GLOBAL REGULARITY OF WAVE MAPS FROM $\mathbf{R^{3+1}}$ TO SURFACES

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Abstract. We consider Wave Maps with smooth compactly supported initial data of small $\dot{H}^{3/2}$-norm from $\mathbf{R^{3+1}}$ to certain 2-dimensional Riemannian manifolds and show that they stay smooth globally in time. Our methods are based on the introduction of a global Coulomb Gauge as in [17], followed by a dynamic separation as in [8]. We then rely on an adaptation of T.Tao's methods used in his recent breakthrough result [24].

1. Introduction

Let M be a Riemannian manifold with metric $(g_{ij}) = g$. A Wave Map $u : \mathbf{R^{n+1}} \to M$, $n \geq 2$ is by definition a solution of the Euler-Lagrange equations associated with the functional $u \to \int_{\mathbf{R^{n+1}}} < \partial_\alpha u, \partial^\alpha u >_g d\sigma$. Here the usual Einstein summation convention is in force, while $d\sigma$ denotes the volume measure on $\mathbf{R^{n+1}}$ with respect to the standard metric. In local coordinates, u is seen to satisfy the equation

$$\Box u_i + \Gamma^i_{jk}(u)\partial_\alpha u_j \partial^\alpha u_k = 0 \tag{1}$$

where Γ^i_{jk} refer to the Riemann-Christoffel symbols associated with the metric g. The relevance of this model problem arises from its connections with more complex nonlinear wave equations of mathematical physics: for example, Einstein's vacuum equations under $U(1)$-symmetry attain the form of a Wave Maps equation coupled with additional elliptic equations. More specifically, Einstein's equations in this case can be cast in terms of a Wave Map $u : (M, g) \to \mathbf{H}^2$, the target being the standard hyperbolic plane with metric h_{ij}, as follows:

$$R_{\alpha\beta} = h_{ij}\partial_\alpha u^i \partial_\beta u^j$$

$$\Box_g u^i = -\Gamma^i_{jk}(u) g^{\alpha\beta}\partial_\alpha u^j \partial_\beta u^k$$

The 2nd equation here is of Wave Maps type, on a curved background. Our model equation deals with the simpler case involving a flat background, but the hope is that the techniques for the latter problem will eventually elucidate the more complicated former problem.

We are interested in the well-posedness of the Cauchy problem for (1) with initial data $u[0]\times\partial_t u[0]$ at time $t=0$ in $H^s\times H^{s-1}$. Classical theory relying on the energy inequality and Sobolev inequalities allows one to deduce local well-posedness in H^s for $s>\frac{n}{2}+1$.
Ideally, one would like to prove local well-posedness in $H^{\frac{n}{2}}$, as this would immediately imply global in time well-posedness. The reason for this is that $\dot{H}^{\frac{n}{2}}$ is the Sobolev space invariant under the natural scaling associated with (1). Unfortunately, it is known that "strong well-posedness" in the sense of analytic or even C^2-dependence on the initial data fails at the $H^{\frac{n}{2}}$-level, $n\geq 3$ [1], [22]. Thus the best result to be hoped for is global regularity of Wave Maps with smooth initial data of small $\dot{H}^{\frac{n}{2}}$-norm.
In two space dimensions, the scale invariant Sobolev space coincides with the classical $\dot{H}^1$, and numerical data as well as the conjectured non-concentration of energy suggest global regularity for Wave Maps with arbitrary smooth initial data, *provided the target is negatively curved*. Non-concentration of energy has been proved by M. Struwe for rotationally symmetric smooth Wave Maps to spheres [20] after earlier work of Christodoulou-Tahvildar-Zadeh[3] establishing the corresponding result for geodesically convex targets. Also, Shatah-Tahvildar-Zadeh [21] showed the corresponding result for smooth equivariant Wave Maps to geodesically convex targets[1]. Moreover, numerical simulations of smooth equivariant Wave Maps to S^2 with large initial data by P.Bizon [2] suggest development of singularities. This underlines the importance of the hyperbolic plane as target manifold.

In the quest for reaching the critical $\frac{n}{2}$ regularity, local well-posedness for (1)with initial data in $H^{\frac{n}{2}+\epsilon}$, $\epsilon>0$ was proved for $n\geq 3$ by Klainerman and Machedon in [6], and for $n=2$ in [11]. Later, Tataru established global in time well-posedness for small data in the Besov space $B^{\frac{n}{2},1}$, [26], [27]. Note that $\dot{B}^{\frac{n}{2},1}$ has the same scaling as $\dot{H}^{\frac{n}{2}}$, but unlike the latter controls L^∞.

An important breakthrough with respect to global regularity was recently achieved by T.Tao in the case of Wave Maps to the sphere [23], [24], proving global regularity for smooth initial data small in $\dot{H}^{\frac{n}{2}}$: Tao's work exemplifies the importance of taking the global geometry of the target into account, an aspect largely ignored by the local formulation (1). Embedding the target sphere in an ambient Euclidean space, the Wave Maps equation considered by Tao takes the form

$$\Box u=-u\partial_\alpha u^t\partial^\alpha u=-(u\partial_\alpha u^t-\partial_\alpha u u^t)\partial^\alpha u \tag{2}$$

α as usual runs over the space-time indices $0,1,...n$. The nonlinearity encodes both geometric (skew-symmetry of $u\partial_\alpha u^t-\partial_\alpha u u^t$) as well as algebraic information

[1]For a nice account of these matters, see [18]

('null-form' structure). Tao manages to analyze all possible frequency interactions of the nonlinearity up to the case in which the derivatives fall on high frequency terms while the undifferentiated term has very low frequency. This bad case is then gauged away, using the skew-symmetric structure. With this method, which served as inspiration for the following developments, as well as sophisticated methods from harmonic analysis, Tao manages to go all the way to $n = 2$(note that the smaller the dimension, the more difficult the problem on account of the increasing scarcity of available Strichartz estimates).

After Tao, Klainerman and Rodnianski [9], extended this result to Wave Maps from $\mathbf{R^{n+1}}, n \geq 5$ to more general and in particular noncompact targets. More precisely, Klainerman and Rodnianski consider parallelizable targets which are well-behaved at infinity. Upon introducing a global orthonormal frame $\{e_i\}$, they define the new variables ϕ^i_α defined by $u_*\partial_\alpha = \phi^i_\alpha e_i$. It turns out that these satisfy the system of equations

$$\partial_\beta \phi^i_\alpha - \partial_\alpha \phi^i_\beta = C^i_{jk}\phi^j_\alpha \phi^k_\beta \tag{3}$$

$$\partial_\alpha \phi^{i\alpha} = -\Gamma^i_{jk}\phi^j_\beta \phi^k_\gamma m^{\beta\gamma} \tag{4}$$

where $m_{\beta\gamma}$ is the standard Minkowski metric on $\mathbf{R^{n+1}}$ and C^i_{jk}, Γ^i_{jk} are defined as follows:

$$[e_j, e_k] = C^i_{jk} e_i \tag{5}$$

$$\nabla_{e_j} e_k = \Gamma^i_{jk} e_i \tag{6}$$

There is again a skew-symmetric structure present in this formulation on account of $\Gamma^i_{jk} = -\Gamma^k_{ji}$. Moreover, by contrast with Tao's formulation (2), the boundedness of ϕ is replaced here by the boundedness of the C^i_{jk}, Γ^i_{jk}. Klainerman and Rodnianski impose in addition the condition that all derivatives of these coefficients be bounded, or in their terminology that M be 'boundedly parallelizable.'
If one now passes to the wave equation satisfied by the vector $\phi_\alpha := \{\phi^i_\alpha\}$, one obtains

$$\Box \phi_\alpha = -R_\mu \partial^\mu \phi_\alpha + E \tag{7}$$

where R_μ is skew-symmetric and moreover depends linearly on ϕ, provided we assume the C^i_{jk}, Γ^i_{jk} to be constant for simplicity's sake. E is a cubic polynomial in ϕ. By contrast with (2), the leading term in the nonlinearity is 'quadratic in ϕ'.

It is now possible to control all possible frequency interactions on the right hand side ($n \geq 5$) except when R_μ is localized to very low frequency while $\partial^\mu \phi$ is at large frequency. However, as Klainerman and Rodnianski observed, the curvature

$$\partial_\nu R_\mu - \partial_\mu R_\nu + [R_\mu, R_\nu] \tag{8}$$

when R is reduced to low frequencies is 'very small', in the sense that it is quadratic in ϕ, hence amenable to good Strichartz estimates. To take advantage of this, they introduce a Coulomb Gauge $\sum_{j=1}^{3} \partial_j \tilde{R}_j = 0$, which allows one to replace the R_μ in (7) by $\tilde{R}_\mu$ which is 'quadratic in ϕ', effectively replacing the nonlinearity by a term which is trilinear in ϕ and hence easily handled by Strichartz estimates. The general philosophy here is that the higher the degree of the nonlinearity, the more room is available to apply Strichartz estimates. Klainerman and Rodnianski's method is thus similar to Tao's in that it utilizes a microlocal Gauge Change to deal with specific bad frequency interactions.

The last result to be mentioned in this development is the simplification and extension of the previous arguments to include the case of $4+1$-dimensional Wave Maps to esssentially arbitrary targets achieved by Shatah-Struwe [17] and (in more restrictive formulation) Uhlenbeck-Stefanov-Nahmod [14]. The former observed that using a Coulomb Gauge, in a similar fashion as above, at the beginning without carrying out a frequency decomposition allows one to reduce the nonlinearity to a form directly amenable to Strichartz estimates. This allows them to avoid the microlocal Gauge Change of Tao and leads to a remarkable simplification of the argument. In addition, they are also able to treat the case of dimension $4+1$.

The methods in [9] and [17] run into serious difficulties for $3+1$-dimensional Wave Maps, and even more so for $2+1$-dimensional Wave Maps. This can be seen intuitively as follows:
The global Coulomb Gauge puts the leading term of the nonlinearity roughly into the form $D^{-1}(\phi^2)D\phi$. In dimensions 4 and higher, Shatah and Struwe can estimate such terms relying on the Strichartz type inequality for Lorentz spaces

$$||\phi||_{L_t^2 L_x^{2n,2}} \leq C||\Box\phi||_{L_t^1 H^\sigma} + C||\phi[0]||_{H^{\frac{n}{2}-1}} \tag{9}$$

where $\sigma = \frac{n}{2} - 2$. This can be used to estimate the $L_t^1 L_x^\infty$-norm of $D^{-1}(\phi^2)$. [2]
However, in three space dimensions, the above estimate fails. In order to handle the case when $D^{-1}(\phi^2)$ has much lower frequency than $D\phi$, one would have to use an endpoint $L_t^2 L_x^\infty$-Strichartz estimate, which is false, even replacing the L_x^∞-norm by BMO, see [25] .

[2]Alternatively, as pointed out by Klainerman and Rodnianski, one can utilize an improved bilinear version of Strichartz estimates in [12] to handle these cases.

The present paper starts with the basic formulation (3), (4) of Klainerman and Rodnianski applied to the simplified context of a 2-dimensional Riemannian manifold (M, g), but utilizes the Coulomb Gauge right at the beginning as do Shatah and Struwe. The main innovation over the preceding then is to introduce a special null-structure into the nonlinearity by way of what we term a *dynamic separation*[3], a method introduced first in [8]: in our context, we introduce 'twisted variables' $\theta^i_\alpha := A^i_k(u)\phi^k_\alpha$ for suitable well-behaved functions $A^i_k(u)$, and utilize the div-curl system satisfied by these to split them into a dynamic part, which has the form of a gradient, and an elliptic part, which satisfies an elliptic div-curl system. Substituting these components into the leading term of the nonlinearity results in a fairly complicated trilinear null-structure[4], as well as error terms at least quadrilinear. These are decomposed into quadrilinear null-forms and error terms at least quintilinear, iterating dynamic separation. In order to estimate the trilinear and quadrilinear null-structures, we have to refer to estimates in [13] which were derived using the technical framework set forth in [24]. Moreover, in order to control the 'twisted variables' we have to prove a sort of 'Gauge Change estimate' (Proposition 3.1) which is new for the spaces introduced in [24]. Part of what distinguishes our setup from Tao's is that we are working at the level of the derivative of the Wave Map. In particular, high-high interactions become more delicate.
The result proved in this paper certainly extends to higher-dimensional targets[5] satisfying similar constraints as the two-dimensional ones considered in this paper.

Our main theorem is the following: Let (M, g) be a 2-dimensional Riemannian manifold, which satisfies one of the following technical conditions:

(1): *M is boundedly parallelizable*[6] *and there exists an isometric embedding $i : (M, g) \hookrightarrow (\mathbf{R}^k, \delta_{ij})$ 'which doesn't twist much' in the following sense: there exists an orthonormal frame $(e_1(x), e_2(x))$, $x \in M$ for TM and an extension $(\tilde{e}_1(x), \tilde{e}_2(x))$ of $(e_1(x), e_2(x))$, $x \in i(M)$ to a neighborhood of $i(M)$ in $\mathbf{R}^k$ such that all the derivatives of the $\tilde{e}_i(x)$ are bounded.*

(2): *M is a compact surface.* Choose an isometric embedding $i : (M, g) \hookrightarrow (\mathbf{R}^k, h)$, where $h = (h_{ij})$ is a metric agreeing with the standard (δ_{ij}) outside of a compact set, such that $i(M)$ is a totally geodesic submanifold of $(\mathbf{R}^k, h)$. That this is possible is shown in [3].

[3]This terminology was suggested by S.Klainerman

[4]This is to be contrasted with the null-structure in [8], which is bilinear

[5]Our restriction on the dimension of the target ensures the commutativity of the Gauge Group. This allows us to avoid certain technicalities related to controlling the Gauge Change. However, the method of Tao ('approximate Gauge Change') as in [24] or in [9] should handle the general case.

[6]This notion was introduced by Klainerman-Rodnianski and means that there exists a global orthonormal frame (e_1, e_2) for TM such that the functions Γ^k_{ij}, C^k_{ij} defined via $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$, $[e_i, e_j] = C^k_{ij} e_k$ have bounded derivatives of all orders

(3): $M = \mathbf{H}^2$, *the hyperbolic plane.* Use the standard coordinates $(\mathbf{x}, \mathbf{y})$, $\mathbf{y} > 0$ with respect to which the metric attains the form $dg = \frac{d\mathbf{x}^2 + d\mathbf{y}^2}{\mathbf{y}^2}$.

Then the following theorem holds true:

Theorem 1.1. *Let M be one of the above. Then there exists a number $\epsilon > 0$ with the following property: Let $(u(0), \partial_t u(0)) : \mathbf{R}^3 \to (M, TM)$ be smooth initial data satisfying the property*[7]

$$\sum_{\alpha=0}^{3} ||\partial_\alpha (i \circ u)(0)||_{\dot{H}^{\frac{1}{2}}} < \epsilon$$

in situations (1), (2), *or*

$$\sum_{\alpha=0}^{3} [||\frac{\partial_\alpha(\mathbf{x} \circ u)}{\mathbf{y}}||_{\dot{H}^{\frac{1}{2}}} + ||\frac{\partial_\alpha(\mathbf{y} \circ u)}{\mathbf{y}}||_{\dot{H}^{\frac{1}{2}}}] < \epsilon$$

in the third situation. Then there exists a global (in time) smooth Wave Map $u : \mathbf{R}^{3+1} \to M$ with these initial conditions.

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2. Outline of the argument

2.1. **Basic formulation of the problem.** This section will serve as outline for the rest of the paper, explaining the strategy for proving the theorem cited at the end of the last section in the case $M = \mathbf{H}^2$.
We translate the problem to the level of the derivative, utilizing the formulation (3), (4) with respect to the global orthonormal frame $\{-\mathbf{y}\partial_{\mathbf{x}}, -\mathbf{y}\partial_{\mathbf{y}}\}$ for $T\mathbf{H}$. More explicitly, we have

$$\phi_\alpha^1 = -\frac{\partial_\alpha \mathbf{x}}{\mathbf{y}}, \, \phi_\alpha^2 = -\frac{\partial_\alpha \mathbf{y}}{\mathbf{y}} \tag{10}$$

The div-curl system satisfied by these quantities is then of the following form:

$$\partial_\beta \phi_\alpha^1 - \partial_\alpha \phi_\beta^1 = \phi_\alpha^1 \phi_\beta^2 - \phi_\alpha^2 \phi_\beta^1 \tag{11}$$

[7] The Sobolev spaces are defined by picking a point $p \in M$ and considering the functions $i \circ u - i \circ p$ instead of $i \circ u$.

$$\partial_\beta \phi^2_\alpha - \partial_\alpha \phi^2_\beta = 0 \tag{12}$$

$$\partial_\alpha \phi^{1\alpha} = -\phi^1_\alpha \phi^{2\alpha} \tag{13}$$

$$\partial_\alpha \phi^{2\alpha} = \phi^1_\alpha \phi^{1\alpha} \tag{14}$$

α, β here vary over the space-time indices $0, 1, 2, 3$, and Einstein's summation convention is in force.

Once we can show that the ϕ^i_α stay smooth globally in time, the actual Wave Map can be obtained by integration from $(-\frac{\partial_t \mathbf{x}}{\mathbf{y}}, -\frac{\partial_t \mathbf{y}}{\mathbf{y}}) = (\phi^1_0, \phi^2_0)$.
Letting ϕ_α denote the column vector with entries $\phi^1_\alpha, \phi^2_\alpha$, we obtain the following wave equations:

$$\Box \phi_\alpha = M_\nu \partial^\nu \phi_\alpha + "\phi^3", \tag{15}$$

where

$$M_\nu = \begin{pmatrix} 0 & -2\phi^1_\nu \\ 2\phi^1_\nu & 0 \end{pmatrix}, \tag{16}$$

and "ϕ^3" refers to a vector with entries that are cubic polynomials in the ϕ^i_α. The fine structure of these entries will actually be relevant later on, but we leave it out for the present discussion.

As explained in the introduction, this formulation does not lend itself to good estimates.

2.2. **Introducing the global Coulomb Gauge.** We now try to modify the matrix M_ν by adding a term of the form $2\partial_\nu A$, in such a way that the resulting matrix $\tilde{M}_\nu = M_\nu + 2\partial_\nu A$ has better properties. More precisely, we want this to depend 'quadratically' on ϕ. This can be achieved by utilizing the Coulomb Gauge condition $\sum_{j=1}^3 \partial_j \tilde{M}_j = 0$, whence $A = -\frac{1}{2} \triangle_x^{-1} \sum_{j=1}^3 \partial_j M_j$.
Indeed, observe that the $\tilde{M}_\nu$ satisfy the following div-curl system:

$$\sum_{i=1}^3 \partial_i \tilde{M}_i = 0,\ \partial_\nu \tilde{M}_\mu - \partial_\mu \tilde{M}_\nu = \begin{pmatrix} 0 & 2(\phi^1_\nu \phi^2_\mu - \phi^1_\mu \phi^2_\nu) \\ -2(\phi^1_\nu \phi^2_\mu - \phi^1_\mu \phi^2_\nu) & 0 \end{pmatrix}, \tag{17}$$

whence

$$\tilde{M}_\nu = \begin{pmatrix} 0 & 2\sum_{i=1}^{3}\triangle^{-1}\partial_i(\phi_\nu^1\phi_i^2 - \phi_i^1\phi_\nu^2) \\ -2\sum_{i=1}^{3}\triangle^{-1}\partial_i(\phi_\nu^1\phi_i^2 - \phi_i^1\phi_\nu^2) & 0 \end{pmatrix}, \quad (18)$$

or in a first approximation $\tilde{M}_\nu = "D^{-1}(\phi^2)"$.
We can now set $U = e^A$ and obtain

$$U^{-1}\Box(U\phi_\alpha) = U^{-1}\Box(U)\phi_\alpha + \tilde{M}_\nu\partial^\nu\phi_\alpha + "\phi^3" \quad (19)$$

Of course, we use the commutativity of the Gauge group for 2-dimensional target. The difference between this wave equation for $U\phi_\alpha$ and (15) is that the nonlinearity here consists of trilinear expressions. In particular, this modification suffices to handle the case of $4+1$-dimensional Wave Maps. For this, observe for example that one can easily estimate the $L_t^1L_x^2$-norm of $\tilde{M}_\nu\partial^\nu\phi_\alpha$ since this is morally $D^{-1}(\phi^2)D\phi$ and

$$||D^{-1}(\phi^2)D\phi||_{L_t^1L_x^2} \leq C||\phi||^2_{L_t^2L_x^{8,2}}||\phi||_{L_t^\infty H_x^1} \quad (20)$$

The right-hand terms are controlled by means of Strichartz' inequalities. Similarly, one can estimate the remaining terms of the nonlinearity in the $L_t^1L_x^2$-norm.
This is Shatah and Struwe's method for $\mathbf{H^2}$. One can also estimate this term using the improved bilinear Strichartz estimate for $D^{-1}(\phi^2)$ in [12], as observed by Klainerman and Rodnianski.

For the 3-dimensional case, Strichartz' estimates alone don't seem sufficient. This can be seen by analyzing the case when $D^{-1}(\phi^2)$ has very low frequency while $D\phi$ has large frequency; in order to recoup the exponential loss caused by D^{-1}, one seems to be forced to employ a $L_t^2L_x^\infty$ Strichartz estimate, which unfortunately doesn't exist. To proceed, we need to take into account more of the special structure of the nonlinear terms.

2.3. **Implementing the dynamic separation.** We use complex notation. Introduce the variables $\phi_\alpha = \phi_\alpha^1 + i\phi_\alpha^2$. Then introduce the 'twisted variables'

$$\psi_\alpha := \psi_\alpha^1 + i\psi_\alpha^2 := e^{-i\Phi}\phi_\alpha$$

where $\Phi := \triangle^{-1}\sum_{k=1}^{3}\partial_k\phi_k^1$ ($\triangle$ stands for $\triangle_x$.) This is of course the same Gauge Change as in the previous subsection, in complex notation. The precise wave equation satisfied by the ψ_α is the following:

$$\Box\psi_\alpha = 2ie^{-i\Phi}\triangle^{-1}(\sum_{k=1}^{3}\partial_k[\phi_k^1\phi_\nu^2 - \phi_k^2\phi_\nu^1])\partial^\nu\phi_\alpha + "\phi^3" - [i\Box\Phi + \partial_\nu\Phi\partial^\nu\Phi]\psi_\alpha$$

The most difficult term on the right-hand side is the first summand, which we also refer to as the 'leading term'. It can be cast into the more concise form (modulo quadrilinear error terms)

$$\triangle^{-1}\sum_{k=1}^{3}\triangle^{-1}\partial_k[\psi_k^1\psi_\nu^2-\psi_k^2\psi_\nu^1]\partial^\nu\psi_\alpha$$

Now observe that the ψ_α satisfy a special curl-system, namely the following:

$$\partial_\alpha\psi_\beta-\partial_\beta\psi_\alpha=i\psi_\beta\triangle^{-1}\sum_{j=1}^{3}(\psi_\alpha^1\psi_j^2-\psi_j^1\psi_\alpha^2)-i\psi_\alpha\triangle^{-1}\sum_{j=1}^{3}(\psi_\beta^1\psi_j^2-\psi_\beta^2\psi_j^1) \tag{21}$$

The *dynamic separation* consists in decomposing

$$\psi_\nu=-R_\nu\Psi+\chi_\nu:=-R_\nu\sum_{k=1}^{3}R_k\psi_k+\chi_\nu$$

where R_ν denotes the Riesz multiplier $\sqrt{-\triangle_x}^{-1}\partial_\nu$, $\nu=0,1,2,3$. The χ_ν ('elliptic part') in turn are determined by the following elliptic div-curl system, which is easily verified:

$$\sum_{j=1}^{3}\partial_j\chi_j=0$$

$$\partial_i\chi_\nu-\partial_\nu\chi_i=\partial_i\psi_\nu-\partial_\nu\psi_i$$

This in addition to (21) implies that

$$\chi_\nu=i\sum_{k,j=1}^{3}\triangle^{-1}\partial_i(\psi_\nu\triangle^{-1}\partial_j[\psi_i^1\psi_j^2-\psi_j^1\psi_i^2]-\psi^i\triangle^{-1}\partial_j[\psi_\nu^1\psi_j^2-\psi_j^1\psi_\nu^2]) \tag{22}$$

Passing to real and imaginary parts, we can write $\psi_\nu^1=-R_\nu\Psi^1+\Re\chi_\nu$, $\psi_\nu^2=-R_\nu\Psi^2+\Im\chi_\nu$, where $\Psi^a=\sum_{k=1}^3 R_k\psi_k^a$.

The *dynamic separation* now enables us to decompose the leading term of the non-linearity into a trilinear term with a special null-structure and an error terms which are at least quintilinear in the ψ_α^i. More precisely, upon substituting the gradient parts $R_\nu\Psi$ for ψ_ν, we modify the leading term to the following:

$$\sum_{j=1}^{3}\triangle^{-1}\partial_j[R_j\Psi^1R_\nu\Psi^2-R_j\Psi^2R_\nu\Psi^1]\partial^\nu\psi_\alpha$$

This expression appears to intertwine what is customarily referred to as a Q_0-structure (referring to $\partial_\nu u\partial^\nu v$) with a $Q_{\nu j}$-structure (referring to $\partial_\nu u\partial_j v-\partial_j u\partial_\nu v$).

The main reason for its being amenable to good estimates (as stated in Proposition 3.5 below) is given by the following simple lemma, which exemplifies the precise underlying null-structure:

Lemma 2.4. *Let f, g, h be Schwartz functions. Then we have*

$$2\sum_{j=1}^{3}\triangle^{-1}\partial_j[R_\nu f R_j g - R_j f R_\nu g]\partial^\nu h$$
$$\sum_{j=1}^{3}\Box[\triangle^{-1}\partial_j[\nabla^{-1} f R_j g]h] - \sum_{j=1}^{3}\Box\triangle^{-1}\partial_j[\nabla^{-1} f R_j g]h$$
$$-\sum_{j=1}^{3}\triangle^{-1}\partial_j[\nabla^{-1} f R_j g]\Box h - \nabla^{-1} f\Box((\nabla^{-1}g)h)$$
$$+\nabla^{-1} f\Box(\nabla^{-1}g)h + \nabla^{-1} f(\nabla^{-1}g)\Box h$$

Proof : Use the identities

$$R_\nu f R_j g - R_j f R_\nu g = \partial_\nu(\sqrt{-\triangle}^{-1} f R_j g) - \partial_j(\sqrt{-\triangle}^{-1} f R_\nu g)$$

$$2\partial_\nu f\partial^\nu g = \Box(fg) - \Box f g - f\Box g$$

■

Remark: The bilinear null form in [8] exhibits similar structure, though our formulation, which avoids the Fourier transform, is more simple and explicit.

Now consider the terms arising upon substituting at least one 'elliptic term' χ_ν for ψ_ν in the leading term. Schematically, they can be represented by either of the following:

$$\nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\psi^2)\psi)\psi)\nabla_{x,t}\psi$$

$$\nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\psi^2)\psi)\nabla^{-1}(\nabla^{-1}(\psi^2)\psi))\nabla_{x,t}\psi$$

Both of these turn out to be significantly easier to treat than the preceding null-form term. Indeed, we won't have to refer to an inherent null-structure anymore.

2.5. **The Bootstrapping argument.** In order to prove the global regularity of u, we utilize a bootstrapping argument, quite similar to the one in [24]. More precisely, we introduce certain translation invariant Banach spaces $S[k]([-T,T]\times\mathbf{R^3})$, $N[k]([-T,T]\times\mathbf{R^3}), k\in\mathbf{Z}, T>0$ which enjoy a list of remarkable properties. The norms $||.||_{S[k]([-T,T]\times\mathbf{R^3})}$ will be used to estimate the components at frequency $\sim 2^k$ of the ϕ_α^i [8] which are known to be smooth on the time interval $[-T,T]$, while the norms $||.||_{N[k]([-T,T]\times\mathbf{R^3})}$ will be used to estimate the components at frequency $\sim 2^k$ of the nonlinearity, again restricted to and smooth on the time interval $[-T,T]$. Of course, $||.||_{S[k]}$ will have to majorize the energy $||.||_{\dot{H}^{\frac{1}{2}}}$ as well as a certain range of Strichartz norms, all applied to functions microlocalized at frequency $\sim 2^k$.
Our goal will be to bootstrap each of the norms $||P_k\phi_\alpha^i||_{S[k]([-T,T]\times\mathbf{R^3})}$. As a matter of fact, we will only have to bootstrap $||P_0\phi_\alpha^i||_{S[0]([-T,T]\times\mathbf{R^3})}$, because the $S[k]$ scale appropriately with respect to 'dilations' compatible with the div-curl system (11)-(14): denoting $\phi_\lambda := 2^\lambda\phi(x2^\lambda)$, we will have $||P_{k+\lambda}\phi_\lambda||_{S[k+\lambda]([-T,T]\times\mathbf{R^3})} = ||P_k\phi||_{S[k]([-T,T]\times\mathbf{R^3})}, k,\lambda\in\mathbf{Z}$. Here P_k denotes the Littlewood-Paley projector to frequency $\sim 2^k$. A similar identity holds for $N[k]([-T,T]\times\mathbf{R^3})$.

The $S[k]$ and $N[k]$ (leaving out the time-parameter T for simplicity's sake) will be related by the fundamental energy inequality:

$$||P_k\phi||_{S[k]([-T,T]\times\mathbf{R^3})} \leq C[||\Box P_k\phi||_{N[k]([-T,T]\times\mathbf{R^3})} + ||P_k\phi[0]||_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}}] \tag{23}$$

where C is independent of T. In order to use this inequality, we need to estimate the $N[k]$-norm of the nonlinearity. For this, it will be important to us amongst other things that there are

(1) null-form estimates of the form

$$||P_0[R_\nu P_{k_1}\phi\partial^\nu P_{k_2}\psi]||_{N[0]} \leq C2^{-\delta\max\{k_1,0\}}||P_{k_1}\phi||_{S[k_1]}||P_{k_2}\psi||_{S[k_2]},\ \delta>0 \tag{24}$$

(2) Bilinear estimates that make up for the missing $L_t^2L_x^\infty$-estimates. These come about by using null frame spaces, and have roughly the form

$$||P_{k_1}\phi P_{k_2}\psi||_{L_t^2L_x^2} \leq C2^{\frac{k_1-k_2}{2}}||P_{k_1}\phi||_{S[k_1]}||P_{k_2}\psi||_{S[k_2]} \tag{25}$$

provided ϕ,ψ are microlocalized on small caps whose distance is at least comparable to their radius, and provided their Fourier support lives fairly closely to the cone.

(3) Trilinear estimates:

$$\begin{aligned}&||P_0\sum_{j=1}^3\triangle^{-1}\partial_j[R_\nu P_{k_1}\psi_1R_jP_{k_2}\psi_2 - R_jP_{k_1}\psi_1R_\nu P_{k_2}\psi_2]\partial^\nu P_{k_3}\psi_3||_{N[0]}\\ &\leq C2^{-\delta_1|k_1-k_2|}2^{-\delta_2|k_3|}\prod||P_{k_j}\phi_j||_{S[k_j]},\ \delta_1,\delta_2>0\end{aligned} \tag{26}$$

These are the crucial tool for the paper.

[8] Suitable dilates of these spaces will be used for the frequency components of u

(4) The $S[k]$ have to be well-behaved under the Gauge Change. In particular, we need an assertion of the form that provided $||P_k\phi||_{S[k]}$ are small in a suitable sense, then so are $||P_k[f(\nabla^{-1}\phi)\phi]||_{S[k]}$, where ∇^{-1} stands for a linear combination of operators of the form $\triangle^{-1}\partial_j$, and $f(x)$ is a smooth function all of whose derivatives are bounded.

3. Technical preparations

The spaces $S[k], N[k]$ and many of their properties were considered in Tao's seminal paper [24], although their origins can be traced back to Tataru's [27]. Most of this section(except the trilinear inequality and the Gauge Change result) is due to these 2 authors; we will therefore be rather brief with the definitions.
First, we introduce Tao's concept of **frequency envelope**, as in [23],[24]: for any Schwartz function ψ on $\mathbf{R^3}$, we consider the quantities

$$c_a := (\sum_{k\in\mathbf{Z}} 2^{-\sigma|a-k|}||P_k\psi||^2_{\dot{H}^{\frac{1}{2}}})^{\frac{1}{2}} \tag{27}$$

Here $P_k, k \in \mathbf{Z}$ are the standard Littlewood-Paley operators that localize to frequency $\sim 2^k$, i.e. they are given by Fourier multipliers $m_k(|\xi|) = m_0(\frac{|\xi|}{2^k})$, where $m_0(\lambda)$ is a smooth function compactly supported within $\frac{1}{2} \leq \lambda \leq 2$ with $\sum_{k\in\mathbf{Z}} m_0(\frac{\lambda}{2^k}) = 1$, $\lambda > 0$.
The $\sigma > 0$ is chosen to be smaller than any of the exponential decays occuring later in the paper. E.g. $\frac{1}{1000}$ would suffice. We note that all of the generic constants C occuring in the sequel depend at most on this parameter σ.
Note that

$$c_k 2^{-\sigma|a-k|} \leq c_a \leq 2^{\sigma|a-k|}c_k \tag{28}$$

as well as $\sum_{k\in\mathbf{Z}} c_k^2 \leq C||\psi||^2_{\dot{H}^{\frac{1}{2}}}$.
The main reason for the usefulness of this concept is that provided we know that the frequency localized components $P_k\rho$ for some other Schwartz function ρ on $\mathbf{R^3}$ (think: the time-evolved Wave Map) have $\dot{H}^{\frac{1}{2}}$-norm bounded by a multiple Cc_k, we can immediately bound the $\dot{H}^{\frac{1}{2}+\epsilon}$-norm of ρ for $\epsilon > 0$ small enough. This will allow us later to continue the Wave Map, by referring to local well-posedness of the div-curl system (11)-(14) in $H^{\frac{1}{2}+\epsilon}$, and finite speed of propagation.

We introduce the following norms on frequency localized Schwartz functions on $\mathbf{R^{3+1}}$ for our bootstrapping argument: for every $l > 10$, choose a covering K_l of S^2 by finitely overlapping caps κ of radius 2^{-l}. This is to be chosen such that the set of concentric caps with half the radius still covers the sphere. Now let

$$||\psi||_{S[k]} :=$$
$$||\nabla_{x,t}\psi||_{L_t^\infty \dot{H}_x^{-\frac{1}{2}}} + ||\nabla_{x,t}\psi||_{\dot{X}_k^{-\frac{1}{2},\frac{1}{2},\infty}} + \sup_{\pm}\sup_{l>10}(\sum_{\kappa\in K_l}||\tilde{P}_{k,\pm\kappa}Q^{\pm}_{<k-2l}\psi||^2_{S[k,\kappa]})^{\frac{1}{2}} \tag{29}$$

where it is understood that ψ lives at frequency $\sim 2^k, k \in \mathbf{Z}$. The operators $\tilde{P}_{k,\kappa}$ are given by symbols $\tilde{m}_k(|\xi|)a_\kappa(\frac{\xi}{|\xi|})$, where $a : S^2 \to \mathbf{R}$ is a smooth function with support contained in the concentric cap inside κ with half the radius of κ, and $\tilde{m}_k$ localizes frequency to size $\sim 2^k$ and satisfies $\tilde{m}_k m_k = m_k$, where m_k is the multiplier chosen above. We also require that $\sum_{\kappa\in K_l} \tilde{P}_{k,\kappa} = \tilde{P}_k$, the latter being defined in the obvious way.
$Q^{\pm}_{<k-2l}$ localizes the modulation, i.e. $||\tau|-|\xi||$, to size $< 2^{k-2l}$ and also restricts the Fourier support to $\tau >< 0$, i.e. to the upper or lower half-space. More precisely, it is given by the multiplier $\sum_{i<k-2l} m_i(||\tau| - |\xi||)\chi_{>0}(\pm\tau)$.
The norm $||\phi||_{\dot{X}_k^{-\frac{1}{2},\frac{1}{2},1}}$ refers to $2^{-\frac{k}{2}}\sum_{j\in\mathbf{Z}} 2^{\frac{j}{2}}||Q_j\phi||_{L^2_t L^2_x}$.

The definition of $S[k,\kappa]$ is a scaled-down version of the one in [24]:

$$||\psi||_{S[k,\kappa]} := 2^{\frac{k}{2}}||\psi||_{NFA^*[\kappa]} + |\kappa|^{-\frac{1}{2}}2^{-\frac{k}{2}}||\psi||_{PW[\kappa]} + 2^{\frac{k}{2}}||\psi||_{L^\infty_t L^2_x} \tag{30}$$

The definitions of the individual ingredients in turn are as follows:

(1) $NFA^*[\kappa]$ is the Banach space obtained upon completing $\mathcal{S}(\mathbf{R}^{3+1})$ with respect to the norm

$$||\psi||_{NFA*[\kappa]} := \sup_{\omega\notin 2\kappa} dist(\omega,\kappa)||\phi||_{L^\infty_{t_\omega} L^2_{x_\omega}} \tag{31}$$

Here (t_ω, x_ω) refer to null-frame coordinates, i.e. $t_\omega = (t,x)\cdot\frac{1}{\sqrt{2}}(1,\omega), x_\omega = (t,x) - t_\omega\frac{1}{\sqrt{2}}(1,\omega)$.

(2) $PW[\kappa]$ is the atomic Banach space whose atoms are the set A of all Schwartz functions ψ with $||\psi||_{L_{t^2_\omega L^\infty_{x_\omega}}} \le 1$ for some $\omega \in \kappa$. In other words,

$$\begin{aligned} ||\psi||_{PW[\kappa]} = &\inf\{|\lambda| | \exists\{0 \le \lambda_i \le 1\}, \{\psi_i\} \subset A, 1 \le i \le N\, s.t. \sum_i \lambda_i = 1, \\ &\lambda\sum_i \lambda_i\psi_i = \psi\} \end{aligned} \tag{32}$$

Of course, the Banach space $S[k]$ is obtained by completing the Schwartz functions on $\mathbf{R}^{3+1}$ with respect to $||.||_{S[k]}$.

Next, we will place frequency localized pieces of the nonlinearity into the following spaces $N[k]$, again introduced by Tao and implicitly present in Tataru's work: they are the atomic Banach spaces whose atoms are

(1) Schwartz functions F at frequency between 2^{k-4} and 2^{k+4} with $||F||_{L_t^1 L_x^2} \leq 2^{\frac{k}{2}}$.
(2) Schwartz functions F with frequency between 2^{k-4} and 2^{k+4} and modulation between 2^{j-5} and 2^{j+5} such that $||F||_{L_t^2 L_x^2} \leq 2^{\frac{j}{2}} 2^{\frac{k}{2}}$.
(3) Schwartz functions F for which there exists a number $l > 10$ and Schwartz functions F_κ with Fourier support in the region $\{(\tau, \xi) | \pm\tau > 0, ||\tau| - |\xi|| \leq 2^{k-2l-100}, 2^{k-4} \leq |\xi| \leq 2^{k+4}, \Theta \in \frac{1}{2}\kappa\}$ such that $F = \sum_{\kappa \in K_l} F_\kappa$ and $(\sum_{\kappa \in K_l} ||F_\kappa||^2_{NFA[\kappa]})^{\frac{1}{2}} \leq 2^{\frac{k}{2}}$. Here $\Theta = \frac{\tau\xi}{|\tau||\xi|}$ and $NFA[\kappa]$ is the dual space of $NFA[\kappa]^*$, i.e. the atomic Banach space whose atoms are Schwartz functions F which satisfy

$$\frac{1}{\mathrm{dist}(\omega, \kappa)} ||F||_{L^1_{t_\omega} L^2_{x_\omega}} \leq 1$$

for some $\omega \notin 2\kappa$.

We try to briefly explain the reason for introducing these spaces: the $PW[\kappa]$ component of $S[k]$ is to be though of as a substitute for the missing $L_t^2 L_x^\infty$-estimate. This is directly exemplified by the following **first fundamental bilinear inequality**:

$$||\phi\psi||_{NFA[\kappa]} \leq C \frac{2^{\frac{k'}{2}} |\kappa'|^{\frac{1}{2}}}{dist(\kappa, \kappa')} ||\phi||_{L_t^2 L_x^2|} ||\psi||_{S[k', \kappa']} \tag{33}$$

which is a direct consequence of the inclusion $S[k, \kappa] \subset 2^{\frac{k}{2}} |\kappa|^{\frac{1}{2}} PW[\kappa]$. This inequality also suggests that $NFA[\kappa]$ is to be seen as a substitute for $L_t^1 L_x^2$, the energy space. This may seem odd, as we are substituting a null-frame analogue for the customary version, and there is no Duhamel's formula in that context. However, we shall only place pieces of the nonlinearity into $NFA[\kappa]$ which are microlocalized along an angular sector contained in κ, and it turns out that there is an analogue of the energy inequality then.
The $NFA*[\kappa]$-component of $S[k]$ makes certain algebra estimates work and will in particular enable us to obtain a general Gauge Change estimate cited below. This shall be a consequence of the following **2nd fundamental bilinear inequality**, which is essentially dual to the first:

$$||\phi\psi||_{L_t^2 L_x^2} \leq C \frac{2^{\frac{k'}{2}} |\kappa'|^{\frac{1}{2}}}{dist(\kappa, \kappa') 2^{\frac{k}{2}}} ||\phi||_{S[k, \kappa]} ||\psi||_{S[k', \kappa']} \tag{34}$$

This is again an immediate consequence of the definitions, viz. also [24]. Finally, we also note that **truncated free waves are naturally embedded into these spaces**, which is of course crucial for an 'energy inequality'(see below, (38)) to work. We exemplify this by the following inequality [9] valid for all Schwartz functions $\phi \in \mathcal{S}(\mathbf{R^{3+1}})$:

[9]Keep in mind that elements of $\dot{X}_k^{\frac{1}{2}, \frac{1}{2}, 1}$ are weighted averages of free waves.

$$||\phi||_{S[k,\kappa]} \leq C||\phi||_{\dot{X}_k^{\frac{1}{2},\frac{1}{2},1}} \tag{35}$$

In the sequel, it will be important to have some Strichartz norms of the form $L_t^p L_x^q$ at our disposal. Unfortunately, the author was unable to build sharp Strichartz norms(satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$) into the $S[k]$, on account of difficulties related to the energy inequality (38). This means we have to make do with a certain range of non-sharp Strichartz norms, which can be seen to be controlled by the $S[k]$. This will be the content of a theorem below.

Since we will be implementing a bootstrapping argument, we can only assume the a priori existence of a solution on a finite time interval $[-T,T]$. We therefore need to localize the above (frequency-localized) norms to this interval. To wit

$$||P_k\phi||_{S[k]([-T,T]\times\mathbf{R^3})} := \inf_{\psi\in\mathcal{S}(\mathbf{R^{3+1}}),\,\psi|_{[-\mathbf{T},\mathbf{T}]}=\phi} ||P_k\psi||_{S[k](\mathbf{R^{3+1}})} \tag{36}$$

$$||P_k\phi||_{N[k]([-T,T]\times\mathbf{R^3})} := \inf_{\psi\in\mathcal{S}(\mathbf{R^{3+1}}),\,\psi|_{[-\mathbf{T},\mathbf{T}]}=\phi} ||P_k\psi||_{N[k](\mathbf{R^{3+1}})} \tag{37}$$

We can now formulate the following **energy inequality**, which is the essential link between the $N[k]$ and $S[k]$-norm that will allow us to finish the bootstrapping argument:

$$||P_k\phi||_{S[k]([-T,T]\times\mathbf{R^3})} \leq C[||\Box P_k\phi||_{N[k]([-T,T]\times\mathbf{R^3})} + ||\phi[0]||_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}}] \tag{38}$$

where C is independent of T. This is proved as in [24]; the only difference between our $S[k,\kappa]$ norm and Tao's $S[k,\kappa]$-norm is their scaling, which doesn't affect the proof.

It is important that the $S[k]([-T,T]\times\mathbf{R^3})$-norms of the frequency localized components of a Schwartz function are in a sense uniformly lower semicontinuous with respect to T, as demonstrated in [24]. In particular, we may assume that $T>0$ has been chosen such that the component functions ϕ of our Wave Map satisfy

$$||P_k\phi||_{S[k]([-T,T]\times\mathbf{R^3})} \leq Cc_k \tag{39}$$

where c_k is a frequency envelope associated with the initial conditions $\phi[0]\times\partial_t\phi[0]$ as above, i.e.

$$c_k := (\sum_{k'} 2^{-\delta|k'-k|}(||P_{k'}\phi||_{\dot{H}^{\frac{1}{2}}} + ||P_{k'}\partial_t\phi||_{\dot{H}^{-\frac{1}{2}}})^2)^{\frac{1}{2}} \tag{40}$$

Moreover, since we assume that ϕ is rapidly decaying in space directions, we can construct a Schwartz function $\tilde{\phi}$ with $\tilde{\phi}|_{[-T,T]}=\phi$ and such that $||P_k\tilde{\phi}||_{S[k]} \leq 2Cc_k$.

This is achieved by using a partition of unity. We will always substitute $\tilde{\phi}$ for ϕ when making actual estimates.

Notation The Riesz operators R_ν, $\nu \in \{0, 1, 2, 3\}$, refer to operators $\partial_\nu (\sqrt{-\triangle_x})^{-1}$. We usually omit the subscript for operators like $\nabla_x, \triangle_x$, understanding that they refer only to space variables.
The symbol ∇^{-1} is either a shorthand for an operator $\triangle^{-1}\partial_i$, or else refers to $(\sqrt{-\triangle})^{-1}$, depending on the context.
We use the notation $P_{k+O(1)} = \sum_{k_1=k+O(1)} P_{k_1}, Q_{j+O(1)} = \sum_{j_1=j+O(1)} Q_{j_1}$. Also, $||\phi||_{S[k+O(1)]} = \sum_{k_1=k+O(1)} ||P_{k_1}\phi||_{S[k_1]}$ etc.
The following terminology, introduced by T.Tao in [24], shall be useful in the future: we call a Fourier multiplier **disposable** if it is given by convolution with a translation invariant measure of mass $\leq O(1)$. In particular, operators such as $P_k, P_k Q_{<>j}$ where $j \geq k + O(1)$ are disposable, see above reference. By contrast, Q_j is not disposable. However, it acts boundedly on Lebesgue spaces of the form $L_t^p L_x^2$.
Whenever we consider an expression of the form $P_0(AB[CD])$, for example, we shall refer to A, B, C, D as **inputs** and the whole expression as **output**. Also, when referring to $[,]$, we mean $[CD]$, while $(,)$ would refer to $P_0(AB[CD])$; thus the shape of brackets matters in the discussion. When considering a part of the whole expression such as $[CD]$, we may also refer to this as output, and C, D as inputs, depending on the context. In the proof of the Gauge Change estimate, we shall use the term **modulation** to refer to the distance of the (space time) Fourier support of a function to the light cone.

Summary of the key properties satisfied by these spaces

The paradifferential Calculus approach chosen in this paper enables us to divide the nonlinearity into different pieces (obtained upon microlocalizing all the inputs as well as the output) which can be controlled individually. However, the fact that we start out with refined information about the frequency localized components of the Wave Map *forces us to retrieve the refined information* via the bootstrapping argument. Thus while on the one hand we gain from the fact that we can subdivide the nonlinearity into many pieces each of which is amenable to an individual attack, we lose in that we have to recover the original frequency envelope from our estimates. For example, whenever enacting a Gauge Change of the form $\psi := f(\triangle^{-1}\sum_{k=1}^{3}\partial_k\tilde{\phi}_k^1)\tilde{\phi}$ where $\tilde{\phi}, \tilde{\phi}_k^1$ are Schwartz functions (the latter real valued[10]) agreeing with ϕ, ϕ_k^1 on $[-T, T]$ and for which the $S[k]$-norms of the frequency localized pieces sit under approximately the same frequency envelope, we shall need to know that the frequency modes of ψ are controlled by a dilate of the same frequency envelope. Moreover, we shall have to rely on refined multilinear estimates which allow us to sum over all possible frequency interactions contributing to a fixed frequency mode of the nonlinearity, as well as to recover the

[10]Note that the $S[k]$, $N[k]$ are conjugation invariant. Thus we can always find real-valued extensions of our component functions with the required properties.

original frequency envelope. We summarize here the key properties to be referred to throughout the rest of the paper:

(1): The Gauge Change Estimate

Proposition 3.1. *Let $f(x)$ be a smooth function all of whose derivatives are bounded. Also, let ϕ_i, $i = 1, 2, 3, 4$ be Schwartz functions satisfying the condition $\max_i ||P_k\phi_i||_{S[k]} \leq c_k$ for a 'sufficiently flat' frequency envelope $\{c_k\}$(i.e. σ in the definition sufficiently small). Then*

$$||P_k(f(\triangle^{-1}\sum_{j=1}^{3}\partial_j\phi_j)\phi_4)||_{S[k]} \leq Cc_k$$

We shall give the proof later in the paper.

(2): Bilinear estimates.

Q_0 null-form estimates

Theorem 3.2. *Let ϕ,ψ be Schwarz functions on $\mathbf{R^{3+1}}$. We have*

$$||P_k[R_\nu P_{k_1}\phi\partial^\nu P_{k_2}\psi]||_{N[k]} \leq C2^{-\delta\max\{k_1-k,0\}}||P_{k_1}\phi||_{S[k_1]}||P_{k_2}\psi||_{S[k_2]}$$

for some $\delta > 0$. Also, we have

$$||P_k\nabla_x[R_\nu P_{k_1}\phi R^\nu P_{k_2}\psi]||_{N[k]} \leq C||P_{k_1}\phi||_{S[k_1]}||P_{k_2}\psi||_{S[k_2]}$$

Finally

$$||R_\nu\phi R^\nu\psi||_{L_t^2L_x^2} \leq C(\sum_{k_1}||P_{k_1}\phi||^2_{S[k_1]})^{\frac{1}{2}}(\sum_{k_2}||P_{k_2}\psi||^2_{S[k_2]})^{\frac{1}{2}}$$

The first two inequalities are due (in somewhat modified form) to T.Tao [24]. We present proofs for the above versions(our spaces being scaled down with respect to Tao's) in [13].

Theorem 3.3. *Let ϕ, F be Schwartz functions, and $k_1 = k_2 + O(1)$. Then we have*

$$||P_0(P_{k_1}\phi P_{k_2}F)||_{N[0]} \leq C2^{-\delta k_1}||P_{k_1}\phi||_{S[k_1]}||\nabla_x(P_{k_2}F)||_{N[k_2]}$$

for some $\delta > 0$.
Moreover, we have the estimate

$$||P_0\nabla_x(\phi P_{k_2}F)||_{N[0]} \leq C(||\phi||_{L_t^\infty L_x^\infty} + \sup_k||P_k\nabla_x\phi||_{S[k]})||\nabla_x(P_{k_2}F)||_{N[k_2]}$$

This is again due to Tao [24] in slightly different form. Proofs may be found in [24], [13].

Bilinear algebra and $Q_{\nu j}$-estimate

Theorem 3.4. *Let ϕ_1, ϕ_2 be Schwartz functions. Then if $j \leq k$, we have $\forall \epsilon > 0$ and $0 < \delta < \epsilon$*

$$||P_k Q_j (P_{k_1}\phi_1 P_{k_2}\phi_2)||_{\dot{X}^{-\epsilon,\epsilon,\infty}} \leq C_{\epsilon,\delta} 2^{\delta \min\{j-\min\{k_1,k_2,k\},0\}} 2^{-\frac{|k_1-k_2|}{2}} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]}$$

$$||P_k Q_j (P_{k_1}\phi_1 P_{k_2}\phi_2)||_{\dot{X}^{-\frac{1}{2},\frac{1}{2},\infty}} \leq C_{\epsilon} 2^{\frac{1}{2+\epsilon}\min\{j-\min\{k_1,k_2,k\},0\}} 2^{-|k_1-k_2|} \prod_{i=1,2} ||P_{k_i}\phi_i||_{S[k_i]}$$

Also, one has the inequality

$$||P_k (P_{k_1}\phi P_{k_2}\psi)||_{L_t^2 L_x^{2+\mu}} \leq C_\mu 2^{\frac{\mu}{4+2\mu}k} 2^{-\frac{|k_1-k_2|}{2}} \prod_{i=1,2} ||P_{k_i}\psi_i||_{S[k_i]}$$

for any $\mu > 0$. In particular, we can control the $L_t^4 L_x^p$-norm, $p > 4$, of the k-th frequency component in terms of $S[k]$, and by interpolation with $L_t^\infty L_x^2$, one controls all norms of the form $L_t^p L_x^q$, $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, $p \geq 4$, at that frequency[11]. Finally, we have

$$\begin{aligned} &||P_k (R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2)||_{L_t^2 L_x^2} \\ &\leq C 2^{-\frac{|k_1-k_2|}{2}} 2^{-|k-\max\{k_1,k_2\}|} \prod_{i=1,2} ||P_{k_i}\psi_i||_{S[k_i]} \end{aligned}$$

This theorem, proved in [13], would be essentially superfluous if $S[k]$ could be customized in such a way as to be included in $L_t^4 L_x^4$.

(3): Trilinear null-form estimates

Proposition 3.5. *Let ψ_l, $l = 1, 2, 3$ be Schwartz functions on $\mathbf{R^{3+1}}$. We then have the estimate*

$$\begin{aligned} &||P_0 (\sum_{j=1}^{3} \triangle^{-1}\partial_j [R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2]\partial^\nu P_{k_3}\psi_3)||_{N[0]} \\ &\leq C 2^{-\delta_1|k_1-k_2|} 2^{\delta_2(\min\{k_3-\max\{k_1,k_2\},0\})} 2^{-\delta_3|k_3|} \prod_{l=1}^{3} ||P_{k_l}\psi_l||_{S[k_l]} \end{aligned} \tag{41}$$

[11] One can also majorize $||P_0 R_0 \phi||_{L_t^4 L_x^p}$ by $C||P_0\phi||_{S[k]}$. For $P_0 Q_{<0}\phi$, this follows from the immediately preceding, whereas for $P_0 Q_{\geq 0}\phi$, this is a consequence of Bernstein's inequality.

for appropriate constants $\delta_1, \delta_2, \delta_3 > 0$. As a corollary, we have

$$||P_0(\sum_{j=1}^{3} \triangle^{-1}\partial_j[R_\nu\psi_1 R_j\psi_2 - R_j\psi_1 R_\nu\psi_2]\partial^\nu\psi_3)||_{N[0]} \leq C(\sum_{k\in\mathbf{Z}} c_k^2)c_0$$

provided $\max_{i=1,2,3}||P_k\psi_i|| \leq c_k$ *for some frequency envelope* $\{c_k\}$ *which is 'sufficiently flat', i.e.* $\sigma << \min\{\delta_i\}$.

Proposition 3.6. *Let ψ_i be as above. Then we have the inequalities*

$$||P_0[R_\nu P_{k_1}\psi_1 R^\nu P_{k_2}\psi_2 P_{k_3}\psi_3]||_{N[0]} \leq C2^{-\delta_1|k_1-k_2|}2^{\delta_2(\min\{k_3-\max\{k_1,k_2\},0\})}2^{-\delta_3|k_3|}\prod_{l=1}^{3}||P_{k_l}\psi_l||_{S[k_l]}$$

$$||P_0[\nabla^{-1}(R_\nu P_{k_1}\psi_1\partial^\nu P_{k_2}\psi_2)P_{k_3}\psi_3]||_{N[0]} \leq C2^{-\delta_1|k_1-k_2|}2^{\delta_2(\min\{k_3-\max\{k_1,k_2\},0\})}2^{-\delta_3|k_3|}\prod_{l=1}^{3}||P_{k_l}\psi_l||_{S[k_l]}$$

for appropriate $\delta_1, \delta_2 > 0$. One obtains a similar corollary as in the preceding Proposition.

Both of these are proved in [13]. The 2nd Proposition is a simpler variant of an inequality in [24].

(4): Quadrilinear null-form estimates.

Proposition 3.7. *Let ψ_i, $i = 1, 2, 3, 4$ be Schwartz functions satisfying $||P_k\psi_i||_{S[k]} \leq c_k$ 'for a sufficiently flat frequency envelope $\{c_k\}$'. Then we have the inequality*

$$\begin{aligned}&||P_0[\sum_{i,j=1}^{3} \triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_\nu\psi_1 R_i\psi_2 - R_i\psi_1 R_\nu\psi_2)R_j\psi_3)\partial^\nu\psi_4\\ &- \sum_{i,j=1}^{3} \triangle^{-1}\partial_j(\triangle^{-1}\partial_i(R_j\psi_1 R_i\psi_2 - R_i\psi_1 R_j\psi_2)R_\nu\psi_3)\partial^\nu\psi_4]||_{N[0]}\\ &\leq C(\sum_{k\in\mathbf{Z}} c_k^2)^{\frac{3}{2}}c_0\end{aligned}$$

The proof of this, which implicitly relies on an identity similar to but more complicated than the one recorded in Proposition 3.5, can also be found in [13].

4. Proof of the Proposition 1.1

We shall present the detailed argument provided (M, g) falls into the first category. The other cases are handled more or less identically. For a given Wave Map u, we introduce the variables ϕ_α^i, $i=1,2$, $\alpha=0,1,2,3$, as follows:

$$\sum_{i=1,2}\phi_\alpha^i e_i(u)=u_*(\partial_\alpha)$$

Then recall the fundamental div-curl system

$$\partial_\beta\phi_\alpha^i-\partial_\alpha\phi_\beta^i=C_{jk}^i(u)\phi_\alpha^j\phi_\beta^k \tag{42}$$

$$\partial_\alpha\phi^{i\alpha}=-\Gamma_{jk}^i(u)\phi_\beta^j\phi_\gamma^k m^{\beta\gamma} \tag{43}$$

We pass from these to the corresponding wave equations, which take the form

$$\Box\phi_\alpha^i=-2\Gamma_{kj}^i(u)\phi_\beta^k\partial^\beta\phi_\alpha^j+A_{jkl}^i(u)\phi_\beta^j\phi^{k\beta}\phi_\alpha^l \tag{44}$$

where we have used the fact that $C_{jk}^i=\Gamma_{jk}^i-\Gamma_{kj}^i$, as well as

$$\partial_\lambda(f(u))=\sum_{i=1,2}e_i(f)(u)\phi_\lambda^i$$

for any smooth function $f:M\rightarrow\mathbf{R}$ and $\lambda=0,1,2,3$. Our assumptions in **(1)** imply that we can extend the A_{jkl}^i to an open neighborhood of M in $\mathbf{R}^k$, where all their derivatives are bounded. We shall prove Theorem 1.1 via the following *Bootstrapping Proposition:*

Proposition 4.1. *Let $T>0$, let $u:\mathbf{R}^{3+1}\rightarrow M$ be a smooth Wave Map on a time interval $[-T,T]$, , and let the notation be as above; then there exist a number $\epsilon>0$ and a large constant $M>0$ independently of T,u, such that the following holds:*

$$||P_k\nabla_x u||_{S[k]([T,-T]\times\mathbf{R}^3)}+\sup_{i,\alpha}||P_k\phi_\alpha^i||_{S[k]([-T,T]\times\mathbf{R}^3)}<Mc_k\Longrightarrow$$
$$||P_k\nabla_x u||_{S[k]([-T,T]\times\mathbf{R}^3)}+\sup_{i,\alpha}||P_k\phi_\alpha^i||_{S[k]([-T,T]\times\mathbf{R}^3)}<\frac{M}{2}c_k$$

for all sufficiently flat[12] *frequency envelopes c_k satisfying $(\sum_{k\in\mathbf{Z}}c_k^2)^{\frac{1}{2}}<\epsilon$.*

The Theorem 1.1 follows from this and the subcritical result of Klainerman-Machedon [8][13]

Proof We employ roughly the same strategy as the one outlined in section 2. The first step consists in changing the Gauge in order to improve the leading term of

[12]In the sense that the σ used in its defining property is small enough.

[13]Note that the Wave Maps equation in terms of u is $\Box(i\circ u)^l=B_{jk}^l(u)(\partial_\nu(i\circ u),\partial^\nu(i\circ u))$, where B_{jk}^i is the 2nd fundamental form of the embedding i. This is structurally identical to the local formulation of Wave Maps studied in [8].

the nonlinearity. For this, we employ a Coulomb Gauge of the following form:

$$\psi_\alpha := \psi_\alpha^1 + \sqrt{-1}\psi_\alpha^2 = e^{\sqrt{-1}\triangle^{-1}\sum_{j=1}^3 \partial_j(\Gamma_{l2}^1(u)\phi_j^l)}(\phi_\alpha^1 + \sqrt{-1}\phi_\alpha^2)$$

Upon introducing the notation $\triangle^{-1}\sum_{j=1}^3 \partial_j(\Gamma_{l2}^1(u)\phi_j^l) = \Phi$, we deduce the following wave equation

$$\Box\psi_\alpha = M_\mu\partial^\mu\psi_\alpha + \sqrt{-1}[\Box\Phi + \sqrt{-1}\partial_\nu\Phi\partial^\nu\Phi]\psi_\alpha + e^{i\Phi}(A_{jkl}^1(u)\phi_\beta^j\phi^{k\beta}\phi_\alpha^l + \sqrt{-1}A_{jkl}^2(u)\phi_\beta^j\phi^{k\beta}\phi_\alpha^l) - M_\mu\sqrt{-1}\triangle^{-1}\sum_{j=1}^3\partial_j\partial_\mu(\Gamma_{l2}^1(u)\phi_j^l)\psi_\alpha \tag{45}$$

The M_μ in turn satisfy the following *elliptic div-curl system*:

$$\sum_{j=1}^3 \partial_j M_j = 0$$

$$\partial_l M_\alpha - \partial_\alpha M_l = -\sqrt{-1}[\partial_l(\Gamma_{k2}^1(u)\phi_\alpha^k) - \partial_\alpha(\Gamma_{k2}^1(u)\phi_l^k)] := E_{jk}(u)\phi_l^j\phi_\alpha^k$$

where the $E_{jk}(.)$ are skew-symmetric in j,k and extend as smooth functions with bounded derivatives of all orders to a neighborhood of M in $\mathbf{R}^k$ [14]. This system allows us easily to solve for the M_α, as follows:

$$M_\alpha = \sum_{l=1}^3 \triangle^{-1}\partial_l(\sum_{j,k=1,2} E_{jk}(u)\phi_l^j\phi_\alpha^k)$$

The conclusion upon substituting these expressions into (45) is that the new leading term of the nonlinearity is the following:

$$\Box\psi_\alpha = \sum_{l=1}^3 \triangle^{-1}\partial_l(\sum_{j,k=1,2} E_{jk}(u)\phi_l^j\phi_\mu^k)\partial^\mu\phi_\alpha + ...$$

We need to make one more substitution, namely $E_{12}(u)\phi_\lambda^2 = \theta_\lambda^1$. Note that by virtue of Proposition 3.1, the k-th frequency mode of ψ_α as well as the k-th frequency mode of θ_λ^1 have their $S[k]$-norm bounded by a suitable dilate of $\{c_k\}$. We reformulate the wave equation as follows:

$$\Box\psi_\alpha = \sum_{l=1}^3 \triangle^{-1}\partial_l(\theta_\mu^1\phi_l^1 - \theta_l^1\phi_\mu^1)\partial^\mu\phi_\alpha + ...$$

In order to render the null-structure visible, we implement the dynamic separation

[14]We shall from now on omit such qualifications as they are automatic from our assumptions.

associated with the curl equation(42) to decompose the ϕ_α^i into a 'dynamic' (gradient) part and an 'elliptic'part (determined via an elliptic divergence curl system). It is easily checked that the θ_α^1 satisfy an analogous curl-system, and can be similarly decomposed. More specifically, we write

$$\phi_\alpha^i = R_\alpha \Phi^i + \tilde{\phi}_\alpha^i,\ i = 1, 2$$

$$\theta_\alpha^1 = R_\alpha \Theta^1 + \tilde{\theta}_\alpha^1$$

where the R_α are Riesz operators as in section 2, and we have set

$$\Phi^i = -\sum_{k=1}^{3} R_k \phi_k^i,\ \Theta^1 = -\sum_{k=1}^{3} R_k \theta_k^1$$

These 'potentials' satisfy similar estimates (up to constants) as the ϕ_α. The trilinear null-form arising upon substituting the gradient parts is of an identical nature as the one discussed in section 2. Moreover, taking into account the fact that we have identities of the form

$$\tilde{\phi}_\alpha^i = \sum_{l=1}^{3} \triangle^{-1}\partial_l \Big(\sum_{j,k=1,2} D_{jk}^i(u) \phi_\alpha^j \phi_l^k \Big)$$

for skew-symmetric $D_{jk}^i(u)$, and similar identities for the $\tilde{\theta}_\alpha^1$, reveals that substituting an 'elliptic part' for either ϕ_α^i or θ_α^i results in terms at least quadrilinear of the following structure:

$$\sum_{l=1}^{3} \triangle^{-1}\partial_l \Big(\theta_l^1 \sum_{r=1}^{3} \triangle^{-1}\partial_r \big(D_{12}^1(u)(\phi_r^1\phi_\mu^2 - \phi_r^2\phi_\mu^1)\big)\Big)\partial^\mu \phi_\alpha$$
$$- \sum_{l=1}^{3} \triangle^{-1}\partial_l \Big(\theta_\mu^1 \sum_{r=1}^{3} \triangle^{-1}\partial_r \big(D_{12}^1(u)(\phi_r^1\phi_l^2 - \phi_r^2\phi_l^1)\big)\Big)\partial^\mu \phi_\alpha$$

$$\nabla^{-1}(\nabla^{-1}(C(u)\phi^2)\nabla^{-1}(D(u)\theta^2))\nabla_{x,t}\phi \tag{46}$$

where the latter term[15] is of course only recorded in schematic form (we don't need its fine structure). As to the quadrilinear terms, we simply repeat the previous step of introducing new variables

$$\xi_\lambda = D_{12}^1(u)\phi_\lambda^2$$

[15]Recall that we use the shorthand ∇^{-1} for operators of the type $\triangle^{-1}\partial_j$; occasionally, we shall also use this notation to denote the multiplier $\sqrt{-\triangle}^{-1}$.

These satisfy similar (frequency localized) estimates as the ϕ^i_α and also a similar curl system, which allows us to apply dynamic separation

$$\xi_\lambda = R_\lambda \Xi + \tilde{\xi}_\lambda,\ \tilde{\xi}_\lambda = \nabla^{-1}(A(u)(\phi^2))$$

Carrying out the substitution leads to a quadrilinear null-form

$$\sum_{l=1}^{3} \triangle^{-1}\partial_l (R_l \Theta^1 \sum_{r=1}^{3} \triangle^{-1}\partial_r (R_r \Phi^1 R_\mu \Xi^1 - R_\mu \Phi^1 R_r \Xi^1))\partial^\mu \phi_\alpha$$
$$-\sum_{l=1}^{3} \triangle^{-1}\partial_l (R_\mu \Theta^1 \sum_{r=1}^{3} \triangle^{-1}\partial_r (R_r \Phi^1 R_l \Xi^1 - R_l \Phi^1 R_r \Xi^1)\partial^\mu \phi_\alpha$$

as well as error terms of the following schematic form[16]:

$$\nabla^{-1}(\phi\nabla^{-1}(\phi\nabla^{-1}(A(u)\phi^2)))\nabla_{x,t}\phi$$

$$\nabla^{-1}(\nabla^{-1}(C(u)\phi^2)\nabla^{-1}(D(u)\phi^2))\nabla_{x,t}\phi$$

and similar terms of higher degree of linearity(up to degree 7.)For future reference, we note that on account of Proposition 3.1, one can always replace $A(u)\phi$ by ϕ. Thus, to summarize the preceding discussion we state

Observation 1: *The leading term $M_\mu \partial^\mu \psi$ can be decomposed into the sum of trilinear null-forms*[17] *of the type in Proposition 3.5 , quadrilinear null-forms of the type contained in Proposition 3.7 and error terms at least quintilinear of the schematic form:*

$$\nabla^{-1}(\phi\nabla^{-1}(\phi\nabla^{-1}(\phi^2)))\nabla_{x,t}\phi$$

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla^{-1}(\phi^2))\nabla_{x,t}\phi$$

and similar terms of higher degree of linearity.

[16]We are fudging the distinction between the variables $\phi^i_\alpha, \theta^i_\alpha, \xi^i_\alpha$, since they are essentially equivalent as far as estimates are concerned.

[17]whose inputs have frequency modes satisfying the same inequalities as the original ϕ^i_α but with respect to a dilate of the frequency envelope $\{c_k\}$

The remaining terms in the nonlinearity of (45) are handled similarly. The third, fourth and fifth term lead to trilinear null-forms of the type contained in Proposition 3.6 upon enacting dynamic separation, as well as quadrilinear terms of the form

$$\nabla^{-1}(\phi^2)\phi^2$$

These in turn are decomposed into quadrilinear null-forms of the schematic type

$$\nabla^{-1}(R_\nu\phi_1 R_j\phi_2 - R_j\phi_1 R_\nu\phi_2)\phi^2$$

where ϕ_1, ϕ_2 refer to suitable expressions $A_{1,2}(u)\phi$, as well as terms at least quintilinear of the type

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\phi)\phi^2$$

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla^{-1}(\phi^2)\phi^2$$

The sixth term of the nonlinearity is decomposed into terms of the exact same type as in the immediately preceding. What remains is the expression

$$\Box\Phi\psi_\alpha$$

contained in the 2nd term of the nonlinearity. We reformulate it using (44). One obtains the expression

$$\sum_{l=1}^{3}\triangle^{-1}\partial_l(\Gamma^i_{jk}\phi^j_\nu\partial^\nu\phi^k_l + A^i_{jkl}(u)\phi^j_\beta\phi^{k\beta}\phi^l_\alpha)\psi_\alpha$$

which, upon introducing the new variables $\eta^i_{k\nu} := \Gamma^i_{jk}\phi^j_\nu$ and implementing dynamic separation with respect to these variables(as well as the ϕ^j_β for the 2nd summand), turns into a trilinear null-form (whose fine structure we have suppressed)

$$\nabla^{-1}(R_\nu E\partial^\nu\phi)\psi$$

as well as quadrilinear terms of the rough form

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla_{x,t}\phi)\phi$$

$$\nabla^{-1}(R_\beta\phi_1 R^\beta\phi_2\phi_3)\phi_4$$

and error terms of the form

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\phi^2)\phi$$

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla^{-1}(\phi^2)\phi)\phi$$

The first kind of quadrilinear expression needs to be further decomposed into quadrilinear null-forms and error terms at least quintilinear. Reiterating dynamic separation with respect to suitable variables allows one to decompose such terms into the sum of schematically written quadrilinear null-forms:

$$\nabla^{-1}(\sum_{l=1}^{3}\triangle^{-1}\partial_l(R_l\phi_1 R_\nu\phi_2 - R_\nu\phi_1 R_l\phi_2)\partial^\nu\phi_3)\psi_\alpha$$

as well as error terms of the schematic form

$$\nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\phi^2)\phi)\nabla_{x,t}\phi)\phi$$

$$\nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla^{-1}(\phi^2))\nabla_{x,t}\phi)\phi$$

We summarize this discussion as follows:

Observation 2: *The remaining terms of the nonlinearity can be expressed as a sum of trilinear null-forms of the types contained in Proposition 3.6, quadrilinear null-forms of the type*

$$\nabla^{-1}(\sum_{l=1}^{3}\triangle^{-1}\partial_l(R_l\phi_1 R_\nu\phi_2 - R_\nu\phi_1 R_l\phi_2)\partial^\nu\phi_3)\phi_4$$

$$\nabla^{-1}(R_\nu\phi_1 R_j\phi_2 - R_j\phi_1 R_\nu\phi_2)\phi^2$$

$$\nabla^{-1}(R_\beta\phi_1 R^\beta\phi_2\phi_3)\phi_4$$

as well as error terms at least quintilinear of the schematic form

$$\nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\phi^2)\phi)\nabla_{x,t}\phi)\phi$$

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\phi)\phi^2$$

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\phi^2)\phi$$

$$\nabla^{-1}(\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla^{-1}(\phi^2))\nabla_{x,t}\phi)\phi$$

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla^{-1}(\phi^2))\phi^2$$

$$\nabla^{-1}(\nabla^{-1}(\phi^2)\nabla^{-1}(\phi^2)\phi)\phi$$

In order to proceed with the proof of Proposition 4.1, we need to estimate the 0-th frequency component of each of the expressions recorded in **Observation 1,2**, and close by means of the energy inequality (38). More precisely, for any expression $F(\phi_1, \phi_2, \cdots, \phi_k)$ occuring in **Observation 1, 2**, we need to establish an inequality

$$||P_0 F(\phi_1, \phi_2, \cdots, \phi_k)||_{N[0]} \leq CM(M(\sum_k c_k^2)^{\frac{1}{2}})^l c_0$$

for some $l > 0$, provided the ϕ_i are Schwartz functions satisfying $||P_k\phi_i||_{S[k]} \leq CMc_k$ for a sufficiently flat frequency envelope $\{c_k\}$. This has already been achieved for the trilinear null-forms as well as the quadrilinear null-form in **Observation 1** by means of Proposition 3.5, Proposition 3.6, Proposition 3.7. For the following computations, we shall make frequent use of the basic *Bernstein's inequality*[18], which states that for any measurable set $R \subset \mathbf{R}^n$ and $\infty \geq p \geq 2$, we have

$$||\mathcal{F}^{-1}(\chi_R \mathcal{F}\phi)||_{L_x^p} \leq C|R|^{\frac{1}{2}-\frac{1}{p}}||\phi||_{L_x^2}$$

The 2nd quadrilinear null-form in **Observation 2***:*

Use the shorthand $\nabla^{-1}(R_\nu\phi_1 R_l\phi_2 - R_l\phi_1 R_\nu\phi_2) = Q_{\nu,j}(\phi_1, \phi_2)$. Then we decompose

[18] as well as simple variations thereof

$$P_0[Q_{\nu,j}(\phi_1,\phi_2)\phi_3\phi_4] = \sum_{k,k_{1,2,3,4}|\max\{k_1,k_2\}>k+O(1)} P_0[Q_{\nu,j}P_k(P_{k_1}\phi_1,P_{k_2}\phi_2)P_{k_3}\phi_3P_{k_4}\phi_4]$$

Now we use Theorem 3.4. Choose 2+ close to 2 and let $\frac{1}{M}+\frac{1}{2+}=\frac{1}{2}$. Then

$$\begin{aligned}
&||\sum_{k,k_{1,2,3,4}|\max\{k_1,k_2\}>k+O(1)} P_0[Q_{\nu,j}P_k(P_{k_1}\phi_1,P_{k_2}\phi_2)P_{k_3}\phi_3P_{k_4}\phi_4]||_{L_t^1L_x^2}\\
&\leq C\sum_{k\geq 0,\,k_{1,2,3,4}|\max\{k_1,k_2\}>k+O(1)} ||P_{k_3}\phi_3P_{k_4}\phi_4||_{L_t^2L_x^{2+}}||Q_{\nu,j}P_k(P_{k_1}\phi_1,P_{k_2}\phi_2)||_{L_t^2L_x^2}\\
&+\sum_{k<0,\,k_{1,2,3,4}|\max\{k_1,k_2\}>k+O(1)} ||P_{k_3}\phi_3P_{k_4}\phi_4||_{L_t^2L_x^{2+}}||Q_{\nu,j}P_k(P_{k_1}\phi_1,P_{k_2}\phi_2)||_{L_t^2L_x^M}\\
&\leq CM^4\sum_{k,k_{1,2,3,4}|\max\{k_1,k_2\}>k+O(1)} 2^{-\frac{(1-\epsilon)}{2}|k|}2^{k-\max\{k_1,k_2\}}2^{-\frac{|k_1-k_2|}{2}}2^{-\frac{|k_3-k_4|}{2}}\prod_i c_i
\end{aligned}$$

It is straightforward to verify that the summation can be carried out to provide the desired estimate for any sufficiently flat envelope.

The first quadrilinear null-form in **Observation 2***:*

Use the shorthand

$$\triangle^{-1}\sum_{j=1}^{3}\partial_j(R_\nu\phi_1R_j\phi_2-R_j\phi_1R_\nu\phi_2)\partial^\nu\phi_3 = N(\phi_1,\phi_2,\phi_3)$$

We use the following *Littlewood-Paley trichotomy*:

$$\begin{aligned}
&P_0[\nabla^{-1}N(\phi_1,\phi_2,\phi_3)\phi_4]\\
&=\sum_{k>10,\,k=k_4+O(1)} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,\phi_3)P_{k_4}\phi_4]\\
&+\sum_{k\in[-10,10],\,k_4\leq 15} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,\phi_3)P_{k_4}\phi_4]\\
&+\sum_{k<-10,\,k_4\in[-5,5]} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,\phi_3)P_{k_4}\phi_4]
\end{aligned}\tag{47}$$

The first summand on the right-hand side is estimated by means of Proposition 3.5 as well as Theorem 3.3:

$$
\begin{aligned}
&||\sum_{k>10,\, k=k_4+O(1)} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,\phi_3)P_{k_4}\phi_4]||_{N[0]}\\
&\leq C\sum_{k>10,\, k=k_4+O(1)} 2^{-\delta k_4}||P_kN(\phi_1,\phi_2,\phi_3)||_{N[k]}||P_{k_4}\phi_4||_{S[k_4]}\\
&\leq CM^4(\sum_r c_r^2)\sum_{k>10,\, k=k_4+O(1)} 2^{-\delta k_4}c_kc_{k_4}\leq CM^4(\sum_r c_r^2)c_0^2
\end{aligned}
$$

provided we choose the frequency envelope sufficiently flat, i.e. $\sigma << \delta$.
The 2nd summand on the right-hand side of (47) is more of the same. As to the third, we decompose it further as follows:

$$
\begin{aligned}
&\sum_{k<-10,\, k_4\in[-5,5]} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,\phi_3)P_{k_4}\phi_4]\\
&=\sum_{k<-10,\, k_4\in[-5,5]} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,P_{<k+C}\phi_3)P_{k_4}\phi_4]\\
&+\sum_{k<-10,\, k_4\in[-5,5]} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,P_{\geq k+C}\phi_3)P_{k_4}\phi_4]
\end{aligned}
$$

Observe that the first summand in the immediately preceding can be schematically written as a sum of terms of the following form:

$$
\begin{aligned}
&\sum_{k<-10,\, k_4\in[-5,5]} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,P_{<k+C}\phi_3)P_{k_4}\phi_4]\\
&=\sum_{k<-10,\, k_4\in[-5,5]} P_0[P_k\nabla^{-1}(Q_{\nu,l}(\phi_1,\phi_2)\nabla_{x,t}P_{<k+C}\phi_3)P_{k_4}\psi_4]
\end{aligned}
\tag{48}
$$

This is estimated by means of Theorem 3.4: let $\frac{2}{4+}+\frac{1}{M}=\frac{1}{2}$.

$$
\begin{aligned}
&||\sum_{k<-10,\, k_4\in[-5,5]} P_0[P_k\nabla^{-1}(Q_{\nu,l}(\phi_1,\phi_2)\nabla_{x,t}P_{<k+C}\phi_3)P_{k_4}\psi_4]||_{N[0]}\\
&\leq C\sum_{k<-10,\, k_4\in[-5,5]} 2^{-k}||P_{<k+C'}Q_{\nu,l}(\phi_1,\phi_2)||_{L_t^2L_x^M}\\
&\qquad\qquad ||\nabla_{x,t}P_{<k+C}\phi_3||_{L_t^4L_x^{4+}}||P_{k_4}\phi_4||_{L_t^4L_x^{4+}}\\
&\leq CM^4(\sum_r c_r^2)\sum_{k<-10,\, k_4\in[-5,5]} 2^{\frac{k}{2+}}c_kc_{k_4}\leq CM^4(\sum_r c_r^2)c_0^2
\end{aligned}
$$

The 2nd term in (48) is estimated by means of the precise formulation of Proposition 3.5:

$$
\begin{aligned}
&||\sum_{k<-10,\, k_4\in[-5,5]} P_0[P_k\nabla^{-1}N(\phi_1,\phi_2,P_{\geq k+C}\phi_3)P_{k_4}\phi_4]||_{N[0]}\\
&\leq \sum_{k_i,\, i\in\{1,2,3\}|\max\{k_1,k_2\}>k_3+O(1),\, k_3\geq k+C}\ \sum_{k_4\in[-5,5]}\\
&\qquad\qquad ||P_0[P_k\nabla^{-1}N(P_{k_1}\phi_1,P_{k_2}\phi_2,P_{k_3}\phi_3)P_{k_4}\phi_4]||_{N[0]}\\
&\leq CM^4 \sum_{k_i,\, i=1,2,3|\max\{k_1,k_2\}>k_3+O(1)}\ \sum_{k_4\in[-5,5]} 2^{-\delta_1|k_1-k_2|}2^{\delta_2(k_3-\max\{k_1,k_2\})}\prod_{i=1}^{4}c_{k_i}
\end{aligned}
$$

This summation can again be carried out, provided the frequency envelope is sufficiently flat.

The third quadrilinear null-form in **Observation 2**

This is treated similarly to the preceding by means of Proposition 3.6 and left out.

The first quintilinear term of **Observation 1**

We note the following elementary estimates: on account of Theorem 3.4, we have

$$||\nabla^{-\epsilon}P_a(P_b\phi_1\nabla^{-1}P_c(\phi_2\phi_3))||_{L_t^{\frac{4}{3}}L_x^p} \leq C_\epsilon 2^{\mu(\epsilon)(\min\{a,b,c\}-\max\{a,b,c\})}||P_b\phi_1||_{S[b]}$$

where $\frac{1}{p}=\frac{5}{12}-\frac{\epsilon}{3}$, $\epsilon>0$ very small and $\mu(\epsilon)>0$.[19]
Next, we note that

$$
\begin{aligned}
||P_a\nabla^{-1}(P_b\phi\nabla^{-(1-\epsilon)}P_cF)||_{L_t^1L_x^\infty} &\leq C_\epsilon 2^{\lambda(\epsilon)(\min\{a,b,c\}-\max\{a,b,c\})}\\
&\qquad ||P_b\phi||_{S[b]}||P_cF||_{L_t^{\frac{4}{3}}L_x^p}
\end{aligned}
$$

$$
\begin{aligned}
||P_a\nabla^{-2\epsilon}(P_b\phi\nabla^{-(1-\epsilon)}P_cF)||_{L_t^1L_x^{3+}} &\leq C_\epsilon 2^{\lambda(\epsilon)(\min\{a,b,c\}-\max\{a,b,c\})}\\
&\qquad ||P_b\phi||_{S[b]}||P_cF||_{L_t^{\frac{4}{3}}L_x^p}
\end{aligned}
$$

where p is as before and $\lambda(\epsilon)>0$, $\frac{1}{3+}=\frac{1}{3}-\frac{\epsilon}{3}$. Now use the *trichotomy*

[19]Use the fact that $||P_0\phi||_{L_t^4L_x^{4+}}\leq C||P_0\phi||_{S[0]}$

$$\begin{aligned}
&||P_0[\nabla^{-1}(\phi\nabla^{-1}(\phi\nabla^{-1}(\phi^2)))\nabla_{x,t}\phi]||_{L_t^1L_x^2}\\
&\leq \sum_{k_1>10,\,k_1=k_2+O(1)}||P_0[\nabla^{-1}P_{k_1}(\phi\nabla^{-1}(\phi\nabla^{-1}(\phi^2)))\nabla_{x,t}P_{k_2}\phi]||_{L_t^1L_x^2}\\
&+\sum_{k_1\in[-10,10],\,k_2<15}||P_0[\nabla^{-1}P_{k_1}(\phi\nabla^{-1}(\phi\nabla^{-1}(\phi^2)))\nabla_{x,t}P_{k_2}\phi]||_{L_t^1L_x^2}\\
&+\sum_{k_1<-10,\,k_2\in[-5,5]}||P_0[\nabla^{-1}P_{k_1}(\phi\nabla^{-1}(\phi\nabla^{-1}(\phi^2)))\nabla_{x,t}P_{k_2}\phi]||_{L_t^1L_x^2}
\end{aligned} \tag{49}$$

Using the preceding calculations, we compute

$$\begin{aligned}
&\sum_{k_1>10,\,k_1=k_2+O(1)}||P_0[\nabla^{-1}P_{k_1}(\phi\nabla^{-1}(\phi\nabla^{-1}(\phi^2)))\nabla_{x,t}P_{k_2}\phi]||_{L_t^1L_x^2}\\
&\leq \sum_{k_1>10,\,k_1=k_2+O(1)}\sum_{a_i,\,i=1,\dots 4}C2^{-k_1}\\
&\qquad ||P_{k_1}(P_{a_1}\phi\nabla^{-1}P_{a_2}(P_{a_3}\phi\nabla^{-1}P_{a_4}(\phi^2)))||_{L_t^1L_x^{3+}}||\nabla_{x,t}P_{k_2}\phi||_{L_t^\infty L_x^2}\\
&\leq \sum_{k_1>10,\,k_1=k_2+O(1)}\sum_{a_i}CM^5(\sum_r c_r^2)\\
&\qquad \sum 2^{\mu(\epsilon)(\min\{k_1,a_1\dots a_4\}-\max\{k_1,a_1,\dots a_4\})}2^{-(\frac{1}{2}-\epsilon)k_1}c_{a_1}c_{a_3}c_{k_1}\\
&\leq CM^5(\sum_k c_k^2)^2c_0
\end{aligned}$$

The remaining terms in (49) are estimated similarly, and left to the reader.

The first quintilinear expression in **Observation 2**

First assume that there is a high-high interaction within the outermost bracket (,), i.e. consider the contribution

$$\sum_{k_1>>k}P_0[\nabla^{-1}P_k(\nabla^{-1}(\nabla^{-1}(\phi^2)\phi)\nabla_{x,t}P_{k_1}\phi)\phi]$$

This term is morally equivalent to

$$\sum_{k_1>>k}P_0[\nabla^{-1}P_k(\nabla^{-1}(\phi^2)\phi P_{k_1}\phi)\phi]$$

It is easy to see upon using Theorem 3.4 as well as an additional *frequency trichotomy* that for fixed k

$$\sum_{k_1>>k}||P_k(\nabla^{-1}(\phi^2)\phi P_{k_1}\phi)||_{L_t^1\dot{H}_x^{-\frac{1}{2}}}\leq CM^3(\sum_r c_r^2)^{\frac{3}{2}}$$

This implies that

$$\sum_{k_1>>k}\min\{||P_k\nabla^{-1}(\nabla^{-1}(\phi^2)\phi P_{k_1}\phi)||_{L_t^1L_x^2},\ ||P_k\nabla^{-1}(\nabla^{-1}(\phi^2)\phi P_{k_1}\phi)||_{L_t^1L_x^\infty}\}$$
$$\leq CM^4(\sum_r c_r^2)^2 2^{-\frac{|k|}{2}}$$

From this the desired estimate follows easily. Next, assume that there is no high-high interaction in the outermost (,), i.e. $k\geq k_1+O(1)$. This contribution is seen to be morally equivalent to

$$\sum_{k_1\leq k+O(1)}P_0[P_k(\nabla^{-1}(\nabla^{-1}(\phi^2)\phi)P_{k_1}\phi)\phi]$$

Now use reasoning similar to the previous quintilinear estimate to obtain

$$\sum_{k_1<k+O(1)}||P_k(\nabla^{-1}(\nabla^{-1}(\phi^2)\phi)P_{k_1}\phi)||_{L_t^1L_x^{3+}}\leq CM^4(\sum_r c_r^2)^2 2^{\delta k}$$

This in conjunction with another frequency trichotomy easily implies the desired inequality.

The remaining error terms of degree five or higher are either similar or simpler and left out.

Having estimated all expressions in **Observations 1, 2**, we can now close the bootstrapping argument. Fix $M>>1$, then choose $\epsilon<<1$ such that (38) as well as Proposition 3.1[20] imply

$$||P_0\nabla_x u||_{S[k]}+||P_0\phi_\alpha^i||_{S[0]}\leq\frac{M}{2}c_k$$

■

5. Proof of the Gauge Change estimate

We commence with the following simple lemma:

Lemma 5.1. *Let $j\geq k+O(1)$. Then provided $f(x):\mathbf{R}\to\mathbf{C}$ as well as ϕ_i, $i=1,2,3$ are as in the statement of Proposition 3.1, we have*

$$||P_kQ_jf(\sum_j\triangle^{-1}\partial_j\phi_j)||_{L_t^2L_x^2}\leq C2^{-\frac{3j}{2}-\frac{k}{2}}$$

[20]recall the definition of ϕ_α^i via u

Proof : Note that the operator $P_k Q_j \Box^{-1}$ with symbol $\frac{m_k(|\xi|)m_j(||\xi|-|\tau||)}{|\tau|^2-|\xi|^2}$ is bounded on $L_t^2 L_x^2$ with norm $\leq C 2^{-2j}$. Thus it suffices to show that

$$||P_k Q_j (\sum_{j,k} \triangle^{-1}\partial^j \partial_\nu \phi_j \triangle^{-1}\partial_k \partial^\nu \phi_k f''(\sum_l \triangle^{-1}\partial_l \phi_l)||_{L_t^2 L_x^2} \leq C$$

$$||P_k Q_j (\sum_j \triangle^{-1}\Box \partial_j \phi_j f''(\sum_l \triangle^{-1}\partial_l \phi_l))||_{L_t^2 L_x^2} \leq C 2^{\frac{j-k}{2}}$$

The first inequality is immediate from Theorem 3.2. The 2nd is proved by invoking a *frequency as well as modulation trichotomy*. In particular, one uses the fact that provided $l >> \max\{k_1, k_2, k_3\}$, we have

$$P_{k_1} Q_{<l-C}(P_{k_2} Q_l f P_{k_3} g) = P_{k_1} Q_{<l-C}(P_{k_2} Q_l f P_{k_3} Q_{l+O(1)} g)$$

as well as (letting $\nabla^{-1} = \sqrt{-\triangle}^{-1}$)

$$P_{k_1} Q_l f(\nabla^{-1}\phi) = P_{k_1} Q_l D_t^{-1}(R_0 \phi f'(\nabla^{-1}\phi))$$

where D_t^{-1} is the operator associated with the multiplier τ^{-1}; of course, the operator $P_k Q_l D_t^{-1}$ is disposable with norm $\sim 2^{-l}$. The proof then boils down to a mechanical exercise in Paradifferential Calculus left for the reader. ■

We also mention the *improved Bernstein's inequality* which states that for any $p \geq 2$, $\epsilon > 0$:

$$||P_k Q_j \phi||_{L_t^2 L_x^p} \leq C_\epsilon 2^{(1-\frac{2}{p})\min\{\frac{j-k}{2+\epsilon}, 0\}} ||P_k Q_j \phi||_{L_t^2 L_x^2}$$

For a proof of this see [24].

Proceeding with the proof of the Proposition, we use the *frequency trichotomy*

$$P_0[\phi f(\nabla^{-1}\phi)] = \sum_{k_1 > 10,\, k_1 = k_2 + O(1)} P_0[P_{k_1}\phi P_{k_2} f(\nabla^{-1}\phi)]$$
$$+ \sum_{k_1 \in [-10,10],\, k_2 < 15} P_0[P_{k_1}\phi P_{k_2} f(\nabla^{-1}\phi)] + \sum_{k_1 < -10,\, k_2 \in [-5,5]} P_0[P_{k_1}\phi P_{k_2} f(\nabla^{-1}\phi)]$$

where we have used a schematic presentation for the exact expression in the statement of Proposition 3.1. We shall only deal with the first and 2nd summand on the right-hand side, the third being much simpler.

(1): High-High interactions: the first term.

Output restricted to small modulation:

$$\sum_{k_1>10,\, k_1=k_2+O(1)} P_0 Q_{<10}[P_{k_1}\phi P_{k_2} f(\nabla^{-1}\phi)]$$

Freeze the output to modulation 2^j, $j<10$. Also, freeze $k_{1,2}$ for the time being. We replace $P_0 Q_j[P_{k_1}\phi P_{k_2} f(\nabla^{-1}\phi)]$ by

$$\sum_{l=1}^{3}\int_{\mathbf{R}^3} a_l(y) P_0 Q_j[P_{k_1}\phi(x) P_{k_2}(R_l\phi f'(\nabla^{-1}\phi))(x-y)]dy$$

where $a_l(y)$ is the convolution kernel of the operator $\triangle^{-1}\partial_l \tilde{P}_{k_2}$[21] Then we observe that

$$||P_{k_2}(R_l\phi P_{\geq j-20} f'(\nabla^{-1}\phi))||_{L_t^4 L_x^{4-}} \leq C_\epsilon 2^{\epsilon k_2} 2^{-\delta(\epsilon) j} c_{k_2}$$

for suitable(small) $\epsilon, \delta(\epsilon)$. Also, using the preceding lemma as well as Bernstein's inequality, we have

$$||P_{k_2}(R_l\phi P_{<j-20} Q_{\geq j-20} f'(\nabla^{-1}\phi))||_{L_t^4 L_x^{4-}} \leq C_\epsilon 2^{\epsilon k_2} 2^{-\delta(\epsilon) j} c_{k_2}$$

The preceding pair of inequalities implies that

$$2^{\frac{j}{2}}||\sum_{l=1}^{3}\int_{\mathbf{R}^3} a_l(y) P_0 Q_j[P_{k_1}\phi(x) P_{k_2}(R_l\phi(f'(\nabla^{-1}\phi) - P_{<j-20} Q_{<j-20} f'(\nabla^{-1}\phi)))(x-y)]dy||_{L_t^2 L_x^2}$$
$$\leq C_\epsilon 2^{(\frac{1}{2}-\delta(\epsilon))j} 2^{(\epsilon-1)k_2} c_{k_2}$$

Provided we choose $\epsilon>0$ small enough, we can sum this over $j<10$, and also obtain the required exponential decay in k_2. This in particular implies that we control the $\dot{X}_0^{\frac{1}{2},\frac{1}{2},1}$-norm of this contribution, which is all we need, on account of the inequality

$$||P_k Q_{<k+O(1)}\phi||_{S[k]} \leq C||P_k\phi||_{\dot{X}_0^{\frac{1}{2},\frac{1}{2},1}}$$

For the remaining term, we introduce the notation $(T_y f)(x) := f(x-y)$ and observe

[21] Recall that $\tilde{P}_{k_2}$ is like P_{k_2} but with $P_{k_2}\tilde{P}_{k_2} = P_{k_2}$.

that

$$\sum_{l=1}^{3}\int_{\mathbf{R}^3} a_l(y)P_0Q_j[P_{k_1}\phi(x)P_{k_2}(R_l\phi P_{<j-20}Q_{<j-20}f'(\nabla^{-1}\phi))(x-y)]dy$$
$$=\sum_{l=1}^{3}\int_{y\in\mathbf{R}^3}\int_{z\in\mathbf{R}^3} a_l(y)b(z)P_0Q_j[Q_{j+O(1)}(P_{k_1}\phi(x)P_{k_2+O(1)}R_lT_{y+z}\phi(x))$$
$$P_{<j-20}Q_{<j-20}T_{y+z}f'(\nabla^{-1}\phi)(x))]dydz$$

where $b(z)$ is the kernel representing the disposable operator P_{k_2}. Then we use Theorem 3.4, as well as the translation invariance of the $S[k]$:

$$2^{\frac{j}{2}}||\int_{y\in\mathbf{R}^3}\int_{z\in\mathbf{R}^3} a_l(y)b(z)P_0Q_j[Q_{j+O(1)}(P_{k_1}\phi(x)P_{k_2+O(1)}R_lT_{y+z}\phi(x))$$
$$P_{<j-20}Q_{<j-20}T_{y+z}f'(\nabla^{-1}\phi)(x))]dydz||_{L_t^2L_x^2}$$
$$\leq C2^{-k_1}\sup_{y,\,z\in\mathbf{R}^3}||Q_{j+O(1)}[P_{k_1}\phi P_{k_2+O(1)}R_lT_{y+z}\phi]||_{\dot{X}^{\frac{1}{2},\frac{1}{2},\infty}}\leq C2^{-k_1}2^{\frac{j}{2+}}c_{k_1}c_{k_2}$$

This can be summed over $j<O(1)$ and furnishes the required exponential gain in $-k_1$.
We now turn to the case when the output is at very large modulation $2^j, j\geq 10$. We decompose into the case $j+10\geq k_1$ and its opposite. Also, we shall only consider the $\dot{X}_0^{\frac{1}{2},\frac{1}{2},\infty}$-component of $S[0]$, since the Proposition in the case of the energy component is standard.

(1.1): $j+10\geq k_1$. We apply another *trichotomy* with respect to modulation:

$$P_0Q_j(P_{k_1}\phi P_{k_2}f(\nabla^{-1}\phi))=P_0Q_j(P_{k_1}Q_{<j-10}\phi P_{k_2}Q_{<j-10}f'(\nabla^{-1}\phi))$$
$$+P_0Q_j(P_{k_1}Q_{\geq j-10}\phi P_{k_2}f'(\nabla^{-1}\phi))+P_0Q_j(P_{k_1}Q_{<j-10}\phi P_{k_2}Q_{\geq j-10}f'(\nabla^{-1}\phi))$$

We observe that the 2nd and third summand on the right-hand side are rather easy to treat on account of lemma 5.1. For the first, note that both inputs may be assumed to be microlocalized on the same half space $\tau><0$, and $k_1=j+O(1)$. We need to estimate

$$2^{\frac{3j}{2}}||P_0Q_j(P_{k_1}Q_{<j-10}\phi P_{k_2}Q_{<j-10}f(\nabla^{-1}\phi))||_{L_t^2L_x^2}$$
$$\sim 2^{\frac{3j}{2}-k_1}||P_0Q_j(P_{k_1}Q_{<j-10}\phi P_{k_2}Q_{<j-10}(\phi f'(\nabla^{-1}\phi))||_{L_t^2L_x^2}$$

We may assume $f'(\nabla^{-1}\phi)$ to be at frequency $<2^{j-10}$, since otherwise, we can use

$$||P_{k_2}Q_{<j-10}(\phi P_{\geq j-10}f'(\nabla^{-1}\phi))||_{L_t^4L_x^{\frac{4}{3}+}}\leq C2^{-(1-\epsilon)k_1}c_{k_1}^2$$

We can also assume $f'(\nabla^{-1}\phi)$ to be at modulation $<2^{j-10}$, on account of lemma 5.1; of course this immediately restricts ϕ to modulation $<2^{j+O(1)}$. Next, assume

$f'(\nabla^{-1}\phi)$ to be at frequency 2^l, $0 \leq l < j-10$. Then we have

$$P_0 Q_j (P_{k_1} Q_{<j-10}\phi P_{k_2} Q_{<j-10}\nabla^{-1}(Q_{<j+O(1)}\phi P_l Q_{<j-10} f'(\nabla^{-1}\phi))$$
$$= \sum_{\kappa_{1,2}\in K_{l-k_1},\, \mathrm{dist}(\kappa_1,-\kappa_2)\leq 2^{l-k_1+O(1)}} P_0 Q_j (P_{k_1,\kappa_1} Q_{<j-10}\phi$$
$$P_{k_2} Q_{<j-10}\nabla^{-1}(P_{k_2+O(1),\kappa_2} Q_{<j+O(1)}\phi P_l Q_{<j-C} f'(\nabla^{-1}\phi))$$

We discard the disposable operator $P_{k_2} Q_{<j-10}\nabla^{-1}$ of L^1-norm $< 2^{-k_1+O(1)}$, and obtain:

$$2^{\frac{3j}{2}} ||P_0 Q_j (P_{k_1} Q_{<j-10}\phi P_{k_2} Q_{<j-10}\nabla^{-1}(\phi P_l f'(\nabla^{-1}\phi)))||_{L_t^2 L_x^2}$$
$$\leq C 2^{\frac{j}{2}} 2^{l-k_1} \sum_{\kappa_{1,2}\in K_{l-k_1},\, \mathrm{dist}(\kappa_1,-\kappa_2)\leq 2^{l-k_1+O(1)}}$$
$$||P_{k_1,\kappa_1} Q_{<j-10}\phi||_{S[k_1,\kappa_1]} ||P_{k_2+O(1),\kappa_2} Q_{<j+O(1)}\phi||_{S[k_2,\kappa_2]} ||P_l f'(\nabla^{-1}\phi)||_{L_t^\infty L_x^3}$$

Using the inequality $||P_l f'(\nabla^{-1}\phi)||_{L_t^\infty L_x^3} \leq C 2^{-l}$, as well as Cauchy-Schwarz and the following inequality[22]:

$$\Big(\sum_{\kappa\in K_{l-k_1}} ||P_{k_1,\kappa} Q_{<j-10}\phi||^2_{S[k_1,\kappa]}\Big)^{\frac{1}{2}} \leq C|k_1| ||P_{k_1}\phi||_{S[k_1]}$$

we obtain the estimate

$$2^{\frac{3j}{2}} ||P_0 Q_j (P_{k_1} Q_{<j-10}\phi P_{k_2} Q_{<j-10}\nabla^{-1}(\phi P_l f'(\nabla^{-1}\phi)))||_{L_t^2 L_x^2}$$
$$\leq C|k_1| 2^{\frac{k_1}{2}} 2^{-k_1} ||P_{k_1}\phi||_{S[k_1]} ||P_{k_2+O(1)}\phi||_{S[k_2]} \leq C 2^{-\frac{k_1}{2+}} c_{k_1}^2$$

This can be summed over $k_1 + O(1) > l \geq 0$ and is acceptable. The case when $f'(\nabla^{-1}\phi)$ is at frequency < 0 is almost identical, by placing $f'(\nabla^{-1}\phi)$ into $L_t^\infty L_x^\infty$.

(1.2): We are left to estimate

$$\sum_{k_1,k_2>j+10} P_0 Q_j [P_{k_1}\phi P_{k_2} f(\nabla^{-1}\phi)] = \sum_{k_1,k_2>j+10} (P_{k_1} Q_{<j-10}\phi P_{k_2} Q_{\geq j-10} f(\nabla^{-1}\phi)$$
$$+ \sum_{k_1,k_2>j+10} (P_{k_1} Q_{\geq j-10}\phi P_{k_2} Q_{<j-10} f(\nabla^{-1}\phi)$$

The 2nd summand on the right-hand side is straightforward on account of the definition of $S[k]$. As to the first, we need a simple modification of lemma 5.1, proved similarly:

$$||P_k Q_j f(\nabla^{-1}\phi)||_{L_t^2 L_x^2} \leq C 2^{-j-k},\ j < k + O(1)$$

---

[22]Which follows easily from the definitions and Plancherel

The desired inequality follows easily form this.

High-Low interactions.

We leave the estimate of the energy of the output to the reader. We commence by estimating the $\dot{X}_0^{\frac{1}{2},\frac{1}{2},\infty}$-norm of the output *provided the modulation is low*. We use the following *mixed trichotomy*: let $j < -10$, say.

$$\begin{aligned}
&P_0 Q_j [P_{[-5,5]}\phi P_{<-10} f(\nabla^{-1}\phi)] \\
&= P_0 Q_j [P_{[-5,5]}\phi P_{-10>.\geq j-10} f(\nabla^{-1}\phi)] \\
&+ P_0 Q_j [P_{[-5,5]} Q_{\geq j-10}\phi P_{<j-10} Q_{<j-10} f(\nabla^{-1}\phi)] \\
&+ P_0 Q_j [P_{[-5,5]}\phi P_{<j-10} Q_{\geq j-10} f(\nabla^{-1}\phi)]
\end{aligned}$$

The 2nd and third summand are easy on account of lemma 5.1. For the first summand, we reformulate it as follows:

$$\begin{aligned}
&P_0 Q_j [P_{[-5,5]}\phi P_{-10>.\geq j-10} f(\nabla^{-1}\phi)] \\
&= \sum_{-10>\tilde{j}\geq j-10} P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} \nabla^{-1}(\phi f(\nabla^{-1}\phi))]
\end{aligned}$$

Freezing $\tilde{j}$ for the moment, we decompose further

$$\begin{aligned}
&P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} f(\nabla^{-1}\phi)] \\
&= P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} \nabla^{-1}(Q_{<j-10}\phi P_{<j-20} Q_{<j-20} f'(\nabla^{-1}\phi))] \\
&+ P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} \nabla^{-1}(Q_{\geq j-10}\phi P_{<j-20} Q_{<j-20} f'(\nabla^{-1}\phi))] \\
&+ P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} \nabla^{-1}(\phi P_{<j-20} Q_{\geq j-20} f'(\nabla^{-1}\phi))] \\
&+ P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} \nabla^{-1}(\phi P_{j-20\leq.\leq\tilde{j}+10} f'(\nabla^{-1}\phi))] \\
&+ P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} \nabla^{-1}(\phi P_{>\tilde{j}+10} f'(\nabla^{-1}\phi))]
\end{aligned} \tag{50}$$

For the first term on the right-hand side, we observe that

$$\begin{aligned}
&P_0 Q_j [P_{[-5,5]}\phi P_{\tilde{j}} \nabla^{-1}(Q_{<j-10}\phi P_{<j-20} Q_{<j-20} f(\nabla^{-1}\phi))] \\
&= \int_{\mathbf{R}^3} a_{\tilde{j}}(y) P_0 Q_j [Q_{j+O(1)}(P_{[-5,5]}\phi P_{\tilde{j}+O(1)} Q_{<j-10} T_y\phi) \\
&\qquad\qquad P_{<j-20} Q_{<j-20} T_y f(\nabla^{-1}\phi)] dy
\end{aligned}$$

where $a_{\tilde{j}}$ is the kernel associated with the multiplier $\nabla^{-1}P_{\tilde{j}}$ of L^1-mass $\sim 2^{-\tilde{j}}$. Using Theorem 3.4 as well as translation invariance of the $S[k]$, we conclude that

$$||P_0 Q_j[P_{[-5,5]}\phi P_{\tilde{j}}\nabla^{-1}(Q_{<j-10}\phi P_{<j-20}Q_{<j-20}f(\nabla^{-1}\phi))]||_{\dot{X}_0^{\frac{1}{2},\frac{1}{2},\infty}}$$
$$\leq C2^{\frac{j-\tilde{j}}{2+}}c_0 c_{\tilde{j}}$$

This can be summed over $O(1) > \tilde{j} > j$ to yield the desired inequality.
For the 2nd term on the right-hand side of (50), we use the improved Bernstein's inequality:

$$2^{\frac{j}{2}}||P_0 Q_j[P_{[-5,5]}\phi P_{\tilde{j}}\nabla^{-1}(Q_{\geq j-10}\phi P_{<j-20}Q_{<j-20}f'(\nabla^{-1}\phi))]||_{L_t^2 L_x^2}$$
$$\leq C\sum_{\tilde{j}>l>j-10}||[P_{[-5,5]}\phi||_{L_t^\infty L_x^2}||P_{\tilde{j}+O(1)}Q_l\phi||_{L_t^2 L_x^\infty}$$
$$+\sum_{l\geq\tilde{j}}||[P_{[-5,5]}\phi||_{L_t^\infty L_x^2}||P_{\tilde{j}+O(1)}Q_l\phi||_{L_t^2 L_x^\infty}$$
$$\leq \sum_{\tilde{j}>l>j-10}C2^{\frac{l-\tilde{j}}{2+}}2^{\frac{j-l}{2}}c_0 c_{\tilde{j}}+\sum_{l\geq\tilde{j}}C2^{\frac{j-l}{2}}c_0 c_{\tilde{j}}\leq C2^{\frac{j-\tilde{j}}{2+}}c_0 c_{\tilde{j}}$$

This can again be summed over $\tilde{j}$.
For the third summand of (50), we invoke lemma 5.1, of course, as well as Bernstein's inequality. One computes

$$2^{\frac{j}{2}}||P_0 Q_j[P_{[-5,5]}\phi P_{\tilde{j}}\nabla^{-1}(\phi P_{<j-20}Q_{\geq j-20}f'(\nabla^{-1}\phi))]||_{L_t^2 L_x^2}$$
$$\leq C2^{\frac{j}{2}-\tilde{j}}||P_{[-5,5]}\phi||_{L_t^\infty L_x^2}||P_{\tilde{j}}\phi||_{L_t^4 L_x^\infty}||P_{<j-20}Q_{\geq j-20}f'(\nabla^{-1}\phi)||_{L_t^4 L_x^\infty}$$
$$\leq C2^{\frac{j-\tilde{j}}{4}}c_0 c_{\tilde{j}}$$

This can again be summed over $\tilde{j} > j-10$.
The fourth term is similar to the third and left out(one can place $f(\nabla^{-1}\phi)$ into $L_t^4 L_x^\infty$). Finally, for the fifth term, one places $\phi P_{>\tilde{j}+10}f'(\nabla^{-1}\phi)$ into $L_t^2 L_x^\infty$, using

$$||P_{\tilde{j}}[\phi P_{>\tilde{j}+10}f'(\nabla^{-1}\phi)]||_{L_t^2 L_x^\infty}\leq C2^{\frac{\tilde{j}}{2}}c_{\tilde{j}}$$

The simple details are left out. This finishes the treatment of the $\dot{X}_0^{\frac{1}{2},\frac{1}{2},\infty}$ component of $||.||_{S[0]}$, provided the output is at small modulation. The case when the modulation is large is dealt with similarly to the analogous situation in the high-high case.

Now we estimate the 'null-frame component' of $||.||_{S[0]}$, i.e.

$$\sup_{\pm}\sup_{l<-10}(\sum_{\kappa\in K_l}||P_{0,\pm\kappa}Q_{<2l}^{\pm}(\phi f(\nabla^{-1}\phi))||_{S[0,\kappa]}^2)^{\frac{1}{2}}$$

Fix $l < -10$. We decompose

$$P_0Q^{\pm}_{<2l}[P_{[-5,5]}\phi f(\nabla^{-1}\phi)] = P_0Q^{\pm}_{<2l}[P_{[-5,5]}Q^{\pm}_{<2l}\phi P_{<l-10}Q_{<-10}f(\nabla^{-1}\phi)]$$
$$+ P_0Q^{\pm}_{<2l}[P_{[-5,5]}Q_{\geq 2l}\phi f(\nabla^{-1}\phi)] + P_0Q^{\pm}_{<2l}[P_{[-5,5]}\phi P_{\geq l-10}f(\nabla^{-1}\phi)]$$

We treat each term on the right-hand side:

First term: Use the disposability of $P_{0,\kappa}Q^{\pm}_{<2l}$, see [24]

$$\sum_{\kappa\in K_l}(||P_{0,\kappa}Q^{\pm}_{<2l}[P_{[-5,5]}Q^{\pm}_{<2l}\phi P_{<l-10}Q_{<-10}f(\nabla^{-1}\phi)]||^2_{S[0,\kappa]})$$
$$= \sum_{\kappa\in K_l}\sum_{\kappa'\in K_{l-5},\,\kappa'\subset\kappa}||P_{0,\kappa}Q^{\pm}_{<2l}[P_{[-5,5],\kappa'}Q^{\pm}_{<2l}\phi P_{<l-10}Q_{<-10}f(\nabla^{-1}\phi)]||^2_{S[0,\kappa']})$$
$$\leq C(\sum_{\kappa'\in K_{l-5}}||P_{[-5,5],\kappa'}Q^{\pm}_{<2l}\phi||^2)^{\frac{1}{2}} \leq Cc_0$$

2nd term: This follows easily from the inequality

$$||P_kQ_{<k}\phi||_{S[k]} \leq C||P_k\phi||_{\dot{X}_k^{\frac{1}{2},\frac{1}{2},1}} \tag{51}$$

as well as the definition of $S[k]$.

Third term: We reformulate it as follows:

$$P_0Q^{\pm}_{<2l}[P_{[-5,5]}\phi P_{\geq l-10}f(\nabla^{-1}\phi)]$$
$$= \sum_{r<2l}\sum_{k\geq l-10}\int_{\mathbf{R}^3}a_k(y)P_0Q_r[P_{[-5,5]}\phi(x)P_k(\phi f'(\nabla^{-1}\phi))(x-y)dy$$

where $a_k(y)$ is the kernel representing the operator $P_k\nabla^{-1}$. Next, one decomposes

$$P_k(\phi(x-y)f'(\nabla^{-1}\phi))(x-y) = P_k(\phi P_{\geq r-10}f'(\nabla^{-1}\phi))(x-y)$$
$$+ P_k(P_{k+O(1)}\phi P_{<r-10}Q_{<r-10}f'(\nabla^{-1}\phi))(x-y)$$
$$+ P_k(P_{k+O(1)}\phi P_{<r-10}Q_{\geq r-10}f'(\nabla^{-1}\phi))(x-y)$$

The 2nd term provides the following contribution to the output:

$$\sum_{r<2l}\sum_{k\geq l-10}\int_{\mathbf{R}^3} a_k(y)P_0Q_r[P_{[-5,5]}\phi(x)P_k(P_{k+O(1)}\phi \\ P_{<r-10}Q_{<r-10}f'(\nabla^{-1}\phi))(x-y)]dy \\ =\sum_{r<2l}\sum_{k\geq l-10}\int_{\mathbf{R}^3}\int_{z\in\mathbf{R}^3} a_k(y)b_k(z)P_0Q_r[Q_{r+O(1)}(P_{[-5,5]}\phi(x)P_{k+O(1)}T_{y+z}\phi(x)) \\ P_{<r-10}Q_{<r-10}T_{y+z}f'(\nabla^{-1}\phi)(x)]dy$$

where $b_k(z)$ is the kernel representing the operator P_k. This is easily estimated by means of Theorem 3.4 as well as the inequality (51). The first and third summand yield contributions estimated by placing $P_{<r-10}Q_{\geq r-10}f'(\nabla^{-1}\phi)(x-y)$ and $P_{\geq r-10}f'(\nabla^{-1}\phi)(x-y)$ into $L_t^4L_x^p$ for suitable $p>4$, as in earlier instances. This is left to the reader, and concludes the proof of Proposition 3.1.

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