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Geodesic flows on semidirect-product Lie groups: geometry of singular measure-valued solutions

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The EPDiff equation (or the dispersionless Camassa–Holm equation in one dimension) is a well-known example of geodesic motion on the Diff group of smooth invertible maps (diffeomorphisms). Its recent two-component extension governs geodesic motion on the semidirect product Diff$\circledS\mathcal{F}$, where $\mathcal{F}$ denotes the space of scalar functions. This paper generalizes the second construction to consider geodesic motion on Diff$\circledS\mathfrak{g}$, where $\mathfrak{g}$ denotes the space of scalar functions that take values on a certain Lie algebra (e.g. $\mathfrak{g}=\mathcal{F}\otimes\mathfrak{so}(3)$). Measure-valued delta-like solutions are shown to be momentum maps possessing a dual pair structure, thereby extending previous results for the EPDiff equation. The collective Hamiltonians are shown to fit into the Kaluza–Klein theory of particles in a Yang–Mills field and these formulations are shown to apply also at the continuum partial differential equation level. In the continuum description, the Kaluza–Klein approach produces the Kelvin circulation theorem.

Keywords: geodesic flow; semidirect product; Kaluza–Klein construction

1. Introduction

(*a*) *Singular solutions in continuum mechanics*

Singular measure-valued solutions arise in the study of many continuum systems. A famous example of singular solutions in ideal fluids is the point vortex solution for the Euler vorticity equation on the plane. Point vortices are delta-like solutions that follow a multi-particle dynamics. In three dimensions one can extend this concept to vortex filaments or vortex sheets, for which the vorticity is supported on a lower dimensional submanifold (one- or two-dimensional, respectively) of the Euclidean space $\mathbb{R}^3$. These solutions form an invariant manifold. However, they are not expected to be created by fluid motion from smooth initial conditions. Singular solutions also exist in plasma physics as magnetic vortex lines and in kinetic theory as single-particle solutions (see Gibbons *et al.* 2008 for discussion and references).

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Another well-known fluid model admitting singular solutions is the Camassa–Holm (CH) equation, which is an integrable Hamiltonian fluid equation describing shallow-water solitons moving in one dimension (Camassa & Holm 1993). In the dispersionless limit, the CH solitons are singular solutions whose velocity profile has a sharp peak at which the derivative is discontinuous. These are the *peakon* solutions of the CH equation. In higher dimensions the dispersionless CH equation is often called EPDiff, which is short for *Euler–Poincaré* equation *on the diffeomorphisms* (Holm *et al.* 1998; Holm & Marsden 2004). EPDiff also has applications in other areas such as turbulence (Foias *et al.* 2001) and imaging (Holm *et al.* 2004; Holm *et al.* submitted *b*). The EPDiff and the Euler fluid equations share the important property of being geodesic motions on diffeomorphism groups (Arnold 1997). Moreover, the EPDiff equation has the additional interesting feature of showing the *spontaneous* emergence of singular solutions from any confined initial velocity configuration. In one dimension, this result follows from the *steepening lemma* (Camassa & Holm 1993). Recent work has proven that the singular solutions of EPDiff represent a certain kind of momentum map (Holm & Marsden 2004). This result answers the fundamental question concerning the geometric nature of these solutions.

The CH equation in one dimension also possesses a two-component integrable extension (CH2) (Chen *et al.* 2006; Falqui 2006; Kuz'min 2007), which represents geodesic motion on the semidirect product Diff Ⓢ $\mathcal{F}$, where $\mathcal{F}$ denotes the space of real scalar functions. This system of equations involves both fluid density and momentum, and it also possesses singular solutions in the latter variable. CH peakon solutions form an invariant subspace of CH2 solutions for the case that the density vanishes identically. A change in the metric allows delta-like singularities in *both* variables, not just the fluid momentum, although such a change may also destroy integrability (Holm *et al.* submitted *b*).

(*b*) *The CH and EPDiff equations*

The dispersionless CH equation is a geodesic flow on the infinite-dimensional Lie group of smooth invertible maps (diffeomorphisms) of either the real line or a periodic interval. This geodesic flow exhibits the spontaneous emergence of singularities from *any* confined smooth initial velocity profile. The CH equation is a 1+1 partial differential equation for the one-dimensional fluid velocity vector u as a function of position $x\in\mathbb{R}$ and time $t\in\mathbb{R}$ and is written as (Camassa & Holm 1993)

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.1}$$

The present work focuses on the CH equation (1.1) in the dispersionless case for which $\kappa=0$ and considers periodic boundary conditions or sufficiently rapid decay at infinity, so that boundary terms do not contribute to integrations by parts. Being geodesic, the CH equation can be recovered from Hamilton's principle with a quadratic Lagrangian,

$$\delta\int_{t_0}^{t_1} L(u)\mathrm{d}t = 0 \quad \text{with } L(u) = \frac{1}{2}\int u(x)(1-\partial_x^2)u(x)\mathrm{d}x.$$

In particular, it follows from the Euler–Poincaré variational principle defined on $\mathfrak{X}(\mathbb{R}) = T_e\mathrm{Diff}(\mathbb{R})$, the Lie algebra of the Diff group, consisting of vector fields on the line (Holm *et al.* 1998). The corresponding Lie–Poisson Hamiltonian formulation follows from the Legendre transform

$$m = \frac{\delta L}{\delta u} = u - u_{xx} \Rightarrow u = (1-\partial_x^2)^{-1} m,$$

in which $m \in \mathfrak{X}^*(\mathbb{R})$. Here, $\mathfrak{X}^*(\mathbb{R})$ is the space of one-form densities, which are dual to the vector fields on the line under the L^2 pairing. After the Legendre transformation, the Hamiltonian becomes

$$H(m) = \frac{1}{2}\int m(1-\partial_x^2)^{-1} m \,\mathrm{d}x \Rightarrow u = \frac{\delta H}{\delta m},$$

and the corresponding Lie–Poisson form of equation (1.1), with $\kappa = 0$, is (Camassa & Holm 1993)

$$m_t + u m_x + 2u_x m = 0.$$

The main result for this equation is its complete integrability, which is guaranteed by its bi-Hamiltonian structure (Camassa & Holm 1993). Another important feature, called the *steepening lemma* (Camassa & Holm 1993), provides the mechanism for the spontaneous emergence of the singular solutions (the peaked solitons, or *peakons* mentioned earlier) from any confined initial velocity distribution.

EPDiff. Except for integrability, these one-dimensional results can be generalized to two or three dimensions, in which the equation becomes EPDiff, namely

$$\partial_t \boldsymbol{m} - \boldsymbol{u} \times \operatorname{curl} \boldsymbol{m} + \nabla(\boldsymbol{u}\cdot\boldsymbol{m}) + \boldsymbol{m}(\operatorname{div} \boldsymbol{u}) = 0. \tag{1.2}$$

The EPDiff Hamiltonian on $\mathfrak{X}^*(\mathbb{R}^3)$ arising from this generalization is given by

$$H(\boldsymbol{m}) = \frac{1}{2}\int \boldsymbol{m}\cdot(1-\alpha^2 \Delta)^{-1} \boldsymbol{m}\,\mathrm{d}\boldsymbol{x} =: \frac{1}{2}\|\boldsymbol{m}\|^2, \tag{1.3}$$

in which α is the length scale over which the velocity is smoothed relative to the momentum via the relation

$$\boldsymbol{u} = (1-\alpha^2\Delta)^{-1}\boldsymbol{m}. \tag{1.4}$$

The singular solutions in higher dimensions are written in the momentum representation as

$$\boldsymbol{m}(\boldsymbol{x}, t) = \sum_{i=1}^{N} \int \boldsymbol{P}_i(s,t)\delta(\boldsymbol{x} - \boldsymbol{Q}_i(s,t))\mathrm{d}s, \tag{1.5}$$

where s is a variable of dimension $k<3$. These solutions represent moving filaments or sheets, when s has dimension 1 or 2, respectively.

A further generalization replaces the kernel that defines the norm of $\|\boldsymbol{m}\|$ in (1.3) via the relation (1.4) with the general convolution

$$G * \boldsymbol{m} = \int G(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{m}(\boldsymbol{x}')\mathrm{d}\boldsymbol{x}', \tag{1.6}$$

involving an arbitrary Green's function, or kernel, G. This generalization produces the metric,

$$H(\boldsymbol{m}) = \frac{1}{2}\int \boldsymbol{m}\cdot(G * \boldsymbol{m})\mathrm{d}\boldsymbol{x} =: \frac{1}{2}\|\boldsymbol{m}\|_G^2, \tag{1.7}$$

which is the Lie–Poisson Hamiltonian for EPDiff (1.2) and is, thus, invariant under its evolution. The dynamics of $(\boldsymbol{Q}_i, \boldsymbol{P}_i)$, with $i=1, \ldots, N$, for the singular pulson solutions in (1.5) is given by canonical Hamiltonian dynamics, with the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{i,j} \iint \boldsymbol{P}_i(s,t) \cdot \boldsymbol{P}_j(s',t)\, G(\boldsymbol{Q}_i(s,t) - \boldsymbol{Q}_j(s',t))\,\mathrm{d}s\,\mathrm{d}s',$$

obtained by evaluating the Lie–Poisson Hamiltonian for EPDiff on its singular solution set.

Theorem 1.1 (Holm & Marsden 2004). *The singular solution (1.5) is a momentum map.*

To illustrate this theorem, fix a k-dimensional manifold S immersed in $\mathbb{R}^n$ and consider the embedding $\boldsymbol{Q}_i: S \to \mathbb{R}^n$. Such embeddings form a smooth manifold $\mathrm{Emb}(S, \mathbb{R}^n)$, and thus one can consider its cotangent bundle $(\boldsymbol{Q}_i, \boldsymbol{P}_i) \in T^*\mathrm{Emb}(S, \mathbb{R}^n)$. Consider $\mathrm{Diff}(\mathbb{R}^n)$ acting on $\mathrm{Emb}(S, \mathbb{R}^n)$ on the left by composition of functions ($g\boldsymbol{Q} = g \circ \boldsymbol{Q}$) and lift this action to $T^*\mathrm{Emb}(S, \mathbb{R}^n)$. This procedure constructs the singular solution momentum map for EPDiff,

$$\boldsymbol{J} : T^*\mathrm{Emb}(S, \mathbb{R}^n) \to \mathfrak{X}^*(\mathbb{R}^n) \quad \text{with } \boldsymbol{J}(\boldsymbol{Q}, \boldsymbol{P}) = \int \boldsymbol{P}(s,t)\delta(\boldsymbol{x} - \boldsymbol{Q}(s,t))\,\mathrm{d}s.$$

This construction was extensively discussed in Holm & Marsden (2004), where proofs were given in various cases. A key result is that the momentum map constructed in this way is *equivariant*, which means it is also a Poisson map. This explains why the coordinates $(\boldsymbol{Q}, \boldsymbol{P})$ undergo Hamiltonian dynamics. There is also a right action ($\boldsymbol{Q}g = \boldsymbol{Q} \circ g$), whose momentum map corresponds to the canonical one-form $\boldsymbol{P} \cdot \mathrm{d}\boldsymbol{Q}$ on $T^*\mathrm{Emb}$.

(*c*) *The two-component CH system*

In recent years, the CH equation has been extended (Chen *et al.* 2006; Falqui 2006; Kuz'min 2007) in order to combine the integrability property with compressibility, which introduces a pressure term in the equation for the fluid momentum. The resulting system (CH2) is a geodesic motion equation on $\mathrm{Diff} \circledS \mathcal{F}$, given as an Euler–Poincaré equation on the semidirect-product Lie algebra $\mathfrak{X} \circledS \mathcal{F}$. In the general case, the Euler–Poincaré equations are written on the dual of a semidirect-product Lie algebra $\mathfrak{g} \circledS V$ as (Holm *et al.* 1998)

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta L}{\delta(\xi, a)} = -\mathrm{ad}^*_{(\xi,a)} \frac{\delta L}{\delta(\xi, a)} \qquad (\xi, a) \in \mathfrak{g} \circledS V,$$

whose components are

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta L}{\delta \xi} = -\mathrm{ad}^*_\xi \frac{\delta L}{\delta \xi} + \frac{\delta L}{\delta a} \diamond a, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta L}{\delta a} = -\xi \frac{\delta L}{\delta a},$$

where the notation $\xi \delta L / \delta a$ stands for the (left) Lie algebra action of $\mathfrak{g}$ on V^*.

The integrable CH2 equations are derived from the following variational principle on $\mathfrak{X}Ⓢ\mathcal{F}$:

$$\delta\int_{t_0}^{t_1} L(u,\rho)\mathrm{d}t = 0 \quad \text{with } L(u) = \frac{1}{2}\int u(1-\partial_x^2)u\,\mathrm{d}x + \frac{1}{2}\int \rho^2\,\mathrm{d}x.$$

Explicitly, the CH2 equations are

$$\rho_t = -(\rho u)_x, \qquad u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} - \rho\rho_x.$$

These equations describe geodesic motion with respect to the H^1 metric in u and the L^2 metric in ρ. Legendre transforming yields the metric Hamiltonian

$$H(m,\rho) = \frac{1}{2}\int m(1-\partial^2)^{-1}m\,\mathrm{d}x + \frac{1}{2}\int \rho^2\,\mathrm{d}x.$$

Extending to more dimesions and more general metrics (Green's functions) yields the Hamiltonian

$$H(\boldsymbol{m},\rho) = \frac{1}{2}\iint \boldsymbol{m}(\boldsymbol{x})G_1(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{m}(\boldsymbol{x}')\mathrm{d}^n\boldsymbol{x}\,\mathrm{d}^n\boldsymbol{x}' + \frac{1}{2}\iint \rho(\boldsymbol{x})G_2(\boldsymbol{x}-\boldsymbol{x}')\rho(\boldsymbol{x}')\mathrm{d}^n\boldsymbol{x}\,\mathrm{d}^n\boldsymbol{x}', \tag{1.8}$$

which represents the two-component extension of EPDiff and is denoted by EP(DiffⓈ$\mathcal{F}$). In three dimensions, the corresponding Lie–Poisson equations assume the form

$$\rho_t = -\nabla\cdot(\rho\boldsymbol{u}), \qquad \boldsymbol{m}_t = \boldsymbol{u}\times\mathrm{curl}\,\boldsymbol{m} - \nabla(\boldsymbol{u}\cdot\boldsymbol{m}) - \boldsymbol{m}(\mathrm{div}\,\boldsymbol{u}) - \rho\nabla\lambda,$$

where $\boldsymbol{m}=\delta H/\delta\boldsymbol{m}=G_1*\boldsymbol{u}$ and $\rho=\delta H/\delta\rho=G_2*\lambda$. These expressions can also be written in a covariant form by using the Lie derivative $\pounds_u$ with respect to the velocity vector field $u\in\mathfrak{X}(\mathbb{R}^n)$ of a one-form density $m=\boldsymbol{m}\cdot\mathrm{d}\boldsymbol{x}\otimes\mathrm{d}^n\boldsymbol{x}\in\mathfrak{X}^*(\mathbb{R}^n)$, whose co-vector components are $\boldsymbol{m}$. The *diamond operator* ($\diamond$) corresponding to the Lie derivative $\pounds_u$ is defined by

$$\langle\rho, -\pounds_{\boldsymbol{u}}\lambda\rangle_{V\times V^*} := \langle\rho\diamond\lambda, \boldsymbol{u}\rangle_{\mathfrak{X}^*\times\mathfrak{X}}, \tag{1.9}$$

where, in the present case, $\lambda\in V^*=\mathcal{F}$ is a scalar function while $\rho\in V=\mathrm{Den}$ is a density variable and $\langle\cdot,\cdot\rangle$ is the L^2 pairing in the corresponding spaces. In this notation, one can write the general case for $\rho\in V$ and $\lambda\in V^*$ as

$$\rho_t + \pounds_u\rho = 0, \qquad m_t + \pounds_u m = \rho\diamond\lambda. \tag{1.10}$$

Specializing the kernel G_2 to $G_2=\Delta^{-1}$ allows this system to be physically interpreted as an incompressible charged fluid. In this case, ρ is interpreted as the charge density, rather than the mass density, and λ is the electrostatic interaction potential, satisfying $\Delta\lambda=\rho$ for the Coulomb potential. The pure CH2 case ($G_2=\delta$) corresponds to a delta-like interaction potential, which also appears in the integrable Benney system (Benney 1973; Gibbons 1981). The application of the full semidirect-product framework to the imaging method called *metamorphosis* was studied in Holm *et al.* (submitted *b*). The relation of the latter work to the current investigation will be traced further in the following sections.

(d) Plan of the paper

This paper follows Holm & Marsden (2004) in studying the singular solutions of geodesic flows on semidirect-product Lie groups by identifying them with momentum maps. Here, the new features are: (i) we explain their geometric nature in the context of the Kaluza–Klein formulation of a particle in a Yang–Mills field, and (ii) we extend the applications of this approach to systems of equations governing geodesic flows on semidirect-product Lie groups with a non-Abelian gauge group.

First, §2 first shows how this works when considering the Abelian gauge group of functions $\mathcal{F}$. Momentum maps for the left and right actions of the diffeomorphisms and the Lie group of gauge symmetry are derived, which recover the corresponding conservation laws. The key observation is that the collective Hamiltonian for the singular solution momentum map is a Kaluza–Klein Hamiltonian, thereby recovering the conservation of the gauge charge. In §3, we show how this observation extends to the consideration of a non-Abelian gauge group G for particles carrying a spin-like variable.

Section 4 is devoted to illustrating how the continuum geodesic equations on semidirect products also arise naturally from a Kaluza–Klein formulation thereby recovering the Kelvin circulation theorem in the continuum description.

2. The singular solution momentum map for EP(Diff Ⓢ $\mathcal{F}$)

Given the semidirect-product Lie–Poisson equations (1.10), a direct substitution shows that they allow the singular solutions

$$(\boldsymbol{m}, \rho) = \sum_{i=1}^{N} \int (\boldsymbol{P}_i(s,t), w_i(s))\delta(\boldsymbol{x} - \boldsymbol{Q}_i(s,t))\mathrm{d}^k s, \tag{2.1}$$

where s is a coordinate on a submanifold S of $\mathbb{R}^n$, exactly as in the case of EPDiff. If dim $S=1$, then this corresponds to fluid variables supported on a filament, while dim $S=2$ yields sheets of fluid density and momentum. The dynamics of $(\boldsymbol{Q}_i, \boldsymbol{P}_i, w_i)$ is given by

$$\frac{\partial \boldsymbol{Q}_i(s,t)}{\partial t} = \sum_j \int \boldsymbol{P}_j(s',t) G_1(\boldsymbol{Q}_i(s,t) - \boldsymbol{Q}_j(s',t))\mathrm{d}^k s',$$

$$\begin{aligned}\frac{\partial \boldsymbol{P}_i(s,t)}{\partial t} = &-\sum_j \int \boldsymbol{P}_i(s,t)\cdot\boldsymbol{P}_j(s',t)\nabla_{\boldsymbol{Q}_i} G_1(\boldsymbol{Q}_i(s,t) - \boldsymbol{Q}_j(s',t))\mathrm{d}^k s' \\ &-\sum_j \int w_i(s) w_j(s') \nabla_{\boldsymbol{Q}_i} G_2(Q_i(s,t) - \boldsymbol{Q}_j(s',t))\mathrm{d}^k s',\end{aligned} \tag{2.2}$$

with $\partial_t w_i(s) = 0, \ \forall \ i$.

Recalling the geometric nature of the pulson solution of EPDiff and following the reasoning in Holm & Marsden (2004), one can interpret $\boldsymbol{Q}_i$ as a smooth embedding in $\mathrm{Emb}(S,\mathbb{R}^n)$ and $P_i = \boldsymbol{P}_i \cdot \mathrm{d}\boldsymbol{Q}_i$ (no sum) as the canonical one-form on $T^*\mathrm{Emb}(S,\mathbb{R}^n)$ for the ith pulson. In the case of EP(Diff Ⓢ $\mathcal{F}$), the weights w_i for $i=1,\ldots,N$, are considered as maps $w_i: S \to \mathbb{R}^*$. That is, the weights w_i are distributions on S, so that $w_i \in \mathrm{Den}(S)$, where $\mathrm{Den} := \mathcal{F}^*$. In particular, we consider the triple

$$(\boldsymbol{Q}_i, \boldsymbol{P}_i, w_i) \in T^*\mathrm{Emb}(S, \mathbb{R}^n) \times \mathrm{Den}(S)$$

and prove the following.

Theorem 2.1 (Singular solution momentum map). *The singular solutions (2.1) of the semidirect-product Lie–Poisson equations (1.10) are given by*

$$(\boldsymbol{m}, \rho) = \sum_{i=1}^{N} \int (\boldsymbol{P}_i(s,t), w_i(s))\delta(\boldsymbol{x} - \boldsymbol{Q}_i(s,t))\mathrm{d}^k s.$$

These expressions for $(\boldsymbol{m}, \rho) \in \mathfrak{X}^*(\mathbb{R}^n) \circledS \mathrm{Den}(\mathbb{R}^n)$ identify a momentum map

$$\boldsymbol{J} : \times_{i=1}^{N} (T^*\mathrm{Emb}(S, \mathbb{R}^n) \times \mathrm{Den}(S)) \to \mathfrak{X}^*(\mathbb{R}^n) \circledS \mathrm{Den}(\mathbb{R}^n).$$

Proof. For convenience, we fix label i and suppress the summations in our singular solution ansatz. In order to define a momentum map, we first need to establish a Poisson structure on $T^*\mathrm{Emb} \times \mathrm{Den}$. Since the weights have no temporal evolution, it is reasonable to propose the canonical Poisson bracket on the new phase space, so that

$$\{F, G\}(\boldsymbol{Q}, \boldsymbol{P}, w) := \sum_{k=1}^{n} \int \left(\frac{\delta F}{\delta Q^k} \frac{\delta G}{\delta P_k} - \frac{\delta F}{\delta P_k} \frac{\delta G}{\delta Q^k} \right) \mathrm{d}^k s.$$

Now, if $\beta = (\boldsymbol{\beta}_1, \beta_0) \in \mathfrak{X}(\mathbb{R}^n) \circledS \mathcal{F}(\mathbb{R}^n)$, then the pairing $\langle \boldsymbol{J}, \beta \rangle$ is naturally written as

$$\langle \boldsymbol{J}(\boldsymbol{Q}, \boldsymbol{P}, w), \beta \rangle = \int (\boldsymbol{P}(s) \cdot \boldsymbol{\beta}_1(\boldsymbol{Q}(s)) + w(s)\beta_0(\boldsymbol{Q}(s)))\mathrm{d}^k s. \tag{2.3}$$

Consequently, one can calculate

$$\begin{aligned} &\{F, \langle \boldsymbol{P}, \boldsymbol{\beta}_1(\boldsymbol{Q}) \rangle\} + \{F, \langle w, \beta_0(\boldsymbol{Q}) \rangle\} \\ &\quad = \int \left(\frac{\delta F}{\delta \boldsymbol{Q}} \cdot \boldsymbol{\beta}_1(\boldsymbol{Q}(s)) - \left(\frac{\mathrm{d}\boldsymbol{\beta}_1}{\mathrm{d}\boldsymbol{Q}}^T \cdot \boldsymbol{P} + w \frac{\mathrm{d}\beta_0}{\mathrm{d}\boldsymbol{Q}} \right) \cdot \frac{\delta F}{\delta \boldsymbol{P}} \right) \mathrm{d}^k s. \end{aligned} \tag{2.4}$$

This may be written equivalently as

$$\{F, \langle \boldsymbol{J}, \beta \rangle\} = X_\beta[F],$$

in which the vector field X_β has $(\boldsymbol{Q}, \boldsymbol{P}, w)$ components

$$X_\beta := \left(\boldsymbol{\beta}_1(\boldsymbol{Q}), -\left(\frac{\mathrm{d}\boldsymbol{\beta}_1}{\mathrm{d}\boldsymbol{Q}}^T \cdot \boldsymbol{P} + w \frac{\mathrm{d}\beta_0}{\mathrm{d}\boldsymbol{Q}} \right), 0 \right). \tag{2.5}$$

This vector field is identified with a Hamiltonian vector field corresponding to the Hamiltonian

$$H = \int (w(s)\beta_0(\boldsymbol{Q}(s)) + P(s) \cdot \boldsymbol{\beta}_1(\boldsymbol{Q}(s)))\mathrm{d}^k s = \langle \boldsymbol{J}, \beta \rangle.$$

This Hamiltonian corresponds to compositions of cotangent lifts $T^*\mathrm{Diff}$ generated by $\boldsymbol{\beta}_1$ with fibre translations

$$\tau_{-\mathrm{d}(w\beta_0)} \cdot (q, p) := (q, p - w\, \mathrm{d}\beta_0),$$

generated by $-w\beta_0$ (note that w is independent of $\boldsymbol{Q}$, so that $\mathrm{d}(w\beta_0) = w\mathrm{d}\beta_0$). Thus, X_β is an infinitesimal generator. ■

(a) A left group action for the singular solution momentum map

The proof of theorem 2.1 shows that the momentum map $\boldsymbol{J}$ in equation (2.3) is obtained by the following left action of the semidirect-product group $\mathrm{Diff}\,Ⓢ\mathcal{F}$:

$$
\begin{aligned}
(\boldsymbol{Q}^{(t)}, \boldsymbol{P}^{(t)}, w^{(t)}) &= \Big(\eta_t(\boldsymbol{Q}^{(0)}), \boldsymbol{P}^{(0)} \cdot T\eta_t^{-1}(\boldsymbol{Q}^{(0)}) - d\Big(w\beta_0(\boldsymbol{Q}^{(0)})\Big), w^{(0)}\Big) \\
&= \Big(\eta_t(\boldsymbol{Q}^{(0)}), \boldsymbol{P}^{(0)} \cdot \nabla\eta_t^{-1}(\boldsymbol{Q}^{(0)}) - \nabla(w\beta_0(\boldsymbol{Q}^{(0)})), w^{(0)}\Big) \\
&= \tau_{-\mathrm{d}(w\beta_0)} \circ \eta_t\Big(\boldsymbol{Q}^{(0)}, \boldsymbol{P}^{(0)}, w^{(0)}\Big),
\end{aligned}
$$

where $(\eta_t, \beta_0) \in \mathrm{Diff}\,Ⓢ\mathcal{F}$. It is worth noting that the order of operations in the composition is not relevant, since

$$
\eta_t^{-1} \circ \tau_{w\mathrm{d}\beta_0}\Big(\boldsymbol{Q}^{(0)}, \boldsymbol{P}^{(0)}, w^{(0)}\Big) = (\eta_t, \beta_0)^{-1}(\boldsymbol{Q}^{(0)}, \boldsymbol{P}^{(0)}, w^{(0)}),
$$

so that $(\eta^{-1}, \beta_0) = (\eta, -\beta_0)^{-1} \in \mathrm{Diff}\,Ⓢ\mathcal{F}$. This can be easily seen from the following calculation, where we take $w=1$ for simplicity:

$$
\begin{aligned}
(\eta, \beta_0)^{-1} \circ (\eta, \beta_0) \cdot (\boldsymbol{Q}, \boldsymbol{P}) &= (\eta^{-1}, -\beta_0) \cdot (\eta_t(\boldsymbol{Q}), \boldsymbol{P} \cdot T\eta_t^{-1}(\boldsymbol{Q}) + \mathrm{d}\beta_0(\boldsymbol{Q})) \\
&= T^*\eta^{-1} \cdot (\eta_t(\boldsymbol{Q}), \boldsymbol{P} \cdot T\eta_t^{-1}(\boldsymbol{Q})) = T^*\eta^{-1} \circ T^*\eta \cdot (\boldsymbol{Q}, \boldsymbol{P}) \\
&= (\boldsymbol{Q}, \boldsymbol{P}).
\end{aligned}
$$

Consequently, exchanging the order of operations in the composition simply yields another element of $\mathrm{Diff}\,Ⓢ\mathcal{F}$ and the arguments above are still valid. Such group operations are useful in deriving fluid descriptions from kinetic equations, where the so-called *plasma-to-fluid momentum map* determines the Hamiltonian structure of the fluid system (Marsden *et al.* 1983).

Another important property of the momentum map $\boldsymbol{J}$ in (2.2) is its equivariance (Marsden & Ratiu 1999), which guarantees that $\boldsymbol{J}$ is also a Poisson map. In this case, the infinitesimal equivariance of $\boldsymbol{J}$ follows by directly verifying that the definition below is satisfied:

$$
X_\beta[\langle \boldsymbol{J}, \gamma \rangle] = \langle \boldsymbol{J}, \mathrm{ad}_\beta \gamma \rangle \quad \forall\ \beta, \gamma \in \mathfrak{X}\,Ⓢ\mathcal{F}
$$

Theorem 2.2 (Equivariance). *The singular solution momentum map* $\boldsymbol{J}$ *in* (2.3),

$$
\boldsymbol{J}(\boldsymbol{Q}, \boldsymbol{P}, w) = \int (\boldsymbol{P}(s,t), w(s))\delta(\boldsymbol{x} - \boldsymbol{Q}(s,t))\mathrm{d}^k s,
$$

is infinitesimally equivariant.

(b) Collective Hamiltonian and Kaluza–Klein formulation

From the expression of the vector field (2.5), we can immediately write the equations of motion for $\boldsymbol{Q}$ and $\boldsymbol{P}$. Moreover, if we insert the singular solution momentum map into the Hamiltonian (1.8), we recover the following collective

Hamiltonian $H_N : \times_{i=1}^N (T^*\mathrm{Emb}(S,\mathbb{R}^n) \times \mathrm{Den}(S)) \to \mathbb{R}$:

$$H_N = \frac{1}{2}\sum_{i,j}^N \iint \boldsymbol{P}_i(s,t)\cdot \boldsymbol{P}_j(s',t)\, G_1\big(\boldsymbol{Q}_i(s,t) - \boldsymbol{Q}_j(s',t)\big)\mathrm{d}^k s\, \mathrm{d}^k s' + \frac{1}{2}\sum_{i,j}^N \iint w_i(s) w_j(s')\, G_2(\boldsymbol{Q}_i(s,t) - \boldsymbol{Q}_j(s\prime,t))\mathrm{d}^k s\, \mathrm{d}^k s'.$$

The corresponding canonical Hamiltonian equations of motion are (2.2).

It is interesting that the collective Hamiltonian allows for a Kaluza–Klein formulation similar to the usual treatment of a particle in a magnetic field (Marsden & Ratiu 1999). In order to see this, first, one can observe that the equations of motion for ($\boldsymbol{Q}_i$, $\boldsymbol{P}_i$, w_i) may be recovered on the Lagrangian side via the Legendre transform,

$$\boldsymbol{V}^i = G_1^{ij}\boldsymbol{P}_j, \qquad \dot\theta^i = G_2^{ij} w_j.$$

This yields the following Lagrangian $L : \times_{i=1}^N (T\mathrm{Emb}(S,\mathbb{R}^n) \times \mathcal{F}(S)) \to \mathbb{R}$:

$$L_N = \frac{1}{2}\sum_{i,j}^N \iint \boldsymbol{V}^i(s,t)\cdot \boldsymbol{V}^j(s',t)\, G^1\big(\boldsymbol{Q}_i(s,t) - \boldsymbol{Q}_j(s',t)\big)\mathrm{d}^k s\, \mathrm{d}^k s' + \frac{1}{2}\sum_{i,j}^N \iint \dot\theta^i(s)\dot\theta^j(s')\, G^2\big(\boldsymbol{Q}_i(s,t) - \boldsymbol{Q}_j(s',t)\big)\mathrm{d}^k s\, \mathrm{d}^k s',$$

where G^i with a raised index is the inverse metric associated to G_i. (If G_i is given by a convolution kernel, then G^i becomes a differential operator.) As our notation may suggest, we now enlarge our configuration space so that the Lagrangian L_N becomes defined on

$$TQ_{KK} := T(\mathrm{Emb}(S,\mathbb{R}^n) \times \mathcal{F}(S)) = T\mathrm{Emb}(S,\mathbb{R}^n) \times T\mathcal{F}(S)$$

and Q_{KK} is called the *Kaluza–Klein configuration space*. Now, since the coordinates θ_i in the Lagrangian $L_N = L(\boldsymbol{Q}^i, \boldsymbol{V}^i, \theta^i, \dot\theta^i)$ are ignorable, its conjugate momenta w_i will be constants of motion. This allows the collective Hamiltonian to be naturally written in the Kaluza–Klein formulation on T^*Q_{KK}. In this framework, it is well known (Ratiu *et al.* 2005) that the weights w_i are another type of conserved momentum map.

Remark 2.3 (Physical interpretation). The physical system described by the Kaluza–Klein Hamiltonian H_N turns out to be related to the motion of electrical charges whose mutual interaction is given by the potential term in G_2. This relation is evident by noting that for the case of a single particle, the Hamiltonian H_N reduces to the Kaluza–Klein Hamiltonian $H_1 = (P^2/2) + (w^2/2)$ of a free charge (Marsden & Ratiu 1999). In the multi-particle case, the momentum of a single charge is affected not only by the momenta of the remaining particles in the system, but also by their charges involved in the potential term.

(*c*) *A right-action momentum map*

As shown in §2*b*, the singular solutions identify a momentum map, which is determined by a left action of the group Diff($\mathbb{R}^n$) on $\boldsymbol{Q} \in \mathrm{Emb}(S,\mathbb{R}^n)$, i.e. $\eta\boldsymbol{Q} = \eta \circ \boldsymbol{Q}$. However, as in the case of EPDiff, one may also construct a right action by $\boldsymbol{Q}\eta = \boldsymbol{Q} \circ \eta$, which is defined through the group Diff(S), rather than Diff($\mathbb{R}^n$).

In order to perform such a construction, we consider the Kaluza–Klein formulation from §2*b*, so that the configuration space is now Q_{KK}=Emb(S,ℝn)× Den(s). The group Diff(S) acts on Q_{KK} from the right according to

$$\left(\boldsymbol{Q}^{(0)}, \theta^{(0)}\right)\eta_t = \left(\boldsymbol{Q}^{(0)}\circ\eta_t, \theta\circ\eta_t\right).$$

By standard arguments (Holm & Marsden 2004), the cotangent lift of this action to T^*Q_{KK} yields the following.

Theorem 2.4. *The map*

$$\boldsymbol{J}_S(\boldsymbol{Q}, \boldsymbol{P}, \theta, w) = \boldsymbol{P}(s)\cdot\mathrm{d}\boldsymbol{Q}(s) + w(s)\mathrm{d}\theta(s) \tag{2.6}$$

is a momentum map

$$\boldsymbol{J}_S : T^*(\mathrm{Emb}(S,\mathbb{R}^n)\times\mathcal{F}(S))\rightarrow\mathfrak{X}^*(S),$$

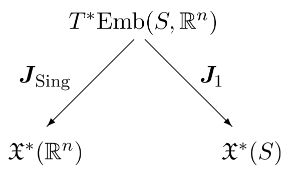
corresponding to the cotangent lift of the right action of Diff(S) *on* Emb(S,ℝ^n)× $\mathcal{F}(S)$. *This quantity is preserved by the flow generated by the Hamiltonian* H_N.

The last part of the statement follows from the fact that the Hamiltonian H_N is invariant under the cotangent lift of the right action of Diff(S), which amounts to the invariance of the integral over S under reparametrization using the change of variables formula.

Remark 2.5 (Dual pair structures). Upon recalling from Holm & Marsden (2004) that the term $\boldsymbol{J}_1 := \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{Q}$ in (2.6) is a momentum map $\boldsymbol{J}_1: T^*\mathrm{Emb}\rightarrow\mathfrak{X}^*(S)$ for the right action of Diff(S), one may construct the same dual pair structure as in the geometric description of the EPDiff equation. Indeed, we may introduce the map

$$\boldsymbol{J}_{\mathrm{Sing}}(\boldsymbol{Q}, \boldsymbol{P}) = \int \boldsymbol{P}(s,t)\delta(\boldsymbol{x}-\boldsymbol{Q}(s,t))\mathrm{d}^k s,$$

i.e. the $\mathfrak{X}^*$-component of the singular solution momentum map $\boldsymbol{J}$ in theorem 2.1. This is well known (Holm & Marsden 2004) to be a momentum map $\boldsymbol{J}_{\mathrm{Sing}}: T^*\mathrm{Emb}\rightarrow\mathfrak{X}^*(\mathbb{R}^n)$ that generates the left leg of the following dual pair picture:



which is the standard dual pair picture associated to the EPDiff equation (Holm & Marsden 2004). Dual pair structures have been used in Marsden & Weinstein (1983) to explore the geometric nature of Clebsch variables in fluid systems. In the present context the variables ($\boldsymbol{Q}$, $\boldsymbol{P}$) form the Clebsch representation associated to the diffeomorphism group Diff(S). We refer to the original works (Marsden & Weinstein 1983; Weinstein 1983) for deeper discussions on the geometric nature of dual pairs.

In the present case, the right-action momentum map $\boldsymbol{J}_S$ also takes into account the extra term $\boldsymbol{J}_2 := w\,\mathrm{d}\theta$, which is associated with the space of scalar functions $\mathcal{F}(S)$. This term is again a momentum map $\boldsymbol{J}_2: T^*\mathcal{F}\rightarrow\mathfrak{X}^*(S)$, which will be used later in this paper for the construction of another dual pair, associated to the $\mathcal{F}^*$-component of the singular solution momentum map in theorem 2.1.

3. Extension to anisotropic interactions

This section extends the previous results on singular solutions to the case when the fluid motion depends on an extra degree of freedom, such as the fluid particle orientation. This occurs, for example, in the theory of liquid crystals (Holm 2002*a*; Gay-Balmaz & Ratiu in press). A geometric fluid theory for such systems is already present in the literature regarding Yang–Mills charged fluids and quark–gluon plasmas (Gibbons *et al.* 1982). This work formulates the equations for the fluid momentum $\boldsymbol{m}(\boldsymbol{x})$, the mass density $\rho(\boldsymbol{x})$ and the charge density $C(\boldsymbol{x})$, where the charge is considered as an extra degree of freedom of each fluid particle. (This is the colour charge in the case of chromo-hydrodynamics for quark–gluon plasmas.) These equations are written as

$$\left.\begin{aligned} &\frac{\partial\rho}{\partial t}+\operatorname{div}\left(\rho\frac{\partial H}{\partial\boldsymbol{m}}\right)=0, \qquad \frac{\partial C}{\partial t}+\operatorname{div}\left(C\frac{\partial H}{\partial\boldsymbol{m}}\right)=\operatorname{ad}^*_{\partial H/\partial C}C,\\ &\frac{\partial\boldsymbol{m}}{\partial t}+\nabla\cdot\left(\frac{\partial H}{\partial\boldsymbol{m}}\otimes\boldsymbol{m}\right)+\left(\nabla\otimes\frac{\partial H}{\partial\boldsymbol{m}}\right)\cdot\boldsymbol{m}=-\rho\nabla\frac{\partial H}{\partial\rho}-\left\langle C,\nabla\frac{\partial H}{\partial C}\right\rangle_{\mathfrak{g}^*\times\mathfrak{g}},\end{aligned}\right\} \tag{3.1}$$

where C takes values on the dual Lie algebra $\mathfrak{g}^*$, whose corresponding coadjoint operation is denoted by ad^*. Thus the charge variable C belongs to the space of $\mathfrak{g}^*$-valued densities, which we denote by

$$\mathfrak{g}^*(\mathbb{R}^n):=\operatorname{Den}(\mathbb{R}^n)\otimes\mathfrak{g}^*,$$

so that $C\in\mathfrak{g}^*(\mathbb{R}^n)$. In the following, we use the elementary fact that the space $\mathfrak{g}^*(\mathbb{R}^n)$ is dual to $\mathfrak{g}(\mathbb{R}^n)=\mathcal{F}\otimes\mathfrak{g}$, and we use the same notation for $\mathfrak{g}$ and $\mathfrak{g}(\mathbb{R}^n)$. The distinction should be clear from the different contexts.

Now, the equations above are known to possess a Lie–Poisson Hamiltonian structure dual to the Lie algebra of the semidirect-product Lie group $\operatorname{Diff}Ⓢ(\mathcal{F}\oplus\mathfrak{g})$ (Gibbons *et al.* 1982). One may also consider *geodesic* Euler–Poincaré equations on this semidirect-product Lie group. This problem has already been considered in Gibbons *et al.* (2008) in terms of its singular solutions, although not in relation to momentum maps. For the sake of simplicity, we consider the semidirect product $\operatorname{Diff}Ⓢ\mathfrak{g}$ and denote the corresponding geodesic equations by $\operatorname{EP}(\operatorname{Diff}Ⓢ\mathfrak{g})$. (The commutative case $\mathfrak{g}=\mathcal{F}\times\mathbb{R}$ reduces to the case studied in §2.)

In order to construct the $\operatorname{EP}(\operatorname{Diff}Ⓢ\mathfrak{g})$ equations, one writes a purely quadratic Hamiltonian

$$\begin{aligned} H(\boldsymbol{m},C)&=\frac{1}{2}\iint\boldsymbol{m}(\boldsymbol{x})\cdot G_1(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{m}(\boldsymbol{x}')\mathrm{d}^n\boldsymbol{x}\,\mathrm{d}^n\boldsymbol{x}'\\ &\quad+\frac{1}{2}\iint\langle C(\boldsymbol{x}),G_2(\boldsymbol{x}-\boldsymbol{x}')C(\boldsymbol{x}')\rangle_{\mathfrak{g}^*\times\mathfrak{g}}\mathrm{d}^n\boldsymbol{x}\,\mathrm{d}^n\boldsymbol{x}',\end{aligned} \tag{3.2}$$

which yields the geodesic equations in a covariant form

$$C_t+\pounds_{G_1*\boldsymbol{m}}C=\operatorname{ad}^*_{G_2*C}C, \qquad \boldsymbol{m}_t+\pounds_{G_1*\boldsymbol{m}}\boldsymbol{m}=C\diamond(G_2*C), \tag{3.3}$$

where the Lie derivative and diamond operations were introduced in §1. In order to simplify the discussion, one can specialize to the case when particles have an orientation (or spin) in space and think of the charge density as the distribution of the local particle orientation in space, so that $\boldsymbol{C} \in \mathrm{Den}(\mathbb{R}^n) \otimes \mathfrak{so}(3) \simeq \mathrm{Den}(\mathbb{R}^n) \otimes \mathbb{R}^3$ and the Lie bracket is given by the usual cross product. However, the following result applies in general.

Theorem 3.1. *The* EP(Diff Ⓢ $\mathfrak{g}$) *equations admit singular solutions of the form*

$$(\boldsymbol{m}, C) = \sum_{i=1}^{N} \int (\boldsymbol{P}_i(s,t), \mu_i(s,t))\delta(\boldsymbol{x} - \boldsymbol{Q}_i(s,t))\mathrm{d}^k s, \tag{3.4}$$

associated with the momentum map

$$\boldsymbol{J} : \mathop{\times}_{i=1}^{N} (T^* \, \mathrm{Emb}(S, \mathbb{R}^n) \times \mathfrak{g}^*(S)) \rightarrow \mathfrak{X}^*(\mathbb{R}^n) \circledS \mathfrak{g}^*(\mathbb{R}^n)$$

Proof. Again, we fix i for convenience and suppress it in the notation. Substitution of the solution ansatz (3.4) into the EP(Diff Ⓢ $\mathfrak{g}$) equations yields

$$\frac{\partial \boldsymbol{Q}}{\partial t} = \boldsymbol{\beta}_1(\boldsymbol{Q}), \qquad \frac{\partial \boldsymbol{P}}{\partial t} = -\boldsymbol{P} \cdot \nabla_{\boldsymbol{Q}} \boldsymbol{\beta}_1 - \langle \mu, \nabla_{\boldsymbol{Q}} \beta_0 \rangle, \qquad \frac{\partial \mu}{\partial t} = \mathrm{ad}^*_{\beta_0} \mu. \tag{3.5}$$

In order to define a momentum map, first, we need to establish a Poisson structure on $T^*\mathrm{Emb} \times \mathfrak{g}^*$. For this purpose, we use the following Poisson bracket (Gibbons *et al.* 1982; Montgomery *et al.* 1984):

$$\{F, G\}(\boldsymbol{Q}, \boldsymbol{P}, \mu) := \int \left(\frac{\delta F}{\delta \boldsymbol{Q}} \cdot \frac{\delta G}{\delta \boldsymbol{P}} - \frac{\delta F}{\delta \boldsymbol{P}} \cdot \frac{\delta G}{\delta \boldsymbol{Q}} \right) \mathrm{d}^k s - \int \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle \mathrm{d}^k s, \tag{3.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing on $\mathfrak{g}^* \times \mathfrak{g}$. Now, if $\beta = (\boldsymbol{\beta}_1, \beta_0) \in \mathfrak{X}(\mathbb{R}^n) \circledS \mathfrak{g}(\mathbb{R}^n)$, then the functional $\langle \boldsymbol{J}, \beta \rangle$ may be defined as

$$\langle \boldsymbol{J}(\boldsymbol{Q}, \boldsymbol{P}, \mu), \beta \rangle = \int (\langle \mu, \beta_0(\boldsymbol{Q}) \rangle + \boldsymbol{P} \cdot \boldsymbol{\beta}_1(\boldsymbol{Q}))\mathrm{d}^k s,$$

where $\langle \cdot, \cdot \rangle$ may now denote the pairing on either $\mathfrak{g}^* \times \mathfrak{g}$ or $\mathfrak{X}^* \circledS \mathfrak{g}^* \times \mathfrak{X} \circledS \mathfrak{g}$. (No confusion should arise from this notation.) At this point, one can calculate the Poisson bracket using equation (3.6) as, cf. equation (2.4),

$$\begin{aligned} \{F, \langle \boldsymbol{J}, \beta \rangle\} = &\int \left(\frac{\delta F}{\delta \boldsymbol{Q}} \cdot \boldsymbol{\beta}_1(\boldsymbol{Q}) - \left(\left\langle \mu, \frac{\mathrm{d}\beta_0}{\mathrm{d}\boldsymbol{Q}} \right\rangle + \frac{\mathrm{d}\boldsymbol{\beta}_1}{\mathrm{d}\boldsymbol{Q}}^T \cdot \boldsymbol{P} \right) \cdot \frac{\delta F}{\delta \boldsymbol{P}} \right) \mathrm{d}^k s \\ &+ \int \left\langle \mathrm{ad}^*_{\beta_0} \mu, \frac{\delta F}{\delta \mu} \right\rangle \mathrm{d}^k s. \end{aligned} \tag{3.7}$$

Thus, one can find the Hamiltonian vector field $\{F, \langle \boldsymbol{J}, \beta \rangle\} = X_\beta[F]$, with $(\boldsymbol{Q}, \boldsymbol{P}, \mu)$ components

$$X_\beta := \left(\boldsymbol{\beta}_1(\boldsymbol{Q}), -\left\langle \mu, \frac{\mathrm{d}\beta_0}{\mathrm{d}\boldsymbol{Q}} \right\rangle - \frac{\mathrm{d}\boldsymbol{\beta}_1}{\mathrm{d}\boldsymbol{Q}}^T \cdot \boldsymbol{P}, \mathrm{ad}^*_{\beta_0} \mu \right).$$

The first two components of this vector field identify a Hamiltonian vector field on $T^*\mathrm{Emb}$ corresponding to the Hamiltonian

$$H = \int (\langle \mu, \beta_0(\boldsymbol{Q}) \rangle + \boldsymbol{P} \cdot \boldsymbol{\beta}_1(\boldsymbol{Q}))\mathrm{d}^k s = \langle \boldsymbol{J}, \beta \rangle,$$

which generates compositions of cotangent lifts $T^*\mathrm{Diff}$ generated by $\boldsymbol{\beta}_1$ with fibre translations

$$\tau_{-\mathrm{d}\langle \mu, \beta_0 \rangle} \cdot (q, p) = (q, p - \langle \mu, d\beta_0 \rangle),$$

generated by $-\langle\mu,\beta_0\rangle$. The third component generates pure coadjoint motion of the charge variable μ on $\mathfrak{g}^*$ according to

$$\mu^{(t)} = \mathrm{Ad}^*_{\exp(-t\beta_0)}\mu^{(0)},$$

under the action of the Lie group G whose underlying Lie algebra is $\mathfrak{g} := T_eG$. Thus, the three-component vector field X_β is an infinitesimal generator. ■

Remark 3.2 (Left action). Just as in the case of EP(Diff Ⓢ $\mathcal{F}$), we can write a similar left group action of Diff Ⓢ $\mathfrak{g}$ on $T^*\mathrm{Emb}\times\mathfrak{g}^*$. Indeed, by the arguments in the proof above, one can see that the momentum map $\boldsymbol{J}$ derives from the following left action:

$$\left(\boldsymbol{Q}^{(t)}, \boldsymbol{P}^{(t)}, \mu^{(t)}\right) = \left(\eta_t\circ\boldsymbol{Q}^{(0)}, \boldsymbol{P}^{(0)}\cdot T\left(\eta_t^{-1}\circ\boldsymbol{Q}^{(0)}\right) - \mathrm{d}\left(\langle\mu,\beta_0\rangle\circ\boldsymbol{Q}^{(0)}\right), \mathrm{Ad}^*_{\exp(-t\beta_0)}\mu^{(0)}\right),$$

where $\eta_t = \exp(t\boldsymbol{\beta}_1)\in\mathrm{Diff}$. The equivariance of $\boldsymbol{J}$ is proved again by direct verification.

(*a*) *Kaluza–Klein collective Hamiltonian*

As in the isotropic case, again, we see that the collective Hamiltonian

$$H_N = \frac{1}{2}\sum_{i,j}^{N}\iint \boldsymbol{P}_i(s)\cdot\boldsymbol{P}_j(s')\,G_1(\boldsymbol{Q}_i(s)-\boldsymbol{Q}_j(s'))\mathrm{d}^ks\,\mathrm{d}^ks'$$

$$+\frac{1}{2}\sum_{i,j}^{N}\iint\langle\mu_i(s),\,G_2(\boldsymbol{Q}_i(s)-\boldsymbol{Q}_j(s'))\mu_j(s')\rangle\mathrm{d}^ks\,\mathrm{d}^ks',$$

obtained by the direct substitution of $\boldsymbol{J}(\boldsymbol{Q},\boldsymbol{P})$ in the EP(Diff Ⓢ $\mathfrak{g}$) Hamiltonian, allows for a Kaluza–Klein formulation. However, in this case the gauge group is not Abelian and we need to proceed more carefully. In the Kaluza–Klein picture of the motion of a coloured particle in a Yang–Mills field, the particle motion is a geodesic on a principal G-bundle B. The metric on B is G-invariant and its geodesics are determined by the G-invariant quadratic Hamiltonian on T^*B, where the Poisson bracket is canonical (Montgomery 1984). Specializing to our case yields

$$Q_{KK} := B = \mathop{\times}_{i=1}^{N}(\mathrm{Emb}\times G)$$

and since the second term in H_N is G-invariant, it may be lifted to T^*Q_{KK} as follows:

$$H_N(\boldsymbol{Q}^i,\boldsymbol{P}_i,g^i,p_i) = \frac{1}{2}\sum_{i,j}^{N}\iint \boldsymbol{P}_i(s)\cdot\boldsymbol{P}_j(s')\,G_1(\boldsymbol{Q}^i(s)-\boldsymbol{Q}^j(s'))\mathrm{d}^ks\,\mathrm{d}^ks'$$

$$+\frac{1}{2}\sum_{i,j}^{N}\iint\langle p_i(s),\,G_2(\boldsymbol{Q}^i(s)-\boldsymbol{Q}^j(s'))p_j(s')\rangle\mathrm{d}^ks\,\mathrm{d}^ks',$$

where p_i is the conjugate momentum of the group coordinate $g^i \in G$, so that $(g^i, p_i) \in T^*G$, and $\langle \cdot, \cdot \rangle$ is now the pairing between the tangent and cotangent vectors on G. The momentum p_i is conserved, since it is conjugate to the cyclic variable g^i. Thus the Hamiltonian H_N is Kaluza–Klein and thereby recovers the conservation of p_i. Such a conservation law becomes coadjoint motion on the dual Lie algebra $\mathfrak{g}^*$, such that

$$\dot{\mu}_i = \mathrm{ad}^*_{\delta H_N/\delta \mu_i} \mu_i \qquad (\text{no sum}),$$

where $\mu_i = {g^i}^{-1} p_i \; \forall \, i = 1, \ldots, N$ (no sum over i), exactly as happens for the motion of a rigid body, when $G = \mathrm{SO}(3)$. As a consequence of the above arguments, it is clear how the dynamics (3.5) of the singular solutions is Hamiltonian with respect to the Poisson bracket in (3.6), which is the sum of a canonical term and a Lie–Poisson term.

(*b*) *The right-action momentum map and its implications*

One may also consider the *right* action through the group $\mathrm{Diff}(S)$. Upon following the same procedure as for the Abelian case, one can find the momentum map corresponding to the right action of $\mathrm{Diff}(S)$:

$$\boldsymbol{J}_S(\boldsymbol{Q}, \boldsymbol{P}, g, p) = \boldsymbol{P}(s) \cdot \mathrm{d}\boldsymbol{Q}(s) + \langle p(s), \mathrm{d}g(s) \rangle \in \mathfrak{X}^*(S), \tag{3.8}$$

where the Kaluza–Klein phase space is now

$$T^* Q_{KK} = T^*\mathrm{Emb} \times T^* G.$$

This momentum map is again conserved owing to the evident symmetry of the Hamiltonian H_N under relabelling s by a change of variables.

Remark 3.3 (Kelvin–Noether theorem). Upon seeking a Kelvin–Noether theorem for the non-Abelian system, one recognizes that this system does not provide any conserved density variable that could be used to construct the loop integral of a differential one form. For this purpose, it suffices to fix a weight $w \in \mathrm{Den}(S)$ preserved by the flow to obtain the following circulation theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} w^{-1}(s) (\boldsymbol{P}(s) \cdot \mathrm{d}\boldsymbol{Q}(s) + \langle p(s), \mathrm{d}g(s) \rangle) = 0.$$

In the Abelian case, the group and its Lie algebra are identified. Moreover the weight w is intrinsically given by the singular solution (2.1) and one has $w(s) = p(s)$. Thus, upon denoting $\theta(s) = g(s)$, the Kelvin–Noether theorem reduces to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} w^{-1}(s) (\boldsymbol{P}(s) \cdot \mathrm{d}\boldsymbol{Q}(s) + w(s) \mathrm{d}\theta(s)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \left(w^{-1}(s) \boldsymbol{P}(s) \cdot \mathrm{d}\boldsymbol{Q}(s) + \mathrm{d}\theta \right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} w^{-1}(s) \boldsymbol{P}(s) \cdot \mathrm{d}\boldsymbol{Q}(s) = 0. \end{aligned}$$

As in §2, it is interesting to note that the momentum map $\boldsymbol{J}_S$ for relabelling symmetry by the right action is determined by the sum $\boldsymbol{J}_1 + \boldsymbol{J}_2$ of two *distinct* momentum maps, one for Diff and the other for the gauge symmetry,

$$\boldsymbol{J}_1(\boldsymbol{Q},\boldsymbol{P}) = \boldsymbol{P}(s)\cdot \mathrm{d}\boldsymbol{Q}(s) \qquad \boldsymbol{J}_1 : T^*\mathrm{Emb} \to \mathfrak{X}^*(S),$$

$$\boldsymbol{J}_2(g,p) = \langle p(s), \mathrm{d}g(s)\rangle \qquad \boldsymbol{J}_2 : T^*G \to \mathfrak{X}^*(S).$$

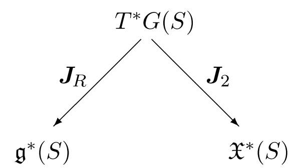
These momentum maps have the same target space, but different image spaces. Now, since the pairing $\langle p(s),\mathrm{d}g(s)\rangle$ is invariant under the (left or right) G-action by cotangent lifts, it is possible to re-express it as

$$\langle p(s), \mathrm{d}g(s)\rangle = \langle g^{-1}p, g^{-1}\nabla_s g\rangle \mathrm{d}s =: \langle \mu(s), \boldsymbol{\mathcal{A}}(s)\rangle \mathrm{d}s = \langle \mu(s), \mathcal{A}(s)\rangle,$$

for $\mu = g^{-1}p$ and a $\mathfrak{g}$-valued one form $\mathcal{A}(s) = g^{-1}\mathrm{d}g = \boldsymbol{\mathcal{A}}(s)\mathrm{d}s$ (i.e. a pure gauge connection). This result does not depend on the particular choice of left or right G-action, since the invariance property is not affected by this choice. Consequently, with the definitions $\mu := g^{-1}p$ and $\mathcal{A} := g^{-1}\mathrm{d}g$ one may rewrite $\boldsymbol{J}_2$ as

$$\boldsymbol{J}_2(g,p) = \langle p(s), \mathrm{d}g(s)\rangle = \langle \mu(s), \mathcal{A}(s)\rangle.$$

The momentum map $\boldsymbol{J}_2$ can now be used to construct another dual pair, describing the geometry of the dynamics of the gauge charge μ. Indeed, it is well known (Marsden & Ratiu 1999) that the expression $\mu = g^{-1}p =: \boldsymbol{J}_R(g,p)$ is a momentum map $\boldsymbol{J}_R: T^*G \to \mathfrak{g}^*$ associated to cotangent lifts of right translation. One may use this map to construct the following dual pair picture:



According to the general definition (Weinstein 1983), a pair of momentum maps $\mathfrak{h}^* \xleftarrow{J_1} P \xrightarrow{J_2} \mathfrak{g}^*$ is called a *dual pair* if and only if $\mathrm{Ker}\, TJ_1$ and $\mathrm{Ker}\, TJ_2$ are symplectically orthogonal to one another. As explained in Holm & Marsden (2004), a necessary condition for $\mathfrak{h}^* \xleftarrow{J_1} P \xrightarrow{J_2} \mathfrak{g}^*$ to be a dual pair is that each Lie group G_i associated to J_i acts transitively on the level sets of J_k, with $k \neq i$. Now, $\mathrm{Diff}(S)$ acts transitively on the level sets of $\boldsymbol{J}_R = \mu(s)$, owing to the parametrization freedom. Moreover, the action of $G(S)$ on the level sets of $\boldsymbol{J}_2$ is transitive too, since it is given by cotangent lifts. Thus, similar arguments to those in Holm & Marsden (2004) allow one to conclude that the above dual pair is properly defined.

4. Kaluza–Klein equations for semidirect products

As we have seen from the previous sections, the Kaluza–Klein construction explains how collective motion on semidirect-product Lie groups arises under the singular solution momentum map. In this section, we extend the Kaluza–Klein formulation to continuum equations on a semidirect-product Lie group. The resulting equations apply to the method of *metamorphosis* in the problem of matching shapes using active templates in imaging science (Holm *et al.* submitted *b*). In this application, the zero level set of the momentum map for the right action plays a crucial role. Below we shall present only the non-Abelian case and the specialization to Abelian groups is always possible.

From the theory of semidirect-product reduction (Holm *et al.* 1998), one knows that the reduced Lagrangian $L:\mathfrak{X}Ⓢ\mathfrak{g}\to\mathbb{R}$ arises form an invariant function $\mathcal{L}(\eta,\dot\eta,g,\dot g)$ on $T(\mathrm{Diff}\,Ⓢ\,G)$. The reduction process underlying the continuum dynamics on $\mathfrak{X}Ⓢ\mathfrak{g}$ proceeds as follows:

$$(T(\mathrm{Diff}Ⓢ G)/G)/\mathrm{Diff}\simeq(T\mathrm{Diff}\times TG/G)/\mathrm{Diff}\simeq\mathfrak{X}\times\mathfrak{g}/\mathrm{Diff}\simeq\mathfrak{X}Ⓢ\mathfrak{g}.$$

However, for G-invariant Lagrangians on $\mathfrak{X}Ⓢ\mathfrak{g}$, we may interpret the dynamics as occurring on the space $\mathfrak{X}\times TG$, so that the Kaluza–Klein Lagrangian

$$L_{KK}:\mathfrak{X}\times TG\to\mathbb{R},$$

with $(g,\dot g)\in TG$ and $\eta\in\mathrm{Diff}$ is written after the right reduction by Diff as

$$\mathcal{L}(\dot\eta\eta^{-1},g\eta^{-1},\dot g\eta^{-1})=L_{KK}(\boldsymbol{u},n,\nu)=\frac{1}{2}\int\boldsymbol{u}Q_1\boldsymbol{u}\,\mathrm{d}^n\boldsymbol{x}+\frac{1}{2}\int\langle Q_2\nu,\nu\rangle\,\mathrm{d}^n\boldsymbol{x},$$

with definitions $\boldsymbol{u}:=\dot\eta\eta^{-1}$, $n:=g\eta^{-1}$ and $\nu=\dot g\eta^{-1}$. Now, since such a Lagrangian is also G-invariant, then one may write

$$L_{KK}(\boldsymbol{u},n^{-1}n,n^{-1}\nu)=L(\boldsymbol{u},\chi),$$

where $n=g\eta^{-1}\in G$ as before and $L(\boldsymbol{u},\chi)$, with $\chi=n^{-1}\nu$, is the Lagrangian on $\mathfrak{X}Ⓢ\mathfrak{g}$. The Legendre transformation of this Lagrangian produces the Hamiltonian $H(\boldsymbol{m},\boldsymbol{C})$ in (3.2). This construction yields the conservation of the conjugate variable $p=\delta L_{KK}/\delta\nu$ along the flow of the group of diffeomorphisms, since n is an ignorable coordinate. The reduction process involved in such a system proceeds as follows:

$$T(\mathrm{Diff}Ⓢ G)/\mathrm{Diff}\simeq(T\mathrm{Diff}/\mathrm{Diff})\times(TG/\mathrm{Diff})\simeq\mathfrak{X}\times(TG/\mathrm{Diff}),$$

where TG is the group of tangent lifts of G, which is itself acted on by the diffeomorphisms. Thus, again, the two Lagrangians L and L_{KK} may be derived from the *same unreduced Lagrangian* $\mathcal{L}$. consequently, the geodesic motion on semidirect-product Lie groups of the kind $\mathrm{Diff}Ⓢ G$ *always* possesses a Kaluza–Klein construction.

(*a*) *Application to metamorphosis*

Lagrangian formulations on $\mathrm{Diff}Ⓢ G$ have recently been considered in Holm *et al.* (submitted *b*), where the whole theory is extensively studied in the context of imaging science. The Euler–Poincaré equations corresponding to a Lagrangian $L(\boldsymbol{u},n,\nu)$ carrying the cyclic variable $n=g\eta^{-1}\in G$ are found to be

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+\pounds_{\boldsymbol{u}}\right)\frac{\delta L}{\delta\boldsymbol{u}}=-\left\langle\frac{\delta L}{\delta\nu},\mathrm{d}\nu\right\rangle,\qquad\left(\frac{\partial}{\partial t}+\pounds_{\boldsymbol{u}}\right)\frac{\delta L}{\delta\nu}=0,\\ \left(\frac{\partial}{\partial t}+\pounds_{\boldsymbol{u}}\right)n=\nu,\end{aligned}\right\}\tag{4.1}$$

in which the last equation arises from the partial time derivative of the definition $n = g\eta^{-1} \in G$. These equations imply that the Legendre-transformed variable $p = \delta L/\delta \nu$ is preserved by the flow, which does *not* occur in the general case, when L depends also on n.

In the general case, one may obtain the dynamics directly from a constrained variational principle $\delta S = 0$ with

$$S = \int \left[L(\boldsymbol{u}, n, \nu) + \left\langle p, \frac{\partial n}{\partial t} + \pounds_{\boldsymbol{u}} n - \nu \right\rangle \right] \mathrm{d}t,$$

where the angle bracket denotes the L^2 pairing. Stationary variations produce

$$\begin{aligned} 0 = \delta S = \int \bigg[& \left\langle \frac{\delta L}{\delta \boldsymbol{u}} - p \diamond n, \delta \boldsymbol{u} \right\rangle + \left\langle \delta p, \frac{\partial n}{\partial t} + \pounds_{\boldsymbol{u}} n - \nu \right\rangle + \left\langle \frac{\delta L}{\delta \nu} - p, \delta \nu \right\rangle \\ & + \left\langle \frac{\delta L}{\delta n} - \frac{\partial p}{\partial t} + \pounds^{\dagger}_{\boldsymbol{u}} p, \delta n \right\rangle \bigg] \mathrm{d}t, \end{aligned} \tag{4.2}$$

in which the diamond operator ($\diamond$) is defined via the natural generalization of (1.9) and $\pounds_{\boldsymbol{u}}$ with the superscript dagger denotes the L^2 adjoint of the Lie derivative so that, in particular, $\langle \pounds^{\dagger}_{\boldsymbol{u}} p, \delta n \rangle = \langle p, \pounds_{\boldsymbol{u}} \delta n \rangle$. The standard Euler–Poincaré theory for the case that n is a scalar function and its dual p is a density then implies the following system after a brief calculation:

$$\left. \begin{aligned} \left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{u}} \right) \left(\frac{\delta L}{\delta \boldsymbol{u}} - p \diamond n \right) = 0, \qquad \left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{u}} \right) \frac{\delta L}{\delta \nu} = \frac{\delta L}{\delta n}, \\ \frac{\delta L}{\delta \nu} = p, \qquad \left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{u}} \right) n = \nu. \end{aligned} \right\} \tag{4.3}$$

System (4.3) possesses an exchange symmetry between the variables $(n, \nu) \in TG$ and their dual variables $(\delta L/\delta n, \delta L/\delta \nu) \in TG^*$, and it satisfies the following proposition.

Proposition 4.1. *System (4.3) is equivalent to system (4.1) when* $\delta L/\delta n = 0$.

Proof. This proposition follows from a direction calculation using the chain rule for the diamond operation and substituting the last three equations in system (4.3), namely

$$0 = \left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{u}} \right) \left(\frac{\delta L}{\delta \boldsymbol{u}} - p \diamond n \right) = \left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{u}} \right) \frac{\delta L}{\delta \boldsymbol{u}} - \frac{\delta L}{\delta n} \diamond n - p \diamond \nu.$$

When n is a scalar and p is a density, then $-p \diamond \nu = \langle p, \mathrm{d}\nu \rangle$ and setting $\delta L/\delta n = 0$ recovers the first equation in the system (4.1). ■

Remark 4.2. The Legendre transformation of the constrained Lagrangian defines the Hamiltonian

$$H(\boldsymbol{m}, p, n) = \langle \boldsymbol{m}, \boldsymbol{u} \rangle + \langle p, \nu \rangle - L(\boldsymbol{u}, n, \nu), \tag{4.4}$$

in terms of the fibre derivatives

$$\boldsymbol{m} := \frac{\delta L}{\delta \boldsymbol{u}} \quad \text{and} \quad p := \frac{\delta L}{\delta \nu}. \tag{4.5}$$

(*b*) *The Kelvin circulation theorem*

Proposition 4.1 allows the Kelvin circulation theorem for these semidirect-product systems, with $\delta L/\delta n = 0$, to be expressed in two ways, upon introducing a conserved density variable ρ, satisfying

$$\left(\frac{\partial}{\partial t} + \pounds_{\boldsymbol{u}}\right)\rho = 0.$$

On the one hand, the first equation in the system (4.1) and the general theory of dynamics on semidirect-product Lie groups (Holm *et al.* 1998) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t} \frac{\boldsymbol{m}}{\rho} = -\oint_{\gamma_t} \frac{1}{\rho}\langle p, \mathrm{d}\nu\rangle, \tag{4.6}$$

where γ_t is a closed loop moving with the flow of the velocity vector field. On the other hand, the first equation in the system (4.3) implies that the Kelvin circulation theorem may also be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma_t} \frac{1}{\rho}(\boldsymbol{m} + \langle p, \mathrm{d}n\rangle) = 0, \tag{4.7}$$

where the sum $\boldsymbol{m}_{\mathrm{tot}} := \boldsymbol{m} + \langle p, \mathrm{d}n\rangle$ is the total momentum. This circulation theorem for the total momentum is the natural extension to the continuum description of formula (3.8) for preservation of the right-invariant momentum map. The two circulation laws (4.6) and (4.7) are shown to be equivalent in the following.

Proposition 4.3. *The two forms of the Kelvin circulation theorem in equations* (*4.9*) *and* (*4.10*) *are equivalent.*

Proof. This statement is recovered by observing that $(\partial/\partial t + \pounds_u)\langle p, \mathrm{d}n\rangle = \langle p, \mathrm{d}\nu\rangle$. This proposition extends the arguments in §2 to the non-Abelian case in the continuum fluid description. ■

Remark 4.4. The zero level set of the total momentum, cf. equation (3.8),

$$\boldsymbol{m} + \langle p, \mathrm{d}n\rangle = 0, \tag{4.11}$$

is preserved by the first equation in the system (4.1). The preservation of zero total momentum is a key step in the metamorphosis approach using active templates in imaging science, because the zero value is imposed by the requirement that an initial image would evolve to match a prescribed final image at a certain end-point in time (Holm *et al.* submitted *b*).

The zero level set condition (4.11) for the total momentum imposes the relation

$$\boldsymbol{m} = -\langle p, \mathrm{d}n\rangle. \tag{4.12}$$

This is the equivariant momentum map obtained from the cotangent lift of the right action of Diff on the gauge group G, defined by

$$\langle \boldsymbol{m}, \boldsymbol{u}\rangle_{\mathfrak{g}^*\times\mathfrak{g}} = \langle p \diamond n, \boldsymbol{u}\rangle_{\mathfrak{g}^*\times\mathfrak{g}} = -\langle p, \pounds_{\boldsymbol{u}} n\rangle_{T^*G}. \tag{4.13}$$

Thus, the zero level set condition (4.11) for the total momentum is itself a momentum map. This particular momentum map also appears in the application of the classical Clebsch method of introducing canonical variables for fluid dynamics (e.g. Holm & Kupershmidt 1983).

5. Conclusions and open questions

We have shown how continuum equations on a certain class of semidirect-product Lie groups allow for singular solution momentum maps arising from the left action of diffeomorphisms on the G-bundle $\mathrm{Emb}(S, \mathbb{R}^n) \times G(S)$. On the other hand, the right action on $\mathrm{Emb} \times G$ has been shown to yield another momentum map that recovers the Kelvin–Noether theorem.

These results arose from the observation that the collective dynamics on $\mathrm{Emb} \times G$ was generated by a Kaluza–Klein Hamiltonian, thereby recovering the conservation of a gauge charge from a cyclic coordinate in the gauge group G.

The Kaluza–Klein construction for the collective motion was implemented in the continuum description by considering the semidirect product $\mathrm{Diff}\,Ⓢ\, G$ as the product of the diffeomorphisms with a gauge group G. The Kaluza–Klein construction implies the Kelvin–Noether theorem.

An important open question is whether the singular solutions (2.1) and (3.4) emerge from smooth initial conditions. This question is being pursued elsewhere (Holm *et al.* submitted *a*).

Another open question concerns more general semidirect products. In fact, the present discussion has considered only semidirect products of the Diff group with G-valued scalar functions. However, in physical applications, one may also find semidirect products of the form $\mathrm{Diff}\,Ⓢ\,\mathcal{T}$, where $\mathcal{T}$ denotes tensor fields in physical space. The most important example is probably ideal magnetohydrodynamics, where $\mathcal{T}$ is the space of exact two forms (cf. Holm *et al.* 1998; Holm 2002*b*). The dynamics on such products differs substantially from the cases considered here and deserves further investigation.

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