ON THE FOCUSING CRITICAL SEMI-LINEAR WAVE EQUATION

J. KRIEGER, W. SCHLAG

Abstract. The wave equation

$$\partial_{tt}\psi - \Delta\psi - \psi^5 = 0$$

in $\mathbb{R}^3$ is known to exhibit finite time blowup for data of negative energy. It also admits the special static solutions

$$\phi(x,a) = (3a)^{\frac{1}{4}}(1 + a|x|^2)^{-\frac{1}{2}}$$

for all $a > 0$ which are linearly unstable. We view these functions as a curve in the energy space $\dot{H}^1 \times L^2$. We show that in a small neighborhood of itself, which lies on a stable hyper-surface of radial data, this curve acts as a one-dimensional attractor.

1. Introduction

We consider the equation

$$\Box\psi - \psi^5 = \partial_{tt}\psi - \Delta\psi - \psi^5 = 0 \tag{1}$$

in $\mathbb{R}^3$. By an argument of Levine [Lev], this equation can blow up in finite time. In fact, negative energy leads to finite-time blowup, see Strauss [Str]. On the other hand, this equation admits the static solutions

$$\phi(r,a) = (3a)^{\frac{1}{4}}(1 + ar^2)^{-\frac{1}{2}} \tag{2}$$

for all $a > 0$, i.e., these functions satisfy

$$-\Delta\phi - \phi^5 = 0$$

Note that although $\phi(\cdot,a) \notin L^2(\mathbb{R}^3)$, we have $\phi(\cdot,a) \in \dot{H}^1$. In particular, $\phi(\cdot,a)$ is a finite-energy solution of (1). We remark that Aubin [Aub] identified the family $\phi(\cdot,a)$ as extremal functions for the norm of the imbedding $\dot{H}^1(\mathbb{R}^3) \to L^6(\mathbb{R}^3)$. Our goal is to understand small perturbations of ϕ under the evolution of (1), more precisely, to show that stable manifolds of finite co-dimension exist. As we shall see later, the point here is that these special solutions are linearly unstable (i.e., the linearized operator has an unstable eigenvalue). Seeking solutions of the form $\psi(x,t) = \phi(r,a(t)) + u(x,t)$ leads to the linearized problem

$$\partial_{tt}u + H(a(t))u = -\partial_{tt}\phi(r,a(t)) + N(u,\phi(\cdot,a(t))) \tag{3}$$

where $H(a(t)) = -\Delta - 5\phi^4(\cdot,a(t)) = -\Delta + V(\cdot,a(t))$ and

$$N(\phi,u) = 10\phi^3u^2 + 10\phi^2u^3 + 5\phi u^4 + u^5$$

By definition,

$$H(a)\,\partial_a\phi(\cdot,a) = 0, \qquad H(a)\nabla\phi(\cdot,a) = 0$$

so that $H(a)$ has a resonance (due to $\partial_a\phi \notin L^2(\mathbb{R}^3)$), as well as an eigenvalue at zero energy. Moreover, since $\langle H(a)\phi,\phi\rangle < 0$, $H(a)$ has negative eigenvalues.

In this paper, we will only consider radial solutions of (1). This precludes any movement of the solution $\phi(\cdot,a)$. More precisely, due to the translation and Lorentz invariance of (1), a small, non-radial perturbation

---

The first author was supported by the NSF grant DMS-0401177. The second author was partially supported by the NSF grant DMS-0300081 and a Sloan Fellowship. He wishes to thank Sigmund Selberg from NTNU in Trondheim, Norway, for his hospitality and the support by the Research Council of Norway through the research project "Partial Differential Equations and Harmonic Analysis". Also, he wishes to thank Fritz Gesztesy for many helpful discussions. Both authors are grateful to Carlos Kenig and Kaihua Cai for their interest in this work and comments on a preliminary version of this paper. They also gratefully acknowledge the support of the MSRI, Berkeley, where this work was presented during a workshop on dispersive PDE in August of 2005. Finally, the authors wish to thank Herbert Koch for pointing out an error in a preliminary formulation of Theorem 1.

of $\phi(\cdot, a)$ will typically impart a non-zero momentum on ϕ and thus lead to a moving bulk-term, whereas in our radial case only the dilation factor $a(t)$ can move with time.

Let $a > 0$ be fixed. If we consider $H(a)$ on the invariant subspace $L^2_r(\mathbb{R}^3)$ of radial functions, then it is easy to describe the spectrum as well as the resonances rigorously: there is exactly one negative eigenvalue and a unique resonance at zero. To see this, write the equation $H(a)f = -k^2 f$ with a radial $f \in L^2(\mathbb{R}^3)$ in the form $-g'' + V(a)g = -k^2 g$ on $L^2(0,\infty)$ where $V(a) = V(r,a) = -5\phi^4(r,a)$ and $g(r) = rf(r)$. Then $g_1 := r\partial_a \phi$ has a unique positive zero, which implies that there is a simple ground-state $g > 0$ with negative energy $-k^2$, and $g_1 \in L^\infty$ is the unique resonance function at zero. Note that $k > 0$ depends on the parameter a through scaling: $k(a) = a^{\frac{1}{2}} k(1)$. This follows from $V(r,a) = aV(\sqrt{a}, 1)$. Moreover, g decays exponentially by Agmon's estimate. In what follows, we will denote the ground state eigenfunction of $H(a)$ with $a = 1$ by g_0 and the associated eigenvalue by $-k_0^2$, with $k_0 > 0$.

Our main result is as follows:

Theorem 1. *Fix $R > 1$ and let*[1]

$$X_R = \{(f_1, f_2) \in H^3_{\mathrm{rad}}(\mathbb{R}^3) \times H^2_{\mathrm{rad}}(\mathbb{R}^3) \mid \mathrm{supp}(f_j) \subset B(0,R)\}$$

Define

$$\Sigma_0 := \{(f_1, f_2) \in X_R \mid \langle k_0 f_1 + f_2, g_0\rangle = 0\}$$

and let $B_\delta(0) \subset \Sigma_0$ denote a δ-ball in the topology of X_R. Then there exists $\delta = \delta(R) > 0$ and a Lipschitz function $h : B_\delta(0) \subset \Sigma_0 \to \mathbb{R}$ with the following properties:

$$\begin{aligned} &|h(f_1,f_2)| \lesssim \|(f_1,f_2)\|^2_{X_R}, \qquad \forall\, (f_1,f_2) \in B_\delta(0) \\ &|h(f_1,f_2) - h(\tilde f_1, \tilde f_2)| \lesssim \delta \|(f_1,f_2) - (\tilde f_1, \tilde f_2)\|_{X_R} \qquad \forall\, (f_1,f_2), (\tilde f_1, \tilde f_2) \in B_\delta(0) \end{aligned} \tag{4}$$

and for any $(f_1, f_2) \in B_\delta(0)$ the Cauchy problem

$$\begin{aligned} &\Box\psi - \psi^5 = 0 \\ &\psi(\cdot, 0) = \phi(\cdot, 1) + f_1 + h(f_1,f_2) g_0, \quad \partial_t \psi(\cdot, 0) = f_2 \end{aligned} \tag{5}$$

has a unique global solution of the form

$$\psi(\cdot, t) = \phi(\cdot, a(\infty)) + v(\cdot, t) \tag{6}$$

where $|a(\infty) - 1| \lesssim \delta$. The radiative term $v(\cdot,t)$ disperses like a free wave, i.e., $\|v(\cdot,t)\|_\infty \lesssim \delta\langle t\rangle^{-1}$ for all $t > 0$, and it also scatters like a free wave with energy data:

$$(v, \partial_t v)(\cdot, t) = (\tilde v, \partial_t \tilde v)(\cdot, t) + o_{\dot H^1 \times L^2}(1) \quad \text{as} \quad t \to \infty$$

with $\Box \tilde v = 0$, $(\tilde v(\cdot,0), \partial_t \tilde v(\cdot, 0)) \in \dot H^1 \times L^2$.

The point of (5) is that it describes a parametric surface $\phi(\cdot,1) + \Sigma$ where[2] $\Sigma \subset H^3 \times H^2$ is parameterized by $(f_1, f_2) \in B_\delta(0)$. In view of the estimates (4), Σ can also be realized as a Lipschitz graph of a function with domain $B_\delta(0)$. Thus, Theorem 1 states the existence of a codimension one Lipschitz graph which is the distorted image of the ball $B_\delta(0) \subset \Sigma_0$. Moreover, Σ_0 is the tangent plane to Σ at zero. Theorem 1 states that this manifold Σ is a stable manifold for the NLW (1) centered at the special solution $\phi(\cdot, 1)$. Furthermore, the theorem implies that the curve $a \mapsto \phi(\cdot, a)$ acts as a one-dimensional attractor for data on this stable manifold. In other words, if $(\psi(\cdot,0), \partial_t\psi(\cdot,0)) \in \phi(\cdot,1) + \Sigma$, then there is a global solution of the form

$$\psi(\cdot,t) = \phi(\cdot, a(\infty)) + O(\delta\langle t\rangle^{-1})_{L^\infty} \quad \text{as } t \to \infty$$

with some $|a(\infty) - 1| \lesssim \delta$. The proof of Theorem 1 proceeds by making an ansatz

$$\psi(\cdot, t) = \phi(\cdot, a(t)) + u(\cdot, t) \tag{7}$$

where $a(t)$ is a path that starts with $a(0) = 1$ and a dispersive part $u(\cdot,t)$, see Section 2. The solution (a, u) as well as the correction $h(f_1, f_2)$ are obtained by a Banach iteration. The significance of the linear hyperplane Σ_0 is that it serves as the stable manifold for the linearized operator at the first step of the iteration. At

[1] The compact support assumption can be replaced with power-like decay, but we ignore this issue

[2] It is perhaps desirable to replace g by a cut-off of itself to $|x| \lesssim R$ in (5). Because of the exponential decay of g we do not make this distinction.

each subsequent step of the iteration we then need to make corrections to these data perpendicular to Σ_0. Eventually, these corrections add up to the quadratically small function $h(f_1, f_2)$.
Theorem 1 was motivated by the stable manifold papers [Sch1], [KriSch1] and [KriSch2], as well as by the numerical evidence presented in Bizoń, Chmaj, Tabor [BizChmTab] and Szpak [Szp]. It is a rigorous expression of the heuristic principle that instability results from the simple, negative eigenvalue. It remains an open problem to determine what happens to data in $B_\delta^+(0)$ and $B_\delta^-(0)$, which are the two halves of $B_\delta(0) \setminus \Sigma$. According to some numerical evidence it is conjectured in [BizChmTab] that these two halves should display a blow-up/scattering dichotomy. However, our understanding of these issues is very limited[3]. Even the question of (nonlinear) orbital stability of $\phi(\cdot, a)$ in the radial energy space appears to be difficult. Of course, we expect that they are orbitally unstable, but the fundamental theory of orbital stability by Grillakis, Shatah, Strauss [GriShaStr1], [GriShaStr2] does not apply to this case.

Nevertheless, it is possible to make a statement about the unique role that the manifold Σ plays in $B_\delta(0)$: If data from $B_\delta(0)$ lead to a solution of the form (7), and $\dot{a}$ and $u(x,t)$ satisfy the estimates (37)–(41) below, then the data are necessarily from Σ. This follows from the proof of Proposition 4.

Our paper is organized as follows: In Section 2, we develop the formalism of the linearized equation on which Theorem 1 is based. This will be accomplished by writing the wave equation as a Hamiltonian equation, and by analyzing the spectrum of the linearized Hamiltonian. Readers familiar with modulation theory will recognize some aspects of it in Section 2. However, we would like to emphasize that modulation theory does not apply to the case of resonances, since they are not associated with L^2 projections. In other words, it is not possible to project $u(\cdot, t)$ as in Theorem 1 onto $\partial_a \phi(\cdot, a)$ as a means of deriving the ODE for $a(t)$. However, the basic principle of modulation theory still applies here: *the ODE for $a(t)$ ensures that the non-dispersive part of the linearized operator resulting from the resonance is removed.*

The remedy here will be a careful analysis of the evolution operator $\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c$ for a Schrödinger operator $H = -\Delta + V$ which has a resonance, but no eigenvalue at zero energy and with P_c being the projection onto the continuous spectrum. Recall that this means that

$$(-\Delta + V - z^2)^{-1} = \frac{B_{-1}}{z} + B_0(z) \quad \text{as} \quad \mathrm{Im} z > 0, \; z \to 0$$

where B_{-1} is a rank-one operator and $B_0(z)$ is uniformly bounded as $z \to 0$ in the operator norm $L^{2,-1-\varepsilon}(\mathbb{R}^3) \to L^{2,1+\varepsilon}(\mathbb{R}^3)$. Equivalently, in three dimensions, a zero energy resonance (but no zero eigenvalue) means that there is a solution f of

$$(-\Delta + V) f = 0 \quad \text{with} \quad f \in \bigcap_{\sigma > \frac{1}{2}} L^{2,-\sigma}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$$

In our application, $f = \partial_a \phi(\cdot, a)$ plays this role. This analysis will be carried out in Section 5 and one of our main results there is the representation formula

$$\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c = c_0\,(\psi \otimes \psi) + \mathcal{S}(t), \qquad \|\mathcal{S}(t) f\|_\infty \lesssim t^{-1} \|f\|_{W^{1,1}(\mathbb{R}^3)}$$

where P_c is the projection onto the continuous spectral subspace, $c_0 \neq 0$, and ψ is the resonance function of H at zero normalized to $\int V\psi\, dx = 1$. The point here is that the singularity of the spectral measure of H at zero produces the contribution $c_0\,(\psi \otimes \psi)$, which is formally a projection, but *not* in L^2. Particular attention here needs to be paid to the fact that the decay of the potential is exactly $\langle x \rangle^{-4}$. For more details, as well as other results (e.g., it is essential to understand the kernel of $\cos(t\sqrt{H})$ and estimates on it) we refer the reader to Section 5. As explained above, our H does have eigenvalues at zero when considered as an operator on $L^2(\mathbb{R}^3)$. However, its restriction to the subspace of radial functions $L^2_{\mathrm{rad}}(\mathbb{R}^3)$ does not. Since that subspace is an invariant subspace of H, the results from Section 5 apply as long as we restrict the evolution to radial functions. Equipped with the formalism of Section 2 and the estimates from Section 5, we find $a(t)$, $u(x,t)$, as well as $h(f_1, f_2)$ by means of a contraction argument in Sections 3 and 4.

---

[3]The recent work of Merle and Zaag [MerZaa1]–[MerZaa3] investigates the question of blow-up for semi-linear wave equations. However, their work does not concern the H^1 critical case which we study here.

2. The linearized problem

Our goal is to solve the Cauchy problem

$$\partial_{tt}u + H(a(\infty))u = -\partial_{tt}\phi(\cdot, a(t)) + (V(\cdot, a(\infty)) - V(\cdot, a(t)))u + N(u, \phi(\cdot, a(t))) \tag{8}$$

$$u(\cdot, 0) = w_1, \quad \partial_t u(\cdot, 0) = w_2 \tag{9}$$

globally in time for small radial data (w_1, w_2) that lie on a suitable manifold. Of course, (w_1, w_2) will eventually equal the expression on the right-hand side of (5). We are assuming here that $a(t) \to a(\infty)$ as $t \to \infty$ with $0 < a(\infty) < \infty$. We will use H_∞ and $H(a(\infty))$ interchangeably. As noted previously, H_∞ has a (unique) negative (radial) ground-state $g_\infty > 0$ with $\|g_\infty\|_2 = 1$. It decays exponentially by Agmon's estimate. There is no other negative spectrum. Let $H_\infty g_\infty = -k_\infty^2 g_\infty$. Here $k_\infty = \sqrt{a(\infty)}\, k_0$ because of scaling, see above. We recast the linearized equation (8) as a Hamiltonian system, with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$:

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = J \begin{bmatrix} H_\infty & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\partial_{tt}\phi(\cdot, a(t)) + (V(\cdot, a(\infty)) - V(\cdot, a(t)))u_1 + N(u_1, \phi(\cdot, a(t))) \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(0) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

which we write more succinctly as

$$\dot{U} = J\mathcal{H}_\infty U + W, \quad U(0) = U_0$$

It is easy to check that[4]

$$\mathrm{spec}(J\mathcal{H}_\infty) = i\mathbb{R} \cup \{\pm k_\infty\}$$

with $\pm k_\infty$ being simple eigenvalues (there are no other eigenvalues). We write k instead of k_∞ for simplicity. The eigenfunctions are[5]

$$G_\pm := (2k)^{-\frac{1}{2}} \begin{pmatrix} g_\infty \\ \pm k g_\infty \end{pmatrix}$$

They satisfy

$$J\mathcal{H}_\infty G_\pm = \pm k G_\pm$$

Similarly, one easily checks that $G_\pm^* := JG_\mp$ are the dual eigenfunctions, i.e.,

$$(J\mathcal{H}_\infty)^* G_\pm^* = \pm k G_\pm^*$$

Since we have normalized $G_\pm$ such that $\langle G_\pm, JG_\mp \rangle = \mp 1$, this implies that the Riesz projections $P_\pm$ onto the discrete spectrum are

$$P_\pm = \mp \langle \cdot, G_\pm^* \rangle G_\pm = \mp \langle \cdot, JG_\mp \rangle G_\pm$$

Now define $n_\pm = n_\pm(t)$ by

$$n_\pm(t) G_\pm(\cdot) := P_\pm U(\cdot, t) \tag{11}$$

where $U(\cdot, a)$ solves (10). Then one checks that $n_\pm = \mp \langle U, JG_\mp \rangle$ solve

$$\dot{n}_\pm(t) \mp k n_\pm(t) = \pm \langle W, JG_\mp \rangle =: F_\pm(t) \tag{12}$$

We now recall a trivial fact from ODEs:

Lemma 2. *Consider the two-dimensional ODE*

$$\dot{x}(t) - A_0 x(t) = y(t), \qquad x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

where $y = \binom{y_1}{y_2} \in L^\infty([0, \infty), \mathbb{C}^2)$ *and* $A_0 = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}$ *where* $k > 0$*. Then* $x(t) = \binom{x_1(t)}{x_2(t)}$ *remains bounded for all times iff*

$$0 = x_1(0) + \int_0^\infty e^{-kt} y_1(t)\, dt. \tag{13}$$

[4]Recall that we are only considering radial functions.

[5]Recall $\|g_\infty\|_2 = 1$

Moreover, in that case

$$x_1(t) = -\int_t^\infty e^{-(s-t)k} y_1(s)\,ds, \quad x_2(t) = e^{-tk}x_2(0) + \int_0^t e^{-(t-s)k} y_2(s)\,ds. \tag{14}$$

for all $t \geq 0$. In particular, if $y_1(s), y_2(s)$ decay like $\langle s\rangle^{-\beta}$ with some $\beta > 0$, then $x_1(t), x_2(t)$ decay at least as fast.

Proof. Clearly, $x_1(t) = e^{tk}x_1(0) + \int_0^t e^{(t-s)k}y_1(s)\,ds$ and $x_2(t) = e^{-tk}x_2(0) + \int_0^t e^{-(t-s)k}y_2(s)\,ds$. If

$$\lim_{t\to\infty} e^{-tk}x_1(t) = 0$$

then

$$0 = x_1(0) + \int_0^\infty e^{-sk} y_1(s)\,ds$$

which is (13). Conversely, if this holds, then

$$x_1(t) = -e^{tk}\int_t^\infty e^{-sk}y_1(s)\,ds$$

and the lemma is proved. □

This lemma leads to the *stability condition*

$$0 = n_+(0) + \int_0^\infty e^{-sk}F_+(s)\,ds \tag{15}$$

for the linearized problem. We pass to the decomposition

$$U = n_+G_+ + n_-G_- + \tilde{U} \tag{16}$$

where $\tilde{U}$ is the projection of U onto the essential spectrum of $J\mathcal{H}$, i.e., $\tilde{U} = (I - P_+ - P_-)U =: P_eU$ (here "e" refers to the essential spectrum). Written out in components,

$$\begin{aligned}
(P_+ + P_-)\binom{u}{v} &= (2k)^{-1}\Big\langle \binom{u}{v}, \binom{kg_\infty}{g_\infty}\Big\rangle \binom{g_\infty}{kg_\infty} + (2k)^{-1}\Big\langle \binom{u}{v}, \binom{kg_\infty}{-g_\infty}\Big\rangle \binom{g_\infty}{-kg_\infty} \\
&= \binom{\langle u, g_\infty\rangle g_\infty}{\langle v, g_\infty\rangle g_\infty} = \binom{P_{g_\infty}u}{P_{g_\infty}v} \\
P_e\binom{u}{v} &= \binom{u - \langle u, g_\infty\rangle g_\infty}{v - \langle v, g_\infty\rangle g_\infty} = \binom{P_{g_\infty}^\perp u}{P_{g_\infty}^\perp v}
\end{aligned}$$

Projecting (10) yields[6]

$$\partial_t\tilde{U} = J\mathcal{H}_\infty\tilde{U} + P_eW, \quad \tilde{U}(0) = P_eU(0) \tag{17}$$

In view of the preceding, the propagator takes the form

$$e^{tJ\mathcal{H}_\infty}P_e = \begin{bmatrix} \cos(t\sqrt{H_\infty})\,P_{g_\infty}^\perp & \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}\,P_{g_\infty}^\perp \\ -\sqrt{H_\infty}\sin(t\sqrt{H_\infty})\,P_{g_\infty}^\perp & \cos(t\sqrt{H_\infty})\,P_{g_\infty}^\perp \end{bmatrix} \tag{18}$$

with $I - P_{g_\infty} = P_{g_\infty}^\perp$ being the projection onto the continuous spectrum of H_∞. The solution of (17) is

$$\tilde{U}(t) = e^{tJ\mathcal{H}_\infty}P_e\tilde{U}(0) + \int_0^t e^{(t-s)J\mathcal{H}_\infty}P_eW(s)\,ds \tag{19}$$

If we set $\tilde{U} = \binom{\tilde{u}}{\tilde{v}}$ in (17), then $\tilde{v} = \partial_t\tilde{u}$ and (17) is equivalent with

$$\begin{aligned}
\partial_{tt}\tilde{u} + H_\infty\tilde{u} &= P_{g_\infty}^\perp[-\partial_{tt}\phi(\cdot, a(t)) + (V(\cdot, a(\infty)) - V(\cdot, a(t)))u + N(u, \phi(\cdot, a(t)))] \\
\tilde{u}(0) &= P_{g_\infty}^\perp w_1, \ \partial_t\tilde{u}(0) = P_{g_\infty}^\perp w_2
\end{aligned} \tag{20}$$

[6]Note that $\mathcal{H}_\infty$ and $P_\pm$ are associated with $a = a(\infty)$ and thus do not depend on time. In particular, $P_{g_\infty}^\perp\phi_a(\cdot, a(\infty)) = \phi_a(\cdot, a(\infty))$.

Here u is the solution of the full equation (8), i.e.,

$$u(\cdot,t) = (2k)^{-\frac{1}{2}}(n_+(t)+n_-(t))g_\infty + \tilde u(\cdot,t)$$

and H_∞ as well as g_∞ are to be taken relative to $a = a(\infty)$.

The solution of (20) is

$$\tilde u(\cdot,t) = \cos(t\sqrt{H_\infty})P^\perp_{g_\infty} w_1 + \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P^\perp_{g_\infty} w_2$$

(21)

$$+ \int_0^t \frac{\sin((t-s)\sqrt{H_\infty})}{\sqrt{H_\infty}} P^\perp_{g_\infty}[-\partial_s(\dot a(s)\phi_a(\cdot,a(s))) + (V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s) + N(u,\phi(\cdot,a(s)))]\,ds$$

Now

$$\int_0^t \frac{\sin((t-s)\sqrt{H_\infty})}{\sqrt{H_\infty}} P^\perp_{g_\infty}\partial_s(\dot a(s)\phi_a(\cdot,a(s)))\,ds$$

$$= -\dot a(0)\,\frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P^\perp_{g_\infty}\phi_a(\cdot,a(0)) + \int_0^t \cos((t-s)\sqrt{H_\infty})P^\perp_{g_\infty}\,\phi_a(\cdot,a(s))\,\dot a(s)\,ds \tag{22}$$

Let us first continue with a model case. Since (with $\psi = \phi_a(\cdot,a(\infty))$)

$$(\partial_{tt}+H_\infty)\psi = 0, \qquad (\partial_{tt}+H_\infty)\,t\psi = 0$$

we obtain

$$\cos(t\sqrt{H_\infty})\,\psi = \psi, \quad \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}\,\psi = t\psi$$

from which we conclude that

$$-\dot a(0)\,\frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P^\perp_{g_\infty}\psi + \int_0^t \cos((t-s)\sqrt{H_\infty})P^\perp_{g_\infty}\,\psi\,\dot a(s)\,ds$$

$$= -t\dot a(0)\,\psi + \int_0^t \psi\,\dot a(s)\,ds = -t\dot a(0)\,\psi + (a(t)-a(0))\psi$$

To pass from the model case to the real one, we use that for all Schwartz functions f

$$\frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P^\perp_{g_\infty} = c_0\,(\psi\otimes\psi) + \mathcal{S}(t), \quad \|\mathcal{S}(t)f\|_\infty \lesssim t^{-1}\|f\|_{W^{1,1}(\mathbb{R}^3)} \tag{23}$$

$$\|\cos(t\sqrt{H_\infty})P^\perp_{g_\infty} f\|_\infty \lesssim t^{-1}\|f\|_{W^{2,1}(\mathbb{R}^3)} \tag{24}$$

We will need the following more precise statement concerning the L^1 norm on the right-hand side of (24): There exists a kernel $K_t(x,y)$ so that

$$|K_t(x,y)| \lesssim (\chi_{[|x|+|y|>t]} + \langle t\rangle^{-1})(\langle x\rangle\langle y\rangle)^{-1} \tag{25}$$

and

$$\|(\cos(t\sqrt{H_\infty})P^\perp_{g_\infty} - K_t)f\|_\infty \lesssim t^{-1}(\|\nabla f\|_{L^1(\mathbb{R}^3)} + \|D^2 f\|_{L^1(\mathbb{R}^3)})$$

Notice that $\|K_t f\|_\infty \lesssim t^{-1}\|f\|_1$, but we will also need to apply K_t to functions not in L^1. More precisely, we shall use that

$$\sup_{x\in\mathbb{R}^3}\Big|\int_{\mathbb{R}^3} K_t(x,y)\,\langle y\rangle^{-3}\,dy\Big| \lesssim \langle t\rangle^{-1}$$

The bounds (23) and (24) are proved in Section 5. Since

$$\phi_a(r,a) = -\frac{1}{4}\,3^{\frac{1}{4}}a^{-\frac{5}{4}}r^{-1} + O(r^{-3}) \qquad \text{as } r\to\infty$$

we conclude that (at least if $a_1, a_2 \in (1/2, 2)$)

$$\phi_a(r, a_1) = (a_2/a_1)^{\frac{5}{4}} \phi_a(r, a_2) + O(|a_1 - a_2| \langle r \rangle^{-3}) \qquad \text{as } r \to \infty$$

and the O-term satisfies symbol-type estimates under differentiation. Hence, returning to (22), we obtain

$$\begin{aligned}
\frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P_{g_\infty}^{\perp} \phi_a(\cdot, a(0)) &= \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P_{g_\infty}^{\perp} [(a(\infty)/a(0))^{\frac{5}{4}} \psi + O(\langle r \rangle^{-3})] \\
&= t(a(\infty)/a(0))^{\frac{5}{4}} \psi + (c_0\, (\psi \otimes \psi) + \mathcal{S}(t)) O(\langle r \rangle^{-3}) \\
&= t(a(\infty)/a(0))^{\frac{5}{4}} \psi + \Omega_1(t)
\end{aligned}$$

In view of (23) and the bounds on K_t,

$$\sup_{t \geq 0} \|\Omega_1(t)\|_\infty < \infty \tag{26}$$

Similarly,

$$\cos((t-s)\sqrt{H_\infty}) P_{g_\infty}^{\perp}\, \phi_a(\cdot, a(s)) = (a(\infty)/a(s))^{\frac{5}{4}} \psi + \Omega_2(t, s)$$

where

$$\Omega_2(t, s) = \cos((t-s)\sqrt{H_\infty}) P_{g_\infty}^{\perp} [\phi_a(\cdot, a(s)) - (a(\infty)/a(s))^{\frac{5}{4}} \psi]$$

is again bounded (and small) since

$$|\phi_a(x, a(s)) - (a(\infty)/a(s))^{\frac{5}{4}} \psi(x)| \lesssim |a(s) - a(\infty)| \langle x \rangle^{-3}$$

Therefore,

$$\begin{aligned}
\tilde{u}(\cdot, t) &= \cos(t\sqrt{H_\infty}) P_{g_\infty}^{\perp} w_1 + \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}} P_{g_\infty}^{\perp} w_2 \\
&\quad + \dot{a}(0)(t(a(\infty)/a(0))^{\frac{5}{4}} \psi + \Omega_1(t)) - \int_0^t [(a(\infty)/a(s))^{\frac{5}{4}} \psi + \Omega_2(t, s)] \dot{a}(s)\, ds \\
&\quad - \int_0^t \frac{\sin((t-s)\sqrt{H_\infty})}{\sqrt{H_\infty}} P_{g_\infty}^{\perp} [(V(\cdot, a(\infty)) - V(\cdot, a(s))) u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\, ds
\end{aligned}$$

which we can further rewrite as

$$\begin{aligned}
\tilde{u}(\cdot, t) &= \cos(t\sqrt{H_\infty}) P_{g_\infty}^{\perp} w_1 + \mathcal{S}(t) P_{g_\infty}^{\perp} w_2 - \int_0^t \cos((t-s)\sqrt{H_\infty}) P_{g_\infty}^{\perp} [\phi_a(\cdot, a(s)) - (a(\infty)/a(s))^{\frac{5}{4}} \psi]\, \dot{a}(s)\, ds \\
&\quad - \int_0^t \mathcal{S}(t-s) P_{g_\infty}^{\perp} [(V(\cdot, a(\infty)) - V(\cdot, a(s))) u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\, ds \\
&\quad + \dot{a}(0)(t(a(\infty)/a(0))^{\frac{5}{4}} \psi + \Omega_1(t)) + \psi \Big\{ c_0 \langle \psi, w_2 \rangle + 4(a(\infty))^{\frac{5}{4}} (a(t)^{-\frac{1}{4}} - a(0)^{-\frac{1}{4}}) \qquad (27) \\
&\qquad - c_0 \int_0^t \langle \psi, (V(\cdot, a(\infty)) - V(\cdot, a(s))) u(\cdot, s) + N(u, \phi(\cdot, a(s))) \rangle\, ds \Big\} \qquad (28)
\end{aligned}$$

The equation for $a(t)$ is now determined from the requirement that (27) and (28) need to decay in time. In particular, this forces $\dot{a}(0) = 0$. The equation for $a(t)$ can be determined by setting the expression in braces equal to zero. However, this would impose compatibility conditions at $t = 0$ which we cannot fulfil without further assumptions on the data (e.g., $\langle\psi, w_2\rangle = 0$). Instead, we only require that it vanishes for times > 1. More precisely, the equation for $a(t)$ is

$$m_1\,\omega(t) + m_2\,t\omega(t) = c_0\langle\psi, w_2\rangle + 4\,a(\infty)^{\frac{5}{4}}(a^{-\frac{1}{4}}(t) - a^{-\frac{1}{4}}(0))$$

$$- c_0\int_0^t \langle\psi, (V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s) + N(u(\cdot, s), \phi(\cdot, a(s)))\rangle\, ds \tag{29}$$

where $\omega(t)$ is a fixed smooth function on $[0,\infty)$ with $\omega(t) = 1$ for $0 \le t \le \frac{1}{2}$, and $\omega(t) = 0$ if $t \ge 1$. The constants m_1 and m_2 will be specified shortly. Despite the fact that ψ only decays like r^{-1}, the scalar product on the right-hand side of (29) is well-defined due to the decay of V, ψ, u. Indeed, we will show below that

$$|u(x,t)| \lesssim \delta\langle x\rangle^{-1}$$

see (40).

The constants m_1 and m_2 are determined as follows. First, setting $t = 0$ leads to the condition

$$m_1 = c_0\langle w_2, \psi\rangle \tag{30}$$

Second, the requirement $\dot a(0) = 0$ implies

$$m_2 = -c_0\,\langle\psi, (V(\cdot, a(\infty)) - V(0))w_1 + N(w_1, \phi(\cdot, a(0)))\rangle \tag{31}$$

Hence the equation for $\tilde u$ now reads

$$\tilde u(\cdot, t) = \cos(t\sqrt{H_\infty})P_{g_\infty}^\perp w_1 + \mathcal{S}(t)P_{g_\infty}^\perp\, w_2 - \int_0^t \dot a(s)\,\cos((t-s)\sqrt{H_\infty})P_{g_\infty}^\perp[\phi_a(\cdot, a(s)) - (a(\infty)/a(s))^{\frac{5}{4}}\psi]\,ds$$

$$- \int_0^t \mathcal{S}(t-s)\,P_{g_\infty}^\perp[(V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\,ds \tag{32}$$

$$+ c_0\big[\langle w_2, \psi\rangle - t\,\langle\psi, (V(\cdot, a(\infty)) - V(0))w_1 + N(w_1, \phi(\cdot, a(0)))\rangle\big]\omega(t)\psi \tag{33}$$

and we need to prove that the right-hand side here is dispersive in a suitable sense. This will require proving the basic collection of estimates for the wave equation, i.e., energy, dispersive, and possibly also Strichartz (for finer results than the ones presented here). Energy is the same as usual: Let $(\partial_{tt} + H_\infty)u = 0$. Then

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\,[\|\partial_t u(\cdot, t)\|_2^2 + \|\sqrt{H_\infty}\,u(\cdot, t)\|_2^2] &= \langle u_{tt}, u_t(t)\rangle + \langle\sqrt{H_\infty}\,u(\cdot, t), \sqrt{H_\infty}\,u_t(t)\rangle \\ &= \langle u_{tt} + H_\infty\,u, u_t(t)\rangle = 0\end{aligned}$$

so that for all $t \ge 0$

$$\|\partial_t u(\cdot, t)\|_2^2 + \|\sqrt{H_\infty}\,u(\cdot, t)\|_2^2 = \text{const}$$

We further remark that for any f, with $H_\infty = -\Delta + V_\infty$,

$$\begin{aligned}\|\sqrt{H_\infty} f\|_2^2 &= \langle H_\infty\, f, f\rangle = \|\nabla f\|_2^2 + \langle V_\infty\, f, f\rangle \\ &\lesssim \|\nabla f\|_2^2 + \|V_\infty\|_{\frac{3}{2}}\|f\|_6^2 \lesssim \|\nabla f\|_2^2\end{aligned} \tag{34}$$

by Sobolev imbedding. Note, however, that the reverse inequality here cannot hold because of the resonance function. However, there is a replacement:

$$\begin{aligned}\|\nabla f\|_2^2 &= \langle -\Delta f, f\rangle = \langle H_\infty\, f, f\rangle - \langle V_\infty\, f, f\rangle \\ &\lesssim \|\sqrt{H_\infty}\, f\|_2^2 + \||V_\infty|^{\frac{1}{2}} f\|_2^2\end{aligned} \tag{35}$$

Similarly,

$$\|D^2 f\|_2 \lesssim \|H_\infty\, f\|_2 + \|V_\infty\, f\|_2, \qquad \|H_\infty f\|_2 \lesssim \|D^2 f\|_2 + \|V_\infty\, f\|_2 \tag{36}$$

The dispersive estimates are proved in Section 5, but the Strichartz estimates are still lacking. It seems that one needs to develop a suitable Littlewood-Paley theorem in the perturbed setting. For the radial case and H_∞ this can be done, albeit only in the range $3/2 < p < 3$, see [Sch3]. This range is optimal due to the resonance at zero energy. On the other hand, we should be looking for a Littlewood-Paley theory not relative to all of H_∞, but only for functions which are orthogonal to the resonance, or more precisely, which correspond to the regular part of the spectral measure. In that case, the full range $1 < p < \infty$ should again be available.

3. The contraction scheme: stability

We will keep the radius $R > 1$ in Theorem 1 fixed. Constants will be allowed to depend on it.

Definition 3. *Let $Y_{R,\delta}$ denote the metric space of $a \in C^1([0,\infty),\mathbb{R}^+)$ and $u \in C([0,\infty), H^2_{\rm rad}(\mathbb{R}^3))$ satisfying the following properties: For all $t \geq 0$,*

$$\|u(\cdot,t)\|_\infty \leq \delta\langle t\rangle^{-1} \tag{37}$$

$$\|\nabla u(\cdot,t)\|_{2+\infty} \leq \delta\langle t\rangle^{-\varepsilon} \tag{38}$$

$$\|\nabla u(\cdot,t)\|_2 + \|D^2 u(\cdot,t)\|_2 \leq \delta \tag{39}$$

$$|u(x,t)| \leq C_1\,\delta\langle x\rangle^{-1} \tag{40}$$

$$|\dot a(t)| \leq \delta\langle t\rangle^{-2} \tag{41}$$

Here $\delta > 0, \varepsilon > 0$ are small[7], and $C_1 > 1$ is some constant that does not depend on δ. In addition, $a(0) = 1$ and $\dot a(0) = 0$.

Let $u_0 := (f_1, f_2) \in B_{\delta_0}(0) \subset \Sigma_0$, where $B_{\delta_0}(0)$ is a δ_0-ball in Σ_0 centered at zero of radius $\delta_0 \ll \delta$, see Theorem 1. Our goal is to find a fixed point for the map

$$\Phi = \Phi_{u_0} : Y_{R,\delta} \to Y_{R,\delta},\ (u,a) \mapsto (v,b)$$

which we now describe (and which depends on the choice of u_0). We intend to show that for a suitable – and unique – choice of $h(u_0; u, a)$ the solution v, b of (with $\psi_0 = \partial_a\phi(\cdot,1)$ and c_0 as in (23))

$$\partial_{tt}v + H(a(\infty))v = -\partial_{tt}\phi(\cdot,b(t)) + (V(\cdot,a(\infty)) - V(\cdot,a(t)))u + N(u,\phi(\cdot,a(t))) \tag{42}$$

$$\begin{aligned} &4\,a(\infty)^{\frac54}(b^{-\frac14}(t)-1) = \\ &= c_0\int_0^t \langle \phi_a(\cdot,a(\infty)), (V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s) + N(u(\cdot,s),\phi(\cdot,a(s)))\rangle\, ds - c_0\langle f_2, \phi_a(\cdot,a(\infty))\rangle \\ &\quad + c_0\big[\langle f_2,\phi_a(\cdot,a(\infty))\rangle - t\,\langle \phi_a(\cdot,a(\infty)), (V(\cdot,a(\infty)) - V(\cdot,a(0)))w_1 + N(w_1,\phi(\cdot,a(0)))\rangle\big]\omega(t) \end{aligned} \tag{43}$$

$$v(0) = f_1 + h(u_0;u,a)g_0 =: w_1, \quad \partial_t v(0) = f_2 \tag{44}$$

$$b(0) = 1, \quad \dot b(0) = 0 \tag{45}$$

satisfies the same bounds (37)–(41) as u, a. It is with this choice of h that the map Φ_{u_0} is being defined. This system should be compared to (29) and (32). The choice of h will be based on the stability condition (15).

Proposition 4. *There exists $0 < \delta_0 \ll \delta$ small[8] so that for any $u_0 = (f_1, f_2) \in B_{\delta_0}(0) \subset \Sigma_0$ the following holds: For any $(u,a) \in Y_{R,\delta}$ there is a unique choice of $h(u_0;u,a)$ so that $\Phi_{u_0}(u,a) \in Y_{R,\delta}$. Moreover,*

$$|h(u_0;u,a)| \lesssim \delta^2$$

as well as

$$|h(u_0;u,a) - h(\tilde u_0;u,a)| \lesssim \delta\|u_0 - \tilde u_0\|_{H^3\times H^2} \tag{46}$$

for all $u_0, \tilde u_0 \in B_{\delta_0}(0)$ and $(u,a) \in Y_{R,\delta}$. In particular,

$$\|\Phi_{u_0}(u,a) - \Phi_{\tilde u_0}(u,a)\|_{Y_{R,\delta}} \lesssim \|u_0 - \tilde u_0\|_{H^3\times H^2} \tag{47}$$

for all $u_0, \tilde u_0 \in B_{\delta_0}(0)$ and $(u,a) \in Y_{R,\delta}$.

---

[7] $\varepsilon > 0$ is a small positive constant that is fixed once and for all, whereas $\delta > 0$ is small but arbitrary.

[8] This means that $\delta_0 = c\delta$ where c is a small absolute constant.

Proof. We begin by checking that (43) reproduces the decay of $\dot a$ under these assumptions. In view of (37) and (40), as well as the definition of ϕ and V,

$$|(V(\cdot,a(\infty))-V(\cdot,a(t)))u(x,t)|\lesssim \delta^2\langle t\rangle^{-2}\langle x\rangle^{-4} \tag{48}$$

as well as

$$|N(u(\cdot,t),\phi(\cdot,a(t)))|\lesssim C_1^3\,\delta^2\langle t\rangle^{-2}\,\langle x\rangle^{-3} \tag{49}$$

Let us first discuss the solvability of (43). The equation (in this proof $\psi=\phi_a(\cdot,a(\infty))$)

$$\begin{aligned}&a(\infty)^{\frac54}(b^{-\frac14}(\infty)-1)\\&=\frac{c_0}{4}\int_0^\infty\langle\phi_a(\cdot,a(\infty)),(V(\cdot,a(\infty))-V(\cdot,a(s)))u(\cdot,s)+N(u(\cdot,s),\phi(\cdot,a(s)))\rangle\,ds-c_0\langle f_2,\psi\rangle\end{aligned}$$

has a unique solution $b(\infty)=1+O(\delta)$ since the right-hand side here is $O(\delta)$. Hence, (43) has a well-defined solution $b(t)$ for all $t\ge 0$ with the property that $|b(t)-1|\lesssim\delta$ for all $t\ge0$. By construction, $b(0)=1$ and $\dot b(0)=0$. Second,

$$\begin{aligned}\dot b(t)=-c_0\,(b(t)/a(\infty))^{\frac54}\Big\{&\langle\psi,(V(\cdot,a(\infty))-V(\cdot,a(t)))u(\cdot,t)+N(u(\cdot,t),\phi(\cdot,a(t)))\rangle\\&-\big[\langle f_2,\psi\rangle-t\,\langle\psi,(V(\cdot,a(\infty))-V(\cdot,a(0)))w_1+N(w_1,\phi(\cdot,a(0)))\rangle\big]\dot\omega(t)\\&+\langle\psi,(V(\cdot,a(\infty))-V(\cdot,a(0)))w_1+N(w_1,\phi(\cdot,a(0)))\rangle\omega(t)\Big\}\end{aligned} \tag{50}$$

The bounds (48), (49) in conjunction with (50) reproduce (41) for small δ. Observe that on the support of ω the equation for $\dot b$ contains the unknown h. However, only a very crude bound $|h|\lesssim\delta$, say, is required to obtain the estimate we need. We will comment on this issue later, when we solve for h. We remark that $|\dot b(t)|\lesssim\delta^2$ outside of the support of $\dot\omega$. It is because of $|\langle f_2,\psi\rangle|\lesssim\delta$ that we only obtain δ on the support of $\dot\omega$. In passing, we also remark that we cannot replace t^{-1} by $t^{-\alpha}$ with $\alpha<1$ in (37) since that would mean that at best $|\dot a(t)|\lesssim\delta\langle t\rangle^{-2\alpha}$. However, that rate of decay cannot be reproduced from (29) because of the term $(V(\cdot,a(\infty))-V(\cdot,a(s)))u(\cdot,s)$. Indeed, that term would contribute $s^{1-3\alpha}$ which is worse than $s^{-2\alpha}$ because of $\alpha<1$. So we work with t^{-1} for the decay of $\|u(\cdot,t)\|_\infty$ (which is the best possible rate of decay).
Next, we turn to v. As in Section 2 we write

$$v(t)=(2k_\infty)^{-\frac12}(n_+(t)+n_-(t))g(\cdot,a(\infty))+\tilde v(t) \tag{51}$$

with $\tilde v(t)=P^\perp_{g(\cdot,a(\infty))}\,v(t)$ (recall that $k_\infty=k(a(\infty))=a(\infty)k_0$). The finite-dimensional part satisfies, cf. (12),

$$\dot n_\pm(t)\mp k_\infty n_\pm(t)=\mp\big\langle-\partial_{tt}\phi(\cdot,b(t))+(V(\cdot,a(\infty))-V(\cdot,a(t)))u+N(u,\phi(\cdot,a(t))),g(\cdot,a(\infty))\big\rangle=:\mp F(t) \tag{52}$$

whereas the remaining part $\tilde v$ satisfies (with $H=-\Delta+V(\cdot,a(\infty))$ throughout this proof)

$$\begin{aligned}\tilde v(t)&=\cos(t\sqrt H)P^\perp_{g(\cdot,a(\infty))}\,w_1+\mathcal S(t)P^\perp_{g(\cdot,a(\infty))}\,f_2\\&-\int_0^t\dot b(s)\,\cos((t-s)\sqrt H)P^\perp_{g(\cdot,a(\infty))}\,[\phi_b(\cdot,b(s))-(a(\infty)/b(s))^{\frac54}\psi]\,ds\\&-\int_0^t\mathcal S(t-s)\,P^\perp_{g(\cdot,a(\infty))}[(V(\cdot,a(\infty))-V(\cdot,a(s)))u(\cdot,s)+N(u,\phi(\cdot,a(s)))]\,ds\\&+c_0\big[\langle f_2,\psi\rangle-t\,\langle\psi,(V(\cdot,a(\infty))-V(\cdot,a(0)))w_1+N(w_1,\phi(\cdot,a(0)))\rangle\big]\omega(t)\psi\end{aligned} \tag{53}$$

cf. (32). We remark that the final term on the right-hand side, which is a multiple of ψ of size $\ll\delta$, does not affect any of the estimates (37)–(40). Hence, we will ignore it from now on.

In order to avoid exponential growth of $n_+(t)$ it is both necessary and sufficient that

$$0 = n_+(0) + \int_0^\infty e^{-tk_\infty} F(t)\, dt \tag{54}$$

see (15). We need to transform (54) into a condition on (w_1, f_2). To do so, note that (51) and (44) imply that

$$n_+(0) + n_-(0) = (2k_\infty)^{\frac{1}{2}} \langle w_1, g(\cdot, a(\infty))\rangle$$
$$\dot n_+(0) + \dot n_-(0) = (2k_\infty)^{\frac{1}{2}} \langle f_2, g(\cdot, a(\infty))\rangle$$

whereas we deduce from (52) that

$$\dot n_+(0) + \dot n_-(0) = k_\infty(n_+(0) - n_-(0))$$

The conclusion is that

$$\begin{aligned} n_+(0) &= (2k_\infty)^{-\frac{1}{2}} \big[\langle g(\cdot, a(\infty)), k_\infty f_1 + f_2\rangle + k_\infty h(u_0; u, a)\langle g_0, g(\cdot, a(\infty))\rangle\big] \\ &= k_\infty \int_0^\infty \dot b(t)\, e^{-tk_\infty} \langle \phi_b(\cdot, b(t)), g(\cdot, a(\infty))\rangle\, dt \\ &\quad - \int_0^\infty e^{-tk_\infty} \big\langle (V(\cdot, a(\infty)) - V(\cdot, a(t)))u + N(u, \phi(\cdot, a(t))), g(\cdot, a(\infty))\big\rangle\, dt \end{aligned} \tag{55}$$

Observe that $(f_1, f_2) \in \Sigma_0$ implies that

$$|\langle g(\cdot, a(\infty)), k_\infty f_1 + f_2\rangle| \le |\langle g(\cdot, a(\infty)) - g_0, k_0 f_1 + f_2\rangle| + |\langle g(\cdot, a(\infty)), (k_\infty - k_0) f_1\rangle| \lesssim \delta^2$$

and

$$\langle g_0, g(\cdot, a(\infty))\rangle = 1 + O(\delta), \qquad |\langle \phi_b(\cdot, b(t)), g(\cdot, a(\infty))\rangle| = |\langle \phi_b(\cdot, b(t)) - \phi_a(\cdot, a(\infty)), g(\cdot, a(\infty))\rangle| \lesssim \delta$$

As mentioned above, $\dot b$ depends on $h(u_0; u, a)$ through w_1. Thus, $h(u_0; u, a)$ also appears on the right-hand side of (55), and not just on the left. However, this dependence occurs either with a small coefficient (in fact, δ), or to higher order. Hence, we can still solve for $h(u_0; u, a)$ by means of the implicit function theorem.

In view of our assumptions on u, a and the bounds we proved on b, it follows for small δ that there exists a unique choice of $h(u_0; u, a)$ so that (55) is satisfied. Moreover,

$$|h(u_0; u, a)| \lesssim \delta^2$$

as well as

$$|n_+(t)| + \delta |n_-(t)| \lesssim \delta^2 \langle t\rangle^{-2} \tag{56}$$

see (14). To bound $n_-(t)$ we use that $|n_+(0)| \lesssim \delta^2$ which implies that $|n_-(0)| \lesssim \delta$. It is also an easy matter to check that (46) holds, which we leave to the reader.

We now turn to estimating $\tilde v$. First, let $0 < t \lesssim 1$. Then, using $\|H^{-\frac{1}{2}} \sin(t\sqrt{H})\ P_c\|_{2\to 2} \le t$, we read off from the equation for $\tilde v$, viz.

$$\begin{aligned} \tilde v(t) &= \cos(t\sqrt{H}) P^\perp_{g(\cdot, a(\infty))}\, w_1 + \Big[\frac{\sin(t\sqrt{H})}{\sqrt{H}} - c_0(\psi \otimes \psi)\Big] P^\perp_{g(\cdot, a(\infty))}\, f_2 \\ &\quad - \int_0^t \dot b(s)\, \cos((t-s)\sqrt{H}) P^\perp_{g(\cdot, a(\infty))} [\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}} \psi]\, ds \\ &\quad - \int_0^t \Big[\frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} - c_0(\psi \otimes \psi)\Big]\, P^\perp_{g(\cdot, a(\infty))} [(V(\cdot, a(\infty)) - V(\cdot, a(s))) u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\, ds \\ &\quad + c_0 \big[\langle f_2, \psi\rangle - t\, \langle \psi, (V(\cdot, a(\infty)) - V(\cdot, a(0)) w_1 + N(w_1, \phi(\cdot, a(0)))\rangle\big] \omega(t) \psi \end{aligned}$$

that

$$\tilde v(t) = w(t) + \psi\,\eta(t), \quad \|w(t)\|_2 + |\eta(t)| \ll \delta$$

for $0 < t \lesssim 1$. In particular,

$$\sup_{0<t\lesssim 1} \|\tilde v(t)\|_{2+\infty} \ll \delta \tag{57}$$

Later we will show that (57), combined with the equation for $\tilde v$, yields

$$\|\nabla \tilde v(t)\|_2 + \|D^2 \tilde v(t)\|_2 \ll \delta$$

for all $0 < t < 1$. Since $\nabla\psi \in L^2$ and $D^2\psi \in L^2$ we conclude that in fact $w(t) \in H^2$ for small times. Therefore,

$$\|\tilde v(t)\|_\infty \lesssim \|w(t)\|_\infty + |\eta(t)| \lesssim \|w(t)\|_{H^2} + |\eta(t)| \ll \delta \quad \forall\, 0 < t \lesssim 1$$

by Sobolev imbedding. Hence, it suffices to consider $t \gg 1$. Using the dispersive decay of $\cos((t-s)\sqrt{H})$ in the integral involving $\dot b$, we obtain the bound

$$\delta^2 \int_0^t \langle t-s\rangle^{-1} \langle s\rangle^{-2}\, ds \lesssim \delta^2 t^{-1}$$

see Remark 20. Next, the dispersive bound on $\mathcal{S}(t-s)$ yields

$$\delta^2 \int_0^{t-t^{-10}} (t-s)^{-1} \langle s\rangle^{-1-\varepsilon/2}\, ds \lesssim \delta^2 t^{-1}$$

We are using here that

$$\|(V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s)\|_{W^{1,1}(\mathbb{R}^3)} \lesssim \delta^2 \langle s\rangle^{-1-\varepsilon}$$

as well as

$$\|N(u, \phi(\cdot, a(s)))\|_{W^{1,1}(\mathbb{R}^3)} \lesssim \delta^2 \langle s\rangle^{-1-\varepsilon/2}$$

This latter bound in turn reduces to four terms of which we consider only the extreme cases $u^2\phi^3$ and u^5 (the fact that there is $\varepsilon/2$ and not ε is due to a small loss through interpolation). Since $\phi^3 \in L^p$ for all $p > 1$ and because of (38), we have

$$\|u^2(s)\phi^3\|_1 + \|\nabla u^2(s)\phi^3\|_1 \lesssim \delta^2 \langle s\rangle^{-1-\varepsilon/2}$$

as claimed. The u^5 term requires more care due to the possible growth of $\|u(\cdot, s)\|_2$ in time. However, in view of (40) we have $u^3 \in L^p$ with $p > 1$ so that the same arguments apply as in the case of ϕ^3. For the integral over $[t - t^{-10}, t]$ we use Sobolev imbedding. More precisely, we write

$$\int_{t-t^{-10}}^t \mathcal{S}(t-s)\, P^\perp_{g(\cdot, a(\infty))} [(V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\, ds$$

$$= \int_{t-t^{-10}}^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}}\, P^\perp_{g(\cdot, a(\infty))} [(V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\, ds \tag{58}$$

$$- c_0 \int_{t-t^{-10}}^t (\psi \otimes \psi)\, [(V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\, ds \tag{59}$$

The final integral (59) is estimated directly in L^∞, leading to a bound of $\delta^2 t^{-12}$. The one in (58) is estimated in H^2 by means of the same L^2 based arguments as above, leading to a contribution of $\delta^2 t^{-22}$. The conclusion is that (37) is regained for $\tilde v$. To extend to v, simply use the bounds (56). Next, we deal with the L^2 estimates (39). In view of (35),

$$\|\nabla \tilde v\|_2 \lesssim \|\sqrt{H}\tilde v\|_2 + \|V\|_1 \|\tilde v\|_\infty \lesssim \|\sqrt{H}\tilde v\|_2 + \delta\langle t\rangle^{-1}$$

at least for $t > 1$. If $0 < t < 1$, then we use $\|\tilde v(t)\|_{2+\infty} \ll \delta$, see (57) instead of $\|\tilde v(t)\|_\infty$ (since we used the energy bound above to control $\|\tilde v(t)\|_\infty$ for small times and thus have to avoid going in circles). Bounding $\|\sqrt{H}\tilde v(t)\|_2$ amounts to checking that

$$\int_0^t |\dot b(s)|\, \|[\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac54}\psi]\|_{H^1}\, ds + \int_0^t \|[(V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\|_2\, ds$$
$$\lesssim \delta^2 \int_0^\infty \langle s\rangle^{-2}\, ds \lesssim \delta^2$$

Here we used (34) as well as $\sqrt{H}\mathcal{S}(t) = \mathcal{S}(t)\sqrt{H} = \sin(t\sqrt{H})$ which is bounded on L^2. This relation is of course a consequence of $\sqrt{H}\psi = 0$. However, since ψ does not lie in $\mathrm{Dom}(\sqrt{H}) = W^{1,2}(\mathbb{R}^3)$, this latter claim requires some care and needs to be interpreted weakly. More precisely, we claim that

$$\sqrt{H}(\chi(\cdot/R)\psi) \rightharpoonup 0$$

in the sense of weak convergence on $L^2(\mathbb{R}^3)$ as $R \to \infty$ (here χ is a smooth cut-off around zero). First, note that

$$\|\sqrt{H}(\chi(\cdot/R)\psi)\|_2 \lesssim \|\nabla(\chi(\cdot/R)\psi)\|_2 + \|\,|V|^{\frac12}\,(\chi(\cdot/R)\psi)\|_2 < \infty$$

uniformly in $R > 1$. Consequently, it suffices to check the weak convergence against a family of functions which is dense in L^2. One such family is $\mathrm{Ran}(\sqrt{H})$. This is dense in L^2 since $\ker(\sqrt{H}) = \{0\}$ and since $\sqrt{H}$ restricted to $\{g\}^\perp$ is self-adjoint. Now, by the explicit decay of ψ and $H\psi = 0$ it follows that for any $f \in \mathrm{Dom}(\sqrt{H})$

$$\langle \sqrt{H}(\chi(\cdot/R)\psi), \sqrt{H}f\rangle = \langle H(\chi(\cdot/R)\psi), f\rangle \to 0$$

as $R \to \infty$. Hence the claim.

For $\|D^2\tilde v\|_2$ we face an additional derivative on the right-hand side. More precisely, using (36) as well as (34), the main estimate is

$$\int_0^t |\dot b(s)|\, \|P^\perp_{g(\cdot,a(\infty))}[\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac54}\psi]\|_{H^2}\, ds$$
$$+ \int_0^t \|\nabla P^\perp_{g(\cdot,a(\infty))}[(V(\cdot, a(\infty)) - V(\cdot, a(s)))u(\cdot, s) + N(u, \phi(\cdot, a(s)))]\|_{L^2}\, ds$$
$$\lesssim \delta^2 \int_0^\infty \langle s\rangle^{-2}\, ds + \delta^2 \int_0^\infty \langle s\rangle^{-1} s^{-\varepsilon}\, ds \lesssim \delta^2$$

Hence, (39) follows. Finally, for (38), we of course use the dispersive estimate as before, but with one extra derivative. More precisely, we invoke the estimates

$$\|\nabla \mathcal{S}(t) P_c f\|_\infty \lesssim t^{-1}\|f\|_{W^{2,1}(\mathbb{R}^3)} \tag{60}$$

$$\|[\nabla \cos(t\sqrt{H})P_c - K_t]f\|_\infty \lesssim t^{-1} \sum_{1\le|\alpha|\le3} \|D^\alpha f\|_{L^1(\mathbb{R}^3)} \tag{61}$$

where the kernel K_t satisfies (25). These inequalities are obtained by passing the gradient through the various expansions in Section 5. Doing so leads to commutators between the gradient and the potential, which requires smoothness of the potential — which we have in our case (it is also convenient that V has a definite sign). Hence these commutators are harmless. More details will be presented in Section 5, and we now use (60), (61). We start with the initial data f_2 in (53) (we leave the analogous estimation of w_1 to the reader). If $t > 1$, then

$$\|\nabla \mathcal{S}(t) P^\perp_{g(\cdot,a(\infty))}\, f_2\|_{2+\infty} \lesssim \|\nabla \mathcal{S}(t) P^\perp_{g(\cdot,a(\infty))}\, f_2\|_\infty \lesssim t^{-1}\|f_2\|_{W^{2,1}(\mathbb{R}^3)} \lesssim t^{-1}\|f_2\|_{H^2}$$

by the compact support assumption on the data. Next, if $0 < t < 1$, then

$$\begin{aligned}
&\|\nabla \mathcal{S}(t) P^{\perp}_{g(\cdot,a(\infty))} f_2\|_{2+\infty} \lesssim \|\nabla \mathcal{S}(t) P^{\perp}_{g(\cdot,a(\infty))} f_2\|_2 \\
&\lesssim \|\sqrt{H}\mathcal{S}(t) P^{\perp}_{g(\cdot,a(\infty))} f_2\|_2 + \||V|^{\frac12} \mathcal{S}(t) P^{\perp}_{g(\cdot,a(\infty))} f_2\|_2 \\
&\lesssim \|\sin(t\sqrt{H}) P^{\perp}_{g(\cdot,a(\infty))} f_2\|_2 + \||V|^{\frac12} (H^{-\frac12}\sin(t\sqrt{H}) - c_0(\psi\otimes\psi)) P^{\perp}_{g(\cdot,a(\infty))} f_2\|_2 \\
&\lesssim \|f_2\|_2
\end{aligned}$$

The $\cos((t-s)\sqrt{H})$ term is treated basically in the same way as in the dispersive estimate for u, so it will suffice to bound the $\mathcal{S}(t-s)$ integral. First, suppose that $0 < t \lesssim 1$. In that case we use only L^2:

$$\begin{aligned}
&\Big\|\nabla \int_0^t \mathcal{S}(t-s) P^{\perp}_{g(\cdot,a(\infty))} [(V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s) + N(u,\phi(\cdot,a(s)))]\, ds\Big\|_2 \\
&\lesssim \Big\|\sqrt{H} \int_0^t \mathcal{S}(t-s) P^{\perp}_{g(\cdot,a(\infty))} [(V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s) + N(u,\phi(\cdot,a(s)))]\, ds\Big\|_2 \\
&+ \Big\||V|^{\frac12} \int_0^t \mathcal{S}(t-s) P^{\perp}_{g(\cdot,a(\infty))} [(V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s) + N(u,\phi(\cdot,a(s)))]\, ds\Big\|_2
\end{aligned}$$

This can be further bounded by

$$\begin{aligned}
&\lesssim \Big\| \int_0^t \sin((t-s)\sqrt{H}) P^{\perp}_{g(\cdot,a(\infty))} [(V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s) + N(u,\phi(\cdot,a(s)))]\, ds\Big\|_2 \\
&+ \Big\||V|^{\frac12} \int_0^t (\sin((t-s)\sqrt{H}) - c_0\psi\otimes\psi) P^{\perp}_{g(\cdot,a(\infty))} [(V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s) + N(u,\phi(\cdot,a(s)))]\, ds\Big\|_2 \\
&\lesssim \delta^2
\end{aligned}$$

Next, let $t \gg 1$. To bound the integral over $[0, t - t^{-10}]$ we use the estimates

$$\|D^2(V(\cdot,a(\infty)) - V(\cdot,a(s)))u(\cdot,s)\|_{L^1(\mathbb{R}^3)} + \|D^2 N(u,\phi(\cdot,a(s)))\|_{L^1(\mathbb{R}^3)} \lesssim \delta^2 \langle s\rangle^{-3\varepsilon/2}$$

We discuss the contribution by u^5:

$$\||Du|^2 u^3(s)\|_1 + \|u^4 D^2 u(\cdot,s)\|_1 \lesssim \|Du(\cdot,s)\|^2_{2+\infty} \|u^3(s)\|_{1\cap\infty} + \|u^3(s)\|_2 \|D^2 u(\cdot,s)\|_2 \|u(\cdot,s)\|_\infty \lesssim \delta^5 \langle s\rangle^{-3\varepsilon/2}$$

Strictly speaking, (40) only gives $u^3 \in L^p$ for $p > 1$. This, however, suffices since we can lower the ∞ in $Du \in L^2 + L^\infty$ by interpolation (this explains the loss in going from 2ε to $3\varepsilon/2$). Hence, we are dealing with the integral

$$\delta^2 \int_0^{t-t^{-10}} (t-s)^{-1} \langle s\rangle^{-3\varepsilon/2}\, ds \lesssim \delta^2 t^{-\varepsilon}$$

as desired. Finally, the contribution of $[t - t^{-10}, t]$ is dealt with in basically the same way as the case of small times. We skip the details.
It remains to prove the important decay estimate (40). In view of (37) it will suffice to consider the case $|x| > A\langle t\rangle$ with A large depending on R. We write the equation for v in the form

$$\begin{aligned}
&\Box v = -V(\cdot,a(\infty))v - \partial_t(\dot{b}\phi_b(\cdot,b(t))) + (V(\cdot,a(\infty)) - V(\cdot,a(t)))u + N(u,\phi(\cdot,a(t))) \\
&v(0) = w_1, \qquad \partial_t v(0) = f_2
\end{aligned} \tag{62}$$

Then, with $S_0(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}$, we have

$$u(x,t) = S_0(t)u_0(x) + \int_0^t S_0(t-s)\big[- V(\cdot, a(\infty))v(\cdot,s) - \partial_s(\dot b\phi_b(\cdot, b(s))) \\ + (V(\cdot,a(\infty)) - V(\cdot,a(s))u(\cdot,s) + N(u,\phi(\cdot,a(s)))\big]\, ds$$

By our assumption on x, $S_0(t)u_0(x) = 0$. Due to the nature of S_0 as an averaging operator, and our assumption on x, we obtain the bound

$$|S_0(t-s)\big[- V(\cdot,a(s))v(\cdot,s) + (V(\cdot,a(\infty)) - V(\cdot,a(s))u(\cdot,s) + N(u,\phi(\cdot,a(s)))\big]|(x) \\ \lesssim (t-s)(\delta\langle x\rangle^{-4}\langle s\rangle^{-1} + \delta^2\langle s\rangle^{-2}\langle x\rangle^{-4} + C_1^5\,\langle x\rangle^{-5}\delta^2)$$

Here we used the estimate $\|v(\cdot,s)\|_\infty \le \delta$, which we derived above independently of the point-wise decay on v being proven now. Thus,

$$\Big|\int_0^t S_0(t-s)\big[- V(\cdot,a(s))v(\cdot,s) + (V(\cdot,a(\infty)) - V(\cdot,a(s))u(\cdot,s) + N(u,\phi(\cdot,a(s)))\big]\, ds\Big|(x) \\ \lesssim C_1\,\delta t^2\langle x\rangle^{-4} + C_1^5\,\delta^2 t^2\langle x\rangle^{-5} \\ \lesssim C_1\,\delta t^2(A\langle t\rangle)^{-3}\langle x\rangle^{-1} + C_1^5\,\delta^2 t^2(A\langle t\rangle)^{-4}\langle x\rangle^{-1} \\ \ll C_1\,\delta\langle x\rangle^{-1}$$

provided A (and thus C_1) is large and δ is small. Furthermore, by the nature of $\cos((t-s)\sqrt{-\Delta})$,

$$\Big|\int_0^t \dot b(s)\,\cos((t-s)\sqrt{-\Delta})\,\phi_b(\cdot,b(s))\,ds\Big|(x) \lesssim \int_0^t \delta\langle s\rangle^{-2}\langle x\rangle^{-1}\,ds \lesssim \delta\langle x\rangle^{-1} \ll C_1\,\delta\langle x\rangle^{-1}$$

where we have again exploited that $|x|$ is large relative to t. It is also important to note that the bound on $\dot b$ does not deteriorate when C_1 becomes large, provided we make δ small. Thus, the conclusion is that

$$|u(x,t)| \le C_1\,\delta\langle x\rangle^{-1}$$

as desired. The proposition is proved. □

4. The contraction scheme: the fixed-point

We now show that the map Φ_{u_0}, which we constructed in the previous section, has a fixed-point. We will show that Φ_{u_0} contracts relative to the following distance:

Definition 5. *For any two points $p^{(1)} = (u^{(1)}, a^{(1)}), p^{(2)} = (u^{(2)}, a^{(2)}) \in Y_{R,\delta}$ we define their distance to be*

$$d(p^{(1)},p^{(2)}) := \sup_{t\ge 0}\big[\|Du^{(1)}(\cdot,t) - Du^{(2)}(\cdot,t)\|_2 + \langle t\rangle^{\varepsilon}\|u^{(1)}(\cdot,t) - u^{(2)}(\cdot,t)\|_{2+\infty}\big] \\ + \sup_{t\ge 0}\langle t\rangle^{1+\varepsilon}|\dot a^{(1)}(t) - \dot a^{(2)}(t)| \tag{63}$$

where $\varepsilon > 0$ is small and fixed (and no larger than the one in (38)*).*

We record a simple technical fact which will need in the main argument. Throughout this section, we will make use of the bounds (37) – (40) without further mention.

Lemma 6. *Fix $p^{(1)} = (u^{(1)}, a^{(1)}), p^{(2)} = (u^{(2)}, a^{(2)}) \in Y_{R,\delta}$ and define*

$$W(\cdot,s) := [V(\cdot,a^{(1)}(\infty)) - V(\cdot,a^{(1)}(s))] - [V(\cdot,a^{(2)}(\infty)) - V(\cdot,a^{(2)}(s))]$$

where $V(x,a) = -5\phi(x,a)^4$. *Then*

$$\sup_{s\ge 0} |a^{(1)}(s) - a^{(2)}(s)| \lesssim d(p^{(1)}, p^{(2)}) \tag{64}$$

$$|W(x,s)| \lesssim \langle x\rangle^{-4}\langle s\rangle^{-\varepsilon} d(p^{(1)}, p^{(2)}) \tag{65}$$

for all $x \in \mathbb{R}^3$, $s \ge 0$.

Proof. First, for all $s \ge 0$,

$$|a^{(1)}(s) - a^{(2)}(s)| \le \int_0^s \langle\sigma\rangle^{1+\varepsilon} |\dot a^{(1)}(\sigma) - \dot a^{(2)}(\sigma)|\, \langle\sigma\rangle^{-1-\varepsilon}\, d\sigma \lesssim d(p^{(1)}, p^{(2)})$$

Second,

$$\begin{aligned} V(\cdot, a^{(1)}(s)) - V(\cdot, a^{(2)}(s)) &= \int_0^1 \frac{d}{d\tau} V(\cdot, \tau a^{(1)}(s) + (1-\tau) a^{(2)}(s))\, d\tau \\ &= \int_0^1 \partial_a V(\cdot, \tau a^{(1)}(s) + (1-\tau) a^{(2)}(s))\, d\tau\, (a^{(1)}(s) - a^{(2)}(s)) \end{aligned}$$

Therefore,

$$\begin{aligned} W(\cdot, s) = &\int_0^1 [\partial_a V(\cdot, \tau a^{(1)}(\infty) + (1-\tau) a^{(2)}(\infty)) - \partial_a V(\cdot, \tau a^{(1)}(s) + (1-\tau) a^{(2)}(s))]\, d\tau\, (a^{(1)}(s) - a^{(2)}(s)) \\ &+ \int_0^1 \partial_a V(\cdot, \tau a^{(1)}(\infty) + (1-\tau) a^{(2)}(\infty))\, d\tau \int_s^\infty (\dot a^{(1)}(\sigma) - \dot a^{(2)}(\sigma))\, d\sigma \end{aligned}$$

which implies that

$$\begin{aligned} |W(x,s)| &\lesssim \delta \langle x\rangle^{-4}\langle s\rangle^{-1} |a^{(1)}(s) - a^{(2)}(s)| + \langle x\rangle^{-4} \int_s^\infty |\dot a^{(1)}(\sigma) - \dot a^{(2)}(\sigma)|\, d\sigma \\ &\lesssim \langle x\rangle^{-4}\langle s\rangle^{-\varepsilon} d(p^{(1)}, p^{(2)}) \end{aligned}$$

The lemma follows. □

The main result of this section is

Proposition 7. *Let* $u_0 = (f_1, f_2) \in B_{\delta_0}(0) \subset \Sigma_0$ *with some* $\delta_0 \ll \delta$. *Then there exists a unique fixed-point* $(u,a) \in Y_{R,\delta}$ *of* Φ_{u_0}. *Moreover, if we define* $h(u_0) := h(u_0; u, a)$ *with this choice of* (u,a), *then*

$$|h(u_0)| \lesssim \|u_0\|_{H^3\times H^2}^2, \qquad |h(u_0) - h(\tilde u_0)| \lesssim \delta \|u_0 - \tilde u_0\|_{H^3\times H^2}$$

for all $u_0, \tilde u_0 \in B_{\delta_0}(0)$.

Proof. Let $p^{(1)} = (u^{(1)}, a^{(1)}) \in Y_{R,\delta}$ and $p^{(2)} = (u^{(2)}, a^{(2)}) \in Y_{R,\delta}$. We set

$$q^{(1)} = (v^{(1)}, b^{(1)}) = \Phi_{u_0}(u^{(1)}, a^{(1)}), \qquad q^{(2)} = (v^{(2)}, b^{(2)}) = \Phi_{u_0}(u^{(2)}, a^{(2)})$$

as well as

$$h^{(1)} = h(u_0; u^{(1)}, a^{(1)}), \qquad h^{(2)} = h(u_0; u^{(2)}, a^{(2)})$$

In other words, we have the following equations: First, the equation for $v^{(j)}$

$$\begin{aligned} &\partial_{tt} v^{(j)} + H(a^{(j)}(\infty)) v^{(j)} \\ &= -\partial_{tt}\phi(\cdot, b^{(j)}(t)) + (V(\cdot, a^{(j)}(\infty)) - V(\cdot, a^{(j)}(t))) u^{(j)} + N(u^{(j)}, \phi(\cdot, a^{(j)}(t))) \end{aligned}$$

Next, the equation for $b^{(j)}$

$$\begin{aligned}0 &= 4\,(a^{(j)}(\infty))^{\frac{5}{4}}((b^{(j)})^{-\frac{1}{4}}(t)-1) - c_0\langle f_2, \phi_a(\cdot, a^{(j)}(\infty))\rangle \\ &\quad - c_0\int_0^t \langle \phi_a(\cdot, a^{(j)}(\infty)), (V(\cdot, a^{(j)}(\infty)) - V(\cdot, a^{(j)}(s)))u^{(j)}(\cdot, s) + N(u^{(j)}(\cdot, s), \phi(\cdot, a^{(j)}(s)))\rangle\, ds \\ &\quad + c_0\big[\langle f_2, \phi_a(\cdot, a^{(j)}(\infty))\rangle - t\,\langle \phi_a(\cdot, a^{(j)}(\infty)), (V(\cdot, a^{(j)}(\infty)) - V(\cdot, a^{(j)}(0)))w_1^{(j)} + N(w_1^{(j)}, \phi(\cdot, a^{(j)}(0)))\rangle\big]\omega(t)\end{aligned}$$

and finally the initial conditions:

$$\begin{aligned}&v^{(j)}(0) = f_1 + h^{(j)}g_0 =: w_1^{(j)}, \quad \partial_t v^{(j)}(0) = f_2 \\ &b^{(j)}(0) = 1, \quad \dot{b}^{(j)}(0) = 0\end{aligned}$$

for $j = 1, 2$. In order to compare $\dot{b}^{(1)}(t)$ with $\dot{b}^{(2)}(t)$, we need to following estimate

$$\begin{aligned}&\Big|\langle \phi_a(\cdot, a^{(1)}(\infty)), (V(\cdot, a^{(1)}(\infty)) - V(\cdot, a^{(1)}(t)))u^{(1)}(\cdot, t) + N(u^{(1)}(\cdot, t), \phi(\cdot, a^{(1)}(t)))\rangle \\ &- \langle \phi_a(\cdot, a^{(2)}(\infty)), (V(\cdot, a^{(2)}(\infty)) - V(\cdot, a^{(2)}(t)))u^{(2)}(\cdot, t) + N(u^{(2)}(\cdot, t), \phi(\cdot, a^{(2)}(t)))\rangle\Big| \\ &\lesssim \Big|\langle \phi_a(\cdot, a^{(1)}(\infty)) - \phi_a(\cdot, a^{(2)}(\infty)), (V(\cdot, a^{(1)}(\infty)) - V(\cdot, a^{(1)}(t)))u^{(1)}(\cdot, t) + N(u^{(1)}(\cdot, t), \phi(\cdot, a^{(1)}(t)))\rangle\Big| \\ &\quad + \big|\langle \phi_a(\cdot, a^{(2)}(\infty)), W(\cdot, t)u^{(1)}(\cdot, t)\rangle\big| + \big|\langle \phi_a(\cdot, a^{(2)}(\infty)), (V(\cdot, a^{(2)}(\infty)) - V(\cdot, a^{(2)}(t)))(u^{(1)}(\cdot, t) - u^{(2)}(\cdot, t))\rangle\big| \\ &+ \big|\langle \phi_a(\cdot, a^{(2)}(\infty)), N(u^{(1)}(\cdot, t), \phi(\cdot, a^{(1)}(t))) - N(u^{(2)}(\cdot, t), \phi(\cdot, a^{(2)}(t)))\rangle\big| \\ &=: (A + B + C + D)(t)\end{aligned}$$

where W is as in Lemma 6. By (64),

$$|\phi_a(\cdot, a^{(1)}(\infty)) - \phi_a(\cdot, a^{(2)}(\infty))| \lesssim \langle x\rangle^{-1} d(p^{(1)}, p^{(2)})$$

Hence,

$$A \lesssim \delta^2\langle t\rangle^{-2} d(p^{(1)}, p^{(2)})$$

Furthermore, in view of (65),

$$B \lesssim \delta\langle t\rangle^{-1-\varepsilon} d(p^{(1)}, p^{(2)})$$

as well as

$$C \lesssim \delta\langle t\rangle^{-1-\varepsilon} d(p^{(1)}, p^{(2)})$$

Next, we have

$$\begin{aligned}&\big|\langle \phi_a(\cdot, a^{(2)}(\infty)), (u^{(1)}(\cdot, t))^2\phi(\cdot, a^{(1)}(t))^3 - (u^{(2)}(\cdot, t))^2\phi(\cdot, a^{(2)}(t))^3\rangle\big| \\ &\lesssim \big|\langle \phi_a(\cdot, a^{(2)}(\infty)), ((u^{(1)}(\cdot, t))^2 - (u^{(2)}(\cdot, t))^2)\phi(\cdot, a^{(2)}(t))^3\rangle\big| \\ &\qquad + \big|\langle \phi_a(\cdot, a^{(2)}(\infty)), (u^{(2)}(\cdot, t))^2(\phi(\cdot, a^{(1)}(t))^3 - \phi(\cdot, a^{(2)}(t))^3)\rangle\big| \\ &\lesssim \delta\langle t\rangle^{-1-\varepsilon} d(p^{(1)}, p^{(2)})\end{aligned}$$

and for the quintic term

$$\begin{aligned}&\big|\langle \phi_a(\cdot, a^{(2)}(\infty)), (u^{(1)}(\cdot, t))^5 - (u^{(2)}(\cdot, t))^5\rangle\big| \\ &\lesssim \delta^5\langle t\rangle^{-1-\varepsilon} d(p^{(1)}, p^{(2)})\end{aligned}$$

In conclusion,

$$(A + B + C + D)(t) \lesssim \delta\langle t\rangle^{-1-\varepsilon} d(p^{(1)}, p^{(2)})$$

From this and the bound

$$|\langle f_2, \phi_a(\cdot, a^{(1)}(\infty))\rangle - \langle f_2, \phi_a(\cdot, a^{(2)}(\infty))\rangle| \lesssim \delta\, d(p^{(1)}, p^{(2)})$$

we infer that

$$|b^{(1)}(\infty) - b^{(2)}(\infty)| \lesssim \delta d(p^{(1)}, p^{(2)})$$

as well as

$$\sup_{t\geq 0}|b^{(1)}(t)-b^{(2)}(t)|\lesssim \delta d(p^{(1)},p^{(2)})$$

The latter requires some care, as on the support of ω it also involved the difference $h^{(1)}-h^{(2)}$. However, if we borrow the estimate (68) for now, then we obtain our desired result. Using these bounds we obtain furthermore that

$$\sup_{t\geq 0}\langle t\rangle^{1+\varepsilon}|\dot b^{(1)}(t)-\dot b^{(2)}(t)|\,dt\lesssim \delta d(p^{(1)},p^{(2)}) \tag{66}$$

Next, we turn to the initial conditions. In view of (55),

$$\begin{aligned}&(2k^{(j)})^{-\frac12}\big[\langle g(\cdot,a^{(j)}(\infty)),k^{(j)}w_1^{(j)}+f_2\rangle+k^{(j)}h^{(j)}\langle g_0,g(\cdot,a^{(j)}(\infty))\rangle\big]\\ &=k^{(j)}\int_0^\infty \dot b^{(j)}(t)\,e^{-tk^{(j)}}\langle \phi_b(\cdot,b^{(j)}(t)),g(\cdot,a^{(j)}(\infty))\rangle\,dt\\ &\quad-\int_0^\infty e^{-tk^{(j)}}\big\langle (V(\cdot,a^{(j)}(\infty))-V(\cdot,a^{(j)}(t)))u^{(j)}(\cdot,t)+N(u^{(j)}(\cdot,t),\phi(\cdot,a^{(j)}(t))),g(\cdot,a^{(j)}(\infty))\big\rangle\,dt\end{aligned} \tag{67}$$

with $j=1,2$. It is important to note that $h^{(j)}$ also appears on the right-hand side of (67) because of the dependence of $\dot b^{(j)}$ on $h^{(j)}$. However, as remarked in the proof of Proposition 4, this dependence occurs with a small factor of δ, or to a higher order. Hence, we can solve for $h^{(j)}$ via the implicit function theorem, and then estimate the difference. Alternatively, we can solve for the difference and then estimate it. Either way, we obtain that

$$|h^{(1)}-h^{(2)}|\lesssim \delta d(p^{(1)},p^{(2)}) \tag{68}$$

It follows from Lemma 2 and (67) that

$$\begin{aligned}n_+^{(j)}(t)&=k^{(j)}\int_t^\infty \dot b^{(j)}(s)\,e^{-(s-t)k^{(j)}}\langle \phi_b(\cdot,b^{(j)}(s)),g(\cdot,a^{(j)}(\infty))\rangle\,ds+k^{(j)}\dot b^{(j)}(t)\,\langle \phi_b(\cdot,b^{(j)}(t)),g(\cdot,a^{(j)}(\infty))\rangle\\ &\quad-\int_t^\infty e^{-(s-t)k^{(j)}}\big\langle (V(\cdot,a^{(j)}(\infty))-V(\cdot,a^{(j)}(s)))u^{(j)}(\cdot,s)+N(u^{(j)}(\cdot,s),\phi(\cdot,a^{(j)}(s))),g(\cdot,a^{(j)}(\infty))\big\rangle\,ds\end{aligned}$$

Note that $k^{(j)}=\sqrt{a^{(j)}(\infty)}\,k_0$ and thus

$$|k^{(1)}-k^{(2)}|\lesssim d(p^{(1)},p^{(2)}),\qquad |e^{-tk^{(1)}}-e^{-tk^{(2)}}|\lesssim d(p^{(1)},p^{(2)})\,e^{-tk^{(1)}/2}\qquad \forall\, t\geq 0$$

By this and the estimates which we have just derived,

$$\langle t\rangle^{1+\varepsilon}|n_+^{(1)}(t)-n_+^{(2)}(t)|\lesssim \delta d(p^{(1)},p^{(2)})$$

In a similar fashion, we derive

$$\langle t\rangle^{1+\varepsilon}|n_-^{(1)}(t)-n_-^{(2)}(t)|\lesssim \delta d(p^{(1)},p^{(2)})$$

from the representations

$$\begin{aligned}&n_-^{(j)}(t)=e^{-tk^{(j)}}\,n_-^{(j)}(0)\\ &+k^{(j)}\int_0^t \dot b^{(j)}(s)\,e^{-(t-s)k^{(j)}}\langle \phi_b(\cdot,b^{(j)}(s)),g(\cdot,a^{(j)}(\infty))\rangle\,ds+k^{(j)}\dot b^{(j)}(t)\,\langle \phi_b(\cdot,b^{(j)}(t)),g(\cdot,a^{(j)}(\infty))\rangle\\ &-\int_0^t e^{-(t-s)k^{(j)}}\big\langle (V(\cdot,a^{(j)}(\infty))-V(\cdot,a^{(j)}(s)))u^{(j)}(\cdot,s)+N(u^{(j)}(\cdot,s),\phi(\cdot,a^{(j)}(s))),g(\cdot,a^{(j)}(\infty))\big\rangle\,ds\end{aligned}$$

and the fact that

$$|n_-^{(1)}(0)-n_-^{(2)}(0)| \lesssim (2k^{(1)})^{-\frac12}|\langle g(\cdot,a^{(1)}(\infty)),w_1^{(1)}\rangle - \langle g(\cdot,a^{(2)}(\infty)),w_1^{(2)}\rangle| + |k^{(1)}-k^{(2)}||\langle g(\cdot,a^{(2)}(\infty)),w_1^{(2)}\rangle| + |n_+^{(1)}(0)-n_+^{(2)}(0)| \lesssim \delta d(p^{(1)},p^{(2)})$$

In view of these bounds, estimate (56), and the fact that

$$v^{(j)}(\cdot,t) = (2k^{(j)})^{-\frac12}(n_+^{(j)}(t)+n_-^{(j)}(t))g(\cdot,a^{(j)}(\infty)) + \tilde v^{(j)}(\cdot,t)$$

it will suffice to estimate the difference of the $\tilde v^{(j)}$ which are given by

$$\tilde v^{(j)}(t) = \cos(t\sqrt{H_j})P^\perp_{g(\cdot,a^{(j)}(\infty))}\, w_1^{(j)} + \mathcal{S}_j(t)P^\perp_{g(\cdot,a^{(j)}(\infty))}\, f_2$$

$$- \int_0^t \dot b^{(j)}(s)\, \cos((t-s)\sqrt{H_j})P^\perp_{g(\cdot,a^{(j)}(\infty))}\,[\phi_b(\cdot,b^{(j)}(s)) - (a^{(j)}(\infty)/b^{(j)}(s))^{\frac54}\phi_a(\cdot,a^{(j)}(\infty))]\,ds \tag{69}$$

$$- \int_0^t \mathcal{S}_j(t-s)\,P^\perp_{g(\cdot,a^{(j)}(\infty))}[(V(\cdot,a^{(j)}(\infty)) - V(\cdot,a^{(j)}(s)))u^{(j)}(\cdot,s) + N(u^{(j)},\phi(\cdot,a^{(j)}(s)))]\,ds \tag{70}$$

$$+ c_0\big[\langle f_2,\psi\rangle - t\,\langle\psi,(V(\cdot,a^{(j)}(\infty)) - V(\cdot,a^{(j)}(0)))w_1^{(j)} + N(w_1^{(j)},\phi(\cdot,a^{(j)}(0)))\rangle\big]\omega(t)\psi \tag{71}$$

Here, $H_j := -\Delta - 5\phi^4(\cdot,a^{(j)}(\infty))$ and $\mathcal{S}_j(t) := \frac{\sin(t\sqrt{H_j})}{\sqrt{H_j}}P^\perp_{g(\cdot,a^{(j)}(\infty))} - c_0(\psi_j\otimes\psi_j)$.
By the bounds (68),

$$\|w_1^{(1)} - w_1^{(2)}\|_{H^3} \lesssim \delta d(p^{(1)},p^{(2)})$$

and also

$$\|P^\perp_{g(\cdot,a^{(1)}(\infty))}\,w_1^{(1)} - P^\perp_{g(\cdot,a^{(2)}(\infty))}\,w_1^{(2)}\|_{H^3} \lesssim \delta d(p^{(1)},p^{(2)})$$

By the estimates from the proof of Proposition 4 we conclude that

$$\|\cos(t\sqrt{H_1})[P^\perp_{g(\cdot,a^{(1)}(\infty))}\,w_1^{(1)} - P^\perp_{g(\cdot,a^{(2)}(\infty))}\,w_1^{(2)}]\|_\infty \lesssim \delta\langle t\rangle^{-1}d(p^{(1)},p^{(2)})$$
$$\|\nabla\cos(t\sqrt{H_1})[P^\perp_{g(\cdot,a^{(1)}(\infty))}\,w_1^{(1)} - P^\perp_{g(\cdot,a^{(2)}(\infty))}\,w_1^{(2)}]\|_2 \lesssim \delta d(p^{(1)},p^{(2)})$$
$$\|\mathcal{S}_1(t)[P^\perp_{g(\cdot,a^{(1)}(\infty))}\,f_2 - P^\perp_{g(\cdot,a^{(2)}(\infty))}\,f_2]\|_\infty \lesssim \delta\langle t\rangle^{-1}d(p^{(1)},p^{(2)})$$
$$\|\nabla\mathcal{S}_1(t)[P^\perp_{g(\cdot,a^{(1)}(\infty))}\,f_2 - P^\perp_{g(\cdot,a^{(2)}(\infty))}\,f_2]\|_2 \lesssim \delta d(p^{(1)},p^{(2)})$$

We also need to consider terms which involve the difference of the evolutions:

$$\cos(t\sqrt{H_1})P^\perp_{g(\cdot,a^{(1)}(\infty))} - \cos(t\sqrt{H_2})P^\perp_{g(\cdot,a^{(2)}(\infty))} \quad\text{and}\quad \mathcal{S}_1(t)-\mathcal{S}_2(t)$$

However, these operators lead to the desired bounds because of Corollary 23. The difference of the integrals in (69) is of the form

$$\int_0^t (\dot b^{(1)}(s)-\dot b^{(2)}(s))\,\cos((t-s)\sqrt{H_1})P^\perp_{g(\cdot,a^{(1)}(\infty))}\,[\phi_b(\cdot,b^{(1)}(s)) - (a^{(1)}(\infty)/b^{(1)}(s))^{\frac54}\phi_a(\cdot,a^{(1)}(\infty))]\,ds \tag{72}$$

$$+ \int_0^t \dot b^{(2)}(s)\,\cos((t-s)\sqrt{H_1})P^\perp_{g(\cdot,a^{(1)}(\infty))}\,\Big\{[\phi_b(\cdot,b^{(1)}(s)) - (a^{(1)}(\infty)/b^{(1)}(s))^{\frac54}\phi_a(\cdot,a^{(1)}(\infty))]$$
$$- [\phi_b(\cdot,b^{(2)}(s)) - (a^{(2)}(\infty)/b^{(2)}(s))^{\frac54}\phi_a(\cdot,a^{(2)}(\infty))]\Big\}\,ds \tag{73}$$

$$+ \int_0^t \dot b^{(2)}(s)\,[\cos((t-s)\sqrt{H_1})P^\perp_{g(\cdot,a^{(1)}(\infty))} - \cos((t-s)\sqrt{H_2})P^\perp_{g(\cdot,a^{(2)}(\infty))}]\cdot$$
$$\cdot\,[\phi_b(\cdot,b^{(2)}(s)) - (a^{(2)}(\infty)/b^{(2)}(s))^{\frac54}\phi_a(\cdot,a^{(2)}(\infty))]\,ds \tag{74}$$

First,

$$\|(72)\|_\infty \lesssim \delta\, d(p^{(1)},p^{(2)}) \int_0^t \langle s\rangle^{-1-\varepsilon}\langle t-s\rangle^{-1}\langle s\rangle^{-1}\, ds \lesssim \delta\,\langle t\rangle^{-1}\, d(p^{(1)},p^{(2)})$$

$$\|(73)\|_\infty \lesssim \int_0^t \delta^2\,\langle s\rangle^{-2}\langle t-s\rangle^{-1}\delta\, d(p^{(1)},p^{(2)})\, ds \lesssim \delta\langle t\rangle^{-1}\, d(p^{(1)},p^{(2)})$$

and second,

$$\|\nabla(72)\|_2 \lesssim \delta\, d(p^{(1)},p^{(2)}) \int_0^t \langle s\rangle^{-1-\varepsilon}\langle s\rangle^{-1}\, ds \lesssim \delta\, d(p^{(1)},p^{(2)})$$

$$\|\nabla(73)\|_2 \lesssim \int_0^t \delta^2\langle s\rangle^{-2}\delta d(p^{(1)},p^{(2)})\, ds \lesssim \delta\, d(p^{(1)},p^{(2)})$$

As far as the term (74) is concerned, we remark that it, too, satisfies the desired bounds due to the stability result in Section 5, see Corollary 23. Finally, we turn to the difference

$$\begin{aligned} E(t) := &\int_0^t \mathcal{S}_1(t-s)\, P^\perp_{g(\cdot,a^{(1)}(\infty))}[(V(\cdot,a^{(1)}(\infty)) - V(\cdot,a^{(1)}(s)))u^{(1)}(\cdot,s) + N(u^{(1)},\phi(\cdot,a^{(1)}(s)))]\, ds \\ &- \int_0^t \mathcal{S}_2(t-s)\, P^\perp_{g(\cdot,a^{(2)}(\infty))}[(V(\cdot,a^{(2)}(\infty)) - V(\cdot,a^{(2)}(s)))u^{(2)}(\cdot,s) + N(u^{(2)},\phi(\cdot,a^{(2)}(s)))]\, ds \end{aligned}$$

By the same type of arguments which we have used repeatedly up to this point the reader will check that

$$\|\nabla E(t)\|_2 + \langle t\rangle^\varepsilon \|E(t)\|_{2+\infty} \lesssim \delta\, d(p^{(1)},p^{(2)})$$

for all $t\ge 0$. This concludes the proof of the estimate

$$d(\Phi_{u_0}(p^{(1)}),\Phi_{u_0}(p^{(2)})) \lesssim \delta\, d(p^{(1)},p^{(2)})$$

and therefore of the existence of a fixed-point

$$(u,a)(u_0) \in Y_{R,\delta}$$

Since Φ_{u_0} is Lipschitz in u_0 by Proposition 4, we conclude that the fixed-point is also Lipschitz in u_0, see Lemma 8 below. Let $u_0,\tilde u_0 \in B_\delta(0)$ and denote their fixed-points by $(u,a),(\tilde u,\tilde a)$. Then, by (46) and (68),

$$\begin{aligned} |h(u_0;u,a) - h(\tilde u_0;\tilde u,\tilde a)| &\lesssim |h(u_0;u,a) - h(\tilde u_0;u,a)| + |h(\tilde u_0;u,a) - h(\tilde u_0;\tilde u,\tilde a)|| \\ &\lesssim \delta\|u_0-\tilde u_0\|_{H^2} + \delta d(p^{(1)},p^{(2)}) \\ &\lesssim \delta\|u_0-\tilde u_0\|_{H^2} \end{aligned}$$

and we are done. □

The following lemma is completely standard, we present it for the sake of completeness.

Lemma 8. *Let S be a complete metric space and T an arbitrary metric space. Suppose that $A : S\times T\to S$ so that with some $0<\gamma<1$*

$$\sup_{t\in T} d_X(A(x,t),A(y,t)) \le \gamma\, d_X(x,y) \quad \textit{for all} \quad x,y\in S,$$

$$\sup_{x\in S} d_X(A(x,t_1),A(x,t_2)) \le C_0\, d_Y(t_1,t_2) \quad \textit{for all} \quad t_1,t_2\in T.$$

Then for every $t \in T$ there exists a unique fixed-point $x(t) \in S$ such that $A(x(t), t) = x(t)$. Moreover, these points satisfy the bounds

$$d_X(x(t_1), x(t_2)) \leq \frac{C_0}{1-\gamma}\, d_Y(t_1, t_2)$$

for all $t_1, t_2 \in T$.

Proof. Clearly, $x(t) = \lim_{n\to\infty} A(x_n(t), t)$ where for some fixed (i.e., independent of t) x_0

$$x_0(t) := x_0, \quad x_{n+1}(t) = A(x_n(t), t).$$

Then inductively,

$$\begin{aligned} d_X(x_{n+1}(t_1), x_{n+1}(t_2)) &\leq d_X(A(x_n(t_1), t_1), A(x_n(t_2), t_1)) + d_X(A(x_n(t_2), t_1), A(x_n(t_2), t_2)) \\ &\leq \gamma d_X(x_n(t_1), x_n(t_2)) + C_0 d_Y(t_1, t_2) \\ &\leq C_0 \sum_{k=0}^{n} \gamma^k \, d_Y(t_1, t_2) \end{aligned}$$

for all $n \geq 0$. Passing to the limit $n \to \infty$ proves the lemma. □

It is now easy to prove Theorem 1.

Proof of Theorem 1: Let $u_0 = (f_1, f_2) \in B_\delta(0) \subset \Sigma_0$. By Proposition 7 there exists a fixed-point $(u, a) \in Y_{R,\delta}$ of the map Φ_{u_0}. By construction, this means that there exists $h(u_0)$ as in Proposition 7 so that the modified initial data (5) lead to a global solution of (42). I.e.,

$$\partial_{tt}u + H(a(\infty))u = -\partial_{tt}\phi(\cdot, a(t)) + (V(\cdot, a(\infty)) - V(\cdot, a(t)))u + N(u, \phi(\cdot, a(t)))$$

which is the same as

$$\partial_{tt}u + H(a(t))u = -\partial_{tt}\phi(\cdot, a(t)) + N(u, \phi(\cdot, a(t)))$$

This in turn implies that

$$\psi(\cdot, t) = \phi(\cdot, a(t)) + u(\cdot, t) = \phi(\cdot, a(\infty)) + v(\cdot, t)$$

with $v(\cdot, t) := \phi(\cdot, a(t)) - \phi(\cdot, a(\infty)) + u(\cdot, t)$ solves

$$\Box\psi - \psi^5 = 0$$

with initial conditions (5). Finally, $a(0) = 1$, $\dot{a}(0) = 0$ by construction and $h(u_0)$ and u satisfy the bounds from Propositions 4 and 7. Therefore, we also have

$$\|v(\cdot, t)\|_\infty \lesssim \delta\langle t\rangle^{-1}$$

To derive the scattering statement, we write the equation for $u(x, t)$ as a Hamiltonian systems with Hamiltonian $J\mathcal{H}_\infty$, see (10). Thus, set $U = \binom{u}{\partial_t u}$ and write

$$U(\cdot, t) = n_+(t)G_+(\cdot) + n_-(t)G_-(\cdot) + \tilde{U}(\cdot, t)$$

where $\tilde{U} = P_e U$, see (16). Our goal is to find initial data $(\tilde{f}_1, \tilde{f}_2) \in \dot{H}^1 \times L^2$ so that

$$U(t) = U_0(t) + \begin{pmatrix} 0 \\ -\dot{a}(t)\phi_a(\cdot, a(t)) \end{pmatrix} + o_{\mathcal{E}}(1) \quad \text{as} \quad t \to \infty \tag{75}$$

where U_0 is the solution vector of the free wave equation with data $U_0(0) = \binom{\tilde{f}_1}{\tilde{f}_2}$. Here $\mathcal{E} = \dot{H}^1 \times L^2$ refers to the energy space with norm

$$\left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{\mathcal{E}}^2 := \|\nabla\, u_1\|_2^2 + \|u_2\|_2^2$$

Notice that (75) yields the scattering claim of Theorem 1 simply because

$$v(\cdot, t) = u(\cdot, t) + \phi(\cdot, a(t)) - \phi(\cdot, a(\infty))$$

satisfies, see (75),

$$\begin{aligned}\begin{pmatrix} v \\ \partial_t v\end{pmatrix}(t) &= \begin{pmatrix} u \\ \partial_t u\end{pmatrix}(t) + \begin{pmatrix} \phi(\cdot,a(t)) - \phi(\cdot,a(\infty)) \\ \dot a(t)\phi_a(\cdot,a(t))\end{pmatrix} \\ &= U_0(t) + \begin{pmatrix} \phi(\cdot,a(t)) - \phi(\cdot,a(\infty)) \\ 0\end{pmatrix} + o_{\mathcal{E}}(1) \\ &= U_0(t) + o_{\mathcal{E}}(1) \quad \text{as} \quad t\to\infty\end{aligned}$$

Since we have shown in Proposition 4 that the coefficients $n_\pm(t)$ decay like $\langle t\rangle^{-2}$, it will suffice to prove (75) with $\tilde U = P_e U$ instead of U. Thus, we need to find initial data $(\tilde f_1, \tilde f_2)\in\mathcal{E}$ so that

$$\|\tilde U(t) - (0, -\dot a(t)\phi_a(\cdot,a(t)))^\dagger - U_0(t)\|_{\mathcal{E}} \to 0 \quad \text{as} \quad t\to\infty \tag{76}$$

Here † means transposition. This will be done in two steps. First, we will find initial data $(f_1', f_2')\in\mathcal{E}$ so that

$$\|\tilde U(t) - (0, -\dot a(t)\phi_a(\cdot,a(t)))^\dagger - e^{tJ\mathcal{H}_\infty} P_e\,(f_1',f_2')^\dagger\|_{\mathcal{E}} \to 0 \quad \text{as} \quad t\to\infty \tag{77}$$

see (18). Because of the dispersive estimates on $\tilde U(t)$ and $e^{tJ\mathcal{H}_\infty}P_e$, and in view of (34), (77) will follow from

$$\|\tilde U(t) - (0, -\dot a(t)\phi_a(\cdot,a(t)))^\dagger - e^{tJ\mathcal{H}_\infty} P_e\,(f_1',f_2')^\dagger\|_{\mathcal{E}_\infty} \to 0 \quad \text{as} \quad t\to\infty \tag{78}$$

where

$$\Big\|\begin{pmatrix} u_1 \\ u_2\end{pmatrix}\Big\|_{\mathcal{E}_\infty}^2 := \|\sqrt{H_\infty}\,u_1\|_2^2 + \|u_2\|_2^2$$

for all $(u_1,u_2)\in P^\perp_{g(\cdot,a(\infty))}[\dot H^1\times L^2]$. Strictly speaking, we have only derived dispersive estimates on $\tilde u$, and not on $\partial_t\tilde u$. However, it follows from the explicit form of $\tilde u$, see (27), that for large times

$$\begin{aligned}\partial_t\tilde u(\cdot,t) = &-\sin(t\sqrt{H_\infty})P^\perp_{g_\infty}\sqrt{H_\infty}\,w_1 + \cos(t\sqrt{H_\infty})P^\perp_{g_\infty}w_2 - \dot a(t)P^\perp_{g_\infty}[\phi_a(\cdot,a(t)) - (a(\infty)/a(t))^{\frac54}\psi] \\ &+ \int_0^t \dot a(s)\sin((t-s)\sqrt{H_\infty})P^\perp_{g_\infty}\sqrt{H_\infty}[\phi_a(\cdot,a(s)) - (a(\infty)/a(s))^{\frac54}\psi]\,ds \\ &- \psi(\cdot)\,\langle\psi, (V(\cdot,a(\infty)) - V(\cdot,a(t)))u(\cdot,t) + N(u,\phi(\cdot,a(t)))\rangle \\ &- \int_0^t \cos((t-s)\sqrt{H_\infty})P^\perp_{g_\infty}[(V(\cdot,a(\infty)) - V(\cdot,a(t)))u(\cdot,t) + N(u,\phi(\cdot,a(t)))]\,ds\end{aligned}$$

where $\psi(\cdot) = \phi_a(\cdot,a(\infty))$ and $g_\infty = g(\cdot,a(\infty))$ as usual. By the estimates which we have derived we easily conclude that

$$\|\partial_t\tilde u(\cdot,t)\|_\infty \lesssim \delta\langle t\rangle^{-1}$$

Thus $\tilde U$ is dispersive as claimed and (77) reduces to (78). Now we remark that the group $e^{tJ\mathcal{H}_\infty}$ is unitary on $P^\perp_{g(\cdot,a(\infty))}[\dot H^1\times L^2]$ relative to the norm $\mathcal{E}_\infty$. Hence, (78) is the same as showing that

$$\|e^{-tJ\mathcal{H}_\infty}[\tilde U(t) - (0, -\dot a(t)\phi_a(\cdot,a(t)))^\dagger] - P_e\,(f_1',f_2')^\dagger\|_{\mathcal{E}_\infty} \to 0 \quad \text{as} \quad t\to\infty \tag{79}$$

Note that

$$\|\dot a(t)[\phi_a(\cdot,a(t)) - (a(\infty)/a(t))^{\frac54}\phi_a(\cdot,a(\infty))]\|_2 \lesssim \delta^4 t^{-3}$$

as $t\to\infty$ since the term in brackets decays like $\langle x\rangle^{-3}$. Thus, in view of the unitarity of $e^{-tJ\mathcal{H}_\infty}$, it follows that (79) is equivalent with

$$\|e^{-tJ\mathcal{H}_\infty}[\tilde U(t) - (a(\infty)/a(t))^{\frac54}(0, -\dot a(t)\phi_a(\cdot,a(\infty)))^\dagger] - P_e\,(f_1',f_2')^\dagger\|_{\mathcal{E}_\infty} \to 0 \quad \text{as} \quad t\to\infty \tag{80}$$

First, we note that by (18)

$$e^{-tJ\mathcal{H}_\infty}\begin{pmatrix} 0 \\ -\dot a(t)(a(\infty)/a(t))^{\frac54}\psi\end{pmatrix} = \dot a(t)(a(\infty)/a(t))^{\frac54}\begin{pmatrix} t\psi \\ -\psi\end{pmatrix}$$

Hence, (80) is equivalent with

$$\|e^{-tJ\mathcal{H}_\infty}\tilde{U}(t) - \dot{a}(t)(a(\infty)/a(t))^{\frac{5}{4}}(t\psi, -\psi)^\dagger - P_e\,(f_1', f_2')^\dagger\|_{\mathcal{E}_\infty} \to 0 \ \text{ as } \ t\to\infty \tag{81}$$

Second, by (19),

$$e^{-tJ\mathcal{H}_\infty}\tilde{U}(t) = P_e\tilde{U}(0) + \int_0^t e^{-sJ\mathcal{H}_\infty}P_e W(s)\,ds$$

with

$$\begin{aligned} W(s) &= \begin{pmatrix} 0 \\ -\partial_s(\dot{a}(s)\phi_a(\cdot, a(s))) + (V(\cdot, a(\infty)) - V(\cdot, a(s)))u + N(u, \phi(\cdot, a(s))) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -\partial_s(\dot{a}(s)\phi_a(\cdot, a(s))) \end{pmatrix} + \tilde{W}(s) \end{aligned}$$

see (10) (we define $\tilde{W}$ by the second line). Integrating by parts and using (18) yields

$$\begin{aligned} &\int_0^t e^{-sJ\mathcal{H}_\infty}P_e\begin{pmatrix} 0 \\ -\partial_s(\dot{a}(s)\phi_a(\cdot, a(s))) \end{pmatrix} ds \\ &= \dot{a}(t)(a(\infty)/a(t))^{\frac{5}{4}}\begin{pmatrix} t\psi \\ -\psi \end{pmatrix} + \dot{a}(t)\begin{pmatrix} \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}P^\perp_{g_\infty}[\phi_a(\cdot, a(t)) - (a(\infty)/a(t))^{\frac{5}{4}}\psi] \\ -\cos(t\sqrt{H_\infty})P^\perp_{g_\infty}[\phi_a(\cdot, a(t)) - (a(\infty)/a(t))^{\frac{5}{4}}\psi] \end{pmatrix} \\ &\quad - \int_0^t \dot{a}(s)\begin{pmatrix} \cos(s\sqrt{H_\infty})P^\perp_{g_\infty}\phi_a(\cdot, a(s)) \\ \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}P^\perp_{g_\infty}H_\infty\phi_a(\cdot, a(s)) \end{pmatrix} ds \end{aligned} \tag{82}$$

Observe that the first term in (82) is identical with the middle term in (81). The second term in (82) is $o(1)$ in the energy norm $\mathcal{E}_\infty$ as $t\to\infty$ and can hence be ignored for the purposes of (81). We define

$$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} := P_e\tilde{U}(0) - \int_0^\infty \dot{a}(s)\begin{pmatrix} \cos(s\sqrt{H_\infty})P^\perp_{g_\infty}\phi_a(\cdot, a(s)) \\ \frac{\sin(t\sqrt{H_\infty})}{\sqrt{H_\infty}}P^\perp_{g_\infty}H_\infty\phi_a(\cdot, a(s)) \end{pmatrix} ds + \int_0^\infty e^{-sJ\mathcal{H}_\infty}P_e\tilde{W}(s)\,ds$$

This definition is justified, since the integrals are absolutely convergent in the norm of $\mathcal{E}_\infty$. In addition, $\binom{f_1'}{f_2'} \in \mathcal{E}_\infty \cap L^\infty \subset \mathcal{E}$. Having established (81) and therefore (77), we now carry out the second step. It consists of finding initial data $(\tilde{f}_1, \tilde{f}_2) \in \mathcal{E}$ so that

$$\|e^{tJ\mathcal{H}_\infty}P_e\,(f_1', f_2')^\dagger - e^{tJ\mathcal{H}_{\text{free}}}\,(\tilde{f}_1, \tilde{f}_2)^\dagger\|_{\mathcal{E}} \to 0 \ \text{ as } \ t\to\infty \tag{83}$$

However, this latter property is a consequence of the asymptotic completeness of $-\Delta + V$, which is standard. The theorem is proved. □

5. Linear theory: Point-wise decay

In this section $H = -\Delta + V$ where V is real-valued and decays faster than a third power: $|V(x)| \lesssim \langle x\rangle^{-\kappa}$ with $\kappa > 3$. Although we are of course only interested in the special potential V from the previous sections, we will keep this discussion more general. We emphasize that we work on all of $L^2(\mathbb{R}^3)$ here and assume that H has no eigenvalue at zero. In the case of H as above this is false, but as explained there, it is true when H is restricted to the radial functions. Hence, in order to apply the results from this section we need to restrict H to the invariant subspace of radial functions.

5.1. **The sine evolution.** We study the evolution

$$\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c = \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} E(d\lambda) \tag{84}$$

where E is the spectral resolution of H and $P_c = \chi_{[0,\infty)}(H)$ the projection onto the continuous spectrum. It arises as solution of the Cauchy problem

$$(\partial_{tt} + H)u = 0, \quad u(0) = 0, \partial_t u(0) = f$$

In this section our goal is to prove Proposition 9. The question of dispersive decay for the wave equation with a potential has received much attention in recent years, see the papers by Beals, Strauss, Cuccagna, Georgiev, Visciglia, Yajima, d'Ancona, Pierfelice in the references. However, none of these refences apply here since they either assume that $V \geq 0$, V small, or that zero is neither an eigenvalue nor a resonance.

Proposition 9. *Assume that V is a real-valued potential such that $|V(x)| \lesssim \langle x \rangle^{-\kappa}$ for some $\kappa > 3$. If H has neither a resonance nor an eigenvalue at zero, then*

$$\Big\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f \Big\|_\infty \lesssim t^{-1} \|f\|_{W^{1,1}(\mathbb{R}^3)}$$

for all $t > 0$. Now assume that zero is a resonance but not an eigenvalue of $H = -\Delta + V$. Let ψ be the unique resonance function normalized so that $\int V\psi(x)\, dx = 1$. Then there exists a constant $c_0 \neq 0$ such that

$$\Big\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f - c_0(\psi \otimes \psi) f \Big\|_\infty \lesssim t^{-1} \|f\|_{W^{1,1}(\mathbb{R}^3)} \tag{85}$$

for all $t > 0$.

We will only prove the second part which is harder. The case when zero is neither a resonance nor an eigenvalue is implicit in our proof below and we will henceforth assume that we are in the second case. In (85)

$$(\psi \otimes \psi) f(x) = \psi(x) \int f(y)\psi(y)\, dy$$

which is well-defined for all $f \in L^1$ since $\psi \in L^\infty(\mathbb{R}^3)$. Indeed, one has $\psi + (-\Delta)^{-1} V\psi = 0$ which is the same as

$$\psi(x) = -\int_{\mathbb{R}^3} \frac{V(y)\psi(y)}{4\pi|x-y|}\, dy \tag{86}$$

Hence, ψ is bounded provided we can show that $V\psi \in L^{\frac{3}{2}-\varepsilon} \cap L^{\frac{3}{2}+\varepsilon}(\mathbb{R}^3)$ for some $\varepsilon > 0$. This, however, follows from the decay of V and

$$\|V\psi\|_{\frac{3}{2}\pm} \lesssim \|\langle x \rangle^\sigma V\|_{6\pm} \|\langle x \rangle^{-\sigma}\psi\|_2$$

where $\sigma = \frac{1}{2}+$. We also remark that (86) implies that well-known fact that

$$\int_{\mathbb{R}^3} V\psi(x)\, dx \neq 0 \tag{87}$$

Indeed, if this vanished, then we could write

$$\psi(x) = -\int_{\mathbb{R}^3} \Big(\frac{V(y)\psi(y)}{4\pi|x-y|} - \frac{V(y)\psi(y)}{4\pi|x|} \Big)\, dy$$

which would imply that $|\psi(x)| \lesssim \langle x \rangle^{-2}$ in contradiction to $\psi \notin L^2(\mathbb{R}^3)$.

We now start with the detailed argument for the proposition. The evolution (84) can be written as

$$\frac{1}{i\pi} \int_0^\infty \frac{\sin(t\lambda)}{\lambda} [R_V^+(\lambda^2) - R_V^-(\lambda^2)]\, \lambda d\lambda = \frac{1}{i\pi} \int_{-\infty}^\infty \sin(t\lambda) R(\lambda) d\lambda \tag{88}$$

where we have set $R(\lambda) := R_V^+(\lambda^2)$ if $\lambda > 0$ and $R(\lambda) = \overline{R(-\lambda)}$ if $\lambda < 0$. For the free resolvent, we write this as R_0. Then, by the usual resolvent expansions,

$$R = \sum_{k=0}^{2n-1} (-1)^k R_0(VR_0)^k + (R_0V)^n R(VR_0)^n \tag{89}$$

As illustration, let us consider the first term in this expansion which leads to the free evolution. It is of the form (for $t > 0$)

$$\begin{aligned}&\Big|\int_{\mathbb{R}^6}\int_{-\infty}^{\infty} \sin(t\lambda)\frac{e^{i\lambda|x-y|}}{|x-y|}\,d\lambda\, f(x)g(y)\,dxdy\Big| = \frac{1}{2}t^{-1}\Big|\int_{\mathbb{R}^3}\int_{[|x-y|=t]} f(x)\,\sigma(dx)\,g(y)\,dy\Big|\\ &\lesssim t^{-1}\|\nabla f\|_1\|g\|_1\end{aligned}$$

To pass to the second line we used the standard divergence theorem trick (see eg. Strauss [Str])

$$\begin{aligned}&\Big|\int_{[|x-y|=t]} f(x)\,\sigma(dx)\Big| = \Big|\int_{[|x-y|=t]} f(x)\frac{x-y}{t}\cdot\vec{n}\,\sigma(dx)\Big| = t^{-1}\Big|\int_{[|x-y|\le t]} \nabla(f(x)(y-x))\,dx\Big|\\ &= t^{-1}\Big|\int_{[|x-y|\le t]} [(y-x)\nabla f(x) - 3f(x)]\,dx\Big| \lesssim \int_{\mathbb{R}^3} |\nabla f(x)|\,dx + \Big(\int_{\mathbb{R}^3} |f(x)|^{\frac{3}{2}}\,dx\Big)^{\frac{2}{3}}\\ &\lesssim \int_{\mathbb{R}^3} |\nabla f(x)|\,dx\end{aligned} \tag{90}$$

where the final inequality follows from Sobolev imbedding.

We distinguish between small energies and all other energies. For the latter, we use (89). Let $\chi_0(\lambda) = 0$ for all $|\lambda| \le \lambda_0$ and $\chi_0(\lambda) = 1$ if $|\lambda| > 2\lambda_0$. Here $\lambda_0 > 0$ is some small parameter. Fix some k as in (89) and consider the contribution of the corresponding Born term (ignoring a factor of $(4\pi)^{-k-1}$):

$$\begin{aligned}&\int_{\mathbb{R}^{3(k+2)}}\int_{-\infty}^{\infty} \chi_0(\lambda)\sin(t\lambda)e^{i\lambda\sum_{j=0}^k |x_j-x_{j+1}|}\frac{\prod_{j=1}^k V(x_j)}{\prod_{j=0}^k |x_j-x_{j+1}|}\, f(x_0)g(x_{k+1})\,d\lambda\,dx_0\ldots dx_{k+1}\\ &= \int\int_{\mathbb{R}^{3(k+1)}} \widehat{\chi_0}(\xi) \int_{[|x_0-x_1|=t-\xi-\sum_{j=1}^k |x_j-x_{j+1}|>0]} \frac{f(x_0)}{|x_0-x_1|}\,\sigma(dx_0)\,\frac{\prod_{j=1}^k V(x_j)}{\prod_{j=1}^k |x_j-x_{j+1}|}\,g(x_{k+1})\,dx_1\ldots dx_{k+1}\,d\xi\end{aligned}$$

By definition, $\chi_1 = 1 - \chi_0 \in C_c^\infty$ so that $\widehat{\chi_0} = \delta_0 - \widehat{\chi_1}$ with $\widehat{\chi_1}$ a Schwartz function. We start with the argument for δ_0. Write $\mathbb{R}^{3(k+1)} = A(t) \cup B(t)$ where

$$A(t) = \Big\{t > \sum_{j=1}^k |x_j - x_{j+1}| > t/2\Big\}, \qquad B(t) = \Big\{\sum_{j=1}^k |x_j - x_{j+1}| \le t/2\Big\} \tag{91}$$

Then, using the divergence theorem as above and with $\rho = t - \sum_{j=1}^k |x_j - x_{j+1}|$,

$$\begin{aligned}&\Big|\int_{A(t)}\int_{[|x_0-x_1|=\rho]} \frac{f(x_0)}{|x_0-x_1|}\,\sigma(dx_0)\,\frac{\prod_{j=1}^k V(x_j)}{\prod_{j=1}^k |x_j-x_{j+1}|}\,g(x_{k+1})\,dx_1\ldots dx_{k+1}\Big|\\ &\lesssim \int_{A(t)} \Big[\rho^{-1}\int_{[|x_0-x_1|<\rho]} |\nabla f(x_0)|\,dx_0 + \rho^{-2}\int_{[|x_0-x_1|<\rho]} |f(x_0)|\,dx_0\Big]\frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=1}^k |x_j-x_{j+1}|}\,|g(x_{k+1})|\,dx_1\ldots dx_{k+1}\end{aligned}$$

We now use the definition of $A(t)$ to obtain a decay factor of t^{-1} from one of the denominators $|x_j - x_{j+1}|$. This allows us to further estimate the previous expression by

$$\lesssim kt^{-1}\sum_{\ell=1}^{k}\int_{\mathbb{R}^{3(k+2)}} |\nabla f(x_0)|\,\frac{\prod_{j=1}^k |V(x_j)|}{\prod_{0\le j<\ell}|x_j - x_{j+1}|\,\prod_{\ell<j\le k}|x_j - x_{j+1}|}\,|g(x_{k+1})|\,dx_0\ldots dx_{k+1}$$

$$+\,kt^{-1}\sum_{\ell=1}^{k}\Big\|\int_{\mathbb{R}^3}\frac{|f(x_0)|}{|x_0-x_1|^2}\,dx_0\Big\|_{L^3_{x_1}}\Big\|\int_{\mathbb{R}^{3k}}\frac{\prod_{j=1}^k |V(x_j)|}{\prod_{1\le j<\ell}|x_j - x_{j+1}|\,\prod_{\ell<j\le k}|x_j - x_{j+1}|}\,|g(x_{k+1})|\,dx_2\ldots dx_{k+1}\Big\|_{L^{\frac32}(dx_1)}$$

$$\lesssim kt^{-1}\|V\|_{\mathcal{K}}^k\|\nabla f\|_1\|g\|_1 + kt^{-1}\|V\|_{L^{\frac32}}\|V\|_{\mathcal{K}}^{k-1}\|\nabla f\|_1\|g\|_1$$

where

$$\|V\|_{\mathcal{K}} = \sup_y \int \frac{|V(x)|}{|x-y|}\,dx$$

is the global Kato norm from [RodSch], which is finite in our case. We also used the bound

$$\Big\|\int_{\mathbb{R}^3}\frac{|f(x_0)|}{|x_0-x_1|^2}\,dx_0\Big\|_{L^3(dx_1)} \lesssim \|f\|_{L^{\frac32}(\mathbb{R}^3)} \lesssim \|\nabla f\|_1$$

which is obtained by fractional integration and Sobolev imbedding. The estimate for the integral over $B(t)$ is similar. Here one uses that $\rho(t) > t/2$ to gain a factor of t^{-1}. More precisely,

$$\Big|\int_{B(t)}\int_{[|x_0-x_1|=\rho]}\frac{f(x_0)}{|x_0-x_1|}\,\sigma(dx_0)\,\frac{\prod_{j=1}^k V(x_j)}{\prod_{j=1}^k |x_j-x_{j+1}|}\,g(x_{k+1})\,dx_1\ldots dx_{k+1}\Big|$$

$$\lesssim \int_{B(t)}\rho^{-1}\int_{\mathbb{R}^3}|\nabla f(x_0)|\,dx_0\,\frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=1}^k |x_j-x_{j+1}|}\,|g(x_{k+1})|\,dx_1\ldots dx_{k+1}$$

$$\lesssim t^{-1}\int_{\mathbb{R}^{3(k+2)}}|\nabla f(x_0)|\,\frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=1}^k |x_j-x_{j+1}|}\,|g(x_{k+1})|\,dx_0\ldots dx_{k+1}$$

$$\lesssim t^{-1}\|V\|_{\mathcal{K}}^k\|\nabla f\|_1\|g\|_1$$

We are done with $A(t)\cup B(t)$ and the δ_0-part of $\widehat{\chi_0}$. Parenthetically, we remark that these arguments (via the infinite expansion in (89)) prove the following small potential result.

Proposition 10. *Assume that the real-valued potential V satisfies $\|V\|_{\mathcal{K}} < 4\pi$ and $\|V\|_{L^{\frac32}} < \infty$. Then one has the bound*

$$\Big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}f\Big\|_\infty \lesssim t^{-1}\|\nabla f\|_{L^1(\mathbb{R}^3)}$$

for all $t > 0$.

In the general case (i.e., large V as needed in our application), we need to work with the finite expansion (89). Recall that we yet have to deal with the contribution by $\widehat{\chi_1}$, which means obtaining the same estimate as above for

$$\int\int_{\mathbb{R}^{3(k+1)}}\widehat{\chi_1}(\xi)\int_{[|x_0-x_1|=t-\xi-\sum_{j=1}^k |x_j-x_{j+1}|>0]}\frac{f(x_0)}{|x_0-x_1|}\,\sigma(dx_0)\,\frac{\prod_{j=1}^k V(x_j)}{\prod_{j=1}^k |x_j-x_{j+1}|}\,g(x_{k+1})\,dx_1\ldots dx_{k+1}\,d\xi$$

Here we need to split the ξ-integral into the regions $\{|\xi| < t/10\}$ and $\{|\xi| > t/10\}$. In the former region the same argument applies as before, whereas in the latter one uses that

$$\int_{[|\xi|>t/10]}|\widehat{\chi_0}(\xi)|\,d\xi < C_N\langle t\rangle^{-N}$$

for any N by the rapid decay of $\widehat{\chi_0}$.

It remains to bound the contribution by the final term in (89), the kernel $K(x,y)$ of which can be reduced to the form

$$\begin{aligned}&\int e^{\pm it\lambda}\chi_0(\lambda)\langle R(\lambda)(VR_0(\lambda))^n(\cdot,x),(VR_0(-\lambda))^n(\cdot,y)\rangle\,d\lambda\\ &=\int e^{i\lambda[\pm t+(|x|+|y|)]}\chi_0(\lambda)\langle R(\lambda)(VR_0(\lambda))^{n-1}VG_x(\lambda,\cdot),(VR_0(-\lambda))^{n-1}VG_y(-\lambda,\cdot)\rangle\,d\lambda\end{aligned} \tag{92}$$

Here

$$G_x(\lambda,u):=\frac{e^{i\lambda(|x-u|-|x|)}}{4\pi|x-u|}$$

and the scalar product appearing in (92) is just another way of writing the composition of the operators. In [GolSch] the following bounds were proved. For the sake of completeness, we reproduce the simple proof.

Lemma 11. *The derivatives of $G_x(\lambda,\cdot)$ satisfy the estimates*

$$\begin{aligned}&\sup_{x\in\mathbb{R}^3}\Big\|\frac{d^j}{d\lambda^j}G_x(\lambda,\cdot)\Big\|_{L^{2,-\sigma}}<C_{j,\sigma}\quad\textit{provided}\quad\sigma>\frac12+j\\ &\sup_{x\in\mathbb{R}^3}\Big\|\frac{d^j}{d\lambda^j}G_x(\lambda,\cdot)\Big\|_{L^{2,-\sigma}}<\frac{C_{j,\sigma}}{\langle x\rangle}\quad\textit{provided}\quad\sigma>\frac32+j\end{aligned} \tag{93}$$

for all $j\geq 0$.

Proof. This follows from the explicit formula

$$\begin{aligned}\Big\|\frac{d^j}{d\lambda^j}\frac{e^{i\lambda(|u-x|-|x|)}}{|x-u|}\langle u\rangle^{-\sigma}\Big\|_{L^2_u}&=\Big(\int_{\mathbb{R}^3}\frac{(|u-x|-|x|)^{2j}}{|x-u|^2}\langle u\rangle^{-2\sigma}\,du\Big)^{\frac12}\\ &\leq\Big(\int_{\mathbb{R}^3}\frac{\langle u\rangle^{2(j-\sigma)}}{|x-u|^2}\,du\Big)^{\frac12}\end{aligned}$$

The final estimate on this integral is obtained by dividing $\mathbb{R}^3$ into the regions $|u|<\frac{|x|}{2}$, $|x-u|<\frac{|x|}{2}$, and the complement of these two. If $\frac12<(\sigma-j)<\frac32$, then each of these regions contributes $\langle x\rangle^{\frac12+j-\sigma}$ to the total. If $\sigma>\frac32+j$, the first region instead contributes $\langle x\rangle^{-2}$, making it the dominant term. □

Let

$$a_{x,y}(\lambda)=\chi_0(\lambda)\langle R(\lambda)(VR_0(\lambda))^{n-1}VG_x(\lambda,\cdot),(VR_0(-\lambda))^{n-1}VG_y(-\lambda,\cdot)\rangle$$

Then in view of the preceding one concludes that $a_{x,y}(\lambda)$ has two derivatives in λ and for large n,

$$\Big|\frac{d^j}{d\lambda^j}a_{x,y}(\lambda)\Big|\lesssim(1+\lambda)^{-2}(\langle x\rangle\langle y\rangle)^{-1}\quad\text{for}\quad j=0,1,\quad\text{and all}\quad\lambda>1 \tag{94}$$

We need to take n sufficiently large (say, $n>10$) in order to obtain sufficiently fast decay of $a_{x,y}$ from the limiting absorption principle. The latter here refers to the bounds for the free and perturbed resolvents due to Agmon [Agm]:

$$\begin{aligned}&\|R_V(\lambda^2\pm i0)\|_{L^{2,\sigma}\to L^{2,-\sigma}}\lesssim\lambda^{-1+},\qquad\sigma>\frac12\\ &\|\partial_\lambda^\ell R_V(\lambda^2\pm i0)\|_{L^{2,\sigma}\to L^{2,-\sigma}}\lesssim 1,\qquad\sigma>\frac12+\ell,\qquad\ell\geq1\end{aligned} \tag{95}$$

for λ separated from zero. Analogous estimates of course hold for the free resolvent.

Let us assume first that $t>1$. To estimate (92) we distinguish between $|t-(|x|+|y|)|<t/10$ and the opposite case. In the former case, we conclude that

$$\max(|x|,|y|)\gtrsim t$$

so that due to (94) we obtain

$$\Big|\int e^{i\lambda[\pm t+(|x|+|y|)]}a_{x,y}(\lambda)\,d\lambda\Big|\lesssim\chi_{[|x|+|y|>t]}(\langle x\rangle\langle y\rangle)^{-1}\lesssim t^{-1}$$

In the latter case we integrate by parts once which also gains t^{-1}, and we obtain the bound

$$\Big| \int e^{i\lambda[\pm t+(|x|+|y|)]} a_{x,y}(\lambda)\, d\lambda \Big| \lesssim t^{-1}(\langle x\rangle\langle y\rangle)^{-1} \lesssim t^{-1}$$

Finally, if $0 < t < 1$, then we simply put the absolute values inside. The conclusion is that in all cases

$$\begin{aligned}&\sup_{x,y} \Big| \int e^{i\lambda[\pm t+(|x|+|y|)]} \chi_0(\lambda) \langle R(\lambda)(VR_0(\lambda))^{n-1} VG_x(\lambda,\cdot), (VR_0(-\lambda))^{n-1} VG_y(-\lambda,\cdot)\rangle\, d\lambda \Big| \\ &\lesssim (\chi_{[|x|+|y|>t]} + \langle t\rangle^{-1})(\langle x\rangle\langle y\rangle)^{-1}\end{aligned}$$

Careful inspection of these bounds reveals that they only require $|V(x)| \lesssim \langle x\rangle^{-\kappa}$ with $\kappa > 3$. For example, consider the term which contains

$$\partial_\lambda R_0(\lambda) V R_0(\lambda)$$

Then the resolvent on the left requires a weight of $\langle x\rangle^{\frac{3}{2}+\varepsilon}$, whereas the one the right requires $\langle x\rangle^{\frac{1}{2}+\varepsilon}$, see (95). Hence, V needs to absorb the weight $\langle x\rangle^{2+\varepsilon}$. On the other hand, by the lemma the term

$$R_0(\lambda) V \partial_\lambda G_x(\lambda,\cdot)$$

requires weights of $\langle x\rangle^{\frac{5}{2}+\varepsilon}$ for $\partial_\lambda G_x(\lambda,\cdot)$ (in order to gain $\langle x\rangle^{-1}$), whereas $R_0(\lambda)$ needs $\langle x\rangle^{\frac{1}{2}+\varepsilon}$. In total, this means that V has to absorb the weight $\langle x\rangle^{3+\varepsilon}$, whence our assumption $|V(x)| \lesssim \langle x\rangle^{-\kappa}$ with $\kappa > 3$. See [GolSch] for similar details. In summary, we have obtained

Lemma 12. *Assume $|V(x)| \lesssim \langle x\rangle^{-\kappa}$ with $\kappa > 3$. Then there exists a kernel $K_t(x,y)$ so that*

$$|K_t(x,y)| \lesssim (\chi_{[|x|+|y|>t]} + \langle t\rangle^{-1})(\langle x\rangle\langle y\rangle)^{-1}$$

and such that

$$\Big\| \Big[\frac{\sin(t\sqrt{H})}{\sqrt{H}} \chi_0(H) - K_t\Big] f \Big\|_\infty \lesssim t^{-1} \|\nabla f\|_{L^1(\mathbb{R}^3)}$$

for all $t > 0$. In particular,

$$\Big\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} \chi_0(H) f \Big\|_\infty \lesssim t^{-1} \|f\|_{W^{1,1}(\mathbb{R}^3)}$$

for all $t > 0$.

We will not make direct use of the more refined bound involving K_t for the evolution $\frac{\sin(t\sqrt{H})}{\sqrt{H}}$. However, we will need such a refined bound for the case of $\cos(t\sqrt{H})$.

In passing, we remark that the previous argument can be easily adapted to accommodate a gradient. More precisely, we have

Corollary 13. *Assume $|V(x)| + |\nabla V(x)| \lesssim \langle x\rangle^{-\kappa}$ with $\kappa > 3$. Then there exists a kernel $K_t(x,y)$ so that*

$$|K_t(x,y)| \lesssim (\chi_{[|x|+|y|>t]} + \langle t\rangle^{-1})(\langle x\rangle\langle y\rangle)^{-1}$$

and such that

$$\Big\| \Big[\nabla \frac{\sin(t\sqrt{H})}{\sqrt{H}} \chi_0(H) - K_t\Big] f \Big\|_\infty \lesssim t^{-1}(\|\nabla f\|_{L^1(\mathbb{R}^3)} + \|D^2 f\|_{L^1(\mathbb{R}^3)})$$

for all $t > 0$. In particular,

$$\Big\| \nabla \frac{\sin(t\sqrt{H})}{\sqrt{H}} \chi_0(H) f \Big\|_\infty \lesssim t^{-1} \|f\|_{W^{2,1}(\mathbb{R}^3)}$$

for all $t > 0$.

Proof. The point here is that the proof of the previous lemma is based on a finite Born series expansion. We can commute the gradient through the free resolvents in the terms of this series, which leads to commutators of the gradient and the potential. These are harmless, though, because of our decay assumption on ∇V. Indeed, the Born terms involving ∇V are of the same nature as those arising in Lemma 12. As far as the final term in the Born series (which involves a perturbed resolvent) is concerned, we note that

$$\nabla R - R\nabla = -R(\nabla V)R$$

But the right-hand side here does not make any difference to the way we treated the function $a_{x,y}(\lambda)$ above. Hence the corollary. □

It remains to deal with small energies. Recall that we are assuming that zero energy is a resonance but not an eigenvalue. Expansions for the perturbed resolvent around zero energy were obtain by Jensen and Kato [JenKat] in that case. Here we will follow [ErdSch] in which the method of Jensen and Nenciu [JenNen] was implemented for the case of $\mathbb{R}^3$. Let us recall the main steps: For $j = 0, 1, 2, ...$, let G_j be the operator with the kernel

$$G_j(x,y) = \frac{1}{4\pi j!}|x-y|^{j-1}.$$

For each $J = 0, 1, 2, ...$,

$$R_0(\lambda) = \sum_{j=0}^{J}(i\lambda)^j G_j + o(\lambda^J), \text{ as } \lambda \to 0.$$

This expansion is valid in the space, $HS_{L^{2,\sigma}\to L^{2,-\sigma}}$, of Hilbert-Schmidt operators between $L^{2,\sigma}$ and $L^{2,-\sigma}$ for $\sigma > \max((2J+1)/2, 3/2)$. Let $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, $v = |V|^{1/2}$ and $w = vU$. We have

$$V = Uv^2 = wv.$$

We use the symmetric resolvent identity, valid for $\lambda \neq 0$,

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)vA(\lambda)^{-1}vR_0(\lambda), \tag{96}$$

with the operator

$$\begin{aligned} A(\lambda) &= U + vR_0(\lambda)v = (U + vG_0v) + \lambda\frac{v[R_0(\lambda) - G_0]v}{\lambda} \\ &=: A_0 + \lambda A_1(\lambda) \end{aligned}$$

Due to the compactness of vG_0v on $L^2(\mathbb{R}^3)$ we remark that the essential spectrum of A_0 is the same as that of U. In particular, if zero is in the spectrum of A_0, then it is an isolated eigenvalue of finite multiplicity. By inspection, $A_1(\lambda)$ has the kernel

$$A_1(\lambda)(x,y) = \frac{1}{\lambda}v(x)\frac{e^{i\lambda|x-y|}-1}{4\pi|x-y|}v(y),$$

$$|A_1(\lambda)(x,y)| \leq \frac{1}{4\pi}|v(x)|\,|v(y)| \tag{97}$$

Therefore, $A_1(\lambda) \in HS := HS_{L^2\to L^2}$ provided $\langle x\rangle^{\frac{3}{2}+}v(x) \in L^\infty$. Also note that

$$A_1(0) = ivG_1v = \frac{i\alpha}{4\pi}P_v, \quad \alpha = \|V\|_1,$$

where P_v is the orthogonal projection onto span(v).

In our case the operator $A(\lambda)$ is not invertible at $\lambda = 0$ due to the resonance. In fact, let $\psi \in L^{2,-\frac{1}{2}-} \setminus \{0\}$ solve $-\Delta\psi + V\psi = 0$. Then $\ker(A_0) = \mathbb{R}\cdot w\psi$ (where ker is relative to L^2). Let S_1 be the orthogonal projection onto $\ker(A(0))$, i.e.,

$$S_1 = \|w\psi\|_2^{-2} w\psi \otimes w\psi =: \tilde{\psi} \otimes \tilde{\psi} \tag{98}$$

and set $A_0 := A(0)$. Then $A_0 + S_1$ is invertible on $L^2(\mathbb{R}^3)$, and $A(\lambda) + S_1$ is invertible, too, for small λ (see below). The following abstract lemma explains how to obtain the singular power of λ by inverting $A(\lambda)$.

Lemma 14. [JenNen] *Let $F \subset \mathbb{C} \setminus \{0\}$ have zero as an accumulation point. Let $A(z)$, $z \in F$, be a family of bounded operators on some Hilbert space of the form*

$$A(z) = A_0 + zA_1(z)$$

with $A_1(z)$ uniformly bounded as $z \to 0$ and A_0 self-adjoint. Suppose that 0 *is an isolated point of the spectrum of A_0, and let S be the corresponding Riesz projection. Assume that* $\mathrm{rank(S)} < \infty$*. Then for sufficiently small $z \in F$ the operators*

$$B(z) := \frac{1}{z}(S - S(A(z)+S)^{-1}S)$$

are well-defined and bounded on $\mathcal{H}$. Moreover, due to $A_0 = A_0^*$, they are uniformly bounded as $z \to 0$. The operator $A(z)$ has a bounded inverse in $\mathcal{H}$ if and only if $B(z)$ has a bounded inverse in $S\mathcal{H}$, and in this case

$$A(z)^{-1} = (A(z)+S)^{-1} + \frac{1}{z}(A(z)+S)^{-1}SB(z)^{-1}S(A(z)+S)^{-1}. \tag{99}$$

For the proof see [ErdSch]. It follows from this lemma that

$$A(\lambda)^{-1} = (A(\lambda)+S_1)^{-1} + \frac{1}{\lambda}(A(\lambda)+S_1)^{-1}S_1 m(\lambda)^{-1}S_1(A(\lambda)+S_1)^{-1}, \tag{100}$$

provided

$$m(\lambda) = \lambda^{-1}[S_1 - S_1(A(\lambda)+S_1)^{-1}S_1] = m(0) + \lambda m_1(\lambda)$$

is invertible for small λ on S_1L^2. However, this is indeed the case due to our assumption that zero is not an eigenvalue of H. In fact,

$$m(0) = S_1A_1(0)S_1 = \frac{i\alpha}{4\pi}S_1P_vS_1 = \frac{i}{4\pi}\langle v,\tilde{\psi}\rangle^2\tilde{\psi}\otimes\tilde{\psi} = \frac{i}{4\pi}\Big(\int_{\mathbb{R}^3} V\psi\,dx\Big)^2\|w\psi\|_2^{-2}S_1$$

is invertible on S_1L^2 because of (87). Our claims about invertibility of these operators for small λ will follow once we justify the L^2-convergence of the Neumann series

$$\begin{aligned}(A(\lambda)+S_1)^{-1} &= (A_0+S_1)^{-1} + \sum_{k=1}^{\infty}(-1)^k\lambda^k(A_0+S_1)^{-1}\left[A_1(\lambda)(A_0+S_1)^{-1}\right]^k \\ &=: (A_0+S_1)^{-1} + \lambda E_1(\lambda),\end{aligned} \tag{101}$$

$$\begin{aligned}S_1m(\lambda)^{-1}S_1 &= S_1m(0)^{-1}S_1 + \sum_{k=1}^{\infty}(-1)^k\lambda^kS_1m(0)^{-1}S_1\left[m_1(\lambda)m(0)^{-1}S_1\right]^k S_1 \\ &=: S_1m(0)^{-1}S_1 + \lambda E_2(\lambda).\end{aligned} \tag{102}$$

We shall return to this issue later. Thus, using $S_1(A_0+S_1) = (A_0+S_1)S_1 = S_1$, we obtain

$$\begin{aligned}A(\lambda)^{-1} &= \frac{1}{\lambda}S_1m(0)^{-1}S_1 + (A(\lambda)+S_1)^{-1} \\ &\quad + E_1(\lambda)S_1m(0)^{-1}S_1\,(A(\lambda)+S_1)^{-1} \\ &\quad + S_1m(0)^{-1}S_1E_1(\lambda) \\ &\quad + (A(\lambda)+S_1)^{-1}\,S_1E_2(\lambda)S_1\,(A(\lambda)+S_1)^{-1} \\ &=: -i\frac{\beta}{\lambda}S_1 + E(\lambda)\end{aligned} \tag{103}$$

with $\beta = 4\pi\Big(\int_{\mathbb{R}^3} V\psi\,dx\Big)^{-2}\|w\psi\|_2^2$. Plugging (103) into (96), we have

$$R(\lambda) = i\frac{\beta}{\lambda}R_0(\lambda)vS_1vR_0(\lambda) + R_0(\lambda) - R_0(\lambda)vE(\lambda)vR_0(\lambda). \tag{104}$$

Next, we describe the contribution of each of these terms to the sine-transform (88). We can ignore the second one, since it leads to the free case. The first term on the right-hand side of (104) yields the following expression in (88):

$$\begin{aligned}
\mathcal{S}_0(t)(x,y) &:= \frac{\beta}{\pi}\int \frac{\sin(t\lambda)}{\lambda}\chi_1(\lambda)\big[R_0(\lambda)vS_1vR_0(\lambda)\big](x,y)\,d\lambda \\
&= \|w\psi\|_2^{-2}\frac{\beta}{2\pi}\int_{\mathbb{R}^6}\int_{-t}^{t} e^{i\lambda\tau}\,d\tau\,\chi_1(\lambda)e^{i\lambda[|x-x'|+|y'-y|]}\frac{V(x')\psi(x')\,V(y')\psi(y')}{4\pi|x-x'|\,4\pi|y'-y|}\,d\lambda\,dx'dy' \\
&= \|w\psi\|_2^{-2}\frac{\beta}{2\pi}\int_{\mathbb{R}^6}\int_{-t}^{t} \widehat{\chi}_1(\tau+|x-x'|+|y'-y|)\frac{V(x')\psi(x')\,V(y')\psi(y')}{4\pi|x-x'|\,4\pi|y'-y|}\,d\tau\,dx'dy' \\
&= \|w\psi\|_2^{-2}\beta\int_{\mathbb{R}^6}\frac{V(x')\psi(x')\,V(y')\psi(y')}{4\pi|x-x'|\,4\pi|y'-y|}\,dx'dy'\int\widehat{\chi_1}(\xi)\,d\xi \\
&\quad - \|w\psi\|_2^{-2}\frac{\beta}{2\pi}\int_{\mathbb{R}^6}\int_{[|\tau|>t]} \widehat{\chi}_1(\tau+|x-x'|+|y'-y|)\frac{V(x')\psi(x')\,V(y')\psi(y')}{4\pi|x-x'|\,4\pi|y'-y|}\,d\tau\,dx'dy'
\end{aligned}$$

Using that $(-\Delta)^{-1}V\psi=\psi$, we further conclude that

$$\begin{aligned}
&\frac{\beta}{\pi}\int \frac{\sin(t\lambda)}{\lambda}\chi_1(\lambda)\big[R_0(\lambda)vS_1vR_0(\lambda)\big](x,y)\,d\lambda = \|w\psi\|_2^{-2}\beta\,\psi(x)\psi(y) \\
&- \|w\psi\|_2^{-2}\frac{\beta}{2\pi}\int_{\mathbb{R}^6}\int_{[|\tau|>t]} \widehat{\chi}_1(\tau+|x-x'|+|y'-y|)\frac{V(x')\psi(x')\,V(y')\psi(y')}{4\pi|x-x'|\,4\pi|y'-y|}\,d\tau\,dx'dy' \qquad (105)
\end{aligned}$$

We claim that the integral in (105) is bounded by t^{-1} as an operator from $L^1(\mathbb{R}^3)\to L^\infty(\mathbb{R}^3)$. To see this, estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^6}\int_{\mathbb{R}^6}\int_{[|\tau|>t]} \widehat{\chi}_1(\tau+|x-x'|+|y'-y|)\frac{V(x')\psi(x')\,V(y')\psi(y')}{4\pi|x-x'|\,4\pi|y'-y|}\,d\tau\,dx'dy'\,f(x)g(y)\,dxdy\Big| \\
&\lesssim \int_{[|x-x'|+|y-y'|<t/2]}\int_{[|\tau|>t]} |\widehat{\chi}_1(\tau+|x-x'|+|y'-y|)|\,d\tau\,\frac{|V(x')\psi(x')|\,|V(y')\psi(y')|}{|x-x'|\,|y'-y|}\,dx'dy'\,|f(x)g(y)|\,dxdy \\
&\quad + \int_{[|x-x'|+|y-y'|>t/2]}\int |\widehat{\chi}_1(\tau+|x-x'|+|y'-y|)|\,d\tau\,\frac{|V(x')\psi(x')|\,|V(y')\psi(y')|}{|x-x'|\,|y'-y|}\,dx'dy'\,|f(x)g(y)|\,dxdy \\
&\lesssim \langle t\rangle^{-N}\Big(\sup_x\int\frac{|V(x')\psi(x')|}{|x-x'|}\,dx'\Big)^2\|f\|_1\|g\|_1 + t^{-1}\sup_x\int\frac{|V(x')\psi(x')|}{|x-x'|}\,dx'\ \|V\psi\|_1\|f\|_1\|g\|_1
\end{aligned}$$

Since the expressions involving V are finite constants (note $\psi\in L^\infty$), we have proved our claim. The conclusion is that

$$\begin{aligned}
&\mathcal{S}_0(t)(x,y) = \frac{\beta}{\pi}\int \frac{\sin(t\lambda)}{\lambda}\chi_1(\lambda)\big[R_0(\lambda)vS_1vR_0(\lambda)\big](x,y)\,d\lambda = c(V)\,\psi(x)\psi(y) + K_t(x,y), \\
&\|K_t\|_{1\to\infty}\lesssim t^{-1}
\end{aligned}$$

Finally, we turn to the third term on the right-hand side of (104). We start with the convergence of the Neumann series (101). In view of (97) we see that the series for $(A(\lambda)+S_1)^{-1}$ converges for small λ provided V decays faster than a third power. By definition,

$$m(\lambda) = -S_1E_1(\lambda)S_1$$

so that

$$\begin{aligned}
m_1(\lambda) &= \lambda^{-1}[m(\lambda)-m(0)] = -\lambda^{-1}S_1(E_1(\lambda)-E_1(0))S_1 \\
&= -\sum_{k=0}^{\infty}(-1)^k\lambda^kS_1\left[A_1(\lambda)(A_0+S_1)^{-1}\right]^{k+2}S_1 + \lambda^{-1}S_1(A_1(\lambda)-A_1(0))S_1 \qquad (106)
\end{aligned}$$

The infinite series here again converges in L^2 for small λ, whereas

$$\lambda^{-1}S_1(A_1(\lambda)-A_1(0))S_1 = \lambda^{-2}S_1v(R_0(\lambda)-G_0-i\lambda G_1)vS_1$$

admits the point-wise bound on its kernel

$$\begin{aligned}&\sup_{\lambda}\Big|\lambda^{-1}S_1(A_1(\lambda)-A_1(0))S_1(x,y)\Big|\\ &\lesssim |v(x)\psi(x)|\int_{\mathbb{R}^6}|\psi(x')V(x')||x'-y'||\psi(y')V(y')|\,dx'dy'\;|v(y)\psi(y)|\\ &\lesssim |v(x)\psi(x)||v(y)\psi(y)|\end{aligned}$$

which holds provided $|V(x)|\lesssim\langle x\rangle^{-\kappa}$ with $\kappa>3$. Finally, the kernel $v(x)\psi(x)v(y)\psi(y)$ has finite Hilbert-Schmidt norm. In summary, $m_1(\lambda)$ is an L^2-bounded operator uniformly for small λ. This proves that the Neumann series (102) for $m(\lambda)^{-1}$ converges for small λ, as claimed.

We will need to control the contribution that each term in these Neumann series makes to the sine-transform (88). We start with the constant term, viz.

$$E(0) = (A_0+S_1)^{-1} + E_1(0)S_1m(0)^{-1}S_1 + S_1E_2(0)S_1 + S_1m(0)^{-1}S_1E_1(0)$$

see (103). Thus,

$$\begin{aligned}&\int_{-\infty}^{\infty}\sin(t\lambda)\chi_1(\lambda)[R_0(\lambda)vE(0)vR_0(\lambda)](x,y)\,d\lambda\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}\sin(t\lambda)e^{i\lambda[|x-x'|+|y'-y|]}\,\chi_1(\lambda)\,d\lambda\,\frac{v(x')E(0)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'\\ &=\frac{1}{32i\pi^2}\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}\delta(t+\xi+[|x-x'|+|y'-y|])\,\widehat{\chi}_1(\xi)\,d\xi\,\frac{v(x')E(0)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy' \qquad (107)\\ &\quad-\frac{1}{32i\pi^2}\int_{\mathbb{R}^6}\int_{-\infty}^{\infty}\delta(-t+\xi+[|x-x'|+|y'-y|])\,\widehat{\chi}_1(\xi)\,d\xi\,\frac{v(x')E(0)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy' \qquad (108)\end{aligned}$$

We start with the kernel $K_-(x,y;t)$ given by (108). Let $\rho(t,\xi,y-y')=t-\xi-|y-y'|$ and, as usual, $t>0$. Using arguments similar to those in the large frequency case, we conclude that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^6}K_-(x,y;t)f(x)g(y)\,dxdy\Big|\\ &\lesssim\Big|\int_{\mathbb{R}^9}\int_{[|\xi|<t/10]}\int_{[|x-x'|=\rho(t,\xi,y-y')]}\frac{f(x)}{|x-x'|}\,\sigma(dx)\,\widehat{\chi}_1(\xi)\,d\xi\,\frac{v(x')E(0)(x',y')v(y')}{|y-y'|}\,dx'dy'\,g(y)\,dy\Big| \qquad (109)\\ &\quad+\Big|\int_{\mathbb{R}^9}\int_{[|\xi|>t/10]}\int_{[|x-x'|=\rho(t,\xi,y-y')]}\frac{f(x)}{|x-x'|}\,\sigma(dx)\,\widehat{\chi}_1(\xi)\,d\xi\,\frac{v(x')E(0)(x',y')v(y')}{|y-y'|}\,dx'dy'\,g(y)\,dy\Big| \qquad (110)\end{aligned}$$

Using the usual divergence theorem argument yields

$$
\begin{aligned}
(109) \lesssim & \int\limits_{[|y-y'|<t/2]} \int\limits_{[|\xi|<t/10]} \rho(t,\xi,y-y')^{-1} \int\limits_{\mathbb{R}^3} |\nabla f(x)|\,dx\,|\widehat{\chi}_1(\xi)|\,d\xi\,\frac{|v(x')E(0)(x',y')v(y')|}{|y-y'|}\,dx'dy'\,g(y)\,dy \\
&+ \int\limits_{[|y-y'|>t/2]} \int\limits_{[|\xi|<t/10]} \int\limits_{\mathbb{R}^3} \Big[\frac{|\nabla f(x)|}{|x-x'|} + \frac{|f(x)|}{|x-x'|^2}\Big]\,dx\,|\widehat{\chi}_1(\xi)|\,d\xi\,\frac{|v(x')E(0)(x',y')v(y')|}{|y-y'|}\,dx'dy'\,g(y)\,dy \\
&\lesssim t^{-1}\|\nabla f\|_1\|\widehat{\chi}\|_1\|v\|_2\|\,|E(0)(\cdot,\cdot)|\,\|_{2\to 2}\sup_y\Big\|\frac{v(y')}{|y-y'|}\Big\|_{L^2(dy')}\|g\|_1 \\
&\quad + t^{-1}\|\widehat{\chi}\|_1\Big\|\int\frac{|f(x)||v(x')|}{|x-x'|^2}\,dx\Big\|_{L^2(dx')}\|\,|E(0)(\cdot,\cdot)|\,\|_{2\to 2}\,\|v\|_2\,\|g\|_1 \\
&\lesssim t^{-1}\|\nabla f\|_1\|g\|_1
\end{aligned}
$$

To pass to the last line, we used the fact that $E(0)$ is an absolutely bounded operator (in the terminology of [Sch1]), see Lemma 15 below. Furthermore, we used that $\|v\|_2 < \infty$, as well as that

$$
\Big\|\int\frac{|f(x)||v(x')|}{|x-x'|^2}\,dx\Big\|_{L^2(dx')} \lesssim \|v\|_{L^6}\Big\|\int\frac{|f(x)|}{|x-x'|^2}\,dx\Big\|_{L^3(dx')} \lesssim \|v\|_{L^6}\|f\|_{L^{\frac{3}{2}}} \lesssim \|\nabla f\|_1
$$

via fractional integration and Sobolev imbedding. The argument for (110) is similar. Indeed, due to the rapid decay of $\widehat{\chi_1}$ one obtains the bound

$$
(110) \lesssim \langle t\rangle^{-N}\|\nabla f\|_1\|g\|_1
$$

for arbitrary $N \geq 1$. In a similar vein, note that in (107) necessarily $\xi \leq -t$. Hence this integral contributes $\langle t\rangle^{-N}$. The conclusion is that

$$
\Big|\int\limits_{\mathbb{R}^6}\int\limits_{-\infty}^{\infty}\sin(t\lambda)\chi_1(\lambda)[R_0(\lambda)vE(0)vR_0(\lambda)](x,y)\,d\lambda\,f(x)g(y)\,dxdy\Big| \lesssim t^{-1}\|\nabla f\|_1\|g\|_1 \tag{111}
$$

For the following lemma, we call an operator with kernel K *absolutely bounded* on L^2, provided the operator with kernel $|K(x,y)|$ is also L^2 bounded. The following lemma is quite standard, see [GolSch] and [Sch1] for similar considerations.

Lemma 15. *The operator $(A_0 + S_1)^{-1}$ is absolutely bounded. In particular, $E(0)$ is also absolutely bounded.*

Proof. Since $U^2 = I$ and

$$
A_0 + S_1 = U(I + UvG_0v + US_1)
$$

we also have

$$
(A_0 + S_1)^{-1} = (I + UvG_0v + US_1)^{-1}U
$$

The operator in parentheses on the right-hand side is a Hilbert-Schmidt perturbation of the identity. Hence

$$
(I + UvG_0v + US_1)^{-1} - I
$$

is again Hilbert-Schmidt, which implies that $(A_0 + S_1)^{-1} - U$ is also Hilbert-Schmidt and thus absolutely bounded. Since U is absolutely bounded, we are done. □

The same argument which lead to (111) also yields the following bound

$$
\Big|\int\limits_{\mathbb{R}^6}\int\limits_{-\infty}^{\infty}\sin(t\lambda)\chi_1(\lambda)[R_0(\lambda)vF(\lambda)vR_0(\lambda)](x,y)\,d\lambda\,f(x)g(y)\,dxdy\Big| \lesssim t^{-1}\|\nabla f\|_1\|g\|_1
$$

provided we replace $E(0)$ with an operator-valued function $F(\lambda)$ satisfying[9]

$$\int_{-\infty}^{\infty} \Big\| |\widehat{\chi_1 F}(\xi)(\cdot,\cdot)| \Big\|_{2\to 2} d\xi < \infty \tag{112}$$

$$\Big\| |\widehat{\chi_1 F}(\xi)(\cdot,\cdot)| \Big\|_{2\to 2} \lesssim \langle \xi \rangle^{-1} \tag{113}$$

The second estimate (113) is needed for the case $|\xi| > t/10$ to obtain t-decay, whereas (112) suffices in case $|\xi| < t/10$. In fact,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \sin(t\lambda)\chi_1(\lambda)[R_0(\lambda)vF(\lambda)vR_0(\lambda)](x,y)\,d\lambda \\
&= \frac{1}{32i\pi^2} \int_{[|\eta|<t/10]} \int_{\mathbb{R}^6} \int_{-\infty}^{\infty} \delta(t+\xi+\eta+[|x-x'|+|y'-y|])\,\widehat{\chi}_1(\xi)\,d\xi\, \frac{v(x')\hat{F}(\eta)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'd\eta \\
&- \frac{1}{32i\pi^2} \int_{[|\eta|<t/10]} \int_{\mathbb{R}^6} \int_{-\infty}^{\infty} \delta(-t+\xi+\eta+[|x-x'|+|y'-y|])\,\widehat{\chi}_1(\xi)\,d\xi\, \frac{v(x')\hat{F}(\eta)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'd\eta \\
&+ \frac{1}{32i\pi^2} \int_{[|\eta|>t/10]} \int_{\mathbb{R}^6} \widehat{\chi}_1(t+\eta+[|x-x'|+|y'-y|])\, \frac{v(x')\hat{F}(\eta)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'd\eta \\
&- \frac{1}{32i\pi^2} \int_{[|\eta|>t/10]} \int_{\mathbb{R}^6} \widehat{\chi}_1(-t+\eta+[|x-x'|+|y'-y|])\, \frac{v(x')\hat{F}(\eta)(x',y')v(y')}{|x-x'|\,|y-y'|}\,dx'dy'd\eta
\end{aligned}$$

In the first two integrals, we distinguish $|\xi| > t/10$ from $|\xi| < t/10$ and use (112), whereas in the third and fourth, we use (113) and the integrability of $\hat{\chi}_1$. The reader should not be confused by the fact that in the first two integrals the ξ-integration has not been carried out whereas in the final two it has. This is due to the fact that in the first two integrals the method from before using spherical integration is needed in order to gain a factor of t^{-1}, whereas in the final two it is not. Indeed, for the final two integrals a gain of t^{-1} is obtained from (113), and the integral in η is reduced to integrating out $\hat{\chi}_1$.

We shall now prove (112) and (113) for $F(\lambda) = E(\lambda)$ where

$$\begin{aligned}
E(\lambda) &= (A(\lambda)+S_1)^{-1} + E_1(\lambda)S_1 m(0)^{-1} S_1\,(A(\lambda)+S_1)^{-1} + S_1 m(0)^{-1} S_1 E_1(\lambda) \\
&\quad + (A(\lambda)+S_1)^{-1}\,S_1 E_2(\lambda) S_1\,(A(\lambda)+S_1)^{-1}
\end{aligned} \tag{114}$$

We start with the case of $F(\lambda) = (A(\lambda)+S_1)^{-1}$. In view of the Neumann series of $(A(\lambda)+S_1)^{-1}$ it will further suffice to prove[10] that

$$\int_{-\infty}^{\infty} \Big\| |[\lambda\chi_1(\lambda)A_1(\lambda)]^{\wedge}(\xi)(\cdot,\cdot)| \Big\|_{2\to 2} d\xi \lesssim \lambda_0 \tag{115}$$

$$\Big\| |[\lambda\chi_1(\lambda)A_1(\lambda)]^{\wedge}(\xi)(\cdot,\cdot)| \Big\|_{2\to 2} \lesssim \lambda_0 \langle \xi \rangle^{-1} \tag{116}$$

More precisely, to pass from (115) to (112) for $F(\lambda) = (A(\lambda)+S_1)^{-1}$ we use the following lemma from [ErdSch], applied to each of the terms in the Neumann series of $(A(\lambda)+S_1)^{-1}$. For small λ_0 we then obtain a summable series.

---

[9] Below we will refer to this approach, which is based on checking estimates (112), (113), as the $F(\lambda)$ method.

[10] Recall that $[-2\lambda_0, 2\lambda_0]$ is the support of χ_1

Lemma 16. *For each $\lambda \in \mathbb{R}$, let $F_1(\lambda)$ and $F_2(\lambda)$ be bounded operators from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ with kernels $K_1(\lambda)$ and $K_2(\lambda)$. Suppose that K_1, K_2 both have compact support in λ and that $K_j(\cdot)(x,y) \in L^1(\mathbb{R})$ for a.e. $x, y \in \mathbb{R}^3$. Let $F(\lambda) = F_1(\lambda) \circ F_2(\lambda)$ with kernel $K(\lambda)$. Then*

$$\int_{-\infty}^{\infty} \Big\| \big|\widehat{K}(\xi)\big| \Big\|_{2\to 2} d\xi \le \Big[\int_{-\infty}^{\infty} \Big\| \big|\widehat{K_1}(\xi)\big| \Big\|_{2\to 2} d\xi\Big] \Big[\int_{-\infty}^{\infty} \Big\| \big|\widehat{K_2}(\xi)\big| \Big\|_{2\to 2} d\xi\Big].$$

This is basically just an operator version of the fact that convolution does not increase L^1-norms. For more details, see Lemma 8 in [ErdSch]. To pass from (116) to (113), we use the same idea, but in addition we need to take the supports of the convolutions into account. More precisely, we use that the support of a convolution is the arithmetic sum of the supports of the individual convolution factors. Hence, if a k-fold convolution is being evaluated at $|\xi| > L$, then at least one of the factors needs to be evaluated at $|\xi| > L/k$. On this factor we use the point-wise estimate. This leads to a loss of a polynomial factor k^2 in our estimate (113) of a k-fold convolution, which can then be absorbed into the exponential gain λ_0^k.

Let us now prove (115), (116). First, we write $\chi_1(\lambda) = \chi(\lambda/\lambda_0)$. Then

$$\begin{aligned}[\chi(\lambda/\lambda_0)\lambda A_1(\lambda)]^\wedge(\xi)(x,y) &= [v(y)\chi(\lambda/\lambda_0)(R_0(\lambda) - G_0)v(x)]^\wedge(\xi) \\ &= \lambda_0 \frac{v(x)v(y)}{4\pi|x-y|}\big(\hat\chi(\lambda_0(\xi + |x-y|)) - \hat\chi(\lambda_0\xi)\big)\end{aligned}$$

Hence,

$$\begin{aligned}\Big|[\chi(\lambda/\lambda_0)\lambda A_1(\lambda)]^\wedge(\xi)(x,y)\Big| &\lesssim \lambda_0 \frac{|v(x)v(y)|}{|x-y|}\Big|\int_{\lambda_0\xi}^{\lambda_0(\xi+|x-y|)} \hat\chi'(u)\,du\Big| \\ &\lesssim \lambda_0^2 |v(x)||v(y)|\langle\lambda_0\xi\rangle^{-N} + \lambda_0\frac{|v(x)v(y)|}{|x-y|}\chi_{[|x-y|>|\xi|]}\end{aligned}$$

and thus, with HS denoting the Hilbert-Schmidt norm,

$$\begin{aligned}&\int \Big\|[\chi(\lambda/\lambda_0)\lambda A_1(\lambda)]^\wedge(\xi)\Big\|_{HS} d\xi \\ &\lesssim \lambda_0^2 \int \|v(x)v(y)\|_{L^2(\mathbb{R}^6)}\langle\lambda_0\xi\rangle^{-N}\,d\xi + \lambda_0\int\Big\|\frac{|v(x)v(y)|}{|x-y|}\chi_{[|x-y|>|\xi|]}\Big\|_{L^2(\mathbb{R}^6)} d\xi \lesssim \lambda_0\end{aligned}$$

where we used that

$$\Big(\int_{[|x-y|>|\xi|]} \frac{|v(x)|^2|v(y)|^2}{|x-y|^2}\,dxdy\Big)^{\frac12} \lesssim \langle\xi\rangle^{(1-\kappa)/2} \tag{117}$$

provided $|V(x)| \lesssim \langle x\rangle^{-\kappa}$. So as long as $\kappa > 3$, which is also needed for $\|v\|_2 < \infty$, we obtain a bound which is integrable in ξ. This proves (115). Moreover, the point-wise bounds also hold:

$$\Big\|[\chi(\lambda/\lambda_0)\lambda A_1(\lambda)]^\wedge(\xi)\Big\|_{HS} \lesssim \lambda_0^2\langle\lambda_0\xi\rangle^{-N} + \lambda_0\langle\xi\rangle^{(1-\kappa)/2} \lesssim \lambda_0\langle\xi\rangle^{-1}$$

if we choose $N = 1$.

It remains to deal with the rank-one pieces of $E(\lambda)$, which are the last three terms in (114):

$$\begin{aligned}F_1(\lambda) &:= E_1(\lambda)S_1 m(0)^{-1}S_1\,(A(\lambda)+S_1)^{-1} \\ F_2(\lambda) &:= S_1 m(0)^{-1}S_1 E_1(\lambda) \\ F_3(\lambda) &:= (A(\lambda)+S_1)^{-1}\,S_1E_2(\lambda)S_1\,(A(\lambda)+S_1)^{-1}\end{aligned}$$

To analyze F_1, F_2, we write

$$E_1(\lambda) = -(A_0+S_1)^{-1}A_1(\lambda)(A_0+S_1)^{-1}G_1(\lambda), \qquad G_1(\lambda) := \sum_{\ell=0}^{\infty}(-1)^\ell[\lambda A_1(\lambda)(A_0+S_1)^{-1}]^\ell$$

The advantage of this representation lies with the fact that $G_1(\lambda)$ satisfies (112) and (113). This follows by the same arguments that established the same property for $(A(\lambda)+S_1)^{-1}$. To show that $F_{1,2}$ satisfy (112) and (113) it will therefore suffice to show that $A_1(\lambda)$ does. To this end, compute

$$\widehat{A_1}(\xi)(x,y) = (4\pi)^{-1}v(y)\Big[\int_0^1 e^{i\lambda|x-y|b}\,db\Big]^\wedge(\xi)\,v(x) = (4\pi)^{-1}v(y)\int_0^1 \delta(\xi-b|x-y|)\,db\,v(x)$$
$$= (4\pi)^{-1}v(y)\chi_{[0<\xi<|x-y|]}\frac{v(x)}{|x-y|}$$

Hence,

$$\Big\|\widehat{A_1}(\xi)\Big\|_{HS} \lesssim \Big\|\chi_{[0<\xi<|x-y|]}\frac{v(x)v(y)}{|x-y|}\Big\|_{L^2(\mathbb{R}^6)} \lesssim \langle\xi\rangle^{(1-\kappa)/2},$$

see (117). Since $\kappa>3$, this implies all the desired properties of A_1.

To deal with F_3, we use the representation

$$E_2(\lambda) = -S_1m(0)^{-1}S_1m_1(\lambda)S_1m(0)^{-1}S_1G_2(\lambda), \qquad G_2(\lambda) := \sum_{\ell=0}^\infty(-1)^\ell S_1[\lambda m_1(\lambda)m(0)^{-1}]^\ell S_1$$

This shows that it suffices to prove (112) and (113) for $m_1(\lambda)$. Indeed, the infinite series for G_2 will inherit these properties due to the geometric factors λ^ℓ. Returning to $m_1(\lambda)$, we use the representation (106) which allows us to write

$$m_1(\lambda) = -S_1A_1(\lambda)(A_0+S_1)^{-1}A_1(\lambda)(A_0+S_1)^{-1}G_1(\lambda)S_1 + \lambda^{-2}S_1v(R_0(\lambda)-G_0-i\lambda G_1)vS_1$$

In view of the preceding, the first term on the right-hand side here has the desired properties. The kernel of the second term here equals

$$c\,w(x)\psi(x)\int_{\mathbb{R}^6}\psi(x')V(x')\frac{e^{i\lambda|x'-y'|}-1-i\lambda|x'-y'|}{\lambda^2|x'-y'|}\psi(y')V(y')\,dx'dy'\;\; w(y)\psi(y)$$

To compute the Fourier transform in λ, we use

$$\frac{e^{i\lambda a}-1-ia\lambda}{\lambda^2 a} = -a\int_0^1(1-b)e^{i\lambda ab}\,db$$

Therefore, for $a>0$,

$$\Big[\frac{e^{i\lambda a}-1-ia\lambda}{\lambda^2 a}\Big]^\wedge(\xi) = -a\int_0^1(1-b)\delta(\xi-ba)\,db = -(1-\xi/a)\chi_{[0<\xi<a]}$$

from which we conclude that

$$\Big|\int_{\mathbb{R}^6}V(x')\psi(x')\Big[\frac{e^{i\lambda|x'-y'|}-1-i|x'-y'|\lambda}{\lambda^2|x'-y'|}\Big]^\wedge(\xi)V(y')\psi(y')\,dx'dy'\Big|$$
$$= \Big|\chi_{[\xi>0]}\int_{[|x'-y'|>\xi]}V(x')\psi(x')\Big(1-\frac{\xi}{|x'-y'|}\Big)V(y')\psi(y')\,dx'dy'\Big|$$
$$\lesssim \langle\xi\rangle^{2-\kappa+\varepsilon}$$

for arbitrary $\varepsilon>0$. It follows that both (112), (113) hold for m_1.

To summarize our small energy results, we have proved the following lemma.

Lemma 17. *Under the assumptions of Proposition 9 there exists a constant $c_0\neq 0$ such that*

$$\mathcal{S}_0(t)(x,y) = c_0\int_{\mathbb{R}^6}\int_{-t}^t\widehat{\chi}_1(\tau+|x-x'|+|y'-y|)\frac{V(x')\psi(x')\,V(y')\psi(y')}{4\pi|x-x'|\,4\pi|y'-y|}\,d\tau\,dx'dy'$$

satisfies

$$\Big\| \frac{\sin(t\sqrt{H})}{\sqrt{H}} \chi_1(H) P_c f - \mathcal{S}_0(t) f \Big\|_\infty \lesssim t^{-1} \|f\|_{W^{1,1}(\mathbb{R}^3)} \tag{118}$$

for all $t > 0$. *Also,*

$$\|\mathcal{S}_0(t) - c_0\,(\psi \otimes \psi)\|_{1\to\infty} \lesssim t^{-1}$$

In particular, $\mathcal{S}_0(t) \to c_0\,(\psi \otimes \psi)$ *as* $t \to \infty$.

It may be useful to retain $\mathcal{S}_0(t)$ in some cases, since its kernel basically lives on the set

$$\{|x| + |y| \lesssim t\}$$

in a weak sense. Proposition 9 follows by combining Lemmas 12 and Lemma 17.

Next, we state a corollary for the differentiated evolution. In conjunction with Corollary 13 this proves (60).

Corollary 18. *Let* V *satisfy* $|V(x)| + |\nabla V(x)| \lesssim \langle x \rangle^{-\kappa}$ *with* $\kappa > 3$. *Also, assume that* $V < 0$ *point-wise. Then*

$$\Big\| \nabla \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f - c_0((\nabla\psi) \otimes \psi) f \Big\|_\infty \lesssim t^{-1} \|f\|_{W^{2,1}(\mathbb{R}^3)} \tag{119}$$

for all $t > 0$.

Proof. We have already settled the high-energy case, see Corollary 13. To deal with the low energies, we need to pass a gradient through the low energy propagator above. Our assumption $V < 0$ implies that $U = -1$. This allows us to commute ∇ with U, and we can also differentiate v, w, ψ. The term $\nabla \mathcal{S}_0(t)$ does not present a problem: It gives $(\nabla\psi) \otimes \psi$ up to an operator bounded by t^{-1} from $L^1 \to L^\infty$. Next, we need to pass a gradient through the operator

$$\begin{aligned} E(\lambda) &= (A(\lambda) + S_1)^{-1} + E_1(\lambda) S_1 m(0)^{-1} S_1 \,(A(\lambda) + S_1)^{-1} + S_1 m(0)^{-1} S_1 E_1(\lambda) \\ &\quad + (A(\lambda) + S_1)^{-1}\, S_1 E_2(\lambda) S_1 \,(A(\lambda) + S_1)^{-1} \end{aligned}$$

First,

$$[\nabla, (A(\lambda) + S_1)^{-1}] = (A(\lambda) + S_1)^{-1}([S_1, \nabla] + [vR_0(\lambda)v, \nabla])(A(\lambda) + S_1)^{-1}$$

and

$$[S_1, \nabla] = (\nabla\tilde{\psi}) \otimes \tilde{\psi} + \tilde{\psi} \otimes (\nabla\tilde{\psi})$$

as well as

$$[vR_0(\lambda)v, \nabla] = -(\nabla v) R_0(\lambda) v - v R_0(\lambda)(\nabla v)$$

Clearly, similar expressions arise when commuting ∇ with $A_0 + S_1$. Second, using these relations allows us to commute ∇ through $m(0)^{-1}$, $E_1(\lambda)$, $A_1(\lambda)$, and $m_1(\lambda)$. In order to apply the $F(\lambda)$-method from above, we need to check that each of the terms arising as a commutator satisfies (112), (113). For example, consider the term

$$(A(\lambda) + S_1)^{-1}[vR_0(\lambda)v, \nabla](A(\lambda) + S_1)^{-1} = (A(\lambda) + S_1)^{-1}[(\nabla v)R_0(\lambda)v + vR_0(\lambda)(\nabla v)](A(\lambda) + S_1)^{-1}$$

We already know that $(A(\lambda) + S_1)^{-1}$ satisfy these bounds, so by Lemma 16 it suffices to check this for $(\nabla v)R_0(\lambda)v$ and $vR_0(\lambda)(\nabla v)$. Ignoring rapidly decaying tails, this amounts to showing that

$$h(\xi) := \Big(\int\!\!\int \frac{|V(x)||V(y)|}{|x-y|^2}\, \chi_{[||x-y|-|\xi|<1]}\, dxdy \Big)^{\frac12}$$

satisfies

$$\int_{-\infty}^{\infty} h(\xi)\, d\xi < \infty, \qquad h(\xi) \lesssim \langle \xi \rangle^{-1} \tag{120}$$

Since

$$h(\xi) \lesssim \langle \xi \rangle^{-1} \Big(\int\!\!\int |V(x)||V(y)|\, \chi_{[||x-y|-|\xi|<1]}\, dxdy \Big)^{\frac12}$$

the latter bound follows immediately since $V \in L^1$ and the former follows by applying Cauchy-Schwarz. The other commutators can be checked similarly. $\square$

5.2. **The cosine evolution.** Next, we state an estimate for $\cos(t\sqrt{H})P_c$.

Proposition 19. *Under the assumptions of Proposition 9, there is a kernel K_t satisfying* (25) *so that the bound*

$$\Big\| [\cos(t\sqrt{H})P_c - K_t]f \Big\|_\infty \lesssim t^{-1} \sum_{1\le|\alpha|\le 2} \|D^\alpha f\|_{L^1(\mathbb{R}^3)} \tag{121}$$

holds for all $t > 0$. Moreover, if V is as in Corollary 18, then there is another kernel $\tilde{K}_t$ satisfying (25) *so that*

$$\Big\| [\nabla \cos(t\sqrt{H})P_c - \tilde{K}_t]f \Big\|_\infty \lesssim t^{-1} \sum_{1\le|\alpha|\le 3} \|D^\alpha f\|_{L^1(\mathbb{R}^3)} \tag{122}$$

holds for all $t > 0$.

Remark 20. The main application of this proposition is the following estimate, which does require using K_t rather than an L^1 norm of f on the right-hand side:

$$\| \cos((t-s)\sqrt{H})P_g^\perp[\phi_a(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\phi_a(\cdot, a(\infty))]\|_\infty \lesssim \delta\, \langle t-s\rangle^{-1} \tag{123}$$

see Section 3. First, if $0 < t-s < 1$, then via Sobolev's imbedding, with $\psi = \phi_a(\cdot, a(\infty))$,

$$\begin{aligned}
&\| \cos((t-s)\sqrt{H})P_g^\perp[\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\psi]\|_\infty \\
&\lesssim \| \cos((t-s)\sqrt{H})P_g^\perp[\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\psi]\|_2 \\
&\qquad + \|D^2 \cos((t-s)\sqrt{H})P_g^\perp[\phi_a(\cdot, a(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\psi]\|_2 \\
&\lesssim \|P_g^\perp[\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\psi]\|_2 + \|H\cos((t-s)\sqrt{H})P_g^\perp[\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\psi]\|_2 \\
&\lesssim \|\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\psi]\|_2 + \|H[\phi_b(\cdot, b(s)) - (a(\infty)/b(s))^{\frac{5}{4}}\psi]\|_2 \\
&\lesssim \delta\, \langle s\rangle^{-1}
\end{aligned}$$

Second, let $t-s > 1$. The problem here is that the function in brackets decays only like $\langle x\rangle^{-3}$ and is therefore not in L^1, but only in weak L^1. However, its first and second derivatives are in L^1, so the bound on K_t saves us, see (25). Indeed, since

$$\int (\chi_{[|x|+|y|>t]} + \langle t\rangle^{-1})\langle x\rangle^{-1}\langle y\rangle^{-1}\, \langle y\rangle^{-3}\, dy \lesssim \langle t\rangle^{-1},$$

we obtain (123) as desired.

Proof of Proposition 19. As before, we shall only deal with the case when a resonance is present. The other case is implicit in what we are doing. We first discuss the estimate (121) without the gradient. The proof proceeds by making appropriate changes to the preceding proof. The logic here is that $\cos(t\sqrt{H})P_c = \partial_t \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c$, as in the free case. Since we have just written

$$\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c = \mathcal{S}_0(t) + \mathcal{S}_1(t)$$

where $\mathcal{S}_1(t)$ is dispersive, we obtain that

$$\cos(t\sqrt{H})P_c = \dot{\mathcal{S}}_0(t) + \dot{\mathcal{S}}_1(t)$$

with

$$\dot{\mathcal{S}}_0(t)(x,y) = c \int_{\mathbb{R}^6} [\widehat{\chi}_1(t + |x-x'| + |y-y'|) + \widehat{\chi}_1(-t + |x-x'| + |y-y'|)] \frac{V(x')\psi(x')\, V(y')\psi(y')}{4\pi|x-x'|\, 4\pi|y'-y|}\, dx'dy'$$

Via Lemma 22 below one can read off here that

$$|\dot{\mathcal{S}}_0(t)(x,y)| \lesssim (\langle t\rangle^{-1} + \chi_{[|x|+|y|>t]})(\langle x\rangle\langle y\rangle)^{-1}$$

which implies that $\dot{\mathcal{S}}_0(t)$ can be included into the kernel K_t. In particular,

$$\|\dot{\mathcal{S}}_0(t)f\|_\infty \lesssim \langle t\rangle^{-1}\|f\|_1$$

Moreover, it is implicit in the arguments for $\sin(t\sqrt{H})/\sqrt{H}$ that $\dot{\mathcal{S}}_1(t)$ dispersive. To make this explicit, first recall the estimate for the free propagator:

$$\|\cos(t\sqrt{-\Delta})f\|_\infty \lesssim t^{-1}\|D^2 f\|_{L^1(\mathbb{R}^3)}$$

This can be seen by an argument similar to (90) above:

$$\begin{aligned}
\cos(t\sqrt{H})f(x) &= \partial_t\, t\int_{S^2} f(x+ty)\,\sigma(dy)\\
&= \int_{S^2}\big[f(x+ty) + t(\nabla f)(x+ty)\cdot y\big]\,\sigma(dy)\\
&= t^{-2}\int_{[|x-y|\le t]}\Big[\nabla f(y)\cdot\frac{y-x}{t} + f(y)\frac{3}{t}\Big]\,dy + t^{-1}\int_{[|x-y|\le t]}\Delta f(y)\,dy
\end{aligned}$$

Hence,

$$\begin{aligned}
|\cos(t\sqrt{H})f(x)| &\lesssim t^{-2}\int_{[|x-y|\le t]}\Big[|\nabla f(y)| + t^{-1}|f(y)|\Big]\,dy + t^{-1}\int_{[|x-y|\le t]}|\Delta f(y)|\,dy\\
&\lesssim t^{-1}\Big(\int_{\mathbb{R}^3}|\nabla f(y)|^{\frac32}\,dy\Big)^{\frac23} + t^{-1}\Big(\int_{\mathbb{R}^3}|f(y)|^3\,dy\Big)^{\frac13} + t^{-1}\|D^2 f\|_1\\
&\lesssim t^{-1}\|D^2 f\|_1
\end{aligned}$$

To pass to the final inequality we used there the Sobolev imbedding

$$\|\nabla f\|_{\frac32} \lesssim \|D^2 f\|_1, \quad \|f\|_3 \lesssim \|\nabla f\|_{\frac32} \lesssim \|D^2 f\|_1$$

For the perturbed case, we distinguish small energies from all other energies. We start by indicating the changes to the argument for energies $\lambda\in[\lambda_0,\infty)$ that we presented above for $\frac{\sin(t\lambda)}{\lambda}$. First, recall (91). As before, with $\rho = t - \sum_{j=1}^k |x_j - x_{j+1}|$, we now need to estimate

$$\Big|\partial_t \int_{A(t)}\int_{[|x_0-x_1|=\rho(t)]}\frac{f(x_0)}{|x_0-x_1|}\,\sigma(dx_0)\,\frac{\prod_{j=1}^k V(x_j)}{\prod_{j=1}^k |x_j-x_{j+1}|}\,g(x_{k+1})\,dx_1\ldots dx_{k+1}\Big|$$

The effect of the ∂_t is twofold: We either need to replace f with $\nabla f(\cdot)\cdot\vec{n}$ (with an outward pointing normal vector $\vec{n}$), or we keep f but replace $|x_0-x_1|$ in the denominator with $|x_0-x_1|^2 = \rho(t)^2$. In the former case, the only change needed is that the final bound is in terms of $\|D^2 f\|_{L^1(\mathbb{R}^3)}$. In the latter case, the same approach to $A(t)$ as above leads to two new terms, viz.

$$\int \frac{|\nabla f(x_0)|}{|x_0-x_1|^2}\,dx_0 \tag{124}$$

$$\rho^{-3}\int_{[|x_0-x_1|<\rho]}|f(x_0)|\,dx_0 \tag{125}$$

To deal with (124) we apply fractional integration and Sobolev imbedding

$$\Big\|\int_{\mathbb{R}^3}\frac{|\nabla f(x_0)|}{|x_0-x_1|^2}\,dx_0\Big\|_{L^3(dx_1)} \lesssim \|\nabla f\|_{L^{\frac32}(\mathbb{R}^3)} \lesssim \|D^2 f\|_1$$

As for (125), it is dominated by a maximal function so that

$$\Big\|\rho^{-3}\int_{[|x_0-x_1|<\rho]} |f(x_0)|\,dx_0\Big\|_{L^3} \lesssim \|Mf\|_3 \lesssim \|f\|_3 \lesssim \|\nabla f\|_{\frac{3}{2}} \lesssim \|D^2 f\|_1$$

In passing, we remark that these arguments lead to a small potential result which is analogous to Proposition 10, see Proposition 21 below. If the potential is no longer small, then one can only sum a finite Born series, and the remainder is controlled by the exact same argument as above (which does not distinguish between $\frac{\sin(t\lambda)}{\lambda}$ and $\cos(t\lambda)$ or $e^{it\lambda}$). The conclusion is that we have proved the desired bound for all energies $\lambda \in [\lambda_0, \infty)$ with $\lambda_0 > 0$ fixed.
As for the small energies, we remark that the changes which need to be made to the analogous argument for $\frac{\sin(t\lambda)}{\lambda}$ are in the spirit of (124) and (125). We skip these details.
Finally, for the gradient estimate (122) we argue similarly to the sin-case by passing the gradient through. As explained previously, this leads to commutator terms that can be treated by the $F(\lambda)$-method. □

For the sake of completeness, we record the small potential result for the cosine.

Proposition 21. *Assume that the real-valued potential V satisfies $\|V\|_{\mathcal{K}} < 4\pi$ and $\|V\|_{L^{\frac{3}{2}}} < \infty$. Then one has the bound*

$$\Big\|\cos(t\sqrt{H})f\Big\|_\infty \lesssim t^{-1}\|D^2 f\|_{L^1(\mathbb{R}^3)}$$

for all $t > 0$.

The following technical lemma was needed in the proof of Proposition 19.

Lemma 22. *Let $0 \le w(x) \le \langle x\rangle^{-4-\varepsilon}$ for all $x \in \mathbb{R}^3$. Then*

$$I(x,y;t) := \int_{[t=|x-x'|+|y-y'|]} \frac{w(x')w(y')}{|x-x'||y-y'|}\,dx'dy' \lesssim (\langle t\rangle^{-1} + \chi_{[|x|+|y|>t/4]})(\langle x\rangle\langle y\rangle)^{-1}$$

for all $t \ge 0$, $x, y \in \mathbb{R}^3$.

Proof. For the sake of simplicity, we assume that $\min(t,|x|,|y|) > 1$. We leave it to the reader to make the necessary modifications in case this fails. First, we consider the contribution of $|x'| < \frac{1}{2}|x|$ and $|y'| < \frac{1}{2}|y|$ to the integral. It is easy to see that this is bounded by

$$\begin{aligned}
&\lesssim (\langle x\rangle\langle y\rangle)^{-1} \int_{[t=|x-x'|+|y-y'|]} w(x')w(y')\chi_{[|x'|<\frac{1}{2}|x|,\ |y'|<\frac{1}{2}|y|]}\,dx'dy' \\
&\lesssim (\langle x\rangle\langle y\rangle)^{-1}\,\chi_{[|x|+|y|>t/2]}
\end{aligned}$$

Now let us suppose that $|x'| > \frac{1}{2}|x|$ and $|y'| > \frac{1}{2}|y|$. The contribution from this regime is

$$\begin{aligned}
&\lesssim (\langle x\rangle\langle y\rangle)^{-1}\langle t\rangle^{-1} \int_{[t=|x-x'|+|y-y'|]} \frac{\langle x'\rangle w(x')}{|x-x'|}\chi_{[|x-x'|<t/2]}\ \langle y'\rangle w(y')\,dx'dy' \\
&= (\langle x\rangle\langle y\rangle)^{-1}\langle t\rangle^{-1} \int_{[|x-x'|<t/2]} \frac{\langle x'\rangle w(x')}{|x-x'|} \int_{[|y-y'|=t-|x-x'|]} \langle y'\rangle w(y')\,\sigma(dy')dx' \\
&\lesssim (\langle x\rangle\langle y\rangle)^{-1}\langle t\rangle^{-1}
\end{aligned}$$

Finally, we turn to the contribution of the regime $|x'| > \frac{1}{2}|x|$ and $|y'| < \frac{1}{2}|y|$. It is bounded by

$$\langle x\rangle^{-1}\langle y\rangle^{-1} \int_{[t=|x-x'|+|y-y'|]} \frac{\langle x'\rangle w(x')}{|x-x'|}\ \langle y'\rangle w(y')\,dx'dy'$$

If $|x-x'| > t/2$ in this integral, then we gain a factor of $\langle t\rangle^{-1}$ as desired. So assume that $|x-x'| < t/2$. Then $|y-y'| > t/2$, which in view of $|y'| < \frac{1}{2}|y|$ forces that $|y| > t/4$. This is again an allowed contribution, and the lemma follows. □

5.3. **Stability in the potential.** In this final subsection, we comment on the dependence of the bounds of this section on the potential. The bounds of this subsection are helpful in obtaining the contraction step in the nonlinear analysis of Section 4. Although that contraction step can also be carried out by other means, we chose this path since it seems novel and of independent interest. For the sake of simplicity, we will restrict ourselves to the operator

$$H_a := -\Delta + V_a, \qquad V_a(x) = -5\phi^4(x,a)$$

rather than trying to formulate this for general potentials. We also set

$$\mathcal{S}_a(t) := \frac{\sin(t\sqrt{H_a})}{\sqrt{H_a}} P^{\perp}_{g(\cdot,a)} - c_0(\psi_a \otimes \psi_a)$$

where $\psi_a := \partial_a \phi(\cdot, a)$. Then we have the following bounds:

Corollary 23. *For any $\frac{1}{2} < a, b < 2$,*

$$\begin{aligned} \big\|[\mathcal{S}_a(t) - \mathcal{S}_b(t)]f\big\|_\infty &\lesssim |a-b|\langle t\rangle^{-1}\|f\|_{W^{1,1}(\mathbb{R}^3)} \\ \big\|\nabla[\mathcal{S}_a(t) - \mathcal{S}_b(t)]f\big\|_\infty &\lesssim |a-b|\langle t\rangle^{-1}\|f\|_{W^{2,1}(\mathbb{R}^3)} \\ \big\|\big\{[\cos(t\sqrt{H_a})P^{\perp}_{g(\cdot,a)} - \cos(t\sqrt{H_b})P^{\perp}_{g(\cdot,b)}] - K_t\big\}f\big\|_\infty &\lesssim |a-b|\langle t\rangle^{-1} \sum_{1\le|\alpha|\le 2} \|D^\alpha f\|_1 \\ \big\|\big\{\nabla[\cos(t\sqrt{H_a})P^{\perp}_{g(\cdot,a)} - \cos(t\sqrt{H_b})P^{\perp}_{g(\cdot,b)}] - \tilde{K}_t\big\}f\big\|_\infty &\lesssim |a-b|\langle t\rangle^{-1} \sum_{1\le|\alpha|\le 3} \|D^\alpha f\|_1 \end{aligned} \tag{126}$$

where

$$|K_t(x,y)| \lesssim |a-b|(\langle t\rangle^{-1} + \chi_{[|x|+|y|>t/4]})(\langle x\rangle\langle y\rangle)^{-1}$$

for all $t > 0$, $x, y \in \mathbb{R}^3$, and similarly for $\tilde{K}_t$.

Proof. We first remark that

$$|V_a(x) - V_b(x)| \lesssim \langle x\rangle^{-4}|a-b|$$

as well as

$$|v_a(x) - v_b(x)| \lesssim \langle x\rangle^{-2}|a-b|$$

where $-v_a^2 = V_a$. Let us consider the difference of two typical Born-series terms that arise in the analysis of the sine evolution:

$$\begin{aligned} &\int_{\mathbb{R}^{3(k+2)}} \int_{-\infty}^{\infty} \chi_0(\lambda)\sin(t\lambda) e^{i\lambda\sum_{j=0}^k |x_j - x_{j+1}|} \frac{\prod_{j=1}^k V_a(x_j)}{\prod_{j=0}^k |x_j - x_{j+1}|} f(x_0)g(x_{k+1})\, dx_0 \ldots dx_{k+1} \\ &- \int_{\mathbb{R}^{3(k+2)}} \int_{-\infty}^{\infty} \chi_0(\lambda)\sin(t\lambda) e^{i\lambda\sum_{j=0}^k |x_j - x_{j+1}|} \frac{\prod_{j=1}^k V_b(x_j)}{\prod_{j=0}^k |x_j - x_{j+1}|} f(x_0)g(x_{k+1})\, dx_0 \ldots dx_{k+1} \end{aligned} \tag{127}$$

Since

$$\prod_{j=1}^k V_a(x_j) - \prod_{j=1}^k V_b(x_j) = \sum_{\ell=1}^k \prod_{j=1}^{\ell-1} V_a(x_j)\ (V_a(x_\ell) - V_b(x_\ell)) \prod_{j=\ell+1}^k V_a(x_j)$$

we can rewrite the difference in (127) as a sum of terms, each of which contains the difference of V_a and V_b. This allows us to gain a factor of $|a-b|$ by the same arguments we used to derive the dispersive estimate above.

The final term of the Born series, which we used to derive the high-energy estimate of Lemma 12 for H_a, involves the perturbed resolvent $(-\Delta + V_a - \lambda + i0){-}1$. Hence, we now face the difference of the two resolvents

$$(-\Delta + V_a - \lambda + i0){-}1 - (-\Delta + V_b - \lambda + i0){-}1 = (-\Delta + V_a - \lambda + i0){-}1[V_b - V_a](-\Delta + V_b - \lambda + i0){-}1$$

Hence, the analysis which involves the bounds in (94) can now be repeated, and we gain a factor of $|a-b|$ here as well using the limiting absorption bounds (95).

As far as the low energies are concerned, we need to compare the (regular) parts of the resolvents as in (104). In other words, we need to compute the sine-transform of the difference

$$R_0(\lambda)v_aE^{(a)}(\lambda)v_aR_0(\lambda) - R_0(\lambda)v_bE^{(b)}(\lambda)v_bR_0(\lambda)$$

where $E^{(a)}, E^{(b)}$ are the operators arising in (103). The only really novel term here is

$$R_0(\lambda)v_a(E^{(a)}(\lambda) - E^{(b)}(\lambda))v_aR_0(\lambda)$$

To understand the difference $E^{(a)}(\lambda) - E^{(b)}(\lambda)$, we start with the observation that

$$A_a(\lambda) - A_b(\lambda) = v_aR_0(\lambda)v_a - v_bR_0(\lambda)v_b$$

which implies that

$$\begin{aligned} A_a(\lambda)^{-1} - A_b(\lambda)^{-1} &= -A_a(\lambda)^{-1}[v_aR_0(\lambda)v_a - v_bR_0(\lambda)v_b]A_b(\lambda)^{-1} \\ &= -A_a(\lambda)^{-1}[(v_a - v_b)R_0(\lambda)v_a - v_bR_0(\lambda)(v_b - v_a)]A_b(\lambda)^{-1} \end{aligned}$$

We already now that $A_a(\lambda)^{-1}$ and $A_b(\lambda)^{-1}$ satisfy the bounds (112) and (113), i.e., they are amenable to the $F(\lambda)$ method. However, by (120) the middle piece $(v_a - v_b)R_0(\lambda)v_a$ (and its symmetric counterpart) satisfies the same bounds. Moreover, these bounds come with a factor of $|a - b|$ which is the desired gain.

The other constituents of $E^{(a)}$ (and of $E^{(b)}$) are one-dimensional, and they involve the following basic building blocks:

$$S_1^{(a)},\ m^{(a)}(0),\ E_1^{(a)}(\lambda),\ E_2^{(a)}(\lambda)$$

and the latter two can be further reduced to the pieces

$$A_1^{(a)}(\lambda),\ m_1^{(a)}(\lambda)$$

in view of the Neumann series (101) and (102). Thus, in order to show that the low-frequency dispersive estimate gains a factor of $|a - b|$, it will suffice to show that the difference of any two of these pieces with parameter values a and b satisfies the bounds (112) and (113) with a gain of $|a - b|$. By (98),

$$S_1^{(a)} = \|v_a\psi_a\|_2^{-2}\ w_a\psi_a \otimes w_a\psi_a$$

so that

$$\|S_1^{(a)} - S_1^{(b)}\|_{2\to 2} \lesssim |a - b|$$

The operators $m^{(a)}(0) - m^{(b)}(0)$ are equally easy to deal with. We leave it to the reader to verify the $F(\lambda)$ property for the pair-wise (i.e., relative to a and b) differences of each the operators $A_1^{(a)}(\lambda),\ m_1^{(a)}(\lambda)$, gaining a factor $|a - b|$ in the process. As for the former, the logic is that

$$A_1^{(a)}(\lambda) - A_1^{(b)}(\lambda) = \frac{1}{\lambda}(v_a - v_b)(x)\frac{e^{i\lambda|x-y|} - 1}{4\pi|x - y|}v_a(y) - \frac{1}{\lambda}v_b(x)\frac{e^{i\lambda|x-y|} - 1}{4\pi|x - y|}(v_b(y) - v_a(y)) \tag{128}$$

and the two constituents on the right-hand side are again of the same form as $A_1(\lambda)$. Since we have established the $F(\lambda)$ property for this operator in the proof of Lemma 17 above, it follows that the same holds for the difference (128). Moreover, we gain a factor $|a-b|$ in the constants. The same type of logic applies to $m_1(\lambda)$, and we leave the details to the reader. To conclude the proof of (126), it only remains to control the difference of $\mathcal{S}_0 - c_0(\psi \otimes \psi)$ for a and b, see (105). However, this is an explicit multi-linear expression in the potentials, and can be estimated just like above.

The remaining bounds stated in this corollary can be proved by the basically identical arguments, which we skip. $\square$

References

[Agm] Agmon, S. *Spectral properties of Schrödinger operators and scattering theory.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.

[Aub] Aubin, T. *Nonlinear analysis on manifolds. Monge-Ampère equations.* Grundlehren der Mathematischen Wissenschaften, 252. Springer-Verlag, New York, 1982.

[Bea] Beals, M. *Optimal L^∞ decay for solutions to the wave equation with a potential.* Comm. Partial Differential Equations 19 (1994), no. 7-8, 1319–1369.

[BeaStr] Beals, M., Strauss, W. *L^p estimates for the wave equation with a potential.* Comm. Partial Differential Equations 18 (1993), no. 7-8, 1365–1397.

[BizChmTab] Bizoń, P., Chmaj, T., Tabor, Z. *On blowup for semilinear wave equations with a focusing nonlinearity.* Nonlinearity 17 (2004), no. 6, 2187–2201.

[Cuc] Cuccagna, S. *On the wave equation with a potential.* Comm. Partial Differential Equations 25 (2000), no. 7-8, 1549–1565.

[D'AnPie] D'Ancona, P., Pierfelice, V. *Dispersive estimate for the wave equation with a rough potential,* to appear in JFA

[ErdSch] Erdoğan, M. B., Schlag, W. *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: I*, Dynamics of PDE, vol. 1, no. 4 (2004), 359–379.

[GeoVis] Georgiev, V., Visciglia, N. *Decay estimates for the wave equation with potential.* Comm. Partial Differential Equations 28 (2003), no. 7-8, 1325–1369.

[GolSch] Goldberg, M., Schlag, W. *Dispersive estimates for Schrödinger operators in dimensions one and three.* Comm. Math. Phys. 251 (2004), no. 1, 157–178.

[GriShaStr1] Grillakis, M., Shatah, J., Strauss, W. *Stability theory of solitary waves in the presence of symmetry. I.* J. Funct. Anal. 74 (1987), no. 1, 160–197.

[GriShaStr2] Grillakis, M., Shatah, J., Strauss, W. *Stability theory of solitary waves in the presence of symmetry. II.* J. Funct. Anal. 94 (1990), 308–348.

[JenKat] Jensen, A., Kato, T. *Spectral properties of Schrödinger operators and time-decay of the wave functions.* Duke Math. J. 46 (1979), no. 3, 583–611.

[JenNen] Jensen, A., Nenciu, G. *A unified approach to resolvent expansions at thresholds.* Rev. Math. Phys. 13 (2001), no. 6, 717–754.

[KriSch1] Krieger, J., Schlag, W. *Stable manifolds for all monic supercritical NLS in one dimension*, preprint 2005.

[KriSch2] Krieger, J., Schlag, W. *Non-generic blow-up solutions for the critical focusing NLS in 1-d*, preprint 2005.

[Lev] Levine, H. *Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$.* Trans. Amer. Math. Soc. 192 (1974), 1–21.

[MerZaa1] Merle, F., Zaag, H. *On growth rate near the blowup surface for semilinear wave equations.* Int. Math. Res. Not. 2005, no. 19, 1127–1155.

[MerZaa2] Merle, F., Zaag, H. *Determination of the blow-up rate for a critical semilinear wave equation.* Math. Ann. 331 (2005), no. 2, 395–416.

[MerZaa3] Merle, F., Zaag, H. *Determination of the blow-up rate for the semilinear wave equation.* Amer. J. Math. 125 (2003), no. 5, 1147–1164.

[Pie] Pierfelice, V. *Decay estimate for the wave equation with a small potential.* preprint 2003, to appear.

[RodSch] Rodnianski, I., Schlag, W. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials.* Invent. Math. 155 (2004), 451–513.

[Sch1] Schlag, W. *Dispersive estimates for Schrödinger operators in dimension two*, Comm. Math. Phys. 257 (2005), 87–117.

[Sch2] Schlag, W. *Stable manifolds for an orbitally unstable NLS*, preprint 2004.

[Sch3] Schlag, W. *A remark on Littlewood-Paley theory for the distorted Fourier transform*, preprint 2005.

[SofWei1] Soffer, A., Weinstein, M. *Multichannel nonlinear scattering for nonintegrable equations.* Comm. Math. Phys. 133 (1990), 119–146

[SofWei2] Soffer, A., Weinstein, M. *Multichannel nonlinear scattering, II. The case of anisotropic potentials and data.* J. Diff. Eq. 98 (1992), 376–390

[Str] Strauss, W. *Nonlinear wave equations.* CBMS Regional Conference Series in Mathematics, 73. AMS, Providence, RI, 1989.

[Szp] Szpak, N. *Relaxation to intermediate attractors in nonlinear wave equations*, Theoretical and Mathematical Physics, Vol. 127 (2001), No. 3, pp. 817-826.

[Yaj1] Yajima, K. *The $W^{k,p}$-continuity of wave operators for Schrödinger operators.* J. Math. Soc. Japan 47 (1995), no. 3, 551–581.

Harvard University, Dept. of Mathematics, Science Center, 1 Oxford Street, Cambridge, MA 02138, U.S.A.
E-mail address: `jkrieger@math.harvard.edu`

Department of Mathematics, University of Chicago, 5734 South University Ave., Chicago, IL 60637, U.S.A.
E-mail address: `schlag@math.uchicago.edu`