Densities with spherical level sets in the Gauss-exponential domain [1] [2]

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Abstract

For a sequence of independent observations $\mathbf{Z}_n = (X_n, Y_n)$ from the bivariate standard normal density there are at least three asymptotic descriptions of the sample clouds:

- Let $r_n = \sqrt{2 \log n}$. The sample clouds $\{\mathbf{Z}_1/r_n, \ldots, \mathbf{Z}_n/r_n\}$ converge onto the closed unit disk E. The probability of a sample point outside a disk of radius $r > 1$ tends to zero, as does the probability for any $m \geq 1$ of less than m sample points in a disk centered in a point of E.
- The two components of $\mathbf{Z}_1$ are independent, and hence asymptotically independent. The margins lie in the domain of the Gumbel distribution. Hence the sample clouds converge in distribution to a Poisson point process on the space $\mathcal{X} = [-\infty, \infty]^2 \setminus \{(-\infty, -\infty)\}$. The mean measure lives on the two boundary lines in $-\infty$.
- Let H_n be halfplanes such that $\mathbb{P}\{\mathbf{Z} \in H_n\} = 1/n$. The high risk scenario $\mathbf{Z}^{H_n}$ describes the vector $\mathbf{Z}$ conditioned to lie in the halfplane H_n. There exist affine transformations α_n mapping the upper halfplane $\{v \geq 0\}$ onto H_n such that $\alpha_n^{-1}(\mathbf{Z}^{H_n}) \Rightarrow \mathbf{W} = (U, V)$. The components U and V are independent variables, Gaussian and exponential. Under the same normalization the sample clouds converge in distribution to a Poisson point process N on the plane with Gauss-exponential intensity.

This paper looks at sample clouds from light-tailed unimodal densities with spherical level sets. What conditions will give the asymptotic behaviour above? We do not assume that the level sets are concentric.

1 Introduction

The *Gauss-exponential point process* is a Poisson point process on $\mathbb{R}^d$ with intensity $e^{-\chi}$ where

$$\chi(\mathbf{w}) = \mathbf{u}^T\mathbf{u}/2 + v \qquad \mathbf{w} = (\mathbf{u}, v) \in \mathbb{R}^{d-1} \times \mathbb{R}. \tag{1.1}$$

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This point process plays the role of the Gaussian distribution among the point processes. Its intensity is more symmetric then the d-dimensional Gaussian density. The level sets of the intensity are parabolas, vertical translates of the standard parabola P,

$$\{e^{-\chi} > e^{-t}\} = P + t\mathbf{e}_d \qquad P = \{v < -\mathbf{u}^T\mathbf{u}/2\}. \tag{1.2}$$

The mean measure ρ has the property that

$$\rho(A + s\mathbf{e}_d) = e^{-s}\rho(A) \qquad s \in \mathbb{R},\ A \text{ a Borel set in } \mathbb{R}^d.$$

Densities with spherical level sets are simple to describe. The set $\{g > e^{-t}\}$ is an open ball of radius r_t centered in $\mathbf{p}_t$, and may be written as $B_t = \mathbf{p}_t + r_t B$, where B denotes the open unit ball in $\mathbb{R}^d$. We assume that the densities g are continuous. Hence $\mathrm{cl}(B_s) \subset B_t$ for $s < t$. This implies $\|\mathbf{p}_t - \mathbf{p}_s\| < r_t - r_s$ for $s < t$. The set $\{g > 0\} = B_\infty = \bigcup B_n$ is an open ball, an open halfspace or $\mathbb{R}^d$.

Definition 1 *($\mathcal{S}, \mathcal{S}_0$): $\mathcal{S}$ is the set of all continuous functions $g \geq 0$ with spherical level sets, $B_t = \{g > e^{-t}\} = \mathbf{p}_t + r_t B$, and $r_{n+1}/r_n \to 1$. The subset $\mathcal{S}_0$ consists of those functions which are determined by the sequence of balls B_n. The balls B_t for $n < t = n + \theta < n + 1$ are defined by linear interpolation:*

$$\mathbf{p}_t = \mathbf{p}_n + \theta(\mathbf{p}_{n+1} - \mathbf{p}_n) \qquad r_t = r_n + \theta R_{n+1} \qquad R_n = r_n - r_{n-1}. \tag{1.3}$$

Functions $g \in \mathcal{S}$ are integrable since they are light-tailed: $r_{n+1} \sim r_n$ implies $r_n < e^{\epsilon n}$ eventually for any $\epsilon > 0$. By altering the function g on a compact subset of its domain B_∞ one can make it into a density $g_0 \in \mathcal{S}$ (or $\mathcal{S}_0$). Since we are only concerned with the asymptotic behaviour we drop the assumption that the function g has integral one.

The densities which we are really interested in are unimodal densities with convex level sets. This paper may be regarded as an attempt to gain insight in the asymptotics of such densities. Spherical level sets make it possible to construct examples and counterexamples, while their asymptotic behaviour still is sufficiently complex to warrant interest. A density with smooth convex level sets may be altered in a given direction and still have smooth convex level sets. For densities in $\mathcal{S}$ such changes in the asymptotic behaviour in a particular direction are not possible. This makes $\mathcal{S}$ an ideal class to study the domain $\mathcal{D}$ of Gauss-exponential attraction in the global sense. In Example I it will be shown that the Gauss-exponential asymptotics may break down in a quite complicated way even for functions whose level sets are balls. Example II shows how bad behaviour on the lower halfplane of convex level sets which agree with disks in the upper halfplane may affect the Gauss-exponential asymptotics for horizontal halfplanes.

Definition 2 *($\mathcal{D}, \mathcal{D}(\omega), \mathcal{D}_0(\omega)$): Let g be a continuous integrable non-negative function on $\mathbb{R}^d$ with convex level sets. For any halfspace $H = \{\omega \geq t\}$ let p_H denote the integral of $g1_H$. The function g lies in the* Gauss-exponential domain, *and we write $g \in \mathcal{D}$, if for any sequence of halfspaces H_n for which p_{H_n} is positive and vanishes for $n \to \infty$ there exist affine transformations $\alpha_n : \mathbf{w} \mapsto \mathbf{q}_n + A_n\mathbf{w}$, where the A_n are invertible linear transformations, such that α_n maps the upper halfspace $J_+ = \{(\mathbf{u}, v) \mid v \geq 0\}$ onto H_n, and such that*

$$h_n(\mathbf{w}) := \frac{g(\alpha_n(\mathbf{w}))}{g(\alpha_n(\mathbf{0}))} \to h(\mathbf{w}) = e^{-\chi(\mathbf{u},v)} \qquad \mathbf{w} = (\mathbf{u}, v) \in \mathbb{R}^d. \tag{1.4}$$

Convergence should hold uniformly on compact sets and in $\mathbf{L}^1$ on all halfspaces $\{v \geq v_0 + \mathbf{a}^T\mathbf{u}\}$. We write $g \in \mathcal{D}(\omega)$ for a direction $\omega \in \partial B$ if the limit relation holds for all halfspaces $H_n = \{\omega_n \geq t_n\}$ with $\omega_n \to \omega$, and $g \in \mathcal{D}_0(\omega)$ if it holds for (parallel) halfspaces $\{\omega \geq t_n\}$. ◊

There is a second reason for our interest in densities with spherical level sets. The limit relation

$$(f(t+s) - f(t))/a(t) \to s \qquad t \to \infty,\ s \in \mathbb{R}, \tag{1.5}$$

plays an important role in the de Haan theory of univariate extremes and second order regular variation, see (1.4) in [7]. The same limit relation also crops up in the theory of regular variation for matrices, which describes the asymptotic behaviour of perturbed linear differential equations and the stability of dynamical systems. The function f is increasing with a slope given by $a(t)$ which varies slowly in the additive sense. The level sets of $g \in \mathcal{S} \cap \mathcal{D}$ satisfy an asymptotic relation which may be described as a geometric analogue of (1.5). The function $t \mapsto r_t$ describing the size of the balls B_t satisfies (1.5) with $a(t) = r_t - r_{t-1}$. But in the geometric setting we also need conditions on the change in the direction in which the centers of the balls move, and on the excentricity of the balls, the departure from concentricity. How does one describe in geometric terms the regular variation of the balls $\{g > e^{-t}\}$ which will yield Gauss-exponential asymptotics? That is the question which will be addressed in this paper.

If g is the density of a vector $\mathbf{Z}$ and H is a halfspace for which $p_H = \mathbb{P}\{\mathbf{Z} \in H\}$ is positive then the *high risk scenario* $\mathbf{Z}^H$ is the vector $\mathbf{Z}$ conditioned to lie in H. The high risk scenario has density $g1_H/p_H$. The $\mathbf{L}^1$ convergence in the definition of $\mathcal{D}$ implies convergence of the normalized high risk scenarios

$$\mathbf{W}_n = \alpha_n^{-1}(\mathbf{Z}^{H_n}) \Rightarrow \mathbf{W} = (\mathbf{U}, V), \tag{1.6}$$

where $\mathbf{U}$ is standard Gaussian on $\mathbb{R}^{d-1}$ and V is standard exponential on $[0,\infty)$ and independent of $\mathbf{U}$. We write $\Rightarrow$ for convergence in distribution. If $p_{H_n} \sim c/n$ then sample clouds with the same normalization converge. Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ be independent observations from the density g. Then for $v_0 \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^{d-1}$

$$N_n = \{\alpha_n^{-1}(\mathbf{Z}_1), \dots, \alpha_n^{-1}(\mathbf{Z}_n)\} \Rightarrow N \quad \text{weakly on } J = \{v \geq v_0 + \mathbf{a}^T\mathbf{u}\}. \qquad (1.7)$$

Here N is the Poisson point process with intensity $c_0 e^{-\chi}$ where $c_0 = c/(2\pi)^{(d-1)/2}$.

The main question in this paper is: What conditions should the sequence of balls B_n satisfy in order that the corresponding function g in $\mathcal{S}_0$ lies in $\mathcal{D}$?

Univariate asymptotics

A brief description of the univariate case may help the reader to put the multivariate results into perspective.

There exists a simple procedure for constructing continuous decreasing densities g on $[0,\infty)$ which lie in the domain of attraction of the Gumbel distribution for maxima. Moreover, see [2] Section 6.6, any df in the domain of attraction is tail asymptotic to a df with a density g as above.

Here is the construction. Choose a sequence of positive reals R_n such that $R_{n+1} \sim R_n$. Set $r_n = R_1 + \cdots + R_n$ and $r_\infty = \sup r_n \leq \infty$. Define $g_0 = e^{-\psi}$ where ψ is the continuous piecewise linear function on $[0, r_\infty)$ with the value n in r_n. The function g_0 has level sets $\{g > e^{-n}\} = [0, r_n)$. Let R be the piecewise linear function with the value R_{n+1} in r_n. By construction the slope of the function ψ is asymptotic to $1/R(r)$ on intervals of length $O(R(r))$ around r. This yields an exponential limit:

$$g_0 \in \mathcal{D}^+ \qquad \Longleftrightarrow \qquad g_0(r + vR(r))/g_0(r) \to e^{-v} \qquad r \to r_\infty,\ v \geq 0. \quad (1.8)$$

The limit relation holds uniformly and in $\mathbf{L}^1$ on $[v_0, \infty)$ for $v_0 \in \mathbb{R}$.

What does a equence (r_n) look like whose increments satisfy the condition $R_{n+1} \sim R_n$? One may choose $R_n = n^\rho$ for $\rho \in \mathbb{R}$. Or take a positive C^1 function a on $[0,\infty)$ whose derivative vanishes at ∞. Set $r_0 = 0$ and $r_{n+1} = r_n + a(r_n)$. Then $R_{n+1} = a(r_n) \sim R_n$. The function a may converge or tend to infinity, but it may also fluctuate. One may construct positive C^1 functions a on $[0,\infty)$ with vanishing derivative in ∞ such that $a(s_n) >> \sqrt{s_n}$ for a sequence $s_n \to \infty$, and also $a(t_n) << e^{-t_n}$ for a sequence $t_n \to \infty$.

The limit relation (1.8) implies convergence of the sample clouds N_n from the density g_0. The limit N is a Poisson point process on $\mathbb{R}$ with intensity e^{-v}, which gives a good description of the extremes for the density g_0. The sample maxima converge:

$$M_m = \max N_n = \alpha_n^{-1}(Y_1) \vee \cdots \vee \alpha_n^{-1}(Y_n) \Rightarrow V_0 = \max N.$$

The same holds for the extreme upper order statistics $V_{n,n-k} = \alpha_n^{-1}(Y_{n,n-k})$. If we order the points of N in a decreasing sequence $V_0 > V_1 > \cdots$, then $(V_{n,n}, \ldots, V_{n,n-m}) \Rightarrow (V_0, \ldots, V_m)$ holds for any $m \geq 0$. Similarly for $g \in \mathcal{S}_0 \cap \mathcal{D}_0(\omega)$ the limit Gauss-exponential point process N contains all information about the extremes of the sample clouds in the direction ω. In [4] and [5] the points $\mathbf{W}_n = (\mathbf{U}_n, V_n) \in N$ are ordered in decreasing order of the vertical coordinate. The horizontal parts $\mathbf{U}_n$ may then be regarded as independent marks with a standard Gaussian distribution.

Example 1 In dimension $d = 1$ let $g \in \mathcal{S}_0$ have level sets $\{g > e^{-n}\} = B_n = (-s_n, t_n)$ with $t_n = 2 - 1/n$ for $n \geq 1$ and $s_n = e^{\sqrt{n}}$. Set $T_n = t_n - t_{n-1}$ and $S_n = s_n - s_{n-1}$. Then $T_n \sim 1/n^2$ implies $T_{n+1} \sim T_n$. So too $S_{n+1} \sim S_n$. It follows that $g \in \mathcal{D}$, even though the normalizations for the two tails are completely different. The one-dimensional ball B_n has center $p_n = (t_n - s_n)/2$ and radius $r_n = (t_n + s_n)/2$. Now construct the multivariate function $g \in \mathcal{S}_0$ with level sets $\{g > e^{-n}\} = (\mathbf{0}, p_n) + r_n B$. Does it lie in $\mathcal{D}$? $\diamondsuit$

The three limits

For the Gaussian density the sample clouds converge onto the closed unit ball. This result is due to [6]. It also holds for densities $g \in \mathcal{S}$. The only difference is in the normalization. The points of the sample clouds now have the form $(\mathbf{Z}_i - \mathbf{p}_{t_n})/r_{t_n}$ where $t_n = \log n$. Asymptotic equality $r_{n+1} \sim r_n$ implies that the intensities tend to infinity uniformly on compact subsets of the unit ball, and to zero uniformly and in $\mathbf{L}^1$ on the complement of the ball rB for any $r > 1$.

For the standard Gaussian vector the components are independent and hence asymptotically independent. Asymptotic independence also holds for the densities $g \in \mathcal{S}$. This follows from the previous result. The unit ball is a smooth convex set. Hence, see [3], the random variables $Y_i = \eta_i(\mathbf{Z})$, $i = 1, \ldots, d$, are asymptotically independent if the functionals $\eta_1, \ldots, \eta_d$ are linearly independent. Extremes in different directions come from sample points in disjoint regions.

The space $\mathcal{X} = [-\infty, \infty]^d \setminus \{(-\infty, \ldots, -\infty)\}$ was introduced in [1] and used in [8] to describe the convergence of the exponent measures for coordinatewise extremes. Asymptotic independence will yield a limiting Poisson point process on $\mathcal{X}$ with a mean measure which lives on the d lines in $-\infty$ if the tails of the

d margins of the underlying df lie in the domain of the Gumbel law. Conditions for this are given in Section 2 below.

The Gauss-exponential limit is a different matter. There one needs extra conditions. The Gauss-exponential point process as a description of the asymptotic behaviour of Gaussian sample clouds at the edge is due to [4]. The standard Gaussian density has spherical symmetry. The multivariate function $g(\mathbf{z}) = g_0(\|\mathbf{z}\|)$ lies in $\mathcal{D}$ if $g_0 \in \mathcal{D}^+$. One may replace the level sets $r_t B$ of g by scaled copies $r_t D$ of an egg-shaped set D, see [2], Theorem 9.1. It follows that $g \in \mathcal{S}_0$ lies in $\mathcal{D}$ if the balls B_n are centered $B_n = r_n B$ and the increments $R_n = r_n - r_{n-1}$ are asymptotically equal. The result remains true if $B_n = r_n D$ where D is a translate of the unit ball B which contains the origin.

Let us give a brief sketch of the results of the paper. Section 2 describes $\mathcal{D}_0(\omega)$ for the vertical direction $\omega = \eta = (\mathbf{0}, 1)$. The condition for $\mathcal{D}_0(\eta)$ is simple. The boundaries of the horizontal halfspaces supporting the balls B_n associated with $g \in \mathcal{S}_0$ should be asymptotically equidistant. The sample clouds then converge weakly on all halfspaces $J = \{v \geq v_0 + \mathbf{a}^T\mathbf{u}\}$. We shall derive a simple asymptotic expression for the marginal density. Section 3 characterizes $\mathcal{D} \cap \mathcal{S}$. For $g \in \mathcal{S}_0$ to lie in the domain of attraction $\mathcal{D}$ the proper condition is AED, the boundaries of the three halfspaces supporting successive balls B_{n-1}, B_n, B_{n+1} should be asymptotically equidistant uniformly in the direction. We give a number of alternative formulations of AED. The last section contains two examples. In the first we construct a function $g \in \mathcal{S}_0$ which lies in $\mathcal{D}_0(\omega)$ for all ω, but not in $\mathcal{D}$. In the second we construct a continuous light-tailed bivariate density f with convex level sets for which the high risk scenarios for horizontal halfplanes converge. The corresponding sample clouds N_n converge weakly on all horizontal halfplanes $J = \{v \geq v_0\}$, but $N_n(J) \to \infty$ in probability for any non-horizontal halfplane J. The density has some other nice features. It is strictly positive and on the upper halfplane it agrees with a function $g \in \mathcal{S} \cap \mathcal{D}$.

Notation

We write B for the open unit ball $\{\mathbf{z}^T\mathbf{z} < 1\}$, ∂B for its boundary, the unit sphere, and $|A|$ for the volume (Lebesgue measure) of the Borel set A. For bounded open sets we write $U_0 \Subset U_1$ if U_1 contains the closure $\mathrm{cl}(U_0)$. Halfspaces with direction ω have the form $\{\omega \geq t\}$ for $\omega \in \partial B$ and $t \in \mathbb{R}$. Halfspaces are closed. The bounded open convex set U *supports* the halfspace H in the point $\mathbf{p}$ if H and U are disjoint and $\mathbf{p} \in \partial H \cap \partial U$. *Horizontal halfspaces* have the form $H = \{y \geq y_0\}$; the *upper halfspace* is $J_+ = \{v \geq 0\}$. We work with two sets of coordinates. The original coordinates are $\mathbf{z} = (\mathbf{x}, y)$; the normalized coordinates are $\mathbf{w} = (\mathbf{u}, v)$.

Here y and v denotes the vertical component and $\mathbf{x}$ and $\mathbf{u}$ the horizontal part.

2 High risk scenarios for horizontal halfspaces

For convergence of high risk scenarios on horizontal halfspaces and the associated sample clouds the condition for $g \in \mathcal{S}_0$ is simple. The horizontal tangent planes to the tops of the balls $B_n = \{g > e^{-n}\}$ should be asymptotically equidistant.

Theorem 2.1 *Suppose $g \in \mathcal{S}_0$. Let $\mathbf{z}_t = (\mathbf{x}_t, y_t)$ be the point on the boundary of $B_t = \{g > e^{-t}\}$ with the maximal y-coordinate. Set $a_t := y_t - y_{t-1}$ and $c_t = \sqrt{a_t r_t}$, where r_t is the radius of B_t. If $a_{n+1} \sim a_n$ then*

$$e^t g(\mathbf{x}_t + c_t\mathbf{u}, y_t + a_t v) \to e^{-\chi(\mathbf{u},v)} \qquad t \to \infty \tag{2.1}$$

uniformly on compact sets in $\mathbb{R}^d$ and in $\mathbf{L}^1$ on all halfspaces $J = \{v \geq v_0 + \mathbf{a}^T\mathbf{u}\}$ with $v_0 \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^{d-1}$.

Corollary 2.2 *The integral of g over the horizontal halfspace $H_t = \{y \geq y_t\}$ supporting the ball B_t satisfies*

$$p_t = \int g 1_{H_t} \sim a_t(2\pi a_t r_t)^{(d-1)/2}/e^t \qquad t \to \infty. \tag{2.2}$$

Proof By $\mathbf{L}^1$ convergence the integral of h_t over J_+ converges to $(2\pi)^{(d-1)/2}$, the integral of h over J_+. The density of $\alpha_t^{-1}(\mathbf{Z})$ is $D_t e^{-t} h_t$ where D_t is the absolute value of the determinant of the linear part of α_t. ¶

These results allow us to compute for probability densities in $\mathcal{S}_0$ the asymptotic value of $\mathbb{P}\{\mathbf{Z} \in A\}$ for horizontal halfspaces far up, but also for Borel subsets of such halfspaces whose boundary has zero Lebesgue measure.

We shall prove the theorem in two steps, first proving pointwise convergence uniformly on compact sets, and then proving $\mathbf{L}^1$ convergence on halfspaces of the form $\{v \geq v_0 + \mathbf{a}^T\mathbf{u}\}$. It suffices to prove $h_n \to h$ since

$$h_{n+\theta}(\mathbf{w}) = e^\theta e^n g(\alpha_n \alpha_n^{-1} \alpha_{n+\theta}(\mathbf{w})) = e^\theta h_n(\alpha_n^{-1}\alpha_{n+\theta}(\mathbf{w}))$$

and $\beta_n = \alpha_n^{-1}\alpha_{n+\theta_n} \to \tau^\theta : \mathbf{w} \mapsto \mathbf{w} + \theta\mathbf{e}_d$ if $\theta_n \to \theta \in \mathbb{R}$. This implies

$$e^{\theta_n} h_n(\beta_n(\mathbf{w})) \to e^\theta h(\mathbf{w} + \theta\mathbf{e}_d) = h(\mathbf{w}).$$

This limit relation holds uniformly on compact sets. It holds in $\mathbf{L}^1$ on halfspaces H of the form above since $\mathbf{L}^1$ convergence $f_n \to f$ implies $\mathbf{L}^1$ convergence $f_n \circ$

$\beta_n \to f \circ \beta$ if $\beta_n \to \beta$ and if f is continuous a.e., and since $h_n 1_{H_n} \to h 1_{H_0}$ in $\mathbf{L}^1$ if the halfspaces converge: $H_n = \{v \geq v_n + \mathbf{a}_n^T \mathbf{u}\}$ for $n \geq 0$ and $v_n \to v_0$, $\mathbf{a}_n \to \mathbf{a}_0$. So $\mathbf{L}^1$ convergence of $(h_n 1_{\beta_n H})(\beta_n(\mathbf{w}))$ gives $\mathbf{L}^1$ convergence of $h_n(\beta_n(\mathbf{w}))1_H(\mathbf{w})$.

We start out with an increasing sequence of balls B_n. Choose new coordinates $\mathbf{w} = (\mathbf{u}, v)$ with the origin in the boundary point $\mathbf{z}_n = (\mathbf{x}_n, y_n)$ of B_n with the largest y-coordinate, and rescale the vertical and horizontal coordinates so that B_n becomes an elongated cylinder symmetric ellipsoid which approximates the standard paraboloid P in (1.2). The ellipsoids

$$E_t = \operatorname{diag}(\sqrt{t}, \ldots, \sqrt{t}, t)\tilde{B} \qquad \tilde{B} = B - \mathbf{e}_d \qquad t > 0 \tag{2.3}$$

increase to P as t increases to ∞. Hence so do

$$\operatorname{diag}(\sqrt{r_n/a_n}, \ldots, \sqrt{r_n/a_n}, r_n/a_n)(\tilde{B}_n/r_n) \qquad \tilde{B}_n = B_n - \mathbf{z}_n \tag{2.4}$$

since $a_n \leq 2R_n = o(r_n)$ for $r_{n+1} \sim r_n$, which implies $r_n/a_n \to \infty$.

What does the ball B_{n+1} look like in the new coordinates? Since the radii of the balls are asymptotically equal it will look like a translate of P, and so will the ball B_{n+k} for any fixed integer k. What conditions should one impose on the sequence of balls to ensure that there exist normalizations such that

$$\alpha_n^{-1}(B_{n+k}) \to P + k\mathbf{e}_d = \{e^{-\chi} > e^{-k}\} \qquad n \to \infty, \qquad k \in \mathbb{Z}. \tag{2.5}$$

We first show that variations in the value of the horizontal component of the point $\mathbf{z}_n = (\mathbf{x}_n, y_n)$ are innocuous.

Lemma 2.3 *Let $B_0 \subset B_1$ be disks in the plane of radius r_0 and $r_1 = r_0 + R$ respectively. Suppose B_0 supports the horizontal halfplane $\{y \geq 0\}$ in $(a, 0)$ and B_1 supports the horizontal halfplane $\{y \geq b\}$ in $(0, b)$. The larger disk intersects the horizontal axis in an interval $(-s, s)$ where $a^2/s^2 < R/r_1$.*

Proof Observe that $s^2 = 2r_1 b - b^2$ and that $a^2 + (R-b)^2 \leq R^2$ since the distance between the centers is bounded by R. Hence $a^2 \leq 2Rb - b^2$, and $R < r_1$ implies $a^2/s^2 < R/r_1$. ¶

Proposition 2.4 *Let $B_1 \subset B_2 \subset \cdots$ be open balls in $\mathbb{R}^d$. Suppose B_n has radius r_n and supports the horizontal halfspace $\{y \geq y_n\}$ in a point $\mathbf{z}_n = (\mathbf{x}_n, y_n)$ for $n \geq 1$. Define*

$$\alpha_n : \mathbf{w} = (\mathbf{u}, v) \mapsto (\mathbf{x}_n + c_n \mathbf{u}, y_n + a_n v) \qquad a_n = y_n - y_{n-1}, c_n = \sqrt{a_n r_n}. \tag{2.6}$$

If $r_{n+1} \sim r_n$ and $a_{n+1} \sim a_n$ then (2.5) holds, and $h_n \to e^{-\chi}$ uniformly on compact sets.

Proof The balls $\tilde{B}_n = B_n - \mathbf{z}_n$ converge to P under the linear transformations A_n^{-1} where $A_n(\mathbf{u}, v) = (c_n\mathbf{u}, a_n v)$. Hence so do the balls $\tilde{B}_{n+k} = (r_{n+k}/r_n)\tilde{B}_n$. Now observe that $\alpha_n^{-1}(\mathbf{z}_{n+k} - \mathbf{z}_n) = ((\mathbf{x}_{n+k} - \mathbf{x}_n)/c_n, (y_{n+k} - y_n)/a_n) \to (\mathbf{0}, k)$ since $\|\mathbf{x}_{n+k} - \mathbf{x}_n\| << \sqrt{|k|a_n r_n}$ by Lemma 2.3 and $y_{n+i} - y_{n+i-1} = a_{n+i} \sim a_n$. The inclusion $E_n \subset E_t \subset E_{n+1}$ for the ellipsoids $A_n^{-1}(\tilde{B}_t)$ for $t \in [n, n+1]$ implies $E_t \to P$ uniformly for $|t - n| \leq m$; the relation $\mathbf{z}_{n+\theta} = \mathbf{z}_n + \theta(\mathbf{z}_{n+1} - \mathbf{z}_n)$ holds for $g \in \mathcal{S}_0$ and implies that $\alpha_n^{-1}(B_{n+s_n}) \to P + s\mathbf{e}_d$ for $s_n \to s$. Convergence of the level sets implies $e^n g(\alpha_n(\mathbf{w})) \to e^{-\chi(\mathbf{w})}$ uniformly on compact sets. ¶

We now turn to $\mathbf{L}^1$ convergence on halfspaces $J = \{v \geq v_0 + \mathbf{a}^T\mathbf{u}\}$.

Write $\alpha_n^{-1}(B_{n+m}) = A_n^{-1}(\tilde{B}_{n+m}) + A_n^{-1}(\mathbf{z}_{n+m} - \mathbf{z}_n)$. Then

$$\alpha_n^{-1}(B_{n+m}) = \frac{r_{n+m}}{r_n} E_{r_n/a_n} + (\frac{\mathbf{x}_{n+m} - \mathbf{x}_n}{c_n}, \frac{y_{n+m} - y_n}{a_n}).$$

Let $0 < \epsilon < 1/d \wedge 1/4$ where d is the dimension. There exists an index n_0 such that $a_{n+1} < e^\epsilon a_n$ and $r_{n+1} < e^\epsilon r_n$ for $n \geq n_0$. For $n \geq n_0$ the quotients a_{n+m}/a_n, r_{n+m}/r_n and c_{n+m}/c_n then are bounded by $e^{m\epsilon}$ for $m \geq 1$. Then $R_n/r_n \leq 1 - e^{-\epsilon} < \epsilon$ for $n \geq n_1 = n_0 + 1$, and Lemma 2.3 gives $\|\mathbf{x}_n - \mathbf{x}_{n-1}\| \leq 2\sqrt{\epsilon}c_n \leq c_n$. Hence for $n \geq n_0$ the level set $\{h_n \geq e^{-m}\} = \alpha_n^{-1}(B_{n+m})$ is contained in $e^{m\epsilon}P + (\mathbf{u}(m), v(m))$ where

$$v(m) = \frac{a_{n+m}}{a_n} + \cdots + \frac{a_{n+1}}{a_n} \leq m e^{m\epsilon} \qquad \|\mathbf{u}(m)\| \leq \frac{c_{n+m}}{c_n} + \cdots + \frac{c_{n+1}}{c_n} \leq m e^{m\epsilon}.$$

The following result is well known. It shows the way in which we shall proceed.

Lemma 2.5 *Suppose $h_n \to h$ uniformly on compact sets. Let J be a halfspace. If for any $\epsilon > 0$ there exists $m \geq 1$, a compact set K and an index n_0 such that $\{h_n > e^{-m}\} \cap J \subset K$ and $\int_J h_n(\mathbf{w}) \wedge e^{-m} d\mathbf{w} < \epsilon$ for $n \geq n_0$ then $h_n \to h$ in $\mathbf{L}^1(J)$.*

Proposition 2.6 *Under the conditions of Proposition 2.4 convergence $h_n \to e^{-\chi}$ holds in $\mathbf{L}^1$ on halfspaces $J = \{v \geq v_0 + \mathbf{a}^T\mathbf{u}\}$.*

Proof The function h_n has level sets $\{h_n > e^{-m}\} = \alpha_n^{-1}(B_{n+m})$. This gives the inequality

$$h_n \leq e \sum_m e^{-m} 1_{\alpha_n^{-1}(B_{n+m})}. \tag{2.7}$$

It suffices to prove that there exist $\theta \in (0, 1)$, $p \geq 0$ and $C_0 \geq 1$ such that

$$|J \cap \alpha_n^{-1}(B_{n+m})| \leq C_0(1 + m)^p e^{\theta m} \qquad n \geq n_0, m \geq m_0. \tag{2.8}$$

Let $J=\{v\geq \mathbf{a}^T\mathbf{u}-a_0\}$. The volume of $P\cap\{v>-r\}$ is

$$|P\cap\{v>-s\}|=B(d-1)\int_0^r s^{(d-1)/2}ds=\frac{2B(d-1)}{d+1}s^{(d+1)/2}$$

where $B(m)$ is the volume of the unit ball in $\mathbb{R}^m$. The volume of $P\cap J$ is $|P\cap\{v>-h\}|$ where $h=a_0+\mathbf{a}^T\mathbf{a}/2$ is the distance between ∂J and the tangent plane $\{v=\mathbf{a}^T\mathbf{u}+\mathbf{a}^T\mathbf{a}/2\}$ to P in $-\mathbf{a}$ parallel to ∂J. We shall determine a uniform upper bound for the volume of the set $(e^{m\epsilon}P+me^{m\epsilon}(\mathbf{q},\theta))\cap J$ for $\|\mathbf{q}\|\leq 1$, $0\leq\theta\leq 1$. Write $c=e^{m\epsilon}$. Then

$$|c(P+m(\mathbf{q},\theta))\cap J|=c^d|P\cap(J/c-m(\mathbf{q},\theta))| \tag{2.9}$$

$$=c^d|P\cap\{v\geq\mathbf{a}^T(\mathbf{u}-m\mathbf{q})-a_0/c-m\theta\}|=c^d|P\cap\{v\geq c_0\}| \tag{2.10}$$

with $c_0=\mathbf{a}^T\mathbf{a}/2+m\mathbf{a}^T\mathbf{q}+m\theta+a_0/c$. There is a constant $C_0=C_0(d,a_0,\mathbf{a})$ such that

$$|c(P+m(\mathbf{q},\theta))\cap H|\leq c^dC_0(m+1)^{(d+1)/2}\qquad \|\mathbf{q}\|\leq 1,\ \theta\in[0,1].$$

This yields the bound (2.8). ¶

The proof of Theorem 2.1 follows from the Propositions 2.4 and 2.6. The uniform convergence on compact sets was a consequence of the convegence of the ellipsoids $\alpha_n(B_{n+k})$ to the paraboloids $P+k\mathbf{e}_d$. (All we needed there was $r_{n+1}\sim r_n$ and $a_n\sim a_n$.) For the $\mathbf{L}^1$ convergence one only needs the upper bounds $a_{n+1}<e^\epsilon a_n$ and $r_{n+1}<e^\epsilon r_n$ eventually for a sufficiently small $\epsilon>0$.

From univariate EVT it is known that two functions in $\mathcal{D}^+$ are asymptotic if they assume the values e^{-n} in the same points r_n. In particular if $g_0\in\mathcal{D}^+$ satisfies $g_0(r_n)=e^{-n}$ and ψ is the piecewise linear function with the values $\psi(r_n)=n$, then $e^{-\psi}$ is asymptotic to g_0 in r_∞. Let $g\in\mathcal{S}$. For $\omega\in\partial B$ define

$$g_\omega(t)=\max_{\omega(\mathbf{z})=t}g(\mathbf{z})\qquad t\in\mathbb{R}. \tag{2.11}$$

For the vertical coordinate η $g_\eta(t)=\max_\mathbf{x}g(\mathbf{x},t)$. If $g\in\mathcal{S}_0$ has level sets $B_t=\{g>e^{-t}\}$ and $\mathbf{z}_t=(\mathbf{x}_t,y_t)$ is the point on ∂B_t where the vertical coordinate is maximal, then $t\mapsto y_t$ is linear on each interval $[y_n,y_{n+1}]$, and $g_\eta\in\mathcal{D}^+$ if $a_{n+1}\sim a_n$ for $a_n=y_{n+1}-y_n$. Theorem 2.1 remains valid if we replace the condition $a_{n+1}\sim a_n$ by $g_\eta\in\mathcal{D}^+$.

If g is the density of a vector $\mathbf{Z}=(\mathbf{X},Y)$, then $\mathbf{X}$ conditional on $Y=y_t$ is asymptotically normal $N(\mathbf{x}_t,a_tr_t)$ for $t\to\infty$. The asymptotic behaviour of conditional distributions on halfspaces and hyperplanes has been treated in [2] and [9].

Theorem 2.7 *Let $g \in \mathcal{S}$ and $g_\eta \in \mathcal{D}^+$. Choose C_t such that $\tilde{g}_t : \mathbf{u} \mapsto g(\mathbf{x}_t + c_t\mathbf{u}, y_t)/C_t$ is a probability density on $\mathbb{R}^{d-1}$. Then $C_t \sim (a_t r_t)^{(d-1)/2}$ and*

$$\tilde{g}_t(\mathbf{u}) \to e^{-\mathbf{u}^T\mathbf{u}/2}/(2\pi)^{(d-1)/2} \quad \text{uniformly on compact sets and in } \mathbf{L}^1 \qquad t \to \infty.$$

Proof Convergence on compact sets follows from Theorem 2.1. The $\mathbf{L}^1$ convergence follows as in the proof of Proposition 2.6. One needs only consider integrals over horizontal hyperplanes $\{v = v_0\}$ with $v_0 \geq 0$. ¶

The function g_η is not the vertical marginal g_d, but there is a simple relation.

Theorem 2.8 *Suppose $g \in \mathcal{S}$. If $g_\eta \in \mathcal{D}^+$ then $g_d \in \mathcal{D}^+$ and*

$$g_d(y_t) \sim (a_t r_t)^{(d-1)/2}/e^t \qquad t \to \infty. \tag{2.12}$$

Proof The asymptotic relation 2.12 follows from the asymptotic expression for C_t in Theorem 2.7. This implies that g_d/g_η is asymptotically constant over intervals of length of order a_n around t_n, which ensures that g_d satisfies the basic limit relation (1.8). The remaining two relations follow by the remark preceding the theorem. ¶

The results of this section hold for any direction $\omega \in \partial B$. If $\mathbf{Z}$ has density $g \in \mathcal{S}$ and $g_\omega \in \mathcal{D}^+$ then the rv $\omega(\mathbf{Z})$ has a density in $\mathcal{D}^+$, see (2.12) and $g \in \mathcal{D}_0(\omega)$. The high risk scenarios converge for halfspaces $H_n = \{\omega \geq t_n\}$. Convergence for halfspaces $\{\omega_n \geq t_n\}$ with directions $\omega_n \to \omega$ is a different matter.

Example 2 Let $g \in \mathcal{S}_0$ have level sets $B_n = \{g > e^{-n}\}$ where B_n is the ball with radius r_n centered on the vertical axis, which intersects the vertical axis in the interval $(n - 2r_n, n)$. Then $g \in \mathcal{D}_0(\eta)$ for the vertical direction η. The only condition on $R_n = r_n - r_{n-1}$ is $R_n = o(r_n)$. Choose $r_n = n^4$ for even indices and $r_{2n+1} = r_{2n} + 1$. The balls B_{2n} and B_{2n+1} are concentric. The ball B_{2n-1} is a contraction of B_{2n} from a center $(\mathbf{0}, c_n)$ with $c_n \to 63n/32$. The distance between parallel tangent planes to B_n and B_{n+1} is one. The results of the next section show that the distance between the tangent planes to B_{2n-1} and B_{2n} with a direction ω_n at distance $1/n$ from η is asymptotic to $16n$. The boundaries of the halfspaces of direction ω_n supporting $B_{2n-1}, B_{2n}, B_{2n+1}$ are certainly not asymptotically equidistant. The Gauss-exponential asymptotics fail. ◊

3 The condition AED

For $g \in \mathcal{S}_0$ to lie in $\mathcal{D}$ the balls $B_n = \{g > e^{-n}\}$ should be asymptotically equidistant. More precisely the boundary planes of the halfspaces supporting three successive balls B_{n-1}, B_n, B_{n+1} should be asymptotivcally equidistant uniformly in the direction (of the halfspaces). For a direction $\omega \in \partial B$ let $H_n = \{\omega \geq t_n\}$ support the level set $B_n = \{g > e^{-n}\}$. The constant t_n depends on ω and is strictly increasing in n. Define the ratio

$$\rho_n(\omega) = \frac{t_{n+1} - t_n}{t_n - t_{n-1}} \qquad \omega \in \partial B,\ n > 1. \tag{3.1}$$

If $\rho_n(\omega) = 1$ the planes $\partial H_{n-1}, \partial H_n, \partial H_{n+1}$ are equidistant.

Definition 3 (The condition AED) *The function $g \in \mathcal{S}_0$ with level sets $B_n = \{g > e^{-n}\}$ is* AED *if $\rho_n \to 1$ uniformly on ∂B.* ◊

The condition AED is geometric. It does not depend on the origin, nor on the coordinates. The ratio of the distance between three successive planes does not even depend on the euclidean norm.

One can define the ratio $\rho_n(\omega)$ also for continuous functions with convex level sets D_n. If $D_n = nD$ for a bounded convex open set D containing the origin, then $\rho_n \equiv 1$. In particular this holds if D is an excentric ball $D = B - \mathbf{c}$ for a point $\mathbf{c} \in B$. We find it more convenient to start with the unit ball, and scale it from a point $\mathbf{c} \in B$. This yields a family of balls

$$B(\mathbf{c}, r) = \mathbf{c} + r(B - \mathbf{c}) \qquad r > 0,\ \mathbf{c} \in B. \tag{3.2}$$

Example 3 Let $r_n = R_1 + \cdots + R_n$ with $0 < R_n \sim R_{n+1}$. Then the sets $B(\mathbf{c}, r_n)$ are asymptotically equidistant since $\rho_n(\omega) \equiv R_{n+1}/R_n$. ◊

Theorem 3.1 *If $g \in \mathcal{S}_0$ is* AED *then $g \in \mathcal{D}$.*

Proof Let $H_n = \{\omega_n \geq t_n\}$ be halfspaces supporting the level sets $B_{s_n} = \{g > e^{-s_n}\}$ in $\mathbf{z}_n$. Suppose $s_n \to \infty$. A rotation which maps ω into $\eta = (\mathbf{0}, 1)$ maps H_n into a horizontal halfspace tangent to the ball $\{g_n > e^{-s_n}\}$ where the graph of g_n is a rotation of the graph of g. Since the results of Theorem 2.1 hold uniformly for $r_{n+1} \sim r_n$ and $a_{n+1} \sim a_n$ they apply to the sequence (g_n), and $e^{s_n} g_{s_n}(\alpha_{s_n}(\mathbf{w})) \to e^{-\chi(\mathbf{w})}$ uniformly on compact sets and in $\mathbf{L}^1$ on halfspaces $J = \{v \geq v_0 + \mathbf{a}^T \mathbf{u}\}$. ¶

Let $g \in \mathcal{S}_0$ have level sets $B_t = \{g > e^{-t}\} = \mathbf{p}_t + r_t B$. The level sets B_t for $t \in [n, n+1]$ are defined by linear interpolation. It follows that $B_{t-\epsilon}, B_t, B_{t+\epsilon}$ are equidistant for $n+\epsilon \le t \le n+1-\epsilon$. So too the balls B_{n-1}, B_n, B_{n+1} are equidistant if $\mathbf{p}_{n+1} = \mathbf{p}_n + (\mathbf{p}_n - \mathbf{p}_{n-1})$ and $r_{n+1} = r_n + R_n$. The ratio will satisfy

$$1 - \epsilon \le \rho_n(\omega) \le 1 + \epsilon \qquad \omega \in \partial B$$

if B_{n+1} lies between the balls with centers $\mathbf{p}_n + (1 \pm \epsilon)(\mathbf{p}_n - \mathbf{p}_{n-1})$ and radii $r_n + (1 \pm \epsilon) R_n$. This raises the question of describing all balls which lie in between two given balls. Given the balls $B_0 \Subset B_1$ how does one describe the balls in between?

$$B_0 \Subset \mathbf{p} + rB \Subset B_1. \tag{3.3}$$

Proposition 3.2 *Let $B_i = \mathbf{p}_i + r_i B$ with $\|\mathbf{p}_1 - \mathbf{p}_0\| < r_1 - r_0$. Let $r \in (r_0, r_1)$. Then (3.3) holds if and only if $\mathbf{p} \in (\mathbf{p}_0 + (r - r_0)B) \cap \mathbf{p}_1 + (r_1 - r)B)$.*

Proof The ball $\mathbf{p} + rB$ contains B_0 if and only if $\|\mathbf{p} - \mathbf{p}_0\| \le r - r_0$ and is contained in R_1 if and only if $\mathbf{p}_1 - \mathbf{p}\| \le r_1 - r$. ¶

The condition depends only on the difference $r_1 - r_0$. So too in our next result.

Proposition 3.3 *Let E be the open ellipsoid with focal points $\mathbf{p}_0$ and $\mathbf{p}_1$ consisting of all points $\mathbf{z}$ the sum of whose distance to $\mathbf{p}_0$ and $\mathbf{p}_1$ is less than $r_1 - r_0$. Let $\|\mathbf{p} - \mathbf{p}_i\| = c_i$ for $i = 0, 1$ and let $r \in (r_0 + c_0, r_1 - c_1)$. Then $\mathbf{p} + rB$ satisfies (3.3). Conversely if $\mathbf{p} + rB$ satisfies these inclusions then $\mathbf{p} \in E$ and $r_0 + c_0 < r < r_1 - c_1$.*

Proof For any point $\mathbf{z} \in \partial E$ the sum $\|\mathbf{z} - \mathbf{p}_0\| + \|\mathbf{z} - \mathbf{p}_1\| = r_1 - r_0$. From the description of E it follows that $c_0 + c_1 < r_1 - r_0$ if and only if $\mathbf{p} \in E$. We can then choose $r \in (r_0 + c_0, r_1 - c_1)$, and the previous proposition gives the inclusions (3.3) since $c_i < a_i = |r - r_i|$ for $i = 0, 1$. Similarly (3.3) by the previous proposition implies $c_i := \|\mathbf{p} - \mathbf{p}_i\| < a_i = |r_i - r|$ for $i = 0, 1$, and $a_0 + a_1 = r_1 - r_0$ implies $\mathbf{p} \in E$. ¶

Given two balls $B_0 \Subset B_1$ one can define balls B_t inbetween by linear interpolation. B_t is the ball $\mathbf{p}_t + r_r B$ where

$$\mathbf{p}_t = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0) \qquad r_t = r_0 + t(r_1 - r_0).$$

These balls belong to a family of balls $B(\mathbf{c}, r)$, $r > 0$. One may describe B_1 as an expansion of B_0 by a factor r_1/r_0 from a center $\mathbf{z} \in B_0$. Let us introduce the relative center $\mathbf{c} \in B$, the position of $\mathbf{z}$ in coordinates in which B_0 is the unit ball.

Lemma 3.4 *Let $T > 0$ denote the distance between the spheres ∂B_0 and ∂B_1 with radii r_0 and $r_1 = r_0 + R$. If $\mathbf{p}_1 = \mathbf{p}_0$ then $\mathbf{z} = \mathbf{p}_0$, $T = R$ and $\mathbf{c} = \mathbf{0}$; if $\mathbf{p}_1 \neq \mathbf{p}_0$ then*

$$\mathbf{c} = \frac{\mathbf{z} - \mathbf{p}_0}{r_0} = \frac{\mathbf{z} - \mathbf{p}_1}{r_1} = \frac{\mathbf{p}_0 - \mathbf{p}_1}{R} \qquad \tau := 1 - \|\mathbf{c}\| = \frac{T}{2R}. \tag{3.4}$$

Proof The first equality is the definition of $\mathbf{c}$; the second holds because of the expansion by r_1/r_0; the third holds by subtracting numerator and denominator; the fourth by scaling: $1 - \|\mathbf{c}\| = T/2R$. ¶

It will be convenient to describe the sequence of balls B_n in a dynamic way where B_n is obtained from B_{n-1} by expansion by r_n/r_{n-1} from the center $\mathbf{z}_n = \mathbf{p}_n + r_n\mathbf{c}_n$. It suffices to know B_0, the sequence of radii $r_1, r_2, \ldots$, and the sequence of relative centers $\mathbf{c}_n \in B$ which are used to transform B_{n-1} into B_n.

Proposition 3.5 *Let $\mathbf{x}_n$ and $\mathbf{u}_n$ be points in the open unit ball B and let δ_n and δ'_n in $(0,1)$ converge to zero. Then $B(\mathbf{x}_n, 1-\delta_n), B, B(\mathbf{u}_n, 1+\delta'_n)$ are* AED *if and only if*

$$\delta'_n \sim \delta_n \qquad \tau_n := 1 - \|\mathbf{x}_n\| \sim 1 - \|\mathbf{u}_n\| \qquad \|\mathbf{u}_n - \mathbf{x}_n\|/\sqrt{\tau_n} \to 0. \tag{3.5}$$

Proof We move up one dimension and consider cones $C(\mathbf{u}, v)$ in $\mathbb{R}^d \times \mathbb{R}$ with top $(\mathbf{u}, v)$ with $v < 1$ and $\mathbf{u} \in B$ which intersect the horizontal plane $\mathbb{R}^d \times \{1\}$ in the unit ball. The intersection of the cone $C(\mathbf{u}, 0)$ with the horizontal plane $\{y = t\}$ is the open ball $B(\mathbf{u}, t)$ for $t \neq 0$ and the point set $\{\mathbf{u}\}$ for $t = 0$. The cones $C(\mathbf{u}, 1 - e^{\pm\epsilon})$ intersect the plane $\{y = 1 + s\}$ in the balls $B(\mathbf{u}, 1 + e^{\mp\epsilon}s)$. The intersection of $C(\mathbf{x}, 0)$ with this plane will lie between these two balls if $(\mathbf{x}, 0)$ lies in both cones $C(\mathbf{u}, 1 - e^{\pm\epsilon})$. Equivalently if

$$\mathbf{x} \in B(\mathbf{u}, 1 - e^{-\epsilon}) \cap B(\mathbf{u}, 1 - e^{\epsilon}). \tag{3.6}$$

This intersection is an open convex set which is approximately equal to the set $B(\mathbf{u}, 1) \cap B(\mathbf{u}, -1)$ contracted by a factor ϵ from the center $\mathbf{u}$. The intersection $B(\mathbf{u}, 1) \cap B(\mathbf{u}, -1)$ is the union of two caps of height $1 - \tau$ whose common base is a disk of radius $\sqrt{\tau(2-\tau)}$. Condition (3.5) implies (3.6) for $\mathbf{x}_n$ and $\mathbf{u}_n$ eventually for any $\epsilon > 0$, and conversely if $\mathbf{x}_n$ and $\mathbf{u}_n$ satisfy (3.6) for a seqeunce $\epsilon_n \to 0$ then (3.5) holds. ¶

This yields a condition for AED.

Theorem 3.6 *Let $g \in \mathcal{S}_0$ have level sets $B_n = \{g > e^{-n}\} = \mathbf{p}_n + r_n B$. Suppose B_{n+1} is the expansion of B_n by the factor r_{n+1}/r_n from the center $\mathbf{z}_n = \mathbf{p}_n + r_n \mathbf{c}_{n+1}$. Set $\tau_n = 1 - \|\mathbf{c}_n\|$. Then $g \in \mathcal{D}$ holds if and only if $R_n = r_n - r_{n-1} \sim R_{n+1}$, $\tau_n \sim \tau_{n+1}$ and $\|\mathbf{c}_n - \mathbf{c}_{n-1}\|/\sqrt{\tau_n} \to 0$.*

There are two conditions, one on the size of the balls B_n one on the structure, the position of the relative center in the unit ball. These conditions are independent. The size condition is $R_{n+1} \sim R_n$. If the relative centers $\mathbf{c}_n$ eventually lie in a disk with radius < 1 then the structure condition is simple: $\|\mathbf{c}_n - \mathbf{c}_{n-1}\| \to 0$. If $\tau_n \to 0$ the condition is $\|\gamma_n - \gamma_{n-1}\| = o(\sqrt{\tau_n})$ for the unit vectors $\gamma_n \mathbf{c}_n/(1-\tau_n)$.

In the metric on $B \subset \mathbb{R}^2$ given at the point $r\omega \in B$ by

$$ds^2 = \frac{dr^2}{(1-r)^2} + \frac{r^2 d\omega^2}{1-r}$$

the distance between two points $(0, r_0)$ and $(0, r_1)$ with $0 \le r_0 < r_1 < 1$ by integration is $\log((1-r_0)/(1-r_1))$; the distance along a circle between $r\omega$ and $r\zeta$ is $d(\omega, \zeta)/(1-r)$. Take $\tau = 1 - r = 1/n^4$. It takes n steps to go from $(1-\tau)\omega$ to $(1-\tau)\zeta$ for ω and ζ in ∂B with distance $d(\omega, \zeta) = 1/n$ if one walks along a circle of radius $1-\tau$, but only $4\log n$ steps to reach the origin. So it is faster to go via the origin, at least for $n \ge 30$ since then $8 \log n < n$. In terms of this hyperbolic metric d_{hyp} the condition for AED is simple: the distance $d_{hyp}(\mathbf{c}_{n+1}, \mathbf{c}_n)$ should vanish for $n \to \infty$.

Example 4 In dimension $d = 1$ the function $g \in \mathcal{S}_0$ with level sets $\{g > e^{-n}\} = (-s_n, t_n)$ lies in $\mathcal{D}$ if $S_{n+1} \sim S_n$ and $T_{n+1} \sim T_n$, see Example 1. For $d > 1$ let $g \in \mathcal{S}_0$ have level sets $\{g > e^{-n}\} = B_n$ with center $(\mathbf{0}, (t_n - s_n)/2)$ and radius $r_n = (s_n + t_n)/2$. Assume $T_n \le S_n$. Then $\tau_n = 1 - \|\mathbf{c}_n\| = T_n/R_n \sim \tau_{n+1}$ and AED holds since the centers all lie on the vertical axis. $\diamondsuit$

Example 5 Let B_n be the disk in the plane of radius n which supports the horizontal halfplane $\{y \ge -1/(n-1)$ in the point $(x_n, -1/(n-1))$. Let B_{n+1} be the expansion of B_n from the center $\mathbf{z}_n = (x_n - 1, -2/n)$ by the factor $1 + 1/n$. Then B_{n+1} supports the horizontal halfplane $\{y > -1/n\}$ in $(x_n + 1/n, -1/n)$. We may choose x_n to satisfy $x_n - \log n \to 0$. The union of the disks is the open lower halfplane. The function $g \in \mathcal{S}_0$ with these level sets lies in the Gauss-exponential domain $\mathcal{D}$ since $R_n \equiv 1$, $\tau_n = (2/n - 1/(n-1))/n \sim 1/n^2$ and the angle between the relative center $\mathbf{c}_n$ and the vertical is asymptotic to $1/n$ which implies $\|\mathbf{c}_n - \mathbf{c}_{n-1}\| = o(\tau_n)$. $\diamondsuit$

There are alternative descriptions of AED. We give two, leaving the proof to the reader. The first is formulated in terms of the increments of the radii $R_n = r_n - r_{n-1}$, the distance from the sphere ∂B_{n-1} to the sphere ∂B_n, and to the next sphere, ∂B_{n+1}.

Proposition 3.7 *Let T_n denote the distance from the sphere ∂B_{n_1} to ∂B_n, and $T_n^{(2)}$ the distance from ∂B_{n-1} to ∂B_{n+1}. Then* AED *holds if and only if*

$$R_{n+1} \sim R_n \qquad T_{n+1} \sim T_n \qquad T_n^{(2)} \sim 2T_n.$$

A function $g \in \mathcal{S}_0$ is determined by its level sets $B_n = \{g > e^{-n}\}$. Introduce the function g_e determined by the even level sets B_{2n}, and the function g_o defined by the odd level sets. For $2n < t < 2n+2$ the level sets of g_e are determined by interpolation, and similarly for g_o for $2n-1 < t < 2n+1$. We then have the characterization:

Proposition 3.8 *The function $g \in \mathcal{S}_0$ lies in $\mathcal{D}$ if and only if g_e and g_o are asymptotic.*

Proposition 3.9 *A density $g \in \mathcal{S}_0$ lies in $\mathcal{D}$ if and only if it lies in $\mathcal{D}(\omega)$ for every $\omega \in \partial B$.*

Proof If $g \in \mathcal{D}$ then $g \in \mathcal{D}(\omega)$ for any ω. Conversely if $g \notin \mathcal{D}$ then the ratio ρ_n in (3.1) does not converge uniformly to one on ∂B and there is an $\epsilon > 0$, a sequence of indices and a sequence $\omega_n \in \partial B$ such that $|\rho_{m_n}(\omega_n) - 1| > \epsilon$ for $n = 1, 2, \ldots$. By compactness of ∂B there is a subsequence for which ω_n converges to a point $\omega \in \partial B$. This implies $g \notin \mathcal{D}(\omega)$. ¶

The level sets of $g \in \mathcal{S}_0$ form a piecewise linear curve in the space of balls. At integer time points $t = n$ two changes occur, the rate of growth changes, R_n becomes R_{n+1}, and the relative center changes, $\mathbf{c}_{n-1}$ becomes $\mathbf{c}_n$. There is a change in the direction in which the centers $\mathbf{p}_t$ of the ball B_t move, determined by the change in the unit vector $\gamma_n = \mathbf{c}_n/(1-\tau_n)$, and a change in the excentricity, the value of τ_n which is one for concentric circles (and zero if the ball is expanded from a point on its boundary). These changes can be measured in the two-dimensional plane through the centers of three successive balls. Normalize the middle ball to be the unit ball. First we shall take a closer look at the relation between B and $B(\mathbf{c}, r)$ for r close to one. If $\mathbf{c} = \mathbf{0}$ the distance between tangent planes to the two balls is constant. For $\tau \in (0,1)$ the distance varies with the direction of the plane. It is a function of the angular distance φ between $\mathbf{c}_n$ and

$\mathbf{c}_{n-1}$. For simplicity we take $\mathbf{c}$ to lie on the vertical axis, $\mathbf{c} = (1-\tau)\mathbf{e}_d$. The dependence on r is a factor $|r-1|$.

Proposition 3.10 *The distance δ between the boundaries of the halfspaces with direction ω supporting $B(\mathbf{c}, r)$ and B for $r > 0$ and $\mathbf{c} = (1-\tau)\mathbf{e}_d$ is*

$$\delta = |r-1|(1 - \cos\varphi + \tau\cos\varphi) \qquad \varphi = \delta(\omega, \mathbf{e}_d). \tag{3.7}$$

Proof Take $d = 2$. The line L from $\mathbf{c}$ perpendicular to ω intersects the ray through ω in $\theta\omega$ with $\theta = (1-\tau)\cos\varphi$. The line is parallel to the tangent lines to B and $B(\mathbf{c}, r)$. Draw lines from the point where $B(\mathbf{c}, r)$ supports the halfplane with direction ω to the origin and to $\mathbf{c}$. One has two similar triangles cut off by L and the tangent to B in $\mathbf{w}$ in the proportion $r : 1+r$. Hence $\delta = \delta(r) = r(1-\theta)$. ¶

In many cases we do not need the precise value of the distance.

Lemma 3.11 *For $\epsilon > 0$ there exists $\delta > 0$ such that*

$$e^{-\epsilon}(\varphi^2/2 + \tau) \le 1 - \cos\varphi + \tau\cos\varphi \le e^{\epsilon}(\varphi^2/2 + \tau) \qquad 0 < \varphi \le \delta,\ \tau \in [0,1]$$
$$1 - \cos\varphi + \tau\cos\varphi \asymp \varphi^2 \vee \tau \qquad 0 < \varphi \le \pi,\ \tau \in [0,1].$$

On the sphere ∂B_n one may define an integer valued metric. The distance between any two points is the smallest number of segments in the shell $\mathrm{cl}(B_n) \setminus B_{n-1}$ connecting a point in ∂B_n and a point in ∂B_{n-1} needed to construct a path from the one point to the other. So thedistance is an even number. It may be defined for any pair of bounded open convex sets $D_0 \Subset D_1$. The metric does not depend on the coordinates. It is geometric and invariant for affine transformations. For the balls B_n and B_{n-1} this integer metric is roughly equal to the Euclidean metric on ∂B_n divided by $\sqrt{n}$. For the shell $\mathrm{cl}(B) \setminus B(\mathbf{c}, 1-\delta)$ the metric is no longer isotropic. If $\mathbf{c} = (1-\tau)\mathbf{e}_d$ with τ small, then close to the North pole one needs many more steps to cover the same Euclidean distance. In dimension $d = 2$ for $\delta \to 0$ the metric is asymptotic to the metric $d\varphi/\sqrt{2\delta(1 - \cos\varphi + \tau\cos\varphi)}$, where $d\varphi$ is the Euclidean metric on the unit circle. For $\mathbf{c} = 0$ the balls are concentric and the distance from North pole to South pole is approximately $\pi/\sqrt{2\delta}$; for $\mathbf{c} = (1-\tau)\mathbf{e}_d$ and τ is small the distance between the poles is of the order of $\log(1/\tau)/\sqrt{\delta}$ by Lemma 3.11.

The function δ in (3.7) yields an explicit formula for the ratio ρ_n in (3.1). Take the plane through the centers of the balls B_{n-1}, B_n, B_{n+1}. Let the line through $\mathbf{p}_n$ and $\mathbf{p}_{n+1}$ be the vertical axis, and let the line through $\mathbf{p}_n$ and $\mathbf{p}_{n-1}$ make an angle ψ with the vertical axis. The ratio of the distance between halfplanes whose normal makes an angle φ with the vertical by (3.7) is

$$\frac{R_{n+1}}{R_n}\frac{1-\cos\varphi+\tau_n\cos\varphi}{1-\cos(\varphi-\psi)+\tau_{n-1}\cos(\varphi-\psi)}.$$

The effect of ψ on $\rho_n(\varphi)$ becomes clear if one takes $\tau_n=\tau_{n-1}$ close to zero and $R_{n+1}=R_n$.

For later use we prove:

Lemma 3.12 *Suppose $r_0=1-\epsilon_0$ and $r_1=1+\epsilon_1$ with ϵ_0 and ϵ_1 non-negative and $\epsilon:=\epsilon_0+\epsilon_1\leq 4/\pi^2$. If $\varphi\geq\sqrt{2\tau}$ then the cap cut off from the ball $B(\mathbf{c}.r_1)$ by the halfspace with direction ω supported by $B(\mathbf{c},r_0)$ is disjoint from the cone $C=\{r\zeta \mid r>0,\ d(\zeta,\mathbf{e})<\varphi/2\}$.*

Proof Assume $d=2$. The cap has height $h=\epsilon(1-\cos\varphi+\tau\cos\varphi)$ and base $2s\leq 2\sqrt{2hr_1}$. The top of the cap lies on the ray with angle $\varphi_1\geq\varphi$, and the cap subtends an angle 2ψ on the larger disk. It suffices to prove that $\psi\leq\varphi/2$. Note that $2\psi<\pi/2$ since ϵ is small, and this implies

$$\psi\leq\pi s/(2\sqrt{2}r_1)\leq(\pi/2)\sqrt{h}\leq(\pi/2)\sqrt{\epsilon}\varphi$$

since the condition $\varphi\geq\sqrt{2\tau}$ implies $1-\cos\varphi+\tau\cos\varphi\leq\varphi^2$. ¶

4 Examples

Example I. Pointwise convergence

Recall that $\mathcal{D}$ is the intersection of $\mathcal{D}(\omega)$ over $\omega\in\partial B$ by Proposition 3.9. It is possible that g lies in $\mathcal{D}_0(\omega)$ for all $\omega\in\partial B$ but not in $\mathcal{D}$. To understand what goes wrong, consider $g\in\mathcal{S}_0$ with level sets $\{g>e^{-n}\}=nB$. Alter g into a density $\tilde{g}$ by cutting off caps of height a half with direction ζ_i from the balls B_{m_i} for an increasing sequence m_i. What is the effect on the asymptotics as one alters the sequence ζ_i and the rate at which m_i increases? We shall do something similar for non-concentric level sets, translating the ball B_{m_i+1} in the directions ζ_i so that the translate still lies between the balls B_{m_i} and B_{m_i+2}. The new function $\tilde{g}$ lies in $\mathcal{S}_0$ and has the same integral as g, but the asymptotics may have changed.

Let U be a dense open set in the unit sphere ∂B. The complement of U may be a finite set or a finite union of smooth closed curves. Such sets have no area.

One can easily construct for any $\epsilon > 0$ dense open sets $U \subset \partial B$ whose area is less than ϵ. Choose a dense sequence of points in the unit sphere and let U be the union of open disks in ∂B centered in these points with area less than $\epsilon/2^n$.

For such a dense open set U we shall construct a density $\tilde{g} \in \mathcal{S}_0$ with the following properties:

- the high risk scenarios $\mathbf{Z}^{H_t}$ converge in distribution for $t \to \infty$ to the Gauss-exponential limit vector for the halfspaces $H_t = \{\omega \geq t\}$ for each $\omega \in \partial B$, and
- the high risk scenarios $\mathbf{Z}^{H_n}$ converge to the Gauss-exponential limit vector for all sequences of halfspaces $H_n = \{\omega_n \geq t_n\}$ with $t_n \to \infty$ and $\omega_n \to \omega$ for $\omega \in U$, and
- for each $\omega \in \partial B \setminus U$ there exists a sequence of halfspaces $H_n = \{\omega_n \geq t_n\}$ with $t_n \to \infty$ and $\omega_n \to \omega$ such that the high risk scenarios $\mathbf{Z}^{H_n}$ converge, but the limit is not Gauss-exponential.

Theorem 4.1 *For any dense open set $U \subset \partial B$ there exists a density $\tilde{g} \in \mathcal{S}_0$ which lies in $\mathcal{D}_0(\omega)$ for all $\omega \in \partial B$, but in $\mathcal{D}(\omega)$ only for $\omega \in U$.*

Construction and proof. There are nine steps.

1) We shall assume that U^c is non-empty. For $\omega \in \partial B$ let $\varphi_U(\omega)$ denote the distance of ω to U^c. The function φ_U is continuous on ∂B and $U = \{\varphi_U > 0\}$. There is a sequence of points $\zeta_n \in U$ such that every point in U^c is limit of a subsequence. We may arrange that $\varphi_U(\zeta_n) = 1/n$. Let $A \subset \mathbb{R}^d$ be the union of the sets $A_n = \{r\omega \mid 0 \leq r \leq m_n + 2n; 0 < \varphi_U(\omega) < 1/n\}$ for some strictly increasing sequence of integers m_n to be specified in step 3). The set A is small: for each unit vector ω there exists an $r_0 > 1$ such that $r\omega \in A^c$ for $r \geq r_0$.

2) Let $g \in \mathcal{S}_0$ satisfy AED. With each halfspace $H = \{\omega \geq t\}$ with $t \in \mathbb{R}$ is associated an affine transformation α_H mapping the upper halfspace $J_+ = \{v \geq 0\}$ onto H such that for any sequence of halfspaces H_n the associated transformations α_n satisfies

$$h_n(\mathbf{w}) := g(\alpha_n(\mathbf{w}))/g(\alpha_n(\mathbf{0}) \to e^{-\chi(\mathbf{w})},$$

provided $\max g 1_{H_n}$ is positive and vanishes for $n \to \infty$.

3) We assume that $B_n = \{g > e^{-n}\}$ has radius $r_n = n$ for all $n \geq 1$. There is a strictly increasing sequence of integers $m_i \geq i^3$ such that B_n has relative center $\mathbf{c}_i = (1 - 1/i^3)\zeta_i$ for $|n - m_i| \leq i$ where $\zeta_i \in \partial B$ is defined in 1) above, and such that B_{m_i} is centered. (Start out with two successive centered balls B_{n_0-1} and

B_{n_0}. Now define B_{n_0+k} for $k = 1, 2, \dots$ to be balls which are symmetric around the line $\mathbb{R}\zeta_i$. The boundary contains a point $y_k\zeta_i$ with $y_k > 0$, and the increments $y_k - y_{k-1} = a_k$ lie in $(0, 2)$ since $r_n = n$. Moreover a_k/a_{k+1} is close to one by the condition AED. It is possible to construct such a sequence of increments a_k with $a_1 = 1$ such that $a_k = 1/i^3$ for $k = k_0 - i, \dots, k_0 + i$ and such that $a_1 + \cdots + a_{k_0} = k_0$, and (by symmetry) such that $a_1 + \cdots + a_{2k_0} = 2k_0$ and $a_{2k_0} = 1$, and $|\log(a_k/a_{k-1}| < 1/i$ for all k.)

4) Define $\tilde{g} \in \mathcal{S}_0$ to have level sets $\tilde{B}_n = \{\tilde{g} > e^{-n}\}$ where $\tilde{B}_{m_i+1} = B_{m_i+1} + \zeta_i/2i^3$ and $\tilde{B}_n = B_n$ for all n which are not of the form m_i+1. Hence $\tilde{g} \asymp g$. If α_i is the affine normalization associated with the halfspace H_i supporting B_{m_i} in $m_i\zeta_i$ then $\tilde{g}(\alpha_n(\mathbf{w}))/\tilde{g}(\alpha_n(\mathbf{0})) \to \tilde{h}(\mathbf{w})$ where $\tilde{h}$ has parabolic level sets but $\tilde{h} \neq e^{-\chi}$ since $\{\tilde{h} > 1/e\} = P + 3\mathbf{e}_d/2$ instead of $P + \mathbf{e}_d$. Note that H_i has direction ζ_i since B_{m_i} is centered. This proves the third statement in the proposition. Pointwise convergence of $\tilde{g}(\alpha_n(\mathbf{w}))/\tilde{g}(\alpha_n(\mathbf{0})$ yields $\mathbf{L}^1$ convergence since $\tilde{g} \asymp g$.

5) Choose new coordinates so that B_{m_i} is the unit ball (by scaling by $1/m_i$). The balls B_n for $n = m_i + k$ with $|k| \leq i$ are obtained by expansion or contraction from the center $\mathbf{c} = (1 - \tau)\zeta_i$ where $\tau = 1/i^3$. The ratios $\rho_n(\omega)$, see (3.1) are identically one since the center is constant and also the increments of the radii. For the balls $\tilde{B}_n$ for $n = m_i$ we obtain

$$\tilde{\rho}_{m_i}(\omega) = \frac{1 - \gamma + 3\gamma/2i^3}{1 - \gamma + \gamma/i^3} = 1 + \frac{\gamma/2i^3}{1 - \gamma + \gamma/i^3} \qquad \gamma = \cos\varphi,$$

where φ is the angle between ω and ζ_i. Hence $\tilde{\rho}_{m_i}(\omega) - 1 \sim 1/(2 + \varphi^2 i^3)$ for $\varphi \to 0$. Lemma 3.11 gives

$$d(\omega, \zeta_i) \geq 1/2i \Rightarrow |\tilde{\rho}_n(\omega) - 1| \leq C_0/i \qquad |n - m_i| \leq i. \tag{4.1}$$

6) Let A^* be the union of the sets A_n^* where A_i^* is the intersection of the ring $B_{m_i+2} \setminus B_{m_i}$ and the cone $\{r\omega \mid r > 0,\ d(\omega, \zeta_i) < 1/2i\}$. For any sequence $\mathbf{z}_n$ in the complement of A^* for which $g(\mathbf{z}_n)$ is positive and vanishes for $n \to \infty$ the quotient $\tilde{g}(\mathbf{z}_n)/g(\mathbf{z}_n)$ tends to one.

7) Let m be a positive integer. Let H_n be the halfspace of direction ω_n supporting the ball B_{t_n} in the point $\mathbf{z}_n$, and let α_n be the associated affine normalization. Assume $t_n \to \infty$. Let $C_n = C_n(m)$ be the cap $B_{t_n+m} \cap H'_n$ where H'_n is the halfspace with direction ω_n supporting B_{t_n-m}. The image $\alpha_n^{-1}(C_n)$ converges to the parabolic cap $P_m = (P + m\mathbf{e}_d) \cap \{v > -m\}$. If the caps C_n are disjoint from A^* for $n \geq n_0$ then

$$\tilde{h}_n(\mathbf{w}) = \tilde{g}(\alpha_n(\mathbf{w}))/\tilde{g}(\alpha_n(\mathbf{0})) \to e^{-\chi(\mathbf{w})}$$

uniformly on compact subsets of the parabolic cap P_m.

8) Claim: If $i \geq 2m+4 \geq 6$ and $C = C_n(m)$ intersects A_i^* then $\mathbf{z} \in A_i$. First observe that $m_i \in [t_n - m - 2, t_n + m + 2]$. By construction B_{m_i} is centered and $B_{m_i}/m_i = B$. The cap C/m_i has height $\epsilon \leq (2m+4)/m_i \leq 1/i^2 \leq 1/36$. Set $\omega = \mathbf{z}/\|\mathbf{z}\|$. By Lemma 3.12 the inequality $d(w, \zeta_i) = \varphi \geq 1/i$ implies that C and the cone $\{r\omega \mid d(\omega, \zeta_i) < 1/2i\}$ are disjoint. Hence we conclude $d(\omega, \zeta_i) < 1/i$ and $|t_n - m_i| \leq m + 2 < i$. This implies $\|\mathbf{z}\| \leq m_i + 2i$ and hence $\mathbf{z} \in A_i$.

9) If $\mathbf{z}_n$ is a sequence in A^c and $\mathbf{z}_n \in \partial B_{t_n}$ with $t_n \to \infty$, then for any integer $m \geq 1$ the cap $C_n(m)$ is disjoint from A^* eventually. Hence $\tilde{g}(\alpha_n(\mathbf{w}))/\tilde{g}(\alpha_n(\mathbf{0})) \to e^{-\chi(\mathbf{w})}$ holds uniformly on compact sets. Weak asymptotic equality $\tilde{g} \asymp g$ then yields $\mathbf{L}^1$ convergence on all halfspaces J on which $g(\alpha_n(\mathbf{w}))/g(\alpha_n(\mathbf{0})) \to e^{-\chi(\mathbf{w})}$ holds in $\mathbf{L}^1(J)$. ¶

Example II. Convex level sets

$\mathbf{L}^1$ convergence to the Gauss-exponential function on $\{v \geq 0\}$ implies $\mathbf{L}^1$ convergence on all horizontal halfspaces $\{v \geq v_0\}$, $v_0 \in \mathbb{R}$ by the Extension Theorem in [2]. $\mathbf{L}^1$ convergence on non-horizontal halfspaces is a different matter.

There is a continuous strictly positive density f on the plane for which there exist scale constant a_t such that

$$h_t(u,v) := e^t f(a_t u, v+t) \to h(u,v) = e^{-u^2/2-v} \qquad t \to \infty$$

uniformly on compact sets of the plane and in $\mathbf{L}^1$ on horizontal halfplanes, but for which

$$\int_J h_t(u,v)\,du\,dv \to \infty \qquad t \to \infty$$

for any halfplane $J = \{v \geq v_0 + au\}$ with $a \neq 0$.

The density f is unimodal with convex level sets $D_t = \{f > e^{-t}\}$. Moreover it is symmetric around the vertical axis, $f(-x,y) = f(x,y)$, and $f(0,t) = e^{-t}$ for $t \geq t_0$. The density is light-tailed, $e^{-\epsilon t} D_t \to (0,0)$ for any $\epsilon > 0$. On the upper halfplane the density f agrees with a density $g \in \mathcal{S} \cap \mathcal{D}$.

The level sets of f have a simple form. The level sets of g are disks $B_t = \{g > e^{-t}\}$ of radius r_t which support the horizontal halfplane $\{y \geq t\}$ in the point $(0,t)$, and which intersect the horizontal axis in the interval $(-x_t, x_t)$. Let T_t be the circumscribed open isosceles triangle whose sides are tangent to the disk B_t in the points $(\pm x_t, 0)$ and $(0, t - 2r_t)$. The level set $D_t = \{f > e^{-t}\}$ agrees with B_t on the upper halfplane $\{y \geq 0\}$ and with the triangle T_t on the lower halfplane $\{y \leq 0\}$, at least for $t \geq t_0$.

Above the line $v = -n$ the level set $Q_n = \{h_n > 0\}$ of the normalized function is an ellipse which fits nicely into the standard parabola P. The set Q_n is a bounded open convex subset of the plane. The functions h_n converge to $e^{-\chi}$ uniformly on compact subsets of the plane and on halfspaces $\{v \geq v_0 + au\}$ for $v_0, a \in \mathbb{R}$. Let J be the halfplane whose lower boundary is tangent to P with slope -1. Because of convexity the precise shape of the level sets Q_{n+m} below the line $v = -(n+m)$ should have no influence on the behaviour of the function h_n on the halfspace J. Yet we claim that the integral of h_n over J goes to infinity for $n \to \infty$.

Let H_t be the halfplane above the tangent line L_t to B_t in $(0, x_t)$ with slope $-\lambda_t$. Then $f \geq e^{-(t+1)}$ on the trapezium A_t bounded by the horizontal lines $y = 0$ and $y = t - 2r_t$ and the tangent lines L_t and L_{t+1}. This trapezium contains a triangle of height $2r_t - t$ bounded by lines with slopes $-\lambda_t$ and $-\lambda_{t+1}$. Hence

$$\mathbb{P}\{\mathbf{Z} \in H_t\} \geq e^{-(t+1)}|A_t| \qquad |A_t| \geq (2r_t - t)^2(\lambda_{t+1}^{-1} - \lambda_t^{-1})/2. \tag{4.2}$$

The trapezium A_t may be large. The probability above may be much larger than the probability for horizontal halfplanes $\{y \geq t\}$.

Here are the details. The disk B_t intersects the horizontal line $y = t - 1$ in an interval of length $2R_t$ with $R_t \sim \sqrt{2r_t}$. The cap of the ball B_t cut off by the halfplane $y \geq t - 1$ in $\mathbf{z}$-space corresponds to the cap cut off from the parabola $P = \{v < -u^2/2\}$ by the halfplane $v \geq -1$ in $\mathbf{w}$-space. We may take normalizations $\alpha_t(u, v) = (a_t u, t + v)$ with $a_t = \sqrt{r_t}$. We claim that

$$q_t := \mathbb{P}\{\mathbf{Z} \in \alpha_t(J)\}/\mathbb{P}\{\mathbf{Z} \in \alpha_t(J_+)\} \to \infty \qquad J = \{v \geq c_0 - cu\}, c \neq 0. \tag{4.3}$$

It suffices to prove this for c_0 large and c close to zero since $J \cup J_+$ then is small. By symmetry we may assume $c > 0$.

The limit (4.3) holds for $J = J_t = \{v \geq c_{0t} - c_t u\}$ provided that c_{0t} does not increase too fast and c_t does not vanish too fast. If $c_{0t} \equiv c_0$ and $c_t \to 0$ sufficiently fast then $\mathbb{P}\{\mathbf{Z} \in \alpha_t(J_t)\} \sim \mathbb{P}\{Y \geq t + c_0\}$ since the distribution of $\mathbf{Z}$ does not charge hyperplanes, and hence $q_t \to e^{-c_0}$. (The function $H \mapsto p(H) = \mathbb{P}\{\mathbf{Z} \in H\}$ is continuous and positive.) We shall give an example where (4.3) holds with $J = J_t$ if c_{0t} and c_t are positive constants, but also if $c_t \to 0$ sufficiently slowly, say $c_t = 1/t^2$, and $c_{0t} = \log t$. Assume $c_t \in (0, 1/2)$ tends to zero for $t \to \infty$. Then $(0, c_{0t}) \in J_t$ and hence $\alpha_t(J_t)$ contains the point $(0, t + c_{0t})$ and intersects $D_{t+c_{0t}}$. The boundary of $\alpha_t(J_t)$ has slope $-\kappa_t$ where $\kappa_t = c_t/a_t$.

Let us now become more specific. Let $\theta \in (1/2, 1)$. Set $\lambda_t = e^{-t^\theta/\theta}$. There is a disk B_t with radius r_t having a horizontal tangent in $(0, t)$ and a tangent

L_t of slope $-\lambda_t$ in a point $(x_t, 0) \in \partial B_t$. Straightforward computations give $r_t(1 - \cos\varphi_t) = t$ for $\varphi_t = \arctan\lambda_t$ and

$$\dot\lambda_t = -\lambda_t/t^{1-\theta} \qquad \dot\varphi_t \sim \dot\lambda_t \qquad \dot r_t \sim 4t^\theta/\lambda_t^2 \qquad t \to \infty. \tag{4.4}$$

Proposition 4.2 *Probability densities g with level sets $B_t = (0, t - r_t) + r_t B$ as above lie in $\mathcal{D}$.*

Proof This is Example 4 with $T_n = n$ and $S_n = 2r_n - n$. The function $t \mapsto T_t$ obviously lies in $\mathcal{D}^+$, and so does $t \mapsto S_t$ since $2\dot r(t) - 1 \sim 4t^\theta/\lambda_t^2$ by (4.4) and the derivative of $\log(4t^\theta/\lambda_t^2)$ vanishes for $t \to \infty$. ¶

The function λ_t satisfies the limit relations

$$\log\left(\frac{\lambda_t}{\lambda_s}\right) \sim \frac{s-t}{t^{1-\theta}} \qquad \frac{1}{\lambda_{s+1}} - \frac{1}{\lambda_s} \sim \frac{1}{\lambda_s s^{1-\theta}} \qquad s = s_t \sim t \to \infty. \tag{4.5}$$

Recall that $\alpha_t(u, v) = (a_t u, v + t)$ with $a_t = \sqrt{r_t}$ where $r_t \sim 2t/\lambda_t^2 \sim 2te^{2t^\theta/\theta}$ by (4.4). The Gauss-exponential asymptotics give

$$\mathbb{P}\{\mathbf{Z} \in \alpha_t(H_+)\} \le 4\sqrt{t}e^{-t}/\lambda_t \qquad t \ge t_1 \tag{4.6}$$

since this probability is asymptotic to $\sqrt{2\pi}a_t e^{-t}$. The halfspace $\alpha_t(J_t)$ is bounded by a line K_t with slope $-\kappa_t$ where $\kappa_t = c_t/a_t \sim c_t\lambda_t/\sqrt{2t}$. Define $s = s(t)$ by $\lambda_s = \kappa_t$. Then by (4.5)

$$s - t \sim t^{1-\theta}\log(\sqrt{2t}/c_t) \qquad t \to \infty. \tag{4.7}$$

Assume $c_{0t} \le s - t$. Then

$$\alpha_t(J_t) \supset H_s \supset H_s \cap \{y \le 0\} \supset A_s.$$

The size of the trapezium A_s was bounded in (4.2) by the asymptotic expression $4s^{1+\theta}e^{5s^\theta/\theta}$. Hence

$$\mathbb{P}\{\mathbf{Z} \in \alpha_t(J_t)\} \ge e^{-s}s^{1+\theta}e^{5m^\theta/\theta} \qquad s = s(t), t \ge t_2. \tag{4.8}$$

Together with (4.6) this yields the bound

$$\log q_t = \log\frac{\mathbb{P}\{\mathbf{Z} \in \alpha_t(J_t)\}}{\mathbb{P}\{\mathbf{Z} \in \alpha_t(H_+)\}} \le s - t + 4t^\theta/\theta \qquad t \ge t_1 \vee t_2.$$

Take

$$c_t = e^{-4t^{2\theta-1}} \qquad c_{0t} = 3t^\theta. \tag{4.9}$$

Then $s - t \sim 4t^\theta$ by (4.7) which implies $n_{0t} << s - t << 4t^\theta/\theta$ and hence $q_t \to \infty$ for $t \to \infty$.

Proposition 4.3 *Let $\alpha_t(u,v) = (a_t u, t+v)$ with $a_t = \sqrt{2t}e^{t^\theta/\theta}$. Then*

$$e^t f_0(\alpha_t(\mathbf{w})) \to e^{-u^2/2}e^{-v} \qquad \mathbf{w} = (u,v) \qquad t \to \infty$$

uniformly on bounded sets and in $\mathbf{L}^1$ on all horizontal halfplanes $\{v \geq c\}$, $c \in \mathbb{R}$, but on no others. Let $J_t = \{v \geq c_{0t} - c_t u\}$ with c_t and c_{0t} as in (4.9). Then

$$e^t \mathbb{P}\{\mathbf{Z} \in \alpha_t(J_t)\} \to \infty.$$

The difference $J_t \setminus H_+$ is a thin sector S_t with top $(u_t, 0)$ with $u_t = 3t^\theta e^{t^{2\theta-1}}$, and bounded by a horizontal halfline and a halfline with slope $-c_t$. Let $t_n - t_n^\theta/\theta = \log(2n\sqrt{\pi})$. Then $\mathbb{P}\{Y \geq t_n\} \sim 1/n$. Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots$ be independent observations from f. The sample clouds $N_n = \{\beta_n^{-1}(\mathbf{Z}_1), \dots, \beta_n^{-1}(\mathbf{Z}_n)\}$ with $\beta_n = \alpha_{t_n}$ has intensity $nf \circ \beta_n$. The expected number of points in the halfspace H_+ goes to one, but the expected number of points in the sliver S_{t_n} goes to infinity. For $m = 1, 2, \dots$

$$\mathbb{P}\{N_n(S_{t_n}) > m\} \to 1 \qquad n \to \infty.$$

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References

[1] A.A. Balkema & S.I. Resnick (1977) Max-infinite divisibility. *J. Appl. Probability* **14** 309–319.

[2] A.A. Balkema & P. Embrechts (2007) *High risk scenarios and extremes, a geometric approach*. Zurich Lectures in Advanced Mathematics. European Mathematical Society, Zurich.

[3] A.A. Balkema & N. Nolde (2010) Asymptotic independence for unimodal densities. *Adv. in Appl. Probab.* **42** 411–432.

[4] W.F. Eddy (1980) The distribution of a convex hull of a Gaussian sample. *J. Appl. Probab.* **17** 686–695.

[5] W.F. Eddy & J.D. Gale (1981) The convex hull of a spherically symmetric sample. *Adv. in Appl. Probab.* **13** 751–763.

[6] L. Fisher (1966) The convex hull of a sample. *Bull. Amer. Math. Soc.* **72** 555–558.

[7] M.I. Gomes & L. de Haan (1999) Approximation by penultimate extreme value distributions. *Extremes* **2** 71–85.

[8] L. de Haan & S.I. Resnick (1977) Limit theory for multivariate sample extremes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **40** 317–337.

[9] J.E. Heffernan & S.I. Resnick (2007) Limit laws for random vectors with an extreme component. *Ann. Appl. Probab.* **17** 537–571.

[10] M. Sibuya (1960) Bivariate extreme statistics. *Ann. Inst. Stat. Math.* **11** 195–210.

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