# Scale-Invariant Models of Gravity and Particle Physics and their Cosmological Implications

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# Abstract

Currently, the best theoretical description of fundamental matter and its gravitational interaction is given by the Standard Model (SM) of particle physics and Einstein's theory of General Relativity (GR). These theories contain a number of seemingly unrelated scales. While Newton's gravitational constant and the mass of the Higgs boson are parameters in the classical action, the masses of other elementary particles are due to the electroweak symmetry breaking. Yet other scales, like $\Lambda_{QCD}$ associated to the strong interaction, only appear after the quantization of the theory.

We reevaluate the idea that the fundamental theory of nature may contain no fixed scales and that all observed scales could have a common origin in the spontaneous break-down of exact scale invariance. To this end, we consider a few minimal scale-invariant extensions of GR and the SM, focusing especially on their cosmological phenomenology.

In the simplest considered model, scale invariance is achieved through the introduction of a dilaton field. We find that for a large class of potentials, scale invariance is spontaneously broken, leading to induced scales at the classical level. The dilaton is exactly massless and practically decouples from all SM fields. The dynamical break-down of scale invariance automatically provides a mechanism for inflation. Despite exact scale invariance, the theory generally contains a cosmological constant, or, put in other words, flat spacetime need not be a solution.

We next replace standard gravity by Unimodular Gravity (UG). This results in the appearance of an arbitrary integration constant in the equations of motion, inducing a run-away potential for the dilaton. As a consequence, the dilaton can play the role of a dynamical dark-energy component. The cosmological phenomenology of the model combining scale invariance and unimodular gravity is studied in detail. We find that the equation of state of the dilaton condensate has to be very close to the one of a cosmological constant.

If the spacetime symmetry group of the gravitational action is reduced from the group of all diffeomorphisms (Diff) to the subgroup of transverse diffeomorphisms (TDiff), the metric in general contains a propagating scalar degree of freedom. We show that the replacement of Diff by TDiff makes it possible to construct a scale-invariant theory of gravity and particle physics in which the dilaton appears as a part of the metric. We find the conditions under which such a theory is a viable description of particle physics and in particular reproduces the SM phenomenology. The minimal theory with scale invariance and UG is found to be a particular case of a theory with scale and TDiff invariance. Moreover, cosmological solutions in models based on scale and TDiff invariance turn out to generically be similar to the solutions of the model with UG.

In usual quantum field theories, scale invariance is anomalous. This might suggest that results based on classical scale invariance are necessarily spoiled by quantum corrections. We show that this conclusion is not true. Namely, we propose a new renormalization scheme which allows to construct a class of quantum field theories that are scale-invariant to all orders of perturbation theory and where the scale symmetry is spontaneously broken. In this type of theory, all scales, including those related to dimensional transmutation, like $\Lambda_{QCD}$, appear as a consequence of the spontaneous break-down of the scale symmetry. The proposed theories are not renormalizable. Nonetheless, they are valid effective theories below a field-dependent cut-off scale. If the scale-invariant renormalization scheme is applied to the presented minimal scale-invariant extensions of GR and the SM, the goal of having a common origin of all scales, spontaneous breaking of scale invariance, is achieved.

Keywords: Scale Invariance, Unimodular Gravity, TDiff Invariance, Restricted Coordinate Invariance, Quantum Scale Invariance, Dilatational Anomaly.

# Zusammenfassung

Die momentan beste theoretische Beschreibung der elementaren Materie und der Gravitationswechselwirkung ist gegeben durch das Standard Modell (SM) der Teilchenphysik und Einsteins Allgemeine Relativitätstheorie (ART). Diese beiden Theorien enthalten mehrere scheinbar komplett unabhängige dimensionsvolle Grössen. Während Newtons Gravitationskonstante, sowie die Masse des Higgs-Bosons explizit in der klassischen Wirkung enthalten sind, erscheinen die Massen der übrigen Elementarteilchen erst durch den spontanen Bruch der elektroschwachen Symmetrie. Wieder andere dimensionsvollen Grössen, wie zum Beispiel die charakteristische Skala der Quantenchromodynamik ($\Lambda_{\mathrm{QCD}}$), erscheinen infolge der Quantisierung der klassischen Feldtheorie.

Wir wollen uns mit der Idee beschäftigen, dass die fundamentalen Gesetze der Natur überhaupt keine dimensionsvollen Grössen enthalten könnten und dass alle beobachtbaren dimensionsvollen Grössen ihren Ursprung im spontanen Bruch der Skalensymmetrie der fundamentalen Theorie haben könnten.

Das einfachste Modell, das wir betrachten werden, ist skaleninvariant Dank der Präsenz eines zusätzlichen skalaren Feldes, welches wir Dilaton nennen werden. Wir stellen fest, dass eine breite Klasse von Potenzialfunktionen zum spontanen Bruch der Skalensymmetrie führt, wodurch alle dimensionsvollen Grössen der klassischen Theorie generiert werden. In dieser Theorie ist das Dilaton exakt massenlos und praktisch entkoppelt von allen SM-Feldern. Es stellt sich heraus, dass die dynamische Brechung der Skaleninvarianz einen Mechanismus für die kosmologische Inflation liefert. Trotz der exakten Skaleninvarianz, enthält die Theorie im Allgemeinen eine Kosmologische Konstante. In anderen Worten: Die flache Raumzeit ist nicht bedingungslos eine Lösung der Feldgleichungen.

In einem nächsten Schritt ersetzen wir Einsteins Gravitationstheorie durch die sogenannt Unimodulare Gravitationstheorie (UG). Die Konsequenz davon ist das Erscheinen einer unbestimmten Integrationskonstante in den Feldgleichungen. Die Präsenz der neuen Konstante wirkt auf das Dilaton wie eine "Wegrenn"-Potenzialfunktion. Folglich kann das Dilaton eine dynamische Komponente der Dunklen Energie bilden. Wir studieren im Detail die kosmologische Phänomenologie des auf Skaleninvarianz und Unimodulare Gravitation basierten Modells. Es stellt sich heraus, dass die Zustandsgleichung des Dilatonkondensates der Zustandsgleichung einer Kosmologischen Konstante sehr ähnlich ist.

Die der ART zugrunde liegende Wirkung ist invariant bezüglich aller Koordinatentransformationen (Diff). Wenn nun die Symmetriegruppe der Gravitationswirkung reduziert wird auf die Untergruppe derjeniger Koordinatentransformationen, welche das Vier-Volumen unverändert lassen (TDiff), enthält die Metrik im allgemeinen einen dynamischen skalaren Freiheitsgrad. Wir zeigen, dass es das Ersetzen der Diff-Symmetriegruppe durch die TDiff-Symmetriegruppe ermöglicht, eine realistische skaleninvariante Theorie der Gravitation und der Teilchenphysik zu bilden, ohne dass die Teilchentheorie um ein zusätzliches Feld erweitert werden muss. In dieser Art von Theorie erscheint das Dilaton als ein Teil der Metrik. Wir finden die Bedingungen, unter welchen die auf Skalen- und TDiff-Invarianz basierten Theorien eine konsistente und phänomenologisch brauchbare Beschreibung der Teilchenphysik sein können. Im Speziellen finden wir die Bedingungen, unter welchen die betrachteten Theorien die Phänomenologie des Standard Modells reproduzieren. Des weiteren stellt sich die minimale, auf UG und Skaleninvarianz basierte Theorie als Spezialfall der allgemeineren auf Skalen- und TDiff-Invarianz basierten Theorien heraus. Es wird gezeigt, dass die kosmologischen Lösungen letzterer Theorien den Lösungen der minimalen Theorie im Allgemeinen sehr ähnlich sind.

Die Skalensymmetrie erweist sich in allen gängigen Quantenfeldtheorien als anomal. Man könnte daraus folgern wollen, dass alle auf die Skaleninvarianz der klassischen Theorie basierten Resultate durch Quantenkorrekturen ungültig gemacht werden. Wir zeigen, dass solch eine Folgerung falsch wäre. Genauer gesagt, präsentieren wir eine neue Renormalisierungsprozedur, die es ermöglicht, eine Klasse von Quantenfeldtheorien zu konstruieren, welche in allen Ordungen der Störungstheorie skalenivariant sind, und in welchen die Skaleninvarianz spontan gebrochen ist. In einer solchen Theorie erscheinen alle dimensionsvollen Grössen, inklusive derer die durch dimensionale Transmutation entstehen, wie zum Beispiel $\Lambda_{\text{QCD}}$, als Folge der spontanen Brechung der Skaleninvarianz. Die neuen Theorien sind zwar nicht renormalisierbar, dennoch können sie als effektive Theorien, gültig unterhalb einer gewissen Maximalenergie, dienen. Wenn die skaleninvariante Renormalisierungsprozedur auf die obgenannten minimalen Erweiterungen der ART and des SM angewandt wird, ist unser Ziel, nämlich die Konstruktion einer Theorie in welcher alle dimensionsvollen Grössen einen gemeinsamen Ursprung haben, erreicht.

Stichworte: Skaleninvarianz, Unimodulare Gravitation, TDiff Invarianz, Quantenskaleninvarianz, Dilatationsanomalie.

# Contents

# 1. Introduction

## 1.1 Introduction to the subject

The foundation pillars of modern Cosmology are Einstein's theory of General Relativity (GR) [1] and the Standard Model (SM) of particle physics [2–7]. The SM, based on quantum field theory, is supposed to describe the matter and radiation content of the universe. GR describes the gravitational interaction to which all forms of energy are subject. Both these theories are very successful in describing most of the observed phenomena in their respective domain of application.[1] It is understood, however, that neither GR nor the SM can be complete theories and that both of them need to be extended or modified (see e.g. [10, 11]). This fact is supported by strong experimental and theoretical arguments of which we want to name the most prominent ones.

The SM, containing only massless neutrinos, is incompatible with observed neutrino oscillations, which can only occur if neutrinos have a mass [9]. On the theoretical side, we can mention the Landau-pole problem [12]. In the SM the U(1) gauge coupling, the Higgs self coupling as well as the Yukawa couplings grow large at high energies. This means that above a certain energy one can no longer extract predictions from the theory.

Turning our attention to GR, there is at the moment no compelling experimental evidence suggesting that classical GR is incomplete. There is, however, a fundamental theoretical issue. GR is a classical field theory. We know that in order to describe systems in which neither quantum nor gravitational effects are negligible, a quantum theory of gravity is needed. If one attempts to quantize GR in the framework of usual quantum field theory, one notices that perturbative GR is non-renormalizable. It can thus only be considered as an effective low-energy theory. The quest for reconciliation of GR with quantum physics is one of the major challenges in theoretical physics.

Additional reasons that call for an extension of GR and the SM appear in the context of cosmology. Over the past decades the so-called Hot Big Bang model (see e.g. [13]) established itself as the standard model of cosmology. This is mainly due to its accord with the observation of an expanding uni-

[1] For a review of experimental tests of general relativity see for example [8]. For an overview of the experimental status of particle physics see [9].

verse, the discovery of the cosmic microwave background and the observed abundances of light elements. Nevertheless, in the scope of this model, a number of questions remains open. Let us present the most important ones.

- Dark Matter (DM)

  Several independent astrophysical and cosmological observations related to the gravitational interaction on different scales cannot be described on the basis of GR and the SM (for a recent review see [14]). A well-known example is provided by the observed flat rotation curves of spiral galaxies. In these galaxies, most of the visible mass is concentrated in a thin disc rotating around the center. Based on this fact, one would expect the rotation velocities of objects circulating around the galactic center to decrease like $1/\sqrt{r}$ with increasing radial distance $r$. This prediction is in sharp conflict with observations, which show that the rotation velocities remain approximately constant over a large range of distances. The observed rotation curves can be explained if the galaxy is embedded into a large halo of a new type of invisible (dark) matter. Alternatively, one can try to remove the discrepancy by modifying the gravitational force law at galactic distances.

  Cosmology provides another reason for the need of a new type of matter. In fact, for primordial nucleosynthesis to successfully describe the measured abundances of light elements in the universe, the present abundance of baryons should be about $\Omega_B \simeq 0.02$. Comparison of this value with the total abundance of non-relativistic matter $\Omega_M \simeq 0.3$ gained from measurements of the Hubble constant, suggests that the universe should contain a large amount of non-baryonic matter. These numbers also rely on the assumption that gravity is well-described by GR. It can not be excluded that the discrepancies could be accounted for in a modified theory of gravity. In either case, there is a piece missing in the puzzle.

- Baryon asymmetry

  The observed matter content of the universe is mostly composed of baryons and electrons. However, we know from particle physics, that for each charged particle there exists an antiparticle with opposite charge. In spite of the almost exact charge conjugation symmetry between particle and antiparticle properties, antimatter is hardly present in our universe. This unexplained fact is commonly referred to as the baryon asymmetry problem.

  From the particle physics point of view, one is tempted to consider interactions, which can produce matter and antimatter at different rates. Such processes might generate today's observed baryon asymmetry during the evolution of the universe, even if it started off with only a tiny asymmetry. This mechanism is called baryogenesis (for a review see e.g. [15]). In 1967, Andrei Sakharov [16] proposed a set of three necessary conditions for the generation of baryon asymmetry: (a) Baryon number violation, (b) C and CP violation, (c) The corresponding processes should be out of thermal equilibrium. Although the Standard Model in principle fulfills these requirements [17], it is unlikely to produce the observed baryon asymmetry, because CP violation in the Cabibbo-Kobayashi-Maskawa mixing matrix of quarks is too small. Also, for a mass of the Higgs boson in the experimentally allowed range, the electroweak phase transition is not of the first order [18], which would be needed to satisfy condition c). A solution to the baryon asymmetry issue might come from a particle physics theory beyond the Standard Model.

- Inflation

  There are strong reasons to believe that the very early universe went through a phase of almost exponential expansion, called inflation (for a good review see [19]). This would explain the nearly perfect homogeneity and spatial flatness of today's universe. In fact, without inflation, different parts of the observable universe would never have been in causal contact. However, observations of the Cosmic Microwave Background (CMB) show that all these supposedly causally disconnected regions have the same temperature to about one part in $10^5$. Without inflation, this fact could only be explained by a very fine tuning of initial conditions. The situation is similar for the observed spatial flatness of the universe, which, in the absence of an inflationary phase in the early universe, could only be due to very special initial conditions. After the idea of inflation had been introduced, people realized that it can, on top of solving the mentioned problems, give an explanation for the primordial density fluctuations that are at the origin of structure formation in the universe [20–23]. Due to all these nice features, the inflationary paradigm has almost become part of standard cosmology. Still, even though many models and mechanisms for inflation have been proposed, we lack a precise understanding of its fundamental origin.

- Dark Energy (DE)

  There exists compelling experimental evidence [24–26], suggesting that the universe is currently undergoing a phase of accelerated expansion. If one sticks to the hypothesis of a homogeneous universe described by Friedmann's equations (GR), accelerated expansion can only be explained by the presence of an energy component (Dark Energy, DE) with almost constant energy density and negative pressure.[2] Moreover, within the standard cosmological framework, it is found that this exotic energy component should constitute about 70% of today's total energy

[2] This is completely analogous to the inflationary phase in the very early universe.

density of the universe, $\Omega_{DE} \simeq 0.7$ [27]. The fundamental nature of dark energy is up to now unclear. A nearly obvious possibility is that dark energy is due to a cosmological constant $\Lambda$ in Einstein's equations (see (1.2) below). Such a term has exactly the properties of a fluid of constant energy density $\rho_\Lambda = cst.$ and negative pressure $p_\Lambda = -\rho_\Lambda$. And indeed, including this term in the equations, makes it possible to fit all current observations. However, the identification of dark energy with the cosmological constant brings about a serious theoretical issue (cf. next paragraph).

If dark energy is not related to the cosmological constant, this would call for the introduction of a completely new energy component, often called Quintessence [28–31]. To this day, observations cannot tell with certainty, whether dark energy behaves exactly like a cosmological constant, or whether its properties (e.g. equation of state) might be changing in time, as is the case for quintessence [32].

Let us mention that the situation is different if one abandons the hypothesis of a nearly homogeneous universe at very large scales. It has been shown in several works, that the reason for the observed acceleration could be that we live in a huge underdense region (void) of the universe (see e.g. [33]). This scenario, of course, would also need a fundamental explanation.

- <u>The Cosmological Constant Problem</u>

  In accord with the symmetry principles of general relativity, Einstein's equations can contain a cosmological constant $\Lambda$.[3] As mentioned in the previous paragraph, this constant can act as dark energy. If the cosmological constant $\Lambda$ is the sole constituent of dark energy, the observed abundance $\Omega_\Lambda = \Omega_{DE} \simeq 0.7$ corresponds to a value $\Lambda \simeq 10^{-47}\,\text{GeV}^4$. Up to here, there is no problem. However, if one uses the framework of effective field theory to combine GR with particle physics, the cosmological constant is expected to receive many different contributions corresponding to the vacuum energies of the various SM fields. The effective cosmological constant should then be the sum of a bare value $\Lambda_0$ and these contributions, which are proportional to the different particle physics scales of the theory. Technically, this result is obtained by computing the constant term in the quantum effective potential of the particle theory. The problem appears because all particle physics scales are much larger than the scale of dark energy. Suppose, for instance, that the electroweak scale $M_W \simeq 250\,\text{GeV}$ is the largest scale of the particle physics theory. One would then have $\Lambda \sim \Lambda_0 + M_W^4 \sim \Lambda_0 + 10^9\,\text{GeV}^4 \sim 10^{-47}\,\text{GeV}^4$. For this to be satisfied, the bare value $\Lambda_0$ has to cancel the vacuum energy contribution to 56

[3] For detailed reviews of the cosmological constant problems see [31, 34, 35].

decimal places. This seems to be a very unnatural fine-tuning. If the particle theory contains bigger scales associated to physics beyond the standard model, the situation gets even worse.

There is a second problem related to the cosmological constant, commonly referred to as the "Cosmic Coincidence Problem". If dark energy is fully attributed to the cosmological constant, its present abundance $\Omega_\Lambda \simeq 0.7$ is of the same order of magnitude as the abundance of non-relativistic matter $\Omega_m \simeq 0.3$. While the energy density of matter decreases as $\rho_m \propto a^{-3}$ with the expansion of the universe, $\rho_\Lambda$ remains constant.[4] Hence, the ratio of abundances scales like $\Omega_m/\Omega_\Lambda \propto a^{-3}$. The fact that this ratio is close to one just in the present universe, is considered to be an unnatural coincidence that might need further explanation.

Let us note that both these problems are "naturalness" problems and do not express an inconsistency of the underlying theory. If these were the only existing problems, the theory might not need to be modified. However, since modifications of the theory are asked for by many other issues, it seems reasonable to look for those modifications which also alleviate the naturalness problems.

The presented "cosmological" problems are most probably rooted in the underlying theories of gravity (GR) and particle physics (SM).[5] And of course, the different problems might be related.

Proposals for extending the particle physics theory are nearly uncountable. One commonly refers to "Physics Beyond the Standard Model" (BSM). It would go beyond the scope of this work to give a detailed account of BSM physics. For recent reviews and an entry point to the literature see e.g. [11, 36]. Some of the new theories involve fundamental new ideas. In supersymmetric theories, for instance, one introduces a whole range of new particles corresponding to supersymmetric partners of the standard model particles. These theories can improve the situation for several of the above problems. For example, they can contain a new particle that has the right properties to be responsible for dark matter and at the same time provide mechanisms for inflation and baryogenesis. A very minimal extension of the SM, not involving supersymmetry, is given by the $\nu$MSM [37–39]. This model corresponds to the SM supplemented by three singlet neutrinos. Interestingly, already such a minimal approach permits to address and possibly solve several of the above-stated problems. Many of the BSM theories pos-

[4] $a = a(t)$ stands for the scale factor of the homogeneous Friedmann-LeMaitre-Robertson-Walker universe.

[5] This need not be the case if some of the assumptions made in cosmology are wrong. For instance, as already mentioned, it can not be excluded that the observed acceleration of the nearby universe is due to the fact that we live in an underdense region of the universe. This, of course, would not solve the cosmological constant problem.

sess signatures that can be tested with the experiments currently going on at the Large Hadron Collider (LHC). We therefore have the exciting perspective of getting to know much more about particle physics in the very near future.

Given the success of classical general relativity, proposals that solely modify the classical theory of gravitation are fewer. Nevertheless, one should bear in mind that some of the cosmological problems could be due to our misunderstanding of classical gravity. It is interesting, for example, that a modification of Newton's law on galaxy scales can "explain" the flat rotation curves of spiral galaxies. However, it turns out to be difficult to obtain the modified Newton's law as the non-relativistic limit of a fully relativistic theory. Another idea is to modify the equations of gravity such that vacuum energy does not act as dark energy, which would solve part of the cosmological constant problem. A possible way to achieve this is to look for theories in which gravity has a finite range (massive gravity) (e.g. [40, 41]). An alternative possibility is to replace GR by Unimodular Gravity (UG). This idea will be discussed in detail in the present work. Often, attempts to change the theory of classical GR are connected with the hope that the new theory would allow to be quantized in the standard framework of quantum field theory. An interesting recent proposal along these lines is the so-called Hořava Gravity [42, 43].

It could well be that the merger of gravitation with the laws of quantum physics cannot be realized in the scope of quantum field theory. Great efforts are being undertaken, giving the example of superstring theory, to incorporate GR and the SM in a new type of theory that would be applicable at all energy scales (UV complete), but which is no longer a traditional quantum field theory (see e.g. [44, 45]). There is a chance, or hope, that this enterprise, beyond solving a crucial theoretical issue, will also solve the mentioned problems of cosmology. At present, unfortunately, the complexity of string theories makes it hard to extract predictions from them.

Some of the cosmological problems can be approached at a more phenomenological level. In particular, related to inflation, dark energy and the cosmological constant, people try to find simple mechanisms which can describe the observed phenomena. Models of inflation typically introduce one or several new scalar fields, together with a particular potential. These scalar fields need not be fundamental but can be effective descriptions of some unknown new fundamental physics. They model the existence of a new energy component in the early universe that can give rise to a phase of accelerated (almost exponential) expansion. The strategy is to find those models whose predictions agree with observations (e.g. CMB). In a next step, one would try to identify the fundamental theories compatible with the successful phenomenological models. Given the precision of current observations, there are still many different models of inflation that can be fitted to the data [19]. As a consequence, it is rather difficult to apply this strategy to gain some knowledge about the underlying fundamental theory. In a very similar manner, one can introduce new scalar fields to model a possible dynamical dark energy component (e.g. [28, 31]). Related to this are the so-called adjustment mechanisms that try to address the cosmological constant problem [34, 35]. Roughly speaking, the idea is to introduce a scalar field that compensates for the expected large value of the cosmological constant. Up to now, activities in this direction have not proven very successful.

## 1.2 Introduction to the present work

In the present work we consider a few modest modifications of the standard theories of gravitation and particle physics, based on the ideas of Scale Invariance (SI), Unimodular Gravity (UG) and TDiff Gravity. One part of the task is to find theoretically consistent ways to implement these ideas. Secondly, it has to be assured that the modified theories do not violate well-established experimental bounds. Thirdly, as our main objective, we try to see how the new theories can address the aforementioned cosmological problems. The separate ingredients (SI,UG and TDiff Gravity) are not new. What is new, however, is the way we implement them and combine several of these ideas in a same theory. Let us start with a brief presentation of the key ingredients.

- Scale Invariance (SI)

  In classical and quantum field theory scale invariance is a symmetry of theories that do not contain any fixed scales, i.e. dimensional parameters.[6] An action (respectively quantum effective action) describing such a theory is invariant under transformations of the type[7]

  $$\Phi(x) \mapsto \sigma^{d_\Phi} \Phi(\sigma x) \, , \tag{1.1}$$

  where $\Phi(x)$ stands for the different fields, $d_\Phi$ is their associated scaling dimension and $\sigma$ is an arbitrary real parameter. These transformations are called scale transformations or dilatations. A famous example of a scale-invariant theory is given by Maxwell's equations for classical electrodynamics in the absence of charges and currents.

  There are several motivations to study scale invariance in the context of particle physics. An old idea is that at high energies masses of particles and other dimensional parameters could become negligible and the theory would be almost scale-invariant. The associated hope

[6] By dimensional parameters we mean parameters that have non-zero dimensionality when expressed in natural units, where $c = \hbar = 1$. Such parameters can be expressed in units of mass or energy to some power.

[7] There are other ways to parametrize the symmetry associated to the absence of dimensional constants. We will also make use of them in the following sections.

is that the study of exactly scale-invariant theories could also teach us something about theories which are approximately scale-invariant, much like in the case of chiral perturbation theory for QCD [46] (for a review of scale invariance in this spirit see [47]).

One can try to go one step further. Inspecting the standard model Lagrangian, one notices that at the classical level scale invariance is broken by one single term – the mass term of the Higgs field. Knowing that the standard model needs to be extended, this observation can motivate the search for extensions that make the Lagrangian completely scale-invariant. Even more so, as up to now the Higgs sector remains experimentally unexplored. A recent study of this idea is presented in [48, 49].

Now, the fact that we do observe massive particles in nature is enough to know that scale invariance must be broken. As usual, there are two ways in which this symmetry can be broken. Allowing for some non-zero mass terms (or other dimensional parameters) in the action breaks the symmetry explicitly. This option does not represent a real progress, as it leads back to the starting point. The other possibility is that scale invariance is broken spontaneously. In this case the action is exactly scale-invariant, but not the solution of the field equations (in the classical case) or the ground state (in the quantum case). In a scale-invariant quantum field theory, a symmetry-breaking ground state can be responsible for particle masses.[8] This interesting scenario has one big drawback. Scale symmetry appears to be anomalous (e.g. [47]). In other words, a quantum field theory constructed from a scale-invariant classical Lagrangian is in general no longer scale-invariant.[9] This suggests that the idea of an exactly scale-invariant theory where masses are induced by spontaneous symmetry breaking is not realizable. Technically, the scale anomaly appears because all common regularization procedures (dimensional, cut-off, Pauli-Villars, ...) introduce a new scale into the theory. In the present work, we propose a new renormalization scheme [50] (see also [51]), which allows to construct a class of theories that are scale-invariant at the quantum level to all orders in perturbation theory. Hence, we will take the point of view that scale invariance can be a "good" symmetry even at the quantum level.

[8] This is analogous to the Higgs mechanism, where the vacuum expectation value of the Higgs field induces the masses of all other particles.

[9] In more technical words, if $J^\mu(x)$ is the current associated to scale transformations, this current is conserved at the classical level $\partial_\mu J^\mu(x) = 0$. However, at the quantum level, the conservation law no longer holds. In fact, the expectation value of the operator $\partial_\mu J^\mu(x)$ is proportional to the beta function of the theory, which in general is not zero. Only if there exists a point in parameter space where the beta function is zero, the theory at this point is scale-invariant at the quantum level, however with anomalous scaling dimensions.

Including gravity provides additional motivation for the study of scale-invariant theories. Classical GR contains one characteristic scale, Newton's gravitational constant $G$, respectively the Planck mass $M_P = G^{-1/2}$.[10] It is a natural question to ask whether Newton's "constant" might actually change in time. The idea that this and other constants of nature could vary with time goes back to Dirac [52]. Dirac was intrigued (and we are still intrigued) by the fact that nature presents several enormously different scales (Planck scale, proton mass, ...). The hope was, that after promoting the constants to time-dependent quantities, one could find a dynamical mechanism to explain their big differences. In modern words, this corresponds to constructing a scale-invariant theory with spontaneous symmetry breaking.

Also Brans and Dicke, led by Machs' principles [53], promoted $G$ to a dynamical field. The Brans-Dicke theory of gravity [54] is still a viable alternative to GR, although experiments have put severe bounds on it.

The idea that the observed scales could be due to the spontaneous break-down of scale invariance in the underlying theory also motivated the proposals [55–59]. In these works, the combined action for gravity and particle physics is made scale-invariant by the introduction of new scalar fields. It is argued that the symmetry can be broken if the scalar fields take non-trivial background values. While in [55–57] the origin of the non-trivial background is not specified, it is argued in [58, 59] that it can be provided by a cosmological solution. Similar ideas will be discussed in the present work. However, there will be some possibly crucial differences. In particular, in our models the symmetry-breaking backgrounds correspond to constant solutions minimizing the potential energy.

It is rather obvious that scale invariance could play a role in the context of the cosmological constant problem and dark energy. In fact, if one manages to construct a scale-invariant theory for particle physics, its effective potential contains no constant term. Equivalently, one can say that the vacuum expectation value of the energy-momentum tensor vanishes, $\langle T^\mu_\mu \rangle = 0$. If this theory is minimally coupled to gravity, there is no cosmological constant problem. In the absence of a cosmological constant, dark energy would need another explanation. This "solution" to the cosmological constant problem is of course also plagued by the dilatational anomaly. Several authors have proposed to start from a classically scale-invariant theory and let the appearing anomalous terms be responsible for the observed dark energy [30, 60].[11] As mentioned before, we will take the point of view that an exactly scale-invariant quantum theory can exist. Now, if scale invariance is taken

[10] Apart from a possible cosmological constant to which we will come back shortly.

[11] For another interesting related proposal see [61].

seriously, the full action including gravity and particle fields should respect this symmetry. In general, such an action involves so-called non-minimal couplings to gravity. Also in this case, scale symmetry forbids the appearance of a constant term in the action. However, as we will discuss in detail, in the presence of non-minimal couplings the cosmological constant problem reappears in a particular way.

- Unimodular Gravity (UG)

  A very modest modification of GR is given by Unimodular Gravity (UG) [34, 62–70]. The key idea is to remove the determinant of the metric from the dynamical variables by imposing the constraint $|\det g_{\mu\nu}| = 1$. This is an additional constraint on the components of the metric, on top of the usual symmetry requirement $g_{\mu\nu} = g_{\nu\mu}$. Unimodular Gravity mainly appeared in the context of the cosmological constant problem. The reason for this can be understood easily. The Einstein-Hilbert action describing GR in the presence of a cosmological constant is given by[12]

$$S_{GR} = -\int dx^4 \sqrt{-g} \left((8\pi G)^{-1} R + \Lambda\right) \ , \tag{1.2}$$

  where $g \equiv \det g_{\mu\nu}$. Einstein's equations are obtained by requiring $S_{GR}$ to be stationary with respect to variations $\delta g_{\mu\nu}$ of the full metric. The cosmological constant appears in the equations of motion due to its coupling to the metric determinant $g$. Imposing the unimodular constraint $|g| = 1$, the above action reduces to

$$S_{UG} = -\int dx^4 \left((8\pi G)^{-1} \hat{R} + \Lambda\right) \ , \tag{1.3}$$

  Here and from now on a "hat" on a quantity means that it is computed from the unimodular metric $\hat{g}_{\mu\nu}$, i.e. satisfying $|\det \hat{g}_{\mu\nu}| = 1$. The cosmological constant no longer couples to the metric. As a consequence, whatever the value of $\Lambda$ is, it will not appear in the equations of motion. One could expect that this would solve the cosmological constant problem. People immediately realized that things are not quite as simple, and that in fact a new type of cosmological constant appears in the equations of motion. Let us see how this happens. The field equations for UG are derived by varying $S_{UG}$ with respect to $\hat{g}_{\mu\nu}$, keeping its determinant fixed. One finds

$$\hat{R}_{\mu\nu} - \frac{1}{4} \hat{R}\, \hat{g}_{\mu\nu} = 0 \ , \tag{1.4}$$

[12] We use throughout this work the following conventions: $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, $R^{\alpha}{}_{\beta\gamma\delta} = \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\lambda_{\beta\gamma}\Gamma^\alpha_{\lambda\delta} - \ (\gamma \leftrightarrow \delta)$ .

which are the traceless part of Einstein's equations.[13] Applying the contracted Bianchi identity $\hat{\nabla}^\mu(\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu}) = 0$ one finds $\hat{R} = 4(8\pi G)\Lambda_0$, where $\Lambda_0$ is an arbitrary integration constant. Reinserting this into the traceless equations, they can be rewritten as

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\,\hat{g}_{\mu\nu} + (8\pi G)\Lambda_0\,\hat{g}_{\mu\nu} = 0\ . \tag{1.5}$$

This shows that, at least at the classical level, UG is equivalent to GR with a cosmological constant.[14] Nevertheless, there is an important conceptual difference. The cosmological constant in UG is not a parameter of the action but an arbitrary integration constant, which can be thought of as an initial condition. In particular, unlike $\Lambda$, the constant $\Lambda_0$ has nothing to do with vacuum energy. There is therefore no reason to expect $\Lambda_0$ to take a very big value. On the other hand, there is no reason either to expect it to be very small, i.e. to correspond to the observed value. Summing up, one can say that UG puts the cosmological constant problem into a new perspective but does not really solve it. In the present work we will study UG in combination with scale-invariant theories. The appearance of an arbitrary integration constant will spontaneously break the scale symmetry. Moreover, in scale-invariant theories, the integration constant will not play the role of a cosmological constant but rather give rise to a dynamical dark energy component.

UG has also been studied in view of a possible quantization of gravity. One hope is that the simpler form of the UG action might also facilitate the quantization of gravity as a perturbative field theory (in this context see [71–73]). According to a recent work [74], however, UG becomes non-renormalizable at two-loops, just like GR. Another possible advantage of UG compared to GR appears in the canonical quantization formalism [66, 67]. In fact, unlike in GR, the Hamiltonian in UG does not vanish. This could possibly clarify the interpretation of the theory. A recent study of the differences between UG and GR at the quantum level can be found in [70].

[13] The same equations can be obtained from the action $S_{GR}$ by doing variations that respect $\delta g = 0$.

[14] Under very general conditions, a metric solving Einstein's equations can be written in coordinates such that its determinant is equal to one [62].

- TDiff Gravity

  The symmetry group of Einstein's theory of General Relativity is the group of all diffeomorphisms (coordinate changes)

$$x^\mu \mapsto \tilde{x}^\mu(x) \; , \tag{1.6}$$

  whose infinitesimal form is

$$x^\mu \mapsto x^\mu + \xi^\mu(x) \; . \tag{1.7}$$

  We will refer to the group of all diffeomorphisms with the abbreviation Diff. If gravity is described by a symmetric metric $g_{\mu\nu}$, Diff invariance, together with the requirement that the field equations should contain no higher than second derivatives, uniquely fixes the form of the gravitational action. Namely, it has to be the Einstein-Hilbert action (1.2). Diff invariance also dictates how matter fields are coupled to gravity.

  GR satisfies both the Weak Equivalence Principle (WEP) and the Strong Equivalence Principle (SEP).[15] In simple words, the WEP states that all effects of gravity can be eliminated locally by an appropriate change of coordinates (i.e. placing the observer into a freely falling frame). The SEP, on top of this, states that the outcome of a local gravitational experiment does not depend on when and where in the universe the experiment is performed.

  Looking for theoretical alternatives to GR, one can ask whether the group of all diffeomorphisms is the minimal symmetry group that gives rise to a satisfactory theory of gravitation. This is one of the motivations for exploring TDiff gravity [65, 72, 74–78]. The starting point in TDiff gravity is to require invariance not under all coordinate changes but only under those that preserve the volume element, i.e.

$$x^\mu \mapsto \tilde{x}^\mu(x), \;\text{ with } J \equiv \left| \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right| = 1 \; , \tag{1.8}$$

  generated by the subalgebra of *transverse* vectors,

$$x^\mu \mapsto x^\mu + \xi^\mu(x), \;\text{ with } \partial_\mu \xi^\mu = 0 \; . \tag{1.9}$$

  We will refer to this symmetry group as the group of transverse diffeomorphisms (TDiff). A theory which is invariant under TDiff will be called "TDiff theory". If the theory only contains the metric field, we will refer to a theory of "TDiff Gravity".

  Unlike Diff invariance, TDiff invariance does not uniquely fix the form of the gravitational action. In particular, the action can contain arbitrary functions of the metric determinant $g$, since it is a scalar under

[15] For precise definitions see e.g. [8].

TDiff. The most general TDiff invariant action for gravity containing no higher than second derivatives is

$$S_{TD} = \int dx^4 \sqrt{-g} \left( -\frac{1}{2} M^2 f(-g) R - \frac{1}{2} M^2 l(-g) g^{\mu\nu} \partial_\mu g \partial_\nu g - M^4 v(-g) \right) , \quad (1.10)$$

where $f(-g)$, $l(-g)$ and $v(-g)$ are arbitrary functions and $M$ is an a priori arbitrary mass scale. The couplings between gravity and matter based on TDiff symmetry are much less restricted than in the case of Diff invariance. Namely, just like the gravitational part of the action, they can contain arbitrary functions of $g$. We will refer to the arbitrary functions of $g$ as "Theory Defining Functions" (TDF). Ultimately, all TDF will be restricted by theoretical and phenomenological considerations.

The action $S_{TD}$ describes in general three propagating degrees of freedom, the graviton plus a new scalar. There are two particular choices for the arbitrary functions that enhance the TDiff symmetry by an additional local symmetry such that the scalar degree of freedom is absent [76]. The first one obviously corresponds to GR ($f = const.$, $v = const.$, $l = 0$). The second one corresponds to choosing the functions such that the action is invariant under local (Weyl) rescalings of the metric $g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}$, where $\sigma(x)$ is an arbitrary function. In this second case (sometimes called WTDiff) the action depends on the metric only through the combination $g_{\mu\nu}(-g)^{-1/4}$, in other words on the unimodular metric $\hat{g}_{\mu\nu} = g_{\mu\nu}(-g)^{-1/4}$. Therefore, this case exactly corresponds to UG.

For all other choices of the TDF the metric contains a new scalar degree of freedom. The presence of this degree of freedom necessarily violates the SEP and potentially also violates the WEP. Therefore, when constructing a theory based on TDiff invariance, one has to make sure that the theory is not in conflict with experimental bounds on the violation of the SEP and the WEP [8].

Let us note that as soon as one allows for an additional degree of freedom in the metric, the distinction between gravity and matter becomes ambiguous. In fact, we will see in the present work that such a theory can always be reformulated as a theory of GR plus a new type of "matter". This new type of matter can then induce a new interaction between the known types of matter. If the theory is formulated in this way, the observational constraints come from the search of a "fifth" force.

Proposals based on TDiff that allow for the presence of the scalar degree of freedom and give it a physical interpretation include [79, 80]. It is argued, for instance, that the new scalar field could be a candidate for dark matter. In the present work we will propose a model in which the scalar degree of freedom in the metric essentially plays the role of the standard model Higgs field. In that case, the new degree of freedom is a part of "standard" matter. Hence, it does not introduce a new interaction, and experimental bounds come from electroweak precision tests.

Some more theoretical motivation for the study of TDiff gravity comes from the field theoretical approach to gravity [81]. In the language of Lorentz invariant field theory, the gravitational force is mediated by a massless spin-two field. This can be described by a second rank symmetric Lorentz tensor $h_{\mu\nu}$. It is well-known that the Lorentz-invariant action consistently describing a self-interacting symmetric tensor $h_{\mu\nu}$ needs to be invariant under the transformations

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \ , \tag{1.11}$$

where the generating vector $\xi_\mu$ is transverse, i.e. $\partial_\mu \xi^\mu = 0$ [62, 76]. In fact, this invariance guarantees the absence of ghosts (i.e. degrees of freedom with negative sign in front of the kinetic term) and classical instabilities. The most general Lagrangian satisfying this symmetry describes 3 local degrees of freedom, of which two are associated to a massless spin-two mode and one to a potentially massive scalar mode. The scalar degree of freedom disappears if the symmetry group is extended to transformations with $\partial_\mu \xi^\mu \neq 0$. Let us now make the link between this perturbative field theory approach and the geometrical approach described above. One can split the metric $g_{\mu\nu}$ into a Lorentz-invariant background part and a perturbation as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where the background $\bar{g}_{\mu\nu}$ solves the equations of motion. Next, one can expand the actions $S_{GR}$ and $S_{TD}$ in powers of the perturbation $h_{\mu\nu}$. From the Einstein-Hilbert action one obtains the perturbative theory of a self-interacting tensor $h_{\mu\nu}$ invariant under (1.11) with $\partial_\mu \xi^\mu \neq 0$. Starting from the TDiff action, one obtains a theory invariant under (1.11) with $\partial_\mu \xi^\mu = 0$. In this sense, TDiff can be understood as the minimal symmetry group at the non-linear level giving rise to a consistent perturbative theory.

The present work is organized as follows. We start in chapter 2 with the construction of a minimal scale-invariant extension of GR and the SM. In a next step we replace GR by UG and carry out a detailed cosmological study of the constructed model. The content of chapter 2 is based on [82, 83]. In chapter 3 we discuss theories based on scale- and TDiff invariance. The main goal is to find the conditions under which the constructed theories describe a viable particle physics phenomenology. The content of this chapter is based on [84]. While the first two chapters mainly deal with the classical theory, we present in chapter 4 a scale-invariant renormalization scheme that allows to maintain exact scale invariance also at the quantum level. This part of the work is based on [50]. In chapter 5 we give our conclusions together with an outlook on possible ways to extend the presented study.

# 2. Scale Invariance and Unimodular Gravity – Higgs Dilaton Cosmology

At the classical level, the Lagrangian describing the SM minimally coupled to GR contains only three dimensional parameters. These are Newton's constant G, the vacuum expectation value (vev) of the Higgs field and a possible cosmological constant $\Lambda$. The masses of the elementary particles other than the Higgs particle are induced by the value of the Higgs field after electroweak symmetry breaking. At the quantum level, additional scales appear due to dimensional transmutation. They include $\Lambda_{QCD}$ and all other scales related to the running of coupling constants. There are therefore three different ways in which the observed scales enter the theory: as parameters in the classical action, induced by the expectation value of the Higgs field or as a consequence of the quantization procedure. This observation lead us, like many authors before, to look for models in which some or all of these seemingly unrelated scales have a common origin.

In this chapter we propose a minimal extension of the SM and GR that contains no dimensional parameters in the action and is therefore classically scale-invariant (section 2.1). This is achieved by the introduction of a new scalar degree of freedom – the dilaton. We show that for a large class of potentials, all scales at the classical level are induced by the spontaneous break-down of scale invariance (SI). As a consequence of the spontaneously broken symmetry, the physical dilaton is exactly massless. We briefly discuss the influence of the dilaton on particle physics phenomenology, while a more rigorous analysis will be carried out in chapter 3. After a discussion of possible naturalness issues, we give some arguments in favor of the parameter choice for which the cosmological constant is absent.

Replacing GR by Unimodular Gravity (UG) (section 2.2) results in the appearence of an arbitrary integration constant in the equations of motion, representing an additional breaking of the scale symmetry. It is shown that in theories with scalar fields non-minimally coupled to gravity, this constant gives rise to a non-trivial potential for the scalar fields. In the case of the discussed scale-invariant model, the new potential involves the dilaton, which otherwise would not have a potential.

The model combining SI and UG describes an interesting cosmological phenomenology (section 2.3). The dynamical break-down of the scale symmetry can provide a mechanism for inflation in the early universe. On the other hand, the light dilaton, practically decoupled from all SM fields, can act as dynamical dark energy. Under some assumptions it is possible to relate the observables associated to inflation to those associated to dark energy. In particular, we establish a functional relation between the predicted value for the tilt $n_s$ of the primordial scalar power spectrum and the predicted equation of state parameter $\omega_{DE}$ for dark energy (DE).

The motivation of the model relies on the assumption that the structure of the theory is not changed at the quantum level. In other words, the full quantum effective action should still be scale-invariant, and the effective scalar potential should preserve the features of the classical potential. A possible method for constructing a quantum field theory that is scale-invariant to all orders in perturbation theory, and where the symmetry is spontaneously broken, will be presented in chapter 4. In such a scenario, also the scales related to the running of coupling constants are induced by the spontaneous break-down of the scale symmetry.

Summing up, we propose a model in which all scales, classical and quantum, are consequences of the spontaneous breaking of scale invariance, and which entails an interesting cosmological phenomenology.

## 2.1 Minimal scale-invariant extension of GR plus SM

### 2.1.1 Construction of the Model

Let us start by writing down the Lagrangian density that combines GR and the SM

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2}M^2 R + \mathcal{L}_{\mathrm{SM}[\lambda\to 0]} - \lambda\left(\varphi^\dagger\varphi - \nu^2\right)^2 - \Lambda\,, \tag{2.1}$$

where the first term is the usual Einstein-Hilbert action for GR with $M = (8\pi G)^{-1/2}$, the second term is the SM Lagrangian without the Higgs potential, the third term is the Higgs potential, $\nu$ being the vacuum expectation value (vev) of the Higgs field and $\Lambda$ is a cosmological constant. In this standard theory, to which we will refer as "GR plus SM", scale invariance is violated by the presence of the dimensional constants $M, \nu$ and $\Lambda$. Our goal is to let these scales be dynamical, i.e. replace them by a field. The most obvious solution, without introducing new degrees of freedom, would be to let the Higgs field be responsible for all scales. This corresponds to considering the Lagrangian

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\xi\varphi^\dagger\varphi R + \mathcal{L}_{\mathrm{SM}[\lambda\to 0]} - \lambda\left(\varphi^\dagger\varphi\right)^2\,, \tag{2.2}$$

$\xi$ being a new real parameter ("non-minimal coupling"). The associated action is now scale-invariant, i.e. invariant under the global transformations

$$\begin{aligned} g_{\mu\nu}(x) &\mapsto g_{\mu\nu}(\sigma x) , \\ \Phi(x) &\mapsto \sigma^{d_\Phi}\Phi(\sigma x) , \end{aligned} \tag{2.3}$$

where $\Phi(x)$ stands for the different particle physics fields, $d_\Phi$ is their associated scaling dimension and $\sigma$ is an arbitrary real parameter. In the case of a theory invariant under all diffeomorphisms (Diff), the symmetry associated to the absence of dimensional parameters can equivalently be written as an internal transformation.

$$\begin{aligned} g_{\mu\nu}(x) &\mapsto \sigma^{-2} g_{\mu\nu}(x) , \\ \Phi(x) &\mapsto \sigma^{d_\Phi}\Phi(x) . \end{aligned} \tag{2.4}$$

Note that in the case of a theory that is invariant only under transverse diffeomorphisms (TDiff), such as UG, the absence of dimensional parameters will still guarantee invariance under (2.3) but not under (2.4).

Can the Lagrangian (2.2) give a satisfactory phenomenology? Since we are looking for a Lagrangian that eventually should be quantized in the framework of quantum field theory, we want to introduce the requirement that the theory has to possess a "Classical Ground State". The term "Classical Ground State" will be used throughout this work to refer to a solution of the classical equations of motion, which corresponds to constant fields in the particle physics sector of the theory and a maximally symmetric geometry, i.e. Minkowski (flat), de Sitter (dS) or Anti de Sitter (AdS) spacetime. The existence of such a ground state might be essential for a consistent quantization of the theory. At the quantum level, the theory should possess a ground state that breaks scale invariance and in this way induces masses and dimensional couplings for the excitations (particles). The strategy applied here is to require that this spontaneous symmetry breaking already appears in the classical theory due to the existence of a symmetry-breaking classical ground state. It is then assumed that the quantization procedure does not change the essential features of the classical theory, i.e. scale invariance and a symmetry-breaking potential. Let us note that this is exactly the strategy applied in the standard Higgs mechanism.[1]

Let us now look for symmetry-breaking classical ground states in the theory (2.2). If gravity is neglected, i.e. the first term in the Lagrangian is dropped, the classical ground states correspond to the minima of the scalar potential $\lambda\left(\varphi^\dagger\varphi\right)^2$. The only possibility for them to break the scale symmetry, $\varphi = \varphi_0 \neq 0$, is to set $\lambda = 0$. In this case the theory possesses an infinite

[1] The authors of [58, 59] propose that scale symmetry could be broken by the pure presence of a time-dependent cosmological background. The validity of that approach still needs to be confirmed.

family of classical ground states satisfying $2\varphi^\dagger\varphi = h_0^2$ where $h_0$ is an arbitrary real constant. If one includes gravity, the set of possible classical ground states becomes richer. Namely, even if $\lambda \neq 0$ the theory possesses a continuous family of classical ground states satisfying $2\varphi^\dagger\varphi = h_0^2$ and $R = -4\lambda h_0^2/\xi$, where $h_0$ is an arbitrary real constant. The states with $h_0 \neq 0$ break scale invariance spontaneously and induce all scales at the classical level. Hence, the goal of having a classical theory in which all scales have the same origin, spontaneous breakdown of SI, is achieved. However, the above theory is in conflict with experimental constraints. In fact, although the non-zero background value of $\varphi$ gives masses to all other SM particles, the excitations of the Higgs field itself are massless and, moreover, decoupled from the SM fields. This fact is seen most easily if the Lagrangian is written in the Einstein-frame by defining the new metric $\tilde{g}_{\mu\nu} = M^{-2}\xi\varphi^\dagger\varphi g_{\mu\nu}$ and the new canonical Higgs field $\tilde{\varphi} = M\sqrt{1/\xi + 6}\ln(\varphi/M)$. (This type of variable change will be discussed in detail in the upcoming sections.) In the new variables, the SI of the original formulation corresponds to a shift symmetry for the Higgs field $\tilde{\varphi}$, which is the massless Goldstone boson associated to the spontaneous break-down of scale invariance. A Higgs field with these properties is excluded by Electroweak precision tests [9]. Therefore, in order to construct a viable SI theory, it seems unavoidable to introduce new degrees of freedom.

The next simplest possibility is to add a new singlet scalar field $\chi$ to the theory. We will refer to it as the dilaton. The scale-invariant extension for the SM plus GR including the dilaton reads

$$\frac{\mathcal{L}_{SI}}{\sqrt{-g}} = -\frac{1}{2}\left(\xi_\chi\chi^2 + 2\xi_h\varphi^\dagger\varphi\right)R + \mathcal{L}_{\mathrm{SM}[\lambda\to 0]} - \frac{1}{2}(\partial_\mu\chi)^2 - V(\varphi,\chi)\ , \quad (2.5)$$

where the scalar potential is given by

$$V(\varphi,\chi) = \lambda\left(\varphi^\dagger\varphi - \frac{\alpha}{2\lambda}\chi^2\right)^2 + \beta\chi^4\ . \quad (2.6)$$

We will only consider positive values for $\xi_\chi$ and $\xi_h$. As a consequence, the coefficient in front of the scalar curvature is positive, whatever values the scalar fields take. The chosen parametrization of the scalar potential assumes that $\lambda \neq 0$. This only excludes the phenomenologically unacceptable case where a quartic term $(\varphi^\dagger\varphi)^2$ is absent. By construction, the action associated to (2.5) is invariant under (2.3), respectively (2.4). The theory should possess a symmetry-breaking classical ground state with $\varphi = \varphi_0 \neq 0$ and $\chi = \chi_0 \neq 0$. The case $\varphi_0 = 0$ would correspond to a theory with no electroweak symmetry breaking while the case $\chi_0 = 0$ would result in a theory with a massless Higgs field. Both these cases are phenomenologically unacceptable.

Let us again start by neglecting the gravitational part of the action. In that case the ground states correspond to the minima of the potential (2.6). It is easy to see that the only possibility to get a ground state satisfying

$\varphi_0 \neq 0$ and $\chi_0 \neq 0$ is to have a potential with a flat direction, i.e. $\alpha > 0$ and $\beta = 0$, such as $\lambda > 0$ for stability. The corresponding family of classical ground states is given by $2\varphi^\dagger\varphi = h_0^2$ and $\chi^2 = \chi_0^2$ with $h_0^2 = \frac{\alpha}{\lambda}\chi_0^2$, where $\chi_0$ is an arbitrary real constant.

Like before, the inclusion of gravity results in the appearance of additional possible classical ground states for $\beta \neq 0$, given by

$$h_0^2 = \frac{\alpha}{\lambda}\chi_0^2 + \frac{4\beta\xi_h\chi_0^2}{\lambda\xi_\chi + \alpha\xi_h} ,$$
$$R = -\frac{4\beta\lambda\chi_0^2}{\lambda\xi_\chi + \alpha\xi_h} . \tag{2.7}$$

The solutions with $\chi_0 \neq 0$ spontaneously break scale invariance. All scales are induced and proportional to $\chi_0$. For instance, one can directly identify the Planck scale as

$$M^2 = \xi_\chi\chi_0^2 + \xi_h h_0^2 = (\xi_\chi + \xi_h\frac{\alpha}{\lambda} + \frac{4\beta\xi_h^2}{\lambda\xi_\chi + \alpha\xi_h})\chi_0^2 . \tag{2.8}$$

Depending on the value of $\beta$, the background corresponds to flat spacetime ($\beta = 0$), de Sitter (dS) or Anti de Sitter (AdS) spacetime of constant scalar curvature $R$ corresponding to a cosmological constant

$$\Lambda = -\frac{1}{4}M^2R = \frac{\beta M^4}{(\xi_\chi + \frac{\alpha}{\lambda}\xi_h)^2 + 4\frac{\beta}{\lambda}\xi_h^2} . \tag{2.9}$$

The spectrum of perturbations around a symmetry-breaking solution contains the usual massless spin-2 perturbation in the gravitational sector. The scalar sector contains an excitation with mass

$$m^2 = 2\alpha M^2\frac{(1+6\xi_\chi) + \frac{\alpha}{\lambda}(1+6\xi_h)}{\xi_\chi(1+6\xi_\chi) + \xi_h\frac{\alpha}{\lambda}(1+6\xi_h)} + \mathcal{O}(\beta) , \tag{2.10}$$

which will play the role of the physical SM Higgs field, plus a massless Goldstone boson (both perturbations are combinations of the fields $\chi$ and $h$). We use $h$ to denote the field $\varphi$ in the unitary gauge. Like in the standard Higgs mechanism, the perturbations of the standard model fields get masses proportional to $h_0$. If one extends the SM by introducing right-handed neutrinos [37, 38], these neutrinos get induced masses proportional to $\chi_0$.

In the described model, physics is completely independent of the value of $\chi_0$, as long as it is non-vanishing. This is because observable quantities correspond to ratios between scales. Therefore, parameters of the model have to be chosen such that these ratios correspond to the measured ones. For instance, one should reproduce the hierarchies between the cosmological scale and the electroweak scale, i.e. $\Lambda/m^4 \sim \mathcal{O}(10^{-56})$ such as the ratio between the electroweak and the gravitational scale $m^2/M^2 \sim \mathcal{O}(10^{-32})$.

We choose the parameter $\beta$ to be responsible for the first ratio and $\alpha$ for the second ratio. Therefore, these parameters have to take values satisfying $\beta \lll \alpha \lll 1$ and $\beta, \alpha \lll \xi_\chi, \xi_h$. One then gets approximately $\Lambda/m^4 \simeq \beta/\xi_\chi^2$ and $m^2/M^2 \simeq 2\alpha/\xi_\chi$. Note that the order of magnitude relation $\sqrt{\Lambda}/m^2 \sim m^2/M^2$ respectively $\sqrt{\beta} \sim \alpha$ is the big number coincidence pointed out by Dirac [52]. However, the present model does not address the question about the origin of the big differences between theses scales, i.e. the smallness of $\alpha$ and $\beta$, nor does it explain their approximate relation. The non-minimal couplings $\xi_\chi$ and $\xi_h$ will be constrained by cosmological considerations, and $\lambda \lesssim \mathcal{O}(1)$ as it corresponds to the self coupling of the Higgs field. Therefore, one can fix the values of $\beta$ and $\alpha$ that give the correct ratios. In the same fashion one has to choose values for the Yukawa couplings of the standard model that produce the observed mass ratios.

As the theory contains a new massless degree of freedom, one has to make sure that it does not contradict any experimental bounds. In section 3.2 we present a detailed analysis of the interactions between this massless field and the SM fields. Let us give the conclusions already now. It turns out that as a consequence of scale invariance, the massless scalar field completely decouples from all SM fields except the Higgs field. Since the massless field is the Goldstone boson associated to the broken scale symmetry, there exists a set of field variables in terms of which it couples to the physical Higgs field only derivatively. In addition, for an appropriate choice of field variables, these interactions appear as non-renormalizable operators, suppressed by the scale $M/\xi_h$.

Other deviations from the SM appear as a consequence of the non-minimal couplings to gravity. In fact, the physical Higgs field, i.e. the field that couples to the SM degrees of freedom, is not $h$, but a combination of $h$ and $\chi$. It will be shown that the resulting deviations from the SM are suppressed by the ratio $m^2/M^2$ between the physical Higgs mass and the Planck mass, respectively by the small parameter $\alpha$. While the new massless field hardly affects SM phenomenology, we will see that it might play an important role in cosmology. Another cosmological implication will be related to the shape of the potential. We will see in section 2.3 that the dynamical breaking of scale invariance with the background fields evolving towards the symmetry breaking ground state can provide a mechanism for inflation.

At the classical level, the above theory successfully implements the idea that all scales are consequences of the spontaneous breaking of SI. All conclusions remain true if SI and the features of the potential can be maintained at the quantum level (cf. chapter 4). In that case, the presented theory is a viable extension of the SM and GR.

### 2.1.2 Naturalness Issues

The presented theory contains two important fine-tunings related to the very big differences between the Planck scale $M$, the electroweak scale $m$ and the cosmological scale $\Lambda$. At the quantum level, this can lead to two much-discussed naturalness issues. One of them is part of the Cosmological Constant Problem explained in section 1. In standard GR plus SM the effective cosmological constant is the sum of a bare constant and radiative corrections proportional to the particle physics mass scales of the theory, e.g. the electroweak scale. Matching the effective cosmological constant with its observed value, tiny compared to, for instance, the electroweak scale, requires a tremendous fine-tuning of the bare cosmological constant. In the case of the scale-invariant theory discussed here, the situation is somewhat different. Exact scale invariance forbids a term $\sqrt{-g}\Lambda$ in the action. Also, if the quantization procedure respects scale invariance (cf. chapter 4), such a term is not generated radiatively. However, as we saw above, due to the non-minimal couplings of the scalar fields to gravity, the cosmological constant is in fact associated to the term $\beta\chi^4$. Now, this term is not forbidden by scale invariance. Therefore, even if scale invariance can be maintained at the quantum level, the quantum effective potential will contain a term $\beta_{\text{eff}}\chi^4$, where $\beta_{\text{eff}}$ is a combination of the bare value of $\beta$ and other non-dimensional couplings of the theory. These other couplings are generally much bigger than the value of $\beta_{\text{eff}}$ that corresponds to the observed cosmological constant. So, again a strong fine-tuning is needed in order to keep $\beta_{\text{eff}}$ sufficiently small. This tells us that the cosmological constant problem also exists in an exactly scale-invariant theory of the type proposed here.

The second naturalness issue is related to the mass of the Higgs boson and is commonly called "Gauge Hierarchy Problem". The problem is twofold. The effective field theory combining the SM with GR contains two extremely different mass scales, namely, the electroweak scale $v = 246$ GeV ($v$ being the vev of the Higgs field) and the Planck scale $M = (8\pi G)^{-1/2} = 2.44{\cdot}10^{18}$ GeV. It is considered unnatural to have such a huge difference between two scales of the same theory. This is the first part of the gauge hierarchy problem. In the considered type of scale-invariant theories, the big difference between the electroweak and the Planck scale remains unexplained.

The other part of the gauge hierarchy problem is related to the stability of the Higgs mass against radiative corrections (for a recent discussion see e.g. [10]). Much like the cosmological constant, the mass of the Higgs field gets radiative corrections proportional to the other particle physics mass scales of the theory. The logic is the same as in the case of the cosmological constant. If there exists a particle physics scale much bigger than the electroweak scale, the measured value of the electroweak scale can only be explained by an important fine-tuning of parameters. In other words, if there exists a new particle physics scale between the electroweak scale $m$ and the Planck scale

$M$, the "smallness" of the Higgs mass constitutes a serious theoretical issue. This issue still appears in an exactly scale-invariant theory with spontaneous breaking of the scale symmetry.

If the theory possesses no intermediate particle physics scale between $m$ and $M$, the situation is different. In that case, whether or not the Higgs mass should be expected to contain big radiative corrections of the order $M$ depends on the ultraviolet (UV) completion of the theory. At the level of the low-energy effective field theory, the UV properties can be encoded in the choice of the renormalization scheme. We will present in chapter 4 a renormalization scheme based on the assumption that the UV completion should be scale-invariant. If this scheme is applied to the considered minimal scale-invariant extension of GR plus SM, the Higgs mass does not obtain corrections proportional to $M$ (induced by the vacuum expectation value of the dilaton) and there is no problem of stability of the Higgs mass against radiative corrections. Hence, scale invariance makes for the absence of this part of the gauge hierarchy problem.

### 2.1.3 The Speciality of the Case $\beta = 0$

In this paragraph we want to give some arguments in favor of the case $\beta = 0$. This case corresponds to the existence of a flat direction in the Jordan-frame potential (2.6) and hence to the absence of a cosmological constant.

The reasoning of the precedent paragraph tells us that choosing $\beta = 0$ corresponds to a fine-tuning of the parameters, especially at the quantum level, just like putting $\Lambda = 0$ in standard GR plus SM. From this point of view, such a parameter choice should clearly be disfavored. Nevertheless, we think that the case $\beta = 0$ is specially interesting. One reason is that only if $\beta = 0$, SI can be spontaneously broken in the absence of gravity. Put in other words, $\beta = 0$ allows flat spacetime together with $(\varphi, \chi) = (\varphi_0, \chi_0) \neq (0, 0)$ to be a classical solution. This argument will be further discussed in section 4.1.

Another argument is related to the stability of the ground state. As discussed above, a scale-invariant theory with spontaneous symmetry breaking always contains a massless scalar degree of freedom, Goldstone boson, independently of the value of $\beta$. Now, if $\beta \neq 0$, the background spacetime of the theory corresponds to de Sitter (or Anti de Sitter) spacetime. It is known, however, that a massless scalar field is unstable in de Sitter spacetime. Therefore, it is conceivable that a consistent quantization of the theory might rely on the requirement $\beta = 0$ and hence the existence of flat spacetime as a solution (cf. [85–89]). Based on these arguments, we will single out the case $\beta = 0$ and study the associated phenomenology in more detail.

A third aspect appears in the context of cosmology. The theory with $\beta = 0$, not containing a cosmological constant, does not seem to withstand the confrontation with cosmological observations. Just like the case $\beta < 0$ (AdS) it can not explain the observed accelerated expansion of the universe without introduction of a new dark energy component. From this point of view, the only viable option seems to be $\beta > 0$ (dS). This conclusion is correct if gravity is described by GR. However, as we will see in the upcoming section, the situation is very different if GR in (2.5) is replaced by Unimodular Gravity. In that case, the appearance of an arbitrary integration constant will give rise to a potential for the Goldstone boson of broken scale invariance. As a consequence, for appropriate parameter values and initial conditions, the now pseudo-Goldstone boson can act as a dynamical dark energy component. In this new situation, the case $\beta = 0$ will again be peculiar, because it is the only case where dark energy is purely dynamical and has no constant contribution.

## 2.2 Scale-invariant Unimodular Gravity

In this section we want add to the idea of scale invariance the idea of Unimodular Gravity (UG) [34, 62–68]. In UG one reduces the dynamical components of the metric $g_{\mu\nu}$ by one, imposing that the metric determinant $g \equiv \det(g_{\mu\nu})$ takes some fixed constant value. Conventionally one takes $|g| = 1$, hence the name. Fixing the metric determinant to one is not a strong restriction, in the sense that the family of metrics satisfying this requirement can still describe all possible geometries. For pure gravity, things are rather simple and well known. The analog of the Einstein-Hilbert Lagrangian for unimodular gravity is

$$\mathcal{L}_{UG} = -\frac{1}{2} M^2 \hat{R} \,. \tag{2.11}$$

Writing quantities with a hat, like $\hat{R}$, we mean that they depend on the metric satisfying the unimodular constraint $g = -1$. These quantities transform like tensors under the group of transverse diffeomorphisms TDiff, i.e. coordinate transformations $x^\mu \mapsto x^\mu + \xi^\mu(x)$ with the condition $\partial_\mu \xi^\mu = 0$. At this point it is important to distinguish UG from theories constructed on the sole requirement of invariance under TDiff [65, 75, 76] (cf. chapter 3). In the latter theories, the metric determinant is unconstrained and represents in general a third dynamical degree of freedom in the metric. Note, however, that UG can be understood as a particular case of a TDiff theory in which the third metric degree of freedom is absent. Doing variations of the action (2.11) keeping the metric determinant fixed, since it is not a dynamical variable, yields the equations of motion (cf. section 1)

$$\hat{R}_{\mu\nu} - \frac{1}{2} \hat{R} \, \hat{g}_{\mu\nu} = -M^{-2} \Lambda_0 \, \hat{g}_{\mu\nu} \,, \tag{2.12}$$

where $\Lambda_0$ is an integration constant related to initial conditions. Now, these are also the equations for standard Einstein gravity with an added cosmological constant, for a choice of coordinates such that the metric determinant is equal to one, which is always possible [62]. Therefore, the two theories are classically equivalent, except that in the standard theory the cosmological constant appears in the action, whereas in unimodular gravity it is an integration constant. It has been shown [34, 62, 66, 69] that if one adds a matter sector that couples minimally to gravity, and therefore has a covariantly conserved energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$, the application of UG also results in the appearance of an integration constant that plays the role of a cosmological constant. We now want to give an analogous statement for the more general case of UG in combination with arbitrary matter fields that have arbitrary couplings to gravity. The result will then be applied to scale-invariant theories.

A similar proof for the corresponding statement in the context of TDiff gravity was given by the authors of [65]. Since UG can be understood as a particular case of TDiff (in which the scalar metric degree of freedom is absent), the statement we are going to make here is in principle contained in [65]. Still, in the present context of UG without the additional metric degree of freedom, the following proof might be somewhat more transparent.

The action for unimodular gravity and any other fields, which couple to gravity in an arbitrary way, has the following functional dependence:

$$\Sigma = \int d^4x \, \mathcal{L}(\hat{g}_{\mu\nu}, \partial\hat{g}_{\mu\nu}, \Phi, \partial\Phi), \tag{2.13}$$

where $\Phi$ stands for all non-gravitational fields. If we want to derive the equations of motion for this theory, we have to vary the action keeping the constraint on the determinant. This is done using the Lagrange multiplier method. We add an additional variable, whose equation of motion will be the constraint. So, the following action is equivalent to the former one:

$$\tilde{\Sigma} = \underbrace{\int d^4x \sqrt{-g}\Big(\mathcal{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \Phi, \partial\Phi) + \lambda(x)\Big)}_{A} \underbrace{- \int d^4x \, \lambda(x)}_{B} \, . \tag{2.14}$$

Here, apart from the usual symmetry requirement $g_{\mu\nu} = g_{\nu\mu}$, the metric is unconstrained (the initial Lagrangian was multiplied by a factor $\sqrt{-g}$, which does not change the theory because of the unimodular constraint).

The equations of motion are

$$\frac{\delta A}{\delta g_{\mu\nu}} = 0 \, , \tag{2.15}$$

$$\frac{\delta A}{\delta \Phi} = 0 \, , \tag{2.16}$$

$$\frac{\delta(A+B)}{\delta\lambda} = 0 = (\sqrt{-g} - 1) \, . \tag{2.17}$$

A definition of the functional derivative is given in appendix A. We observe that $\int d^4x \, A(x)$ is invariant under the full group of diffeomorphisms (Diff).

The infinitesimal transformations are

$$\begin{aligned} g_{\mu\nu} &\mapsto g_{\mu\nu} + \delta_\xi g_{\mu\nu} \ , \\ \Phi &\mapsto \Phi + \delta_\xi \Phi \ , \\ \lambda &\mapsto \lambda + \delta_\xi \lambda \ , \end{aligned} \tag{2.18}$$

where $\delta_\xi$ depends on the nature of the fields, i.e. scalar, vector, etc. If, for instance, we take $\Phi$ to be a scalar field, the $\delta_\xi$'s are given by

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \ , \\ \delta_\xi \Phi &= \partial_\mu \Phi \xi^\mu \ , \\ \delta_\xi \lambda &= \partial_\mu \lambda \xi^\mu \ . \end{aligned} \tag{2.19}$$

Due to this symmetry, the following relation holds.

$$\int d^4x \Big( \frac{\delta A}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \frac{\delta A}{\delta \Phi} \delta_\xi \Phi + \frac{\delta A}{\delta \lambda} \delta_\xi \lambda \Big) = 0 \ . \tag{2.20}$$

The coefficients of the first two terms are zero because of the equations of motion and the last coefficient yields $\frac{\delta A}{\delta \lambda} = \sqrt{-g}$. The equation reduces to

$$\int d^4x \sqrt{-g} (\partial_\mu \lambda) \xi^\mu = 0 \ . \tag{2.21}$$

Since this holds for all possible functions $\xi^\mu(x)$, we can conclude that

$$\partial_\mu \lambda(x) = 0 \ , \tag{2.22}$$

and hence that $\lambda(x)$ is a constant of motion, $\lambda(x) = \Lambda_0$. Its value should be interpreted as an additional initial condition. This can be understood, for example, by expressing the theory in the Hamiltonian formalism (cf. [66, 67, 90]). Equations (2.15) can now be written as

$$\frac{\delta A}{\delta g_{\mu\nu}} = \frac{\delta \{ \int d^4x \sqrt{-g} \Big( \mathcal{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \Phi, \partial \Phi) + \Lambda_0 \Big) \}}{\delta g_{\mu\nu}} = 0 \ .$$

These equations along with the constraint $\sqrt{-g} = 1$ are the field equations for unimodular gravity plus arbitrary other fields. Now the solutions to these equations are the same as the solutions obtained from the Diff invariant action

$$\Sigma_e = \int d^4x \, \mathcal{L}_e = \int d^4x \sqrt{-g} \Big( \mathcal{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \Phi, \partial \Phi) + \Lambda_0 \Big) \ , \tag{2.23}$$

written in coordinates for which $\sqrt{-g} = 1$, which is always possible [62]. The subscript "$e$" stands for "equivalent Diff invariant theory" and will be used this way throughout the present work. We conclude that the theory given by (2.13) is classically equivalent to a fully diffeomorphism invariant theory described by the action (2.23) apart from the different ways in which the scale $\Lambda_0$ appears. In the theory with explicit Planck mass (minimal coupling) the quantity $\Lambda_0$ plays the role of a cosmological constant. However, as we will see shortly, things are different in a theory where Newton's constant is induced dynamically (non-minimal coupling).

Let us now apply this result to the case of the scale-invariant theory discussed in the previous section. We consider the theory given by (2.5) and let gravity be unimodular.

$$\mathcal{L}_{SI-UG} = -\frac{1}{2}\left(\xi_\chi \chi^2 + 2\xi_h \varphi^\dagger \varphi\right)\hat{R} + \hat{\mathcal{L}}_{\mathrm{SM}[\lambda\to 0]} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\varphi,\chi)\ , \tag{2.24}$$

where $V(\varphi,\chi)$ is given by (2.6) and whose associated action is invariant under (2.3). The result derived above tells us that the classical solutions obtained from this Lagrangian are equivalent to the solutions derived from the equivalent Diff invariant Lagrangian

$$\frac{\mathcal{L}_e^{SI-UG}}{\sqrt{-g}} = -\frac{1}{2}\left(\xi_\chi \chi^2 + 2\xi_h \varphi^\dagger \varphi\right) R + \mathcal{L}_{\mathrm{SM}[\lambda\to 0]} - \frac{1}{2}(\partial_\mu\chi)^2 - V(\varphi,\chi) - \Lambda_0\ . \tag{2.25}$$

$\Lambda_0$ is an arbitrary integration constant related to initial conditions. Therefore, the presence of $\Lambda_0$ should be understood as a specific kind of *spontaneous* breaking of scale invariance, even though at the level of the equivalent Diff invariant theory $\Lambda_0$ appears *explicitly* in the action.

Next, we turn our attention to the physical implications of the term proportional to $\Lambda_0$. In the first instance, let us consider the gravitational and the scalar sector of the theory, i.e.

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2}\left(\xi_\chi\chi^2 + \xi_h h^2\right) R - \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}(\partial_\mu h)^2 - V(h,\chi) - \Lambda_0\ , \tag{2.26}$$

where $h$ is the Higgs field in the unitary gauge. In order to simplify the physical interpretation we define the Einstein-frame (E-frame) metric[2]

$$\tilde{g}_{\mu\nu} = M^{-2}\left(\xi_\chi\chi^2 + \xi_h h^2\right) g_{\mu\nu} \tag{2.27}$$

in terms of which the Lagrangian reads

$$\frac{\mathcal{L}}{\sqrt{-\tilde{g}}} = -M^2\frac{\tilde{R}}{2} - \frac{1}{2}K - \tilde{V}(h,\chi)\ , \tag{2.28}$$

[2] The Lagrangian in terms of the original variables is said to be written in the Jordan-frame (J-frame).

where $K$ is a non-canonical but positive definite kinetic term (given below in (2.46)) and $\tilde{V}(h,\chi)$ is the E-frame potential given by

$$\tilde{V}(h,\chi) = \frac{\lambda}{4}\frac{\left(h^2-\alpha^2\chi^2\right)^2+\beta\chi^4}{(\xi_\chi\chi^2+\xi_h h^2)^2}M^4 + \frac{\Lambda_0}{(\xi_\chi\chi^2+\xi_h h^2)^2}M^4\ . \tag{2.29}$$

Note that the E-frame potential gets singular at $\chi = h = 0$. The reason is that at this point the transformation (2.27) is singular and the change to the E-frame is not allowed. Since for $\chi = h = 0$ scale invariance is not broken, we will not be interested in the theory around this point. Let us discuss the shape of the E-frame potential and the classical ground states of the theory for $\alpha, \lambda, \xi_\chi, \xi_h > 0$ (cf. figure 2.1). If $\Lambda_0 = 0$, the potential is minimal along the two valleys

$$h_0^2 = \frac{\alpha}{\lambda}\chi_0^2 + \frac{4\beta\xi_h\chi_0^2}{\lambda\xi_\chi+\alpha\xi_h}\ , \tag{2.30}$$

They correspond to the infinitely degenerate family of classical ground states found in (2.7). If $\beta = 0$, the potential vanishes at its minimum, while a non-zero $\beta$ gives rise to a cosmological constant

$$\Lambda = -\frac{1}{4}M^2\tilde{R} = \tilde{V}(h_0,\chi_0) = \frac{\beta M^4}{(\xi_\chi+\frac{\alpha}{\lambda}\xi_h)^2+4\frac{\beta}{\lambda}\xi_h^2}\ . \tag{2.31}$$

In other words, spacetime in the classical ground state is Minkowskian, dS or AdS. These are the results we have already discussed section 2.1. As soon as $\Lambda_0 \neq 0$ the valleys get a tilt which lifts the degeneracy of the classical ground states. For $\Lambda_0 < 0$ the valleys are tilted towards the origin. The true classical ground state for this case is the trivial one, $\chi = h = 0$. Hence, we discard this possibility. For $\Lambda_0 > 0$ the potential is tilted away from the origin, it is of the run-away type. In this case the theory has an asymptotic classical ground state, given by (2.30) with $\chi_0 \to \infty$. Again, depending on the value of $\beta$ this asymptotic solution corresponds to Minkowski, dS or AdS spacetime with curvature given by (2.31).

We see that as a consequence of the non-minimal coupling between the scalar fields and gravity, the arbitrary integration constant $\Lambda_0$ does not play the role of a cosmological constant but rather gives rise to a peculiar potential. For $\Lambda_0 > 0$ the potential is of the run-away type. In the following sections we will see that such a potential can have an interesting cosmological interpretation. In fact, the evolution of the scalar fields along the valley can give rise to dynamical dark energy (quintessence). We will focus on the case $\beta = 0$ where dark energy does not contain a constant contribution and is purely due to the term proportional to $\Lambda_0$ (cf. arguments in section 2.1.3).

## 2.3 Cosmological Phenomenology - Higgs Dilaton Cosmology

In this section we turn our attention to the cosmological phenomenology issued by the model presented in the preceding section. We consider the theory described by the Lagrangian (2.24), i.e. the scale-invariant extension of the SM and GR (2.5) where GR is replaced by Unimodular Gravity. In the equivalent Diff invariant formulation, the gravitational and the scalar sector of the theory are given by

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2}\left(\xi_\chi \chi^2 + \xi_h h^2\right) R - \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}(\partial_\mu h)^2 - \frac{\lambda}{4}\left(h^2 - \frac{\alpha}{\lambda}\chi^2\right)^2 - \Lambda_0 \,. \tag{2.32}$$

Here we only consider the case $\beta = 0$, i.e. where the potential has a flat direction in the original variables (Jordan frame). Some arguments in favor of this choice were given in section 2.1.3. Allowing for $\beta \neq 0$ would barely affect the discussion of inflation (section 2.3.1). It would, however, affect the dark energy phenomenology (section 2.3.3). The main effect would be that one part of dark energy would be due to a cosmological constant and the other part due to the scalar fields rolling down the potential valley. It would not be possible to know how the observed abundance of dark energy is distributed between these two contributions. As a consequence, unlike in the case $\beta = 0$, one could not directly relate the observables from inflation to the equation of state parameter of dark energy.

In order to get the standard physical interpretation, we define the E-frame metric

$$\tilde{g}_{\mu\nu} = M^{-2}\left(\xi_\chi \chi^2 + \xi_h h^2\right) g_{\mu\nu} \tag{2.33}$$

in terms of which the Lagrangian reads

$$\frac{\mathcal{L}}{\sqrt{-\tilde{g}}} = -M^2 \frac{\tilde{R}}{2} - \frac{1}{2}K - \tilde{V}(h,\chi) \,, \tag{2.34}$$

where $K$ is a non-canonical but positive definite kinetic term (given below in (2.46)) and $\tilde{V}(h,\chi)$ is the E-frame potential given by

$$\tilde{V}(h,\chi) = \frac{M^4}{\left(\xi_\chi \chi^2 + \xi_h h^2\right)^2}\left(\frac{\lambda}{4}\left(h^2 - \frac{\alpha}{\lambda}\chi^2\right)^2 + \Lambda_0\right) \,. \tag{2.35}$$

The scalar fields $\chi$ and $h$ are now considered to be homogeneous background fields evolving in spatially flat Friedmann-LeMaitre-Robertson-Walker (FLRW) spacetime. Their evolution roughly corresponds to the rolling of a ball in the potential $\tilde{V}(h,\chi)$ with "Hubble"-friction. The motion is affected by the non-canonical nature of the kinetic term. However, as the kinetic term is positive definite, a ball starting at rest will still move downwards in the potential. Hence, to get a qualitative picture, it is enough to look at the features of the potential (cf. figure 2.1). In the absence of $\Lambda_0$, $\tilde{V}$ has its minima along the two valleys $h^2 = \frac{\alpha}{\lambda}\chi^2$. The scalar fields tend to roll into one of these valleys and, due to the Hubble friction, asymptotically come to rest in the valley. The main effect of $\Lambda_0 \neq 0$ is to give a tilt to the valleys. As discussed in the previous section, $\Lambda_0 < 0$ is phenomenologically unviable. We will only consider the case $\Lambda_0 > 0$, in which the valleys are tilted away from the origin. For an appropriate choice of parameters, the crude picture of the role of the cosmological scalar fields is the following. If the fields start off far from the valley, they will in a first phase slowly roll towards it. This roll-down can be responsible for inflation. In a next stage, the fields oscillate around the valley and thereby transfer most of their energy to standard model particles (reheating). After the oscillations are sufficiently damped, the fields start rolling down the valley away from the origin. During this phase the scalar fields play the role of a dark energy component that eventually comes to dominate the universe (quintessence [28, 30]). In this late stage, the fields satisfy $h(t)^2 \simeq \frac{\alpha}{\lambda}\chi(t)^2$. On particle physics time-scales the time-variation of the fields can be neglected. Perturbations around this almost constant symmetry-breaking background can be interpreted as the SM particles plus an additional almost massless and almost decoupled particle. Note that as long as the background is constant, it is equivalent to quantize perturbations in the original (Jordan-) frame or in the Einstein-frame (cf. [91]).

In the following sections we examine these three stages in the evolution of the universe one by one and find the ranges of parameters and initial conditions that successfully produce the described picture. Moreover, under some assumptions, we will be able to find a relation between the scalar spectral index $n_s^*$ and the equation of state parameter of dark energy $\omega_{DE}$, which yields a prediction for $\omega_{DE}$.

### 2.3.1 Two-field inflation

As usual, it is assumed that during inflation all the energy of the system is contained in the inflaton fields and in the gravitational field. Therefore, during this stage, the SM fields can be neglected. Let us rewrite the scalar-tensor part of (2.25) as

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{f(\phi)}{2}R - \frac{1}{2}g^{\mu\nu}\delta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b - V(\phi)\,, \tag{2.36}$$

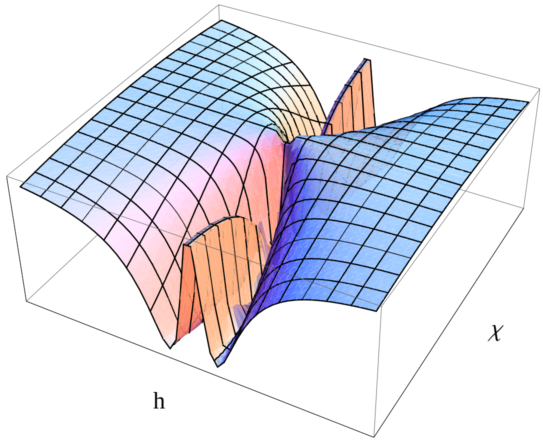
with a non-minimal coupling

$$f(\phi) \equiv \sum_a \xi_a {\phi^a}^2\,, \tag{2.37}$$

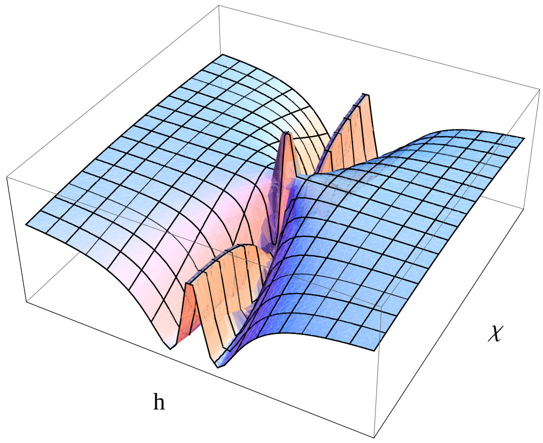
and the potential

$$V(\phi) = V(h,\chi) = \frac{\lambda}{4}\left(h^2 - \frac{\alpha}{\lambda}\chi^2\right)^2 + \Lambda_0\,, \tag{2.38}$$

$\Lambda_0 = 0$




$\Lambda_0 > 0$




$\Lambda_0 < 0$

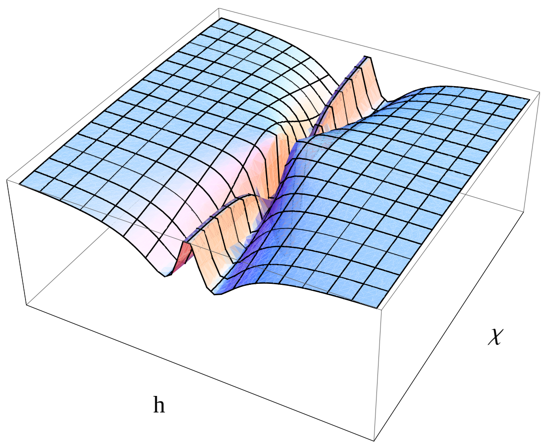



**Fig. 2.1:** In the above figures we show the shape of the E-frame potential $\tilde{V}(h, \chi)$ (equation (2.35)) for $\Lambda_0 = 0$, $\Lambda_0 > 0$ and $\Lambda_0 < 0$ respectively.

including the SI breaking term $\Lambda_0$. As discussed in section 2.1, the parameter $\alpha$ is set to be very tiny $\alpha \sim \mathcal{O}(10^{-30})$, in order to obtain the correct hierarchy between the electroweak and the Planck scale. Greek indices $\mu, \nu, ... = 0, 1, 2, 3$ denote spacetime coordinates while Latin indices are used to label the two real scalar fields present in the model: the dilaton field $\phi^1 = \chi$ and the Higgs field in the unitary gauge $\phi^2 = h$. The abstract notation in terms of $\phi^i$ will in the following allow us to interpret the scalar fields as the coordinates of a two-dimensional sigma-model manifold. We will be able to write expressions and equations that are covariant under variable changes $\phi \mapsto \phi'(\phi)$.

### Scale transformations and their associated current

By construction, all terms in the Lagrangian (2.36), except the one proportional to $\Lambda_0$, are invariant under the scale transformations (2.4). We can write the infinitesimal form of these transformations as

$$g_{\mu\nu} \mapsto g_{\mu\nu} + \sigma \Delta g_{\mu\nu} \, , \tag{2.39}$$

$$\phi^i \mapsto \phi^i + \sigma \Delta \phi^i \, , \tag{2.40}$$

where $\sigma$ is an infinitesimal real parameter. The explicit expressions for $\Delta g_{\mu\nu}$ and $\Delta\phi^i$ depend on the variables chosen. For the original variables, we have $\Delta g_{\mu\nu} = -2g_{\mu\nu}$, $\Delta\chi = \chi$ and $\Delta h = h$. The current associated to this transformation is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu g_{\alpha\beta}\right]} \Delta g_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \phi^i\right]} \Delta\phi^i \tag{2.41}$$

and satisfies

$$\partial_\mu J^\mu = 4\sqrt{-g}\Lambda_0 \, . \tag{2.42}$$

Whenever $\Lambda_0$ vanishes or can be neglected, the scale symmetry becomes exact and the current $J^\mu$ is conserved.

### Lagrangian and equations of motion in the Einstein frame

Whenever the non-minimal coupling is non-zero[3] $f(\phi) \neq 0$, one can define the new metric

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \, , \tag{2.43}$$

with $\Omega^2 = M^{-2} f(\phi)$ to reformulate the Lagrangian in the E-frame. Taking into account that the metric determinant and the Ricci scalar transform as

$$\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}} \, , \tag{2.44}$$

$$R = \Omega^2 \left( \tilde{R} - 6\tilde{\Box} \ln\Omega + 6\tilde{g}^{\mu\nu} \partial_\mu \ln\Omega \, \partial_\nu \ln\Omega \right) \, , \tag{2.45}$$

[3] For our choice of parameters, where $\xi_\chi, \xi_h > 0$ this is the case whenever the scalar fields are away from the origin $(\chi, h) \neq (0, 0)$.

where the action of the covariant d'Alembertian $\tilde{\Box}$ on a scalar field $s(x)$ is given by $\tilde{\Box}s = \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu s\right)$, one obtains

$$\frac{\mathcal{L}}{\sqrt{-\tilde{g}}} = -\frac{M^2}{2}\tilde{R} - \frac{1}{2}l_{ab}\tilde{g}^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b - \tilde{V}(\phi)\ , \tag{2.46}$$

where $l_{ab}$ is the non-diagonal and non-canonical field space metric given by

$$l_{ab} = \frac{1}{\Omega^2}\left(\delta_{ab} + \frac{3}{2}M^2\frac{\Omega^2_{,a}\Omega^2_{,b}}{\Omega^2}\right)\ . \tag{2.47}$$

Unlike in the single-field case, the non-canonical kinetic term can not in general be recast in canonical form by redefining the scalar field variables. In fact, the field-space metric can be brought to canonical form by a variable change if and only if its Riemann tensor identically vanishes. In the present case of a two-dimensional manifold, the Riemann tensor has only one independent component, and it is enough to compute the Ricci scalar $R_l$ associated to the field space metric $l_{ab}$,

$$R_l = (\xi_\chi - \xi_h)\frac{2}{M}\frac{\xi_\chi^2(1+6\xi_\chi)\chi^4 - \xi_h^2(1+6\xi_h)h^4}{\left(\xi_h(1+6\xi_h)h^2 + \xi_\chi(1+6\xi_\chi)\chi^2\right)^2}\ . \tag{2.48}$$

For $R_l$ to vanish globally, one would need to have $\xi_\chi = \xi_h$. As we will see, this case is not allowed by phenomenology.

The E-frame potential is defined as

$$\tilde{V}(\phi) = \frac{V(\phi)}{\Omega^4}. \tag{2.49}$$

In the E-frame, the scale transformation does not act on the metric, $\Delta\tilde{g}_{\mu\nu} = 0$, and is simply given by

$$\phi^i \mapsto \phi^i + \sigma\Delta\phi^i\ . \tag{2.50}$$

The expression for the current simplifies to

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\phi^i\right]}\Delta\phi^i\ , \tag{2.51}$$

while the conservation law becomes

$$\partial_\mu J^\mu = 4\Omega^{-4}\sqrt{-\tilde{g}}\Lambda_0\ . \tag{2.52}$$

Borrowing the notations of [92] we write down the equations of motion derived from the Lagrangian (2.46). Einstein's equations are

$$\tilde{G}_{\mu\nu} = -l_{ab}\left(\partial_\mu\phi^a\partial_\nu\phi^b - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\partial_\rho\phi^a\partial_\sigma\phi^b\right) + \tilde{V}\tilde{g}_{\mu\nu}\ , \tag{2.53}$$

where $\tilde{G}_{\mu\nu}$ is the Einstein tensor computed from the metric $\tilde{g}_{\mu\nu}$. The equations for the scalar fields are

$$\tilde{\Box}\phi^c + \tilde{g}^{\mu\nu}\Upsilon^c_{ab}\partial_\mu\phi^a\partial_\nu\phi^b = \tilde{V}^c \ , \tag{2.54}$$

where $\Upsilon^c_{ab}$ is the Christoffel symbol computed from the field space metric $l_{ab}$,

$$\Upsilon^c_{ab} = \frac{1}{2}l^{cd}\left(l_{da,b} + l_{db,a} - l_{ab,d}\right) \tag{2.55}$$

and where we use the notation

$$\tilde{V}^c = l^{cd}\tilde{V}_d = l^{cd}\tilde{V}_{,d} \ . \tag{2.56}$$

Notice that equations (2.53) and (2.54) are covariant under redefinitions of the scalar field variables $\phi \mapsto \phi'(\phi)$.

We choose to do our analysis of the inflationary stage in the Einstein-, rather than in the Jordan frame. The reason for this choice is that in the literature predictions for measurable quantities are usually computed in the frame in which gravity has the standard GR form. At the classical level there is, apart from such practical arguments, nothing that would privilege one or the other frame. After all, the choice of the frame simply corresponds to a choice of variables.

### Evolution of the homogeneous background

Consider now the homogeneous scalar fields $\phi^i = \phi^i(t)$ in flat FLRW spacetime characterized by the line element

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 d\vec{x}^2 \ . \tag{2.57}$$

For this case, the equations (2.53) and (2.54) reduce to the Friedmann and the Klein-Gordon-like equations

$$H^2 = \frac{1}{3M^2}\left(\frac{1}{2}l_{ab}\dot{\phi}^a\dot{\phi}^b + \tilde{V}\right) \ , \tag{2.58}$$

$$\ddot{\phi}^c + \Gamma^c_{ab}\dot{\phi}^a\dot{\phi}^b + 3H\dot{\phi}^c = -\tilde{V}^c \ , \tag{2.59}$$

where a dot stands for a derivative with respect to $t$ and $H \equiv \dot{a}/a$. To this we can add the equation for the current (2.52), which for homogeneous fields reduces to

$$\frac{d}{dt}\left(a^3 l_{ab}\dot{\phi}^a\Delta\phi^b\right) = 4\Omega^{-4}a(t)^3\Lambda_0 \ . \tag{2.60}$$

This relation is of course not independent of the equations of motion. However, it will prove useful in the following. Let us now inspect equations (2.58) and (2.59) in order to see in which region of field space the effect of a non-zero $\Lambda_0$ will be important. $\Lambda_0$ enters the equations through $\tilde{V}$ and $\tilde{V}^a$, which in terms of the original variables and for $\alpha = 0$ are given by

$$\tilde{V} = M^4\Omega^{-4}\frac{\lambda}{4}h^4\left(1+\upsilon_1\right) \,, \tag{2.61}$$

$$\tilde{V}^\chi = -u\left(\chi, h\right)h\left(1+6\xi_h+\upsilon_1\right) \,, \tag{2.62}$$

$$\tilde{V}^h = u\left(\chi, h\right)\left(1+6\xi_\chi\right)\chi\left(1-\upsilon_2\right) \,, \tag{2.63}$$

where $u\left(\chi, h\right) = \frac{\lambda\xi_\chi M^2\chi h^3}{\Omega^2\left(M^2\Omega^2+6\xi_\chi^2\chi^2+6\xi_h^2h^2\right)}$ and we have defined the scale invariance breaking parameters $\upsilon_1$ and $\upsilon_2$ as

$$\upsilon_1 \equiv \frac{4\Lambda_0}{\lambda h^4} \,, \tag{2.64}$$

$$\upsilon_2 \equiv \frac{4\xi_h\Lambda_0}{\lambda\xi_\chi\left(1+6\xi_\chi\right)\chi^2h^2} \,. \tag{2.65}$$

These parameters characterize the importance of $\Lambda_0$. In the region of field space where $\upsilon_1 \ll 1$ and $\upsilon_2 \ll 1$ the effect of $\Lambda_0$ is negligible and the equations become practically scale-invariant. As a consequence, the scale current is almost conserved in this region and one obtains the almost constant quantity

$$a^3 l_{ab}\dot{\phi}^a\Delta\phi^b \simeq \text{cst.} \tag{2.66}$$

For exact scale-invariance, i.e. $\Lambda_0 = 0$, the equality is exact. If we rewrite the relation as

$$l_{ab}\dot{\phi}^a\Delta\phi^b \simeq \frac{\text{cst.}}{a^3} \,, \tag{2.67}$$

we see the interesting result that if the scale factor grows big, the right-hand side vanishes and one gets

$$l_{ab}\dot{\phi}^a\Delta\phi^b \simeq 0 \,. \tag{2.68}$$

We will refer to the region where $\Lambda_0$ can be neglected as to the "scale-invariant" region. It corresponds to the shaded area in figure 2.2. It will turn out that for phenomenologically viable values of the parameters, all the observable inflation takes place in this region.

### Slow-roll approximation and trajectories

In the present model inflation can occur due to a phase of slow-roll of the scalar fields down the potential. The standard slow-roll parameters can be generalized to the two-field case as (see e.g. [93])

$$\epsilon = \frac{M^2 l^{ab}\tilde{V}_{,a}\tilde{V}_{,b}}{2\tilde{V}^2} \tag{2.69}$$

and

$$\eta_1 \, , \quad \eta_2 \, , \tag{2.70}$$

which stand for the eigenvalues of the matrix $N^a_b$ defined by

$$N^a_b e^b_i = \eta_i e^a_i \, , \quad \text{and} \quad N^a_b = \frac{M^2 l^{ac} \tilde{V}_{;cb}}{\tilde{V}} \, , \tag{2.71}$$

where $\tilde{V}_{;cb} = \tilde{V}_{,cb} - \Upsilon^a_{bc}(\phi)\tilde{V}_{,a}$. Notice that the slow-roll parameters are scalars under redefinition of the field variables. The system describes an inflating universe as long as the slow-roll parameters satisfy

$$\epsilon \ll 1 \quad \text{and} \quad \eta_i \ll 1 \, . \tag{2.72}$$

These are sufficient but in general not necessary conditions for inflation, the necessary and sufficient condition being $\ddot{a}(t) > 0$ , or equivalently $-\dot{H}/H^2 < 1$. For instance, inflation can also occur if one of the eigenvalues $\eta_i$ is much bigger than 1. In that case, phenomenology is very close to that of single-field inflation along the more unstable direction in the potential (cf. e.g. [93]). If the conditions (2.72) are satisfied, the equations (2.58) and (2.59) are well-approximated by the slow-roll equations

$$\tilde{H}^2 \simeq \frac{\tilde{V}}{3M^2} \, , \tag{2.73}$$

$$3\tilde{H}\dot{\phi}^c \simeq -\tilde{V}^c \, . \tag{2.74}$$

Let us now discuss the regions in the $(\chi, h)$-plane for which the slow-roll conditions hold (cf. figure 2.2). The slow-roll region extends to infinity along the potential valleys if $\xi_1 < \frac{1}{2}$. As will be shown in section 2.3.3, only if this condition holds, the scalar fields can constitute a dark energy component in the late stage. During inflation it is safe to neglect the term in the potential proportional to $\alpha$. In fact, for $\alpha = 0$ the potential possesses only one valley which goes along the $\chi$-axis. For $\alpha \neq 0$ this valley splits into two valleys that lie at the angles $\theta = \pm \arctan(\alpha)$ with respect to the $\chi$-axis. For $\alpha \lll 1$ these angles are very small. We will see that inflation in our model occurs far from these valleys where the effect of a non-zero $\alpha$ is irrelevant. Hence, we will put $\alpha = 0$ for the rest of this section. The plot of the slow-roll region for $\xi_1 < \frac{1}{2}$ and $\alpha = 0$ is presented in figure 2.2.

Next, we want to analyze the different trajectories the fields can take if the initial conditions are chosen in the slow-roll region. We will only consider trajectories starting in the first quadrant $\chi, h > 0$. Trajectories starting in other quadrants have exactly the same behaviour. The nature of the potential (2.49) makes that all trajectories tend to approach one of the two potential valleys. There are trajectories (type a) that on their way to the valley never leave the slow-roll region. Numerical computations show that

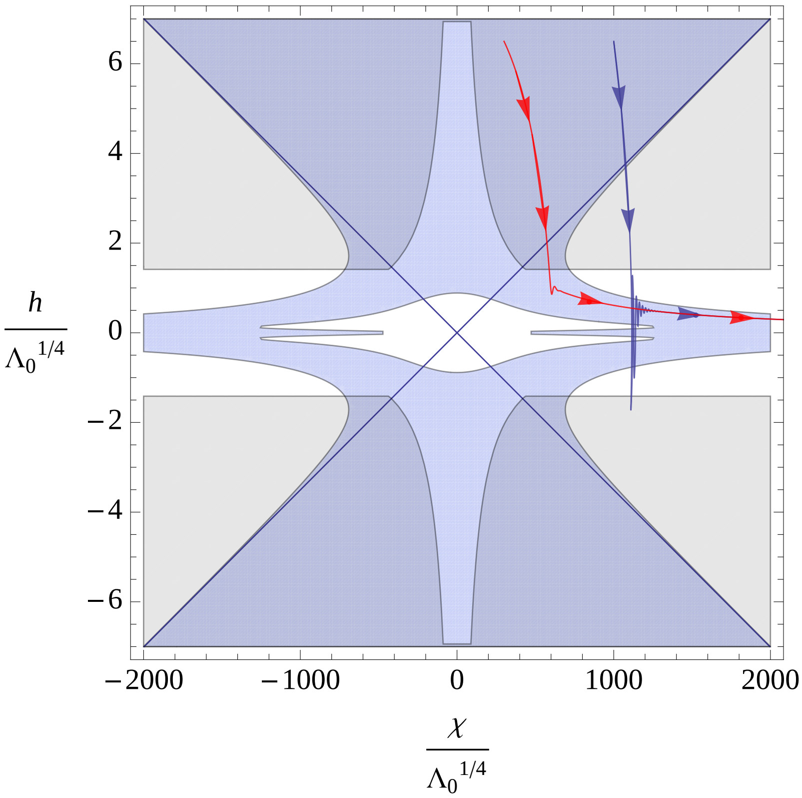



**Fig. 2.2:** The blue region is the slow-roll region for $\xi_1 \ll 1$ and $\xi_2 \gg 1$ limited by $\epsilon = 1$. The general features of the region are the same whenever $\xi_1 < \frac{1}{2}$ and $\xi_2 > \frac{1}{2}$. For $\xi_1 < \frac{1}{2}$ and $\xi_2 < \frac{1}{2}$ the central fast-rolling region vanishes. For $\xi_1 > \frac{1}{2}$ the slow-roll region does not extend to infinity along the $\chi$-axis in which case the scalar fields can not act as dark energy in the late stage of evolution. In the shaded region the potential fulfills $v_1, v_2 < 1$, i.e. the influence of $\Lambda_0 \neq 0$ is small. The presence of the slow roll-region along the $\chi$-axis such as the central fast-roll region are effects of $\Lambda_0 > 0$. For $\Lambda_0 = 0$ the slow-roll region is simply given by the triangles delimited by the two diagonal lines. Note that in this case the units of the axis have to be chosen differently. The red line represents a trajectory of type a, never leaving the slow-roll region. The blue line is a trajectory of the type b, which leaves the slow-roll region and oscillates strongly before rolling down the valley. These trajectories were found by numerically solving the exact equations (2.59).

such trajectories undergo only very few slow oscillations before asymptotically approaching the valley. One can not expect a successful reheating period from this type of behaviour (cf. section 2.3.2). The good trajectories (type a) are those that at some point leave the slow-roll region. After the exit of the slow-roll region, which at the same time marks the end of inflation, these trajectories undergo a fast-roll towards the valley and consequently oscillate strongly around its minimum. Typical examples for both types of trajectories are given in figure 2.2.

In terms of the variables $\chi$ and $h$ and using (2.62) and (2.63) the slow-roll equations (2.74) are (for $\alpha = 0$)

$$\dot{\chi} = \frac{u(\chi, h)}{3H} h \left(1 + 6\xi_h + v_1\right) \ , \tag{2.75}$$

$$\dot{h} = -\frac{u(\chi, h)}{3H} \left(1 + 6\xi_\chi\right) \chi \left(1 - v_2\right) \ . \tag{2.76}$$

Combining the two equations, one can find the equation for the slow-roll trajectories

$$\frac{d\chi}{dh} = \frac{\dot{\chi}}{\dot{h}} = -\frac{h}{\chi} \frac{1 + 6\xi_h + v_1}{\left(1 + 6\xi_\chi\right) \chi \left(1 - v_2\right)} \ . \tag{2.77}$$

Looking at figure 2.2 we observe that all good trajectories (type b) go through the scale-invariant region before leaving the slow-roll region. This can be proven mathematically. It is enough to show that for $\xi_\chi < \frac{1}{2}$ there is no intersection between the line limiting the slow-roll region and the line $v_2 = 1$ limiting the scale-invariant region. Therefore, for trajectories of the type b, the passage from the slow-roll to the fast-roll region will always take place within the scale-invariant region. We will see that if the scalar fields are to act as a dark-energy component in the late phase, the whole period of observable inflation (i.e. the final $\sim 60$ e-folds) takes place in the scale-invariant region. In this region, the equation for the trajectories (2.77) simplifies to

$$\frac{d\chi}{dh} = -\frac{\left(1 + 6\xi_h\right) h}{\left(1 + 6\xi_\chi\right) \chi} \tag{2.78}$$

and can be solved exactly. The solutions satisfy the equation of an ellipse

$$\left(1 + 6\xi_\chi\right) \chi^2 + \left(1 + 6\xi_h\right) h^2 = \mathrm{cst} \ . \tag{2.79}$$

One can get the same relation by integrating (2.68). Hence, these trajectories are a good approximation, even beyond the slow-roll approximation, as long as $\Lambda_0$ is negligible. The existence of this constant of the motion leads us to the definition of the new variables

$$\rho = \frac{M}{2} \ln \left( \frac{\left(1 + 6\xi_\chi\right) \chi^2 + \left(1 + 6\xi_h\right) h^2}{M^2} \right) \ , \tag{2.80}$$

$$\theta = \arctan \left( \sqrt{\frac{1 + 6\xi_h}{1 + 6\xi_\chi}} \frac{h}{\chi} \right) \ , \tag{2.81}$$

where $\rho \in ]-\infty, \infty[$ and $\theta \in ]-\pi/2, \pi/2[$. In the scale-invariant part of the slow-roll region one has $\rho = \rho_0 =$ cst., and the angle $\theta$ obeys the equation

$$\frac{d\theta}{dn} = -\frac{4\xi_\chi}{1+6\xi_\chi} \cot\theta \left(1 + \frac{6\xi_\chi \xi_h}{\xi_\chi \cos^2\theta + \xi_h \sin^2\theta}\right) , \tag{2.82}$$

where we have used the e-fold time variable defined as $n \equiv \ln a(t)$. Note that this equation is independent of $\rho_0$, which is a consequence of scale invariance.[4] The equation can be integrated in order to get the number of e-folds N before the end of inflation as a function of the angle $\theta$,

$$N = \frac{1}{4\xi_\chi}\left[\ln\left(\frac{\cos\theta_{\text{end}}}{\cos\theta}\right) + 3\xi_\chi \ln\left(\frac{\xi_\chi \cos^2\theta_{\text{end}} + \xi_h \sin^2\theta_{\text{end}} + 6\xi_\chi\xi_h}{\xi_\chi\cos^2\theta + \xi_h \sin^2\theta + 6\xi_\chi\xi_h}\right)\right] , \tag{2.83}$$

where $\theta_{\text{end}}$ is the value of $\theta$ at the end of inflation. In our model, the condition marking the end of inflation is $\epsilon = 1$. In the scale-invariant region this condition reads

$$\frac{8\xi_\chi^2(1+6\xi_h)}{1+6\xi_\chi} \frac{\cot^2\theta_{\text{end}}}{\xi_\chi \cos^2\theta_{\text{end}} + \xi_h \sin^2\theta_{\text{end}}} = 1 . \tag{2.84}$$

After inserting values for $\xi_\chi$, $\xi_h$ and requiring a minimal number of inflationary e-folds $N = N_{\text{min}}$, equation (2.83) can be solved to obtain a lower bound $\theta_{\text{initial}} > \theta_{\text{min}}$ on the initial conditions for inflation. In the next section we derive bounds on the parameters $\xi_\chi$ and $\xi_h$, which are related to the spectra of primordial perturbations.

### Linear perturbations

The theory of cosmological perturbations as stemming from quantum fluctuations during inflation was developed in [20–23, 94] (see also [95] and references therein). Including scalar and tensor perturbations and fixing the longitudinal transverse traceless gauge, the line element can be written as

$$d\tilde{s}^2 = a(\eta)^2 \left((1+\Phi)\, d\eta^2 - (1-\Psi)\left(\delta_{ij} + h_{ij}^{TT}\right) dx^i dx^j\right) , \tag{2.85}$$

where $\Phi$ and $\Psi$ are the Bardeen potentials [96]. The comoving curvature perturbation is defined as [97]

$$\zeta \equiv \Psi - \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2}\left(\Psi' + \mathcal{H}\Phi\right) , \tag{2.86}$$

where $\mathcal{H}$ is the comoving Hubble parameter $\mathcal{H} = aH$ and prime stands for derivative with respect to comoving time $\eta$. Following reference [92] we find the evolution equation for $\zeta$ at linear order in perturbations to be

$$\zeta' = \frac{2\mathcal{H}}{\sigma'^2}\Delta\Psi - \frac{2\mathcal{H}}{\sigma'^2} \perp^c_d a^2 \tilde{V}_{,c}\delta\phi^d . \tag{2.87}$$

[4] In fact, when $\Lambda_0$ is neglected, $\rho$ is the massless Goldstone boson related to the scale symmetry. Hence, it only appears derivatively in the equations of motion.

Here, $\delta\phi^d$ are the perturbations to the background field trajectory, $\sigma$ is defined through the relation $\sigma' = \sqrt{l_{ab}\phi^{a\prime}\phi^{b\prime}}$ and $\perp_{ab}$ is the projector orthogonal to the field trajectory, given by

$$\perp_{ab} = l_{ab} - p_{ab} \ , \tag{2.88}$$

where $p_{ab}$ is the projector on the field trajectory,

$$p_{ab} = u_a u_b \ , \tag{2.89}$$

with $u^a \equiv \frac{\phi^{a\prime}}{\sigma'} = \frac{\dot{\phi}^a}{\dot{\sigma}}$. In the long wavelength limit, (2.87) reduces to

$$\zeta' = -\frac{2\mathcal{H}}{\sigma'^2} \perp^c_d a^2 \tilde{V}_{,c} \delta\phi^d \ . \tag{2.90}$$

This is a well-known result, saying that for multi-field inflation $\zeta$ is not in general conserved outside the Hubble horizon. There are two situations in which the source term on the right-hand side of the equation vanishes nevertheless. One of them is if the perturbation vector $\delta\phi^a$ is tangent to the field trajectory, i.e. $\delta\phi^a \| u^a$. This corresponds to the complete absence of isocurvature perturbations during inflation and will not be satisfied in our scenario.[5] The second possibility is to have $\tilde{V}^a \| u^a$. At zeroth order in slow-roll, when the background equations are approximated by (2.74), this is always satisfied. However, it does no longer hold, in general, if one goes beyond the slow-roll approximation. In our model, due to the scale symmetry, one might still have approximately $\tilde{V}^a \| u^a$, to a higher precision than the zeroth order slow-roll approximation. In fact, if $\Lambda_0$ can be neglected, one can apply equation (2.67), which in terms of the variables $(\rho, \theta)$ reads

$$\dot{\rho} \simeq \text{cst.} \cdot l^{\rho\rho} \, e^{-3n} \ . \tag{2.91}$$

Therefore, in the scale-invariant region, $\dot{\rho} = 0$ is an attractor. This means that after a few e-folds, independently of the initial conditions, $\rho$ is very close to constant and hence $u^\rho \simeq 0$. At the same time, still in the scale-invariant region, the potential is a function of $\theta$ only, i.e. $\tilde{V}_{,\rho} \simeq 0$. As a consequence, in terms of arbitrary variables, we have approximately $\tilde{V}^a \| u^a$ and $\zeta$ is almost constant for large wavelengths.

We will from now on suppose that initial conditions are such that the attractor $\rho \simeq cst.$ has been reached before the observable scales cross the horizon. In that case, $\zeta$ is constant outside the Hubble horizon and one can express the primordial spectrum of $\zeta$ to first non-trivial order in the slow-roll parameters as [95, 98, 99]

$$\mathcal{P}_\zeta(k) \simeq \frac{1}{2M^2\epsilon^*}\left(\frac{H^*}{2\pi}\right)^2 \left(1 - 2(C+3)\epsilon^* + 2C\,\eta^*_{eff} - (6\epsilon^* - 2\eta^*_{eff})\ln\frac{k}{k^*}\right) \ , \tag{2.92}$$

[5] Isocurvature perturbations correspond to the component of $\delta\phi^a$ perpendicular to the field trajectory, i.e. for which $\delta\phi^a \perp u^a$ (cf. e.g. [92]).

where $k^*$ is the pivot scale, $C \equiv 2 - \ln 2 - \gamma$, $\gamma$ is the Euler-Mascheroni constant and we have defined

$$\eta_{eff} = p_{ab}N^{ab} = \frac{\tilde{V}_{,a}\tilde{V}_{,b}}{l^{cd}\tilde{V}_{,c}\tilde{V}_{,d}}N^{ab} \ , \tag{2.93}$$

corresponding to the projection of the matrix $N^{ab}$ onto the background trajectory. Quantities marked with a star " * " are evaluated at the moment when the pivot scale $k^*$ leaves the Hubble horizon. The scalar spectral index is found to be

$$n_s(k^*) - 1 \equiv \left.\frac{d\ln \mathcal{P}_\zeta}{d\ln k}\right|_{k=k^*} \simeq -6\epsilon^* + 2\eta^*_{eff} \ , \tag{2.94}$$

while the amplitude of the scalar spectrum is identified as

$$\Delta^2_\zeta(k^*) \equiv \frac{1}{2M^2\epsilon^*}\left(\frac{H^*}{2\pi}\right)^2 \ . \tag{2.95}$$

Now, even if isocurvature perturbations do not affect the evolution of $\zeta$, they are in general present at the end of inflation. So, they could in principle give rise to unobserved entropy perturbations. Though, if one assumes that after reheating the universe reaches a state of local thermal equilibrium, entropy perturbations are efficiently erased [100]. Working with this assumption, we will be able to relate the primordial spectra to CMB observations.

The primordial spectrum of the tensor perturbations is given, to first order in the slow-roll approximation, by [95, 98] [6]

$$\mathcal{P}_h(k) \simeq \frac{8}{M^2}\left(\frac{H^*}{2\pi}\right)^2\left(1 - 2(C+1)\epsilon^* - 2\epsilon^*\ln\frac{k}{k^*}\right) \ , \tag{2.96}$$

which results in a tensorial spectral index

$$n_h(k^*) \equiv \left.\frac{d\ln \mathcal{P}_h}{d\ln k}\right|_{k=k^*} \simeq -2\epsilon^* \ . \tag{2.97}$$

The ratio of the tensor and the scalar spectra to first order in slow-roll is then given by

$$r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} \simeq 16\epsilon^* \ . \tag{2.98}$$

[6] This result is based on the slow-roll approximation and involves no further assumptions.

### Constraints on parameters from CMB observations

In this section we make the assumption that the whole period of observable inflation takes place in the scale-invariant region. As mentioned before, this assumption will automatically be valid if the initial conditions of the fields are chosen such that they act as dark energy in the late stage. We can therefore use equations (2.83) and (2.84) for the background.

We also make the assumptions that $\zeta$ is conserved for large wavelengths during inflation and that after inflation entropy perturbations die away before having an observable effect. This allows us to directly relate the primordial spectra (2.92) and (2.96) to observations of the CMB.

The observational bounds (WMAP7 + BAO + $H_0$) [27] for the tilt and the amplitude of the scalar power spectrum and for the tensor-to-scalar ratio are[7]

$$n_s(k^*) = 0.968 \pm 0.012 \ , \tag{2.99}$$

$$\Delta_\zeta^2(k^*) = (2.43 \pm 0.091) * 10^{-9} \ , \tag{2.100}$$

$$r(k^*) < 0.24 \ , \tag{2.101}$$

where $k^*/a_0 = 0.002$ Mpc$^{-1}$. These values are obtained for the standard $\Lambda CDM$ model. The errors indicate the 68% confidence level. We do not consider the values obtained for a model with varying equation of state parameter for dark energy, since in the here presented model $\omega_{DE}$ is very close to constant.

Let us denote by $N^*$ the number of e-fold between the moment where $k^*$ exits the horizon and the end of inflation. In order to determine $N^*$ we need to know the post-inflationary evolution of the universe including the details of the reheating process. If there are uncertainties related to the post-inflationary history, these can be accounted for by varying the value of $N^*$. We turn back to this issue at the end of this section and in section 2.3.2.

In a first instance we can compute the spectral parameters $n_s(k^*)$, $\Delta_\zeta^2(k^*)$ and $r(k^*)$ as functions of $\xi_\chi$, $\xi_h$, $\lambda$ and of $N^*$. This is done in three steps:
**1)** Equation (2.84) is solved for $\theta_{end} = \theta_{end}(\xi_\chi, \xi_h)$.
**2)** $\theta_{end}$ is inserted into (2.83) from which one determines $\theta^* = \theta^*(\xi_\chi, \xi_h, N^*)$.
**3)** Expressions (2.94), (2.95) and (2.98) are evaluated at $\theta^*$ to find the spectral parameters as functions of of $\xi_\chi$, $\xi_h$, $\lambda$ and of $N^*$.

The value of the parameter $\lambda$ to be used corresponds to the Higgs self-coupling evaluated at the scale of inflation [101]. It contains the uncertainty related to the Higgs mass $m_H^2$. We expect the running of $\lambda$ to be similar as in the case of the Higgs inflation model [101]. As was shown in [101], for $m_H^2 \simeq 130 - 180$GeV, the coupling $\lambda$ evaluated at the scale of inflation lies in the range $\lambda \simeq 0.1 - 1$.

[7] These numbers are taken from the updated preprint http://lambda.gsfc.nasa.gov/product/map/dr4/pub_papers/sevenyear/cosmology/wmap_7yr_cosmology.pdf.

The second of the steps described above cannot be done analytically. Numerical evaluation shows that for the spectral parameters to lie in the allowed region, the parameters have to be such that $\xi_\chi \ll 1$ and $\xi_h \gg 1$ (cf. figure 2.3). With this knowledge, we can derive approximate analytical results. From (2.84) we obtain

$$\theta_{end} = 2 * 3^{\frac{1}{4}} \sqrt{\xi_\chi} \left(1 + \mathcal{O}\left(\xi_\chi, \frac{1}{\xi_h}\right)\right) . \tag{2.102}$$

In order to approximately solve (2.83) for $\theta^*$ we neglect the second term on the right-hand side. The inversion then gives

$$\theta^* \simeq \arccos\left(\cos(\theta_{end}) e^{-4\xi_\chi N^*}\right) . \tag{2.103}$$

Inserting $\theta_{end}$ from (2.102) we obtain

$$\theta^* \simeq \arccos\left(e^{-4\xi_\chi N^*}\right)\left(1 + \mathcal{O}\left(\xi_\chi, \frac{1}{\xi_h}\right)\right) . \tag{2.104}$$

Here, the sign for approximate equality "$\simeq$" refers to the approximation made when inverting (2.83). This approximation constitutes the main source of error. In fact, one can get a much more accurate approximation by reinserting the first approximation into the right-hand side of(2.83) in order to compute the second order approximation of an iterative solution. However, as the expressions get very complicated, we stick to the first order approximation which will already come very close to the numerical results.

We can now evaluate the spectral parameters at the approximate value for $\theta^*$. Inserting (2.104) into (2.94), (2.95) and (2.98) we obtain

$$n_s(k^*) - 1 \simeq -8\xi_\chi \coth(4\xi_\chi N^*) \left(1 + \mathcal{O}\left(\xi_\chi, \frac{1}{\xi_h}\right)\right) , \tag{2.105}$$

$$\Delta_\zeta^2(k^*) \simeq \frac{\lambda \sinh^2(4\xi_\chi N^*)}{1152\pi^2 \xi_\chi^2 \xi_h^2} \left(1 + \mathcal{O}\left(\xi_\chi, \frac{1}{\xi_h}\right)\right) , \tag{2.106}$$

$$r(k^*) \simeq 192\xi_\chi^2 \sinh^{-2}(4\xi_\chi N^*) \left(1 + \mathcal{O}\left(\xi_\chi, \frac{1}{\xi_h}\right)\right) . \tag{2.107}$$

These are the approximate analytical results that we will compare to the results found numerically. Although the quantity $4\xi_\chi N^*$ can be of the order one, $4\xi_\chi N^* \sim \mathcal{O}(1)$, the series expansions of the hyperbolic functions converge rapidly and we can further approximate

$$n_s(k^*) - 1 \simeq -\frac{2}{N^*}\left(1 + \frac{1}{3}(4\xi_\chi N^*)^2 + ...\right) , \tag{2.108}$$

$$\Delta_\zeta^2(k^*) \simeq \frac{\lambda N^{*2}}{72\pi^2 \xi_h^2}\left(1 + \frac{1}{3}(4\xi_\chi N^*)^2 + ...\right) , \tag{2.109}$$

$$r(k^*) \simeq \frac{12}{N^{*2}}\left(1 - \frac{1}{3}(4\xi_\chi N^*)^2 + ...\right) . \tag{2.110}$$

We remark that in the limit $\xi_\chi \to 0$ these results reduce to those found for the Higgs-inflation model [101]. Hence, one can think of $\xi_\chi$ as the deviation of our predictions from those of the Higgs inflation model.

Making a few assumptions about the post-inflationary evolution of the universe, one can also relate $N^*$ to the parameters of the theory. First, during the reheating phase the scale factor is expected to evolve like in a matter dominated universe. The reason is that during this stage the present model behaves much like the Higgs inflation model of [101–103] (cf. section 2.3.2). In this sense matter like scaling of the universe during reheating is not really an assumption but rather a property of the considered model. Next, we make the usual assumptions that reheating is followed by the standard radiation and matter dominated stages. Further assuming that the transitions between the different phases are instantaneous, one can derive the following relation (cf. [98])

$$N^* \simeq -\ln\frac{k^*}{a_0 H_0} - \ln\left(\frac{\rho_0^{cr}/\Omega_0^\gamma}{\tilde{V}(\theta^*)}\right)^{1/4} + \ln\left(\frac{\tilde{V}(\theta^*)}{\tilde{V}(\theta_{end})}\right)^{1/4} - \frac{1}{3}\ln\left(\frac{\tilde{V}(\theta_{end})}{\rho_{rh}}\right)^{1/4}. \tag{2.111}$$

Here, $a_0$, $H_0$, $\rho_0^{cr}$ and $\Omega_0^\gamma$ stand for todays values of the scale factor, the Hubble parameter, the critical density and the abundance of radiation respectively. $\rho_{rh}$ denotes the radiation energy density at the end of reheating, i.e. at the onset of the hot big bang. After inserting the observational values $\rho_0^{cr} h^{-2} \simeq 8.1\cdot 10^{-47}\mathrm{GeV}^4$ and $\Omega_0^\gamma h^2 \simeq 4.3\cdot 10^{-5}$, where $h$ is the dimensionless Hubble parameter, the above formula can be written as

$$N^* \simeq 59 - \ln\frac{k^*\mathrm{Mpc}}{0.002 a_0} - \ln\frac{10^{16}\mathrm{GeV}}{\tilde{V}(\theta^*)^{1/4}} + \ln\left(\frac{\tilde{V}(\theta^*)}{\tilde{V}(\theta_{end})}\right)^{1/4} - \frac{1}{3}\ln\left(\frac{\tilde{V}(\theta_{end})}{\rho_{rh}}\right)^{1/4}. \tag{2.112}$$

Note that the dependence on $h$ has cancelled out.[8] A detailed determination of $\rho_{rh}$ goes beyond the scope of this work. However, we will see in section 2.3.2 that $\rho_{rh}$ has to take a value between $\rho_{rh}^{min} \simeq \frac{\lambda}{4\xi_h^4} M^4$ and $\rho_{rh}^{max} = \frac{\lambda}{\xi_h^2} X M^4$, where $X = 7 - 4\sqrt{3} + \mathcal{O}(\xi_\chi, 1/\xi_h)$. In these two limits we can combine (2.112) with (2.83) and (2.84) to find $N^* = N^*(\xi_\chi, \xi_h, \lambda)$. For $\xi_\chi \ll 1$ and $\xi_h \gg 1$ and using the approximate expressions (2.104) and (2.102) for $\theta^*$ and $\theta^{end}$, we obtain the approximate expressions

$$N^*_{min} \simeq 64.55 - \frac{1}{12}\ln\lambda - \frac{2}{3}\ln\frac{\xi_h}{\sqrt{\lambda}}\,, \tag{2.113}$$

$$N^*_{max} \simeq 64.55 - \frac{1}{2}\ln\frac{\xi_h}{\sqrt{\lambda}}\,, \tag{2.114}$$

where the subscripts "min" and "max" stand for the cases $\rho_{rh} = \rho_{min}$ and $\rho_{rh} = \rho_{max}$ respectively.

[8] In the analog formula of [98] this fact remains somewhat hidden.

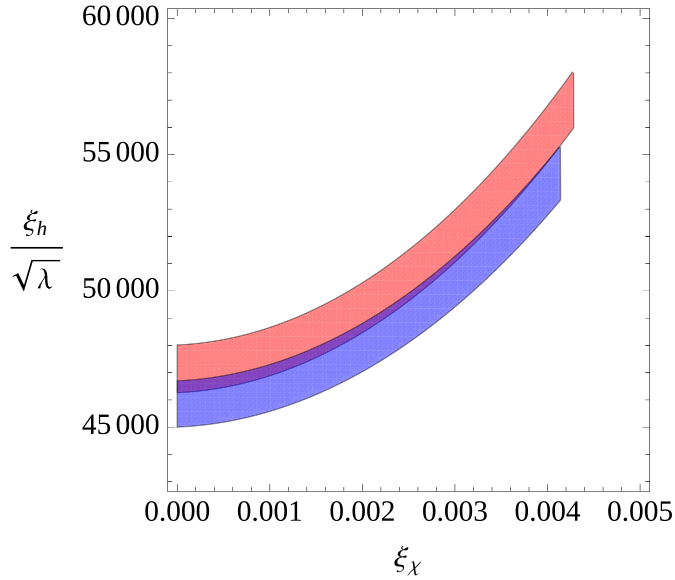



**Fig. 2.3:** This plot shows the parameter regions for which the values of $n_s(k^*)$, $\Delta^2_\zeta(k^*)$ and $r(k^*)$ lie in the observationally allowed region, for $0.1 < \lambda < 1$. The red region is obtained for $\rho_{rh} = \rho_{rh}^{max}$ (instant reheating), while the blue region corresponds to $\rho_{rh} = \rho_{rh}^{min}$ (long reheating).

Accurate predictions for the values of $n_s(k^*)$, $\Delta^2_\zeta(k^*)$ and $r(k^*)$ can be found numerically by combining equations (2.84), (2.83), (2.112) to find the precise value of $\theta^*$ which is then inserted into (2.94), (2.95) and (2.98). The parameter regions yielding observables in the allowed range is plotted in figure 2.3.

The red region is obtained under the assumption of instant reheating ($\rho_{rh} = \rho_{rh}^{max}$). The blue region is gotten for $\rho_{rh} = \rho_{rh}^{min}$. As one can understand from the approximate formula (2.106), the band-shape of the parameter regions is due to the constraint on the amplitude $\Delta^2_\zeta(k^*)$. The fact that the bands are cut on the right comes from the constraint on $n_s(k^*)$ as is clear from (2.105). The constraint on the tensor-to-scalar ratio $r$ is weaker. We obtain the bounds

$$\begin{aligned} 0 &< \xi_\chi < 0.0041\,, \\ 45000 &< \frac{\xi_h}{\sqrt{\lambda}} < 55300\,, \end{aligned} \qquad \text{for} \quad \rho_{rh} = \rho_{min} \tag{2.115}$$

and

$$\begin{aligned} 0 &< \xi_\chi < 0.0043\,, \\ 46300 &< \frac{\xi_h}{\sqrt{\lambda}} < 58100\,, \end{aligned} \qquad \text{for} \quad \rho_{rh} = \rho_{max}\,. \tag{2.116}$$

The approximate formulas (2.105)-(2.107) such as (2.113) and (2.114) are helpful to understand the dependence of the observables on the different parameters. For $\rho_{rh} = \rho_{rh}^{max}$ the observables essentially depend on $\xi_h$ and $\lambda$ only through the ratio $\xi_h/\sqrt{\lambda}$. For $\rho_{rh} = \rho_{rh}^{min}$, the approximate expressions for $N^*_{min}$ contains the term $-\frac{1}{12}\ln\lambda$. This means that if one changes the value of $\lambda$, keeping the ratio $\xi_h/\sqrt{\lambda}$ fixed, $N^*_{min}$ gets a shift. For $0.1 < \lambda < 1$ this shift is $\Delta N \sim 0.2$ leading to negligible changes in the observables. It is therefore sufficient to discuss the dependence of the observables on $\xi_\chi$ and $\xi_h/\sqrt{\lambda}$. Varying $\xi_h/\sqrt{\lambda}$ within the observationally allowed order of magnitude hardly affects the value of $N^*$. We have

$$N^*_{min} \simeq 57.5 \, , \tag{2.117}$$

$$N^*_{max} \simeq 59 \, . \tag{2.118}$$

This means, as can be seen from (2.105), that $n_s(k^*)$ mainly depends on $\xi_\chi$. The same is true for the tensor-to-scalar ratio $r(k^*)$ (2.107). The scalar amplitude $\Delta^2_\zeta(k^*)$ (2.106) depends on both $\xi_\chi$ and the ratio $\xi_h/\sqrt{\lambda}$. Given the insensitivity of the scalar tilt and the tensor-to scalar-ratio on $\xi_h/\sqrt{\lambda}$, we plot them as functions of $\xi_\chi$ (cf. figures 2.4 and 2.5).

In the relevant parameter range the accuracy of the approximate formula for $n_s(k^*)$ is of the per mill level. The approximations of $r(k^*)$ and $\Delta^2_\zeta(k^*)$ are good at the percent level.

We find that the scalar spectral index predicted by the present model has an absolute maximum,

$$n_s(k^*) < 0.97 \simeq 1 - \frac{2}{N^*} \, . \tag{2.119}$$

The extreme value is obtained in the limit where $\xi_\chi \to 0$, and corresponds to the value predicted by the Higgs-Inflation model [102]. It will be very interesting to compare this bound with more stringent observational constraints.

The predicted tensor-to-scalar ratio also has an upper bound, i.e.

$$r(k^*) < 0.0035 \simeq \frac{12}{N^{*2}} \, , \tag{2.120}$$

which is much stronger than the current observational bound (2.101). Also here, the extreme value is gotten for $\xi_\chi \to 0$ and corresponds to the result for Higgs-Inflation.

The upper bounds on $n_s(k^*)$ and on $r(k^*)$ are two non-trivial predictions of our model. They correspond to the values predicted by Higgs-Inflation [102]. We will see in section 2.3.3 that, if the scalar fields constitute a dark-energy component at late times, the parameter $\xi_\chi$ is related to its equation of state. As a consequence, the upper bound on $n_s(k^*)$ will induce a bound on the equation of state parameter $\omega_{DE}$ of dark energy. This is a very non-trivial connection provided by our model.

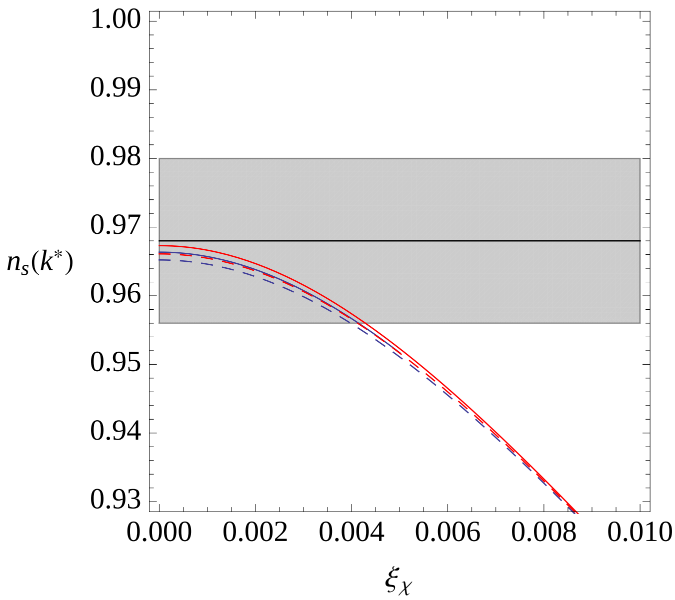



**Fig. 2.4:** The red curve is obtained for $\rho_{rh} = \rho_{rh}^{max}$ (instant reheating), while the blue curve represents the case $\rho_{rh} = \rho_{rh}^{min}$ (long reheating). The dashed curves are obtained from the approximate formula (2.105) for $N^*_{max}$ and $N^*_{min}$ respectively. The horizontal line and the shaded region correspond to the observational mean value and the associated error (2.99).

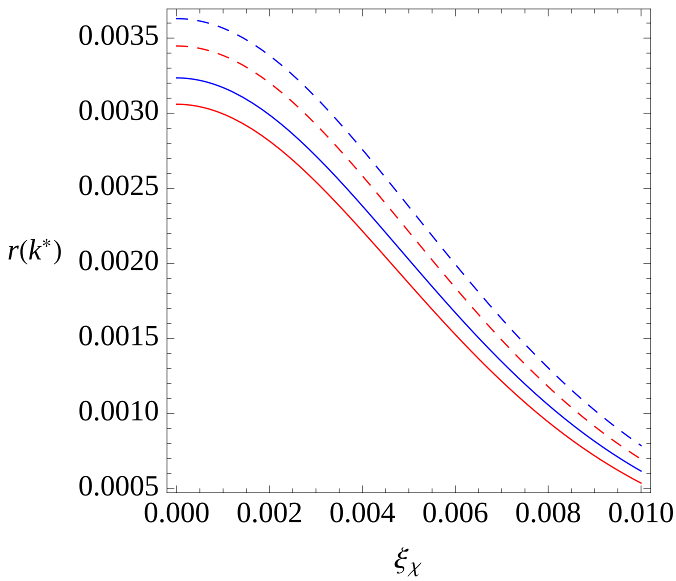



**Fig. 2.5:** The red curve is obtained for $\rho_{rh} = \rho_{rh}^{max}$ (instant reheating), while the blue curve represents the case $\rho_{rh} = \rho_{rh}^{min}$ (long reheating). The dashed curves are obtained from the approximate formula (2.106) for $N^*_{max}$ and $N^*_{min}$ respectively.

The findings of this and the following section allow us to constrain the region of initial conditions for the scalar fields which lead to successful inflation. Based on the assumption that the last $N^*$ e-folds of inflation take place in the scale-invariant region, we have found the field value $\theta^*$ close to which the observable scales exit the Hubble horizon during inflation. The initial conditions for inflation have to be such that $\theta_{in} \geq \theta^*$. In terms of the original variables this condition reads

$$\frac{h_{in}}{\chi_{in}} \geq \sqrt{\frac{1+6\xi_\chi}{1+6\xi_h}} \tan\theta^* \,. \tag{2.121}$$

$\theta^*$ was found to be $\theta^* \simeq \arccos\left(e^{-4\xi_\chi N^*}\right)$ (2.104). For typical parameter values $\xi_\chi = 0.003$, $\xi_h = 50000$ and $N^* = 60$ one obtains $\theta^* \sim 1.2$ and $\frac{h_{in}}{\chi_{in}} \gtrsim 0.004$. Considerations related to dark energy (section 2.3.3) will yield an additional constraint on the initial conditions. The region of acceptable initial conditions satisfying both constraints is shown in figure 2.7.

Let us at this point make an important remark. The fact that a scale-invariant theory including the dilaton can give a mechanism for inflation does not depend on whether gravity is GR or UG. Moreover, since $\Lambda_0$ can be neglected during inflation, the analysis we did also applies to the case $\Lambda_0 = 0$, i.e. SI plus GR. The replacement of GR by UG mainly affects the late universe. Namely, it provides a mechanism for dynamical dark energy.

### 2.3.2 Qualitative picture of the reheating process

After the end of inflation, i.e. after the field trajectory leaves the slow-roll region, the fields undergo damped oscillations around the potential valley. During these oscillations, the scalar fields essentially decay into standard model particles. In this way the initial conditions are set for the hot big bang starting with a radiation dominated phase. The details of the reheating process in the present model will be discussed in an upcoming work. Nevertheless, we can give some simple arguments as to why reheating is expected to be very similar to reheating in the Higgs-inflation model [101–103].

During the oscillations one has $\theta < \theta_{end}$, where $\theta_{end}$ is the solution of (2.84). As we found in the previous section, the parameters giving a successful inflation satisfy $\xi_\chi \ll 1$ and $\xi_h \gg 1$ such that $\theta_{end} \simeq 2 * 3^{\frac{1}{4}}\sqrt{\xi_\chi} \ll 1$. Therefore, during the oscillations, the relation (2.80) becomes approximately $\rho \simeq \frac{M}{2}\ln\frac{\chi^2}{M^2}$. During the oscillations the trajectory passes through the region in which $\Lambda_0$ is not negligible. In this region $\rho$, which was constant during inflation, will no longer be constant. However, numerical simulations show that the relative change in $\rho$ from the beginning to the end of the oscillations is very small. Therefore, in a first approximation, one can consider $\rho$ and consequently $\chi$ as constant during reheating. Now, if $\chi \simeq cst$, one can identify the Planck scale $M^2 = \xi_\chi\chi^2$ and the action (2.32) reduces to the action for the Higgs-inflation model, except for the term $\Lambda_0$. The presence of $\Lambda_0$ gives rise to an effective cosmological constant. As we will see in the next section, this constant has to be negligibly small at the beginning of the radiation dominated phase in order to cope with phenomenology. So, in a first approximation, one gets the same results for reheating as in the case of Higgs-inflation. Here, we only want to reproduce the upper and the lower bounds on the reheating temperature $T_{rh}$, defined as the initial temperature of the homogeneous radiation dominated universe. The energy transferred to radiation has to be lower than the energy scale at the end of inflation, i.e.

$$\rho_{rh} = \frac{\pi^2}{30}g_{eff}(T_{rh})T_{rh}^4 < \rho_{rh}^{max} = \tilde{V}(\theta_{end}) = \frac{\lambda}{\xi_h^2}XM^4 \ , \tag{2.122}$$

where $\rho_{rh}$ stands for the energy density contained in radiation, $g_{eff}(T_{rh})$ is the effective number of relativistic degrees of freedom present in the thermal bath at the temperature $T_{rh}$ and $X$ is a numerical factor given by $X = 7 - 4\sqrt{3} + \mathcal{O}(\xi_\chi, 1/\xi_h) \simeq 0.7$. Inserting the lower bound on $\xi_h/\sqrt{\lambda}$ (2.116) and $g_{eff}(T_{rh}) = 106.75$ this gives the following upper bound on the reheating temperature

$$T_{rh} < 2.4 \times 10^{15} GeV \ . \tag{2.123}$$

As explained in [104], a lower bound on $T_{rh}$ is obtained by the following argument. As soon as the oscillation amplitude of the field $h$ falls below $\frac{M}{\xi_h}$, the non-minimal coupling between $h$ and the Ricci scalar $R$ can be neglected.

The interactions of $h$ reduce to the standard model interactions of the Higgs field and therefore $h$ decays almost immediately into relativistic standard model particles. This means, again at lowest order in $\xi_\chi$ and $\frac{1}{\xi_h}$, that

$$\rho_{rad}(T_{rh}) = \frac{\pi^2}{30} g_{eff}(T_{rh}) T_{rh}^4 > \rho_{rh}^{min} = \frac{\lambda}{4\xi_h^4} M^4 \ . \tag{2.124}$$

Here, we substitute the upper bound on $\xi_h/\sqrt{\lambda}$ (2.115) to obtain

$$T_{rh} > 2.1 \sqrt{\frac{20000}{\xi_h}} 10^{13} GeV \ . \tag{2.125}$$

The more detailed analysis of reheating in the Higgs- inflation model carried out in [101] (cf. also [103]) results in a reheating temperature $T_{rh} \simeq (3-15)\times 10^{13} GeV$.

### 2.3.3 The scalar fields during the hot big bang

After the phase of reheating the system enters the radiation dominated stage, at the beginning of which the total energy density is $\rho_{total}(T_{rh}) \simeq \rho_{rad}(T_{rh}) = \frac{\pi^2}{30} g_{eff}(T_{rh}) T_{rh}^4$. The scalar fields have almost settled down in the valley. i.e. $h \simeq \alpha\chi$. Let us assume that this equality is exact and the fields evolve exactly along the valley (cf. arguments given in [82]).[9] We are then left with only one scalar degree of freedom. After imposing the constraint $h = \alpha\chi$, the scalar-tensor part of the action (2.32) becomes

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\left(\xi_\chi + \alpha^2 \xi_h\right) \chi^2 \frac{R}{2} - \frac{1}{2}\left(1+\alpha^2\right)\left(\partial_\mu \chi\right)^2 - \Lambda_0 \ . \tag{2.126}$$

We can define the Einstein-frame metric as

$$\tilde{g}_{\mu\nu} = \frac{\left(\xi_\chi + \alpha^2 \xi_h\right)\chi^2}{M^2} g_{\mu\nu} \tag{2.127}$$

and the canonical scalar field $\phi$ through[10]

$$\chi = M \exp\left(\frac{\gamma\phi}{4M}\right) \ , \tag{2.128}$$

with

$$\gamma = \frac{4}{\sqrt{\frac{1+\alpha^2}{\xi_\chi + \alpha^2 \xi_h} + 6}} = \frac{4}{\sqrt{\frac{1}{\xi_\chi} + 6}} + \mathcal{O}(\alpha) \ . \tag{2.129}$$

As explained in section 1, $\alpha$ is related to the hierarchy between the electroweak and the Planck scale such that $\alpha \lll 1$ and $\alpha \lll \xi_\chi,\ \xi_h$. Hence,

[9] The validity of this approximation can also be checked numerically.

[10] This field $\phi$ is not to be confounded with the two-component notation $\phi^i$ introduced in section 2.3.1.

the corrections $\mathcal{O}(\alpha)$ in the above expression for $\gamma$ are negligibly small. A non-zero value of $\alpha$ will, however, be relevant for the particle physics phenomenology (cf. section 3.6). In the approximation $\alpha \lll 1$ the field $\phi$ is related to the field $\rho$ used to describe the inflationary phase as

$$\rho \simeq \frac{\phi}{\sqrt{6+\frac{1}{\xi_\chi}}} + \frac{M}{2}\ln\left(1+6\xi_\chi\right) \simeq \sqrt{\xi_\chi}\phi + \mathcal{O}\left(\xi_\chi\right) \ . \tag{2.130}$$

In terms of these new variables the Lagrangian (2.126) reads

$$\frac{\mathcal{L}}{\sqrt{-\tilde{g}}} = -M^2\frac{R}{2} - \frac{1}{2}(\partial_\mu\phi)^2 - \tilde{V}_{QE}(\phi) \ , \tag{2.131}$$

where the potential is given by

$$\tilde{V}_{QE}(\phi) = \frac{\Lambda_0}{\xi_\chi^2}\exp\left(-\gamma\frac{\phi}{M}\right) \ . \tag{2.132}$$

The field $\phi$ does not have direct couplings to the standard model fields except for its coupling to the other scalar field $h$. The latter is a derivative coupling and therefore very weak. As a consequence $\phi$ can be considered as a field minimally coupled to gravity but not interacting with matter and radiation. Let us now look at its influence on standard homogeneous cosmology. The equation of motion for a homogeneous field $\phi = \phi(t)$ in flat FLRW spacetime (2.57) is given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV_{QE}}{d\phi} = 0 \ . \tag{2.133}$$

Defining energy density $\rho_\phi$, pressure $p_\phi$ and the equation of state parameter $\omega_\phi$ of the scalar field as

$$\rho_\phi \equiv \frac{1}{2}\dot{\phi}^2 + V_{QE} \ , \tag{2.134}$$

$$p_\phi \equiv \frac{1}{2}\dot{\phi}^2 - V_{QE} \ , \tag{2.135}$$

$$\omega_\phi \equiv \frac{p_\phi}{\rho_\phi} \ , \tag{2.136}$$

the equation of motion can also be written as

$$\dot{\rho}_\phi = -3H\rho_\phi\left(1+\omega_\phi\right) \ . \tag{2.137}$$

In presence of a fluid of energy density $\rho_m$, for instance relativistic or non-relativistic matter , the Hubble parameter is given by the first Friedmann equation as

$$H^2 = \frac{1}{3M^2}\left(\rho_m + \rho_\phi\right) \ . \tag{2.138}$$

In terms of the relative abundances $\Omega = \frac{\rho}{3M^2H^2}$ the Friedmann equation reads

$$\Omega_m + \Omega_\phi = 1 \ . \tag{2.139}$$

The cosmological model described by equations (2.133) and (2.138) with a scalar field evolving in an exponential potential has been widely studied in the literature (for a recent review see [105]). It is interesting to note that the exponential potential that was proposed for quintessence long time ago [28–30] appears automatically in the present model. We want to recap the main results established in the literature and show how they apply to our model.

For the qualitative analysis of the system we follow [106] and rewrite equations (2.133) and (2.138) in terms of the observable quantities $\Omega_\phi$ and $\delta_\phi \equiv 1 + \omega_\phi$ as

$$\delta'_\phi = -3\delta_\phi(2-\delta_\phi) + \gamma(2-\delta_\phi)\sqrt{3\delta_\phi\Omega_\phi} \ , \tag{2.140}$$

$$\Omega'_\phi = 3(\delta_m - \delta_\phi)\Omega_\phi(1-\Omega_\phi) \ , \tag{2.141}$$

where prime now denotes the derivative with respect to the number of e-folds $n = \ln a$. Further, $\delta_m \equiv 1 + \omega_m$, where $\omega_m$ is the equation of state parameter of the barotropic fluid. For radiation one has $\delta_m = 4/3$ while for non-relativistic matter $\delta_m = 1$. In order to keep the discussion as general as possible, we include the option that in addition to non-relativistic matter, radiation and $\phi$ there exists a dark energy component with constant equation of state. Such a fluid would have $\delta_m < 2/3$. In the scale-invariant model analyzed here, a component of this type is present as soon as $\beta \neq 0$, i.e. if the action (2.32) contains a term $\beta\chi^4$ (cf. section 2.1). Note that the following discussion is not valid if on top of $\phi$ there exists another fluid with varying equation of state. It was shown in [30, 107] that depending on the value of the parameter $\gamma$ the system approaches one of two qualitatively very different attractor solutions.

For $\gamma > \sqrt{3\delta_m}$, the variables go towards the stable fixed point $\Omega_\phi = 3\delta_m/\gamma^2$ and $\delta_\phi = \delta_m$. This means that the scalar field inherits the equation of state parameter of the barotropic fluid. Hence, the energy density of the scalar field scales like the energy density of the fluid. Unless the scalar field gives the dominating contribution to the energy density from the beginning, it will never become dominating. Therefore, these so-called "scaling solutions" can not be responsible for the late-time acceleration of the universe. In this case, the accelerated expansion must be due to another mechanism, e.g. a barotropic dark energy component with $\delta < 2/3$. Hence, a scaling field can at best provide a small contribution to dark energy.

For $\gamma < \sqrt{3\delta_m}$, the situation is very different. The stable fixed point is given by $\Omega_\phi = 1$ and $\delta_\phi = \gamma^2/3$. So, in this case, the asymptotic solution describes a scalar field dominated universe, which is accelerating whenever

$\gamma < \sqrt{2}$, i.e. $\xi_\chi \lesssim \frac{1}{2}$. Hence, the scalar field with exponential potential and $\gamma < \sqrt{3\delta_m}$ can potentially describe the late-time acceleration of the universe, provided that the system has not attained the fixed-point by today.[11]

In the previous section we have found that our model successfully describes inflation if $\xi_\chi \lesssim 4 \times 10^{-3}$. This yields the bound $\gamma \simeq 4\sqrt{\xi_\chi} \lesssim 0.25$. Therefore, in the absence of another dark energy component with $\delta < \gamma/\sqrt{3}$, the system evolves towards the second type of fixed point, corresponding to a scalar-field dominated universe in accelerated expansion. If a barotropic dark energy component with $\delta < \gamma/\sqrt{3}$ is present, the system tends towards the first type of fixed point, describing a universe dominated by this other type of dark energy. This allows us to draw a non-trivial conclusion. Namely, if the parameters of the model are fixed by the requirements of inflation, the late time behaviour of the system necessarily corresponds to an accelerating universe, dominated either by $\phi$ or a barotropic dark energy component, which in our model can be due to $\beta \neq 0$.

Current observations [27] show that at present the abundance of dark energy is $\Omega^0_{DE} \simeq 0.73$. Applied to our model, this constitutes an upper bound on the contribution of the field $\phi$ to the present energy density

$$\Omega^0_\phi \lesssim \Omega^0_{DE} \simeq 0.73 \ . \tag{2.142}$$

The first inequality becomes an equality if $\Omega^0_\phi$ is to be the only component of dark-energy. The observed bound shows that dark energy is not clearly dominating todays universe, which means that the system must not have reached its fixed-point yet.

We now want to qualitatively discuss the scenario in which the field $\phi$ is irrelevant during the radiation and matter dominated stages, but has become important recently and is now responsible for the present acceleration of the universe.[12] During the radiation and matter dominated stages one must have $\Omega_\phi \ll 1$. As long as this is the case, the second term on the right-hand side of (2.141) is small compared to the first one. Hence, $\delta_\phi$ is driven towards a very small value $\delta_\phi \ll 1$ and $\omega_\phi \simeq -1$. This shows that even if initially $\rho_\phi$ were dominated by kinetic energy, the kinetic part would soon die away and $\Omega_\phi$ become potential dominated. [13] As a consequence the value of $\phi$ is almost constant during the radiation and matter dominated epochs and remains practically equal to its value at the end of reheating. However, since $\rho_\phi$ decreases more slowly than the energy densities of radiation and matter ,$\Omega_\phi$ becomes relevant at some point. At this point $\phi$ starts rolling down the

[11] As has been shown in [108], this statement holds even if $\gamma > \sqrt{2}$.

[12] A detailed study of exactly this issue can be found in [108].

[13] In principle one could imagine a scenario in which after reheating $\Omega_\phi$ is non-negligible, as long as $\delta_\phi \simeq 1$. Since the kinetic part of $\rho_\phi$ decreases as $a^{-6}$ it would soon fall below $\rho_{radiation}$ and radiation would start dominating provided that the potential energy of $\phi$ is small enough. However, we do not expect this to happen in our model, because the field $\phi$ is almost constant during reheating.

potential and $\delta_\phi$ starts growing towards its attractor value. The initial value of $\Omega_\phi$ has to be small enough such that $\Omega_\phi$ remains negligible throughout radiation domination and only becomes important in the late matter dominated stage. The described scenario in which the quintessence field remains constant for a long time and then starts rolling down the potential goes under the name of "thawing quintessence" [109]. Two recent studies treating the case of an exponential potential can be found in [106] and [110].

If $\phi$ is not the main contribution to dark energy, $\Omega_\phi$ remains small until today. Hence, in that case $\delta_\phi$ remains closer to zero and the field $\phi$ evolves even less.

Combining (2.140) and (2.141) and in the approximation where $\delta_\phi \ll 1$ one can find the interesting relation [106]

$$\delta_\phi = 1 + \omega_\phi \simeq \frac{\gamma^2}{3} F(\Omega_\phi) \, , \tag{2.143}$$

where

$$F(\Omega_\phi) = \left[ \frac{1}{\sqrt{\Omega_\phi}} - \frac{1}{2} \left( \frac{1}{\Omega_\phi} - 1 \right) \ln \frac{1+\sqrt{\Omega_\phi}}{1-\sqrt{\Omega_\phi}} \right]^2 \, . \tag{2.144}$$

The function $F(\Omega_\phi)$ is monotonously growing. If the scalar field $\phi$ is the sole component of dark energy we can equate $\Omega_\phi^0 = \Omega_{DE}^0 \simeq 0.73$, for which one gets $F(\Omega_{DE}^0 = 0.73) = 0.5$, while $F(\Omega_{DE}^0 < 0.73) < 0.5$. Inserting this into (2.143) and using (2.129) we can obtain the following bound on the present equation of state parameter of $\phi$

$$\delta_\phi^0 = 1 + \omega_\phi^0 \lesssim \frac{8}{3} \frac{1}{\frac{1}{\xi_\chi} + 6} \, . \tag{2.145}$$

From the analysis of equations (2.140) and (2.141) we can understand that if dark energy is mainly due to a barotropic component, its value of $\delta$ cannot be bigger than $\delta_\phi$. Therefore, the bound (2.145) is at the same time a bound on the equation of state parameter of the total dark energy.

$$\delta_{DE}^0 = 1 + \omega_{DE}^0 \lesssim \frac{8}{3} \frac{1}{\frac{1}{\xi_\chi} + 6} \, . \tag{2.146}$$

The last two inequalities become equalities if $\Omega_\phi^0 = \Omega_{DE}^0$. Now, remember that in the analysis of inflation we have derived an upper bound $\xi_\chi < 0.0043$ (cf. eq. (2.116)). This bound coming from inflation implies the following bound on the equation of state parameter of dark energy (cf. figure 2.6)

$$0 \leq 1 + \omega_{DE}^0 \lesssim 0.01 \, . \tag{2.147}$$

In summary, we have found that the parameter bound from inflation implies a very strong bound on the equation of state parameter of dark energy. This is a rather non-trivial result.

Unfortunately, the current observational constraint $-0.24 < 1 + \omega_{DE}^0 < 0.04$ [27] is too weak to compete with this theoretical prediction. From this point of view, the energy density $\rho_{DE}$ is practically indistinguishable from a cosmological constant. Nevertheless, the observational bound is expected to improve considerably in the near future. A measurement precision at the percent level would make it possible to check the prediction of our model.

The theoretical prediction of the model can be further refined if $\phi$ is alone responsible for dark energy, i.e. $\Omega_\phi^0 = \Omega_{DE}^0$ and hence $\omega_\phi^0 = \omega_{DE}^0$. This is what happens in the special case $\beta = 0$. The fact that both the scalar spectral index $n_s^*$ and the equation of state parameter $\omega_{DE}^0$ depend mainly on $\xi_\chi$ makes it possible to establish a functional relation between these two very different observables. Namely, combining (2.146) (where inequality is replaced by equality) with the approximate relation (2.105) allows us to express the scalar tilt $n_s^*$ as a function of $\delta_{DE}^0$ and the number of e-folds $N^*$ as

$$n_s^*(\delta_{DE}^0) \simeq 1 - \frac{12\delta_{DE}^0}{4 - 9\delta_{DE}^0} \coth\left(\frac{6N^*\delta_{DE}^0}{4 - 9\delta_{DE}^0}\right) . \tag{2.148}$$

We plot this relation in figure 2.6 and compare it to the numerical result. The plot is equivalent to the plot of figure 2.4, except that the independent variable is changed from $\xi_\chi$ to $\omega_{DE}^0$ with the help of (2.143). As before, we see that the result is rather insensitive to variations of $N^*$ in the range $N_{min}^* < N^* < N_{max}^*$ (cf. (2.117) and (2.118)).

The formulas (2.147) and (2.148) are the main results of this section. Let us stress again that the link between the observable $n_s(k^*)$ related to inflation and $\omega_{DE}^0$ related to dark energy is a non-trivial prediction of the present model. On the other hand, one should also mention that this result relies on several important assumptions. In particular, the functional relation (2.148) is based on the requirement that the J-frame potential has a flat direction ($\beta = 0$).

Let us now show how the obtained results justify the assumption we made about inflation taking place in the scale-invariant region. From (2.147) we deduce that $\rho_\phi \simeq V_{QE}(\phi)$. This allows us to write the upper bound on $\Omega_\phi^0$ given in (2.142) as a lower bound on todays value of $\phi$

$$\phi_0 \gtrsim -\frac{1}{\gamma} M \ln\left(\xi_\chi^2 \frac{\Lambda_{eff}}{\Lambda_0}\right) , \tag{2.149}$$

where we have defined an effective cosmological constant as

$$\Lambda_{eff} \equiv 3M^2 H_0^2 \Omega_{DE}^0 \simeq 10^{-120} M^4 . \tag{2.150}$$

Numerical simulations show that the field $\phi$, respectively the field $\rho$ have been almost constant from the end of inflation till today. Therefore, the lower bound on $\phi_0$ provides an approximate lower bound on the value of $\rho$ at the end of inflation. In the analysis of inflation we have made the assumption

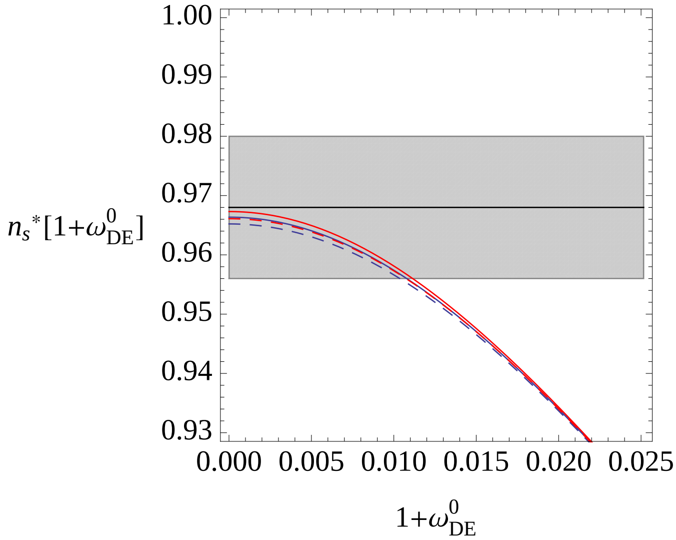



**Fig. 2.6:** This plot shows the approximate functional relationship between $n_s^*$ and $\omega_{DE}^0$. The plain curves are numerical results. The red plain curve is obtained for $\rho_{rh} = \rho_{rh}^{max}$ (instant reheating), while the blue plain curve represents the case $\rho_{rh} = \rho_{rh}^{min}$ (long reheating). The dashed curves are obtained from the approximate relation (2.148). The red one for $N^* = N^*_{max} = 59$ (instant reheating) and the blue one for $N^* = N^*_{min} = 57.5$ (long reheating). The shaded region represents the experimental mean value for $n_s^*$ and the associated errors (2.99).

that the whole period of observable inflation, i.e. the last $\sim 60$ e-folds, took place in the scale-invariant region, where $\upsilon_1, \upsilon_2 \ll 1$ and hence $\rho = cst.$. We can now check this assumption by computing $\upsilon_1$ and $\upsilon_2$ (defined in (2.64) and (2.65)) at $\rho = \rho^* = \rho_{end} \simeq \sqrt{\xi_\chi}\phi_0$. Using (2.149) and working in the usual approximation $\xi_\chi \ll 1$ and $\xi_h \gg 1$, we obtain

$$\upsilon_1 \lesssim 144 \xi_\chi^2 \xi_h^2 \frac{\Lambda_{eff}}{M^4} \sin^{-4}\theta \,, \tag{2.151}$$

$$\upsilon_2 \lesssim \frac{24 \xi_\chi \xi_h^2}{\lambda} \frac{\Lambda_{eff}}{M^4} \sin^{-2}\theta \cos^{-2}\theta \,. \tag{2.152}$$

From (2.102) and (2.104) we have $\theta_{end} \simeq 2 * 3^{\frac{1}{4}} \sqrt{\xi_\chi}$ and $\theta^* \simeq \arccos\left(e^{-4\xi_\chi N^*}\right)$. Evaluating the above bounds for values $\xi_\chi$, $\xi_h$ and $N^*$ of the orders of magnitude found in section 2.3.1 we find that for the whole interval $\theta_{end} < \theta < \theta^*$, $\upsilon_1, \upsilon_2 \lll 1$. This justifies a posteriori the neglecting of $\Lambda_0$ during inflation. Let us note that this conclusion is not altered if one takes into account the slight change of the scalar fields between the end of inflation and today. Both the change of $\rho$ during the oscillations and the change of $\phi$ during the thawing are at the percent level.

In section (2.3.1) we have seen that for our model to successfully describe inflation, the initial conditions for the scalar fields have to satisfy $\theta_{in} > \theta^*$, respectively $\frac{h_{in}}{\chi_{in}} \geq \sqrt{\frac{1+6\xi_\chi}{1+6\xi_h}} \tan\theta^*$, where $\theta^* \simeq \arccos\left(e^{-4\xi_\chi N^*}\right)$. We recall that for typical values $\xi_\chi = 0.003$, $\xi_h = 50000$ and $N^* = 60$ one obtains $\frac{h_{in}}{\chi_{in}} \gtrsim 0.004$. The observational bound on $\Omega_{DE}^0$ (2.142) (respectively (2.149)) together with the knowledge that the field $\rho$ remains almost constant from horizon crossing during inflation until today, allows us to further restrict the region of allowed initial conditions (cf. figure 2.7). Namely, as long as the initial conditions lie in the scale-invariant region ($\upsilon_1, \upsilon_2 \ll 1$), the bound (2.149) translates to

$$\rho_{in} \simeq \rho^* \simeq \rho_{end} \simeq \sqrt{\xi_\chi}\phi_0 \gtrsim -\frac{1}{4} M \ln\left(\xi_\chi^2 \frac{\Lambda_{eff}}{\Lambda_0}\right) \,. \tag{2.153}$$

In terms of the original variables $\chi$ and $h$ the same bound reads

$$\frac{\chi_{in}^2}{\Lambda_0^{1/2}} + 6\xi_h \frac{h_{in}^2}{\Lambda_0^{1/2}} \gtrsim \frac{1}{\xi_\chi} \frac{M^2}{\Lambda_{eff}^{1/2}} \sim 10^{60} \,. \tag{2.154}$$

Together with the bound $h_{in}/\chi_{in} \gtrsim 10^{-3}$ this shows that initial conditions have to approximately satisfy $h_{in}/\Lambda_0^{1/4} \gtrsim 10^{30}$. Hence, the initial value of $h$ has to be much larger than the arbitrary scale $\Lambda_0^{1/4}$. For $\phi$ to exactly produce the observed abundance of dark energy, the inequalities have to be replaced by equalities. In that case, the initial values have to be chosen very precisely on a line in the $(\rho, \theta)$-, respectively the $(\chi, h)$-plane. This tuning of initial conditions corresponds to the Cosmic Coincidence Problem described in section 1. Hence, our model does not alleviate this problem with respect to other quintessence models. If one allows for an additional dark energy component, the set of acceptable initial conditions extends to an infinite region. In our model the additional component is present if $\beta \neq 0$. In that case the fine-tuning issue does not concern the initial conditions, but the parameter $\beta$. Hence, in either case some "fine-tuning" is needed.

At this point it should be recalled that, although the Cosmic Coincidence Problem is an undesirable feature, it is not a consistency problem and therefore does not invalidate this and other models of dynamical dark energy.

Finally, we can briefly comment on the case of initial conditions lying in the region where $\Lambda_0$ cannot be neglected. Initial conditions lie in this region ($\upsilon_2 > 1$) whenever $\theta_{in}$ is sufficiently close to $\pi/2$, respectively when $h_{in}/\chi_{in}$ is sufficiently big. Note, however, that as a consequence of the bound (2.153) this only happens for extreme values $\pi/2 - \theta_{in} \lesssim 10^{-53}$, respectively $h_{in}/\chi_{in} \gtrsim 10^{53}$. In this region, $\rho$ is no longer constant. The E-frame potential (2.35) becomes dominated by the term proportional to $\Lambda_0$, i.e. $\tilde{V}(h,\chi) \simeq \frac{M^4\Lambda_0}{(\xi_\chi\chi^2+\xi_h h^2)^2}$. The effect of this potential is to drive the scalar fields to larger values of $\chi$ and $h$, respectively larger values of $\rho$, before they enter into the scale-invariant region. Qualitatively this means that if initial conditions are chosen in the non-scale-invariant region, the strong bound (2.153) (respectively (2.154)) gets relaxed. Still, the discussion related to the Cosmic Coincidence Problem equally applies to initial conditions in this region.

## 2.4 Summary

Let us recapitulate the results of this section.

- We consider a minimal scale-invariant extension of GR plus SM by introducing a dilaton field $\chi$. We give the general class of potentials for which SI is spontaneously broken and all mass scales at the classical level are induced. The physical dilaton is massless but hardly affects particle physics phenomenology. SI does not guarantee the absence of a cosmological constant. We give arguments as to why the parameter choice that forbids a cosmological constant ($\beta = 0$) is specially interesting.

- Replacing GR by UG gives rise to an arbitrary integration constant in the equations of motion. This constant does not act like a cosmological constant but rather like a non-trivial potential giving a small mass to the dilaton.

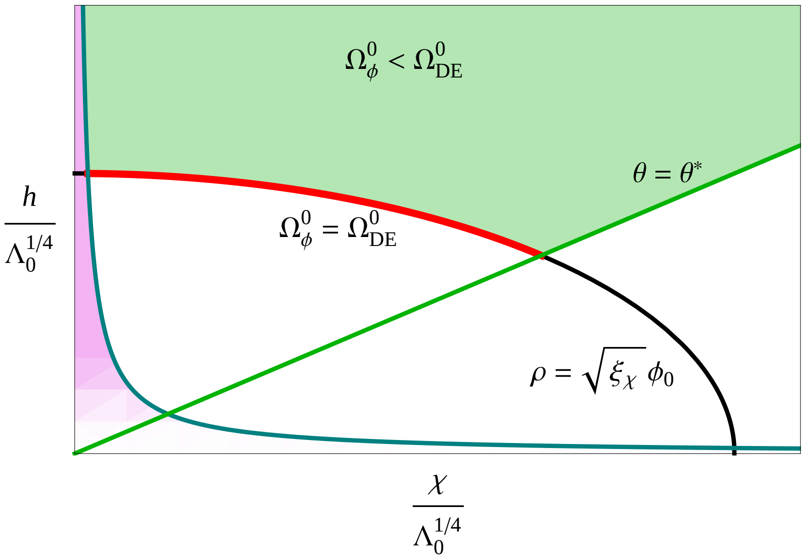



**Fig. 2.7:** This plot shows the different regions of initial conditions for the scalar fields, giving rise to qualitatively different evolutions. For a successful description of inflation, initial conditions have to lie above the green line $\theta = \theta^*$. For $\Lambda_0 > 0$, the scalar fields contribute to dark energy in the late stage. Initial conditions have to lie above the arc of an ellipse $\rho = \sqrt{\xi_\chi}\phi_0$, where $\phi_0 = -\frac{1}{\gamma} M \ln\left(\xi_\chi^2 \frac{\Lambda_{eff}}{\Lambda_0}\right)$, for this contribution not to exceed the observed value of $\Omega_{DE}^0$. Hence, the green region corresponds to initial conditions giving rise to successful inflation and a contribution to dark energy not exceeding $\Omega_{DE}^0$. The red segment of the ellipse corresponds to initial conditions for which the scalar fields yield the total observed dark energy. The blue hyperbola is given by $v_2 = 1$. Initial conditions below the hyperbola lie in the non-scale-invariant region, where $\Lambda_0$ is important. Trajectories starting here tend to move away from the origin before entering the scale-invariant region and following a scale-invariant trajectory. Therefore, such initial conditions (pink region) can also be acceptable as long as the corresponding trajectories enter the scale-invariant region above the line $\rho = \sqrt{\xi_\chi}\phi_0$. Note that, while we only describe the quadrant $\chi/\Lambda_0^{1/4}, h/\Lambda_0^{1/4} > 0$, the reasoning would be completely analog in the other quadrants.

- The constructed model of SI plus UG gives a rich cosmological phenomenology. For appropriate parameters and initial conditions, it provides both a mechanism for inflation and a mechanism for dark energy (The mechanism for inflation also works for $\Lambda_0 = 0$, i.e. SI plus GR). The observational bound on the scalar spectral index $n_s^*$ translates into the bound $-1 \leq \omega_{DE}^0 \lesssim -0.99$ on the equation of state parameter of dark energy, which is very close to a cosmological constant. In the case $\beta = 0$, dark energy is purely dynamical and our model predicts a functional relation between $n_s^*$ and $\omega_{DE}^0$.

- The presented model does neither solve nor alleviate the Cosmic Coincidence Problem.

- The findings of this section rely on the assumption that SI and the features of the potential can be maintained at the quantum level. A method for achieving this is given in section 4. We end up with a theory where all scales, including those coming from dimensional transmutation, are due to spontaneous breaking of SI.

# 3. Scale Invariance and TDiff Gravity – The Dilaton as a Part of the Metric

In the previous chapter we discussed the construction of scale-invariant theories for gravity and particle physics. We found that the spectrum of excitations around a symmetry-breaking classical ground state always contains a massless degree of freedom (Goldstone boson). This role can not be played by the SM Higgs field, as the Higgs boson should be massive. Hence, the realization of a phenomenologically viable scale-invariant extension of GR plus SM requires the addition of new degrees of freedom. We considered the minimal option of just adding one real scalar singlet, the dilaton $\chi$. From the point of view of the SM and standard GR the addition of the dilaton $\chi$ is somewhat ad hoc. In this chapter we want to study the possibility for the new scalar degree of freedom to appear in the gravitational sector of the theory. To this end, we consider the idea that the symmetry group of the gravitational action might not be the group of all diffeomorphisms (Diff) but rather its restriction to transverse diffeomorphisms (TDiff) [72, 74–78] given by

$$x^\mu \mapsto \tilde{x}^\mu(x), \text{ with } J \equiv \left|\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right| = 1, \tag{3.1}$$

which is generated by the subalgebra of transverse vectors,

$$x^\mu \mapsto x^\mu + \xi^\mu(x), \text{ with } \partial_\mu \xi^\mu = 0\ . \tag{3.2}$$

As mentioned in the introduction (section 1), from a field-theoretical point of view TDiff can be understood as the minimal symmetry group needed to construct a consistent theory of a Lorentz invariant symmetric second rank tensor [62, 76]. Under TDiff the metric determinant $g = \det g_{\mu\nu}$ transforms as a scalar. Hence, the action is much less restricted by TDiff than it would be by requiring Diff invariance. In particular, a TDiff theory[1] can in general contain arbitrary functions of the metric determinant in every term. We will refer to these functions as "Theory-Defining Functions" (TDF). For general choices of the TDF, TDiff gravity contains a new scalar degree of freedom,

[1] We will use the terms "Diff theory" and "TDiff theory" to refer to theories invariant under all diffeomorphisms (Diff), respectively invariant only under transverse diffeomorphisms (TDiff). By "TDiff gravity" we mean a TDiff theory containing the metric tensor as the only field.

on top of the massless graviton (the only exceptions are given by GR and UG [76]). Moreover, as has been argued in the past, a TDiff invariant theory is not equivalent to standard scalar-tensor gravity but rather to unimodular gravity plus a new scalar field [71, 76, 82]. This means that the equations of a TDiff theory also contain an arbitrary constant associated to initial conditions.

The goal of this chapter is to study the general properties of scale-invariant TDiff theories and to establish the conditions under which they can be phenomenologically viable. In particular, we show that replacing Diff gravity by TDiff gravity makes it possible to construct theories with spontaneously broken scale invariance, and that contain a massive scalar field, without the need of introducing new fields in the particle sector, as long as this sector contains at least one scalar field, e.g. the SM Higgs. The model discussed in the precedent chapter (SI plus UG, (2.24)) turns out to be a particular case of this new class of theories.

We start the chapter with a discussion of TDiff gravity theories and show that they are equivalent to unimodular scalar-tensor theories of gravity (section 3.1). The TDiff (rather than Diff) invariance leads to the existence of an extra parameter – $\Lambda_0$ – which is fixed as an initial condition. We introduce an equivalent Diff invariant reformulation of TDiff theories, which simplifies their analysis. In this formulation, $\Lambda_0$ represents a new coupling constant of a peculiar potential for the scalar field.

Next, the attention is turned to scale-invariant TDiff theories (SI TDiff), including scalar matter (section 3.2). It is shown that, assuming the metric to be dimensionless, the existence of scalar matter (which may be the Higgs boson of the SM) is necessary for the construction of a scale-invariant TDiff theory. After passing to the Einstein-frame, we identify the massless field (Dilaton) and the potentially massive field (Higgs boson). Loosely speaking, the metric determinant plays the role of the Higgs field, while the scalar matter field takes the part of the dilaton. In the Einstein-frame, the original scale invariance, existing in the Jordan-frame, is replaced by a shift symmetry of the dilaton field. As long as $\Lambda_0 = 0$, this symmetry is unbroken and the dilaton couples only derivatively to the Higgs boson. Hence, it easily avoids experimental bounds on the existence of a long-ranged 5th force. For $\Lambda_0 \neq 0$, the shift symmetry is broken by the presence of a new interaction term between the dilaton and the Higgs field. We want to give $\Lambda_0 \neq 0$ a cosmological interpretation and therefore neglect it in the discussion of the particle physics phenomenology.

Like in the previous chapter, we look for classical ground states (solutions that are constant in the particle physics sector), which may act as symmetry breaking ground states of the theory. In this more general framework, the situation is exactly as in the minimal model considered in the previous chapter. Classical ground states only exist for the choice $\Lambda_0 = 0$. For $\Lambda_0 = 0$ and depending on the TDF, there are three distinct types of solutions which may act as ground states of the theory. They all correspond to constant fields in the particle physics sector and have de Sitter, Minkowski or anti de Sitter geometry. So, even though the initial theory does not contain any dimensional parameters, and there is no explicit breaking of scale invariance for $\Lambda_0 = 0$, the theory generally contains a cosmological constant, associated with a definite dimensionless coupling. For comparison, in the minimal model of section 2.1 it was the parameter $\beta$, coefficient of a quartic interaction in the J-frame, which was responsible for the presence of a cosmological constant.

Most interesting for phenomenology is again the case where the TDF are chosen such that the cosmological constant is absent (analogous to $\beta = 0$).[2] We study it in detail. In particular, we discuss the choice of TDF, which leads to renormalizability in the particle physics sector of the theory. This results in a number of non-trivial constraints.

We then include fermions and gauge fields in our considerations (sections 3.3 and 3.4) and outline how the new framework can be applied to the Standard Model (section 3.5). In Section 3.6 we summarize the conditions to be put on the TDF, which lead to acceptable low energy theories. Moreover, we give two simple examples for TDF that satisfy all conditions. In one of the examples, the theory is exactly equivalent to the theory presented in the previous chapter.

Section 3.7 briefly discusses the case $\Lambda_0 \neq 0$ and cosmological applications. For appropriate choices of the TDF, the cosmological phenomenology is equivalent or very similar to the one of the model of the previous chapter. Namely, the *dynamical* break-down of SI due to the almost flat direction in the scalar potential gives a mechanism for inflation, while the *explicit* break-down of SI due to $\Lambda_0$ leads to dynamical dark energy. We find that also for $\Lambda_0 \neq 0$ the dilaton practically decouples and thus evades all experimental constraints.

## 3.1 TDiff invariant theories

The family of TDiff invariant actions built out of the metric $g_{\mu\nu}$ is very rich (this is to be contrasted with the uniqueness of the Einstein-Hilbert action in GR). The reason is that the metric determinant $g \equiv \det g_{\mu\nu}$ transforms like a scalar quantity under TDiff. As a consequence, the most general Lagrangian density for pure gravity invariant under TDiff and containing up to two derivatives can be written as

$$\mathcal{L}_{\text{TDiff}} = \sqrt{-g}\left(-\frac{1}{2}M^2 f(-g)R - \frac{1}{2}M^2\mathcal{G}(-g)g^{\mu\nu}\partial_\mu g\partial_\nu g - M^4 v(-g)\right) , \tag{3.3}$$

[2] Some theoretical arguments in favor of this choice were given in section 2.1.3 (cf. also [82]).

where $f(-g)$, $\mathcal{G}(-g)$ and $v(-g)$ are arbitrary functions which will be constrained by theoretical and phenomenological considerations (for GR: $f = const$, $v = const$, $\mathcal{G} = 0$; for UG: $f = (-g)^{-1/4}$, $v = 0$, $\mathcal{G} = -\frac{3}{32}(-g)^{-9/4}$). We will refer to them as the "Theory-Defining Functions" – TDF. The constant $M$ appearing in (3.3) can be an arbitrary mass scale. For definiteness we choose it to be the reduced Planck mass $M = (8\pi G)^{-1/2}$. Since $g_{\mu\nu}$ is a metric, its determinant has to be non-vanishing. Hence, here and from now on we adopt the condition $-g > 0$.

Before analyzing in detail the degrees of freedom described by the above theory, we are now going to show how any TDiff theory can equivalently be formulated in a Diff invariant way.

### 3.1.1 Equivalent Diff invariant theories

As we have seen in the particular case of UG (section 2.2), it proves very convenient to reformulate TDiff invariant theories as Diff invariant theories. In this section we will make use of the Stückelberg formalism to achieve this goal. Let us mention that the idea to reformulate a TDiff theory in a Diff invariant way has been considered before (e.g. in [65]). The Stückelberg formalism used here, as compared to other formalisms, makes it very easy to identify and track the new scalar degree of freedom. We start by considering the action (see also [65, 111, 112] for related considerations)

$$\mathcal{S}_e = \int \mathrm{d}^4x\sqrt{-g}\Bigg( -\frac{1}{2}M^2 f(-g/a)R - \frac{1}{2}M^2\mathcal{G}(-g/a)g^{\mu\nu}\partial_\mu(-g/a)\partial_\nu(-g/a) - M^4 v(-g/a) - \frac{\Lambda_0}{\sqrt{-g/a}}\Bigg), \tag{3.4}$$

where $\Lambda_0$ is an *arbitrary* constant and $a(x) > 0$ is a quantity to be explained shortly. This action is obtained by adding an arbitrary constant $\Lambda_0$ to the Lagrangian (3.3), which doesn't change the theory, and then transforming the associated action to an arbitrary coordinate frame. The quantity $a(x)$ is then identified as $a(x) = J(x)^{-2}$, where $J(x)$ is the Jacobian of the coordinate transformation. Hence, the action (3.4) is classically equivalent to (3.3) and the equations of motion for $g_{\mu\nu}$ still hold. Let us now promote $a(x)$ to a field (commonly called Stückelberg, Goldstone or Compensator field) and let it transform under Diff like the determinant of the metric, i.e. like a scalar density of weight 2. As a consequence, we have the identity

$$\int \mathrm{d}^4y\left(\frac{\delta\mathcal{S}_e}{\delta a(y)}\delta_\xi a(y) + \frac{\delta\mathcal{S}_e}{\delta g_{\mu\nu}(y)}\delta_\xi g_{\mu\nu}(y)\right) = 0, \tag{3.5}$$

where

$$\delta_\xi a = \xi^\mu\partial_\mu a + 2a\partial_\mu\xi^\mu, \quad \delta_\xi g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu. \tag{3.6}$$

Using the equations of motion for the metric, the previous identity is reduced to

$$\int \mathrm{d}^4y\,\sqrt{a}\,\xi^\mu\partial_\mu\left(\sqrt{a}\frac{\delta\mathcal{S}_e}{\delta a}\right)=0. \tag{3.7}$$

This identity is valid for arbitrary $\xi^\mu$ and hence

$$\frac{\delta\mathcal{S}_e}{\delta a}=\frac{C_0}{\sqrt{a}}\,, \tag{3.8}$$

where $C_0$ is an arbitrary integration constant. The left-hand side of these equations contains a term proportional to $\Lambda_0$, which has exactly the same form as the term on the right-hand side. Hence, the term of the right-hand side can always be absorbed by a redefinition of the arbitrary constant $\Lambda_0$, resulting in

$$\frac{\delta\mathcal{S}_e}{\delta a}=0\,. \tag{3.9}$$

This is enough to prove that the equations of motion derived from (3.4), considering $g_{\mu\nu}$ and $a$ as independent fields, are equivalent to those derived from (3.3). By construction, the new equations of motion have an additional local (gauge) symmetry. In the gauge $a=1$ the solutions of the new equations are exactly the same as those gotten from (3.3). Solutions derived in a gauge $a\neq 1$ also correspond to the solutions of (3.3), however now written in different coordinates. We will refer to the action (3.4) as the "equivalent Diff invariant theory".

Let us now define the field

$$\sigma\equiv -g/a>0,$$

which is a scalar under all diffeomorphisms, and rewrite the Lagrangian as

$$\mathcal{L}_e=\sqrt{-g}\left(-\frac{1}{2}M^2f(\sigma)R-\frac{1}{2}M^2\mathcal{G}(\sigma)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma-M^4v_{\Lambda_0}(\sigma)\right)\,, \tag{3.10}$$

where

$$v_{\Lambda_0}(\sigma)=M^4v(\sigma)+\frac{\Lambda_0}{\sqrt{\sigma}}.$$

The theory formulated this way reduces to (3.3) after imposing the gauge condition $-g=\sigma$ (corresponding to $a=1$). For any other gauge conditions with $-g\neq\sigma$ (which may be more convenient for other reasons), it still corresponds to the original TDiff theory but written in new coordinates related to the original ones by a transformation with Jacobian $J\neq 1$. Let us note at this point that, by construction, the field $\sigma$ can only take positive values.

The appearence of a new parameter $\Lambda_0$ is a general feature of TDiff theories. For $f(-g)=(-g)^{-1/4}*cst.$ (like in pure UG) it plays the role of a cosmological constant. In all other cases, $\Lambda_0$ leads to a new specific potential term for the scalar field $\sigma$. We would like to stress once more that $\Lambda_0$ is a parameter characterizing the solution of the equations of motion and is not a fundamental coupling constant in the action (3.3). Note that the value of $\Lambda_0$ is not determined by specifying initial conditions for the propagating local degrees of freedom. Instead, it should be understood as an additional initial condition. The one-to-one correspondence between the two formulations (3.10) and (3.3) is given by the fact that the single Lagrangian (3.3) corresponds to a whole family of Lagrangians (3.10) with different values of $\Lambda_0$. It is the appearance of the arbitrary constant $\Lambda_0$ that distinguishes TDiff gravity from a standard scalar-tensor gravity. Let us note that the difference between TDiff gravity and standard scalar-tensor gravity is similar to the difference between UG and GR. This difference, unimportant for local physics (sections 3.2-3.6), can be crucial for cosmology, as will be discussed in section 3.7.

In the following, when analyzing different aspects of TDiff theories, we will often make use of the equivalent Diff invariant formulation.

### 3.1.2 Classical ground states and local degrees of freedom

The Lagrangian (3.10) has the form of a scalar-tensor theory with a non-minimal coupling between the scalar field and the tensorial degrees of freedom. In order to uncover the nature of the local degrees of freedom, it is convenient to write the theory in the Einstein-frame, i.e. to redefine the metric such that the scalar field becomes minimally coupled. Whenever $f(\sigma) \neq 0$, we can define the E-frame metric

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu} \, , \\ \tilde{g}^{\mu\nu} &= \Omega^{-2} g^{\mu\nu} \, , \\ \Omega^2 &= f(\sigma) \, , \end{aligned} \tag{3.11}$$

in terms of which the Lagrangian (3.10) reads

$$\mathcal{L}_e = \sqrt{-\tilde{g}} \left( -\frac{1}{2} M^2 \tilde{R} - \frac{1}{2} \mathcal{K}(\sigma) M^2 \tilde{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V_{\Lambda_0}(\sigma) \right) \, , \tag{3.12}$$

where

$$\mathcal{K}(\sigma) = \frac{\mathcal{G}(\sigma)}{f(\sigma)} + \frac{3}{2} \left( \frac{f'(\sigma)}{f(\sigma)} \right)^2 \, , \quad V_{\Lambda_0}(\sigma) = \frac{v_{\Lambda_0}(\sigma)}{f(\sigma)^2} \, . \tag{3.13}$$

Let us explain the meaning of the hypothesis $f(\sigma) \neq 0$. In the theories given by (3.3), the gravitational coupling is induced by $f(-g)$, respectively $f(\sigma)$ . We will be interested in excitations $h_{\mu\nu}$ around a background solution $\bar{g}_{\mu\nu}$, defined through $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Next, we will require the existence of background solutions $\bar{g}_{\mu\nu}$ for which $f(\bar{g}) \neq 0$, yielding a non-zero induced gravitational coupling. In order to analyze the theory in the vicinity of such a background, one can always perform a well-defined transformation to the Einstein-frame.

As a first step in the analysis, we want to look for the classical ground states of the theory. The term "classical ground state" was introduced in section 2.1 and refers to a solution of the classical field equations, constant in the particle physics sector and with maximally symmetric spacetime, i.e. Minkowski (flat), de Sitter (dS) or Anti de Sitter (AdS) spacetime. As argued in section 2.1, the existence of such solutions might be crucial for a successful quantization of the theory. By doing the analysis in the E-frame, we implicitly make the assumption that all classical ground states fulfill the condition

$$f(\sigma_0) \neq 0 \,.$$

Only if this condition holds, the induced gravitational coupling is non-zero. The classical ground states of (3.12) correspond to the extrema of the potential $V_{\Lambda_0}(\sigma)$. We first consider the particular case where $f(\sigma)$ is such that there exists a value $\sigma_0$, for which $f(\sigma_0) + 4\sigma_0 f'(\sigma_0) = 0$. (Note that if $f(\sigma) = \sigma^{-1/4} * cst.$, like in pure UG, this is true for all values of $\sigma$.) In this case, a classical ground state exists, provided that $v(\sigma)$ satisfies the relation $v(\sigma_0) + 2\sigma_0 v'(\sigma_0) = 0$. The classical ground state is given by $\sigma = \sigma_0$ and $\tilde{R} = -4V_{\Lambda_0}M^{-2}$. Depending on the value of $\Lambda_0$, the corresponding maximally symmetric spacetime is flat, dS or AdS.

All values of $\sigma_0$ for which $f(\sigma_0) + 4\sigma_0 f'(\sigma_0) \neq 0$ correspond to classical ground states, given by

$$\begin{aligned} \sigma = \sigma_0, \quad \tilde{R} &= -4M^2 \frac{v(\sigma_0) + 2\sigma_0 v'(\sigma_0)}{f(\sigma_0)(f(\sigma_0) + 4\sigma_0 f'(\sigma_0))}, \\ \Lambda_0 &= 2M^4 \sigma_0^{3/2} \frac{f(\sigma_0)v'(\sigma_0) - 2f'(\sigma_0)v(\sigma_0)}{f(\sigma_0) + 4\sigma_0 f'(\sigma_0)} \,. \end{aligned} \tag{3.14}$$

Depending on the TDF, these solutions allow for flat, dS or AdS spacetime.

To summarize, we have found that whenever there exists a value $\sigma_0$ for which

$$f(\sigma_0) \neq 0 \tag{3.15}$$

and

$$v(\sigma_0) + 2\sigma_0 v'(\sigma_0) = 0 \,, \tag{3.16}$$

the theory possesses a classical ground state which induces a non-zero gravitational constant and which corresponds to flat spacetime. If in addition to this one has $f(\sigma_0) + 4\sigma_0 f'(\sigma_0) = 0$, the ground state can also be dS or AdS. If condition (3.15) does not hold, there exists no classical ground state with induced gravitational constant. If condition (3.16) does not hold, spacetime in the classical ground state is dS or AdS, except if $f(\sigma) = \sigma^{-1/4} * cst.$ in which case there exists no classical ground state with induced gravitational constant.

Let us now find the propagating degrees of freedom. To this end, we focus on the case where both conditions (3.15) and (3.16) hold, and hence the theory possesses the solution

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} \;, \quad \sigma = \sigma_0, \quad \Lambda_0 = -M^4 v(\sigma_0)\sqrt{\sigma_0} \;. \tag{3.17}$$

We introduce the perturbations around this background as

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \frac{\tilde{h}_{\mu\nu}}{M} \;, \quad \sigma = \sigma_0 + \frac{\varsigma}{M} \;, \tag{3.18}$$

and find the quadratic part of the Lagrangian (3.12) to be

$$\mathcal{L}_e^Q = \frac{1}{2}\tilde{\mathcal{L}}_{GR}^Q - \frac{1}{2}\mathcal{K}^{(0)}\eta^{\mu\nu}\partial_\mu\varsigma\partial_\nu\varsigma - \frac{1}{2}V_{\Lambda_0}^{(2)}M^2\varsigma^2 \;. \tag{3.19}$$

Here, and in the rest of this work, we use the notation

$$F^{(n)} \equiv \frac{d^n F(\sigma)}{d\sigma^n}\Big|_{\sigma=\sigma 0}$$

for the derivatives of functions evaluated at the background field value . The first term in (3.19) is the standard quadratic Einstein-Hilbert Lagrangian

$$\tilde{\mathcal{L}}_{GR}^Q = -\frac{1}{4}\partial_\rho\tilde{h}_{\mu\nu}\partial^\rho\tilde{h}^{\mu\nu} + \frac{1}{2}\partial^\nu\tilde{h}_{\mu\nu}\partial^\rho\tilde{h}^\mu_\rho - \frac{1}{2}\partial_\mu\tilde{h}\partial_\nu\tilde{h}^{\mu\nu} + \frac{1}{4}\partial_\mu\tilde{h}\partial^\mu\tilde{h} \;, \tag{3.20}$$

where indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$ and $\tilde{h} \equiv \tilde{h}^\mu_\mu$. This term describes two massless tensor degrees of freedom. From (3.19) one can see that, whenever $\mathcal{K}^{(0)} \equiv \mathcal{K}(\sigma_0) \neq 0$, the theory contains a propagating scalar degree of freedom.[3] In that case, the scalar part of the Lagrangian can be brought to canonical form by defining the canonical field

$$\varsigma_c = \sqrt{|\mathcal{K}^{(0)}|}\varsigma \;. \tag{3.21}$$

We get

$$\mathcal{L}_e^Q = \frac{1}{2}\tilde{\mathcal{L}}_{EH}^Q - \epsilon_\varsigma\frac{1}{2}\partial_\mu\varsigma_c\partial^\mu\varsigma_c - \frac{m_\varsigma^2}{2}\varsigma_c^2 \;, \tag{3.22}$$

where

$$\epsilon_\varsigma \equiv \mathrm{sign}\left(\mathcal{K}^{(0)}\right), \quad m_\varsigma^2 \equiv \epsilon_\varsigma\frac{V_{\Lambda_0}^{(2)}}{\mathcal{K}^{(0)}}M^2 \;. \tag{3.23}$$

[3] For Einstein's theory of gravity, $f(-g) = 1$ and $\mathcal{G}(-g) = v(-g) = 0$. Hence, $\mathcal{K}^{(0)} = V_{\Lambda_0}^{(2)} = 0$ and the theory only contains the two massless tensor degrees of freedom.

The perturbations around the background (3.17) are well-behaved, provided that:

- The scalar field $\varsigma_c$ has a positive definite kinetic term (absence of ghosts): $\mathcal{K}^{(0)} > 0$.
- The field $\varsigma_c$ has positive or zero mass (absence of tachyons): $V_{\Lambda_0}^{(2)} \geq 0$.

On top of the terms quadratic in the perturbations, there is obviously a series of interaction terms. We will get interested in those terms in the upcoming sections, where we will consider different types of fields coupled to TDiff gravity.

## 3.2 Scale-invariant TDiff theories

In this section, we focus on scale-invariant TDiff theories including scalar matter fields only. Other SM fields will be introduced in the subsequent sections.

Clearly, the Lagrangian (3.3) cannot be used to construct a scale-invariant theory – putting the mass scale $M$ to zero makes it vanish.[4] A minimal way to overcome this difficulty is to introduce a real scalar field $\phi$ coupled to TDiff gravity. Eventually, $\phi$ might be replaced by the SM Higgs field. The generalization of Lagrangian (3.3) to this case is given by

$$\begin{aligned}\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2}\phi^2 f(-g)R - \frac{1}{2}\phi^2\mathcal{G}_{gg}(-g)(\partial g)^2 - \frac{1}{2}\mathcal{G}_{\phi\phi}(-g)(\partial\phi)^2 \\ + \mathcal{G}_{g\phi}(-g)\phi\,\partial g\cdot\partial\phi - \phi^4 v(-g)\ .\end{aligned} \tag{3.24}$$

Here, and in many of the upcoming expressions, in order to shorten notations, we no longer write Lorentz indices explicitly. The implicit contractions of Lorentz indices are done with the metric $g_{\mu\nu}$, when the Lagrangian is written in the J-frame, and with $\tilde{g}_{\mu\nu}$, when it is written in the E-frame. The action (3.24) is invariant, by construction, under scale transformations of the form

$$\phi(x) \mapsto \lambda\phi(\lambda x), \quad g_{\mu\nu}(x) \mapsto g_{\mu\nu}(\lambda x)\ . \tag{3.25}$$

Using the Stückelberg formalism illustrated in section 3.1.1, we can directly write down the equivalent Diff invariant theory of (3.24) as

$$\begin{aligned}\frac{\mathcal{L}_e}{\sqrt{-g}} = -\frac{1}{2}\phi^2 f(\sigma)R - \frac{1}{2}\phi^2\mathcal{G}_{gg}(\sigma)(\partial\sigma)^2 - \frac{1}{2}\mathcal{G}_{\phi\phi}(\sigma)(\partial\phi)^2 \\ - \mathcal{G}_{g\phi}(\sigma)\phi\ \partial\sigma\cdot\partial\phi - \phi^4 v(\sigma) - \frac{\Lambda_0}{\sqrt{\sigma}}\ .\end{aligned} \tag{3.26}$$

[4] We make the assumption that the metric is dimensionless, i.e. has zero scaling-dimension. (See also discussion in [84].)

A non-zero $\Lambda_0$ breaks the invariance of $\mathcal{L}_e$ under (3.25). This type of symmetry breaking is a consequence of replacing Diff gravity by TDiff gravity. The situation is completely analog to the case of scale-invariant models with UG (instead of GR) considered in chapter 2.

As long as $\phi^2 f(\sigma) > 0$, the Lagrangian (3.26) can be transformed to the E-frame with the help of the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu} \,, \quad \Omega^2 = \frac{\phi^2 f(\sigma)}{M^2} \,. \tag{3.27}$$

It takes the form

$$\begin{aligned} \frac{\mathcal{L}_e}{\sqrt{-\tilde{g}}} = &-\frac{1}{2}M^2\tilde{R} - \frac{1}{2}M^2\mathcal{K}_{\sigma\sigma}(\sigma)(\partial\sigma)^2 - \frac{1}{2}M^2\mathcal{K}_{\phi\phi}(\sigma)(\partial\ln(\phi/M))^2 \\ &- M^2\mathcal{K}_{\sigma\phi}(\sigma)\,\partial\sigma\cdot\partial\ln(\phi/M) - M^4V(\sigma) - \frac{M^4\Lambda_0}{\phi^4 f(\sigma)^2\sqrt{\sigma}} \,, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} \mathcal{K}_{\sigma\sigma}(\sigma) &= \frac{\mathcal{G}_{gg}(\sigma)}{f(\sigma)} + \frac{3}{2}\left(\frac{f'(\sigma)}{f(\sigma)}\right)^2 \quad , \quad \mathcal{K}_{\phi\phi}(\sigma) = \frac{\mathcal{G}_{\phi\phi}(\sigma)}{f(\sigma)} + 6 \,, \\ \mathcal{K}_{\sigma\phi}(\sigma) &= \frac{\mathcal{G}_{g\phi}(\sigma)}{f(\sigma)} + 3\frac{f'(\sigma)}{f(\sigma)} \quad , \quad V(\sigma) = \frac{v(\sigma)}{f(\sigma)^2} \,. \end{aligned} \tag{3.29}$$

The kinetic term for the scalar fields can be diagonalized by redefining the fields as[5]

$$\begin{aligned} \tilde{\sigma} &= \int_{\sigma_0}^{\sigma} d\sigma' \sqrt{\left|\frac{\mathcal{K}_{\sigma\sigma}(\sigma')\mathcal{K}_{\phi\phi}(\sigma') - \mathcal{K}_{\sigma\phi}(\sigma')^2}{\mathcal{K}_{\phi\phi}(\sigma')}\right|} \,, \\ \tilde{\phi} &= M\left(\ln\frac{\phi}{M} + \int_{\sigma_0}^{\sigma} d\sigma' \frac{\mathcal{K}_{\sigma\phi}(\sigma')}{\mathcal{K}_{\phi\phi}(\sigma')}\right) . \end{aligned} \tag{3.30}$$

Note that we chose the integration constant such that $\tilde{\sigma}(\sigma_0) = 0$ and kept $\sigma_0$ arbitrary for the moment. After this field redefinition, which is always solvable in perturbation theory, the Lagrangian simplifies to

$$\begin{aligned} \frac{\mathcal{L}_e}{\sqrt{-\tilde{g}}} = &-\frac{1}{2}M^2\tilde{R} - \frac{1}{2}\epsilon_\sigma M^2(\partial\tilde{\sigma})^2 - \frac{1}{2}\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})(\partial\tilde{\phi})^2 \\ &- M^4\tilde{V}(\tilde{\sigma}) - \Lambda_0\,\tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma})\exp\left(-\frac{4\tilde{\phi}}{M}\right) \,, \end{aligned} \tag{3.31}$$

[5] In view of the conditions (3.42), we assume that both $\mathcal{K}_{\phi\phi}(\sigma)$ and $\frac{\mathcal{K}_{\sigma\sigma}(\sigma)\mathcal{K}_{\phi\phi}(\sigma) - \mathcal{K}_{\sigma\phi}(\sigma)^2}{\mathcal{K}_{\phi\phi}(\sigma)}$ are non-zero.

where $\epsilon_\sigma = \text{sign}\left(\frac{\mathcal{K}_{\sigma\sigma}(\sigma)\mathcal{K}_{\phi\phi}(\sigma)-\mathcal{K}_{\sigma\phi}(\sigma)^2}{\mathcal{K}_{\phi\phi}(\sigma)}\right)$, and the different functions are obtained by expressing $\sigma$ as a function of $\tilde{\sigma}$,

$$\tilde{V}(\tilde{\sigma}) = V(\sigma), \quad \tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma}) = \mathcal{K}_{\phi\phi}(\sigma), \quad \tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma}) = \frac{\exp\left(4\int_{\sigma_0}^{\sigma} d\sigma' \frac{\mathcal{K}_{\sigma\phi}(\sigma')}{\mathcal{K}_{\phi\phi}(\sigma')}\right)}{f(\sigma)^2\sqrt{\sigma}} \ . \tag{3.32}$$

The Lagrangian (3.31) is the main result of this section. It is invariant under global shifts of the dilaton field $\tilde{\phi} \mapsto \tilde{\phi}+\lambda$ (except for the term proportional to $\Lambda_0$) which is the Einstein-frame manifestation of scale invariance in the Jordan-frame. For $\Lambda_0 \neq 0$, scale invariance is broken by a new potential term, which is of the "run-away" type in the dilaton direction. Note that the dependence of the potential on the dilaton $\tilde{\phi}$ is uniquely determined by the way scale invariance is broken in TDiff theories. If $\Lambda_0 = 0$, the shift symmetry is exact. Hence, in this case, the dilaton is exactly massless and interacts with the matter field only through derivatives.

### 3.2.1 Classical ground states and local degrees of freedom

In this subsection, we repeat the analysis of Section 3.1.2, but now for the SI TDiff theories given by (3.24). In order to determine the classical ground states, it is most convenient to consider the Lagrangian in the form (3.28). Here, similar to the case of pure TDiff gravity (section 3.1.2), by looking for classical ground states in the E-frame, we implicitly impose the condition that they all satisfy

$$\phi_0^2 f(\sigma_0) \neq 0 \ . \tag{3.33}$$

In scale-invariant TDiff theories, this is the condition for a non-zero induced gravitational coupling. The non-zero value of $\phi_0$ will at the same time induce all other scales of the theory. This fact is easier to see in the Jordan-frame. There, expanding around a constant background, one finds that all dimensional couplings are proportional to $\phi_0$. In the Einstein-frame, the same fact is implicit, since the transformation to the Einstein-frame is only allowed when $\phi_0 \neq 0$.

Constant solutions for the scalar fields have to be extrema of the potential. One can see immediately that such solutions necessarily correspond to $\Lambda_0 = 0$. This is the only case in which the derivative of the potential with respect to $\phi$ vanishes. The second condition for the existence of a constant solution is that there exists a value $\sigma_0$, for which

$$V'(\sigma_0) = 0 \ ,$$

or, in terms of the original TDF,

$$f(\sigma_0)v'(\sigma_0) - 2v(\sigma_0)f'(\sigma_0) = 0 \ . \tag{3.34}$$

Hence, whenever the conditions (3.33) and (3.34) hold, the theory possesses an infinite family of classical ground states satisfying

$$\sigma = \sigma_0, \quad \phi = \phi_0, \quad \Lambda_0 = 0, \quad \tilde{R} = -4M^2 \frac{v(\sigma_0)}{f(\sigma_0)^2} , \tag{3.35}$$

where $\phi_0$ is an arbitrary non-zero constant, inducing all scales of the theory and therefore breaking scale invariance spontaneously. The degeneracy of the classical ground states is a consequence of scale invariance. For $v(\sigma_0) \neq 0$, the classical ground states correspond to dS or AdS spacetime, while for $v(\sigma_0) = 0$, they correspond to flat spacetime. If $v(\sigma_0) = 0$, the condition (3.34) reduces to $v'(\sigma_0) = 0$. In the present case, requiring the existence of a constant solution with flat spacetime imposes two conditions (instead of one for pure TDiff gravity) on the function $v(\sigma)$, namely, that there exists a value $\sigma_0$, for which

$$v(\sigma_0) = v'(\sigma_0) = 0 . \tag{3.36}$$

For the rest of the discussion in this chapter, except for section 3.7, we will impose the *three* conditions (3.33) and (3.36) on the TDF. These conditions guarantee that the theory possesses an infinite family of symmetry-breaking classical ground states, which induce all scales through $\phi_0 \neq 0$, and which correspond to flat spacetime. Let us stress that for the existence of non-flat symmetry-breaking ground states, the *two* conditions (3.33) and (3.34) would be enough. The additional condition $v(\sigma_0) = 0$ corresponds to requiring the absence of a cosmological constant. Imposing this condition is exactly analog to requiring $\beta = 0$ in the minimal scale-invariant model of section 2.1. Some arguments in favor of the case $v(\sigma_0) = 0$ ($\beta = 0$) compared to the cases where $v(\sigma_0) \neq 0$ ($\beta \neq 0$) were given in section 2.1.3 (see also [82]). Still, one should keep in mind that, at the moment, these arguments are rather speculative and that $v(\sigma_0) \neq 0$ might be perfectly acceptable from a theoretical point of view. In either case, for the model to display a viable cosmological phenomenology, the term proportional to $v(\sigma_0)$ has to be extremely small compared to the particle physics scales. Therefore, allowing for $v(\sigma_0) \neq 0$ would not change the discussion of particle physics phenomenology in the upcoming sections.

For (3.33) and (3.36) fulfilled, the theory taken in the form (3.28) possesses the family of constant solutions

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu}, \quad \tilde{\sigma} = 0, \quad \tilde{\phi} = \tilde{\phi}_0, \quad \Lambda_0 = 0 . \tag{3.37}$$

Let us look at the nature of perturbations around such a symmetry-breaking background.

We define the perturbations as

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= \eta_{\mu\nu} + \frac{\tilde{h}_{\mu\nu}}{M} \,, \\
\tilde{\sigma} &= \frac{\varsigma}{M} \,, \\
\tilde{\phi} &= \tilde{\phi}_0 + \frac{\varphi}{\sqrt{\left|\tilde{\mathcal{K}}^{(0)}_{\phi\phi}\right|}} \,.
\end{aligned} \tag{3.38}$$

In the rest of section 3.2, Lorentz indices are raised, lowered and contracted with the Minkowski metric $\eta_{\mu\nu}$. The Lagrangian can be split into a term quadratic in the perturbations and an interaction term as

$$\mathcal{L}_e = \mathcal{L}_e^Q + \mathcal{L}_e^{(\text{int})} \,. \tag{3.39}$$

For the quadratic term we get

$$\mathcal{L}_e^Q = \tilde{\mathcal{L}}^Q_{GR} - \epsilon_\varsigma \frac{1}{2}(\partial\varsigma)^2 - \epsilon_\varphi \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varsigma^2 \varsigma^2 \,, \tag{3.40}$$

where we have defined

$$\begin{aligned}
\epsilon_\varsigma &\equiv \text{sign}\left(\frac{\mathcal{K}^{(0)}_{\sigma\sigma}\mathcal{K}^{(0)}_{\phi\phi} - \left(\mathcal{K}^{(0)}_{\sigma\phi}\right)^2}{\mathcal{K}^{(0)}_{\phi\phi}}\right), \quad \epsilon_\varphi \equiv \text{sign}\left(\mathcal{K}^{(0)}_{\phi\phi}\right) \,, \\
m_\varsigma^2 &\equiv \epsilon_\varsigma \tilde{V}^{(2)} M^2 = \epsilon_\varsigma \left(\frac{\mathcal{K}^{(0)}_{\sigma\sigma}\mathcal{K}^{(0)}_{\phi\phi} - \left(\mathcal{K}^{(0)}_{\sigma\phi}\right)^2}{\mathcal{K}^{(0)}_{\phi\phi}}\right)^{-1} V^{(2)} M^2 \,.
\end{aligned} \tag{3.41}$$

In this case, on top of the two tensorial massless degrees of freedom, the theory contains *two* scalar degrees of freedom, among which at least one is massless. We have the following criteria for the perturbations to be well-behaved:

- For positive definite kinetic terms (absence of ghosts):

$$\mathcal{K}^{(0)}_{\sigma\sigma}\mathcal{K}^{(0)}_{\phi\phi} - \left(\mathcal{K}^{(0)}_{\sigma\phi}\right)^2 > 0 \quad \text{and} \quad \mathcal{K}^{(0)}_{\phi\phi} > 0 \,. \tag{3.42}$$

- For positive or zero mass of $\varsigma_c$ (absence of tachyons):

$$V^{(2)} \geq 0 \,. \tag{3.43}$$

These conditions could equivalently be formulated in a variable-independent way. The first two conditions correspond to a positive definite field space metric to lowest order in the expansion around the constant background.

Requiring that the matrix of second derivatives of the potential evaluated at the constant background solution should be positive semidefinite is the analog of the third condition.

In the following section, we will see that the massless field $\varphi$ is only derivatively coupled to $\sigma$ and, moreover, that these couplings only appear in higher dimensional operators. Hence, the effects of $\varphi$ at low energies are naturally suppressed.

### 3.2.2 Interactions and separation of scales

We now want to include the interactions contained in the Lagrangian (3.31). In general, it contains an infinite series of interaction terms arising from the expansion of the functions $\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})$ and $\tilde{V}(\tilde{\sigma})$ and of the metric tensor around the constant background. The interaction terms obtained from the expansion of the Ricci scalar in (3.31) are suppressed by the Planck mass. We neglect them, as we are only interested in sub-Planckian processes. Let us first consider the terms of dimension up to four,

$$\mathcal{L}_e^{\text{int}\leq 4} = -\frac{1}{3!}\kappa_\varsigma\varsigma^3 - \frac{\lambda_\varsigma}{4!}\varsigma^4 - \frac{1}{4}\frac{m_\varsigma^2}{M}\varsigma^2\tilde{h} - \frac{1}{16}\frac{m_\varsigma^2}{M^2}\varsigma^2\left(\tilde{h}^2 - 2\tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu}\right) - \frac{1}{12}\frac{\kappa_\varsigma}{M}\varsigma^3\tilde{h} \ , \tag{3.44}$$

where

$$\kappa_\varsigma \equiv \tilde{V}^{(3)}M, \qquad \lambda_\varsigma \equiv \tilde{V}^{(4)} \ . \tag{3.45}$$

These are the relevant operators for a scalar field minimally coupled to Einstein gravity. Having in mind the identification of the field $\varsigma$ with a low-energy degree of freedom (such as the Higgs boson of the SM), the TDF must obey several constraints. In particular, one has to require that the scales $m_\varsigma$ and $\kappa_\varsigma$ are much smaller than the gravitational scale $M$, i.e.

$$\left|\frac{m_\varsigma}{M}\right| = \sqrt{\left|\tilde{V}^{(2)}\right|} \ll 1, \qquad \left|\frac{\kappa_\varsigma}{M}\right| = \left|\tilde{V}^{(3)}\right| \ll 1 \ . \tag{3.46}$$

The conditions (3.46) are similar to the ”fine-tuning” conditions of the SM, fixing the hierarchy between the Fermi scale and the Planck scale.

In order to have a weakly coupled theory, we also need to have $\frac{\kappa_\varsigma}{m_\varsigma}, \lambda_\varsigma \lesssim 1$, which means

$$\frac{|\tilde{V}^{(3)}|}{\sqrt{|\tilde{V}^{(2)}|}}, \ \tilde{V}^{(4)} \lesssim 1 \ . \tag{3.47}$$

In general, for a viable effective field theory at energies smaller than the Planck scale, the corrections to Lagrangian (3.44) originating from power expansions of the TDF must be suppressed (see however section 3.2.3). The higher-dimensional operators can be written schematically as

$$\begin{aligned}\mathcal{L}_e^{\text{int}>4} = & \sum_{n_h>0}^{\infty} \frac{1}{M^{n_h}} \left(\mathcal{L}_e^Q + \mathcal{L}_e^{\text{int}\leq 4}\right) \tilde{h}^{n_h} \\ & + \sum_{\substack{n_h\geq 0\\ n_\varsigma>0}}^{\infty} \left(\frac{1}{M_{\phi\phi}(n_h, n_\varsigma)}\right)^{n_h+n_\varsigma} (\partial\varphi)^2 \tilde{h}^{n_h} \varsigma^{n_\varsigma} \\ & + \sum_{\substack{n_h\geq 0\\ n_\varsigma>4}}^{\infty} \left(\frac{1}{M_V(n_h, n_\varsigma)}\right)^{n_h+n_\varsigma-4} \tilde{h}^{n_h} \varsigma^{n_\varsigma} ,\end{aligned} \tag{3.48}$$

where we neglect numerical factors of order one, neglect tensor indices and define

$$M_{\phi\phi}(n_h, n_\varsigma) \sim M \left|\frac{\tilde{\mathcal{K}}_{\phi\phi}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{\phi\phi}^{(0)}}\right|^{\frac{-1}{n_h+n_\varsigma}} , \tag{3.49}$$

$$M_V(n_h, n_\varsigma) \sim M \left|\tilde{V}^{(n_\varsigma)}\right|^{\frac{-1}{n_h+n_\varsigma-4}} . \tag{3.50}$$

The first line of (3.48) represents the standard higher-dimensional operators for Einstein gravity and a minimally coupled scalar field. If the conditions (3.46) hold, all these operators are suppressed at energies below the scale $M$. The remaining terms are new higher-dimensional operators, that appear if the kinetic term is non-canonical and/or if the expansion of the potential does not stop at the fourth order. The suppression scales of these operators are given by $M_{\phi\phi}(n_h, n_\varsigma)$ and $M_V(n_h, n_\varsigma)$, which are at least of the order of the Planck scale $M$, provided that

$$\left|\frac{\tilde{\mathcal{K}}_{\phi\phi}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{\phi\phi}^{(0)}}\right|^{\frac{1}{n_h+n_\varsigma}} \lesssim 1 \quad \text{and} \quad \left|\tilde{V}^{(n_\varsigma)}\right|^{\frac{1}{n_h+n_\varsigma-4}} \lesssim 1 . \tag{3.51}$$

Let us summarize the findings of this section up to now. We have considered a scale-invariant theory of a scalar field coupled to TDiff gravity described by the Lagrangian (3.24). If there exists a value of $\sigma_0$ for which $f(\sigma_0) \neq 0$ (3.33) and $v(\sigma_0) = v'(\sigma_0) = 0$ (3.36), there exists a family of solutions of the equations of motion corresponding to constant scalar fields and flat spacetime. Those solutions for which $\phi_0 \neq 0$ spontaneously break the dilatational symmetry of the theory. Besides, scale invariance can be independently broken by an integration constant $\Lambda_0$, which introduces a run-away potential for the dilaton field. The quadratic analysis of perturbations around the constant solution with $\Lambda_0 = 0$ has shown that, if the conditions $\mathcal{L}_{\sigma\sigma}^{(0)}\mathcal{L}_{\phi\phi}^{(0)} - \left(\mathcal{L}_{\sigma\phi}^{(0)}\right)^2 > 0$ and $\tilde{\mathcal{L}}_{\phi\phi}^{(0)} > 0$ are satisfied, the theory describes two massless tensor degrees of freedom, a massless scalar and a scalar of mass $m_\varsigma^2 = \tilde{V}^{(2)}$. The scale $M$ for gravity and the scales $m_\varsigma$ and $\kappa_\varsigma$ associated to the scalar field are induced by the non-zero value of $\phi_0$. If the theory-defining functions are such that the conditions (3.46), (3.47) and (3.51) are fulfilled, the scalar and the tensor sectors decouple, and all non-renormalizable interactions are suppressed below the scale $M$. In this case, at energies well-below $M$, the scalar field phenomenology issued by (3.24) is indistinguishable from the phenomenology of the corresponding renormalizable scalar-field theory.

### 3.2.3 Dependence on the choice of variables

Under very general conditions, the Lagrangian (3.24) can be brought to the form (3.31), for all choices of field variables. However, the explicit expressions of the functions $\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})$, $\tilde{V}(\tilde{\sigma})$, etc. depend on the chosen variables. For example, for some functions $\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})$, one can make a change of variables $(\tilde{\sigma}, \tilde{\phi}) \mapsto (\tilde{\sigma}', \tilde{\phi}')$, which brings the kinetic term to the canonical form. Note that under such variable changes, the function $\tilde{K}_{\phi\phi}(\tilde{\sigma})$ does not transform as a scalar, but rather like a metric component. Also the functions $\tilde{V}(\tilde{\sigma})$ and $\tilde{\mathcal{L}}_{\Lambda_0}(\tilde{\sigma})$ appearing in the potential take different forms for different choices of variables. For instance, there might exist a set of variables in terms of which the potential is polynomial, whereas expressed in another set of variables it contains exponential functions.

In the previous sections, we expanded the Lagrangian around the constant background (3.37). The idea is that perturbations around this background can be quantized and interpreted as particles. Their tree-level masses and coupling constants are given by the coefficients of the Taylor expansion around the point $\tilde{\sigma} = 0$, i.e. $\tilde{\mathcal{L}}_{\phi\phi}^{(n)}$ and $\tilde{V}^{(n)}$. Since the functions depend on the variable choice, also these coefficients do. This means that for different sets of variables, tree-masses and coupling constants will take different values. The equivalence theorems of [91] show that the so constructed quantum theories are equivalent for all choices of variables, as long as the variable transformations are well-defined perturbatively. A consequence of these theorems is that whenever one takes into account the whole (possibly infinite) series of terms in the Lagrangian to compute S-matrix elements, the result will not depend on the choice of variables. The situations is different, however, if one uses effective field theory arguments to truncate the Lagrangian, because, as already mentioned, the individual terms of the series expansions do depend on the choice of variables. This means that conditions like (3.46), (3.47) and (3.51) depend on the choice of variables. Therefore, applied to arbitrary variables, such conditions should be considered as sufficient but not necessary. It can happen, for instance, that for a certain choice of variables some of the suppression conditions (3.51) do not hold, but that the corresponding terms are nevertheless irrelevant. Technically, this would be due to cancellations between terms of the different series contained in (3.48). In

order to have a variable-independent statement, the ensemble of conditions (3.46), (3.47) and (3.51) should be read in the following way:
*"If there exists a set of variables in terms of which the conditions* (3.46), (3.47) *and* (3.51) *hold, then, at energies well below* $M$*, the scalar-field theory contained in* (3.24) *is indistinguishable from the corresponding renormalizable theory."*
Read this way, the conditions are necessary and sufficient.

### 3.2.4 Conditions for exact renormalizability

One may wonder, whether there exists a set of field variables in terms of which the kinetic part of the Lagrangian (3.31) takes *exactly* the canonical form. The condition for such variables to exist is the vanishing of the Riemann tensor computed from the field space metric [113]

$$\left\{\tilde{\mathcal{K}}_{ij}(\tilde{\sigma},\tilde{\phi})\right\} = \begin{pmatrix} \epsilon_\sigma M^2 & 0 \\ 0 & \tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma}) \end{pmatrix} . \tag{3.52}$$

This condition corresponds to[6]

$$\tilde{\mathcal{K}}'_{\phi\phi}(\tilde{\sigma})^2 - 2\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})\tilde{\mathcal{K}}''_{\phi\phi}(\tilde{\sigma}) = 0 . \tag{3.53}$$

Functions $\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})$ which satisfy this equation have the form

$$\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma}) = c_1 \left(\tilde{\sigma} + c_2\right)^2 , \tag{3.54}$$

where $c_1$ and $c_2$ are arbitrary constants. One can also formulate the conditions which guarantee that, for the same variables that give a canonical kinetic term, the scalar field potential (for $\Lambda_0 = 0$) becomes a polynomial of a maximum order $p$. They read

$$\tilde{V}(\tilde{\sigma})_{;i_1;i_2;i_3;\ldots;i_{p+1}} = 0 , \tag{3.55}$$

where the semicolon stands for the covariant derivative constructed from the metric (3.52). If these conditions hold for $p = 4$ and at the same time condition (3.53) is fulfilled, the scalar part of the Lagrangian describes a tree-unitary and renormalizable quantum field theory [113]. For all conditions to hold, the function $\tilde{V}(\tilde{\sigma})$ must be of the form

$$\tilde{V}(\tilde{\sigma}) = c_3\, \tilde{\sigma}\left(\tilde{\sigma} + 2c_2\right)\left(\tilde{\sigma}^2 + 2c_2\,\tilde{\sigma} + c_4\right) + c_5 , \tag{3.56}$$

where $c_3$, $c_4$ and $c_5$ are arbitrary constants. If we also impose the conditions (3.36), which correspond to $\tilde{V}(0) = \tilde{V}'(0) = 0$, we can further restrict the form of the function $\tilde{V}(\tilde{\sigma})$ to

$$\tilde{V}(\tilde{\sigma}) = c_3\tilde{\sigma}^2\left(\tilde{\sigma} + 2c_2\right)^2 . \tag{3.57}$$

[6] In terms of the functions without tilde, this same condition reads

$$\mathcal{K}'_{\phi\phi}(\sigma)\left(\mathcal{K}_{\phi\phi}(\sigma)\mathcal{K}'_{\sigma\sigma}(\sigma) + \mathcal{K}'_{\phi\phi}(\sigma)\mathcal{K}_{\sigma\sigma}(\sigma) - 2\mathcal{K}_{\sigma\phi}(\sigma)\mathcal{K}'_{\sigma\phi}(\sigma)\right) + 2\left(\mathcal{K}_{\sigma\phi}(\sigma)^2 - \mathcal{K}_{\phi\phi}(\sigma)\mathcal{K}_{\sigma\sigma}(\sigma)\right)\mathcal{K}''_{\phi\phi}(\sigma) = 0 .$$

## 3.3 Including gauge bosons

Having discussed pure scalar theories, we now add extra ingredients, namely, gauge fields and fermions.

In the Higgs mechanism, gauge fields get their masses from a non-zero expectation value of a scalar field. We are going to show how a similar phenomenon can occur due to spontaneous breaking of scale invariance in a scale-invariant TDiff theory, where the massive field will play a role similar to the role of the Higgs field in the SM. For simplicity, we will consider the case of an Abelian gauge group.

If the scalar field $\phi$ in (3.24) is promoted to a complex field, the action is trivially invariant under a global $U(1)$ symmetry. This symmetry can be turned into a gauge symmetry by introducing an Abelian gauge field (remember that gauge fields have scale dimension 1). The generalization of (3.24) to this case reads

$$\begin{aligned}\frac{\mathcal{L}}{\sqrt{-g}} =& -\frac{1}{2}|\phi|^2 f(-g)R - \frac{1}{2}|\phi|^2\mathcal{G}_{gg}(-g)(\partial g)^2 - \frac{1}{2}\mathcal{G}_{\phi\phi}(-g)D\phi\cdot(D\phi)^* \\ &+ \frac{1}{2}\mathcal{G}^*_{g\phi}(-g)\phi^*\ \partial g\cdot D\phi + \frac{1}{2}\mathcal{G}_{g\phi}(-g)\phi\ \partial g\cdot(D\phi)^* - \frac{1}{2}\mathcal{G}_{na}(-g)(\partial|\phi|)^2 \\ &- \frac{1}{4}\mathcal{G}_{AA}(-g)F^2 - \frac{1}{4}\mathcal{G}_{\varepsilon}(-g)F\wedge F - v(-g)|\phi\phi^*|^2\ ,\end{aligned} \tag{3.58}$$

where the covariant derivative is defined as $D_\mu \equiv \partial_\mu - ieA_\mu$ and the function $\mathcal{G}_{g\phi}(-g)$ is complex-valued. To be most general, we have included the non-analytical term $\partial|\phi|$. The wedge product is defined as $F\wedge F = \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, where $\epsilon_{\mu\nu\rho\sigma} \equiv \sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$, with $\varepsilon_{\mu\nu\rho\sigma}$ being the standard Levi-Civita tensor. For simplicity, let us directly choose the unitary gauge $\phi^* = \phi$, in which the Lagrangian reads

$$\begin{aligned}\frac{\mathcal{L}}{\sqrt{-g}} =& -\frac{1}{2}\phi^2 f(-g)R - \frac{1}{2}\phi^2\mathcal{G}_{gg}(-g)(\partial g)^2 - \frac{1}{2}\left(\mathcal{G}_{\phi\phi}(-g) + \mathcal{G}_{na}(-g)\right)(\partial\phi)^2 \\ &+ \mathrm{Re}\left[\mathcal{G}_{g\phi}(-g)\right]\phi\ \partial g\cdot\partial\phi + e\ \mathrm{Im}\left[\mathcal{G}_{g\phi}(-g)\right]\phi^2\ \partial g\cdot A \\ &- \frac{1}{2}e^2\ \mathcal{G}_{\phi\phi}(-g)A^2\phi^2 - \frac{1}{4}\mathcal{G}_{AA}(-g)F^2 - \frac{1}{4}\mathcal{G}_{\varepsilon}(-g)F\wedge F - v(-g)\phi^4\ ,\end{aligned} \tag{3.59}$$

where Re and Im stand for the real and imaginary parts, respectively. Following the formalism developed in Sec. 3.1.1, one can directly write down the equivalent Diff invariant theory in the E-frame as

$$\begin{aligned}\frac{\mathcal{L}_e}{\sqrt{-\tilde{g}}} =& -\frac{1}{2}M^2\tilde{R} - \frac{1}{2}M^2\mathcal{K}_{\sigma\sigma}(\sigma)(\partial\sigma)^2 - \frac{1}{2}M^2\mathcal{K}_{\phi\phi}(\sigma)(\partial\ln(\phi/M))^2 \\ & - M^2\mathcal{K}_{\sigma\phi}(\sigma)\ \partial\sigma\cdot\partial\ln(\phi/M) - eM^2\mathcal{K}_{\sigma A}(\sigma)\ \partial\sigma\cdot A - \frac{1}{2}e^2M^2\mathcal{K}_{int}(\sigma)A^2 \\ & - \frac{1}{4}\mathcal{K}_{AA}(\sigma)F^2 - \frac{1}{4}\mathcal{K}_{\varepsilon}(\sigma)F\wedge F - M^4V(\sigma) - \frac{M^4\Lambda_0}{\phi^4 f(\sigma)^2\sqrt{\sigma}}\ ,\end{aligned} \tag{3.60}$$

where

$$\begin{aligned}\mathcal{K}_{\sigma\sigma}(\sigma) &= \frac{\mathcal{G}_{gg}(\sigma)}{f(\sigma)} + \frac{3}{2}\left(\frac{f'(\sigma)}{f(\sigma)}\right)^2 \quad , \quad \mathcal{K}_{\phi\phi}(\sigma) = \frac{\mathcal{G}_{\phi\phi}(\sigma)+\mathcal{G}_{na}(\sigma)}{f(\sigma)} + 6\ , \\ \mathcal{K}_{\sigma\phi}(\sigma) &= \frac{\mathrm{Re}\left[\mathcal{G}_{g\phi}(\sigma)\right]}{f(\sigma)} + 3\frac{f'(\sigma)}{f(\sigma)} \quad , \quad \mathcal{K}_{\sigma A}(\sigma) = \frac{\mathrm{Im}\left[\mathcal{G}_{g\phi}(\sigma)\right]}{f(\sigma)}\ , \\ \mathcal{K}_{int}(\sigma) &= \frac{\mathcal{G}_{\phi\phi}(\sigma)}{f(\sigma)} \quad , \quad \mathcal{K}_{AA}(\sigma) = \mathcal{G}_{AA}(\sigma)\ , \\ \mathcal{K}_{\varepsilon}(\sigma) &= \mathcal{G}_{\varepsilon}(\sigma) \quad , \quad V(\sigma) = \frac{v(\sigma)}{f(\sigma)^2}\ .\end{aligned}$$

At this point, as in the case without gauge fields, we can make a field redefinition in order to eliminate the derivative couplings between the different fields. This will simplify the interpretation of the theory as a description of interacting particles. The extension of (3.30) is[7]

$$\begin{aligned}&\tilde{\sigma} = \int_{\sigma_0}^{\sigma} d\sigma'\sqrt{\left|\frac{\mathcal{K}_{\sigma\sigma}\mathcal{K}_{\phi\phi}-\mathcal{K}_{\sigma\phi}^2}{\mathcal{K}_{\phi\phi}} - \frac{\mathcal{K}_{\sigma A}^2}{\mathcal{K}_{int}}\right|}\ , \quad \tilde{\phi} = M\left(\ln\frac{\phi}{M} + \int_{\sigma_0}^{\sigma} d\sigma'\frac{\mathcal{K}_{\sigma\phi}}{\mathcal{K}_{\phi\phi}}\right)\ , \\ &\tilde{A}_\mu = A_\mu + \frac{1}{e}\frac{\mathcal{K}_{\sigma A}}{\mathcal{K}_{int}}\partial_\mu\sigma\ ,\end{aligned} \tag{3.61}$$

in terms of which the above Lagrangian reads

$$\begin{aligned}\frac{\mathcal{L}_e}{\sqrt{-\tilde{g}}} =& -\frac{1}{2}M^2\tilde{R} - \frac{1}{2}\epsilon_\sigma M^2(\partial\tilde{\sigma})^2 - \frac{1}{2}\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})(\partial\tilde{\phi})^2 \\ & - \frac{1}{2}e^2\tilde{\mathcal{K}}_{int}(\tilde{\sigma})M^2\tilde{A}^2 - \frac{1}{4}\tilde{\mathcal{K}}_{AA}(\tilde{\sigma})\tilde{F}^2 - \frac{1}{4}\tilde{\mathcal{K}}_{\varepsilon}(\tilde{\sigma})\tilde{F}\wedge\tilde{F} \\ & - \tilde{V}(\tilde{\sigma})M^4 - \Lambda_0\,\tilde{\mathcal{K}}_{\Lambda_0}\exp\left(-\frac{4\tilde{\phi}}{M}\right)\ ,\end{aligned} \tag{3.62}$$

[7] In view of the conditions (3.68), we assume that $\mathcal{K}_{\phi\phi}$, $\mathcal{K}_{int}$ and $\frac{\mathcal{K}_{\sigma\sigma}\mathcal{K}_{\phi\phi}-\mathcal{K}_{\sigma\phi}^2}{\mathcal{K}_{\phi\phi}} - \frac{\mathcal{K}_{\sigma A}^2}{\mathcal{K}_{int}}$ are non-vanishing.

where $\epsilon_\sigma = \text{sign}\left(\frac{\mathcal{K}_{\sigma\sigma}\mathcal{K}_{\phi\phi}-\mathcal{K}^2_{\sigma\phi}}{\mathcal{K}_{\phi\phi}} - \frac{\mathcal{K}^2_{\sigma A}}{\mathcal{K}_{int}}\right)$ and $\tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma})$ is defined in (3.32). Like in the previous case, the role of the field providing the mass scales is played by $\tilde{\sigma}$. Note that $\tilde{\phi}$ is completely decoupled from the vector field $\tilde{A}_\mu$.

### 3.3.1 Local degrees of freedom

Like in the case without gauge fields, the existence of a symmetry-breaking classical ground state with flat spacetime

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu}\ , \quad \tilde{\sigma} = 0\ , \quad \tilde{\phi} = \tilde{\phi}_0\ , \quad \tilde{A}_\mu = 0\ , \quad \Lambda_0 = 0\ , \tag{3.63}$$

is assured by the conditions

$$f(\sigma_0) \neq 0 \quad \text{and} \quad v(\sigma_0) = v'(\sigma_0) = 0\ . \tag{3.64}$$

We again want to examine the nature of the perturbations around this type of solution, which we define as

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \eta_{\mu\nu} + \frac{\tilde{h}_{\mu\nu}}{M} \quad , \quad \tilde{\sigma} = \frac{\varsigma}{M}\ , \\ \tilde{\phi} &= \tilde{\phi}_0 + \frac{\varphi}{\sqrt{\left|\tilde{\mathcal{K}}^{(0)}_{\phi\phi}\right|}} \quad , \quad \tilde{A}_\mu = \frac{\tilde{A}^c_\mu}{\sqrt{\left|\tilde{\mathcal{K}}^{(0)}_{AA}\right|}}\ . \end{aligned} \tag{3.65}$$

In the rest of section 3.3, Lorentz indices are raised, lowered and contracted with the Minkowski metric $\eta_{\mu\nu}$. To quadratic order, the Lagrangian (3.62) reduces to

$$\mathcal{L}^Q_e = \tilde{\mathcal{L}}^Q_{GR} - \epsilon_\varsigma\frac{1}{2}(\partial\varsigma)^2 - \epsilon_\varphi\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2_\varsigma\varsigma^2 - \epsilon_A\frac{1}{4}\tilde{F}^2 - \frac{1}{2}m^2_A\tilde{A}^2\ , \tag{3.66}$$

where

$$\begin{aligned} \epsilon_\varsigma &\equiv \text{sign}\left(\frac{\mathcal{K}^{(0)}_{\sigma\sigma}\mathcal{K}^{(0)}_{\phi\phi} - \left(\mathcal{K}^{(0)}_{\sigma\phi}\right)^2}{\mathcal{K}^{(0)}_{\phi\phi}} - \frac{\left(\mathcal{K}^{(0)}_{\sigma A}\right)^2}{\mathcal{K}^{(0)}_{int}}\right)\ , \quad \epsilon_\varphi \equiv \text{sign}\left(\mathcal{K}^{(0)}_{\phi\phi}\right)\ , \\ \epsilon_A &\equiv \text{sign}\left(\mathcal{K}^{(0)}_{AA}\right)\ , \\ m^2_\varsigma &\equiv \epsilon_\varsigma\tilde{V}^{(2)}M^2 = \epsilon_\varsigma\left(\frac{\mathcal{K}^{(0)}_{\sigma\sigma}\mathcal{K}^{(0)}_{\phi\phi} - \left(\mathcal{K}^{(0)}_{\sigma\phi}\right)^2}{\mathcal{K}^{(0)}_{\phi\phi}} - \frac{\left(\mathcal{K}^{(0)}_{\sigma A}\right)^2}{\mathcal{K}^{(0)}_{int}}\right)^{-1} V^{(2)}M^2\ , \\ m^2_A &\equiv \epsilon_A e^2\frac{\mathcal{K}^{(0)}_{int}}{\mathcal{K}^{(0)}_{AA}}M^2\ . \end{aligned} \tag{3.67}$$

At the level of the quadratic Lagrangian, the following conditions must be satisfied:

- For positive definite kinetic terms (absence of ghosts):

$$\epsilon_\varsigma,\ \epsilon_\varphi,\ \epsilon_A = 1\ . \tag{3.68}$$

- For positive or zero masses (absence of tachyons):

$$m_\varsigma^2,\ m_A^2 \geq 0\ . \tag{3.69}$$

### 3.3.2 Interactions and separation of scales

The terms of dimension up to four are

$$\begin{aligned}\mathcal{L}_e^{\text{int}\leq 4} = &-\frac{1}{3!}\kappa_\varsigma\varsigma^3 - \frac{\lambda_\varsigma}{4!}\varsigma^4 - \frac{1}{2}\kappa_A\eta^{\mu\nu}\tilde{A}^c_\mu\tilde{A}^c_\nu\varsigma - \frac{1}{4}\lambda_A\eta^{\mu\nu}\tilde{A}^c_\mu\tilde{A}^c_\nu\varsigma^2\\ &-\frac{1}{4}\frac{m_\varsigma^2}{M}\varsigma^2\tilde{h} - \frac{1}{16}\frac{m_\varsigma^2}{M^2}\varsigma^2\left(\tilde{h}^2 - 2\tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu}\right) - \frac{1}{12}\frac{\kappa_\varsigma}{M}\varsigma^3\tilde{h}\\ &-\frac{1}{4}\frac{m_A^2}{M}\tilde{A}^c_\mu\tilde{A}^c_\nu\left(\eta^{\mu\nu}\tilde{h} - 2\tilde{h}^{\mu\nu}\right) - \frac{1}{4}\frac{\kappa_A}{M}\varsigma\tilde{A}^c_\mu\tilde{A}^c_\nu\left(\eta^{\mu\nu}\tilde{h} - 2\tilde{h}^{\mu\nu}\right)\\ &-\frac{1}{16}\frac{m_A^2}{M^2}\tilde{A}^c_\mu\tilde{A}^c_\nu\left(\eta^{\mu\nu}\tilde{h}^2 - 4\tilde{h}^{\mu\nu}\tilde{h} - 2\eta^{\mu\nu}\tilde{h}_{\rho\sigma}\tilde{h}^{\rho\sigma} + 8\tilde{h}^\mu_\rho\tilde{h}^{\rho\nu}\right)\ ,\end{aligned} \tag{3.70}$$

where we have defined the parameters

$$\kappa_\varsigma \equiv \tilde{V}^{(3)}M\ ,\quad \lambda_\varsigma \equiv \tilde{V}^{(4)}\ ,\quad \kappa_A \equiv e^2\frac{\tilde{\mathcal{K}}^{(1)}_{int}}{\tilde{\mathcal{K}}^{(0)}_{AA}}M\ ,\quad \lambda_A \equiv e^2\frac{\tilde{\mathcal{K}}^{(2)}_{int}}{\tilde{\mathcal{K}}^{(0)}_{AA}}\ . \tag{3.71}$$

If the scales $m_\varsigma$, $\kappa_\varsigma$, $m_A$ and $\kappa_A$, associated with the scalar and vector field sector are much smaller than the scale $M$, the relevant interactions between these fields and the tensor sector are suppressed, as usual in field theories minimally coupled to gravity. This happens, if the following conditions are met:

$$\begin{aligned}\left|\tfrac{m_\varsigma}{M}\right| = \sqrt{\left|\tilde{V}^{(2)}\right|} \ll 1 \quad &,\quad \left|\tfrac{\kappa_\varsigma}{M}\right| = \left|\tilde{V}^{(3)}\right| \ll 1\ ,\\ \left|\tfrac{m_A}{M}\right| = \sqrt{\left|\tilde{e}^2\mathcal{K}^{(0)}_{int}\right|} \ll 1 \quad &,\quad \left|\tfrac{\kappa_A}{M}\right| = \left|\tilde{e}^2\tilde{\mathcal{K}}^{(1)}_{int}\right| \ll 1\ ,\end{aligned} \tag{3.72}$$

with the definition $\tilde{e}^2 \equiv \frac{e^2}{\mathcal{K}^{(0)}_{AA}}$. In addition, we have the following conditions that prevent the theory from being strongly coupled:

$$\left\{\frac{\kappa_\varsigma}{m_{min}}, \frac{\kappa_A}{m_{min}}, \lambda_\varsigma, \lambda_A,\right\} \lesssim 1\ , \tag{3.73}$$

where $m_{min} \equiv \min\left(m_\varsigma, m_A\right)$.

The higher-dimensional terms can be written schematically as

$$\begin{aligned}\mathcal{L}_e^{\text{int}>4} = & \sum_{n_h>0}^{\infty} \frac{1}{M^{n_h}} \left(\mathcal{L}_e^Q + \mathcal{L}_e^{\text{int}\leq 4}\right) \tilde{h}^{n_h} \\ & + \sum_{\substack{n_h\geq 0 \\ n_\varsigma>0}}^{\infty} \left(\frac{1}{M_{\phi\phi}(n_h,n_\varsigma)}\right)^{n_h+n_\varsigma} (\partial\varphi)^2 \tilde{h}^{n_h}\varsigma^{n_\varsigma} \\ & + \sum_{\substack{n_h\geq 0 \\ n_\varsigma>4}}^{\infty} \left(\frac{1}{M_V(n_h,n_\varsigma)}\right)^{n_h+n_\varsigma-4} \tilde{h}^{n_h}\varsigma^{n_\varsigma} \\ & + \sum_{\substack{n_h\geq 0 \\ n_\varsigma>2}}^{\infty} \left(\frac{1}{M_{int}(n_h,h_\varsigma)}\right)^{n_h+n_\varsigma-2} \tilde{A^c}^2 \tilde{h}^{n_h}\varsigma^{n_\varsigma} \\ & + \left(\sum_{\substack{n_h\geq 0 \\ n_\varsigma>0}}^{\infty} \left(\frac{1}{M_{AA}(n_h,n_\varsigma)}\right)^{n_h+n_\varsigma} + \sum_{\substack{n_h\geq 0 \\ n_\varsigma>0}}^{\infty} \left(\frac{1}{M_\varepsilon(n_h,n_\varsigma)}\right)^{n_h+n_\varsigma}\right) \partial^2 \tilde{A^c}^2 \tilde{h}^{n_h}\varsigma^{n_\varsigma},\end{aligned} \tag{3.74}$$

where, as before, we neglect numerical factors of order one, neglect tensor indices and define the suppression scales

$$\begin{aligned} M_{\phi\phi}(n_h,n_\varsigma) &\sim M \left|\frac{\tilde{\mathcal{K}}_{\phi\phi}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{\phi\phi}^{(0)}}\right|^{\frac{-1}{n_h+n_\varsigma}} , \quad M_V(n_h,n_\varsigma) \sim M \left|\tilde{V}^{(n_\varsigma)}\right|^{\frac{-1}{n_h+n_\varsigma-4}} , \\ M_{int}(n_h,n_\varsigma) &\sim M \left|\tilde{e}^2\, \tilde{\mathcal{K}}_{int}^{(n_\varsigma)}\right|^{\frac{-1}{n_h+n_\varsigma}} , \quad M_{AA}(n_h,n_\varsigma) \sim M \left|\frac{\tilde{\mathcal{K}}_{AA}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{AA}^{(0)}}\right|^{\frac{-1}{n_h+n_\varsigma}} , \\ M_\varepsilon(n_h,n_\varsigma) &\sim M \left|\frac{\tilde{\mathcal{K}}_{\varepsilon}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{AA}^{(0)}}\right|^{\frac{-1}{n_h+n_\varsigma}} . \end{aligned} \tag{3.75}$$

The first term in (3.74) represents the standard higher-dimensional operators of a theory minimally coupled to gravity, which are suppressed at energies below $M$, as soon as the conditions (3.72) hold. The additional operators come with the suppression scales $M_{\phi\phi}(n_h,n_\varsigma)$, $M_V(n_h,n_\varsigma)$, $M_{int}(n_h,n_\varsigma)$, $M_{AA}(n_h,n_\varsigma)$ and $M_\varepsilon(n_h,n_\varsigma)$. These are comparable to or bigger than the scale $M$, whenever

$$\begin{aligned} &\left|\frac{\tilde{\mathcal{K}}_{\phi\phi}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{\phi\phi}^{(0)}}\right|^{\frac{1}{n_h+n_\varsigma}} \lesssim 1 \quad , \quad \left|\tilde{V}^{(n_\varsigma)}\right|^{\frac{1}{n_h+n_\varsigma-4}} \lesssim 1 \quad , \quad \left|\tilde{e}^2\tilde{\mathcal{K}}_{int}^{(n_\varsigma)}\right|^{\frac{1}{n_h+n_\varsigma}} \lesssim 1 \, , \\ &\left|\frac{\tilde{\mathcal{K}}_{AA}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{AA}^{(0)}}\right|^{\frac{1}{n_h+n_\varsigma}} \lesssim 1 \quad , \quad \left|\frac{\tilde{\mathcal{K}}_{\varepsilon}^{(n_\varsigma)}}{\tilde{\mathcal{K}}_{AA}^{(0)}}\right|^{\frac{1}{n_h+n_\varsigma}} \lesssim 1 \quad , \end{aligned} \tag{3.76}$$

for all values $n_\varsigma$ and $n_h$ can take in the sums in (3.74). If the conditions (3.68), (3.69), (3.72), (3.73) and (3.76) are met, the effective Lagrangian describing the scalar and vector sectors at energies far below $M$ is

$$\begin{aligned}\mathcal{L}_e^{\text{eff}} = &-\frac{1}{2}(\partial\varsigma)^2 - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varsigma^2\varsigma^2 - \frac{1}{3!}\kappa_\varsigma\varsigma^3 - \frac{\lambda_\varsigma}{4!}\varsigma^4 \\ &- \frac{1}{4}(\tilde{F}^c)^2 - \frac{1}{2}m_A^2(\tilde{A}^c)^2 - \frac{1}{2}\kappa_A(\tilde{A}^c)^2\varsigma - \frac{1}{4}\lambda_A(\tilde{A}^c)^2\varsigma^2 \;.\end{aligned} \tag{3.77}$$

We would like this Lagrangian to give rise to a consistent quantum field theory at energies small compared to $M$. It has been shown [113] that the only tree-unitary theories containing scalar fields and massive vector particles are those that correspond to a spontaneously broken gauge theory.[8] Thus, for our model to be tree-unitary at energies below $M$, the above effective Lagrangian should correspond to the Abelian Higgs model in the unitary gauge. This means that the six couplings $m_\varsigma$, $\kappa_\varsigma$, $\lambda_\varsigma$, $m_A$, $\kappa_A$ and $\lambda_A$ should satisfy the three relations

$$\frac{\lambda_\varsigma}{\lambda_A} = \frac{\kappa_\varsigma}{\kappa_A}, \quad \frac{\lambda_\varsigma}{\lambda_A} = \frac{3}{2}\frac{m_\varsigma^2}{m_A^2}, \quad m_\varsigma^2 = \frac{1}{3}\frac{\kappa_\varsigma^2}{\lambda_\varsigma} \;. \tag{3.78}$$

In the present model, these relations can be translated to the following conditions on the TDF:

$$\frac{\tilde{\mathcal{K}}_{int}^{(0)}}{\tilde{\mathcal{K}}_{int}^{(1)}} \simeq \frac{1}{2}\frac{\tilde{\mathcal{K}}_{int}^{(1)}}{\tilde{\mathcal{K}}_{int}^{(2)}}\;, \quad \frac{\tilde{V}^{(2)}}{\tilde{V}^{(3)}} \simeq \frac{2}{3}\frac{\tilde{\mathcal{K}}_{int}^{(0)}}{\tilde{\mathcal{K}}_{int}^{(1)}}\;, \quad \frac{\tilde{V}^{(2)}}{\tilde{V}^{(3)}} \simeq \frac{1}{3}\frac{\tilde{V}^{(3)}}{\tilde{V}^{(4)}}\;, \tag{3.79}$$

where by the approximate equalities we mean that the relations should hold up to suppressed terms, i.e. for two quantities $a$ and $b$ one has $a \simeq b$, whenever $a = b\left(1 + \mathcal{O}\left(\frac{m_\varsigma}{M}, \frac{\kappa_\varsigma}{M}, \frac{m_A}{M}, \frac{\kappa_A}{M}\right)\right)$.

We can now draw the following conclusion. If there exists a set of variables in terms of which the conditions (3.64), (3.68), (3.69), (3.72), (3.73), (3.76) and (3.79) hold, then, at energies well below $M$, the theory given by (3.58) is indistinguishable from the renormalizable Abelian Higgs model. Some of the conditions can be satisfied very naturally, for instance by polynomial TDF. The conditions related to the smallness of particle masses with respect to the Planck scale $M$, however, require important fine-tunings (see Sec. 3.6). Hence, SI TDiff theories do not provide an explanation for the the big difference between $M$ and the scales in the particle physics sector. However, as we will see in chapter 4, if scale invariance can be maintained at the quantum level, there is no problem of stability of the Higgs mass against radiative corrections. Therefore, in the proposed theories, the smallness of the particle physics scales is not a naturalness problem from the effective field theory point of view (cf. section 2.1.2).

[8] An exception is given by theories in which the massive vector fields only interact with conserved currents [113] .

## 3.4 Coupling to fermionic matter

Let us finally study the inclusion of fermions into SI TDiff theories. For the sake of illustration, we only consider Dirac spinors. Though, the conclusions are generic, for they only depend on the dimensionality of the fields. (In this context see [77, 114] for the first order formalism of unimodular gravity.) The most general scale-invariant spinor Lagrangian compatible with TDiff can be written as[9]

$$\mathcal{L}_\psi = -b\,\mathcal{G}_\psi(b^2)\,\bar{\psi}\,b^{\mu i}\gamma_i\left(\partial_\mu + \frac{1}{8}[\gamma_j,\gamma_k]\,\omega_\mu{}^{jk}\right)\psi - b\,\phi\,v_\psi(b^2)\bar{\psi}\,\psi, \tag{3.80}$$

where $b^{\mu a}$ represents the inverse vierbein related to the metric through $g_{\mu\nu} = \eta_{ij}\,b_\mu{}^i b_\nu{}^j$ , $\omega_\mu{}^{jk}$ is its spin connection (see e.g. [115]) and $b = det[b_\mu{}^i] = \sqrt{-g}$. Introducing the Stückelberg field as described in section 3.1.1, the Lagrangian can be written as

$$\mathcal{L}_\psi = -b\,\mathcal{G}_\psi(\sigma)\,\bar{\psi}\,b^{\mu i}\gamma_i\left(\partial_\mu + \frac{1}{8}[\gamma_j,\gamma_k]\,\omega_\mu{}^{jk}\right)\psi - b\,\phi\,v_\psi(\sigma)\bar{\psi}\,\psi. \tag{3.81}$$

The change of variables (3.27) corresponds to the transformation $\tilde{b}_\mu{}^i \equiv \Omega b_\mu{}^i$. Together with the field redefinition

$$\tilde{\psi} \equiv \Omega^{-3/2}\,\psi,$$

it yields the Lagrangian in the E-frame (see e.g. [115])

$$\mathcal{L}_\psi = -\tilde{b}\,\mathcal{G}_\psi(\sigma)\,\bar{\tilde{\psi}}\,\tilde{b}^{\mu i}\gamma_i\left(\partial_\mu + \frac{1}{8}[\gamma_j,\gamma_k]\,\tilde{\omega}_\mu{}^{jk}\right)\tilde{\psi} - \tilde{b}\,\frac{M v_\psi(\sigma)}{\sqrt{f(\sigma)}}\bar{\tilde{\psi}}\,\tilde{\psi}. \tag{3.82}$$

We see that the scale invariance of the spinor Lagrangian in the J-frame also leads to the decoupling of fermions from the dilaton field $\phi$ in the E-frame.

## 3.5 Application to the Standard Model

The basics established in the preceding sections can be used to construct a scale-invariant version of the Standard Model of particle physics coupled to gravity. Let us describe how this should be done. The scalar-tensor sector of the theory is given by the Lagrangian (3.24), where $\phi$ is replaced by the complex Higgs-doublet $\Phi$. All fermions and bosons of the SM are then added and coupled to gravity in the way described in sections (3.3) and (3.4), again with $\Phi$ replacing $\phi$. The generalization to the group structure of the SM is straightforward. All theory-defining functions have to be chosen such that they fulfill a series of conditions of the type of (3.36), (3.68),

[9] In this section we use the conventions of [115].

(3.69), (3.72), (3.73), (3.76) and (3.79). In this way, one obtains a model, whose particle phenomenology at energies well below the Planck mass $M$ is indistinguishable from that of the SM. In particular, the massless dilaton decouples from all SM fields except for the Higgs field, to which it couples only through suppressed operators.

## 3.6 Particular choices of the theory-defining functions

In the previous sections we have derived a number of conditions to be satisfied by the theory-defining functions (TDF). These conditions are summarized in Table 3.1. Similar conditions should be imposed in the fermionic sector. For simplicity, we will restrict our considerations to the scalar and gauge field sectors described by the Lagrangian (3.58).

| | *Physical Meaning* | *Formal Conditions* |
|---|---|---|
| 1 | Existence of a constant flat solution | $v(\sigma_0) = v'(\sigma_0) = 0$ |
| 2 | Induced gravitational coupling | $f(\sigma_0) \neq 0$ |
| 3 | Positive definite kinetic terms (absence of ghosts) | $\epsilon_\varsigma,\, \epsilon_\varphi,\, \epsilon_A = 1$ |
| 4 | No negative masses (absence of tachyons) | $m_\varsigma^2,\, m_A^2 \geq 0$ |
| 5 | Decoupling of gravitational interactions | $m_\varsigma,\, m_A \ll M$ |
| 6 | No strong coupling | $\kappa_\varsigma,\, \kappa_A \lesssim \min(m_\varsigma, m_A)$ <br> $\lambda_\varsigma,\, \lambda_A \lesssim 1$ |
| 7 | Suppression of higher-dimensional operators | $M_{\phi\phi},\, M_V,\, M_{int},\, M_{AA},\, M_\varepsilon \gtrsim M$ |
| 8 | Equivalence with Abelian Higgs model | $\frac{\kappa_A}{\lambda_A} \simeq \frac{\kappa_\varsigma}{\lambda_\varsigma} \simeq 3\frac{m_\varsigma^2}{\kappa_\varsigma} \simeq 2\frac{m_A^2}{\kappa_A}$ |

**Tab. 3.1:** Conditions to be imposed on the theory-defining functions (TDF)

The parameters in terms of which the conditions are formulated are defined through the TDF. These definitions are summarized in Table 3.2 (remember that $\tilde{e}^2 \equiv e^2/\mathcal{K}_{AA}^{(0)}$).

It is clear that as long as one does not have a rationale for choosing the arbitrary functions, like e.g. an additional symmetry principle, they can always be chosen such that all conditions are fulfilled. A nearby possibility would be to require invariance of the theory under global Weyl transformations $g_{\mu\nu}(x) \mapsto \lambda\, g_{\mu\nu}(x)$. This would completely fix the functional form of all TDF. In this case, however, condition 1 can only be satisfied if $v(-g) = 0$ (for all values of $-g$). The corresponding theory would contain 2 massless scalars. Hence, if one of the scalar fields is supposed to play the role of

| | | |
|---|---|---|
| i. | Signs of kinetic terms | $\epsilon_\varsigma = \text{sign}\left(\mathcal{K}^{(0)}_{\sigma\sigma} - \frac{\left(\mathcal{K}^{(0)}_{\sigma\phi}\right)^2}{\mathcal{K}^{(0)}_{\phi\phi}} - \frac{\left(\mathcal{K}^{(0)}_{\sigma A}\right)^2}{\mathcal{K}^{(0)}_{int}}\right)$ <br> $\epsilon_\varphi \equiv \text{sign}\left(\mathcal{K}^{(0)}_{\phi\phi}\right)$ <br> $\epsilon_A \equiv \text{sign}\left(\mathcal{K}^{(0)}_{AA}\right)$ |
| ii. | Masses and relevant couplings | $m^2_\varsigma \equiv \epsilon_\varsigma \tilde{V}^{(2)} M^2, \quad \kappa_\varsigma \equiv \tilde{V}^{(3)} M, \quad \lambda_\varsigma \equiv \tilde{V}^{(4)},$ <br> $m^2_A \equiv \epsilon_A \tilde{e}^2 \mathcal{K}^{(0)}_{int} M^2, \quad \kappa_A \equiv \tilde{e}^2 \tilde{\mathcal{K}}^{(1)}_{int} M, \quad \lambda_A \equiv \tilde{e}^2 \tilde{\mathcal{K}}^{(2)}_{int},$ |
| iii. | Suppression scales | $M_{\phi\phi}(n_h, n_\varsigma) \sim M \left\lvert \frac{\tilde{\mathcal{K}}^{(n_\varsigma)}_{\phi\phi}}{\tilde{\mathcal{K}}^{(0)}_{\phi\phi}} \right\rvert^{\frac{-1}{n_h+n_\varsigma}}, \quad n_h \geq 0,\ n_\varsigma > 0$ <br> $M_V(n_h, n_\varsigma) \sim M \left\lvert \tilde{V}^{(n_\varsigma)} \right\rvert^{\frac{-1}{n_h+n_\varsigma-4}}, \quad n_h \geq 0,\ n_\varsigma > 4$ <br> $M_{int}(n_h, n_\varsigma) \sim M \left\lvert \tilde{e}^2\, \tilde{\mathcal{K}}^{(n_\varsigma)}_{int} \right\rvert^{\frac{-1}{n_h+n_\varsigma}}, \quad n_h \geq 0,\ n_\varsigma > 2$ <br> $M_{AA}(n_h, n_\varsigma) \sim M \left\lvert \frac{\tilde{\mathcal{K}}^{(n_\varsigma)}_{AA}}{\tilde{\mathcal{K}}^{(0)}_{AA}} \right\rvert^{\frac{-1}{n_h+n_\varsigma}}, \quad n_h \geq 0,\ n_\varsigma > 0$ <br> $M_\varepsilon(n_h, n_\varsigma) \sim M \left\lvert \frac{\tilde{\mathcal{K}}^{(n_\varsigma)}_{\varepsilon}}{\tilde{\mathcal{K}}^{(0)}_{AA}} \right\rvert^{\frac{-1}{n_h+n_\varsigma}}, \quad n_h \geq 0,\ n_\varsigma > 0$ |

**Tab. 3.2:** Definitions of the parameters appearing in table 3.1.

the Higgs field, this choice is not viable. Up to the present stage, we have not found a plausible and satisfactory rationale for the choice of the TDF. Nevertheless, we will give in this section two explicit *ad hoc* examples for the TDF which satisfy all requirements.

### 3.6.1 Polynomials

The first example is motivated by its simplicity. All theory-defining functions can be taken to be polynomials of the metric determinant. In analogy with the Higgs potential, we choose one of the TDF as

$$v(-g) = \frac{\lambda}{4}\left((-g_0)^2 - (-g)^2\right)^2 , \tag{3.83}$$

which satisfies condition 1. The simplest possibility we find for the choice of the remaining functions is given by

$$\begin{aligned} &f(-g) = \mathcal{G}_{gg}(-g) = \mathcal{G}_{AA}(-g) = 1 , \\ &\mathcal{G}_{g\phi}(-g) = \mathcal{G}_{na}(-g) = \mathcal{G}_{\epsilon}(-g) = 0 , \\ &\mathcal{G}_{\phi\phi}(-g) = (-g)^2 . \end{aligned} \tag{3.84}$$

For this choice of functions, the parameters of the theory are summarized in table 3.3 ($\sigma_0 = -g_0$).

| | | |
|---|---|---|
| i. | Signs of kinetic terms | $\epsilon_\varsigma = \epsilon_\phi = \epsilon_A = 1$ . |
| ii. | Masses and relevant couplings | $m_\varsigma^2 = 2\lambda\sigma_0^2 M^2 \quad \kappa_\varsigma = 6\lambda\sigma_0 M \quad \lambda_\varsigma = 6\lambda$<br>$m_A^2 = e^2\sigma_0^2 M^2 \quad \kappa_A = 2e^2\sigma_0 M \quad \lambda_A = 2e^2$ |
| iii. | Suppression scales | $M_{\phi\phi}(n_h, 1) \sim M\left(\frac{6+\sigma_0^2}{2\sigma_0}\right)^{\frac{1}{1+n_h}}$ ,<br>$M_{\phi\phi}(n_h, 2) \sim M\left(\frac{6+\sigma_0^2}{2}\right)^{\frac{1}{2+n_h}}$ . |

**Tab. 3.3:** Parameters in the case of the polynomial TDF ((3.83) and (3.84)).

Conditions 1-3 and 8 in Table 3.1 are immediately satisfied by this choice of TDF, independently of any parameter values. Conditions 4-7 are satisfied provided that $0 < \sigma_0 \ll 1$, $0 < e^2 \lesssim 1/2$ and $0 < \lambda \lesssim 1/6$. The small value of $\sigma_0$ is responsible for the hierarchy between the Planck scale $M$ and the scales related to the scalar and vector sectors. We observe that the higher-dimensional operators are suppressed below the Planck scale, independently of the value of $\sigma_0$.

We conclude that the theory given by the Lagrangian (3.58) with TDF (3.83) and (3.84) is indistinguishable from the renormalizable Abelian Higgs model at energies well below the Planck scale $M$. The non-renormalizable interactions with the dilaton certainly produce differences between both theories, however, these effects are suppressed both by the Planck scale and by the fact that the dilaton couplings always contain spacetime derivatives.

Let us note that, changing variables, one can easily find other sets of polynomial functions, which describe a theory equivalent to the one given by (3.83) and (3.84) and which also satisfy all conditions 1-8. For example, one can redefine the metric and the scalar field $\phi$ through

$$g_{\mu\nu} \mapsto (-g)^{2a} g_{\mu\nu} , \qquad (3.85)$$
$$\phi \mapsto (-g)^{b} \phi , \qquad (3.86)$$

where $a$ and $b$ are arbitrary constants. In terms of the new variables, the Lagrangian (3.58) keeps its structure. The TDF equivalent to (3.83) and (3.84) are given by

$$\begin{aligned}
v(-g) &= \tfrac{\lambda}{4}\left((-g_0)^{2+16a} - (-g)^{2+16a}\right)^2 (-g)^{4(a+b)} , \\
f(-g) &= (-g)^{2(a+b)} , \\
\mathcal{G}_{gg}(-g) &= \left((1+8a)^2 + b^2\right)(-g)^{18a+2b} - (6a^2+12ab)(-g)^{2(a+b)-2} , \\
\mathcal{G}_{\phi\phi}(-g) &= (-g)^{18a+2b+2} , \\
\mathcal{G}_{g\phi}(-g) &= 6a(-g)^{2(a+b)-1} + b(-g)^{18a+2b+1} , \\
\mathcal{G}_{na}(-g) &= 0 . \\
\mathcal{G}_{AA}(-g) &= 1 , \\
\mathcal{G}_{\epsilon}(-g) &= 0 .
\end{aligned} \qquad (3.87)$$

It is straightforward to check explicitly that for $0 < (-g_0)^{1+8a} \ll 1$, $0 < e^2 \lesssim 1/2$ and $0 < \lambda \lesssim 1/6$ this set of polynomials also satisfies conditions 1-8. The two-parameter family of sets of functions (3.87) describes one and the same theory for different variable choices. For $a = b = 0$ the functions take the simple forms (3.83) and (3.84).

### 3.6.2 Functions that reproduce scale-invariant unimodular gravity

In the previous chapter, we have presented a model (2.24) that combines scale invariance with unimodular gravity. There, a new singlet scalar field $\chi$ was introduced in order to make both the gravitational and the matter part of the action scale-invariant. That scalar field was introduced *ad hoc* and was not related to the restriction of the spacetime symmetries from Diff to TDiff. Scale invariance was spontaneously broken due to the shape of the potential. In the same model standard GR was replaced by UG which resulted in the appearance of an arbitrary integration constant in the equations of motion. As already mentioned in section 3.1, UG can be understood as a particular case of a TDiff theory in which the metric does not contain a propagating scalar degree of freedom. It is therefore no surprise that the model of chapter 2 can be found as a particular case of the scale-invariant TDiff theories discussed here. In other words, one can construct a SI TDiff theory and choose the TDF such as to obtain exactly the model (2.24). In order to fit the model of chapter 2 into the framework of the present chapter, we replace the full SM in (2.24) by the Abelian Higgs model. Let us choose the TDF as follows:[10]

$$\begin{aligned}
v(-g) &= \tfrac{\lambda}{4}\left(2-\tfrac{\alpha}{\lambda}(-g)^{-2}\right)^2 ,\\
f(-g) &= \xi_\chi(-g)^{-2}+2\xi_h ,\\
\mathcal{G}_{gg}(-g) &= \tfrac{49-90\xi_\chi}{64}(-g)^{-4}+\tfrac{1+6\xi_h}{32}(-g)^{-2} ,\\
\mathcal{G}_{\phi\phi}(-g) &= 2 ,\\
\mathcal{G}_{g\phi}(-g) &= -\tfrac{7-6\xi_\chi}{8}(-g)^{-3}+\tfrac{1+6\xi_h}{4}(-g)^{-1} ,\\
\mathcal{G}_{na}(-g) &= (-g)^{-2} ,\\
\mathcal{G}_{AA}(-g) &= 1 ,\\
\mathcal{G}_{\epsilon}(-g) &= 0 ,
\end{aligned} \tag{3.88}$$

For this choice, the Lagrangian (3.58) can be brought to the form

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}(\xi_\chi\chi^2+2\xi_h\Phi\Phi^*)\hat{R}-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi-\hat{g}^{\mu\nu}D_\mu\Phi(D_\nu\Phi)^*\\
&-\frac{1}{4}\hat{g}^{\mu\nu}\hat{g}^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}-\lambda(\Phi\Phi^*-\frac{\alpha}{2\lambda}\chi^2)^2 ,
\end{aligned} \tag{3.89}$$

where we have defined the unimodular metric $\hat{g}_{\mu\nu}=(-g)^{-1/4}g_{\mu\nu}$ and the scalar fields $\Phi=\phi(-g)^{1/8}$ and $\chi=|\phi|(-g)^{-7/8}$. $\hat{R}$ is the Ricci scalar associated to the unimodular metric $\hat{g}_{\mu\nu}$. Now, this is exactly the Lagrangian (2.24) except that the particle physics sector is reduced, for simplicity, to the Abelian Higgs model. Of course, the choice of functions (3.88) is rather peculiar. In particular, the presence of the non-analytic term $\mathcal{G}_{na}\neq 0$, see eq. (3.58), was essential. Nevertheless, it is an interesting fact that the previously suggested model appears as a particular case in the new framework. Also note that with the variable change $\chi=|\phi|(-g)^{-7/8}$, $\chi$ is only allowed to take positive values. However, the theory being symmetric under $\chi\mapsto-\chi$, one can equally allow for negative values of $\chi$. In that part of phase space, the matching of variables is given by $\chi=-|\phi|(-g)^{-7/8}$.

[10] Note that just like in the above example, this set of functions is only one representative of an infinite family of sets of functions that correspond to the same theory.

| | | |
|---|---|---|
| i. | Signs of kinetic terms | $\epsilon_\varsigma = \epsilon_\phi = \epsilon_A = 1$ . |
| ii. | Masses and relevant couplings | $m_\varsigma^2 = 2\lambda \frac{\alpha}{\lambda\xi_\chi} M^2(1+\mathcal{O}(\alpha))$,<br>$\kappa_\varsigma = 6\lambda\sqrt{\frac{\alpha}{\lambda\xi_\chi}} M(1+\mathcal{O}(\alpha))$,<br>$\lambda_\varsigma = 6\lambda(1+\mathcal{O}(\alpha))$,<br>$m_A^2 = e^2 \frac{\alpha}{\lambda\xi_\chi} M^2(1+\mathcal{O}(\alpha))$,<br>$\kappa_A = 2e^2\sqrt{\frac{\alpha}{\lambda\xi_\chi}} M(1+\mathcal{O}(\alpha))$,<br>$\lambda_A = 2e^2(1+\mathcal{O}(\alpha))$. |
| iii. | Suppression scales | $M_{\phi\phi}, M_V, M_{int}, M_{AA} \sim \frac{M}{\xi_h} < M$ . |

**Tab. 3.4:** Parameters for scale-invariant unimodular gravity (TDF (3.88)).

We have seen that the Lagrangian (3.89) (if one adds all SM matter and gauge fields) can describe a rich cosmological phenomenology if the parameters are positive and such that $\alpha \lll 1$, $\xi_\chi \sim \mathcal{O}(10^{-3})$, $\xi_h \sim \mathcal{O}(10^5)$ and $\lambda \lesssim 1$. The formalism of the present chapter makes it easy to check whether the same model yields a viable (SM like) particle physics phenomenology. For the case $\Lambda_0 = 0$, it is enough to check whether the model satisfies conditions 1-8 of table 3.1. To this end, we consider the expansion of the different functions around a constant solution $-g_0 = \sigma_0 = \sqrt{\frac{2\lambda}{\alpha}}$. The different parameters are summarized in Table 3.4. If $\xi_\chi$ and $\xi_h$ are in the phenomenologically interesting range and $\alpha$ is small enough such that it can be responsible for the hierarchy between $M$ and the particle physics scales, conditions 1-6 and 8 are fulfilled with very high accuracy. Note that this time the equations in condition 8 are not exact, but contain small corrections of the order $\mathcal{O}(\alpha)$.

All terms of (3.62), except the one proportional to $\tilde{\mathcal{K}}_\epsilon(\tilde{\sigma}) = 0$ and the one proportional to $\Lambda_0 = 0$, give rise to an infinite number of higher-dimensional operators. Depending on the values of the parameters, their suppression scales can be smaller than the Planck scale $M$. For the phenomenologically interesting parameters $\xi_\chi \sim \mathcal{O}(10^{-3})$ and $\xi_h \sim \mathcal{O}(10^5)$, the lowest suppression scales are of the order $M/\xi_h$. Although significantly smaller than the Planck scale, this scale is much higher then the scales relevant to particle physics. Hence, even though condition 7 is not satisfied, the higher-dimensional operators are still negligible at particle physics energies.

We conclude that at energies well below $M/\xi_h$, the theory given by the Lagrangian (3.58) with the TDF (3.88) (or equivalently the theory (3.89)) is also indistinguishable from the renormalizable Abelian Higgs model.

## 3.7 The case $\Lambda_0 \neq 0$, cosmology and dilaton interactions

So far, we have discussed the phenomenology of SI TDiff theories in the vicinity of constant backgrounds, i.e. around classical solutions with constant scalar fields and flat spacetime. The existence of such backgrounds was guaranteed by imposing the conditions $v(\sigma_0) = v'(\sigma_0) = 0$ (3.36) on the theory defining functions. We will for the moment stick to these conditions and comment on the case $v(\sigma_0) \neq 0$ at the end of this section. In a theory where conditions (3.36) hold, considering a flat background entails the choice $\Lambda_0 = 0$, to be thought of as an initial condition. Choosing $\Lambda_0 \neq 0$, one no longer obtains flat spacetime as a solution. Nevertheless, one can get interesting cosmological solutions. We now want to qualitatively discuss these cosmological solutions and see how the presence of such non-flat backgrounds affects particle physics. This is done most easily by considering the SI TDiff theory (3.24) in its equivalent Diff invariant formulation (3.31),

$$\begin{aligned} \frac{\mathcal{L}_e}{\sqrt{-\tilde{g}}} =& -\frac{1}{2}M^2\tilde{R} - \frac{1}{2}\epsilon_\sigma M^2(\partial\tilde{\sigma})^2 - \frac{1}{2}\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})(\partial\tilde{\phi})^2 \\ & - M^4\tilde{V}(\tilde{\sigma}) - \Lambda_0\tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma})\exp\left(-\frac{4\tilde{\phi}}{M}\right) + \mathcal{L}_m \ , \end{aligned} \tag{3.90}$$

where we have included a matter part $\mathcal{L}_m$. The term $\mathcal{L}_m$ contains all bosonic and fermionic degrees of freedom of the SM coupled to the scalar fields and gravity in the way described in Sections 3.3 and 3.4. Notice that the dependence of the potential on $\tilde{\phi}$ is *uniquely* determined by the way scale invariance is broken in TDiff theories. Consider now the homogeneous fields $\tilde{\sigma} = \tilde{\sigma}(t)$ and $\tilde{\phi} = \tilde{\phi}(t)$ living in spatially flat FLRW spacetime with metric

$$d\tilde{s}^2 = -dt^2 + a(t)^2 d\vec{x}^2 \ , \tag{3.91}$$

where $a(t)$ is the scale factor. The dynamics of the homogeneous scalar fields is mainly determined by the potential

$$\tilde{V}_{\Lambda_0}(\tilde{\sigma}, \tilde{\phi}) = M^4\tilde{V}(\tilde{\sigma}) + \Lambda_0\tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma})\exp\left(-\frac{4\tilde{\phi}}{M}\right) \ . \tag{3.92}$$

As long as the kinetic term of the scalar fields is positive definite, the scalar fields tend to roll down the potential, with some friction caused by the expansion of spacetime. In the $\tilde{\sigma}$-direction, the potential has a minimum at $\tilde{\sigma} = 0$ due to the conditions (3.36).[11] In the $\tilde{\phi}$-direction, the potential is governed by the exponential factor. If $\Lambda_0\tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma}) > 0$, the potential is of the

[11] Note that even if $v(\sigma_0) \neq 0$, the condition (3.34) would guarantee the existence of such a minimum.

run-away type, i.e. it gets minimal for $\tilde{\phi} \to \infty$. In the opposite – pathological – case, the potential is not bounded from below. A typical evolution of the scalar condensates $\tilde{\sigma}$ and $\tilde{\phi}$ will be the following: The first term of the potential $\tilde{V}_{\Lambda_0}$ drives the trajectories towards the "valley" $\tilde{\sigma} = 0$. Due to Hubble friction, the fields undergo damped oscillations around the valley before asymptotically approaching $\tilde{\sigma} = 0$. The second term in $\tilde{V}_{\Lambda_0}$ drives the trajectory towards $\tilde{\phi} \to \infty$. After $\tilde{\sigma}$ has settled down around $\tilde{\sigma} = 0$, this leads to a roll-down along the valley. We assume here that the function $\tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma})$ defined in (3.32) does not play a significant role in the cosmological evolution.

For appropriate choices of the theory-defining functions and initial conditions, the roll-down *towards* the valley $\tilde{\sigma} = 0$ can give a mechanism for inflation. During the subsequent roll-down *along* the valley, the scalar fields play the role of a dynamical dark-energy component (quintessence). A concrete realization of this scenario is given by the model of chapter 2. What we have found in the present chapter is that a potential with a minimum in one direction and run-away shape in the other direction is generic for SI TDiff theories in which scale invariance is broken spontaneously.

Let us now briefly discuss to what extent particle physics phenomenology is different around a typical time-evolving cosmological background ($\Lambda_0 > 0$), as compared to a static flat (Minkowski) background ($\Lambda_0 = 0$). Since the evolution drives $\tilde{\sigma} \to 0$, it seems reasonable to assume that in the present universe one has $\tilde{\sigma} \simeq 0$ (see also comments in [82]). If this is fulfilled, all masses and couplings of the SM particles are like in the case $\Lambda_0 = 0$, described in the above sections. The only effects of the cosmological background on particle physics would then come through $\tilde{\phi}(t)$. However, one can put simple and still very strong bounds on the influence of $\tilde{\phi}(t)$ by requiring that its energy density must not give a too big contribution to the total energy density of the universe. In other words, both the kinetic and the potential energy of the condensate $\tilde{\phi}(t)$ have to be smaller than today's critical energy density $\rho_{cr}^0 = 3M^2H_0^2 \simeq 10^{-120}M^4$, i.e.

$$\frac{1}{2}\tilde{\mathcal{K}}_{\phi\phi}\,(\tilde{\sigma}_0 = 0)\,(\partial_0\tilde{\phi})^2 < \rho_{cr}^0 \,, \tag{3.93}$$

$$\Lambda_0\tilde{V}(\tilde{\sigma}_0 = 0)\exp\left(-\frac{4\tilde{\phi}}{M}\right) < \rho_{cr}^0 \,. \tag{3.94}$$

These constraints, together with the conditions on the derivatives of $\tilde{\mathcal{K}}_{\phi\phi}(\tilde{\sigma})$ and $\tilde{V}(\tilde{\sigma})$ (3.76) plus similar conditions on the derivatives of $\tilde{\mathcal{K}}_{\Lambda_0}(\tilde{\sigma})$ guarantee, that all interactions induced by $\partial_0\tilde{\phi} \neq 0$ are highly suppressed and can be neglected in the description of local particle interactions.

Up to here, we have assumed the theory to be such that both conditions $v(\sigma_0) = v'(\sigma_0) = 0$ (3.36), respectively $v(\sigma_0) = 0$ together with condition (3.34), hold. The condition $v(\sigma_0) = 0$ is the analog of the condition $\beta = 0$ in the model of chapter 2. It makes for the absence of a cosmological constant and consequently for the fact that dark energy can be purely due to the scalar-field dynamics.

The discussion of the $\beta$-term of chapter 2 equally applies to the term $v(\sigma_0)$. In particular, fixing it to zero can be considered as very "unnatural" from a field-theoretical point of view (section (2.1.2)). We were only able to give some speculative arguments that might make such a choice favorable in the context of scale-invariant theories (section 2.1.3). If $v(\sigma_0) \neq 0$ (just like $\beta \neq 0$) the theory contains a cosmological constant. In order not to violate observational bounds, this constant has to be extremely small and hence does not affect particle physics phenomenology. Also the dynamical break-down of scale invariance and the associated mechanism for inflation remain practically unchanged. What changes, as already mentioned for the case $\beta \neq 0$, is the late time behaviour of cosmological solutions. Namely, dark energy gets a constant contribution on top of the constribution due to the scalar fields. Asymptotically, it is this constant contribution that comes to dominate the energy density of the universe.

With our present knowledge, both, the case where $v(\sigma_0)$ (respectively $\beta$) is zero and the case where it is very small such as not to violate the observational bound correspond to strong fine-tunings of the theory. The cosmological constant problem is therefore present in all SI TDiff theories. Still, in view of the arguments of section 2.1.3, scale-invariant theories can provide another perspective towards its solution.

## 3.8 Summary

Let us summarize the findings of this section.

- Reducing the spacetime symmetry group from Diff to TDiff makes it possible to construct scale-invariant theories of gravity and particle physics without introducing a dilaton *by hand*. The dilaton appears naturally as an additional scalar degree of freedom in the metric.

  The minimal model proposed in chapter 2 can be found as a particular case of the SI TDiff theories discussed here.

- SI TDiff theories where SI is spontaneously broken can be a viable alternative to standard GR plus SM. Being the Goldstone boson of spontaneously broken scale invariance, the dilaton practically decouples from all matter fields and hence hardly affects predictions for both gravitational and particle physics phenomena. If the theory-defining functions satisfy a number of conditions, the effective low-energy theory becomes indistinguishable from a renormalizable theory and can in particular reproduce SM phenomenology. While SI theories do not have the naturalness problem related to the stability of the Higgs mass against radiative corrections, the big difference between the gravitational scale $M$ and the particle physics scales remains unexplained and needs a fine-tuning of the parameters.

- In SI TDiff theories, scale invariance can be broken in two ways. The first one is due to the shape of the potential, which allows the scalar fields to take constant non-zero background values. Such symmetry breaking backgrounds lead to induced particle physics scales and an induced gravitational constant. Secondly, SI can be broken due to the appearance of an arbitrary constant $\Lambda_0$ in the equations of motion, respectively in the equivalent Diff invariant action. The effect of this constant is to give a "run-away" potential to the dilaton.

- SI TDiff theories generically have cosmological solutions similar to those of the minimal model discussed in chapter 2. Namely, a mechanism for inflation is provided by the dynamical evolution of the homogeneous scalar fields towards a symmetry-breaking minimum of the potential. For $\Lambda_0 \neq 0$, the dilaton moving down the induced run-away potential plays the role of a dynamical dark energy component.

- For generic choices of the theory defining functions, SI TDiff theories contain a cosmological constant (analog to the $\beta$-term in the previous chapter). Imposing a particular condition on the TDF makes for the absence of the cosmological constant, in which case dark energy is purely due to the dilaton. However, from the effective field theory point of view, imposing this condition is an unnatural fine-tuning (see, however, section 2.1.2). If the cosmological constant is not tuned to zero, it still has to be extremely small, in order not to violate the observational bound on the abundance of dark energy. As a consequence, in the scope of SI TDiff theories, including the particular case of chapter 2, the cosmological constant problem remains an open issue.

# 4. Quantum Scale Invariance

In the previous two chapters, we have considered scale invariance (SI) as a building principle to extend the standard theory of gravitation (GR) and the Standard Model of particle physics (SM). The discussion mostly remained on the level of classical field theory. We have found and discussed several interesting features of scale-invariant theories. First of all, the spontaneous break-down of SI gives a common origin to all scales (at least at the classical level). In other words, all dimensional parameters are induced by the non-zero background value of a scalar field. Then, in the context of cosmology, we found that theories with spontaneously-broken SI automatically provide a mechanism for primordial inflation. Namely, inflation can be due to the dynamical evolution of the background scalar fields towards a symmetry-breaking minimum of the potential. Despite their exact scale invariance, the constructed theories in general contain a cosmological constant. This constant is absent if the Jordan-frame potential is chosen to have a flat direction ($\beta = 0$). As long as gravity is described by GR, a scale-invariant theory necessarily contains a massless dilaton (Goldstone boson of broken SI) which has practically no influence on neither gravitational nor particle physics phenomenology. If GR is substituted by UG or TDiff, the dilaton can get an induced potential and play the role of a dynamical dark energy component.

We know that classical field theory is not the appropriate description of particle physics (and most probably gravity) and that the proposed classical theories should be quantized in the framework of quantum field theory. The findings of the preceding sections remain valid only if scale invariance is a symmetry of the full quantum effective action. However, it is well-known that scale invariance is anomalous for all realistic renormalizable quantum field theories (for a review see [47]). In other words, a quantum field theory built from a scale-invariant classical Lagrangian is in general no longer scale-invariant. Technically speaking, the anomaly appears because all common regularization procedures (cut-off, Pauli-Villars, dimensional regularization) introduce a new scale to the theory.

This suggests that we should expect all reasonings we made in the classical theory, and which were based on scale invariance, to be ruined by quantum corrections. The goal of this section is to show that this conclusion is not true and that scale invariance can still be a viable symmetry at the quantum level. We are going to introduce a modified version of the dimensional regularization of 't Hooft and Veltman [116], which allows us to construct a new class of effective field theories that have the following properties:

- Scale invariance is preserved on the quantum level to all orders of perturbation theory.
- Scale invariance is broken spontaneously, leading to a massless dilaton.
- The effective running of coupling constants is automatically reproduced at low energies.

In other words, the benefits of classical SI theories can all be present on the quantum level. At the same time, the standard results of quantum field theory, such as the running of coupling constants, remain in place.

In section 4.1, we present our idea and apply it to the case of a simple model of two scalar fields. Then, in section 4.2, we describe the generalization to the case of an arbitrary matter sector. In section 4.3, we discuss the inclusion of gravity. Conclusions together with a number of open questions are given in section 4.4.

## 4.1 Scalar field example

We want to explain our idea using the example of a simple system of two scalar fields in flat spacetime described in the classical theory by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}(\partial_\mu h)^2 - \frac{\lambda}{4}\left(h^2 - \zeta^2\chi^2\right)^2 \ . \tag{4.1}$$

Note that this Lagrangian corresponds to the scalar part of the theory (2.5) considered in chapter 2, if gravity is absent, $\beta = 0$ and $\zeta^2 \equiv \alpha/\lambda$. Hence, we will think of $h$ as the Higgs boson (in the unitary gauge) and of $\chi$ as the dilaton. Containing no dimensional parameters, the Lagrangian (4.1) is scale-invariant at the classical level. In fact, the requirement of dilatational symmetry does not forbid the presence of an additional term $\beta\chi^4$ in (4.1). We can briefly repeat the discussion of classical ground states of chapter 2. Namely, if $\beta < 0$, the theory does not have a stable ground state, while for $\beta > 0$, the ground state is unique and corresponds to $h = \chi = 0$. At the classical level, one would conclude that the theory describes two massless scalar excitations around the ground state respecting scale invariance. In the case $\beta = 0$, the potential contains two flat directions $h = \pm\zeta\chi$ and the ground state is infinitely degenerate. A ground state with $\chi = \chi_0 \neq 0$ spontaneously breaks the dilatational invariance. The theory around such a background contains a massive Higgs boson, $m_H^2(\chi_0) = 2\lambda\zeta^2(1+\zeta^2)\chi_0^2$, and a massless dilaton. So, the only choice for $\beta$, interesting for phenomenology, is $\beta = 0$. In the other cases, either a ground state does not exist or the theory does not contain any massive particles. Remember from chapter 2 that as soon as gravity is included (in a scale-invariant way), this conclusion is no longer true. In particular, the theory possesses solutions with constant scalar fields even if $\beta \neq 0$. Nevertheless, in the flat-space theory discussed here, $\beta = 0$ is the only choice yielding spontaneous symmetry breaking, and we will stick to it from now on.[1] The fact that in the absence of gravity $\beta = 0$ is the only viable case was actually presented in section 2.1.3 as an argument in favor of this special choice, even for the theory including gravity.

In what follows, we will also assume that $\zeta \lll 1$. In the model of chapter 2, this is motivated by phenomenological reasons, since $\zeta$ is responsible for the hierarchy between the Planck scale $M = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}$ GeV and the electroweak scale $v = 246$ GeV, hence $\zeta \sim v/M \sim 10^{-16}$. Note, however, that the smallness of $\zeta$ is not essential for the theoretical construction presented here.

It is well known what happens if the theory (4.1) is quantized with the use of the *standard* dimensional regularization procedure (cf. e.g. [117]). In $d$-dimensional spacetime (we use the convention $d = 4 - 2\epsilon$) the mass dimension of the scalar fields is $1 - \epsilon$ and the one of the coupling constant $\lambda$ is $2\epsilon$. Introducing a (finite) dimensionless coupling $\lambda_R$, one can write

$$\lambda = \mu^{2\epsilon} \left[ \lambda_R + \sum_{n=0}^{\infty} \frac{a_n}{\epsilon^n} \right] , \tag{4.2}$$

where $\mu$ is an arbitrary parameter with the dimension of a mass, and the Laurent series in $\epsilon$ corresponds to counter-terms. The parameters $a_n$ are to be fixed by the requirement that renormalized Green's functions are finite in every order of perturbation theory. Similar replacements have to be done with other parameters of the theory, and the factors $Z_\chi$ and $Z_h$, related to the renormalization of the fields, must be introduced (they do not appear at the one-loop level in our scalar theory). Then, in the $\overline{MS}$ subtraction scheme (cf. e.g. [117]), the one-loop contribution to the effective potential along the flat direction has the form

$$V_{\text{eff}}^{1l}(\chi) = \frac{m_H^4(\chi)}{64\pi^2} \left[ \log \frac{m_H^2(\chi)}{\mu^2} - \frac{3}{2} \right] , \tag{4.3}$$

spoiling the degeneracy of the ground state and thus leading to an explicit breaking of the dilatational symmetry. The vacuum expectation value of the field $\chi$ can be fixed by renormalization conditions [118]. The dilaton acquires a non-zero mass. It is the mismatch between the mass dimensions of bare ($\lambda$) and renormalized couplings ($\lambda_R$) that leads to the dilatational anomaly and thus to the explicit breaking of scale invariance (see [49] for a recent discussion). Put in other words, standard dimensional regularization does

[1] Some more comments on the case $\beta > 0$ in the present flat space theory will be given at the end of this section.

not preserve the scale symmetry, because the action analytically continued to $d$ dimensions is no longer scale-invariant.

Let us now introduce another prescription, which we will call the "SI prescription".[2] It consists in replacing $\mu^{2\epsilon}$ in (4.2) and in all other similar relations by (in general different) combinations of the fields $\chi$ and $h$, which have the correct mass dimension:

$$\mu^{2\epsilon} \to \chi^{\frac{2\epsilon}{1-\epsilon}} F_\epsilon(x) \ , \tag{4.4}$$

where $x = h/\chi$ and $F_\epsilon(x)$ is a function depending on the parameter $\epsilon$ with the property $F_0(x) = 1$. In principle, one can use different functions $F_\epsilon(x)$ for the various couplings. The resulting field theory is, by construction, scale-invariant in any number of spacetime dimensions $d$. This means that if, for instance, the $\overline{MS}$ subtraction scheme is used for calculations, the renormalized theory is also scale-invariant in any order of perturbation theory.

The sole requirement of scale invariance does not fix the details of the prescription, i.e. the functions $F_\epsilon(x)$. We can get some guidance by looking at the theory (2.5), which includes gravity. The form of the couplings of the scalar fields $\chi$ and $h$ to gravity indicates that the combination

$$\xi_\chi \chi^2 + \xi_h h^2 \equiv \omega^2 \tag{4.5}$$

plays a special role, being the effective Planck constant. This observation leads us to the simple "GR-SI prescription", in which

$$\mu^{2\epsilon} \to \left[\omega^2\right]^{\frac{\epsilon}{1-\epsilon}} \ , \tag{4.6}$$

corresponding to the choice of the function $F_\epsilon(x) = (\xi_\chi + \xi_h x^2)^{\frac{\epsilon}{1-\epsilon}}$. Below, we will apply the GR-SI prescription to the one-loop analysis of our scalar theory. For comparison, a result based on a different SI prescription is given in appendix B.

The SI construction is entirely perturbative and can be used *only* if scale invariance is spontaneously broken. In other words, in order to use the GR-SI prescription, the ground state has to be such that $(h_0, \chi_0) \neq (0, 0)$, because otherwise it is impossible to perform a perturbative expansion of (4.6). Let us develop the condition, which guarantees the existence of a symmetry-breaking ground state at the quantum level. To this end, consider the exact effective potential $V_{eff}(h, \chi)$ of our theory, constructed using the prescription (4.4) or (4.6), in the limit $\epsilon \to 0$. Because of exact SI, it can be written as

$$V_{\text{eff}}(h, \chi) = \chi^4 V_\chi(x) = h^4 V_h(x) \ . \tag{4.7}$$

For the ground state to exist, we need to have $V_\chi(x) \geq 0$ (or, what is the same, $V_h(x) \geq 0$) for all values of $x$. For the minimum of $V_{eff}(h, \chi)$

[2] A similar procedure was suggested in [51] in connection with the conformal (trace) anomaly.

to lie in the region where $\chi \neq 0$ (or $h \neq 0$), we must have $V_\chi(x_0) = 0$ ($V_h(x_0) = 0$), where $x_0$ is a solution of $V'_\chi(x_0) = 0$ ($V'_h(x_0) = 0$) and prime denotes the derivative with respect to $x$. If these conditions are satisfied, the effective potential has a flat direction corresponding to an infinite set of ground states that spontaneously break scale invariance. The dilaton is massless in all orders of perturbation theory. In this case, one can develop the perturbation theory around the vacuum state corresponding to $\chi_0 \neq 0$, $h_0 = x_0\chi_0$ with arbitrary $\chi_0$ (or $h_0 \neq 0$, $\chi_0 = h/x_0$ with arbitrary $h_0$).

To summarize, the SI prescriptions (4.4) or (4.6) have to be supplemented by the requirement that the quantum effective potential has a flat direction, i.e. that there exists a value $x_0$, for which

$$V_\chi(x_0) = V'_\chi(x_0) = 0 \tag{4.8}$$

or

$$V_h(x_0) = V'_h(x_0) = 0\ . \tag{4.9}$$

This leads to a new class of theories exhibiting spontaneously broken scale invariance, which is exact on quantum level.

The theories we construct are somewhat different from ordinary renormalizable theories. Their physics is determined not only by the values of "classical" coupling constants ($\lambda$ and $\zeta$ in our case), but also by "hidden" parameters contained in the functions $F_\epsilon(x)$. Still, as we will see shortly, for the SI-GR prescription, in the limit $\zeta \lll 1$ and for small energies $E \ll \chi_0$, only "classical" parameters matter. Moreover, they automatically acquire the necessary renormalization group running.

Let us now carry out a one-loop analysis of the theory (4.1) with the GR-SI prescription. We write the $d$-dimensional generalization of the classical potential as[3]

$$V = \frac{\lambda_R}{4}\left[\omega^2\right]^{\frac{\epsilon}{1-\epsilon}}\left[h^2 - \zeta_R^2\chi^2\right]^2 \tag{4.10}$$

and introduce the counter-terms

$$V_{c.t.} = \left[\omega^2\right]^{\frac{\epsilon}{1-\epsilon}}\left[Ah^2\chi^2\left(\frac{1}{\bar\epsilon} + a\right) + B\chi^4\left(\frac{1}{\bar\epsilon} + b\right) + Ch^4\left(\frac{1}{\bar\epsilon} + c\right)\right]\ ,$$

where $\frac{1}{\bar\epsilon} = \frac{1}{\epsilon} - \gamma + \log(4\pi)$, $\gamma$ is the Euler-Mascheroni constant and $a$, $b$, $c$, $A$, $B$, and $C$ are for the moment arbitrary constants. We do not introduce any modification of the kinetic terms, since no wave function renormalization is expected at the one loop level.

The one-loop effective potential for this theory can be computed in the usual way (cf. e.g. [117]). The counter-terms removing the divergences turn

[3] If we define the parameters $\alpha \equiv \sqrt{\lambda}$ and $\beta \equiv \sqrt{\lambda}\zeta^2$, the classical potential takes the form $V = \frac{1}{4}\left(\alpha h^2 - \beta\chi^2\right)^2$. In this notation, the GR-SI prescription corresponds to the substitutions $\alpha \to \left[\omega^2\right]^{\frac{\epsilon}{2(1-\epsilon)}}\alpha_R$ and $\beta \to \left[\omega^2\right]^{\frac{\epsilon}{2(1-\epsilon)}}\beta_R$.

out to coincide with those of the standard prescription and are given by

$$\begin{aligned} A &\to -\lambda_R^2\zeta_R^2\frac{3\zeta_R^4-4\zeta_R^2+3}{32\pi^2}\ , \\ B &\to \lambda_R^2\zeta_R^4\frac{9\zeta_R^4+1}{64\pi^2}\ , \\ C &\to \lambda_R^2\frac{\zeta_R^4+9}{64\pi^2}\ . \end{aligned} \tag{4.11}$$

The one-loop contribution to the effective potential has the generic form $V_{\rm eff}^{1l}=\chi^4 V_\chi^{1l}(x)$ and is given by a rather lengthy expression (presented in appendix C), which also depends on the "hidden" parameters. For a generic choice of $a$, $b$, and $c$, the classical flat direction $x_0=\zeta_R$ is lifted by quantum effects. However, the requirement $V_\chi^{1l}(\zeta_R)=V_\chi^{1l\prime}(\zeta_R)=0$ (4.8) allows to fix two of these parameters in a way such that the one-loop potential has exactly the same flat direction. For $\zeta_R \lll 1$, this requirement leads to[4]

$$\begin{aligned} b &= 3a+2\log\left(\frac{2\lambda_R\zeta_R^2}{\xi_\chi}\right)+\mathcal{O}\left(\zeta_R^2\right)\ , \\ c &= \frac{1}{3}\left[a+2-2\log\left(\frac{2\lambda_R\zeta_R^2}{\xi_\chi}\right)\right]+\mathcal{O}\left(\zeta_R^2\right)\ . \end{aligned} \tag{4.12}$$

The function $V_\chi^{1l}(x)$ is positive near the flat direction, provided that

$$a+2+2\log\left(\frac{2\lambda_R\zeta_R^2}{\xi_\chi}\right)>0\ .$$

It is interesting to look at the one-loop effective potential as a function of $h$, for $\chi=\chi_0$, $h\sim\zeta_R\chi_0\equiv v$ and $\zeta\lll 1$, i.e. $h_0\lll\chi_0$. It reads

$$\begin{aligned} V_{\rm eff}^{1l}\left(h,\chi_0\right) = &\ \frac{m^4(h)}{64\pi^2}\left[\log\frac{m^2(h)}{v^2}+\mathcal{O}\left(\zeta_R^2\right)\right] \\ &+\frac{\lambda_R^2}{64\pi^2}\left[C_0v^4+C_2v^2h^2+C_4h^4\right]+\mathcal{O}\left(\frac{h^6}{\chi^2}\right), \end{aligned} \tag{4.13}$$

where $m^2(h)=\lambda_R(3h^2-v^2)$ and

$$\begin{aligned} C_0 &= \frac{3}{2}\left[2a-1+2\log\left(\frac{\zeta_R^2}{\xi_\chi}\right)+\frac{4}{3}\log 2\lambda_R+O(\zeta_R^2)\right]\ , \\ C_2 &= -3\left[2a-3+2\log\left(\frac{\zeta_R^2}{\xi_\chi}\right)+O(\zeta_R^2)\right]\ , \\ C_4 &= \frac{3}{2}\left[2a-5+2\log\left(\frac{\zeta_R^2}{\xi_\chi}\right)-4\log 2\lambda_R+O(\zeta_R^2)\right]\ . \end{aligned} \tag{4.14}$$

[4] The truncation only serves to shorten the expressions. The exact expressions are given in appendix C.

The first term in (4.13) is exactly the standard one-loop effective potential for the theory (4.1) with the dynamical field $\chi$ replaced by a constant $\chi_0$. The rest is a quartic polynomial of $h$, coming from our GR-SI prescription, which leads to redefinitions of coupling constants, mass and vacuum energy. By construction, the coefficients are exactly such that the potential has its minimum at $h = h_0 = \zeta_R \chi_0$ and $V_{\text{eff}}^{1l}(h_0, \chi_0) = 0$.

One can see from (4.13) that the quantum corrections to the Higgs mass are proportional to $v^2 \propto \zeta_R^2 \chi_0^2$. This means that they are small compared to the classical value. Also in higher orders of perturbation theory, potentially dangerous corrections to the Higgs mass of the type $\lambda^n \chi_0^2$ (remember that $\chi_0 \sim M$) *cannot* appear. This can be understood from the following argument. For $\zeta = 0$, the Higgs field decouples from the dilaton at the classical level, and the dilaton field is described by a free theory. Therefore, if $\zeta = 0$, the (large) value of the field $\chi$ can appear in the effective potential only through logarithms coming from the expansion of $[\omega^2]^{\epsilon/1-\epsilon}$ in eq. 4.10, while for $\zeta \neq 0$, it appears at most as $\zeta_R^2 \chi^2$. Hence, in this theory there is no problem of instability of the Higgs mass against quantum corrections (cf. section 2.1.2).

At this point, an important comment is in order. As we will see in the upcoming formula (4.15), in the present theory the quantity $\sqrt{\xi_\chi}\chi_0$ plays the role of an effective cut-off scale. We have found that if one uses the proposed scale-invariant renormalization scheme, the Higgs mass does not obtain corrections proportional to this scale. Now, it is well-known that also in standard dimensional regularization the Higgs mass does not get this type of corrections. On the other hand, the use of cut-off regularization, for instance, yields corrections proportional to the cut-off. The point is that, unlike in a scale-invariant theory, in a non-scale-invariant theory and in the absence of a concrete hypothesis about the high-energy completion, it is not clear which type of regularization should be used and whether the large corrections to the Higgs mass should be expected (for a discussion of this point see e.g. [10]).

Consider now the high energy ($\sqrt{s} \gg v$ but $\sqrt{s} \ll \chi_0 \sim M$) behavior of scattering amplitudes using the example of Higgs-Higgs scattering (assuming, as usual, that $\zeta_R \lll 1$). In the one-loop approximation, the 4-point function is found to be

$$\Gamma_4 = \lambda_R + \frac{9\lambda_R^2}{64\pi^2}\left[\log\left(\frac{s}{\xi_\chi \chi_0^2}\right) + \text{const}\right] + \mathcal{O}\left(\zeta_R^2\right) \ . \tag{4.15}$$

This implies that at energies $\sqrt{s}$ such that $v \ll \sqrt{s} \ll \chi_0$ the effective Higgs self-coupling runs in the way prescribed by the ordinary renormalization group. Consequently, not only the tree Higgs mass is determined by the vev of the dilaton, but also all $\Lambda_{QCD}$-like parameters. We expect that these results remain valid in higher orders of perturbation theory.

The theories constructed with the SI prescription could be called renormalizable, if the introduction of a finite number of counter-terms were sufficient to remove all divergences and guarantee the existence of a flat direction in the potential. We have seen that at the one-loop level, the only counter-terms needed have the form of the terms already contained in the classical action. However, after the scale-invariant renormalization scheme was proposed, the authors of [119] have shown that starting from the two-loop level new divergences appear, which require the introduction of new types of counter-terms. Hence, unless one can find a way to resum the new terms such that they take the form of the terms contained in the classical action, the new theories are not renormalizable. Nevertheless, they are scale-invariant to all orders of perturbation theory and can be considered as effective theories below the scale $\sqrt{\xi_\chi}\chi_0 \sim M$.

Let us now add a comment on the case where the flat direction does not exist at the quantum level (classically this corresponds to $\beta > 0$). In this case, the ground state of the theory is scale-invariant. Theories of this type do not in general contain asymptotic particle states (for a review see [120]). If they do (this would correspond to anomalous dimensions for the fields equal to zero), the propagators will coincide with the free ones, leading to a theory with a trivial S-matrix [121, 122]. In other words, the requirement that a scale-invariant quantum field theory can be used for the description of interacting particles, existing as asymptotic states, singles out the class of theories with spontaneous breaking of scale invariance.

## 4.2 Scale-invariant quantum field theory: General formulation

It is straightforward to generalize the construction presented above to the case of theories containing fermions and gauge fields, such as the Standard Model. The mass dimension of a fermionic field is $\frac{3}{2} - \epsilon$, leading to the dimension of bare Yukawa couplings $F_B$ equal to $\epsilon$. The mass dimension of gauge fields can be fixed to 1 for any number of space-time dimensions $d$, leading to the dimensionality of the bare gauge coupling $g_B$ equal to $\epsilon$. Hence, in the standard procedure, one would choose $F_B \propto \mu^\epsilon F_R$ and $g_B \propto \mu^\epsilon g_R$, where the index $R$ refers to renormalized couplings. For the SI or GR-SI prescription, one replaces $\mu^\epsilon$ by a combination of scalar fields of appropriate mass dimension, as in (4.4) or in (4.6). For the perturbation theory to make sense, counter-terms have to be chosen such that the full effective potential has a flat direction, allowing for spontaneously-broken dilatational invariance.

## 4.3 Inclusion of gravity

The inclusion of scale-invariant gravity is carried out precisely along the same lines. The metric tensor $g_{\mu\nu}$ is dimensionless for any number of spacetime dimensions, while the mass dimension of the scalar curvature $R$ is always equal to 2. Therefore, the non-minimal couplings $\xi_\chi$, $\xi_h$ (see eq. (2.5)) are dimensionless and thus can only be multiplied by functions $F_\epsilon(x)$ of the type defined in (4.4). In addition to (2.5), the gravitational action may contain the operators $R^2$, $R_{\mu\nu}R^{\mu\nu}$, $\Box R$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, multiplied by $\chi^{\frac{-2\epsilon}{1-\epsilon}}F_\epsilon(x)$ (here $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are the Ricci and Riemann curvature tensors). These operators are actually needed for the renormalization of field theories in curved spacetime (for a review see [123]).

As we have seen in the previous chapters, if gravity is included, the dilaton decouples from all non-scalar particles of the SM (see also [29, 30, 124, 125]), and thus satisfies all laboratory and astrophysical constraints. From this point of view, the inclusion of gravity might be crucial.

If the scalar potential in the Jordan frame has a flat direction ($\beta = 0$), the cosmological constant is absent. However, in the presence of gravity the condition $\beta = 0$ might no longer be needed. Namely, as we have seen in chapters 2 and 3, if gravity is included in a scale-invariant way, then (at least at the classical level) scale invariance can be broken even if the J-frame potential does not have a flat direction, i.e. for $\beta \neq 0$. In fact, in that case a flat direction appears after changing to the E-frame, independently of the value of $\beta$.

We have seen in the previous chapters that if one considers a scale-invariant theory including gravity and transforms it to the E-frame, the original invariance under scale transformations is translated to invariance under global shifts of the dilaton field. This suggests that, taken in the E-frame formulation, the theory could actually be regularized with the usual regularization methods, as they do not violate the shift symmetry. For theories with gravity, this would present an alternative to the scale-invariant procedure discussed here. The main difficulty for the quantization of the E-frame action would be that in general the E-frame kinetic term is non-canonical and the potential non-polynomial.

## 4.4 Summary and open questions

We now want to summarize the results of this section.

- The proposed SI regularization procedure makes it possible to construct a class of effective quantum field theories, which are exactly scale-invariant to all orders in perturbation theory. In this type of theories, all mass scales, including those related to the running of coupling constants, are induced by the spontaneous break-down of scale invariance. The proposed theories are not renormalizable. However, they are valid effective field theories below a cut-off scale related to the background value of the dilaton (in this context see [30]).

- We considered a simple model with two scalar fields (Higgs, Dilaton), corresponding to the scalar part of the minimal SI extension of GR plus SM (2.5) presented in chapter 2. Using the GR-SI prescription, we found that quantum corrections to the Higgs mass depend only logarithmically on the effective cut-off scale, and hence the theory has no problem of instability of the Higgs mass against radiative corrections.

- It was described how the new procedure should be applied to a theory with arbitrary fields including gravity. In particular, it can be applied to the scale-invariant models considered in the previous sections. As a consequence, our findings based on scale invariance can remain true also at the quantum level. More generally, the existence of an SI quantization procedure legitimizes the use of SI as a building principle for new theories.

Several questions remain open. Let us give a partial list of them.

- The consistency of the procedure still needs to be studied more thoroughly. For instance, one should do explicit computations in order to see how the SI prescription makes for the conservation of the scale current.

- We made use of a scale-invariant version of dimensional regularization. However, also other common regularization methods can be made scale-invariant. In cut-off regularization, scale invariance can be achieved by using a field-dependent value for the cut-off scale (see [30]). Similarly, in Pauli-Villars regularization, scale invariance can be maintained if the Pauli-Villars masses are chosen to depend on the fields. In the case of lattice regularization, one can use a dynamical lattice spacing (see [126]). One should try to reproduce the results found in the present work by using these other regularization procedures.

- As the proposed construction is purely perturbative, it would also be interesting to look for a non-perturbative approach. A proposal in this direction, based on lattice regularization, has been discussed in [126].

# 5. Conclusions and Outlook

The Standard Model of particle physics (SM) as a description of matter at the fundamental level and the theory of General Relativity (GR) describing the gravitational interaction have both met great experimental confirmation. Based on the SM and GR is the hot big-bang model of cosmology, whose predictions are also in good agreement with the observable features of the present universe. Nevertheless, there exists a number of open issues, both at the level of the underlying theories, SM and GR, as well as at the level of cosmology, that calls for an extension of the standard theoretical framework.

In the present work, we considered minimal extensions of GR and the SM based on the ideas of Scale Invariance (SI), Unimodular Gravity (UG) and TDiff gravity. The task was twofold. On one hand, we had to implement these ideas in a theoretically consistent way, and such that the modified theories do not violate well-established theoretical bounds. On the other hand, we were interested in the cosmological phenomenology of the constructed theories, especially in their relevance in the context of inflation, dark energy and the cosmological constant problem.

We started in chapter 2 by considering a minimal scale-invariant extension of the classical actions of GR and the SM, where scale invariance was achieved through the introduction of a scalar dilaton field. In this model (SI plus GR) all scales at the classical level have a common origin, namely, they appear due to the spontaneous break-down of scale invariance. The exactly massless dilaton field does not violate experimental bounds, because it decouples from all SM fields except for the Higgs field, to which it only couples derivatively. It turned out that in this type of model, the dynamical break-down of scale invariance automatically provides a mechanism for inflation. We found that in the presence of non-minimally coupled scalar fields, SI does not forbid the presence of a cosmological constant. The cosmological constant is absent, as soon as the Jordan-frame potential has a flat direction ($\beta = 0$). From the effective field theory point of view, this corresponds to an unnatural fine-tuning. Hence, SI does not solve the cosmological constant problem. Still, we gave some theoretical arguments suggesting that the case in which the potential has a flat direction might play a special role. In particular, it is the only case in which scale symmetry can be spontaneously broken also in the absence of gravity.

In a next step we considered the same model with GR replaced by UG. This resulted in the appearance of an arbitrary constant $\Lambda_0$ (related to initial conditions) in the equations of motion, representing an additional spontaneous breaking of scale invariance. We found that in this model, unlike in pure UG, $\Lambda_0$ does not play the role of a cosmological constant, but rather gives rise to a run-away potential for the dilaton. It corresponds exactly to the type of potential introduced for quintessence long time ago [28, 30]. So, the combination of scale invariance and unimodular gravity lead us to a model that automatically provides a mechanism for inflation and moreover contains a dynamical dark energy component in the form of the dilaton. We found the parameters and initial conditions which give rise to successful inflation and late-time acceleration. In the considered minimal model, the parameter bounds coming from inflation were translated to a bound on the equation of state parameter of dark energy. In the case where the cosmological constant is absent ($\beta = 0$), the dilaton is the only component of dark energy. For that case, we found a functional relation between the scalar spectral index and the equation of state parameter of dark energy. Like in other models of quintessence, the observed abundance of dark energy can only be achieved by a precise tuning of initial conditions. Hence, in our model, the cosmic coincidence problem remains unsolved.

The introduction of the dilaton was essential for the construction of the minimal scale-invariant models involving GR or UG. However, simply adding a new scalar field to the theory might seem to be an *ad hoc* solution. In chapter 3, we considered the possibility to reduce the spacetime symmetries of the gravitational action from the group of all diffeomorphisms (Diff) to the group of transverse diffeomorphisms (TDiff). In theories invariant under TDiff, the metric generally contains an additional scalar degree of freedom, except for the particular cases given by GR and UG. Another generic feature of TDiff invariant theories is the appearance of an arbitrary constant $\Lambda_0$ in the equations of motion. We found that replacing Diff by TDiff allows for the construction of scale-invariant theories of gravity and matter, as soon as the matter sector contains a scalar field (which can be the SM Higgs field). In other words, in SI TDiff theories, the dilaton appears as a part of the metric. We found the conditions under which the constructed theories provide a viable description of particle physics and in particular reproduce the SM phenomenology. The minimal model based on scale invariance and UG turned out to be a particular case of a SI TDiff theory. The cosmological phenomenology of SI TDiff theories is generally similar to the phenomenology found in this particular case. To specify, these theories generically provide a mechanism for inflation as well as a dynamical dark energy component associated to the dilaton. Like in the minimal model, the smallness or absence of a cosmological constant requires a fine-tuning of parameters.

In usual quantum field theories, scale invariance is anomalous. Hence, one might expect the results of chapters 2 and 3, based on classical scale invariance, to be invalidated by quantum corrections. We presented in section 4 a new renormalization scheme that permits to construct quantum field theories, which are scale-invariant to all orders in perturbation theory and where SI is spontaneously broken. In these theories, all scales, also the ones related to the running of coupling constants, are consequences of the spontaneous breaking of SI. Although the so-constructed theories are not renormalizable, they are valid effective theories below an effective cut-off scale.

In summary, we found that it is possible to construct viable scale-invariant extensions of GR and the SM in which the gravitational scale as well as all particle physics scales, including those related to running couplings, are induced by the spontaneous break-down of SI. In these theories, the mass of the Higgs field is protected from large radiative corrections due to exact SI. Theories with spontaneously broken SI automatically provide a mechanism for inflation. If GR is replaced by UG or more generally by TDiff gravity, they contain an additional mechanism of spontaneous symmetry breaking, which allows the dilaton to be a dynamical dark energy component. In the considered theories, the naturalness issue related to the cosmological constant remains unsolved. Also, they cannot provide an explanation for the big differences between the Planck scale, the electroweak scale and the cosmological scale.

Related to the present work, there exists a number of questions that should be studied in more detail. Let us give a partial account of them.

- In the cosmological analysis of the model combining SI with UG, we assumed the dynamics of the dilaton field to be negligible during reheating. This assumption should be checked by a quantitative study of the reheating process including the dilaton. Also the assumption that in the same model no entropy perturbations are generated needs further confirmation. We have seen that current observational bounds on the equation of state parameter of dark energy do not allow to distinguish a dilaton condensate from a cosmological constant. One might ask, whether the presence of a dilaton field has other observable signatures.

- The proposed theories based on TDiff invariance contain a big degree of arbitrariness, due to the freedom in the choice of the theory-defining functions. One should try to find criteria (e.g. an additional symmetry) that allow to constrain the functional form of the TDF. One should also further investigate the difference betwen GR and TDiff gravity. For instance, one could try to understand how this difference manifests itself in the canonical quantization procedure based on the Hamiltonian formalism.

- One should try to get a deeper understanding of the constructed scale-invariant quantum field theories. As already mentioned, it would be important to explicitly compute the expectation value of the scale current, in order to see how its conservation arises as a consequence of the SI prescription. Another interesting point concerns the relation between the SI prescription and conformal invariance. In particular, one could check whether, when applied to a theory that is invariant under the full conformal group, the SI prescription allows to maintain the full invariance (or only dilatation invariance) at the quantum level. Yet another question to ask is, whether one can find a way to construct renormalizable scale-invariant theories.

To summarize, many interesting question are awaiting us.

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# Appendix

# A. Definition of the functional derivative

Consider the functional $I$, given by

$$I = \int d^4x \, F(y(x), \partial_\mu y(x), \partial_\mu \partial_\nu y(x), ...) \; , \tag{A.1}$$

where $F(x) = F(y(x), \partial_\mu y(x), \partial_\mu \partial_\nu y(x), ...)$ can be called a functional density. The functional derivative of $I$ with respect to a function $y(x)$ is defined as (cf. e.g. [127])

$$\frac{\delta I}{\delta y(x)} \equiv \int d^4x' \, \frac{\delta F(x)}{\delta y(x')} \; , \tag{A.2}$$

where the functional derivative of the functional density $F(x)$ is defined through

$$\frac{\delta F(x)}{\delta y(x')} = \delta(x-x') \frac{\partial F}{\partial y}(x) + \delta_{,\mu}(x-x') \frac{\partial F}{\partial [\partial_\mu y]}(x) + \delta_{,\mu\nu}(x-x') \frac{\partial F}{\partial [\partial_\mu \partial_\nu y]}(x) + ... \tag{A.3}$$

# B. One-loop analysis with an alternative SI prescription

For the GR-SI prescription considered in the Letter, physics well below the Planck scale associated with the dilaton vev $\chi_0$ was the same as for the ordinary renormalizable scalar theory containing the Higgs field $h$ only. This is not necessarily the case if the SI prescription given by Eq. (4.4) is used. Indeed, consider now a distinct way of continuing the scalar potential to $d$-dimensional space-time:[1]

$$V = \frac{\lambda_R}{4}\left[h^{\frac{2-\epsilon}{1-\epsilon}}x^{a_1\epsilon} - \zeta_R^2\chi^{\frac{2-\epsilon}{1-\epsilon}}x^{b_1\epsilon}\right]^2 \quad , \tag{B.1}$$

and introduce counter-terms for all terms appearing in the potential:

$$\begin{aligned} V_{c.t.} = \Big[ \quad & A\left(\frac{1}{\bar{\epsilon}}+a\right)h^{\frac{2-\epsilon}{1-\epsilon}}\chi^{\frac{2-\epsilon}{1-\epsilon}}x^{(a_1+b_1)\epsilon} + \\ & B\left(\frac{1}{\bar{\epsilon}}+b\right)\chi^{\frac{4-2\epsilon}{1-\epsilon}}x^{2b_1\epsilon} + C\left(\frac{1}{\bar{\epsilon}}+c\right)h^{\frac{4-2\epsilon}{1-\epsilon}}x^{2a_1\epsilon}\Big]. \end{aligned} \tag{B.2}$$

As before, we do not introduce any modification of the kinetic terms. Now we have more freedom in comparison with the GR-SI prescription due to the existence of new arbitrary parameters $a_1$ and $b_1$.

The coefficients $A$, $B$, and $C$ are fixed as in Eq. (4.11). The parameters $a_1$ and $b_1$ can be chosen in such a way that the one-loop effective potential does not contain terms $\chi^6/h^2$ and $h^6/\chi^2$, which are singular at $(0,0)$. These conditions lead to $a_1 = 0,\ b_1 = 0$. Then the requirement that the classical flat direction $x_0 = \zeta$ is not lifted by quantum effects gives (for $\zeta \lll 1$):

$$\begin{aligned} b &= 3a - 7 + 2\log(2\lambda_R) + \mathcal{O}\left(\zeta_R^2\right) \\ c &= \frac{1}{3}\left[a + 7 - 2\log(2\lambda_R)\right] + \mathcal{O}\left(\zeta_R^2\right) \ . \end{aligned} \tag{B.3}$$

With all these conditions satisfied the one-loop effective potential as a function of $h$ for $\chi = \chi_0$ fixed, $h \sim \zeta\chi_0 = v$ and $\zeta \lll 1$ is *different* from that in Eq. (4.13):

$$V_{\text{eff}}^{1l}\,(h,\chi_0) = \frac{m^4(h)}{64\pi^2}\left[\log\frac{m^2(h)}{v^2} + \mathcal{O}\left(\zeta_R^2\right)\right] + P_1\log\frac{h^2}{v^2} + P_2\ , \tag{B.4}$$

[1] In the notation with $\alpha \equiv \sqrt{\lambda}$ and $\beta \equiv \sqrt{\lambda}\zeta^2$, the prescription used here corresponds to the substitutions $\alpha \to h^{\frac{\epsilon}{1-\epsilon}}x^{a_1\epsilon}\alpha_R$ and $\beta \to \chi^{\frac{\epsilon}{1-\epsilon}}x^{b_1\epsilon}\beta_R$.

where $m^2(h) = \lambda_R(3h^2 - v^2)$ and $P_1,\ P_2$ are quadratic polynomials of $h^2$ and $v^2$. Though the first term is exactly the standard effective potential for the theory (4.1) with the dynamical field $\chi$ replaced by a constant $\chi_0$, the rest is not simply a redefinition of the coupling constants of the theory due to the presence of $\log \frac{h^2}{v^2}$. In other words, even the low-energy physics is modified in comparison with ordinary renormalizable theories.

# C. Exact expressions for the one-loop analysis with the GR-SI prescription

The exact expressions for the coefficients $b$ and $c$ of equation (4.12) are given by

$$b = \frac{1}{(1+9\zeta_R^4)(\xi_\chi+\zeta_R^2\xi_h)}\Bigg( a\left(3-4\zeta_R^2+3\zeta_R^4\right)\left(\xi_\chi+\zeta_R^2\xi_h\right) - 6\zeta_R^6\xi_h(-1+\mathrm{Log}[2])$$
$$+2\zeta_R^2(\xi_h(-5+\mathrm{Log}[2]) - \xi_\chi(-8+\mathrm{Log}[4]))$$
$$+\xi_\chi\mathrm{Log}[4] - 2\zeta_R^4(\xi_h(2+\mathrm{Log}[4]) + \xi_\chi(-8+\mathrm{Log}[8]))$$
$$-2\left(1+\zeta_R^2\right)\left(-1+3\zeta_R^2\right)\left(\xi_\chi+\zeta_R^2\xi_h\right)\mathrm{Log}\left[\frac{\zeta_R^2(1+\zeta_R^2)\lambda_R}{\xi_\chi+\zeta_R^2\xi_h}\right]\Bigg).$$

$$c = \frac{1}{(9+\zeta_R^4)(\xi_\chi+\zeta_R^2\xi_h)}\Bigg( \zeta_R^2\xi_h\Big(16 + a\left(3-4\zeta_R^2+3\zeta_R^4\right) - 4\zeta_R^2(-4+\mathrm{Log}[2])$$
$$-6\mathrm{Log}[2] + \zeta_R^4\mathrm{Log}[4]\Big) + \xi_\chi\Big(a\left(3-4\zeta_R^2+3\zeta_R^4\right)$$
$$+2\left(1+\zeta_R^2\right)\left(3+\zeta_R^2(-5+\mathrm{Log}[2]) - \mathrm{Log}[8]\right)\Big)$$
$$+2\left(-3+\zeta_R^2\right)\left(1+\zeta_R^2\right)\left(\xi_\chi+\zeta_R^2\xi_h\right)\mathrm{Log}\left[\frac{\zeta_R^2(1+\zeta_R^2)\lambda_R}{\xi_\chi+\zeta_R^2\xi_h}\right]\Bigg).$$

The exact expression of the one-loop contribution to the effective potential with a flat direction, obtained with the SI-GR prescription, is given by

$$V_{\mathrm{eff}}^{1l} = \frac{\lambda_R^2}{256\pi^2}\chi^4 f(x)\ , \tag{C.1}$$

where $f(x)$ is given by

$$
\begin{aligned}
f(x) = \;& 2\left(\zeta_R^4+9\zeta_R^8+x^4\left(9+\zeta_R^4\right)+x^2\left(-6\zeta_R^2+8\zeta_R^4-6\zeta_R^6\right)\right)\left(-3-2\mathrm{Log}\left[4\pi\right]\right)\\
&+\tfrac{4}{\xi_\chi+\zeta_R^2\xi_h}\Big(-2ax^2\zeta_R^2\left(3-4\zeta_R^2+3\zeta_R^4\right)\left(\xi_\chi+\zeta_R^2\xi_h\right)+a\zeta_R^4\left(3-4\zeta_R^2+3\zeta_R^4\right)\left(\xi_\chi+\zeta_R^2\xi_h\right)\\
&+2\zeta_R^6\left(1+\zeta_R^2\right)\left(8\xi_\chi+\left(-5+3\zeta_R^2\right)\xi_h\right)+x^4\Big(\left(3(2+a)-4(1+a)\zeta_R^2+(-10+3a)\zeta_R^4\right)\xi_\chi\\
&+\zeta_R^2\left(16+3a+4(4-a)\zeta_R^2+3a\zeta_R^4\right)\xi_h\Big)\Big)\\
&+4\left(\zeta_R^4+9\zeta_R^8+x^4\left(9+\zeta_R^4\right)+x^2\left(-6\zeta_R^2+8\zeta_R^4-6\zeta_R^6\right)\right)\mathrm{Log}\left[4\pi\left(\xi_\chi+x^2\xi_h\right)\right]\\
&+8\left(1+\zeta_R^2\right)\left(\zeta_R^4-3\zeta_R^6+x^4\left(-3+\zeta_R^2\right)\right)\mathrm{Log}\left[\tfrac{2\zeta_R^2\left(1+\zeta_R^2\right)\lambda_R}{\xi_\chi+\zeta_R^2\xi_h}\right]\\
&\\
&-p(x)^2\Big(-\mathrm{Log}[p(x)\lambda_R/2]+\tfrac{1}{p(x)r(x)(\xi_\chi+x^2\xi_h)^2}\Big(\\
&\left(x^2-\zeta_R^2\right)\Big(\zeta_R^2\xi_\chi\left(-7\left(r(x)+\zeta_R^2+3\zeta_R^4\right)\xi_\chi+\left(-r(x)+\zeta_R^2+3\zeta_R^4\right)\xi_h\right)-x^6\left(3+\zeta_R^2\right)(\xi_\chi-7\xi_h)\xi_h\\
&+x^4\left(\left(3+\zeta_R^2\right)\xi_\chi^2+\left(r(x)+3\left(9+7\zeta_R^2+4\zeta_R^4\right)\right)\xi_\chi\xi_h+\left(7r(x)-4\zeta_R^2+12\zeta_R^4\right)\xi_h^2\right)\\
&-x^2\left(\left(r(x)-4\zeta_R^2\left(\zeta_R^2-3\right)\right)\xi_\chi^2-3\left(3r(x)-(4+3r(x))\zeta_R^2-7\zeta_R^4-9\zeta_R^6\right)\xi_\chi\xi_h-\zeta_R^2\left(r(x)-\zeta_R^2-3\zeta_R^4\right)\xi_h^2\right)\Big)\\
&+2\left(\xi_\chi+x^2\xi_h\right)^2\mathrm{Log}\left[\xi_\chi+x^2\xi_h\right]\Big(-r(x)\zeta_R^2+\zeta_R^4+3r(x)\zeta_R^4+6\zeta_R^6+9\zeta_R^8+x^4\left(3+\zeta_R^2\right)^2\\
&-x^2\left(-3r(x)+(6+r(x))\zeta_R^2+4\zeta_R^4+6\zeta_R^6\right)\Big)\Big)\Big)\\
&\\
&-q(x)^2\Big(-\mathrm{Log}[-q(x)\lambda_R/2]-\tfrac{1}{q(x)r(x)(\xi_\chi+x^2\xi_h)^2}\Big(\\
&\left(x^2-\zeta_R^2\right)\Big(\zeta_R^2\xi_\chi\left(\zeta_R^2\left(1+3\zeta_R^2\right)(7\xi_\chi-\xi_h)-r(x)(7\xi_\chi+\xi_h)\right)+x^6\left(3+\zeta_R^2\right)(\xi_\chi-7\xi_h)\xi_h\\
&+x^4\left(-\left(3+\zeta_R^2\right)\xi_\chi^2+\left(r(x)-3\left(9+7\zeta_R^2+4\zeta_R^4\right)\right)\xi_\chi\xi_h+\left(7r(x)+4\zeta_R^2-12\zeta_R^4\right)\xi_h^2\right)\\
&-x^2\left(\left(r(x)+4\zeta_R^2\left(\zeta_R^2-3\right)\right)\xi_\chi^2-3\left(3r(x)+(4-3r(x))\zeta_R^2+7\zeta_R^4+9\zeta_R^6\right)\xi_\chi\xi_h-\zeta_R^2\left(r(x)+\zeta_R^2+3\zeta_R^4\right)\xi_h^2\right)\Big)\\
&-2\left(\xi_\chi+x^2\xi_h\right)^2\mathrm{Log}\left[\xi_\chi+x^2\xi_h\right]\Big(\zeta_R^2\Big(r(x)-3r(x)\zeta_R^2+\left(\zeta_R+3\zeta_R^3\right)^2\Big)+x^4\left(3+\zeta_R^2\right)^2\\
&+x^2\left(-3r(x)+(-6+r(x))\zeta_R^2-4\zeta_R^4-6\zeta_R^6\right)\Big)\Big)\Big)\,,
\end{aligned}
\tag{C.2}
$$

with $x=\frac{h}{\chi}$ and the definitions

$$
\begin{aligned}
r(x) &= \sqrt{\zeta_R^4(1+3\zeta_R^2)^2-2x^2\zeta_R^2(3+2\zeta_R^2+3\zeta_R^4)+x^4(3+\zeta_R^2)^2}\,,\\
p(x) &= r(x)-\left(\zeta_R^2-3\zeta_R^4-x^2(3-\zeta_R^2)\right)\,,\\
q(x) &= r(x)+\left(\zeta_R^2-3\zeta_R^4-x^2(3-\zeta_R^2)\right)\,.
\end{aligned}
$$

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