two open sets U_{E_x} and W such that $E_x \subset U_{E_x}$, $F \subset W$, and $U_{E_x} \cap W = \varnothing$. After shrinking, if necessary, we can take $U_{E_x} \subset U$. By the closedness of f there exists an open neighborhood $V_{f(x)}$ of $f(x)$ in V such that $E_x \subset f^{-1}(f(x)) \subset f^{-1}(V_{f(x)}) \subset U_{E_x} \cup W$, by Theorem 2.20 **(ii)**.

The set $A := U_{E_x} \cap f^{-1}(V_{f(x)})$ is a nonempty open subset of X which is also saturated with respect to the equivalence relation that defines π_f or, otherwise stated, ${\pi_f}^{-1}(\pi_f(A)) = A$. Indeed, if a connected component of a fiber of f from $f^{-1}(V_{f(x)})$ intersects U_{E_x}, respectively W, then it is entirely contained either in U_{E_x} or in W since $U_{E_x} \cap W = \varnothing$. We have hence proved that $\pi_f(A)$ is an open neighborhood of $[x]$ since π_f is an open map and ${\pi_f}^{-1}(\pi_f(A)) \subset U$. This shows, via Theorem 2.20 **(ii)**, that π_f is a closed map.
(ii) Since π_f is open, it suffices to prove that the equivalence relation that defines X_f has a closed graph in order to show that X_f is Hausdorff.

For every $x \in X$ there exists a neighborhood U_x that satisfies (VN), (SLO), and (LFC). Notice that the open set $\pi_f^{-1}(\pi_f(U'_x))$ also satisfies the (LFC) condition. By the normality of X there also exists a neighborhood U'_x of x with $\overline{U'_x} \subset U_x$. We shall prove that $\overline{{\pi_f}^{-1}(\pi_f(U'_x))} \subset {\pi_f}^{-1}(\pi_f(U_x))$ which shows that for every connected component E_x of a fiber there exists a saturated neighborhood of it which contains a smaller saturated neighborhood whose closure still satisfies (LFC). In order to prove the above inclusion observe that, since π_f is closed, we have $\overline{\pi_f(U'_x)} = \pi_f(\overline{U'_x}) \subset \pi_f(U_x)$. By the continuity of π_f we obtain the inclusion $\overline{{\pi_f}^{-1}(\pi_f(U'_x))} \subset {\pi_f}^{-1}(\overline{\pi_f(U'_x)}) \subset {\pi_f}^{-1}(\pi_f(U_x))$.

We now prove the closedness of the graph of the equivalence relation that defines X_f. Take $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ to be two convergent sequences in X such that x_n and y_n are in the same equivalence class for all $n \in \mathbb{N}$. Suppose that $x_n \to x$ and $y_n \to y$. The continuity of f guarantees that $f(x) = f(y)$. Additionally, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ all $x_n \in {\pi_f}^{-1}(\pi_f(U'_x))$. Consequently $y_n \in {\pi_f}^{-1}(\pi_f(U'_x))$ since x_n and y_n are in the same equivalence class and ${\pi_f}^{-1}(\pi_f(U'_x))$ is saturated. Since $y \in \overline{{\pi_f}^{-1}(\pi_f(U'_x))}$ and $\overline{{\pi_f}^{-1}(\pi_f(U'_x))}$ satisfies (LFC) we obtain that x and y sit necessarily on the same connected component of the fiber $f^{-1}(f(x))$

and are hence equivalent. This shows that the graph of the equivalence relation is closed, as required. □

Remark 2.22. Let $f : X \to V$ be a continuous map such that f has local convexity data and is locally fiber connected. Then for every point $[x] \in X_f$ there is a neighborhood $\widetilde{U}_{[x]}$ of $[x]$ such that $\widetilde{f} : \widetilde{U}_{[x]} \to C_{x,f(x)}$ is a homeomorphism onto its open image. Eventually, after shrinking $\widetilde{U}_{[x]}$, we can suppose that its image is convex.

Note that X_f is connected since X is connected. Additionally, this remark implies that X_f is locally path connected. Therefore, X_f is path connected.

From now on we take V to be a Banach space. Define a distance d on X_f in the following way: for $[x], [y] \in X_f$ let $d([x], [y])$ be the infimum of all the lengths $l(\widetilde{f} \circ \gamma)$, where γ is a continuous curve in X_f that connects $[x]$ and $[y]$. The length is calculated with respect to the distance d_V defined by the norm on V. From the definition it follows that $d_V(\widetilde{f}([x]), \widetilde{f}([y])) \leq d([x], [y])$ and, by Remark 2.22, equality holds for $[x]$ and $[y]$ sufficiently close. Indeed, we shall prove that for any two points $[x], [y] \in X_f$ that are both contained in an open subset U that satisfies (VN), (SLO), and (LFC), the equality $d_V(\widetilde{f}([x]), \widetilde{f}([y])) = d([x], [y])$ is satisfied. Due to the (VN) condition, the set $\widetilde{f}(U)$ is convex in V and hence there exists a straight line c_0 joining $\widetilde{f}([x])$ and $\widetilde{f}([y])$. Since U satisfies (SLO) and (LFC), the map $\widetilde{f}|_U : U \to \widetilde{f}(U)$ is a homeomorphism and hence the curve $\gamma_0 := \widetilde{f}^{-1} \circ c_0$ is continuous and joins $[x]$ and $[y]$. Consequently,

$$d([x], [y]) = \inf_{\gamma \in \Gamma_{[x],[y]}} \{l(\widetilde{f} \circ \gamma)\} = l(c_0) = d_V(\widetilde{f}([x]), \widetilde{f}([y])),$$

where $\Gamma_{[x],[y]} := \{\gamma : [a, b] \to X_f \mid \gamma \text{ is continuous and } \gamma(a) = [x],\ \gamma(b) = [y]\}$.

Proposition 2.23. *Assume the hypotheses of Lemma* 2.21 *and suppose that V is a Banach space. The distance on X_f introduced above defines a metric topology on X_f that coincides with the quotient topology of X_f.*

Proof. The proof mimics those in [7] or [14]. The symmetry of d and the triangle inequality for d are obvious. It remains to be proved that if $d([x], [y]) = 0$, then $[x] = [y]$. By contradiction, suppose that $d([x], [y]) = 0$ with $[x] \neq [y]$. Then $d_V(\widetilde{f}([x]), \widetilde{f}([y])) = 0$ and hence $\widetilde{f}([x]) = \widetilde{f}([y])$. Since X_f is Hausdorff we can find two disjoint open neighborhoods $\widetilde{U}_{[x]}$ and $\widetilde{U}_{[y]}$ of $[x]$ and $[y]$, respectively, that behave as in Remark 2.22. Take $r > 0$ such that the open disk $D_r(f([x])) := \{v \in V \mid d_V(v, f([x])) < r\}$ satisfies $D_r(f([x])) \cap C_{x,f(x)} \subset \widetilde{f}(\widetilde{U}_{[x]})$. Then, for all the curves γ that connect $[x]$ and $[y]$ we have that $l(\widetilde{f} \circ \gamma) > 2r$. So, in order to have $l(\widetilde{f} \circ \gamma) < 2r$, $[y]$ must be contained in $\widetilde{U}_{[x]}$, which is a contradiction since $\widetilde{U}_{[x]} \cap \widetilde{U}_{[y]} = \varnothing$.

The construction of d and Remark 2.22 show that $\widetilde{f} : X_f \to V$ is a local isometry on the image $\widetilde{f}(\widetilde{U}_{[x]})$ of all the small enough neighborhoods $\widetilde{U}_{[x]}$ of any point $[x] \in X_f$. It follows that the metric topology on X_f coincides with the quotient topology of X_f. □

The proof of the following lemma can be found in [9].

Lemma 2.24. *(Vaĭnšteĭn) If $f : X \to Y$ is a closed mapping from a metrizable space X onto a metrizable space Y, then for every $y \in Y$ the boundary* $\mathrm{bd}(f^{-1}(y)) := \overline{f^{-1}(y)} \cap \overline{(X \setminus f^{-1}(y))}$ *is compact.*

Definition 2.25. Let X be a Hausdorff topological space and $f : X \to Y$ be a continuous map. We call f a **proper map** if f is closed and all fibers $f^{-1}(y)$ are compact subsets of X.

Theorem 2.26 ([9])**.** *If $f : X \to Y$ is a proper map, then for every compact subset $Z \subset Y$ the inverse image $f^{-1}(Z)$ is compact.*

A converse of this theorem is available when Y is a k-space (i.e., Y is a Hausdorff topological space that is the image of a locally compact space under a quotient mapping).

The finite dimensional case. For the next considerations we need that V is a finite dimensional vector space with a fixed chosen inner product that defines the distance d_V. The infinite dimensional case will be discussed later.

Proposition 2.27. *In the hypotheses of Lemma* 2.21 *with V a finite dimensional Euclidean vector space we have that $\widetilde{f} : X_f \to V$ is a proper map and $B_r([x]) := \{[y] \in X_f \mid d([x],[y]) \leq r\}$ is compact.*

Proof. The locally fiber connectedness condition on f and the openness of π_f imply that $\widetilde{f}^{-1}(v)$ is a collection of isolated points in X_f and hence $\widetilde{f}^{-1}(v) = \mathrm{bd}\left(\widetilde{f}^{-1}(v)\right)$, for every $v \in f(X)$. As a consequence of the Vaĭnšteĭn Lemma we obtain that the fibers of $\widetilde{f}$ are all compact. Also, $\widetilde{f}$ is closed since f is closed and hence $\widetilde{f}$ is a proper map.

The set $B_r([x])$ is closed in X_f and, by the definition of the distance d, we have that $\widetilde{f}(B_r([x])) \subset B_r(\widetilde{f}([x])) \subset V$. Since the ball $B_r(\widetilde{f}([x])) \subset V$ is closed and bounded it is necessarily compact in the finite dimensional vector space V. Properness of $\widetilde{f}$ implies that $\widetilde{f}^{-1}(B_r(\widetilde{f}([x])))$ is compact in X_f. Since $B_r([x])$ is a closed subset of $\widetilde{f}^{-1}(B_r(\widetilde{f}([x])))$, it is necessarily compact in X_f. □

We can now prove one of the main results of this section, which is the generalization of the Local-to-Global Principle in [7] or [14] to the case of closed maps that are not necessarily proper.

Theorem 2.28. *Let $f : X \to V$ be a closed map with values in a finite dimensional Euclidean vector space V and X a connected, locally connected, first countable, and normal topological space. Assume that f has local convexity data and is locally fiber connected. Then:*

- **(i):** *All the fibers of f are connected.*
- **(ii):** *f is open onto its image.*
- **(iii):** *The image $f(X)$ is a closed convex set.*

Proof. We begin with the argument of [7, 14]. Let $[x]_0, [x]_1 \in X_f$ be two arbitrary points and $c := d([x]_0, [x]_1)$. By the definition of d, we have that for every $n \in \mathbb{N}$ there exists a curve γ_n defined on the interval $[a, b]$, connecting $[x]_0$ and $[x]_1$, and satisfying $l(\widetilde{f} \circ \gamma_n) \leq c + \frac{1}{n}$. Also, for every $n \in \mathbb{N}$, let $v_n = (\widetilde{f} \circ \gamma_n)(t_0)$ be the point on the curve $\widetilde{f} \circ \gamma_n$ such that $l(\widetilde{f} \circ \gamma_n|_{[a,t_0]}) = \frac{1}{2} l(\widetilde{f} \circ \gamma_n)$. Then there exists a finite set

of points $\{[x]_1^n, ..., [x]_{k_n}^n\}$ in X_f such that $\widetilde{f}^{-1}(v_n) \cap \text{range}(\gamma_n) = \{[x]_1^n, ..., [x]_{k_n}^n\} \subset B_{c+1}([x]_0)$, where $B_{c+1}([x]_0) := \{[x] \in X_f \mid d([x]_0, [x]) \leq c+1\}$ is compact by Proposition 2.27. Relabeling the elements of the set $\bigcup_{n\in\mathbb{N}}\{[x]_1^n, ..., [x]_{k_n}^n\}$ we obtain a sequence included in the compact set $B_{c+1}([x]_0)$ and, consequently, it will have an accumulation point denoted by $[x]_{\frac{1}{2}}$.

The definition of d implies that $d([x]_0, [x]_{\frac{1}{2}}) = d([x]_{\frac{1}{2}}, [x]_1) = \frac{c}{2}$. Repeating this process for the pair of points $([x]_0, [x]_{\frac{1}{2}})$ and $([x]_{\frac{1}{2}}, [x]_1)$ we obtain the points $[x]_{\frac{1}{4}}$ and $[x]_{\frac{3}{4}}$ satisfying $d([x]_0, [x]_{\frac{1}{4}}) = d([x]_{\frac{1}{4}}, [x]_{\frac{1}{2}}) = d([x]_{\frac{1}{2}}, [x]_{\frac{3}{4}}) = d([x]_{\frac{3}{4}}, [x]_1) = \frac{c}{4}$. Inductively, we obtain points $[x]_{n/2^m}$, $[x]_{n'/2^{m'}}$ for $0 \leq n \leq 2^m$, $0 \leq n' \leq 2^{m'}$, such that

$$d([x]_{n/2^m}, [x]_{n'/2^{m'}}) = c\left|\frac{n}{2^m} - \frac{n'}{2^{m'}}\right|. \tag{2.1}$$

We can extend the map $n/2^m \mapsto [x]_{n/2^m}$ to a continuous map $\gamma : [0,1] \to X_f$ such that

$$d(\gamma(t), \gamma(t')) = c|t - t'|. \tag{2.2}$$

To see this, note that every $t \in [0,1]$ can be approximated by a sequence of the type $n_k/2^{m_k}$. The corresponding points $[x]_{n_k/2^{m_k}}$ are contained in the compact set $B_{c+1}([x]_0)$, and hence they have an accumulation point $[x]_t$. It is now easy to see, using (2.1), that $[x]_t$ does not depend on the sequence $n_k/2^{m_k}$ and that the curve γ constructed in this way is continuous.

Remark 2.22 and (2.2) imply that, locally, $d_V((\widetilde{f} \circ \gamma)(t), (\widetilde{f} \circ \gamma)(t')) = c|t - t'|$ which shows that $\widetilde{f} \circ \gamma$ is locally a straight line. Due to (2.2), $\widetilde{f} \circ \gamma$ is necessarily a straight line that goes through $\widetilde{f}([x]_0)$ and $\widetilde{f}([x]_1)$. This proves the convexity of $f(X)$. Since f is a closed map the set $f(X)$ is closed in V which proves **(iii)**.

In order to prove the connectedness of the fibers of f let $[x]_0, [x]_1 \in X_f$ be two arbitrary points such that $v := \widetilde{f}([x]_0) = \widetilde{f}([x]_1)$ and $c := d([x]_0, [x]_1)$. Any curve that connects these two points is mapped by $\widetilde{f}$ into a loop based at v. We shall prove that $c = 0$, which implies that $[x]_0 = [x]_1$ and hence that the fibers of f are connected. Let γ be the curve constructed above. Then the range of $\widetilde{f} \circ \gamma$ is a segment that contains v. We will prove by contradiction that this segment consists of just one point which is v itself.

Suppose that this is not true. Since $\widetilde{f} \circ \gamma$ is a loop based at v and at the same time a straight line, there exists a turning point $v_0 := (\widetilde{f} \circ \gamma)(t_0)$ on the segment $\widetilde{f} \circ \gamma$ such that for $t \leq t_0$ we approach v_0 and for $t' \geq t_0$ we move away from v_0 staying on the same segment which is the range of $\widetilde{f} \circ \gamma$. Otherwise stated, $\text{range}(\widetilde{f} \circ \gamma|_{[t,t_0]}) = \text{range}(\widetilde{f} \circ \gamma|_{[t_0,t']})$ and hence in a neighborhood of $\gamma(t_0)$ the map $\widetilde{f}$ is not injective. However, since f is locally fiber connected the map $\widetilde{f}$ is locally injective, which is a contradiction. This proves **(i)**. Note that from $c := d([x]_0, [x]_1)$ and $d_V(\widetilde{f}([x]_0), \widetilde{f}([x]_1)) = d_V(v, v) = 0$ we cannot conclude that $c = 0$ since the equality between the two metrics holds only locally.

The openness of f is implied by Corollary 2.18. □

Remark 2.29. This result remains true if we replace the vector space V by a convex subset C of V. More specifically, Theorem 2.28 is valid when we apply it to a map $f : X \to C$, with C a convex subset of V. In particular, this allows us to generalize

to the case of closed maps Theorem 4.2 of Lerman et al. [27] and Theorem 3.3 of Weinstein [36], initially stated for proper maps. This remark is also important later on when we generalize several other classical convexity theorems.

The infinite dimensional case. Now let V be an infinite dimensional Banach space and d_V the distance induced by the norm. Analyzing the proof of Proposition 2.27 it can be seen that the compactness of the closed balls in the norm topology was essential. Thus, there is no direct analog of this statement in infinite dimensions. What is needed is a second topology on V whose closed balls are compact. A natural hypothesis is that V is the topological dual space of another Banach space W for then Alaoglu's Theorem guarantees that $B_r(0) := \{v \in V \mid \|v\| \leq r\}$ is weak* compact. We shall denote by $(V, \|\cdot\|)$ the Banach space V endowed with the norm topology (and hence this is the Banach space dual W^*) and by (V, w^*) the space V endowed with the weak* topology of W^*. Since the weak* topology is weaker than the norm topology we have

- if $f : X \to (V, \|\cdot\|)$ is continuous, then $f : X \to (V, w^*)$ is continuous;
- if $f : X \to (V, w^*)$ is closed, then $f : X \to (V, \|\cdot\|)$ is closed.

The analog of Proposition 2.27 is the following.

Proposition 2.30. *Assume the hypotheses of Lemma* 2.21 *with $V = W^*$, for W a Banach space. If $f : X \to (V, \|\cdot\|)$ is continuous and $f : X \to (V, w^*)$ is closed, then $\widetilde{f} : X_f \to V$ is a proper map relative to both topologies on V and $B_r([x]) := \{[y] \in X_f \mid d([x], [y]) \leq r\}$ is compact in X_f.*

Proof. The locally fiber connectedness condition for f and the openness of π_f imply that $\widetilde{f}^{-1}(v)$ is a collection of isolated points in X_f and hence $\widetilde{f}^{-1}(v) = \mathrm{bd}\left(\widetilde{f}^{-1}(v)\right)$, for every $v \in f(X)$. Since $\widetilde{f} : X_f \to (V, w^*)$ is a closed map, then so is $\widetilde{f} : X_f \to (V, \|\cdot\|)$. By Vaĭnšteĭn's Lemma the fibers of $\widetilde{f}$ are all compact. Also $\widetilde{f} : X_f \to (V, w^*)$ is closed and hence $\widetilde{f} : X_f \to (V, w^*)$ is a proper map.

The set $B_r([x])$ is closed in X_f and by the definition of the distance d we have that $\widetilde{f}(B_r([x])) \subset B_r(\widetilde{f}([x])) \subset V$. The ball $B_r(\widetilde{f}([x])) \subset V$ is weak* compact and since $\widetilde{f} : X_f \to (V, w^*)$ is a proper map we have that $\widetilde{f}^{-1}(B_r(\widetilde{f}([x])))$ is compact in X_f. As $B_r([x])$ is a closed subset of $\widetilde{f}^{-1}(B_r(\widetilde{f}([x])))$, it is necessarily compact in X_f. □

Using Proposition 2.30, one can generalize Theorem 2.28 to the context of maps with an infinite dimensional range by virtually copying its proof.

Theorem 2.31. *Let $(V, \|\cdot\|)$ be a Banach space such that $V = W^*$, for W a Banach space. Let $f : X \to (V, \|\cdot\|)$ be a continuous map and $f : X \to (V, w^*)$ closed, where X is a connected, locally connected, first countable, and normal topological space. Assume that f has local convexity data and is locally fiber connected. Then:*

(i): *All the fibers of f are connected.*
(ii): *$f : X \to (V, w^*)$ is open onto its image.*
(iii): *The image $f(X) \subset (V, w^*)$ is a closed convex set.*

Remark 2.32. Since the weak* topology is weaker than the norm topology the previous theorem also implies that $f : X \to (V, \|\cdot\|)$ is open onto its image and that $f(X)$ is closed in $(V, \|\cdot\|)$ (and obviously also convex since the notion of convexity is not related to the topology). Moreover, as the fibers of f are connected the map

$\widetilde{f} : X_f \to f(X)$ is a homeomorphism when $f(X)$ is considered as a topological subspace of either $(V, \|\cdot\|)$ or (V, w^*). This apparently strong conclusion is actually guaranteed to hold automatically for an important class of normed spaces $(V, \|\cdot\|)$. For instance, as $V = W^*$, if W is reflexive, then the weak and the weak* topologies in V coincide. Moreover, Mazur's Theorem [29] guarantees that the weak and the norm closures of a convex set on a normed space coincide. The same interplay between the weak* and norm topology can be found in the infinite dimensional convexity results of Bloch-Flaschka-Ratiu [4] and Neumann [31].

3. Openness and local convexity for momentum maps

In the previous section we presented sufficient conditions for the convexity of the image of a map that has both local convexity data and is locally fiber connected. More specifically, in Theorem 2.13 we saw that if the map is open onto its image, then the image is locally convex and therefore convex if it is also closed. In this section we will characterize the situations in which the momentum map associated to a compact and symplectic Lie group action on a symplectic manifold is an open map onto its image. We will also give several generalizations of convexity results found in the literature. Throughout this paper, all manifolds are assumed to be paracompact; that is, they are Hausdorff spaces and every open cover has a locally finite open refinement.

Definition 3.1. Let V be a finite dimensional vector space.

- **(i)**: A subset $K \subset V$ is called ***polyhedral*** if it is the intersection of a finite family of closed half-spaces of V. Consequently, a polyhedral subset of V is closed and convex.
- **(ii)**: A subset $K \subset V$ is called ***locally polyhedral*** if for every $x \in K$ there exists a polytope P_x in V such that $x \in \mathrm{int}(P_x)$ and $K \cap P_x$ is a polytope.

In order to proceed we need further preparation. First we specialize the Marle-Guillemin-Sternberg Normal Form Theorem to the case of torus actions in the formulation of [14]; see also [28, 13, 32] for general normal form theorems and their proofs.

Theorem 3.2. *Let (M, ω) be a symplectic manifold and let T be a torus acting on M in a globally Hamiltonian fashion with invariant momentum map $J_T : M \to \mathfrak{t}^*$. Let $m \in M$ and $T_0 = (T_m)^0$ be the connected component of the stabilizer T_m. Let $T_1 \subset T$ be a subtorus such that $T = T_0 \times T_1$. Then:*

- **(i)** *There exist a symplectic vector space (V, ω_V), a T-invariant open neighborhood $U \subset M$ of the orbit $T \cdot m$, and a symplectic covering of a T-invariant open subset U' of $T_1 \times \mathfrak{t}_1^* \times V$ onto U under which the T-action on M is modeled by*

$$\begin{array}{rcl} (T_0 \times T_1) \times ((T_1 \times \mathfrak{t}_1^*) \times V) & \to & ((T_1 \times \mathfrak{t}_1^*) \times V) \\ ((t_0, t_1), (t_1^{'}, \beta, v)) & \mapsto & (t_1 t_1^{'}, \beta, \pi(t_0)v), \end{array}$$

  where $\pi : T_0 \to Sp(V)$ is a symplectic representation.
- **(ii)** *There exists a complex structure I on V such that $\langle v, w \rangle := \omega_V(Iv, w)$ defines a positive definite scalar product on V. Then $V = \bigoplus_{\alpha \in \mathcal{P}_V} V_\alpha$, where*

$V_\alpha := \{v \in V \mid Y \cdot v = \alpha(Y)Iv, \textit{for all } Y \in \mathfrak{t}_0\}$ *and* $\mathcal{P}_V := \{\alpha \in \mathfrak{t}_0^* \mid V_\alpha \neq \{0\}\}$. *The corresponding* T*-momentum map* $\Phi : T^*(T_1) \times V \to \mathfrak{t}_1^* \times \mathfrak{t}_0^* \simeq \mathfrak{t}^*$ *is given by*

$$\Phi\left((t_1, \beta), \sum_\alpha v_\alpha\right) = \Phi(1,0,0) + \left(\beta, \frac{1}{2}\sum_{\alpha \in \mathcal{P}_V} ||v_\alpha||^2 \alpha\right).$$

Notice that the original version of the Marle-Guillemin-Sternberg Normal Form Theorem provides the twisted product $(T_0 \times T_1) \times_{T_0} (\mathfrak{t}_1^* \times V)$ as a T-invariant local model for M. This is equivariantly diffeomorphic to $T_1 \times \mathfrak{t}_1^* \times V$ via the map

$$\begin{array}{ccc} (T_0 \times T_1) \times_{T_0} (\mathfrak{t}_1^* \times V) & \longrightarrow & T_1 \times \mathfrak{t}_1^* \times V \\ [(t_1, t_0), \eta, v] & \longmapsto & (t_1, \eta, t_0 \cdot v). \end{array}$$

The next result shows that the momentum maps of globally Hamiltonian torus actions always have local convexity data with closed cones and are locally fiber connected. In fact, the associated cones are closed in $\mathfrak{t}^*$.

Theorem 3.3. *Let* (M, ω) *be a symplectic manifold and let* T *be a torus acting on* M *in a globally Hamiltonian fashion with invariant momentum map* $J_T : M \to \mathfrak{t}^*$. *Then there exist an arbitrarily small neighborhood* U *of* m *and a convex polyhedral cone* $C_{J(m)} \subset \mathfrak{t}^*$ *with vertex* $J_T(m)$ *such that:*

- **(i):** $J_T(U) \subset C_{J_T(m)}$ *is an open neighborhood of* $J_T(m)$ *in* $C_{J_T(m)}$*;*
- **(ii):** $J_T : U \to C_{J_T(m)}$ *is an open map;*
- **(iii):** *if* $\mathfrak{t}_0$ *is the Lie algebra of the stabilizer* T_m *of* m*, then* $C_{J_T(m)} = J_T(m) + \mathfrak{t}_0^\perp + \mathrm{cone}(\mathcal{P}_V)$*;*
- **(iv):** $J_T^{-1}(J_T(m)) \cap U$ *is connected for all* $m \in U$.

Here is a sketch of the proof; for details, see [14]. Recall that $\mathrm{cone}(\mathcal{P}_V) := \{\sum_j a_j \alpha_j \mid a_j \geq 0\}$. By Theorem 3.2, it suffices to work with the momentum map Φ. For small neighborhoods $B_{\mathfrak{t}_1^*}$ and B_V of the origin in $\mathfrak{t}_1^*$ and V, respectively, the restriction of Φ to $U := T_1 \times B_{\mathfrak{t}_1^*} \times B_V$ takes values in the polyhedral closed convex cone $C_{J_T(m)} = J_T(m) + \mathfrak{t}_1^* + \mathrm{cone}(\mathcal{P}_V)$, where $\mathrm{cone}(\mathcal{P}_V)$ denotes the cone generated by the finite set $\mathcal{P}_V = \{\alpha_1, ..., \alpha_n\}$ of T_0-weights for the action on V; $\mathrm{cone}(\mathcal{P}_V)$ is clearly closed. In order to prove that J_T satisfies the conditions of the theorem, we decompose it into two maps $\varphi_1 : (t_1, \beta, \sum_\alpha v_\alpha) \mapsto (\beta, (\|v_\alpha\|^2))$ and $\varphi_2 : (\beta, a_1, ..., a_n) \mapsto (\beta, \frac{1}{2}\sum_j a_j \alpha_j)$. We have $J_T = \varphi_2 \circ \varphi_1 + J_T(m)$. One proves that φ_1, φ_2 are open onto their images and have connected fibers, so J_T has the same properties.

We now state a generalization of the Atiyah-Guillemin-Sternberg Convexity Theorem for noncompact manifolds.

Corollary 3.4. *Let* M *be a paracompact connected symplectic manifold on which a torus* T *acts in a Hamiltonian fashion. Let* $J_T : M \to \mathfrak{t}^*$ *be an associated momentum map which we suppose is closed. Then the image* $J_T(M)$ *is a closed convex locally polyhedral subset in* $\mathfrak{t}^*$. *The fibers of* J_T *are connected and* J_T *is open onto its image.*

Proof. This is a consequence of Theorems 3.3 and 2.28. The image is locally polyhedral since J_T is open onto its image and the associated cones are polyhedral. □

Our approach to convexity also allows us to generalize a result due to Prato [33].

Theorem 3.5. *Let M be a connected symplectic manifold on which a torus T acts in a Hamiltonian fashion with associated invariant momentum map $J_T : M \to \mathfrak{t}^*$.*

- **(i):** *If there exists $\xi \in \mathfrak{t}$ such that the map $J_T^\xi \in C^\infty(M)$ defined by $J_T^\xi := \langle J_T, \xi \rangle$ is proper, then the image $J_T(M)$ is a closed convex locally polyhedral subset in $\mathfrak{t}^*$. Moreover, the fibers of J_T are connected and J_T is open onto its image.*
- **(ii):** *If there exists an integral element $\xi \in \mathfrak{t}$ such that J_T^ξ is a proper function having a minimum as its unique critical value, then $J_T(M)$ is the convex hull of a finite number of affine rays in $\mathfrak{t}^*$ stemming from the images of T-fixed points.*

Proof. **(i)** We will carry out the proof using our generalization of the Local-to-Global Principle (Theorem 2.28). According to this result all that needs to be proved is that in the presence of our hypotheses J_T is a closed map. This will be shown by verifying that $\overline{J_T(A)} \subset J_T(\overline{A})$, for any subset $A \subset M$.

We start by noticing that the map $J_T^\xi \in C^\infty(M)$ can be written as $J_T^\xi = b \circ \pi \circ J_T$, where $\pi : \mathfrak{t}^* \to \operatorname{span}\{\xi\}^*$ is the dual of the inclusion $\operatorname{span}\{\xi\} \hookrightarrow \mathfrak{t}$ and $b : \operatorname{span}\{\xi\}^* \to \mathbb{R}$ is the linear isomorphism obtained as the map that assigns to each element in $\operatorname{span}\{\xi\}^*$ its coordinate in the dual basis of $\{\xi\}$ as a basis of $\operatorname{span}\{\xi\}$. Let $\mu \in \overline{J_T(A)}$ be arbitrary and $\{\mu_n\}_{n\in\mathbb{N}} \subset J_T(A)$ a sequence such that $\mu_n \to \mu$. Let $\{x_n\}_{n\in\mathbb{N}} \subset A$ be a sequence such that $J_T(x_n) = \mu_n$. By continuity we have that $J_T^\xi(x_n) = b \circ \pi \circ J_T(x_n) \to b \circ \pi(\mu)$. Since by hypothesis J_T^ξ is a proper map there exists a convergent subsequence $x_{n_k} \to x \in \overline{A}$ and hence $J_T(x_{n_k}) \to J_T(x) = \mu$, which shows that $\mu \in J_T(\overline{A})$, as required. Part **(ii)** is the original result of Prato [33]. □

We now recall some standard notions from the theory of proper Lie group actions. Let M be a manifold and G a Lie group acting properly on it. The orbit $G \cdot m$ is called ***regular*** if the dimension of nearby orbits coincides with the dimension of $G \cdot m$. Let M^{reg} denote the union of all regular orbits. For every connected component M^0 of M the subset $M^{reg} \cap M^0$ is connected, open, and dense in M^0. Note that if U is a G-invariant connected open submanifold, then the set of regular points for the induced G-action on U equals $U \cap M^{reg}$. This shows that $U \cap M^{reg}$ is open, dense, and connected in U.

Next, we give necessary topological conditions on the image of J_T that ensure that J_T is open onto its image. Later we will show that these conditions are also sufficient. We recall that a ***region*** in a topological space is, by definition, a connected open set.

Proposition 3.6. *Let $J_T : M \to \mathfrak{t}^*$ be the momentum map of a torus action on the connected symplectic manifold (M, ω). Suppose that J_T is open onto its image. Then the complement $\mathbf{C}J_T(M^{reg}) := J_T(M) \setminus J_T(M^{reg})$ does not disconnect any region in $J_T(M)$.*

Proof. Suppose there exists a region $V \subset J_T(M)$ (relative to the induced topology from $\mathfrak{t}^*$) such that $V \backslash \mathbf{C}J_T(M^{reg})$ is disconnected and hence $V \backslash \mathbf{C}J_T(M^{reg}) = A \cup B$, where A and B are open in V and $A \cap B = \varnothing$. Moreover, since $J_T(M^{reg})$ is dense in $J_T(M)$, we have that $J_T(M^{reg}) \cap V = V \backslash \mathbf{C}J_T(M^{reg}) = A \cup B$ is dense in $J_T(M) \cap V = V$. Hence $\overline{(A \cup B)} \cap V = V = (\overline{A} \cap V) \cup (\overline{B} \cap V)$.

We claim that there exists an element $v \in \mathbf{C}J_T(M^{reg}) \cap V$ such that any neighborhood $V_v \subset V$ of v in V is disconnected by $\mathbf{C}J_T(M^{reg}) \cap V$. Suppose that this claim is false. Then for every $v \in \mathbf{C}J_T(M^{reg}) \cap V$ there would exist a neighborhood $V_v \subset V$ of v such that $V_v \backslash \mathbf{C}J_T(M^{reg})$ is connected. Therefore, we must have that either $V_v \backslash \mathbf{C}J_T(M^{reg}) \subset A$ or $V_v \backslash \mathbf{C}J_T(M^{reg}) \subset B$. Thus, either $v \in \overline{A} \cap V$ or $v \in \overline{B} \cap V$. Since $v \in \mathbf{C}J_T(M^{reg}) \cap V$ is arbitrary, this shows that $(\overline{A} \cap V) \cap (\overline{B} \cap V) = \varnothing$. This contradicts the connectivity of V and hence there exist a $v \in \mathbf{C}J_T(M^{reg}) \cap V$ such that any neighborhood $V_v \subset V$ is disconnected by $\mathbf{C}J_T(M^{reg})$.

Take an arbitrary element $x \in J_T^{-1}(v)$ and U_x a small neighborhood of x such that $J_T(U_x) \subset V$ is an open neighborhood of v in $J_T(M)$; this holds because J_T is open onto its image by hypothesis. Then, by assumption, $J_T(U_x) \cap J_T(M^{reg})$ is disconnected. Taking the T-saturation of U_x we get a T-invariant neighborhood whose image is in V since J_T is T-invariant. Thus, we can assume that U_x is T-invariant and then the set of regular points for the induced T-action on U_x equals the set $U_x \cap M^{reg}$, which in turn is open, dense, and connected in U_x.

Let $E := \{z \in U_x \mid J_T(z) \in \mathbf{C}J_T(M^{reg})\}$. Since we can write $U_x = E \cup D$ with $D := U_x \setminus E$, by the construction of E we have $J_T(E) = J_T(U_x) \cap \mathbf{C}J_T(M^{reg})$ and $J_T(D) = J_T(U_x) \cap J_T(M^{reg})$. Now, since $E \subset U_x \setminus M^{reg}$, the inclusion $U_x \cap M^{reg} \subset D$ also holds. Because $U_x \cap M^{reg}$ is dense and connected in U_x, so is D in U_x. But this is a contradiction with the fact that $J_T(D) = J_T(U_x) \cap J_T(M^{reg})$ is disconnected. This proves the result. □

Now we will prove the converse of Proposition 3.6 in the case in which the momentum map has connected fibers. For this we need a preparatory lemma.

Lemma 3.7. *Let (M, ω) be a connected symplectic manifold and $J_T : M \to \mathfrak{t}^*$ be the invariant momentum map associated to a canonical T-action on M. Then $J_T|_{M^{reg}} : M^{reg} \to J_T(M)$ is an open map. In particular $J_T(M^{reg})$ is an open dense subset of $J_T(M)$.*

Proof. We shall prove that for each point in M^{reg} there is an open neighborhood such that the restriction of J_T to this neighborhood is an open map onto its image. Let $x_0 \in M^{reg}$ be an arbitrary point. By the openness and the T-invariance of M^{reg} we can find an open connected T-invariant neighborhood U_{x_0} of x_0 included in M^{reg}. Therefore, for any $x \in U_{x_0}$, we have $\dim T \cdot x = \dim T \cdot x_0 = \dim T/T_0 = \dim T_1$, where $T_0 := (T_{x_0})^0$ and $T = T_0 \times T_1$. Eventually shrinking U_{x_0}, using Theorem 3.2, we can work with the normal form. Recall that the original action is symplectically and T-equivariantly transformed to the action

$$\begin{aligned} (T_0 \times T_1) \times ((T_1 \times \mathfrak{t}_1^*) \times V) &\to ((T_1 \times \mathfrak{t}_1^*) \times V) \\ ((t_0, t_1), (t_1', \beta, v)) &\mapsto (t_1 t_1', \beta, \pi(t_0)v), \end{aligned}$$

where $\pi : T_0 \to Sp(V)$ is a linear symplectic representation. Since the isotropy subgroup of this action at the point (t_1', β, v) equals $\{t_0 \in T_0 \mid \pi(t_0)v = v\} \times \{e\} \subset T_0 \times T_1$, the condition that it be equal to $T_0 \times \{e\}$ implies that the representation π is trivial. Therefore all its weights are zero. By Theorem 3.3 we conclude that $C_{J_T(x_0)} = J_T(x_0) + \mathfrak{t}_1^* + \mathrm{cone}(\mathcal{P}_V) = J_T(x_0) + \mathfrak{t}_1^*$ and that $J_T : U_{x_0} \to C_{J_T(x_0)} = J_T(x_0) + \mathfrak{t}_1^*$ is an open map.

Note that $J_T(M^{reg}) \subset J_T(x_0) + \mathfrak{t}_1^*$ for some (and hence any) $x_0 \in M^{reg}$ and T_1 is the torus whose Lie algebra is $\mathfrak{t}_0^\perp$, where $\mathfrak{t}_0$ is the isotropy algebra of a regular point

in M and the perpendicular is taken relative to an a priori chosen T-invariant inner product on $\mathfrak{t}$. Indeed, using the well-chained property of M^{reg} any two points in M^{reg} can be linked by a finite chain formed by the open neighborhoods constructed above. The image of each such neighborhood lies in a translate of $\mathfrak{t}_1^*$ and since the neighborhoods intersect pairwise, all these affine spaces coincide. Thus, $J_T(M^{reg})$ lies in just one translate of $\mathfrak{t}_1^*$. By the density of M^{reg} in M and the closedness of the affine space in $\mathfrak{t}^*$ it follows that $J_T(M)$ lies in the same affine space.

Hence, we have shown that for any $x_0 \in M^{reg}$ there exists an open neighborhood $U_{x_0} \subset M^{reg}$ such that $J_T(U_{x_0})$ is open in a given translate of $\mathfrak{t}_1^*$. Therefore, $J_T(U_{x_0})$ is open in $J_T(M)$. □

Proposition 3.8. *Let $J_T : M \to \mathfrak{t}^*$ be the momentum map of a torus action. Assume that J_T has connected fibers and that $\mathbf{C}J_T(M^{reg}) := J_T(M) \setminus J_T(M^{reg})$ does not disconnect any region in $J_T(M)$. Then J_T is open onto its image.*

Proof. If M has more than one connected component, then the J_T-images of any two components do not intersect, for otherwise this would contradict the connectedness of the fibers. Since connected components of M are necessarily T-invariant, we can suppose without loss of generality that M is connected. We will establish the openness of J_T onto its image through Proposition 2.19. In order to apply this result it only remains to be shown that for any $v \in J_T(M)$ and any neighborhood V_v, the pre-image $J_T^{-1}(V_v)$ is connected in M.

By Lemma 3.7 we know that $J_T|_{M^{reg}} : M^{reg} \to J_T(M)$ is an open map. Denote, as in Lemma 3.7, by $\mathfrak{t}_1^*$ the dual of the subtorus whose translate contains $J_T(M)$.

Since M is path connected, $J_T(M)$ is also path connected and thus it is also locally connected. Let $J_T(x) \in J_T(M)$ be arbitrary. Choose a small neighborhood $V_{J_T(x)}$ of $J_T(x)$ in $\mathfrak{t}_1^*$ such that $V := V_{J_T(x)} \cap J_T(M)$ is a region in $J_T(M)$. Then $V_0 := V_{J_T(x)} \cap J_T(M^{reg})$ is connected due to the hypothesis that the region $V := V_{J_T(x)} \cap J_T(M)$ cannot be disconnected by removing $\mathbf{C}J_T(M^{reg})$. Now we are in the hypotheses of Lemma 2.5. Indeed, $J_T|_{M^{reg}} : M^{reg} \to J_T(M)$ is an open map and we just showed that $V_0 \subset J_T(M^{reg})$ is connected. Any fiber of J_T is connected by hypothesis. Since such a fiber is T-invariant, the set of its regular points for the induced T-action is open dense and connected in it. If $v \in J_T(M^{reg})$, then $J_T^{-1}(v) \cap M^{reg}$ is connected. Now applying Lemma 2.5 we conclude that $J_T^{-1}(V_0) \cap M^{reg}$ is connected. Since $J_T^{-1}(V_0) \cap M^{reg}$ is dense in $J_T^{-1}(V_0)$ it follows that $U_0 := J_T^{-1}(V_0)$ is connected.

Next we will show that U_0 is dense in $U := J_T^{-1}(V)$. Indeed, if this is not true, then there exist an element $x_0 \in U \backslash U_0$ and a neighborhood U_{x_0} that does not intersect U_0. For the open set $U' := U \cap U_{x_0} \cap M^{reg} \neq \varnothing$ we have that $J_T(U') \subset V_0$ is open in V_0. So there exists an element $v_0 \in J_T(U')$ such that $J_T^{-1}(v_0) \cap U_{x_0} \neq \varnothing$ and $J_T^{-1}(v_0) \cap U_0 \neq \varnothing$. But $J_T^{-1}(v_0) \subset U_0$, which contradicts the assumption that $U_{x_0} \cap U_0 = \varnothing$.

By the connectedness of U_0 and the fact that it is dense in U we obtain that U is connected and hence the result follows by Proposition 2.19. □

We summarize Propositions 3.6 and 3.8 in the following theorem. Since we assume that J_T has connected fibers, we work on a possibly disconnected symplectic manifold and we apply these propositions to each connected component separately.

Theorem 3.9. *Let $J_T : M \to \mathfrak{t}^*$ be the momentum map of a torus action which has connected fibers. Then J_T is open onto its image if and only if $\mathbf{C}J_T(M^{reg})$ does not disconnect any region in $J_T(M)$. Moreover, the image of the momentum map is locally convex and locally polyhedral.*

Remark 3.10. The proof of Theorem 3.9 does not use in an essential way the finite dimensionality of the manifold and the torus. A careful look at the proof shows that the same result holds for continuous maps $f : X \to V$ that have local convexity data with closed cones, where X is a connected Hausdorff topological space and V is a locally convex topological vector space. Additionally, assume that

- X is path connected;
- there exists an open dense connected subset $X' \subset X$ such that $f|_{X'}$ is open in $f(X)$;
- for any $v \in V$, the fiber $f^{-1}(v)$ is connected and $f^{-1}(v) \cap X'$ is connected.

Then f is open onto its image if and only if $\mathbf{C}f(X') := f(X) \setminus f(X')$ does not disconnect any region in $f(X)$. Moreover, the image of f is locally convex. If $f(X)$ is, in addition, closed, then it is convex.

We illustrate the above results with two examples, one in which the momentum map is open onto its image and another one in which it is not. This information is obtained by inspection of the image.

Example 3.11 (Prato [33])**.** Let $M := \mathbb{C}^2 \backslash (D^1 \times D^1)$, where D^1 is the closed unit disc in $\mathbb{C}$. If we consider M as a symplectic submanifold of $\mathbb{C}^2$ with its standard symplectic structure, then we have on M a natural globally Hamiltonian action of T^2 given by $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) := (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$. The momentum map for this action is $(z_1, z_2) \mapsto (|z_1|^2, |z_2|^2)/2$. This momentum map has connected fibers. Denote by $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. The image of the momentum map is $\mathbb{R}^2_+ \backslash \{(x, y) \mid x \leq 1/2 \text{ and } y \leq 1/2\}$ and $\mathbf{C}(J_{T^2}(M^{reg}))$ is the union of $\{(x, 0) \mid x > 1/2\}$ and $\{(0, y) \mid y > 1/2\}$ which does not disconnect any region in the image $J_{T^2}(M)$. Consequently, according to Theorem 3.9, this momentum map is open onto its image and has a locally convex image, in agreement with Theorem 2.13. ♦

Example 3.12 (Karshon and Lerman [18])**.** Let $M_1 := T^2 \times U$, where T^2 is the two dimensional torus and U is the subset of $\mathbb{R}^2$ obtained by removing the origin and the positive x-axis. M_1 is a symplectic manifold when viewed as an open submanifold of the cotangent bundle of T^2. The restriction to M_1 of the lifted action of T^2 on its cotangent bundle has as momentum map the projection onto U. Hence, the image of this momentum map is $\mathbb{R}^2$ minus the origin and the positive x-axis.

Let M_2 be the symplectic manifold $\mathbb{C}^2$ minus the points whose first coordinate is nonzero. The momentum map for the T^2-action on M_2 is given by $(z, w) \mapsto (|z|^2, |w|^2)/2$ and the image is the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y \geq 0\}$.

Gluing these two spaces M_1 and M_2 along the pre-images of the positive quadrant we obtain another globally Hamiltonian T^2-space with a momentum map with connected fibers whose image is $\mathbb{R}^2$ minus the origin. It is easy to see that $\mathbf{C}(J_{T^2}(M^{reg}))$ is the open positive x-axis which disconnects regions in $\mathbb{R}^2$. Theorem 3.9 implies that this momentum map is not open onto its image. ♦

To prove a converse to Proposition 3.6 in the case when the fibers are not connected we need a few more topological facts. A metric space is called a ***generalized continuum*** if it is locally compact and connected. In a topological space a

quasi-component of a point is the intersection of all closed-and-open sets that contain that point. A topological space is called ***totally disconnected*** if the quasi-component of any point consists of the point itself. A continuous map $f : X \to Y$ is called ***light*** if all fibers $f^{-1}(y)$ are totally disconnected. We say that a subset of a topological space is ***nondense*** if and only if it contains no open subsets. Whyburn [37] (page 94) proved the following result called "the extension of openness".

Theorem 3.13 (Whyburn)**.** *Let X and Y be locally connected generalized continua and let $f : X \to Y$ be an onto light mapping which is open on $X \backslash f^{-1}(F)$, where F is a closed nondense set in Y which separates no region in Y and is such that $f^{-1}(F)$ is nondense. Then f is open on X.*

If the fibers of J_T are not connected we still need a control on the connected components of the fibers of J_T in order to have a similar result to that in Proposition 3.8 (which is a converse of Proposition 3.6).

Definition 3.14. Let $J_T : M \to \mathfrak{t}^*$ be the momentum map of a torus action on a connected symplectic manifold (M, ω). We say that J_T satisfies the ***connected component fiber condition***

(CCF): if $J_T(x) = J_T(y)$ and $E_x \cap M^{reg} \neq \varnothing$, then $E_y \cap M^{reg} \neq \varnothing$, where E_x and E_y are the connected components of the fiber $J_T^{-1}(J_T(x))$ that contain x and y respectively.

Recall that M_{J_T} is the quotient topological space whose points are the equivalence classes given by the connected components of the fibers of J_T. Denote by $\pi_{J_T} : M \to M_{J_T}$ the canonical projection.

Proposition 3.15. *Suppose that M_{J_T} is a Hausdorff space, $J_T(M)$ is locally compact, $\mathbf{C}J_T(M^{reg})$ does not disconnect any region in $J_T(M)$, and J_T also satisfies condition (CCF). Then J_T is open onto its image.*

Proof. To prove the result we will show that the conditions of Whyburn's Theorem are satisfied where we take f to be $\widetilde{J}_T : M_{J_T} \to J_T(M) \subset \mathfrak{t}^*$, that is, the quotient map uniquely defined by $J_T = \pi_{J_T} \circ \widetilde{J}_T$ and $F := \mathbf{C}J_T(M^{reg})$, which is closed and nondense in $J_T(M)$ by Lemma 3.7.

By hypothesis M_{J_T} is a Hausdorff space. Using the fact that M is locally compact and π_{J_T} is open (see Lemma 2.16 **(i)**) we obtain that M_{J_T} is locally compact. Since M is connected, its quotient M_{J_T} is connected. Proposition 2.23 guarantees that M_{J_T} is a metric space since it is Hausdorff. Therefore, M_{J_T} is a generalized continuum. The same is true for $J_T(M)$. Both are locally connected since M is path connected.

Now we prove that $\widetilde{J}_T : M_{J_T} \to J_T(M)$ is a light map. For this, take $v \in J_T(M)$. We want to show that $\widetilde{J}_T^{-1}(v)$ is totally disconnected. Let $[x] \in \widetilde{J}_T^{-1}(v)$ be arbitrary and choose $x \in M$ to be a representative of this class. Since J_T has local convexity data and is locally fiber connected (see Theorem 3.3), we can find a small neighborhood U_x of x in M such that $\pi_{J_T}(U_x)$ is open in M_{J_T} and is such that $\widetilde{J}_T^{-1}(v) \cap \pi_{J_T}(U_x) = [x]$. Thus, $\widetilde{J}_T : M_{J_T} \to J_T(M)$ is a light map.

Now we prove that $\widetilde{J}_T^{-1}(F)$ is nondense in M_{J_T}. By contradiction, suppose that this is not true. Then there exists an open set $U \subset \widetilde{J}_T^{-1}(F)$. Because $\pi_{J_T}(M^{reg})$ is dense in M_{J_T} we have that $U \cap \pi_{J_T}(M^{reg}) \neq \varnothing$. Thus there exists an element

$[x] \in \widetilde{J}_T^{-1}(F) \cap \pi_{J_T}(M^{reg})$ and hence there is an $x \in M^{reg}$ such that $J_T(x) = \widetilde{J}_T([x]) \in F$. This contradicts the definition of F.

The last thing to be verified is that $\widetilde{J}_T$ restricted to $M_{J_T} \backslash \widetilde{J}_T^{-1}(F)$ is an open map. To see this note that the inclusion

$$\widetilde{J}_T^{-1}(F) = \widetilde{J}_T^{-1}(\mathbf{C}J_T(M^{reg})) = \mathbf{C}\widetilde{J}_T^{-1}(J_T(M^{reg})) \subset \mathbf{C}\pi_{J_T}(M^{reg})$$

shows that $M_{J_T} \backslash \widetilde{J}_T^{-1}(F) \supset \pi_{J_T}(M^{reg})$. Now we shall prove the reverse inclusion. Let $[x] \in M_{J_T} \backslash \widetilde{J}_T^{-1}(F)$. If the connected component E_x of the fiber $J_T^{-1}(J_T(x))$ intersects M^{reg}, then $[x] \in \pi_{J_T}(M^{reg})$. If not, then we have $J_T(x) = \widetilde{J}_T([x]) \notin F$, that is, $J_T(x) \in \mathbf{C}F = J_T(M^{reg})$. Therefore, there is some $y \in M^{reg}$ such that $J_T(x) = J_T(y)$. By condition (CCF), $E_x \cap M^{reg} \neq \varnothing$ and hence $[x] \in \pi_{J_T}(M^{reg})$, which proves the equality $M_{J_T} \backslash \widetilde{J}_T^{-1}(F) = \pi_{J_T}(M^{reg})$. Thus, since $\widetilde{J}_T$ is open in $\pi_{J_T}(M^{reg})$ the last requirement of Whyburn's Theorem is verified. □

To summarize, in the situation for nonconnected fibers we obtain the following result.

Theorem 3.16. *Let $J_T : M \to \mathfrak{t}^*$ be the momentum map of a torus action on a connected symplectic manifold (M, ω). Suppose that M_{J_T} is a Hausdorff space.*

Then J_T is open onto its image if and only if $J_T(M)$ is locally compact, $\mathbf{C}J_T(M^{reg})$ does not disconnect any region in $J_T(M)$, and J_T satisfies condition (CCF). Moreover, under these hypotheses, the image of the momentum map is locally convex and locally polyhedral.

Proof. The only thing that remains to be shown is that if $J_T : M \to J_T(M)$ is open onto its image, then condition (CCF) holds. Suppose that the condition (CCF) does not hold. So there exists a fiber with at least two connected components E_x and E_y such that $E_x \cap M^{reg} \neq \varnothing$ and $E_y \cap M^{reg} = \varnothing$.

Consequently, we can suppose that $x \in M^{reg}$ and y is contained in a lower stratum of the T-action. Then we have the strict inclusion $C_{J_T(y)} = v + \mathfrak{t}_0^\perp + \mathrm{cone}(\mathcal{P}_V) \subset C_{J_T(x)} = v + \mathfrak{t}_1^*$, where $v = J_T(x) = J_T(y)$ (see Theorem 3.3 and the proof of Lemma 3.7). By condition (VN), there exist open neighborhoods U_x and U_y of x and y, respectively, such that $J_T(U_x)$ is an open ball in $\mathfrak{t}_1^*$ centered at v and $J_T(U_y)$ is the intersection of an open ball in $\mathfrak{t}_1^*$ centered at v with a closed proper cone in $\mathfrak{t}_1^*$ with vertex v. This contradicts the openness onto its image of J_T. □

In the case of a compact, connected, and non-Abelian group, the momentum map $J_G : M \to \mathfrak{g}^*$ is, in general, not open onto its image even if it is a proper map. Nevertheless, we will show that similar results to those obtained in the Abelian case hold for the quotient map $j_G : M \to \mathfrak{g}^*/G \simeq \mathfrak{t}_+^*$, where $j_G = \pi_G \circ J_G$ and $\pi_G : \mathfrak{g}^* \to \mathfrak{t}_+^*$ is the projection map which is always proper if G is compact.

The quotient map j_G has local convexity data due to the following result of Sjamaar [35, Theorem 6.5].

Theorem 3.17 (Sjamaar)**.** *Let M be a connected Hamiltonian G-manifold. Then for every $x \in M$ there exist a unique, closed, polyhedral convex cone C_x in $\mathfrak{t}_+^*$ with vertex at $j_G(x)$ such that for every sufficiently small G-invariant neighborhood U of x the set $j_G(U)$ is an open neighborhood of $j_G(x)$ in C_x.*

Using Lerman's [26] symplectic cut technique, Knop [23, Theorem 5.1] proved that j_G is locally fiber connected. As a consequence of these results we have the following theorem.

Theorem 3.18 (Knop-Sjamaar)**.** *The map j_G is locally fiber connected and has local convexity data.*

Using the above theorem and Theorem 2.28 we obtain the following generalization of Kirwan's convexity result [20]. In the next statement we will use the map $\widetilde{j}_G : M/G \to \mathfrak{t}^*_+$ defined by the identity $j_G = \widetilde{j}_G \circ \pi$, where $\pi : M \to M/G$ is the projection. We will say that the G-equivariant momentum map $J_G : M \to \mathfrak{g}^*$ is ***G-open*** onto its image whenever $\widetilde{j}_G$ is open onto its image.

Theorem 3.19. *Let M be a connected Hamiltonian G-manifold with G a compact connected Lie group. If the momentum map J_G is closed, then $J_G(M) \cap \mathfrak{t}^*_+$ is a closed convex locally polyhedral set. Moreover, J_G is G-open onto its image and all its fibers are connected.*

Proof. As a direct application of Theorem 2.28 we obtain that j_G is open onto its image and consequently that J_G is G-open onto its image. Additionally, the set $j_G(M) = J_G(M) \cap \mathfrak{t}^*_+$ is a closed convex locally polyhedral set and j_G has connected fibers.

It remains to be proved that J_G has connected fibers. To see this, note that since j_G has connected fibers, the pre-images $J_G^{-1}(\mathcal{O}_\mu)$ are connected as topological subspaces of M for every coadjoint orbit $\mathcal{O}_\mu \subset J_G(M)$. Note now that $J_G^{-1}(\mathcal{O}_\mu)$ can also be endowed with the initial topology induced by the map

$$\begin{array}{rccc} J_G^{\mathcal{O}_\mu} : & J_G^{-1}(\mathcal{O}_\mu) & \longrightarrow & \mathcal{O}_\mu \\ & z & \longmapsto & J_G(z), \end{array}$$

where the orbit $\mathcal{O}_\mu$ comes with its orbit smooth structure induced by the homogeneous manifold G/G_μ. Since G is compact, the orbit $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ and hence the initial topology for $J_G(\mathcal{O}_\mu)$ is weaker than the subspace topology. Indeed, the sets of the form $(J_G^{\mathcal{O}_\mu})^{-1}(U \cap \mathcal{O}_\mu) = J_G^{-1}(U) \cap J_G^{-1}(\mathcal{O}_\mu)$, with U open in $\mathfrak{g}^*$, form a subbasis of the initial topology of $J_G^{\mathcal{O}_\mu}$ and since they are open in M by the continuity of J_G, the claim follows. Therefore, $J_G^{-1}(\mathcal{O}_\mu)$ is also connected for the initial topology. Proposition 8.4.1 in [32] states that if $J_G^{-1}(\mathcal{O}_\mu)$ is endowed with its initial topology, then the map

$$\begin{array}{rccc} f : & G \times_{G_\mu} J_G^{-1}(\mu) & \longrightarrow & J_G^{-1}(\mathcal{O}_\mu) \\ & [g, z] & \longmapsto & g \cdot z \end{array}$$

is a homeomorphism, where $G \times_{G_\mu} J_G^{-1}(\mu)$ denotes the orbit space of the free and continuous action $h \cdot (g, z) := (gh, h^{-1} \cdot z)$, $h \in G_\mu$, $g \in G$, $z \in J_G^{-1}(\mu)$, of the compact connected group (see Theorem 3.3.1 in [8]) G_μ on the product $G \times J_G^{-1}(\mu)$. The set $J_G^{-1}(\mu)$ is considered with its subspace topology. Let $\pi_\mu : G \times J_G^{-1}(\mu) \to G \times_{G_\mu} J_G^{-1}(\mu)$ be the continuous and open projection. Since the fibers of π_μ are connected (they are homeomorphic to G_μ) and π_μ is open it follows that the preimage of any connected set is connected by Lemma 2.5. Therefore $G \times J_G^{-1}(\mu)$ is connected and hence so is $J_G^{-1}(\mu)$ since G is connected. □

Remark 3.20. We emphasize that J_G is G-open but not open in general. See [30] for a counterexample.

Analyzing the proofs of the results leading to Theorem 3.9 and using the natural definition of the (CCF) condition for the map j_G we have the following two results (see Remark 3.10) which are the non-Abelian analogs of Theorems 3.9 and 3.16.

Theorem 3.21. *Let G be a compact connected Lie group and M be a connected Hamiltonian G-manifold with equivariant momentum map $J_G : M \to \mathfrak{g}^*$. Suppose that J_G has connected fibers. Then J_G is G-open onto its image if and only if $\mathbf{C}((\pi_G \circ J_G)(M^{reg}))$ does not disconnect any region in $J_G(M) \cap \mathfrak{t}^*_+$. Moreover, in this context, the image $J_G(M) \cap \mathfrak{t}^*_+$ is a locally convex and locally polyhedral set.*

Theorem 3.22. *Let G be a compact connected group and M be a connected Hamiltonian G-manifold with the momentum map $J_G : M \to \mathfrak{g}^*$. Suppose that $(M/G)_{j_G}$ is a Hausdorff space. Then J_G is G-open onto its image if and only if $J_G(M)$ is locally compact, $\mathbf{C}(\pi_G \circ J_G(M^{reg}))$ does not disconnect any region in $J_G(M) \cap \mathfrak{t}^*_+$, and j_G satisfies the (CCF) condition. Moreover, in this context, the image $J_G(M) \cap \mathfrak{t}^*_+$ is a locally convex and locally polyhedral set.*

4. Convexity for Poisson actions of compact Lie groups

In this section we shall prove a generalization of the Flaschka-Ratiu convexity theorem (Theorem 4.39 in [10]) which has as one of its main consequences the convexity theorem for Poisson actions of compact connected Poisson Lie groups on compact connected symplectic manifolds. We hasten to add that the hypotheses of this theorem do *not* imply that the compact Lie group action is necessarily a Poisson action of a Poisson Lie group and that this theorem can be applied in other situations. The generalization below will require only that a certain map is closed (not even properness is assumed) instead of assuming compactness of the symplectic manifold.

Let K be a compact connected semisimple Lie group. Since any compact Lie group K is the commuting product $(Z_K)_0 K_{ss}$ of the connected component of the identity $(Z_K)_0$ of the center Z_K and of a closed semisimple subgroup K_{ss} (see Theorem 4.29 in [22]) we will provide in what follows, without any loss of generality, the proofs for the compact semisimple case, even though the results will be stated for general compact Lie groups. Since K admits a complexification, we can think of it as the compact real form of a connected complex semisimple Lie group G. Denote by $G^{\mathbb{R}}$ the real Lie group underlying G and let $G^{\mathbb{R}} = KAN$ be its Iwasawa decomposition. Denote by $\mathfrak{g}^{\mathbb{R}}, \mathfrak{k}, \mathfrak{a}$, and $\mathfrak{n}$ the real Lie algebras of $G^{\mathbb{R}}, K, A$, and N, respectively. Then $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}^{\mathbb{R}}$. If $\mathfrak{t} = i\mathfrak{a}$, then $T = \exp \mathfrak{t}$ is a maximal torus of K. Define $B := AN$ whose Lie algebra is $\mathfrak{b} := \mathfrak{a} \oplus \mathfrak{n}$.

Let κ be the Killing form of $\mathfrak{g}$. Its imaginary part $\operatorname{Im}\kappa$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}^{\mathbb{R}}$. Since $\operatorname{Im}\kappa(\mathfrak{k}, \mathfrak{k}) = \operatorname{Im}(\mathfrak{b}, \mathfrak{b}) = 0$, the vector spaces $\mathfrak{k}$ and $\mathfrak{b}$ are dual to each other relative to $\langle\, , \rangle := \operatorname{Im}\kappa$. The Cartan decomposition $G^{\mathbb{R}} = PK$ defines the Cartan involution $\tau : G^{\mathbb{R}} \to G^{\mathbb{R}}$. Define $g^* := \tau(g^{-1})$ for any $g \in G$. The derivative of these maps at the identity will be denoted by the same symbols. The map $\tau : \mathfrak{g}^{\mathbb{R}} \to \mathfrak{g}^{\mathbb{R}}$ has eigenvalues ± 1. The $+1$ eigenspace is $\mathfrak{k}$ and the -1 is denoted by $\mathfrak{p}$. Note that $\langle\, , \rangle$ identifies $\mathfrak{k}^*$ with $\mathfrak{p}$. The exponential map is a diffeomorphism from $\mathfrak{p}$ to P.

Let $\mathfrak{a}_+$ be the positive Weyl chamber in $\mathfrak{a} \cong \mathfrak{t}^*$ corresponding to the subgroup B, let $\mathfrak{a}_0$ be the interior of $\mathfrak{a}_+$, and set $A_+ = \exp \mathfrak{a}_+$ and $A_0 = \exp \mathfrak{a}_0$. Let $W^1, W^2, \ldots$ be all the closed walls of varying dimensions of the positive Weyl chamber $\mathfrak{a}_+$; there are only finitely many such walls. If P^i is the subspace of $\mathfrak{a}$ spanned by W^i, then W^i is closed in P^i and W^i_0 denotes the interior of W^i in P^i. We will also use the notation $W^0 = \mathfrak{a}_+$ and $W^0_0 = \mathfrak{a}_0$; note that $P^0 = \mathfrak{a}$.

Theorem 4.1. *Suppose that a compact connected Lie group K acts on a paracompact connected symplectic manifold (M, ω) and that a maximal torus T of K acts in a Hamiltonian fashion with equivariant momentum map $J_T : M \to \mathfrak{t}^*$. Suppose there exists a closed map $\mathcal{P} : M \to \mathfrak{p}$ with the following properties:*

(i): *$\mathcal{P}$ is equivariant (with respect to the adjoint action of K on $\mathfrak{p}$);*

(ii): *for every $x \in M$, $T_x\mathcal{P}(T_xM) = \mathfrak{k}_x^{\mathrm{ann}} := \{\mu \in \mathfrak{p} \mid \langle \mu, \xi \rangle = 0$ for all $\xi \in \mathfrak{k}\}$;*

(iii): *for every $x \in M$, the kernel of $T_x\mathcal{P}$ is*

$$(\mathfrak{k} \cdot x)^\omega := \{v \in T_xM \mid \omega(x)(v, \xi_M(x)) = 0 \text{ for all } \xi \in \mathfrak{k}\};$$

(iv): *the restriction of $\mathcal{P}$ to $\mathcal{P}^{-1}(\mathfrak{a}_+)$ is proportional to J_T.*

Then $\mathcal{P}(M) \cap \mathfrak{a}_+$ is a closed convex set. If M is compact, the set $\mathcal{P}(M) \cap \mathfrak{a}_+$ is a compact convex polytope.

Proof. To prove the statement we shall use the technique of symplectic cross sections for the map $\mathcal{P}$ (see [11, 7, 14]). Two situations may arise: if the intersection $\mathcal{P}(M) \cap \mathfrak{a}_0$ is nonempty, then the symplectic cross section can be taken to be $Y := \mathcal{P}^{-1}(\mathfrak{a}_0)$. Otherwise, $\mathcal{P}(M) \cap \mathfrak{a}_0 = \varnothing$ and $Y := \mathcal{P}^{-1}(W_0^i)$ is a symplectic cross section, where W^i is the unique closed wall such that $\mathcal{P}(M) \cap \mathfrak{a}_+ \subset W^i$ and $\mathcal{P}(M) \cap W_0^i \neq \varnothing$.

The hypotheses in the theorem ensure via Proposition 4.27 in [10] that the set Y defined above is a symplectic cross section and that, additionally, the closure of $\mathcal{P}(Y)$ in $\mathfrak{a}_+$ is $\mathcal{P}(M) \cap \mathfrak{a}_+$. Consequently, it suffices to show that $\mathcal{P}(Y)$ is convex in order to prove the convexity of $\mathcal{P}(M) \cap \mathfrak{a}_+$.

The proportionality of $\mathcal{P}|_Y$ to J_T guarantees that $\mathcal{P}|_Y$ is locally fiber connected and has local convexity data. Notice that $\mathcal{P}|_Y : Y \to W_0^i$ is a closed map. Then, using Remark 2.29, it follows that $\mathcal{P}(Y)$ is convex, closed, and locally polyhedral in W_0^i.

If M is compact, then Proposition 4.38 in [10] states that $\overline{\mathcal{P}(M)}$ is polyhedral and that it has only a finite number of faces. Thus $\mathcal{P}(M) \cap \mathfrak{a}_+$ is a compact convex polytope. □

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