GLOBAL DYNAMICS AWAY FROM THE GROUND STATE FOR THE ENERGY-CRITICAL NONLINEAR WAVE EQUATION

J. KRIEGER, K. NAKANISHI, AND W. SCHLAG

Abstract. We study global behavior of radial solutions for the nonlinear wave equation with the focusing energy critical nonlinearity in three and five space dimensions. Assuming that the solution has energy at most slightly more than the ground states and gets away from them in the energy space, we can classify its behavior into four cases, according to whether it blows up in finite time or scatters to zero, in forward or backward time direction. We prove that initial data for each case constitute a non-empty open set in the energy space.

This is an extension of the recent results [15, 16] by the latter two authors on the subcritical nonlinear Klein-Gordon and Schrödinger equations, except for the part of the center manifolds. The key step is to prove the "one-pass" theorem, which states that the transition from the scattering region to the blow-up region can take place at most once along each trajectory. The main new ingredients are the control of the scaling parameter and the blow-up characterization by Duyckaerts, Kenig and Merle [3, 4].

Contents

1. Introduction

We consider the H^1-critical, focusing nonlinear wave equation

$$\ddot{u} - \Delta u = |u|^{2^*-2}u, \quad u(t,\,x) : \mathbb{R}^{1+d} \to \mathbb{R}, \quad 2^* = \frac{2d}{d-2} \quad (d = 3 \text{ or } 5), \tag{1.1}$$

in the radial context, where 2^* denotes the H^1 Sobolev critical exponent. We remark that the dimensional restriction is needed only for using the blow-up characterization by Duyckaerts-Kenig-Merle [4].

We take the radial energy space as the phase space for the above equation, which can be normalized to L^2 by putting

$$\vec{u} := (|\nabla| u, \dot{u}) \in L^2_{\text{radial}}(\mathbb{R}^d)^2 =: \mathcal{H}, \tag{1.2}$$

at each time $t \in \mathbb{R}$, where $|\nabla| = \sqrt{-\Delta}$ is an isometry from $\dot{H}^1_{\text{radial}}(\mathbb{R}^d)$ onto $L^2_{\text{radial}}(\mathbb{R}^d)$. Thus, to any scalar space-time function $u(t,x)$, we will associate the

The first author wishes to thank the Mathematics Department of the University of Chicago for its hospitality in July 2010, when part of this work was done. Support of the National Science Foundation, DMS-0757278 (JK) and DMS-0617854 (WS) as well as the Sloan Foundation (JK) is gratefully acknowledged. The third author was partially supported by a Guggenheim fellowship.

vector function $\vec{u}(t,x)$ by the above relation. Conversely, for any time independent $\vec{\varphi} = (\varphi_1, \varphi_2) \in \mathcal{H}$, we introduce the following notation

$$\varphi := |\nabla|^{-1}\varphi_1, \quad \dot{\varphi} := \varphi_2. \tag{1.3}$$

The conserved energy of (1.1) is denoted by

$$E(\vec{u}) := \int_{\mathbb{R}^d} \Big[\frac{|\dot{u}|^2 + |\nabla u|^2}{2} - \frac{|u|^{2^*}}{2^*}\Big]\,dx. \tag{1.4}$$

It is well-known that this problem admits the static Aubin solutions of the form

$$W_\lambda = T_\lambda W, \quad W(x) = \left[1 + \frac{|x|^2}{d(d-2)}\right]^{1-\frac{d}{2}}, \tag{1.5}$$

where T_λ denotes the $\dot{H}^1$ preserving dilation

$$T_\lambda \varphi = \lambda^{d/2-1}\varphi(\lambda x). \tag{1.6}$$

These are positive radial solutions of the static equation

$$-\Delta W - |W|^{2^*-2}W = 0, \tag{1.7}$$

which are unique, up to dilation and translation symmetries, amongst the non-negative, non-zero (not necessarily radial) C^2 solutions, see [2]. They also minimize the static energy

$$J(\varphi) := \int_{\mathbb{R}^d} \Big[\frac{1}{2}|\nabla\varphi|^2 - \frac{1}{2^*}|\varphi|^{2^*}\Big]\,dx, \tag{1.8}$$

among all non-trivial static solutions. The work of Kenig, Merle [9, 10] and Duyckaerts, Merle [5, 6] allows for a characterization of the global-in-time behavior of solutions with $E(\vec{u}) \le J(W)$.

In this paper we study the behavior of solutions with

$$E(\vec{u}) < J(W) + \varepsilon_0^2, \tag{1.9}$$

for some small $\varepsilon_0 > 0$. Solutions of subcritical focusing NLKG and NLS equations with radial data in $\mathbb{R}^3$ of energy slightly above that of the ground state were studied by the latter two authors in [15, 16]. Our goal in this paper is to extend those results to the critical case. The key feature of (1.1) by contrast to NLKG is the scaling invariance of (1.1) manifested by

$$u(t,x) \mapsto \lambda^{\frac{d}{2}-1}u(\lambda t, \lambda x) = T_\lambda u(\lambda t) \tag{1.10}$$

which leaves the energy unchanged. In particular, the analogue of the "one pass theorem" proved in [15] needs to be modified, specifically by replacing the discrete set of attractors $\{Q, -Q\}$ there by the one–parameter family of the ground states

$$\mathcal{S} := \{W_\lambda\}_{\lambda>0}. \tag{1.11}$$

Note that in the subcritical NLS case [16], the scaling parameter λ (in frequency) is essentially fixed or at least bounded from above and below by the L^2 conservation law, but in the critical case there is no factor which a priori prevents the scale from going to 0 or $+\infty$.

Introduce the "virial functional"

$$K(\varphi) := \int_{\mathbb{R}^d} [|\nabla\varphi|^2 - |\varphi|^{2^*}]\, dx \tag{1.12}$$

and note that $K(W) = 0$. The following positivity is crucial for the variational structure around W

$$H(\varphi) := \|\nabla\varphi\|_2^2/d = J(\varphi) - K(\varphi)/2^*. \tag{1.13}$$

Note that the derivative of $J(\varphi)$ with respect to any scaling $\varphi(x) \mapsto \lambda^a\varphi(\lambda^b x)$ except for T_λ gives a non-zero constant multiple of $K(\varphi)$. This is a special feature of the scaling critical case, which allows us to work with a single K, whereas in the subcritical case [15] we needed two different functionals and their equivalence.

The main result of this paper is summarized as follows.

Theorem 1.1. *There exist a small $\varepsilon_* > 0$, a neighborhood $\mathcal{B}$ of $\vec{\mathcal{S}}$ within $O(\varepsilon_*)$ distance in $\mathcal{H}$, and a continuous functional*

$$\mathfrak{S} : \{\vec{\varphi} \in \mathcal{H} \setminus \mathcal{B} \mid E(\vec{\varphi}) < J(W) + \varepsilon_*^2\} \to \{\pm 1\}, \tag{1.14}$$

such that the following properties hold: For any solution u with $E(\vec{u}) < J(W) + \varepsilon_*^2$ on the maximal existence interval $I(u)$, let

$$\begin{aligned} I_0(u) &:= \{t \in I(u) \mid \vec{u}(t) \in \mathcal{B}\}, \\ I_\pm(u) &:= \{t \in I(u) \mid \vec{u}(t) \notin \mathcal{B},\ \mathfrak{S}(\vec{u}(t)) = \pm 1\}. \end{aligned} \tag{1.15}$$

Then $I_0(u)$ is an interval, $I_+(u)$ consists of at most two infinite intervals, and $I_-(u)$ consists of at most two finite intervals. $u(t)$ scatters to 0 as $t \to \pm\infty$ if and only if $\pm t \in I_+(u)$ for large $t > 0$. Moreover, there is a uniform bound $M < \infty$ such that

$$\|u\|_{L^q_{t,x}(I_+(u)\times\mathbb{R}^d)} \le M, \quad q := \frac{2(d+1)}{d-2}. \tag{1.16}$$

For each $\sigma_1, \sigma_2 \in \{\pm\}$, let A_{σ_1,σ_2} be the collection of initial data $\vec{u}(0) \in \mathcal{H}$ such that $E(\vec{u}) < J(W) + \varepsilon_*^2$, and for some $T_- < 0 < T_+$,

$$(-\infty, T_-) \cap I(u) \subset I_{\sigma_1}(u), \quad (T_+, \infty) \cap I(u) \subset I_{\sigma_2}(u). \tag{1.17}$$

Then each of the four sets $A_{\pm,\pm}$ is open and non-empty, exhibiting all possible combinations of scattering to zero/finite time blowup as $t \to \pm\infty$, respectively.

The neighborhood $\mathcal{B}$ as well as the sign functional $\mathfrak{S}$ will be defined explicitly, cf. Corollary 4.2. In short, every solution u with energy $E(\vec{u}) < J(W) + \varepsilon_*^2$ can change the sign $\mathfrak{S}(\vec{u}(t))$ at most once, by entering the neighborhood $\mathcal{B}$, and u scatters/blows-up if it keeps $\mathfrak{S} = +1$ / -1. This is the same description as in the subcritical case [15] concerning the dynamics away from the ground states $\mathcal{S}$. Indeed, the part about the sign change seems fairly general, which we called "one-pass" theorem, relying only on the energy and virial type arguments. It will be proved separately as the first step in Theorem 4.1.

However, we do not know at this time how to deal with solutions u which stay in $\mathcal{B}$. Note that the solutions constructed in [12] in the three-dimensional case belong to this tube. Moreover, Duyckaerts, Kenig, Merle [3] showed that all type-II blowup (i.e., blowup with bounded energy norm) under the constraint (1.9) is of the form of those solutions found in [12]. But the tube around $\mathcal{S}$ might also contain solutions which do not blow up but rather scatter to $\mathcal{S}$. This would correspond to the center-stable manifolds in [15, 16]. However, in contrast to [15, 16] we do not address the issue of existence of a center-stable manifold associated with (1.1), nor do we give a complete description of all possible dynamics for solutions as in (1.9). Recall that [11] establishes the existence of such a manifold for the radial three-dimensional critical wave equation, but not in the energy class. It appears to be a delicate question in any dimension to decide whether or not a center-stable manifold associated with the ground states exists in the case of energy critical equations.

The key idea behind the proof is similar to the one in [15], which relies on an interplay between the hyperbolic dynamics of the linearized operator around W with the variational structure of J and K away from $\mathcal{S}$.

Dynamically speaking, the linearization around W is delicate, as one needs to take a time-dependent scaling parameter $\lambda(t)$ into account. This is a major difference from [15]. To address it, we use the observation that the evolution of $\lambda(t)$ is much slower than that of the exponentially unstable mode. Indeed, the evolution of $\lambda(t)$ is governed by the threshold eigenvalue (which lies at zero energy) of the linearized operator and is therefore by nature algebraically unstable rather than exponentially unstable. This will allow us to freeze the dilation parameter in those time intervals during which the trajectories are dominated by the hyperbolic (and unstable) dynamics.

The other major difference, which could be more serious, is the possibility of concentration blow-up in the region $K \geq 0$ and away from $\mathcal{S}$, where the solutions are bounded and so automatically global in the subcritical case. This problem arises after applying the one-pass theorem. Fortunately, we will see that the blow-up analysis by Duyckaerts, Merle, Kenig [3, 4] precludes it, so that we can proceed essentially in the same way as in the subcritical case.

2. Energy distance functional

In this section we define the nonlinear distance functional to the ground state family $\mathcal{S}$, by using the linearized operator, but still keeping the nonlinear structure, so that it will best reflect the hyperbolic nature around $\mathcal{S}$. The main difference from the subcritical radial NLKG [15] is that we need a good choice of the scaling parameter.

Let $\rho > 0$ be the unique L^2-normalized ground-state for the linearized operator

$$\mathcal{L} := -\Delta - (2^* - 1)W^{2^*-2}, \quad \mathcal{L}\rho = -k^2\rho, \quad \|\rho\|_2 = 1. \tag{2.1}$$

Then $\rho_\lambda := T_\lambda \rho$ is a ground state of the rescaled operator

$$\mathcal{L}_\lambda := -\Delta - (2^* - 1)W_\lambda^{2^*-2}, \quad \mathcal{L}_\lambda\rho_\lambda = -k^2\lambda^2\rho_\lambda, \quad \|\rho_\lambda\|_2 = 1/\lambda. \tag{2.2}$$

Expand u around W_λ by

$$u = W_\lambda + v_\lambda = W_\lambda + \mu_\lambda(u)\rho_\lambda + \gamma_\lambda, \quad \gamma_\lambda \perp \rho_\lambda, \tag{2.3}$$

where μ_λ is given by

$$\mu_\lambda(\varphi) := \langle \varphi - W_\lambda | \lambda^2 \rho_\lambda \rangle = \langle \varphi - W_\lambda | T^*_{1/\lambda}\rho \rangle. \tag{2.4}$$

Since $\rho \in \mathcal{S} \subset \dot{H}^{-1}$, we obtain by rescaling

$$|\mu_\lambda(u)| \lesssim \|\nabla v_\lambda\|_{L^2}. \tag{2.5}$$

Note that γ_λ may contain the root mode in the direction

$$\partial_\lambda W_\lambda = \Lambda W_\lambda, \quad \Lambda := r\partial_r + d/2 - 1. \tag{2.6}$$

However, this will not cause any problems in our analysis of the hyperbolic dynamics. The energy is expanded as

$$\begin{aligned} E(\vec{u}) - J(W) &= \frac{1}{2}[\|\dot{u}\|_2^2 + \langle \mathcal{L}_\lambda v_\lambda | v_\lambda \rangle] - C_\lambda(v) \\ &= \frac{1}{2}[\|\dot{u}\|_2^2 - k^2\mu_\lambda(\varphi)^2 + \langle \mathcal{L}_\lambda \gamma_\lambda | \gamma_\lambda \rangle] - C_\lambda(v_\lambda), \end{aligned} \tag{2.7}$$

where C_λ denotes the superquadratic part of the energy, i.e.,

$$\begin{aligned} C_\lambda(v) &:= \int_{\mathbb{R}^d} \Big[\frac{|v + W_\lambda|^{2^*} - |W_\lambda|^{2^*}}{2^*} - W_\lambda^{2^*-1} v - \frac{2^*-1}{2} W_\lambda^{2^*-2} |v|^2 \Big] dx \\ &= O(\|v\|_{\dot{H}^1}^3). \end{aligned} \tag{2.8}$$

In the same way as in [15], we introduce an energy functional

$$\begin{aligned} E_\lambda(\vec{u}) &:= E(\vec{u}) - J(W) + k^2\mu_\lambda(u)^2 \\ &= \frac{1}{2}[\|\dot{u}\|_2^2 + k^2\mu_\lambda(\varphi)^2 + \langle \mathcal{L}_\lambda \gamma_\lambda | \gamma_\lambda \rangle] - C_\lambda(v_\lambda). \end{aligned} \tag{2.9}$$

Now we choose $\lambda = \lambda(u)$ for u close to $\mathcal{S}$ by the orthogonality condition

$$\langle u | \Lambda^* \rho_\lambda \rangle = \langle v_\lambda | \Lambda^* \rho_\lambda \rangle = 0, \tag{2.10}$$

using the fact that

$$\langle W_\lambda | \Lambda^* \rho_\lambda \rangle = \langle \Lambda W_\lambda | \rho_\lambda \rangle = 0, \tag{2.11}$$

which follows from $\mathcal{L}_\lambda \Lambda W_\lambda = 0$ and $\mathcal{L}_\lambda \rho_\lambda = -k^2\lambda^2\rho_\lambda$, and $\rho \in \mathcal{S}$. Such $\lambda(u)$ is uniquely determined at least in the region

$$\|\nabla v_\lambda\|_2 \sim \operatorname{dist}_{\dot{H}^1}(u, \mathcal{S}) \ll 1, \tag{2.12}$$

by the implicit function theorem, since

$$\begin{aligned} \partial_{\lambda=1} \langle W | \Lambda^* \rho_\lambda \rangle = \langle \Lambda W | \Lambda^* \rho \rangle &= \langle \Lambda W | -k^{-2}(\Lambda(2^*-2)W^{2^*-2})\rho \rangle \\ &= -k^{-2}(2^*-1)(2^*-2)\langle W^{2^*-3}(\Lambda W)^2 | \rho \rangle < 0. \end{aligned} \tag{2.13}$$

In order to bound the remainder by the energy, we use the following result.

Lemma 2.1. *For any $\gamma \in \dot{H}^1_{rad}$ such that $\gamma \perp \rho$, we have $\langle \mathcal{L}\gamma | \gamma \rangle \geq 0$ and*

$$\|\nabla \gamma\|_2^2 \sim \langle \gamma | \Lambda^* \rho \rangle^2 + \langle \mathcal{L}\gamma | \gamma \rangle. \tag{2.14}$$

Proof. Let $z = |\nabla|\gamma \in L^2$, then the bilinear form is rewritten

$$\langle \mathcal{L}\gamma | \gamma \rangle = \langle (1 - A)z | z \rangle, \quad A := |\nabla|^{-1} W^{2^*-2} |\nabla|^{-1}. \tag{2.15}$$

A is a positive, compact and self-adjoint operator on L^2. Hence $\operatorname{spec}(1-A)$ is bounded, discrete, with the only accumulation point being 1. Since

$$z \perp |\nabla|^{-1}\rho \perp |\nabla|\Lambda W \in (1-A)^{-1}(0), \tag{2.16}$$

we can decompose

$$z = c|\nabla|\Lambda W + z_+, \quad z_+ \perp \{|\nabla|^{-1}\rho, |\nabla|\Lambda W\}, \tag{2.17}$$

then

$$\langle \mathcal{L}\gamma|\gamma\rangle = \langle (1-A)z_+|z_+\rangle. \tag{2.18}$$

First we prove

$$L^2_{\mathrm{radial}} \ni z \perp \{|\nabla|^{-1}\rho, |\nabla|\Lambda W\} \implies \langle (1-A)z|z\rangle \sim \|z\|_2^2, \tag{2.19}$$

noting that

$$L^2 \ni z \perp |\nabla|^{-1}\rho \implies \langle (1-A)z|z\rangle \geq 0. \tag{2.20}$$

Suppose (2.19) fails. Then there exists a sequence $z_n \in L^2_{\mathrm{radial}}$ such that $\|z_n\|_2 = 1$, $z_n \to z_\infty$ weakly and $\langle (1-A)z_n|z_n\rangle \to 0$ as $n \to \infty$. Since $Az_n \to Az_\infty$ strongly, we have

$$\langle (1-A)z_\infty|z_\infty\rangle \leq 0, \quad z_\infty \perp |\nabla|^{-1}\rho, |\nabla|\Lambda W. \tag{2.21}$$

Then (2.20) implies that $\langle (1-A)z_\infty|z_\infty\rangle = 0$, and $z_n \to z_\infty$ strongly. So there is a Lagrange multiplier $c \in \mathbb{R}$ such that

$$(1-A)z_\infty = c|\nabla|^{-1}\rho. \tag{2.22}$$

On the other hand, $\mathcal{L}\rho = -k^2\rho$ gives $(1-A)|\nabla|\rho = -k^2|\nabla|^{-1}\rho$, whence

$$c = \langle (1-A)z_\infty||\nabla|\rho\rangle = \langle z_\infty|(1-A)|\nabla|\rho\rangle = 0, \tag{2.23}$$

and thus

$$(1-A)z_\infty = 0. \tag{2.24}$$

This implies that for some $b \in \mathbb{R}$,

$$z_\infty = b|\nabla|\Lambda W. \tag{2.25}$$

Since $z_\infty \perp |\nabla|\Lambda W$, we conclude that $z_\infty = 0$, which contradicts the strong convergence and $\|z_n\| = 1$. Thus (2.19) is proved.

It remains to bound c. Since

$$\langle \gamma|\Lambda^*\rho\rangle = c\langle \Lambda W|\Lambda^*\rho\rangle + \langle z_+||\nabla|^{-1}\Lambda^*\rho\rangle, \tag{2.26}$$

$\langle \Lambda W|\Lambda^*\rho\rangle < 0$ by (2.13) and $|\nabla|^{-1}\Lambda^*\rho \in L^2$, we infer that

$$|c| \lesssim |\langle \gamma|\Lambda^*\rho\rangle| + \|z_+\|_2, \tag{2.27}$$

which together with (2.19) implies the desired estimate. □

Thus we deduce that, if u is close enough to $\mathcal{S}$ and $\lambda = \lambda(u)$ then

$$\begin{aligned} E_{\lambda(u)}(\vec{u}) &\sim \|\dot{u}\|_2^2 + |\mu_\lambda(u)|^2 + \|\nabla\gamma_\lambda\|_2^2 + O(\|\nabla v_\lambda\|_2^3) \\ &\sim \|\dot{u}\|_2^2 + \|\nabla v_\lambda\|_2^2 \sim \|\vec{u} - (|\nabla|W_{\lambda(u)}, 0)\|_2^2. \end{aligned} \tag{2.28}$$

For brevity, we write

$$\mu_{\mathcal{S}}(u) := \mu_{\lambda(u)}(u), \quad E_{\mathcal{S}}(\vec{u}) := E_{\lambda(u)}(\vec{u}), \tag{2.29}$$

when u is close to $\mathcal{S}$.

Now we can define our distance function $d_{\mathcal{S}}(\vec{\varphi})$. Let $\chi(r) \in C_0^\infty(\mathbb{R})$ be a symmetric decreasing function such that

$$\chi(r) = \begin{cases} 1 & (|r| \le 1) \\ 0 & (|r| \ge 2). \end{cases} \tag{2.30}$$

Let d_0 denote the linear distance from $\mathcal{S}$

$$d_0(\vec{\varphi}) := \inf_{\nu>0} \|\vec{\varphi} - \vec{W}_\nu\|_2, \tag{2.31}$$

and then define

$$\begin{aligned} d_{\mathcal{S}}(\vec{\varphi}) :=& \chi(d_0(\vec{\varphi})/\delta_E) E_{\mathcal{S}}(\vec{\varphi})^{1/2} + \chi(d_0(-\vec{\varphi})/\delta_E) E_{\mathcal{S}}(-\vec{\varphi})^{1/2} \\ &+ [1 - \chi(d_0(\vec{\varphi})/\delta_E) - \chi(d_0(-\vec{\varphi})/\delta_E)] C_E \min_{\pm} d_0(\pm\vec{\varphi}), \end{aligned} \tag{2.32}$$

for some fixed $0 < \delta_E \ll \min(1, \|\nabla W\|_2)$ and $C_E \gg 1 + \|\nabla W\|_2$, such that for $d_0(\vec{\varphi}) < 2\delta_E$, $\vec{\varphi}$ is close to either $\vec{\mathcal{S}} = \{(|\nabla|W_\lambda, 0)\}_{\lambda>0}$ or $-\vec{\mathcal{S}}$, and

$$d_{\mathcal{S}}(\varphi)^2 = E_{\mathcal{S}}(\pm\vec{\varphi}) = E(\vec{\varphi}) - J(W) + k^2 \mu_{\mathcal{S}}(\pm\varphi)^2. \tag{2.33}$$

Since $\lambda(\varphi)$, $\mu_{\mathcal{S}}$ and $E_{\mathcal{S}}$ have been defined only near $\mathcal{S}$, it is harmless and convenient to extend them evenly around $-\mathcal{S}$:

$$E_{\mathcal{S}}(\vec{\varphi}) := E_{\mathcal{S}}(-\vec{\varphi}), \quad \mu_{\mathcal{S}}(\varphi) := \mu_{\mathcal{S}}(-\varphi), \quad \lambda(\varphi) := \lambda(-\varphi). \tag{2.34}$$

Thus $d_{\mathcal{S}} : \mathcal{H} \to [0, \infty)$ is continuous and even, satisfying

$$d_{\mathcal{S}}(\vec{\varphi}) \sim \min_{\pm} d_0(\vec{\varphi}) = \operatorname{dist}_{L^2}(\vec{\varphi}, \vec{\mathcal{S}} \cup -\vec{\mathcal{S}}). \tag{2.35}$$

The following lemma shows the basic property of the distance: once we are slightly away from $\mathcal{S}$, then the unstable mode $\mu_{\mathcal{S}}(u)$ becomes the dominant part of the distance. Our analysis in this paper is mostly in this region.

Lemma 2.2. *For any $\vec{\varphi} \in \mathcal{H}$ satisfying*

$$E(\vec{\varphi}) - J(W) \le d_{\mathcal{S}}(\vec{\varphi})^2/2, \quad d_{\mathcal{S}}(\vec{\varphi}) \le \delta_E, \tag{2.36}$$

one has $|\mu_{\mathcal{S}}(\varphi)| \sim d_{\mathcal{S}}(\vec{\varphi}) = E_{\mathcal{S}}(\varphi)^{1/2}$.

Proof. By definition of $d_{\mathcal{S}}$, we have

$$d_{\mathcal{S}}(\vec{\varphi})^2 = E_{\mathcal{S}}(\vec{\varphi}) = E(\vec{\varphi}) - J(W) + k^2 |\mu_{\mathcal{S}}(\varphi)|^2, \tag{2.37}$$

and so $d_{\mathcal{S}}(\vec{\varphi})^2 - k^2|\mu_{\mathcal{S}}(\varphi)|^2 < d_{\mathcal{S}}(\vec{\varphi})^2/2$, which implies $|\mu_{\mathcal{S}}(\varphi)| \gtrsim d_{\mathcal{S}}(\vec{\varphi})$, while the other direction of the inequality is always true by (2.28). $\square$

3. Variational structure

In this section, we prove the following crucial variational type lemma, which is used to control the dynamics away from the ground states in the proof of the "one pass theorem". Here the argument is static in the phase space $\mathcal{H}$. Due to the underlying scaling invariance, we need to use the concentration compactness approach.

Lemma 3.1. *There is a continuous increasing function $\varepsilon_V : (0,\infty) \to (0,1)$ such that if $\vec{\varphi} \in \mathcal{H}$, $E(\vec{\varphi}) \le J(W) + \varepsilon_V(\delta)^2$ and $d_{\mathcal{S}}(\vec{\varphi}) \ge \delta$ for some $\delta > 0$, then we have either*

$$K(\varphi) \ge \min\{\kappa(\delta),\, c\|\nabla\varphi\|_{L^2}^2\} \tag{3.1}$$

or else

$$K(\varphi) \le -\kappa(\delta) \tag{3.2}$$

for suitable $\kappa(\delta) > 0$ and an absolute constant $c > 0$.

Proof. We may assume $\varepsilon_V(\delta) \ll \delta \ll \delta_E$. If $\|\dot\varphi\|_2 \ll \delta$, then we have $\delta < d_{\mathcal{S}}(\vec{\varphi}) \sim \operatorname{dist}_{\dot H^1}(\varphi, \mathcal{S}\cup -\mathcal{S})$. Otherwise, $J(u) < J(W) - O(\delta^2)$ and so $\operatorname{dist}_{\dot H^1}(\varphi, \mathcal{S}\cup-\mathcal{S}) \gtrsim \delta^2$. The conclusion is clear for $\|\nabla\varphi\|_2 \ll 1$ by Sobolev. Hence, if the conclusion fails for some $\delta > 0$, then there exists a sequence $\varphi_n \in \dot H^1_{\text{radial}}$ such that $\|\nabla\varphi_n\|_2 \gtrsim 1$ and

$$J(\varphi_n) < J(W) + 1/n,\ |K(\varphi_n)| < 1/n,\ \operatorname{dist}_{\dot H^1}(\varphi_n, \mathcal{S}) \gtrsim \delta^2. \tag{3.3}$$

The first two conditions together with $K(W) = 0$ imply that

$$\limsup_{n\to\infty} H(\varphi_n) \le H(W), \tag{3.4}$$

and so φ_n is bounded in $\dot H^1 \subset L^{2^*} \cap rL^2$ by the Sobolev and Hardy inequalities. We deal with possible concentration by the dyadic decomposition in $x \in \mathbb{R}^d$:

$$D_j^< := \{|x| < 2^j\},\quad D_j := \{2^j < |x| < 2^{j+1}\},\quad D_j^> := \{2^{j+1} < |x|\}. \tag{3.5}$$

First we show that for any $\varepsilon > 0$, there is $\nu > 0$ such that for any $h \in \mathbb{Z}$ and n,

$$\|\varphi_n/r\|_{L^2(D_h^<)} > \varepsilon,\ \|\varphi_n/r\|_{L^2(D_h^>)} > \varepsilon \implies \|\varphi_n/r\|_{L^2(D_h)} > \nu. \tag{3.6}$$

If this fails for some $\varepsilon > 0$, then along a subsequence there exist h_n such that

$$\|\varphi_n/r\|_{L^2(D_{h_n}^<)} > \varepsilon,\ \|\varphi_n/r\|_{L^2(D_{h_n}^>)} > \varepsilon,\quad \|\varphi_n/r\|_{L^2(D_{h_n})} \to 0. \tag{3.7}$$

Let $\varphi_n^0 := \chi(2^{-h_n}|x|)\varphi_n$ and $\varphi_n^1 := \varphi_n - \varphi_n^0$, with χ given in (2.30). Then we have

$$\begin{aligned} &\|\varphi_n/r\|_{L^2(D_{h_n}^<)} \lesssim \|\nabla\varphi_n^0\|_2,\quad \|\varphi_n/r\|_{L^2(D_{h_n}^>)} \lesssim \|\nabla\varphi_n^1\|_2,\\ &\|\nabla\varphi_n\|_2^2 = \|\nabla\varphi_n^0\|_2^2 + \|\nabla\varphi_n^1\|_2^2 + O(\|\nabla\varphi_n\|_2\|\varphi_n/r\|_{L^2(D_{h_n})}),\\ &\|\varphi_n\|_{2^*}^{2^*} = \|\varphi_n^0\|_{2^*}^{2^*} + \|\varphi_n^1\|_{2^*}^{2^*} + O(\|\nabla\varphi_n\|_2^{2^*-2}\|\varphi_n/r\|_{L^2(D_{h_n})}^2), \end{aligned} \tag{3.8}$$

where for the last error estimate, we used the radial Sobolev inequality

$$\|r^{d/2-1}\varphi\|_\infty \lesssim \|\nabla\varphi\|_2 \quad (\varphi \in \dot H^1_{\text{radial}}). \tag{3.9}$$

Then for large n and $j = 0, 1$, we have $H(\varphi_n^j) < H(W) - O(\varepsilon^2)$, and so, by the optimality of W for the Sobolev inequality,

$$K(\varphi_n^j) \gtrsim \varepsilon^2 \|\nabla \varphi_n^j\|_2^2 \gtrsim \varepsilon^4. \tag{3.10}$$

which contradicts

$$o(1) = K(\varphi_n) = K(\varphi_n^0) + K(\varphi_n^1) + o(1) \quad (n \to \infty). \tag{3.11}$$

Thus we obtain (3.6). Its right-hand side can hold only for a limited number $N(\nu) = O(\nu^{-2})$ of $h \in \mathbb{Z}$ for each n, since

$$\sum_{h \in \mathbb{Z}} \|\varphi_n/r\|_{L^2(D_h)}^2 \lesssim \|\nabla \varphi_n\|_2^2 \lesssim 1. \tag{3.12}$$

Hence we can rescale $\varphi_n \mapsto \lambda_n^{d/2-1}\varphi_n(\lambda_n x)$ so that for any $\varepsilon > 0$ there are $j < k \in \mathbb{Z}$ such that for all n

$$\|\varphi_n/r\|_{L^2(r<2^j \cup r>2^k)} < \varepsilon, \tag{3.13}$$

which controls the L^{2^*} norm on the same region, via the radial Sobolev estimate as above. Since φ_n converges strongly in $L^{2^*}(2^j < r < 2^k)$ by the radial Sobolev, we conclude that the rescaled φ_n converges to some φ_∞ in $L^{2^*}(\mathbb{R}^d)$. Since all the functional properties are preserved by the rescaling, we deduce

$$\|\nabla \varphi_\infty\|_2 \le \|\nabla W\|_2, \quad K(\varphi_\infty) \le 0. \tag{3.14}$$

The uniqueness of W as the Sobolev maximizer implies that $\varphi_\infty \in \mathcal{S}$, and then the norm convergence implies the strong convergence in $\dot{H}^1$. However, this implies that $\operatorname{dist}_{\dot{H}^1}(\varphi_\infty, \mathcal{S}) \gtrsim \delta^2$, a final contradiction. $\square$

In order to analyze the behavior of $d_{\mathcal{S}}^2(u) \sim E_{\mathcal{S}}(u)$ and thereby also $K(u)$ close to the ground states, we will crucially employ the following ejection lemma.

Lemma 3.2. *There exists $\delta_H \in (0, \delta_E)$ with the following properties: Let u be a solution on an open interval I such that for some $t_0 \in I$*

$$\delta_0 := d_{\mathcal{S}}(\vec{u}(t_0)) \le \delta_H, \quad E(\vec{u}) - J(W) \le \delta_0^2/2, \tag{3.15}$$

and

$$\partial_t d_{\mathcal{S}}(\vec{u}(t_0)) \ge 0. \tag{3.16}$$

Then for $t > t_0$ in I and as long as $d_{\mathcal{S}}(\vec{u}(t)) \le \delta_H$, $d_{\mathcal{S}}(\vec{u}(t))$ is increasing,

$$\begin{aligned} &d_{\mathcal{S}}(\vec{u}(t)) \sim -\mathfrak{s}\mu_{\mathcal{S}}(u(t)) \sim e^{k(t-t_0)\lambda(u(t_0))}\delta_0, \\ &\mathfrak{s}K(u(t)) \gtrsim (e^{k(t-t_0)\lambda(u(t_0))} - C_*\langle (t-t_0)\lambda(u(t_0))\rangle)\delta_0 \\ &|\lambda(u(t)) - \lambda(u(t_0))| \lesssim (e^{k(t-t_0)\lambda(u(t_0))} - 1)\delta_0\lambda(u(t_0)), \end{aligned} \tag{3.17}$$

for some absolute constant $C_* > 0$ and $\mathfrak{s} = \pm 1$ is fixed on the time interval.

Proof. We will show that the hyperbolic mode $\mu_{\mathcal{S}}$ grows exponentially, dominating the other modes. The main difficulty we encounter by comparison to [15] is that we need to pay attention to the evolution of the root mode, or equivalently the scaling parameter $\lambda(u(t))$, which cannot be controlled by the energy or other conserved quantities. What saves us is that the evolution of $\lambda(u(t))$ is slow enough that it can be ignored compared with the exponential growth. Without loss of generality we rescale to achieve

$$\lambda(u(t_0)) = 1, \tag{3.18}$$

and we work first with this fixed scale. We may also assume that u is close to $\mathcal{S}$ at $t = t_0$, decomposing it by

$$u = W + v_1 = W + \mu_1(u)\rho + \gamma_1. \tag{3.19}$$

We prove exponential upper bounds by a bootstrap argument.

Bootstrap assumption: *We assume, for some large constant $M \gg 1$, and for $t \in I$ such that $M^2\delta_0 e^{k(t-t_0)} \ll 1$,*

$$\begin{aligned} &|\vec{\mu}_1(u(t))| \le M\delta_0 e^{k(t-t_0)}, \\ &\|\vec{\gamma}_1(t)\|_2 \le M\delta_0\langle t - t_0\rangle + M^3\delta_0^2 e^{2k(t-t_0)}, \end{aligned} \tag{3.20}$$

which implies $\|\vec{v}_1(t)\|_2 \lesssim M\delta_0 e^{k(t-t_0)}$. We will show that better bounds hold under the above assumption. Then by the time continuity, we obtain the above bound on any such time interval. We emphasize that in this argument we do not employ any dispersive estimates.

In the following, we abbreviate $\mu_1(t) = \mu_1(u(t))$. Then $v_1 = u - W$ solves

$$\begin{aligned} \ddot{v}_1 + \mathcal{L}v_1 = N(v_1) &:= |W + v_1|^{2^*-2}(W + v_1) - W^{2^*-1} - (2^* - 1)W^{2^*-2}v_1 \\ &= O(W^{2^*-3}v_1^2 + |v_1|^{2^*-1}), \end{aligned} \tag{3.21}$$

and so, the eigenmode solves

$$(\partial_t^2 - k^2)\mu_1 = \langle N(v_1)|\rho\rangle. \tag{3.22}$$

This leads to the integral equation

$$\mu_1(t) = \mu_+(t) + \mu_-(t) + \int_{t_0}^t \frac{\sinh(k(t-s))}{k}\langle N(v_1)(s)|\rho\rangle\, ds, \tag{3.23}$$

where $\mu_\pm(t)$ denote the solutions of the linearized equation

$$\mu_\pm(t) := e^{\pm k(t-t_0)}\frac{1}{2}\big[1 \pm \frac{1}{k}\partial_t\big]\mu_1(t_0). \tag{3.24}$$

Our assumptions at time $t = t_0$ imply that $|\mu_\pm(t_0)| \lesssim \delta_0 \ll 1$. Furthermore, we estimate via the bootstrap assumptions

$$\begin{aligned} \Big|\int_{t_0}^t e^{\pm k(t-s)}\langle N(v_1)(s)|\rho\rangle\, ds\Big| &\lesssim \int_{t_0}^t e^{k(t-s)}\|N(v_1(s))\|_{L^{2d/(d+2)}}\, ds \\ &\lesssim \int_{t_0}^t e^{k(t-s)}\|v_1(s)\|_{\dot{H}^1}^2 ds \lesssim M^2\delta_0^2 e^{2k(t-t_0)}. \end{aligned} \tag{3.25}$$

It is immediate from this that

$$|\vec{\mu}_1(t)| \lesssim \delta_0 e^{k(t-t_0)} + M^2\delta_0^2 e^{2k(t-t_0)} \ll M\delta_0 e^{k(t-t_0)}, \tag{3.26}$$

since the right-hand side is small. To bound the remainder γ_1, we use the energy identity. Multiplying the equation of μ_1 with its time derivative yields

$$\partial_t[\dot{\mu}_1^2/2 - k^2\mu_1^2 - C_1(\mu_1\rho)] = \langle N(v_1) - N(\mu_1\rho)|\dot{\mu}_1\rho\rangle. \tag{3.27}$$

Subtracting it from the energy of v_1

$$E(\vec{u}) - J(W) = \frac{\dot{\mu}_1^2 - k^2\mu_1 + \|\dot{\gamma}_1\|_2^2 + \langle \mathcal{L}\gamma_1|\gamma_1\rangle}{2} - C_1(v_1), \tag{3.28}$$

we obtain

$$\partial_t[\|\dot{\gamma}_1\|_2^2/2 + \langle L\gamma_1|\gamma_1\rangle/2 - C_1(v_1) + C_1(\mu_1\rho)] = \langle N(\mu_1\rho) - N(v_1)|\dot{\mu}_1\rho\rangle. \tag{3.29}$$

The nonlinear terms are estimated by Hölder and Sobolev (using that v_1 is small)

$$\begin{aligned} &|C_1(v_1) - C_1(\mu_1\rho)| \lesssim \|\nabla\gamma_1\|_2\|\nabla v_1\|_2^2, \\ &|\langle N(v_1) - N(\mu_1\rho)|\dot{\mu}_1\rho\rangle| \lesssim \|\nabla\gamma_1\|_2\|\nabla v_1\|_2|\dot{\mu}_1|. \end{aligned} \tag{3.30}$$

Hence by time integration using the bootstrap bounds, one concludes that

$$\|\dot{\gamma}_1\|_2^2 + \langle \mathcal{L}\gamma_1|\gamma_1\rangle \lesssim \delta_0^2 + (M\delta_0\langle t-t_0\rangle + M^3\delta_0^2 e^{2k(t-t_0)})M^2\delta_0 e^{2k(t-t_0)}. \tag{3.31}$$

The orthogonality (2.10) at $t = t_0$ implies that

$$\langle \gamma_1|\Lambda^*\rho\rangle = \int_{t_0}^t \langle \dot{\gamma}_1|\Lambda^*\rho\rangle dt. \tag{3.32}$$

Hence we can estimate $\|\nabla\gamma_1\|_2$ by using Lemma 2.1 and (3.31). Thus we obtain

$$\begin{aligned} &\|\dot{\gamma}_1\|_2 \lesssim \delta_0 + M^{3/2}\delta_0^{3/2}\langle t-t_0\rangle^{1/2}e^{k(t-t_0)} + M^{5/2}\delta_0^2 e^{2k(t-t_0)}, \\ &\|\nabla\gamma_1\|_2 \lesssim \delta_0\langle t-t_0\rangle + M^{3/2}\delta_0^{3/2}\langle t-t_0\rangle^{1/2}e^{k(t-t_0)} + M^{5/2}\delta_0^2 e^{2k(t-t_0)}, \end{aligned} \tag{3.33}$$

which is better by $O(M^{-1/2}) \ll 1$ than the bootstrap assumption. This completes the bootstrap argument, whence the proof of (3.20). Henceforth, we shall regard M as being an absolute constant and ignore it.

Next we utilize the monotonicity assumption (3.16) on $E_{\mathcal{S}}(\vec{u})$ in order to obtain a lower bound on μ_1 in the same form. The technical difficulty we face here is that $E_{\mathcal{S}}(\vec{u})$ is defined with respect to the time-dependent scale $\lambda(u(t))$, while the above estimates are at the fixed scale $1 = \lambda(u(t_0))$. The idea is that $E_{\mathcal{S}}(u)$ should differ from $E_1(u)$ only by $O((t-t_0)^2)$ with a small multiple. To see this, we compare the two decompositions

$$\begin{aligned} u(t) &= W + v_1(t) = W + \mu_1(u(t))\rho + \gamma_1(t) \\ &= W_\lambda + v_\lambda(t) = W_\lambda + \mu_\lambda(u(t))\rho_\lambda + \gamma_\lambda(t), \end{aligned} \tag{3.34}$$

where $\lambda = \lambda(u(t))$ is chosen according to (2.10). Then we have

$$E_1(u) - E_{\mathcal{S}}(u) = k^2[\mu_1(u)^2 - \mu_\lambda(u)^2], \tag{3.35}$$

as long as u remains close to $\mathcal{S}$. The right-hand side is estimated by

$$\begin{aligned} \mu_1(u) - \mu_\lambda(u) &= \langle v_1|\rho\rangle - \langle v_\lambda|T_{1/\lambda}^*\rho\rangle \\ &= \langle v_1 - v_\lambda|\rho\rangle + \langle v_\lambda|(1 - T_{1/\lambda}^*)\rho\rangle = O((\lambda-1)^2), \end{aligned} \tag{3.36}$$

where we used that

$$\begin{aligned} &v_1 - v_\lambda = W_\lambda - W = (\lambda-1)\Lambda W + O((\lambda-1)^2) \\ &(1 - T_{1/\lambda}^*)\rho_\lambda = (\lambda-1)\Lambda^*\rho_\lambda + O((\lambda-1)^2), \end{aligned} \tag{3.37}$$

$\Lambda W \perp \rho$ and $v_\lambda \perp \Lambda^* \rho_\lambda$. On the other hand, using the upper bound (3.20), we have

$$\begin{aligned} |\langle \Lambda^* \rho | v_1(t) \rangle| = |\langle \Lambda^* \rho | v_1(t) - v_1(t_0) \rangle| &= |\langle \Lambda^* \rho, \int_{t_0}^t \dot{v}_1(s)\, ds \rangle| \\ &\lesssim (e^{k(t-t_0)} - 1)\delta_0 \ll 1, \end{aligned} \tag{3.38}$$

hence the implicit function theorem implies that

$$|\lambda(u(t)) - 1| \lesssim (e^{k(t-t_0)} - 1)\delta_0, \tag{3.39}$$

and so,

$$|E_1(u) - E_{\mathcal{S}}(u)| \lesssim (e^{k(t-t_0)} - 1)^2 \delta_0^3. \tag{3.40}$$

This implies in particular that

$$\partial_t E_1(\vec{u}(t_0)) = \partial_t E_{\mathcal{S}}(\vec{u}(t_0)) \geq 0, \tag{3.41}$$

where the last inequality follows from the "exiting assumption" (3.16). From the energy conservation and the equation (3.22) of μ_1, we have

$$\partial_t E_1(\vec{u}(t)) = \partial_t k^2 \mu_1^2 = 2k^2 \mu_1 \dot{\mu}_1, \tag{3.42}$$

hence $\partial_t E_1(\vec{u}(t_0)) \geq 0$ implies $\mu_+(t_0) \sim \mu_1(u(t_0)) \sim \delta_0$, and so via (3.23), finally

$$|\mu_1(u(t))| \sim e^{k(t-t_0)} \delta_0. \tag{3.43}$$

By continuity, there is $\mathfrak{s} = \pm 1$ constant such that $\mathfrak{s}\mu_1(u(t)) < 0$. Expanding K around W, and plugging the above estimates into this expansion yields

$$\begin{aligned} \mathfrak{s} K(u) &= -\mathfrak{s}(2^* - 2)\langle W^{2^*-1} | v \rangle + O(\|\nabla v\|_2^2) \\ &\gtrsim \mu_1(u(t)) - O(\|\vec{\gamma}_1(t)\|_2) \gtrsim (e^{k(t-t_0)} - C_* \langle t - t_0 \rangle)\delta_0. \end{aligned} \tag{3.44}$$

To finish the proof of the lemma, it only remains to establish the monotonicity of $d_{\mathcal{S}}$. At the fixed scale 1, it is immediate from the equation that

$$\partial_t^2 E_1(\vec{u}(t)) = 2k^2(\dot{\mu}_1^2 + \mu_1 \ddot{\mu}_1) \geq 2k^2 \mu_1 (k^2 \mu_1 + \langle N(v) | \rho \rangle) \gtrsim e^{2k(t-t_0)} \delta_0^2. \tag{3.45}$$

Combining this with (3.40), we infer that

$$E_{\mathcal{S}}(\vec{u}(t)) \geq E_{\mathcal{S}}(\vec{u}(t_0))(1 + c(t - t_0)^2), \tag{3.46}$$

with some constant $c > 0$. If $E_{\mathcal{S}}(\vec{u}(t))$ becomes decreasing, or more precisely, $\partial_t E_{\mathcal{S}}(\vec{u}(t_1)) = 0$ at some $t_1 > t_0$ before reaching δ_H^2, then we can apply the above argument backward in time from t_1 to concludes that $E_{\mathcal{S}}(\vec{u}(t_0)) > E_{\mathcal{S}}(\vec{u}(t_1))$. However, this contradicts the above estimate. Hence $d_{\mathcal{S}}(\vec{u}(t))$ is increasing, all the way until it reaches δ_H. $\square$

4. The one-pass theorem

The key step in the proof of Theorem 1.1 consists of the following assertion.

Theorem 4.1. *There exist* $0 < \varepsilon_* \ll \delta_* \ll \delta_H$ *with the following properties: Let* $\vec{u} \in C(I; \mathcal{H})$ *be a solution of* (1.1) *on an open interval* I*, satisfying for some* $\varepsilon \in (0, \varepsilon_*]$*,* $\delta \in (\sqrt{2}\varepsilon, \delta_*]$ *and* $T_1 < T_2 \in I$

$$E(\vec{u}) \le J(W) + \varepsilon^2, \quad d_{\mathcal{S}}(\vec{u}(T_1)) < \delta = d_{\mathcal{S}}(\vec{u}(T_2)). \tag{4.1}$$

Then $d_{\mathcal{S}}(\vec{u}(t)) > \delta$ *for all* $t > T_2$ *in* I*.*

Proof. By increasing T_1 and decreasing T_2 if necessary, we may assume in addition that $\sqrt{2}\varepsilon < d_{\mathcal{S}}(\vec{u}(T_1))$ and $d_{\mathcal{S}}(\vec{u}(t))$ is nondecreasing on $[T_1, T_2]$. Then Lemma 3.2 applies for all $t \in [T_1, T_2]$ and so $d_{\mathcal{S}}(\vec{u}(t))$ is increasing for $t > T_1$ until it reaches δ_H. Arguing by contradiction, we assume that for some $t > T_2$ we have $d_{\mathcal{S}}(\vec{u}(t)) \le \delta$. Such a t can occur only away from T_2 (this will be made more precise shortly), and after $d_{\mathcal{S}}(\vec{u}(t))$ has increased to size $\delta_H \gg \delta$. Moreover, by applying Lemma 3.2 backward in time, we can find $T_3 > T_2$ such that $d_{\mathcal{S}}(\vec{u}(t))$ decreases from δ_H down to δ as $t \nearrow T_3$, and so that $d_{\mathcal{S}}(\vec{u}(t)) > \delta$ for $T_2 < t < T_3$. We may further assume

$$\lambda(u(T_2)) = 1 \le \lambda(u(T_3)), \tag{4.2}$$

by rescaling and reversing time, if necessary.

The theorem is now proved by deducing a contradiction from a localized virial identity, as in [15]. Our argument differs from that in [15] in the following two points:

(1) The estimates in the hyperbolic regime incorporate the scaling changes.
(2) The degeneration $\|\nabla u(t)\|_2 \ll 1$ is treated by the equipartition of energy.

(1) already appeared in (3.17), which is essential in the critical case. (2) seems to be a more general argument than that used in [15]. The latter relies on the time oscillation at zero frequency as well as the subcriticality of the equation, neither of which is available for the critical wave equation.

Following [15], introduce a space-time cutoff function

$$w(t,x) = \chi\Big(\frac{|x|}{t - T_2 + \tau_2}\Big)\chi\Big(\frac{|x|}{T_3 - t + \tau_3}\Big), \tag{4.3}$$

where χ is the same cut-off function as in (2.32), and define[1]

$$\tau_2 = \tau_3 = \delta^{-1}. \tag{4.4}$$

Then from the equation of u we obtain the localized virial identity

$$\frac{d}{dt}V(t) = 2K(u(t)) + O(E_{\mathrm{ext}}(t)), \quad V(t) := \langle w u_t | (x \cdot \nabla + \nabla \cdot x) u \rangle, \tag{4.5}$$

where the exterior free energy is denoted by

$$E_{\mathrm{ext}}(t) := \frac{1}{2}\int_{|x| \ge R(t)} [|\nabla u|^2 + u_t^2]\, dx, \quad R(t) := \max(t - T_2 + \tau_2, T_3 - t + \tau_3), \tag{4.6}$$

[1] We are going to recycle this argument with $\tau_2 \ne \tau_3$.

so that $\operatorname{supp} \partial_{t,x} w \subset \{|x| \geq R(t)\}$. By the finite speed of propagation, we have

$$\sup_{T_2<t<T_3} E_{\rm ext}(t) \lesssim \max_{j=2,3} E_{\rm ext}(T_j) \lesssim \delta, \tag{4.7}$$

where the last estimate follows from $d_{\mathcal{S}}(\vec{u}(T_j)) = \delta$, $\lambda(T_j) \geq 1$, $\tau = \delta^{-1}$, and

$$\|\nabla W_\lambda\|_{L^2(|x|>R)} \lesssim (R/\lambda)^{1-d/2}, \quad 1 - d/2 \leq -1/2. \tag{4.8}$$

The left-hand inequality in (4.7) is proved as follows. For each T_j, we can find a free solution u_j^0 so that $\partial_{t,x} u_j^0(T_j, x) = \partial_{t,x} u(T_j, x)$ on $|x| > R(T_j) = \tau$, and

$$\|\vec{u}_j^0\|_2^2 \lesssim E_{\rm ext}(T_j) \ll 1, \tag{4.9}$$

by a suitable extension to $|x| < \tau$. Since $\delta \ll 1$, the small data wellposedness theory implies that there exists[2] a global solution u_j of (1.1) with the same initial data as u_j^0 at $t = T_j$, which moreover satisfies

$$\|\vec{u}_j\|_{L_t^\infty L_x^2} \lesssim \|\vec{u}_j^0\|_{L_t^\infty L_x^2} \lesssim E_{\rm ext}(T_j)^{1/2} \ll 1. \tag{4.10}$$

The propagation property of the linear wave together with the uniqueness for the nonlinear equation implies that $u_2^0 = u$ for $|x| > R(t)$ and $T_2 < t < (T_2 + T_3)/2$, and $u_3^0 = u$ for $|x| > R(t)$ and $(T_2 + T_3)/2 < t < T_3$. Thus we obtain (4.7), and so

$$\dot{V}(t) = -2K(u(t)) + O(\delta). \tag{4.11}$$

On the other hand, the decay property of W_λ together with $d_{\mathcal{S}}(\vec{u}(T_j)) = \delta$ and our choice of cut-off $\tau = \delta^{-1}$ implies that

$$|V(T_2)| + |V(T_3)| \lesssim \delta(1 + \tau^{2-d/2}) + \delta^2\tau \lesssim \delta^{1/2}, \tag{4.12}$$

hence we have

$$\left| \int_{T_2}^{T_3} [2K(u(t)) - O(\delta)]\, dt \right| \lesssim \delta^{1/2}, \tag{4.13}$$

which we are going to lead to a contradiction.

Now we choose two parameters $0 < \delta_M \ll \delta_H$ and $0 < \nu \ll 1$, and set

$$0 < \delta_*^{1/2} \ll \min(\delta_M, \kappa(\delta_M), \nu^2), \quad 0 < \varepsilon_* \ll \min(\delta_*, \varepsilon_V(\delta_M)), \tag{4.14}$$

where $\kappa(\cdot)$ and $\varepsilon_V(\cdot)$ are as in Lemma 3.1.

First we consider the hyperbolic region in $[T_2, T_3]$. Let $\mathfrak{m}$ be the collection of local minimal points of $d_{\mathcal{S}}(\vec{u}(t))$ in $[T_2, T_3]$, with the respective local minima less than δ_M. Since $\delta \ll \delta_M$, we have $T_2, T_3 \in \mathfrak{m}$. For each $t_* \in \mathfrak{m}$, applying Lemma 3.2 forward and/or backward in time, we obtain a subinterval $\hat{I}(t_*) \ni t_*$ of $[T_2, T_3]$ such that for $t \in \hat{I}(t_*)$

$$d_{\mathcal{S}}(\vec{u}(t)) \sim e^{k|t-t_*|\lambda(u(t_*))} d_{\mathcal{S}}(\vec{u}(t_*)), \tag{4.15}$$

and on the boundary $\partial\hat{I}(t_*)$, either $d_{\mathcal{S}}(\vec{u}(t)) = \delta_H$, $t = T_2$ or $t = T_3$. In the latter cases, $d_{\mathcal{S}}(\vec{u}(t)) = \delta_H$ at the other endpoint. Hence we have

$$|\hat{I}(t_*)| \gtrsim \lambda(u(t_*))^{-1} \log(\delta_H/\delta_M(\vec{u}(t_*)) \geq \lambda(u(t_*))^{-1} \log(\delta_H/\delta_M), \tag{4.16}$$

[2]Here we do not need the global Strichartz estimate or the scattering property, but the global existence follows from the local wellposedness combined with conservation of the small energy $E(\vec{u}_j) \sim \|\vec{u}_j\|_2^2$ as well as $K(u_j) \geq 0$.

and those intervals are mutually disjoint. Let[3]

$$I_H := \bigcup_{t_*\in\mathfrak{m}} \hat{I}(t_*) \subset [T_2, T_3], \quad I_V := [T_2, T_3] \setminus I_H. \tag{4.17}$$

Then by the definition of $\mathfrak{m}$, we have $d_{\mathcal{S}}(\vec{u}(t)) \geq \delta_M$ on I_V, and so Lemma 3.1 implies

$$\mathfrak{s}K(u(t)) \geq \min(\kappa(\delta_M), c\|\nabla u(t)\|_2^2), \tag{4.18}$$

with $\mathfrak{s} = \pm 1$ constant on each connected component of I_V. Moreover, (3.17) implies that $K(u(t))$ has the same sign at the two endpoints for each internal $\hat{I}(t_*)$. Since $K(u(t))$ cannot change its sign while $t \in I_V$, we deduce that $\mathfrak{s}$ in (4.18) for $t \in I_V$ and $\mathfrak{s}$ in (3.17) for $t \in I_H$ are the same constant sign on the whole $[T_2, T_3]$.

Using the estimate on K in (3.17), we obtain for each $t_* \in \mathfrak{m}$,

$$\begin{aligned}&\mathfrak{s}\int_{\hat{I}(t_*)} [2K(u(t)) - O(\delta)]\, dt \\ &\gtrsim \int_{\hat{I}(t_*)} (e^{k|t-t_*|\lambda(u(t_*))} - 2C_*\langle (t-t_*)\lambda(u(t_*))\rangle) d_{\mathcal{S}}(u(t_*))\, dt \gtrsim \frac{\delta_H}{\lambda(u(t_*))},\end{aligned} \tag{4.19}$$

where the $O(\delta)$ error was absorbed by the linearly growing factor. Moreover, the latter is absorbed by the exponentially growing factor after integration, since $d_{\mathcal{S}}(\vec{u}(t_*)) \leq \delta_M \ll \delta_H$. Combining this and (4.16), we infer that

$$-\mathfrak{s}\int_{\hat{I}(t_*)} [2K(u(t)) - O(\delta)]dt \gtrsim \frac{\delta_H}{\log(\delta_H/\delta_M)} |\hat{I}(t_*)| \gtrsim \delta_M |\hat{I}(t_*)|. \tag{4.20}$$

Further, since $T_2 \in \mathfrak{m}$ where $\lambda(u(T_2)) = 1$, one has

$$|\hat{I}(T_2)| \gtrsim \log(\delta_H/\delta_M) \gg 1. \tag{4.21}$$

On I_V, we use the variational bound (4.18). If $\mathfrak{s} = -1$, the bound is uniform and

$$-K(u(t)) \geq \kappa(\delta_M) \gg \delta_* > \delta. \tag{4.22}$$

Hence

$$\mathfrak{s}\int_{I_V} [2K(u(t)) - O(\delta)]dt \gtrsim \kappa(\delta_M)|I_V|. \tag{4.23}$$

Combining it with the above estimate on I_H, we obtain

$$-\mathfrak{s}\int_{T_2}^{T_3} [2K(u(t)) - O(\delta)]\, dt \gtrsim \delta_M \gg \delta_*^{1/2} > \delta^{1/2}, \tag{4.24}$$

which contradicts (4.13), concluding the proof in the case $\mathfrak{s} = -1$.

If $\mathfrak{s} = +1$, then the lower bound degenerates as $\|\nabla u(t)\|_2 \to 0$. This scenario can occur along some trajectory, since our equation is of the second order in time. We are going to show that it does not essentially affect the time integral by using energy equipartition. Decompose I_V into

$$I_0 := \{t \in I_V \mid d_{\mathcal{S}}(\vec{u}(t)) < \nu\}, \quad I_1 := I_V \setminus I_0. \tag{4.25}$$

[3]We chose the decomposition into I_H and I_V to maximize the use of the hyperbolic dynamics. One can also use the variational estimate in the overlapping region.

The above argument implies that on I_1 we have

$$K(u(t)) \geq \min(\kappa(\delta_M), \nu^2) \gg \delta_* > \delta, \tag{4.26}$$

and so

$$\int_{I_1} [2K(u(t)) - O(\delta)]\, dt \gtrsim \min(\kappa(\delta_M), \nu^2)|I_1|, \tag{4.27}$$

whereas on I_0 we have $K(u(t)) \sim \|\nabla u(t)\|_2^2$ and so

$$\int_{I_0} [2K(u(t)) - O(\delta)]\, dt \gtrsim \int_{I_0} \|\nabla u(t)\|_2^2\, dt - O(\delta)|I_0|. \tag{4.28}$$

In order to control this, we consider the energy equipartition with the same space-time cut-off as above for the virial identity: from the equation for u,

$$\begin{aligned} \partial_t \langle w u_t | u \rangle &= \|\dot{u}(t)\|_2^2 - K(u(t)) + O(E_{\rm ext}(t)) \\ &= \|\dot{u}(t)\|_2^2 - K(u(t)) + O(\delta). \end{aligned} \tag{4.29}$$

Then in the same way as for (4.13), we obtain

$$\left| \int_{T_2}^{T_3} [\|\dot{u}(t)\|_2^2 - K(u(t)) + O(\delta)]\, dt \right| \lesssim \delta^{1/2}. \tag{4.30}$$

On the other hand, (4.20), (4.27) and (4.28) together with (4.13) imply

$$\min(\delta_M, \kappa(\delta_M), \nu^2) \int_{T_2}^{T_3} \|\nabla u(t)\|_2^2 dt - O(\delta)|I_0| \lesssim \delta^{1/2}, \tag{4.31}$$

where we used the fact that $\vec{u}(t)$ is uniformly bounded in the case $\mathfrak{s} = +1$; this is obvious in the hyperbolic region $d_{\mathcal{S}}(\vec{u}(t)) < \delta_H$, while in the exterior it follows from that $K(u(t)) \geq 0$, since

$$E(\vec{u}) - K(u(t))/2^* = \|\nabla u(t)\|_2^2/d + \|\dot{u}(t)\|_2^2/2. \tag{4.32}$$

Using (4.31) and (4.30) as well as (4.14), we deduce

$$\int_{T_2}^{T_3} [\|\dot{u}(t)\|_2^2 + \|\nabla u(t)\|_2^2] dt \ll 1 + \delta^{1/2}|T_3 - T_2|, \tag{4.33}$$

which contradicts the energy conservation

$$\int_{T_2}^{T_3} E(\vec{u}) dt = |T_3 - T_2| E(\vec{u}) > |T_3 - T_2| J(W)/2, \tag{4.34}$$

since $|T_3 - T_2| > |I_H| + |I_0| \gg 1$. This concludes the proof in the case $\mathfrak{s} = +1$. $\square$

The above result can be restated in terms of the sign functional as in the subcritical case [15]. Let

$$\begin{aligned} \mathcal{H}_* &= \{\vec{\varphi} \in \mathcal{H} \mid E(\vec{\varphi}) \leq J(W) + \varepsilon_*^2\}, \\ \mathcal{H}_X &= \{\vec{\varphi} \in \mathcal{H}_* \mid E(\vec{\varphi}) < J(W) + d_{\mathcal{S}}^2(\vec{\varphi})/2\}. \end{aligned} \tag{4.35}$$

It is easy to see that $\mathcal{H}_* \setminus \mathcal{H}_X$ is a small neighborhood of $\vec{\mathcal{S}} \cup -\vec{\mathcal{S}}$.

Corollary 4.2. *There exists a continuous function $\mathfrak{S} : \mathcal{H}_X \to \{\pm 1\}$ with the following properties.*

(1) *Every solution u in $\mathcal{H}_*$ can change $\mathfrak{S}(\vec{u}(t))$ at most once. Moreover, it can enter or exit the region $d_{\mathcal{S}}(\vec{u}) < \delta_*$ at most once.*
(2) *The region $\mathfrak{S} = +1$ is bounded in $\mathcal{H}$, while the region $\mathfrak{S} = -1$ is unbounded.*
(3) *If $\vec{\varphi} \in \mathcal{H}_X$ and $E(\vec{\varphi}) \le J(W) + \varepsilon_V^2(d_{\mathcal{S}}(\vec{\varphi}))$, then $\mathfrak{S}(\vec{\varphi}) = \mathrm{sign} K(\varphi)$, with the convention* $\mathrm{sign} 0 = +1$.
(4) *If $\vec{\varphi} \in \mathcal{H}_X$ and $d_{\mathcal{S}}(\vec{\varphi}) \le \delta_M$, then $\mathfrak{S}(\vec{\varphi}) = -\mathrm{sign}\mu_{\mathcal{S}}(\varphi)$.*

Note that $\mathcal{H}_* \setminus \mathcal{H}_X$ is included in $d_{\mathcal{S}} < \delta_*$, and that (3)–(4) completely determine $\mathfrak{S}(\vec{\varphi})$, since we have chosen $\varepsilon_* < \varepsilon_V(\delta_M)$. Moreover, $\mathfrak{S}(\vec{\varphi})$ depends only on φ.

Proof. Since (3) and (4) are overdetermining $\mathfrak{S}$, we need the consistency of the conditions. However, this is provided by the ejection lemma 3.2, starting from any solution in the overlapping region, where $d_{\mathcal{S}} < \delta_M \ll \delta_H$. The second estimate in (3.17) implies that the two definitions coincide at least at the endpoint of the ejection $d_{\mathcal{S}}(\vec{u}(t)) = \delta_H$. Since both signs are invariant along the continuous trajectory $\vec{u}$, they must be the same all the way from the starting point. Thus $\mathfrak{S}$ is well defined, and then (1) is the conclusion of the one-pass theorem 4.1. The boundedness in (2) has been shown between (4.31) and (4.32), while it is obvious that the $\mathfrak{S} = -1$ region is unbounded, since it contains all $\vec{\varphi}$ with negative energy. □

It remains to determine the fate of the solutions in $\mathcal{H}_*$ with $d_{\mathcal{S}} \ge \delta_*$. We will do this in the following two sections for $\mathfrak{S} = \pm 1$, respectively.

5. Blow-up after ejection

Proposition 5.1. *No solution can stay strongly continuous in $\mathcal{H}_*$ with $\mathfrak{S} = -1$ and $d_{\mathcal{S}} \ge \delta_*$ for all $t > 0$.*

Proof. Suppose towards a contradiction that there is a solution u on $0 < t < \infty$ in $\mathcal{H}_*$ with $\mathfrak{S}(\vec{u}(t)) = -1$ and $d_{\mathcal{S}}(\vec{u}(t)) \ge \delta_*$. Here we use the identity for $|u|^2$, localized in the same way as for the virial identity.

We may assume $E(\vec{u}) > J(W)$, since otherwise the conclusion follows from [10, 3, 4]. We choose a time-dependent cut-off function and the localized L^2 norm

$$w(t,x) = \chi(|x|/(t+\tau)), \quad y(t) = \langle wu|u\rangle, \tag{5.1}$$

for a fixed large $\tau > 0$ to be determined later. Using that $\dot{w} \ge 0$, we have

$$\dot{y} = \langle \dot{w}u + 2w\dot{u}|u\rangle \ge 2\langle w\dot{u}|u\rangle, \tag{5.2}$$

and using the equation and Hardy's inequality,

$$\begin{aligned}\ddot{y} &= \langle 2w|\dot{u}^2 - |\nabla u|^2 + |u|^{2^*}\rangle + \langle \ddot{w}u|u\rangle + \langle 4\dot{w}u|\dot{u}\rangle + 2\langle u\nabla w|\nabla u\rangle \\ &= 2(\|\dot{u}\|_2^2 - K(u)) + O(E_{\mathrm{ext}}(t)),\end{aligned} \tag{5.3}$$

where

$$E_{\mathrm{ext}}(t) := \int_{|x|>t+\tau} [|\dot{u}|^2 + |\nabla u|^2]\, dx \lesssim E_{\mathrm{ext}}(0) \ll \varepsilon_*, \tag{5.4}$$

by the same argument as for (4.7), provided that we choose τ sufficiently large.

In order to control the right-hand side of (5.3), we follow the argument in the previous section, below (4.13). Note that in the $\mathfrak{S} = -1$ case, the contradiction assumption at $t = T_3$ was used only for the upper bound on $|V(T_3) - V(T_2)|$, and so the rest of the argument is still valid.

Let $I = (T_2, \infty) = I_H \cup I_V$ and $I_H = \bigcup_{t_* \in \mathfrak{m}} \hat{I}(t_*)$ as before, see (4.17). We have $-K(u(t)) \gg \delta_* \gg \varepsilon_*$ on the variational region I_V, while $\int_{\hat{I}(t_*)} -K(u(t))dt \gg \varepsilon_* |\hat{I}(t_*)|$ on each hyperbolic interval $\hat{I}(t_*)$. Hence $\dot{y}(t) \to \infty$ and $y(t) \nearrow \infty$ as $t \to \infty$. Moreover, we can rewrite

$$\|\dot{u}\|_2^2 - K(u) = (1 + 2^*/2)\|\dot{u}\|_2^2 + (2^* - 2)\|\nabla u\|_2^2 - 2^* E(\vec{u}). \tag{5.5}$$

In the variational region I_V, using that $K(u(t)) < 0$ we have

$$E(\vec{u}) < J(W) + \varepsilon_*^2 = \frac{2^* - 2}{2^*}\|\nabla W\|_2^2 + \varepsilon_*^2 < \frac{2^* - 2}{2^*}\|\nabla u\|_2^2 + \varepsilon_*^2, \tag{5.6}$$

which implies

$$\|\dot{u}\|_2^2 - K(u) > (1 + 2^*/2)\|\dot{u}\|_2^2 - 2^*\varepsilon_*^2. \tag{5.7}$$

Interpolating it with the other lower bound $\|\dot{u}\|_2^2 + \delta_*$ and using $\delta_* \gg \varepsilon_*^2$, we get

$$\ddot{y} > 4(1 + c)\|\dot{u}\|_2^2 + \varepsilon_*^2 \quad (t \in I_V), \tag{5.8}$$

for some constant $c > 0$ (say $1/d$). In the other region I_H, the last inequality of (5.6) may fail, but the smallness $|K(u(t))| \lesssim \delta_H \ll 1$ allows us to replace it by

Lemma 5.2. *For any nonzero $\varphi \in \dot{H}^1$, we have*

$$\|\nabla W\|_2^2 \le \|\nabla\varphi\|_2^2 + (d/2 - 1)K(\varphi) + O(K(\varphi)^2/\|\nabla\varphi\|_2^2). \tag{5.9}$$

Proof. Since $\varphi \neq 0$, there is a unique $\lambda > 0$ such that $K(\lambda\varphi) = 0$, that is

$$\lambda^{2^*-2} = \|\nabla\varphi\|_2^2 / \|\varphi\|_{2^*}^{2^*}. \tag{5.10}$$

Since W is the Sobolev optimizer with $K(W) = 0$, we have

$$\|\nabla W\|_2^2 \le \|\nabla\lambda\varphi\|_2^2. \tag{5.11}$$

Inserting (5.10), we obtain the desired conclusion after Taylor expansion. □

Since $\|\nabla u\|_2^2 \sim \|\nabla W\|_2^2$ in the hyperbolic region I_H, we thus replace (5.6) with

$$E(\vec{u}) < \frac{2^* - 2}{2^*}\|\nabla u\|_2^2 + \frac{d - 2}{d}K(u) + O(K(u)^2 + \varepsilon_*^2), \tag{5.12}$$

and so from (5.5) we obtain

$$\ddot{y} > 4(1 + c)\|\dot{u}\|_2^2 - 2K(u) - O(K(u)^2 + \varepsilon_*^2) \quad (t \in I_H). \tag{5.13}$$

The leading term is bounded from below via Cauchy-Schwarz:

$$4(1 + c)\|\dot{u}\|_2^2 \ge (1 + c)\frac{|\dot{y}|^2}{y}. \tag{5.14}$$

Hence

$$\ddot{y} \ge (1 + c)(\dot{y})^2/y + \begin{cases} \varepsilon_*^2 & (t \in I_V) \\ -2K(u) - O(K(u)^2 + \varepsilon_*^2) & (t \in I_H). \end{cases} \tag{5.15}$$

Hence y is convex on I_V, while on each interval $\hat{I}(t_*)$ in I_H, we have in the same way as for (4.20),

$$\int_{\hat{I}(t_*)} [-2K(u) - O(K(u)^2 + \varepsilon_*^2)]dt \gtrsim \delta_M |\hat{I}(t_*)|, \tag{5.16}$$

since $|K(u)| \lesssim \delta_H \ll 1$. Moreover, if $d_{\mathcal{S}}(\vec{u}(t)) = \delta_H$ at both ends of $\hat{I}(t_*)$ (which is the case except for the first interval), then the above integral on $\hat{I}(t_*) \cap (-\infty, T)$ is positive for any T; indeed, the main contribution comes from the region where $d_{\mathcal{S}}(\vec{u}(t)) \sim \delta_H$, and it is much bigger than the negative contribution. Therefore $\dot{y} \to \infty$ as $t \to \infty$. In particular, $\dot{y} > 0$ and $y \nearrow \infty$ for large $t \gg 1$. Since

$$\partial_t y^{-c} = -cy^{-1-c}\dot{y}, \quad \partial_t^2 y^{-c} = -cy^{-1-c}[\ddot{y} - (1+c)(\dot{y})^2/y], \tag{5.17}$$

and y^{-1-c} is decreasing for large t, the same logic as above implies that $\partial_t y^{-c}$ does not become bigger in each $\hat{I}(t_*)$ than its value at the left end of the interval. Hence $\partial_t y^{-c} < -a$ for some $a > 0$ uniformly for large t, which leads to a blow-up by contradiction. □

6. Scattering after ejection

In the other region $\mathfrak{S} = +1$, we already know that all solutions are uniformly bounded in $\mathcal{H}$, but that is not sufficient for the global existence of strongly continuous solutions in the critical case. Now we resort to the recent result by Duyckaerts-Kenig-Merle [3, 4] to preclude concentration (type II) blow-up. This is the only place where we have to restrict the dimensions[4] to 3 or 5

Proposition 6.1. *No solution blows up in $\mathcal{H}_X$ with $\mathfrak{S} = +1$.*

Proof. First, the ejection lemma 3.2 precludes blow-up in the hyperbolic region, since the scaling parameter is a priori bounded during the ejection process, which is valid when reversing the time direction. Hence a blow-up may happen only when $d_{\mathcal{S}}(\vec{u}(t)) > \delta_H$, where $K(u(t)) \geq 0$ and so (4.32) implies

$$\|\dot{u}(t)\|_2^2/2 + \|\nabla u(t)\|_2^2/d < J(W) + \varepsilon_*^2 = \|\nabla W\|_2^2/d + \varepsilon_*^2. \tag{6.1}$$

This allows us to employ the main result in [3, 4], after reducing ε_* if necessary. Suppose u is a solution on $[0, T_+)$ in $\mathcal{H}_X$ with $\mathfrak{S} = +1$ and $d_{\mathcal{S}}(\vec{u}(t)) > \delta_H$ with the blow-up time $T_+ < \infty$. According to their result, we can then write for t sufficiently near T_+

$$\vec{u}(t) = \vec{W}_{\lambda(t)} + \vec{\varphi} + o(1) \quad \text{in } \mathcal{H}, \tag{6.2}$$

for some $0 < \lambda(t) \to 0$ and some fixed $\vec{\varphi} \in \mathcal{H}$. It is then easily checked that as $t \to T_+ - 0$ we have

$$K(u(t)) = K(W_{\lambda(t)}) + K(\varphi) + o(1) = K(\varphi) + o(1), \tag{6.3}$$

from which we infer in particular that $K(\varphi) \geq 0$. Similarly, we obtain

$$J(W) + \varepsilon_*^2 > E(\vec{u}) = J(W) + E(\vec{\varphi}), \tag{6.4}$$

[4]Strictly speaking, the long-time perturbation argument should be also modified for $d > 6$ in the scattering proof of Proposition 6.2, but it is a minor issue. See [14, 8] for the solution.

which implies via (4.32) and $K(\varphi) \ge 0$,

$$\|\dot\varphi\|_2^2/2 + \|\nabla\varphi\|_2^2/d < \varepsilon_*^2. \tag{6.5}$$

This however contradicts $d_{\mathcal{S}}(\vec u(t)) > \delta_H \gg \varepsilon_*$ near T_+. □

Next we employ the Kenig-Merle scheme from [9, 10] to improve the above result. The one-pass theorem will be incorporated in the same way as in the subcritical case [15]. Extinction of the critical element requires a little extra work due to the possibility of concentration, which will be however reduced to the above proposition.

Proposition 6.2. *Every solution staying in $\mathcal{H}_X$ with $\mathfrak{S} = +1$ and $d_{\mathcal{S}} \ge \delta_*$ for $t > 0$ scatters to 0 as $t \to +\infty$ with uniformly bounded Strichartz norms on $[0, \infty)$.*

The restriction $d_{\mathcal{S}} \ge \delta_*$ is essential for the uniform Strichartz bound, since the latter does not hold for all scattering solutions, even for $E(\vec u) < J(W)$.

Proof. We argue by contradiction. Let u_n be solutions on $[0, \infty)$ in $\mathcal{H}_X$ satisfying

$$\begin{aligned} &E(\vec u_n) \to E_* \le J(W) + \varepsilon_*^2, \quad \|u_n\|_{L^q_{t,x}(0,\infty)} \to \infty, \\ &d_S(\vec u_n(t)) \ge \delta_*, \quad \mathfrak{S}(\vec u_n(t)) = +1, \quad (t > 0) \end{aligned} \tag{6.6}$$

where we choose $q = 2(d+1)/(d-2)$ so that $L^q_{t,x}$ is an admissible Strichartz norm for the wave equation on $\mathbb{R}^d$. Henceforth, $X(I)$ denotes the restriction to $I \times \mathbb{R}^d$ of the Banach function space X on $\mathbb{R} \times \mathbb{R}^d$. It is well-known that $L^q_{t,x}$ and the energy norm are sufficient to control all the other Strichartz norms, such as $L^p_t \dot B^{1/2}_{p,2}$ with $p = 2(d+1)/(d-1)$, as well as the nonlinear term in some dual admissible norm such as in $L^{p'}_t \dot B^{1/2}_{p',2}$ (see, for example, [7]).

We may assume that E_* is the minimum for the above property. Following the Kenig-Merle argument, the proof consists of two parts: construction and exclusion of a critical element.

Part I: Construction of a critical element.

Assuming the existence of (6.6), we are going to show that there is a critical element u_*, that is a solution on $[0, \infty)$ in $\mathcal{H}_X$ satisfying

$$E(\vec u_*) = E_*, \quad \|u_*\|_{L^q_{t,x}(0,\infty)} = \infty, \quad d_S(\vec u_*(t)) \ge \delta_*, \quad \mathfrak{S}(\vec u_*(t)) = +1, \tag{6.7}$$

and that its trajectory is precompact modulo dilations in $\mathcal{H}$.

If $d_{\mathcal{S}}(\vec u_n(0)) < \delta_H$, then by the ejection lemma 3.2, we have $d_{\mathcal{S}}(\vec u_n(t)) \ge \delta_H$ at some later $t > 0$. Since the Strichartz norm on the ejection time interval is uniformly bounded, we may translate each u_n so that

$$d_{\mathcal{S}}(\vec u_n(0)) \ge \delta_H, \tag{6.8}$$

without losing (6.6). The translation time is bounded by $\lambda(u_n(0))$: the scaling at $t = 0$, which remains the same order after the translation.

Since we chose $\varepsilon_* \ll \varepsilon_V(\delta_H)$, Lemma 3.1 implies

$$K(u_n(0)) \ge \min(\kappa(\delta_H), c\|\nabla u_n(0)\|_2^2). \tag{6.9}$$

Now apply[5] the Bahouri-Gerard decomposition from [1], see also Lemma 4.3 in [10], to $\{\vec{u}_n(0)\}_{n\geq 1}$. Let $U(t)$ denotes the free wave propagator. We conclude that there exist $\lambda_n^j > 0$, $t_n^j \in \mathbb{R}$, $\vec{\varphi}^j \in \mathcal{H}$ and free waves w_n^J such that for any $J \geq 1$

$$U(t)\vec{u}_n(0) = \sum_{j=1}^{J} \vec{V}_n^j(t) + \vec{w}_n^J(t), \quad \vec{V}_n^j(t) := U(t+t_n^j)T_n^j\vec{\varphi}^j, \tag{6.10}$$

where T_n^j is the operator defined by $T_n^j f := (\lambda_n^j)^{d/2} f(\lambda_n^j x)$, such that

$$|\log(\lambda_n^j/\lambda_n^k)| + |t_n^j - t_n^k|/\lambda_n^j \to \infty \tag{6.11}$$

for each $j \neq k$,

$$\begin{aligned} &\lim_{n\to\infty}\Big[\|\vec{u}_n(0)\|_2^2 - \sum_{j=1}^{J}\|\vec{V}_n^j(0)\|_2^2 - \|\vec{w}_n^J(0)\|_2^2\Big] = 0, \\ &\lim_{n\to\infty}\Big[E(\vec{u}_n(0)) - \sum_{j=1}^{J} E(\vec{V}_n^j(0)) - E(\vec{w}_n^J(0))\Big] = 0 \end{aligned} \tag{6.12}$$

for each J, and

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|w_n^J\|_{L_t^\infty L_x^{2^*}(\mathbb{R})\cap L_{t,x}^q(\mathbb{R})} = 0. \tag{6.13}$$

The last property applies to any other non-sharp Strichartz norm by interpolation, since those free waves are all uniformly bounded.

First we check that all components retain $K \geq 0$ at $t = 0$. Using (4.32), we get

$$E(\vec{u}_n) - \frac{1}{2^*}K(u_n(0)) \geq \frac{1}{d}\|\vec{u}_n(0)\|_2^2 = \sum_{j=1}^{J}\frac{1}{d}\|\vec{V}_n^j(0)\|_2^2 + \frac{1}{d}\|\vec{w}_n^J(0)\|_2^2 + o(1). \tag{6.14}$$

Hence if $\|\nabla u_n(0)\|_2^2 \lesssim \varepsilon_*^2$, then $\|\nabla V_n^j(0)\|_2^2 \lesssim \varepsilon_*^2 \ll 1$, and so $K(V_n^j(0)) \geq 0$. Otherwise, the lower bound in (6.9) is much bigger than ε_*^2, so for large n, we get from the above inequality

$$H(V_n^j(0)) < J(W), \tag{6.15}$$

which implies $K(V_n^j(0)) \geq 0$, by the variational property of W. The same argument implies $K(w_n^J(0)) \geq 0$ as well. Thus, each component has non-negative energy E. We may assume that $j = 1$ gives the maximum among $E(\vec{V}_n^j(0))$, and so

$$E(\vec{V}_n^j(0)) < \frac{2}{3}J(W), \quad (j > 1). \tag{6.16}$$

Now let U^j be the nonlinear profile associated with V_n^j, that is the nonlinear solution satisfying as $n \to \infty$,

$$\|\vec{U}^j(s_n^j) - U(s_n^j)\vec{\varphi}^j\|_2 \to 0, \quad s_n^j := \lambda_n^j t_n^j, \tag{6.17}$$

defined uniquely around $t = s_\infty^j := \lim_{n\to\infty} s_n^j$, such that

$$\|\vec{U}_n^j(0) - \vec{V}_n^j(0)\|_2 \to 0 \quad \vec{U}_n^j(t) := (T_n^j\vec{U}^j)(\lambda_n^j(t+t_n^j)). \tag{6.18}$$

[5]In what follows, we will pass to subsequences without any further mention. Also note that Merle, Vega independently obtained a decomposition of this type for NLS, [13].

By the scaling invariance of the equation, each U_n^j is also a solution, defined locally around $t=0$. Hence the above property of $\vec{V}_n^j(0)$ is transferred to U_n^j:

$$\begin{aligned} &K(U_n^j(0)) \geq 0, \quad 0 \leq E(\vec{U}_n^j) = E(\vec{U}^j) \sim \|\vec{U}_n^j(0)\|_2^2, \\ &\sum_{j=1}^J E(U^j) \lesssim J(W), \quad \sup_{j>1} E(\vec{U}^j) \leq \frac{2}{3} J(W), \end{aligned} \tag{6.19}$$

and so, by [10], each U^j for $j>1$ exists globally and scatters with

$$\sum_{j=2}^J \|U^j\|_{L^q_{t,x}(\mathbb{R})}^2 \lesssim 1. \tag{6.20}$$

Note that only a bounded number of profiles can escape from the small energy scattering theory, where all Strichartz norms are bounded by the energy norm.

Now assume the same for U^1 and thus for all $j \geq 1$, which is the case if $E(U^1) < J(W)$. Then from the long-time perturbation theory, cf. Theorem 2.20 in [10], one obtains the *nonlinear profile decomposition* for the solutions $u_n(t)$, provided J is large and fixed, and $n \geq n_0(J)$ is sufficiently large:

$$u_n = \sum_{j=1}^J U_n^j + w_n^J + R_n^J, \quad \lim_{J\to\infty} \limsup_{n\to\infty} \|\vec{R}_n^J\|_{(L_t^\infty \mathcal{H} \cap L^q_{t,x})(\mathbb{R})} = 0, \tag{6.21}$$

which implies u_n is bounded in $L^q_{t,x}$, contradicting (6.6). Thus we have obtained

$$\|U^1\|_{L^q_{t,x}(\mathbb{R})} = \infty, \quad J(W) \leq E(U^1) \leq E_*, \quad \sum_{j=2}^J E(U^j) + \|\vec{w}_n^J\|_2^2 \lesssim \varepsilon_*^2. \tag{6.22}$$

We now distinguish three cases (a)–(c) by means of $s_\infty^1 = \lim_{n\to\infty} \lambda_n^1 t_n^1$:

(a) $s_\infty^1 = \infty$. Then by definition (6.17), U^1 is a local solution around $t=\infty$ with finite Strichartz norms, and

$$\|U_n^1\|_{L^q_{t,x}(0,\infty)} = \|U^1\|_{L^q_{t,x}(s_n^1,\infty)} \to 0. \tag{6.23}$$

Hence we can use the long-time perturbation argument on $(0,\infty)$, which gives a contradiction via (6.21) as above.

(b) $s_{1,\infty} = -\infty$. In this case U^1 scatters at $t=-\infty$ by definition. Let $I = (-\infty, T_+)$ be the maximal interval of existence of U^1.

If $d_{\mathcal{S}}(U^1(t)) > \delta_*/2$ for all $t < T_+$, then U^1 remains in $\mathcal{H}_X$ with $\mathfrak{S} = +1$ from $t=-\infty$. Hence $T_+ = \infty$ by Proposition 6.1, and $\|U^1\|_{L^q_{t,x}(0,\infty)} = \infty$. Moreover, the one-pass theorem 4.1 together with the ejection lemma 3.2 implies that $d_{\mathcal{S}}(U^1(t)) \geq \delta_*$ for large t. Hence U^1 is a critical element after some time translation.

Otherwise, $d_{\mathcal{S}}(U^1(t_*)) = \delta_*/2$ at some minimal $t_* < T_+$, until which U^1 remains in $\mathcal{H}_X$ with $\mathfrak{S} = +1$, and $\|U^1\|_{L^q_{tx}(-\infty,t_*)} < \infty$. Hence one can apply the nonlinear profile decomposition on the interval $\lambda_n^1(t + t_n^1) \leq t_*$, which yields in particular

$$d_{\mathcal{S}}(\vec{u}_n((t_* - s_n^1)/\lambda_n^1)) \leq d_{\mathcal{S}}(\vec{U}_1(t_*)) + O(\varepsilon_*) + o(1) \leq \frac{2}{3}\delta_* + o(1), \tag{6.24}$$

as $n \to \infty$, provided J is large enough. However, since $t_* - s_n^1 \to \infty$, this contradicts our assumption $\inf_{t\ge 0} d_{\mathcal{S}}(\vec{u}_n(t)) \ge \delta_*$. To obtain the $O(\varepsilon_0)$-term in (6.24), one uses the bound, valid for J large and all $n \ge n_0$,

$$\sup_{\lambda_n^1(t+t_n)\le t_*} \|\vec{R}_n^J(t)\|_2 \lesssim \varepsilon_* \tag{6.25}$$

which follows from the main estimate of Theorem 2.20 in [10].

(c) $s_\infty^1 \in \mathbb{R}$. Let $(T_-, T_+) \ni s_\infty^1$ be the maximal interval of existence for U^1. We know that $K(U^1(s_\infty^1)) \ge 0$. Moreover, by the same perturbative arguments as above, the nonlinear profile decomposition (6.21) holds on $(T_-, T_+)/\lambda_n^1 - t_n^1$. Thus, as in the case (b), we deduce from $\inf_{t\ge 0} d_{\mathcal{S}}(\vec{u}_n(t)) \ge \delta_*$ that

$$\inf_{s_\infty^1 \le t < T_+} d_{\mathcal{S}}(\vec{U}^1(t)) \ge \delta_*/2. \tag{6.26}$$

Then the same argument as in (b) implies that $T_+ = \infty$ and U^1 is a critical element after time translation, provided that $\|U^1\|_{L^q_{t,x}(s_\infty^1,\infty)} = \infty$. Otherwise U^1 scatters and the nonlinear profile decomposition holds on $[0,\infty)$, contradicting (6.6).

Thus we arrive at the conclusion that $s_\infty^1 < \infty$ and U^1 is a critical element after time translation. This implies $E(U^1) = E_*$ by the minimality, which extinguishes the other profiles U^j $(j > 1)$ as well as the remainder w_n^J as $n \to \infty$, through the nonlinear energy decomposition.

Having a critical element u_*, we apply the above argument to the sequence

$$u_n(t) = u_*(t - t_n), \quad t_n \to \infty. \tag{6.27}$$

The vanishing of all but one profile implies that for some continuous $\lambda(t) > 0$

$$\{\lambda(t)^{-d/2} \vec{u}_*(t, x/\lambda(t))\}_{t\ge 0} \subset \mathcal{H} \tag{6.28}$$

is precompact, concluding the first part of the proof.

Part II: Exclusion of a critical element.

Let u_* be a critical element (6.7), hence

$$\vec{w}_*(t) := \varrho(t)^{d/2} \vec{u}_*(t, \varrho(t)x), \quad \varrho(t) := 1/\lambda(t) \tag{6.29}$$

for $t \ge 0$ is precompact in $\mathcal{H}$. We proceed in three steps.

Step 1: $\limsup_{t\to\infty} \varrho(t)/t < \infty$. To see this, note that by finite propagation speed, we have

$$\lim_{R\to\infty} \sup_{t\ge 0} \|\vec{u}_*(t)\|_{L^2(|x|>t+R)} = 0, \tag{6.30}$$

whence we have

$$\lim_{R\to\infty} \sup_{t\ge 0} \|\vec{w}_*(t)\|_{L^2(|x|>(t+R)/\varrho(t))} = 0. \tag{6.31}$$

If for some sequence of times $\{s_n\}_{n\ge 1}$ we had $\varrho(s_n)/s_n \to 0$, then by pre-compactness of $\{\vec{w}_*(t)\}_{t\ge 0}$, we get $\|\vec{w}_*(s_n)\|_{L^2} \to 0$, whence also $\|\vec{u}_*(s_n)\|_{L^2} \to 0$, which would force $E_* = 0$, a contradiction.

Step 2: $\liminf_{t\to\infty} \varrho(t)/t > 0$. This follows from the localized virial identity (4.5) as in the proof of Theorem 4.1. By the precompactness, there is $R > 0$, depending on u_*, such that for all $t \geq 0$

$$\int_{|x|>R\varrho(t)} |\dot{u}_*|^2 + |\nabla u_*|^2 dx < \delta. \tag{6.32}$$

Suppose for contradiction that $\liminf_{t\to\infty} \varrho(t)/t = 0$. Choose $T_3 \gg T_2 \gg 1$ and $\tau_2, \tau_3 > 0$ such that

$$\varrho(T_j) \ll \delta T_j/R, \quad \tau_j = R\varrho(T_j). \tag{6.33}$$

Then we have

$$\begin{aligned} &|\langle wu_t | x \cdot \nabla u + \nabla \cdot xu\rangle| + |\langle wu_t|u\rangle| \lesssim R\varrho(T_j) \ll \delta_* T_j \quad (t = T_j,\ j = 2, 3), \\ &\sup_{T_2<t<T_3} E_{\rm ext}(t) \lesssim \max_{t=T_2,T_3} E_{\rm ext}(t) < \delta, \end{aligned} \tag{6.34}$$

where w and $E_{\rm ext}$ are as in (4.3) and (4.6). Then we have in place of (4.30)–(4.31),

$$\begin{aligned} &\int_{T_2}^{T_3} [\|\dot{u}\|_2^2 - K(u(t)) + O(\delta_*)]dt \ll \delta_* T_3, \\ &\int_{T_2}^{T_3} [\delta_*^{1/2} \|\nabla u(t)\|_2^2 - O(\delta_*)]dt \ll \delta_* T_3, \end{aligned} \tag{6.35}$$

which leads to

$$|T_3 - T_2| J(W) \leq \int_{T_2}^{T_3} E(u)dt \ll \delta_*^{1/2} T_3, \tag{6.36}$$

a contradiction. Here again we assumed $E(u) \geq J(W)$ since in the other case one can easily get a simpler bound, as was done in [9].

Step 3: Construction of a blow up solution via re-scaling u_*. Pick a sequence $s_n \to \infty$ with $\lim_{n\to\infty} \varrho(s_n)/s_n = c \in (0, \infty)$, as well as $\vec{w}_*(s_n) \to \exists\vec{\varphi}$ in L^2. Define a sequence of solutions

$$u_n(t, x) := s_n^{d/2-1} u_*(s_n t, s_n x) \tag{6.37}$$

whence we have $\vec{u}_n(1) \to c^{-d/2}\vec{\varphi}(x/c)$ in L^2.

The above two steps imply that $\vec{u}_n$ is precompact in $C([\tau, 1]; L^2)$ for any $0 < \tau < 1$, and so, after passing to a subsequence, it converges to some $\vec{u}_\infty$ in $C((0, 1]; L^2)$. By the local wellposedness theory, it has finite Strichartz norms locally in time, and so u_∞ is the unique strong solution on $(0, 1]$ with the initial condition $\vec{u}_\infty(1) = \vec{\varphi}$. Clearly we also have $d_{\mathcal{S}}(\vec{u}_\infty(t)) \geq \delta_*$ and $\mathfrak{S}(\vec{u}_\infty(t)) = +1$ for $0 < t \leq 1$.

We now show that u_∞ is a solution blowing up at $t = 0$, which contradicts Proposition 6.1. The fact that u_∞ blows up at $t = 0$ follows from

Claim: $u_\infty(t, x) = 0$ on $|x| > t$. To see this, pick $0 < \varepsilon \ll 1$ arbitrary, let m large enough such that $\|\vec{w}_*(s_m) - \vec{\varphi}\|_{L^2} \ll \varepsilon$ and further pick $R > 0$ such that $\|\vec{\varphi}\|_{L^2(|x|>R)} \ll \varepsilon$. Then for $n > m$, we have

$$\|\vec{u}_n(s_m/s_n)\|_{L^2(|x|>R\varrho(s_m)/s_n)} = \|w_*(s_m)\|_{L^2(|x|>R)} \ll \varepsilon. \tag{6.38}$$

From this and the finite propagation speed, we deduce that for $s_m/s_n \leq t \leq 1$

$$\|\vec{u}_n(t)\|_{L^2(|x|>R\varrho(s_m)/s_n+t-s_m/s_n)} \ll \varepsilon. \tag{6.39}$$

Letting $n \to \infty$, we infer that for $0 < t \le 1$

$$\|\vec{u}_n(t))\|_{L^2(|x|>t)} \ll \varepsilon. \tag{6.40}$$

Since $\varepsilon > 0$ is arbitrary, this implies that u_∞ is supported on $|x| \le t$, as claimed. This completes the proof of Proposition 6.2. $\square$

In order to complete the proof of Theorem 1.1, we now exhibit open data sets at time $t = 0$ such that we have blow up/scattering at $t = \pm\infty$, four possibilities in all. For this, we use the representation

$$u = W + v_1 = W + \mu_1(u)\rho + \gamma_1, \tag{6.41}$$

used in the proof of Lemma 3.2, see (3.19). We pick data of the form

$$u(0) = W + a\rho + f, \quad \dot{u}(0) = b\rho + g, \tag{6.42}$$

for some $a, b \in \mathbb{R}$, $f \in \dot{H}^1$ and $g \in L^2$ radial, with the conditions

$$\|\nabla f\|_2 + \|g\|_2 \ll |a| + |b| \ll \delta_*. \tag{6.43}$$

It then follows from the same argument as below (3.23) that we have

$$\begin{aligned} &\mu_1(t) = e^{kt}\mu_+ + e^{-kt}\mu_- + O(e^{2k|t|}(a^2 + b^2)), \\ &\|\vec{\gamma}_1(t)\|_2 \lesssim \langle t\rangle(|a| + |b|) + e^{2k|t|}(a^2 + b^2), \end{aligned} \tag{6.44}$$

as long as $e^{k|t|}(|a| + |b|) \lesssim \delta_H$, where δ_H is as in Lemma 3.2, and further

$$\mu_+ := \frac{1}{2}\left(a + \frac{1}{k}b\right), \quad \mu_- := \frac{1}{2}\left(a - \frac{1}{k}b\right). \tag{6.45}$$

Using the expansion of K in (3.44) as well, it is now easy to see that under the conditions (6.43) we obtain 4 disjoint open sets, depending on the signs of a and b, such that $K(u) \lesseqgtr 0$ at the ejection times, i.e. the endpoints of the time interval around 0 where $d_{\mathcal{S}}(\vec{u}) \le \delta_H$. This completes the proof of Theorem 1.1. $\square$

References

[1] Bahouri, H., Gérard, P. *High frequency approximation of solutions to critical nonlinear wave equations.* Amer. J. Math. **121** (1999), no. 1, 131–175.

[2] Caffarelli, L., Gidas, B., Spruck, J. *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth.* Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297.

[3] Duyckaerts, T., Kenig, C., Merle, F. *Universality of blow-up profile for small radial type II blow-up solutions of energy-critical wave equation,* preprint, arXiv:0910.2594.

[4] Duyckaerts, T., Kenig, C., Merle, F. *Universality of the blow-up profile for small type II blow-up solutions of energy-critical wave equation: the non-radial case*, preprint, arXiv:1003.0625.

[5] Duyckaerts, T., Merle, F. *Dynamic of threshold solutions for energy-critical NLS.* Geom. Funct. Anal. **18** (2009), no. 6, 1787–1840.

[6] Duyckaerts, T., Merle, F. *Dynamic of threshold solutions for energy-critical wave equation.* Int. Math. Res. Pap. IMRP **2008**.

[7] Ginibre, J., Soffer, A., Velo, G. *The global Cauchy problem for the critical non-linear wave equation*, J. Funct. Anal. **110** (1992), 96–130.

[8] Ibrahim, S., Masmoudi, N., Nakanishi, K. *Scattering threshold for the focusing nonlinear Klein-Gordon equation*, preprint, arXiv:1001.1474, to appear in Analysis & PDE.

[9] Kenig, C., Merle, F. *Global well-posedness, scattering, and blow-up for the energy-critical focusing nonlinear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), no. 3, pp. 645–675.
[10] Kenig, C., Merle, F. *Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation.* Acta Math. **201** (2008), no. 2, 147–212.
[11] Krieger, J., Schlag, W. *On the focusing critical semi-linear wave equation.* Amer. J. Math. **129** (2007), no. 3, 843–913.
[12] Krieger, J., Schlag, W., Tataru, D. *Slow blow-up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semilinear wave equation.* Duke Math. J. **147** (2009), no. 1, 1–53.
[13] Merle, F., Vega, L. *Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D.* Internat. Math. Res. Notices 1998, no. 8, 399–425.
[14] Nakanishi, K. *Scattering theory for the nonlinear Klein-Gordon equation with Sobolev critical power.* Internat. Math. Res. Notices **1999**, no. 1, 31–60.
[15] Nakanishi, K., Schlag, W. *Global dynamics above the ground state energy for the focusing nonlinear Klein-Gordon equation*, preprint, arXiv:1005.4894.
[16] Nakanishi, K., Schlag, W. *Global dynamics above the ground state energy for the cubic NLS equation in 3D*, preprint, arXiv:1007.4025.
[17] Shatah, J., Struwe, M. *Geometric Wave Equations*, Courant Lecture Notes, AMS, 1998.

Department of Mathematics, The University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104, U.S.A.

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

Department of Mathematics, The University of Chicago, 5734 South University Avenue, Chicago, IL 60615, U.S.A.