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Adaptive control for nonlinear uncertain systems with actuator amplitude and rate saturation constraints

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SUMMARY

A direct adaptive nonlinear tracking control framework for multivariable nonlinear uncertain systems with actuator amplitude and rate saturation constraints is developed. To guarantee asymptotic stability of the closed-loop tracking error dynamics in the face of amplitude and rate saturation constraints, the control signal to a given reference (governor or supervisor) system is modified to effectively robustify the error dynamics to the saturation constraints. Illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In light of the increasingly complex and highly uncertain nature of dynamical systems requiring controls, it is not surprising that reliable system models for many high-performance engineering applications are unavailable. In the face of such high levels of system uncertainty, robust controllers may unnecessarily sacrifice system performance, whereas adaptive controllers are clearly appropriate since they can tolerate far greater system uncertainty levels to improve system performance. However, an implicit assumption inherent in most adaptive control frameworks is that the adaptive

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control law is implemented without any regard to actuator amplitude and rate saturation constraints. Of course, any electromechanical control actuation device is subject to amplitude and/or rate constraints leading to saturation nonlinearities enforcing limitations on control amplitudes and control rates. As a consequence, actuator nonlinearities arise frequently in practice and can severely degrade closed-loop system performance, and in some cases drive the system to instability. These effects are even more pronounced for adaptive controllers that continue to adapt when the feedback loop has been severed due to the presence of actuator saturation causing unstable controller modes to drift, which in turn leads to severe windup effects.

The research literature on adaptive control with actuator saturation effects is rather limited. Notable exceptions include [1–9]. However, the results reported in [1–7, 9] are confined to linear plants with amplitude saturation. Of particular interest is the positive μ-modification framework given in [9], which was applied to a specific class of nonlinear systems in [8]. The result presented in [8], however, does not consider rate saturation.

Many practical applications involve nonlinear dynamical systems with simultaneous control amplitude and rate saturation. The presence of control rate saturation may further exacerbate the problem of control amplitude saturation. For example, in advanced tactical fighter aircraft with high maneuverability requirements, pilot-induced oscillations [10, 11] can cause actuator amplitude and rate saturation in the control surfaces, leading to catastrophic failures.

In this paper, we develop a direct adaptive control framework for adaptive tracking of multivariable nonlinear uncertain systems with amplitude and rate saturation constraints. In particular, we extend the Lyapunov-based direct adaptive control framework developed in [12, 13] to guarantee asymptotic stability of the closed-loop tracking system; that is, asymptotic stability with respect to the closed-loop system states associated with the tracking error dynamics in the face of actuator amplitude and rate saturation constraints. Specifically, a reference (governor or supervisor) dynamical system is constructed to address tracking and regulation by deriving adaptive update laws that guarantee that the error system dynamics are asymptotically stable and the adaptive controller gains are Lyapunov stable. In the case where the actuator amplitude and rate are limited, the adaptive control signal to the reference system is modified to effectively robustify the error dynamics to the saturation constraints, thus guaranteeing asymptotic stability of the error states.

2. ADAPTIVE TRACKING FOR NONLINEAR UNCERTAIN SYSTEMS

In this section, we consider the problem of characterizing adaptive feedback tracking control laws for nonlinear uncertain systems. Specifically, we consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + Bu(t), \quad x(0) = x_0, \quad t \geqslant 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $t \geqslant 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geqslant 0$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$, the matrix $B \in \mathbb{R}^{n \times m}$ is of the form $B = [0_{m \times (n-m)} \ \ B_s^{\mathrm{T}}]^{\mathrm{T}}$, with $B_s \in \mathbb{R}^{m \times m}$ full rank and such that there exists $\Lambda \in \mathbb{R}^{m \times m}$ for which $B_s \Lambda$ is positive definite. The control input $u(\cdot)$ in (1) is restricted to the class of *admissible controls* so that (1) has a unique solution forward in time. Here, we assume that a *desired* trajectory (command) $x_{\mathrm{d}}(t)$, $t \geqslant 0$, is given and the aim is to determine the control input $u(t)$, $t \geqslant 0$, so that $\lim_{t \to \infty} \|x(t) - x_{\mathrm{d}}(t)\| = 0$.

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In order to achieve this, we construct a reference system $\mathcal{G}_r$ given by

$$\dot{x}_{r1}(t)=A_r x_{r1}(t)+B_r r(t), \quad x_{r1}(0)=x_{r1_0}, \quad t\geqslant 0 \tag{2}$$

where $x_{r1}(t)\in\mathbb{R}^n$, $t\geqslant 0$, is the reference state vector, $r(t)\in\mathbb{R}^m$, $t\geqslant 0$, is the reference input, and $A_r\in\mathbb{R}^{n\times n}$ and $B_r\in\mathbb{R}^{n\times m}$ are such that the pair (A_r, B_r) is stabilizable. Now, we design $u(t)$, $t\geqslant 0$, and a bounded piecewise continuous reference function $r(t)$, $t\geqslant 0$, such that $\lim_{t\to\infty}\|x(t)-x_{r1}(t)\|=0$ and $\lim_{t\to\infty}\|x_{r1}(t)-x_{\mathrm{d}}(t)\|=0$, respectively, so that $\lim_{t\to\infty}\|x(t)-x_{\mathrm{d}}(t)\|=0$. The following result provides a control architecture that achieves tracking error convergence in the case where the dynamics in (1) are known. The case where $\mathcal{G}$ is unknown is addressed in Theorem 2.2. For the statement of this result, define the tracking error $e(t)\triangleq x(t)-x_{r1}(t)$, $t\geqslant 0$, whose dynamics are described by

$$\dot{e}(t)=(f(x(t))+Bu(t))-(A_r x_{r1}(t)+B_r r(t)), \quad e(0)=x_0-x_{r_0}\triangleq e_0, \quad t\geqslant 0 \tag{3}$$

Theorem 2.1

Consider the nonlinear system $\mathcal{G}$ given by (1) and the reference system $\mathcal{G}_r$ given by (2). Assume that there exist gain matrices $\Theta^*\in\mathbb{R}^{m\times s}$ and $\Theta_r^*\in\mathbb{R}^{m\times m}$ and a continuously differentiable function $F:\mathbb{R}^n\to\mathbb{R}^s$ such that

$$0=f(x)+B\Lambda\Theta^* F(x)-A_r x, \quad x\in\mathbb{R}^n \tag{4}$$

$$0=B\Lambda\Theta_r^*-B_r \tag{5}$$

Furthermore, let $K\in\mathbb{R}^{m\times n}$ be given by

$$K=-R_2^{-1}B_r^{\mathrm{T}}P \tag{6}$$

where the $n\times n$ positive-definite matrix P satisfies

$$0=A_r^{\mathrm{T}}P+PA_r-PB_rR_2^{-1}B_r^{\mathrm{T}}P+R_1 \tag{7}$$

and $R_1\in\mathbb{R}^{n\times n}$ and $R_2\in\mathbb{R}^{m\times m}$ are arbitrary positive-definite matrices. Then the feedback control law

$$u(t)=\Lambda(\Theta_1^*\varphi_1(t)+\Theta_r^* r(t)), \quad t\geqslant 0 \tag{8}$$

where

$$\Theta_1^*\triangleq[\Theta^*, \Theta_r^*, \Lambda^{\mathrm{T}}B^{\mathrm{T}}]\in\mathbb{R}^{m\times(m+n+s)} \tag{9}$$

$$\varphi_1(t)\triangleq[F^{\mathrm{T}}(x(t)), e^{\mathrm{T}}(t)K^{\mathrm{T}}, -\tfrac{1}{2}k_\lambda e^{\mathrm{T}}(t)P]^{\mathrm{T}}\in\mathbb{R}^{m+n+s}, \quad t\geqslant 0 \tag{10}$$

with $k_\lambda>0$ guarantees that the zero solution $e(t)\equiv 0$, $t\geqslant 0$, of the error dynamics given by (3) is globally asymptotically stable.

Proof

Substituting the feedback control law given by (8) into (3), we obtain

$$\dot{e}(t)=f(x(t))+B\Lambda\Theta_1^*\varphi_1(t)+B\Lambda\Theta_r^* r(t)-A_r x_{r1}(t)-B_r r(t), \quad e(0)=e_0, \quad t\geqslant 0 \tag{11}$$

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which, using (9) and (10), can be rewritten as

$$\dot{e}(t) = (A_r + B\Lambda\Theta_r^* K - \tfrac{1}{2}k_\lambda B\Lambda\Lambda^{\mathrm{T}} B^{\mathrm{T}} P)e(t) + (f(x(t)) + B\Lambda\Theta^* F(x(t)) - A_r x(t))$$
$$+ (B\Lambda\Theta_r^* - B_r)r(t), \quad e(0) = e_0, \quad t \geqslant 0 \tag{12}$$

Now, using (4) and (5), it follows from (12) that

$$\dot{e}(t) = (A_r + B_r K - \tfrac{1}{2}k_\lambda B\Lambda\Lambda^{\mathrm{T}} B^{\mathrm{T}} P)e(t), \quad e(0) = e_0, \quad t \geqslant 0 \tag{13}$$

Next, consider the Lyapunov function candidate

$$V(e) = e^{\mathrm{T}} P e \tag{14}$$

where $P > 0$ satisfies (7). Note that $V(0) = 0$ and, since P is positive definite, $V(e) > 0$ for all $e \neq 0$. Now, letting $e(t)$, $t \geqslant 0$, denote the solution to (13) and using (7), it follows from (13) that the Lyapunov derivative along the closed-loop system trajectories of (13) is given by

$$\dot{V}(e(t)) = -e^{\mathrm{T}}(t)(R_1 + K^{\mathrm{T}} R_2 K + K_e)e(t) < 0, \quad t \geqslant 0 \tag{15}$$

where $K_e \triangleq k_\lambda PB\Lambda\Lambda^{\mathrm{T}} B^{\mathrm{T}} P \geqslant 0$. Hence, the zero solution $e(t) \equiv 0$ of the error dynamics given by (13) is globally asymptotically stable. □

Theorem 2.1 provides sufficient conditions for characterizing tracking controllers for a given nominal nonlinear dynamical system $\mathscr{G}$. In the next result we show how to construct adaptive gains $\Theta_1(t) \in \mathbb{R}^{m\times(m+n+s)}$, $t \geqslant 0$, and $\Theta_r(t) \in \mathbb{R}^{m\times m}$, $t \geqslant 0$, for achieving tracking control in the face of system uncertainty. For this result we do *not* require explicit knowledge of the gain matrices Θ^* and Θ_r^*; all that is required is the existence of Θ^* and Θ_r^* such that the matching conditions (4) and (5) hold.

Theorem 2.2
Consider the nonlinear system $\mathscr{G}$ given by (1) and the reference system $\mathscr{G}_r$ given by (2). Assume that there exist unknown gain matrices $\Theta^* \in \mathbb{R}^{m\times s}$ and $\Theta_r^* \in \mathbb{R}^{m\times m}$ and a continuously differentiable function $F: \mathbb{R}^n \to \mathbb{R}^s$ such that (4) and (5) hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by (6), where $P = [P_1 \;\; P_2] > 0$ satisfies (7) with $P_1 \in \mathbb{R}^{n\times(n-m)}$ and $P_2 \in \mathbb{R}^{n\times m}$. In addition, let $\Gamma_1 \in \mathbb{R}^{(m+n+s)\times(m+n+s)}$ and $\Gamma_r \in \mathbb{R}^{m\times m}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \Lambda(\Theta_1(t)\varphi_1(t) + \Theta_r(t)r(t)), \quad t \geqslant 0 \tag{16}$$

where $\Theta_1(t) \in \mathbb{R}^{m\times(m+n+s)}$, $t \geqslant 0$, and $\Theta_r(t) \in \mathbb{R}^{m\times m}$, $t \geqslant 0$, are estimates of Θ_1^* and Θ_r^*, respectively, with update laws

$$\dot{\Theta}_1(t) = -P_2^{\mathrm{T}} e(t)\varphi_1^{\mathrm{T}}(t)\Gamma_1, \quad \Theta_1(0) = \Theta_{10}, \quad t \geqslant 0 \tag{17}$$

$$\dot{\Theta}_r(t) = -P_2^{\mathrm{T}} e(t) r^{\mathrm{T}}(t)\Gamma_r, \quad \Theta_r(0) = \Theta_{r0} \tag{18}$$

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guarantees that the closed-loop system given by (3), (17), and (18), with control input (16), is Lyapunov stable and $e(t)\to 0$ as $t\to\infty$.

Proof
With $u(t),\ t\geqslant 0$, given by (16) it follows from (4) and (5) that the error dynamics $e(t),\ t\geqslant 0$, are given by

$$\dot{e}(t)=(A_r+B_rK-\tfrac{1}{2}k_\lambda B\Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}P)e(t)+B\Lambda(\Theta_1(t)-\Theta_1^*)\varphi_1(t)$$
$$+B\Lambda(\Theta_r(t)-\Theta_r^*)r(t),\quad e(0)=e_0,\ t\geqslant 0 \tag{19}$$

Now, consider the Lyapunov function candidate

$$V(e,\Theta_1,\Theta_r)=e^{\mathrm{T}}Pe+\mathrm{tr}[B_s\Lambda(\Theta_1-\Theta_1^*)\Gamma_1^{-1}(\Theta_1^{\mathrm{T}}-\Theta_1^{*\mathrm{T}})]$$
$$+\mathrm{tr}[B_s\Lambda(\Theta_r-\Theta_r^*)\Gamma_r^{-1}(\Theta_r^{\mathrm{T}}-\Theta_r^{*\mathrm{T}})] \tag{20}$$

where $P>0$ satisfies (7) and Γ_1 and Γ_r are positive definite. Note that $V(0,\Theta_1^*,\Theta_r^*)=0$ and, since P, Γ_1, Γ_r, and $B_s\Lambda$ are positive definite, $V(e,\Theta_1,\Theta_r)>0$ for all $(e,\Theta_1,\Theta_r)\neq(0,\Theta_1^*,\Theta_r^*)$. Now, letting $e(t),\ t\geqslant 0$, denote the solution to (19) and using (7), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}\dot{V}(e(t),\Theta_1(t),\Theta_r(t))&=e^{\mathrm{T}}(t)P\dot{e}(t)+\dot{e}^{\mathrm{T}}(t)Pe(t)+2\,\mathrm{tr}[B_s\Lambda(\Theta_1(t)-\Theta_1^*)\Gamma_1^{-1}\dot{\Theta}_1^{\mathrm{T}}(t)]\\&\quad+2\,\mathrm{tr}[B_s\Lambda(\Theta_r(t)-\Theta_r^*)\Gamma_r^{-1}\dot{\Theta}_r^{\mathrm{T}}(t)]\\&=2e^{\mathrm{T}}(t)PB\Lambda(\Theta_1(t)-\Theta_1^*)\varphi_1(t)+2\,\mathrm{tr}[B_s\Lambda(\Theta_1(t)-\Theta_1^*)\Gamma_1^{-1}\dot{\Theta}_1^{\mathrm{T}}(t)]\\&\quad+2e^{\mathrm{T}}(t)PB\Lambda(\Theta_r(t)-\Theta_r^*)r(t)+2\,\mathrm{tr}[B_s\Lambda(\Theta_r(t)-\Theta_r^*)\Gamma_r^{-1}\dot{\Theta}_r^{\mathrm{T}}(t)]\\&\quad+e^{\mathrm{T}}(t)P(A_r+B_rK)e(t)\\&\quad+e^{\mathrm{T}}(t)(A_r+B_rK)^{\mathrm{T}}Pe(t)-e^{\mathrm{T}}(t)K_ee(t)\end{aligned}\tag{21}$$

Next, using (17) and (18) and the fact that $PB=P_2B_s$, it follows that

$$\begin{aligned}\dot{V}(e(t),\Theta_1(t),\Theta_r(t))&=-e^{\mathrm{T}}(t)(R_1+K^{\mathrm{T}}R_2K+K_e)e(t)\\&\quad+2\,\mathrm{tr}[B_s\Lambda(\Theta_1(t)-\Theta_1^*)(\varphi_1(t)e^{\mathrm{T}}(t)P_2+\Gamma_1^{-1}\dot{\Theta}_1^{\mathrm{T}}(t))]\\&\quad+2\,\mathrm{tr}[B_s\Lambda(\Theta_r(t)-\Theta_r^*)(r(t)e^{\mathrm{T}}(t)P_2+\Gamma_r^{-1}\dot{\Theta}_r^{\mathrm{T}}(t))]\\&=-e^{\mathrm{T}}(t)(R_1+K^{\mathrm{T}}R_2K+K_e)e(t)\\&\leqslant 0,\quad t\geqslant 0\end{aligned}\tag{22}$$

Hence, the closed-loop system given by (3) and (16)–(18) is Lyapunov stable, and, by the LaSalle–Yoshizawa theorem [14], $\lim_{t\to\infty}e^{\mathrm{T}}(t)(R_1+K^{\mathrm{T}}R_2K+K_e)e(t)=0$ and, hence, $\lim_{t\to\infty}e(t)=0$. □

Remark 2.1
Note that the conditions in Theorem 2.2 imply that $e(t)\to 0$ as $t\to\infty$ and, hence, it follows from (17) and (18) that $\dot{\Theta}_1(t)\to 0$ and $\dot{\Theta}_r(t)\to 0$ as $t\to\infty$.

It is important to note that the adaptive law (16)–(18) does *not* require explicit knowledge of the gain matrices Θ^* and Θ_r^*. Furthermore, no specific knowledge of the structure of the nonlinear dynamics $f(x)$ or the input matrix B is required to apply Theorem 2.2; all that is required is the existence of $F(x)$ and Λ such that the matching conditions (4) and (5) hold for a given reference system $\mathcal{G}_r$. However, if (1) is in normal form with asymptotically stable internal dynamics [15], then we can always construct $F:\mathbb{R}^n\to\mathbb{R}^s$, $\Lambda\in\mathbb{R}^{m\times m}$, and a stabilizable pair (A_r, B_r) such that (4) and (5) hold *without* requiring knowledge of the system dynamics. In order to see this, assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t)=f_{ui}(q(t))+b_i u(t), \quad q(0)=q_0, \ \ t\geqslant 0, \ \ i=1,\dots,m \tag{23}$$

where $q_i^{(r_i)}$ denotes the r_ith derivative of q_i, r_i denotes the relative degree with respect to the output q_i, $f_{ui}(q)=f_{ui}(q_1,\dots,q_1^{(r_1-1)},\dots,q_m,\dots,q_m^{(r_m-1)})$, the row vector $b_i\in\mathbb{R}^m$, and $q\in\mathbb{R}^{\hat{r}}$, where $\hat{r}=r_1+\cdots+r_m$ is the (vector) relative degree of (23). Furthermore, since (23) is in a form where it does not possess internal dynamics, it follows that $\hat{r}=n$. The case where (23) possesses input-to-state stable internal dynamics can be handled as shown in [12].

Next, define $x_i\triangleq[q_i,\dots,q_i^{(r_i-2)}]^{\mathrm{T}}, i=1,\dots,m$, $x_{m+1}\triangleq[q_1^{(r_1-1)},\dots,q_m^{(r_m-1)}]^{\mathrm{T}}$, and $x\triangleq[x_1^{\mathrm{T}},\dots, x_{m+1}^{\mathrm{T}}]^{\mathrm{T}}$ so that (23) can be described as (1) with

$$f(x)=\tilde{A}x+\tilde{f}_u(x), \quad B_s=[b_1^{\mathrm{T}} \ \dots \ b_m^{\mathrm{T}}]^{\mathrm{T}} \tag{24}$$

where

$$\tilde{A}=\begin{bmatrix} A_0 \\ 0_{m\times n}\end{bmatrix}, \quad \tilde{f}_u(x)=\begin{bmatrix} 0_{(n-m)\times 1} \\ f_u(x)\end{bmatrix}$$

$A_0\in\mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [16] and $f_u:\mathbb{R}^n\to\mathbb{R}^m$ and $B\in\mathbb{R}^{m\times m}$ are unknown. In addition, we introduce the parametrization $f_u(x)=\Theta_\ell x+\Theta_{n\ell}f_{n\ell}(x)$, where $f_{n\ell}:\mathbb{R}^n\to\mathbb{R}^q$, $x\in\mathbb{R}^n$, and $\Theta_\ell\in\mathbb{R}^{m\times n}$ and $\Theta_{n\ell}\in\mathbb{R}^{m\times q}$ are uncertain constant matrices.

Next, to apply Theorem 2.2 to the uncertain system (1) with $f(x)$ and B_s given by (24), let $B_r=[0_{m\times(n-m)}, B_{rs}^{\mathrm{T}}]^{\mathrm{T}}$, where $B_{rs}\in\mathbb{R}^{m\times m}$, let $A_r=[A_0^{\mathrm{T}},\Theta_n^{\mathrm{T}}]^{\mathrm{T}}$, where $\Theta_n\in\mathbb{R}^{m\times n}$ is a known matrix, let $\Theta^*\in\mathbb{R}^{m\times(n+q)}$ be given by

$$\Theta^*=(B_s\Lambda)^{-1}[\Theta_n-\Theta_\ell, -\Theta_{n\ell}] \tag{25}$$

and let

$$F(x)=\begin{bmatrix} x \\ f_{n\ell}(x)\end{bmatrix} \tag{26}$$

In this case, it follows that, with $\Theta_r^*=(B_s\Lambda)^{-1}B_{rs}$,

$$B\Lambda\Theta_r^*=B_r \tag{27}$$

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and

$$f(x)+B\Lambda\Theta^* F(x) = \tilde{A}x+\tilde{f}_u(x)+\begin{bmatrix} 0_{(n-m)\times m} \\ B_s \end{bmatrix}\Lambda(B_s\Lambda)^{-1}[\Theta_n x-\Theta_\ell x-\Theta_{n\ell} f_{n\ell}(x)]$$

$$= \tilde{A}x+\begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_n x \end{bmatrix} = A_r x \tag{28}$$

where A_r is in multivariable controllable canonical form. Hence, choosing A_r and B_r such that (A_r, B_r) is stabilizable and choosing $R_1>0$ and $R_2>0$, it follows that there exists a positive-definite matrix P satisfying the Riccati equation (7).

3. DYNAMIC ADAPTIVE TRACKING FOR NONLINEAR UNCERTAIN SYSTEMS

In this section, we extend the results of Section 2 by constructing an adaptive, dynamic controller for (1), with stability properties identical to those given by Theorem 2.2. The ultimate objective here is to be able to account for both amplitude and rate saturation constraints in the control input. In order to be able to account for rate saturation, it is necessary to consider the time derivative of the command $u(t),\ t\geqslant 0$. Amplitude saturation is accounted for by setting this time derivative to zero. To that end, the command $u(t),\ t\geqslant 0$, will be generated by a dynamic compensator of the form

$$\dot{x}_c(t) = f_c(x(t), x_r(t), x_c(t)), \quad x_c(0)=x_{c0}, \quad t\geqslant 0 \tag{29}$$

$$u(t) = x_c(t) \tag{30}$$

where $x_c(t)\in\mathbb{R}^m,\ t\geqslant 0$, is the compensator state, $x(t)\in\mathbb{R}^n,\ t\geqslant 0$, is the system state, $x_r(t)\in\mathbb{R}^{m+n},\ t\geqslant 0$, is a reference state, and $f_c:\mathbb{R}^n\times\mathbb{R}^{m+n}\times\mathbb{R}^m\to\mathbb{R}^m$. In order to account for the compensator state, we modify the reference system (2) as

$$\dot{x}_r(t) = \begin{bmatrix} A_r & B_r \\ 0_{m\times n} & -T_r^{-1} \end{bmatrix} x_r(t) + \begin{bmatrix} 0_{n\times m} \\ T_r^{-1} \end{bmatrix} r(t), \quad x_r(0) = \begin{bmatrix} x_{r1_0} \\ x_{r2_0} \end{bmatrix}, \quad t\geqslant 0 \tag{31}$$

where $x_r(t)=[x_{r1}^{\mathrm{T}}(t)\ \ x_{r2}^{\mathrm{T}}(t)]^{\mathrm{T}},\ t\geqslant 0$, with $x_{r1}(t)\in\mathbb{R}^n$, $x_{r2}(t)\in\mathbb{R}^m,\ t\geqslant 0$, and $T_r\in\mathbb{R}^{m\times m}$ is positive definite.

Next, we introduce an expression for the desired control input, which is identical to the expression of $u(t),\ t\geqslant 0$, obtained in the previous section, given by

$$u_d^*(t) \triangleq \Lambda(\Theta_1^*\varphi_1(t)+\Theta_r^* x_{r2}(t)), \quad t\geqslant 0 \tag{32}$$

where $u_d^*(t),\ t\geqslant 0$, is such that for $u(t)=u_d^*(t),\ t\geqslant 0$, Theorem 2.1 guarantees that $e(t),\ t\geqslant 0$, converges to zero. In the classical backstepping literature (see, for example [14]), this desired control input is referred to as a 'virtual command.' Note that $r(t),\ t\geqslant 0$, in (8) is replaced by $x_{r2}(t),\ t\geqslant 0$, in (32) to account for the modification to the reference system. With this definition for $u_d^*(t),\ t\geqslant 0$, the error dynamics (3) become

$$\dot{e}(t) = (A_r+B_r K+K_e)e(t)+B(u(t)-u_d^*(t)), \quad e(0)=e_0, \quad t\geqslant 0 \tag{33}$$

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Defining the error $e_u^*(t) \triangleq u(t) - u_{\rm d}^*(t),\ t \geqslant 0$, the remaining problem is to find the appropriate expression for $f_{\rm c}(\cdot)$ such that $e_u^*(t),\ t \geqslant 0$, converges to zero.

Note that a number of constant parameters in (32) are uncertain and need to be estimated, with appropriate update laws similar to those in Theorem 2.2. Ultimately, the expression we desire $u(t),\ t \geqslant 0$, to converge to is given by

$$u_{\rm d}(t) = \Lambda(\Theta_1(t)\varphi_1(t) + \Theta_r(t)x_{r2}(t)), \quad t \geqslant 0 \tag{34}$$

where $\Theta_1(t) \in \mathbb{R}^{m\times(m+n+s)}$ and $\Theta_r(t) \in \mathbb{R}^{m\times m},\ t \geqslant 0$, are estimates of Θ_1^* and Θ_r^*, respectively. To this end, we define the tracking error $e_u(t) \triangleq u(t) - u_{\rm d}(t),\ t \geqslant 0$.

For the statement of the next result, we need the expression for $\dot{u}_{\rm d}(t),\ t \geqslant 0$. Using the update laws given by Theorem 2.2, and $\Theta_1(t) = [\Theta_{11}(t)\ \ \Theta_{12}(t)],\ t \geqslant 0$, with $\Theta_{11}(t) \in \mathbb{R}^{m\times s}$ and $\Theta_{12}(t) \in \mathbb{R}^{m\times(m+n)},\ t \geqslant 0$, we obtain

$$\begin{aligned}\dot{u}_{\rm d}(t) = \Lambda(&-P_2^{\rm T}e(t)(\varphi_1^{\rm T}(t)\Gamma_1\varphi_1(t) + x_{r2}^{\rm T}(t)\Gamma_r x_{r2}(t)) + \Theta_{11}(t)F'(x(t))(f(x(t)) + Bu(t)) \\ &+ \Theta_{12}(t)\begin{bmatrix} K \\ -\frac{1}{2}k_\lambda P \end{bmatrix}\dot{e}(t) + \Theta_r(t)T_r^{-1}(r(t) - x_{r2}(t))), \quad t \geqslant 0\end{aligned} \tag{35}$$

where $F'(x(t))$ denotes the Fréchet derivative of $F(\cdot)$ at $x(t),\ t \geqslant 0,\ \Gamma_1 \in \mathbb{R}^{(m+n+s)\times(m+n+s)}$, $\Gamma_r \in \mathbb{R}^{m\times m}$, $\Gamma_1 > 0$, and $\Gamma_r > 0$. Note that the above expression can be rewritten as

$$\dot{u}_{\rm d}(t) = g(t) + h(t)\Theta_2^*\varphi_2(t), \quad u_{\rm d}(0) = \Lambda(\Theta_{10}\varphi_1(0) + \Theta_{r0}x_{r2_0}),\ \ t \geqslant 0 \tag{36}$$

where

$$h(t) \triangleq \Lambda\Theta_1(t)\begin{bmatrix} F'(x(t)) \\ K \\ -\frac{1}{2}k_\lambda P \end{bmatrix}, \quad \Theta_2^* \triangleq B[-\Lambda\Theta^*, I_m], \quad \varphi_2(t) \triangleq \begin{bmatrix} F(x(t)) \\ u(t) \end{bmatrix} \tag{37}$$

and

$$\begin{aligned} g(t) \triangleq \Lambda(&-P_2^{\rm T}e(t)(\varphi_1^{\rm T}(t)\Gamma_1\varphi_1(t) + x_{r2}^{\rm T}(t)\Gamma_r x_{r2}(t)) - \Theta_{12}(t)[K^{\rm T}, -\tfrac{1}{2}k_\lambda P]^{\rm T}\dot{x}_{r1}(t) \\ &+ \Theta_r(t)T_r^{-1}(r(t) - x_{r2}(t))) + h(t)A_r x(t)\end{aligned} \tag{38}$$

Note that (36) allows us to isolate the unknown term Θ_2^* in $\dot{u}_{\rm d}(t),\ t \geqslant 0$. In addition, from (29), (30), and (36), we obtain the following expression for the time derivative of the new tracking error $e_u(t),\ t \geqslant 0$:

$$\dot{e}_u(t) = f_{\rm c}(t) - g(t) - h(t)\Theta_2^*\varphi_2(t), \quad e_u(0) = e_{u0},\ \ t \geqslant 0 \tag{39}$$

By choosing $f_{\rm c}(t),\ t \geqslant 0$, we can reshape the error dynamics (39) as we desire. In particular, if $\dot{e}_u(t),\ t \geqslant 0$, is assigned, it follows from (39) that the corresponding dynamic compensator (29) is given by

$$f_{\rm c}(t) = g(t) + h(t)\Theta_2^*\varphi_2(t) + \dot{e}_u(t), \quad t \geqslant 0 \tag{40}$$

The following result presents an expression for $f_{\rm c}(t),\ t \geqslant 0$, which is similar to (40), with $\dot{e}_u(t),\ t \geqslant 0$, replaced by an appropriate function that guarantees the convergence of $e_u(t),\ t \geqslant 0$, to zero.

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Theorem 3.1
Consider the controlled nonlinear system $\mathscr{G}$ given by (1) and reference system (31). Assume that there exist unknown gain matrices $\Theta^* \in \mathbb{R}^{m\times s}$ and $\Theta_r^* \in \mathbb{R}^{m\times m}$ and a continuously differentiable function $F:\mathbb{R}^n \to \mathbb{R}^s$ such that (4) and (5) hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by (6), where $P=[P_1 \;\; P_2]>0$ with $P_1 \in \mathbb{R}^{n\times(n-m)}$ and $P_2 \in \mathbb{R}^{n\times m}$ satisfies (7). Consider the control input $u(t),\ t\geqslant 0$, generated by (29) and (30), where

$$f_{\mathrm{c}}(t)=g(t)+h(t)\Theta_2(t)\varphi_2(t)-2\Theta_3(t)Pe(t)-K_u e_u(t),\quad t\geqslant 0 \tag{41}$$

where $\varphi_2(t)$ and $h(t)\in\mathbb{R}^{m\times n},\ t\geqslant 0$, are given by (37), $g(t)\in\mathbb{R}^m,\ t\geqslant 0$, is given by (38), $K_u \in \mathbb{R}^{m\times m}$ is positive definite, and $\Theta_2(t)$ and $\Theta_3(t),\ t\geqslant 0$, are estimates of Θ_2^* and $\Theta_3^* \triangleq B^{\mathrm{T}} \in \mathbb{R}^{m\times n}$, respectively. The estimates $\Theta_1(t)\in\mathbb{R}^{m\times(m+n+s)}$, $\Theta_r(t)\in\mathbb{R}^{m\times m}$, $\Theta_2(t)\in\mathbb{R}^{n\times(m+s)}$, and $\Theta_3(t)\in\mathbb{R}^{m\times n},\ t\geqslant 0$, are given by

$$\dot\Theta_1(t)=-P_2^{\mathrm{T}}e(t)\varphi_1^{\mathrm{T}}(t)\Gamma_1,\quad \Theta_1(0)=\Theta_{10},\ \ t\geqslant 0 \tag{42}$$

$$\dot\Theta_r(t)=-P_2^{\mathrm{T}}e(t)x_{r2}^{\mathrm{T}}(t)\Gamma_r,\quad \Theta_r(0)=\Theta_{r0} \tag{43}$$

$$\dot\Theta_2(t)=-h(t)^{\mathrm{T}}e_u(t)\varphi_2^{\mathrm{T}}(t)\Gamma_2,\quad \Theta_2(0)=\Theta_{20} \tag{44}$$

$$\dot\Theta_3(t)=e_u(t)e^{\mathrm{T}}(t)P\Gamma_3,\quad \Theta_3(0)=\Theta_{30} \tag{45}$$

where $\Gamma_1\in\mathbb{R}^{(m+n+s)\times(m+n+s)}$, $\Gamma_r\in\mathbb{R}^{m\times m}$, $\Gamma_2\in\mathbb{R}^{(m+s)\times(m+s)}$, and $\Gamma_3\in\mathbb{R}^{n\times n}$ are positive-definite matrices. In this case, the control input $u(t),\ t\geqslant 0$, generated by (29) and (30) with (41) guarantees that the closed-loop system given by (3), (42)–(45), is Lyapunov stable and $(e(t),e_u(t))\to(0,0)$ as $t\to\infty$.

Proof
From (34) and the definition of $e_u(t),\ t\geqslant 0$, it follows that

$$u(t)=\Lambda(\Theta_1(t)\varphi_1(t)+\Theta_r(t)x_{r2}(t))+e_u(t),\quad t\geqslant 0 \tag{46}$$

or, equivalently, using (10),

$$\begin{aligned} u(t) &= \Lambda(\Theta^* F(x(t))+\Theta_r^*(x_{r2}(t)+Ke(t)))-\tfrac{1}{2}k_\lambda\Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}Pe(t)+\Lambda(\Theta_1(t)-\Theta_1^*)\varphi_1(t)\\ &\quad +\Lambda(\Theta_r(t)-\Theta_r^*)x_{r2}(t)+e_u(t) \end{aligned} \tag{47}$$

Substituting (4), (5), and (47) into (3), we obtain

$$\begin{aligned} \dot e(t) &= (A_r+B_rK-\tfrac{1}{2}k_\lambda B\Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}P)e(t)+B\Lambda(\Theta_1(t)-\Theta_1^*)\varphi_1(t)+B\Lambda(\Theta_r(t)-\Theta_r^*)x_{r2}(t)\\ &\quad +Be_u(t),\quad e(0)=e_0,\ \ t\geqslant 0 \end{aligned} \tag{48}$$

Similarly, from (39), (41), and $\Theta_3^*=B^{\mathrm{T}}$, we obtain

$$\begin{aligned} \dot e_u(t) &= -2B^{\mathrm{T}}Pe(t)-K_u e_u(t)+h(t)(\Theta_2(t)-\Theta_2^*)\varphi_2(t)\\ &\quad +2(\Theta_3^*-\Theta_3(t))Pe(t),\quad e_u(0)=e_{u0},\ \ t\geqslant 0 \end{aligned} \tag{49}$$

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Now, consider the Lyapunov function candidate

$$V(e, e_u, \Theta_1, \Theta_2, \Theta_3, \Theta_r) = e^{\mathrm{T}} P e + \tfrac{1}{2} e_u^{\mathrm{T}} e_u + \mathrm{tr}[B_s \Lambda (\Theta_1 - \Theta_1^*) \Gamma_1^{-1} (\Theta_1^{\mathrm{T}} - \Theta_1^{*\mathrm{T}})]$$
$$+ \mathrm{tr}[(\Theta_2 - \Theta_2^*) \Gamma_2^{-1} (\Theta_2^{\mathrm{T}} - \Theta_2^{*\mathrm{T}})] + \mathrm{tr}[(\Theta_3 - \Theta_3^*) \Gamma_3^{-1} (\Theta_3^{\mathrm{T}} - \Theta_3^{*\mathrm{T}})]$$
$$+ \mathrm{tr}[B_s \Lambda (\Theta_r - \Theta_r^*) \Gamma_r^{-1} (\Theta_r^{\mathrm{T}} - \Theta_r^{*\mathrm{T}})] \tag{50}$$

where $P > 0$ satisfies (7). Note that $V(0, 0, \Theta_1^*, \Theta_2^*, \Theta_3^*, \Theta_r^*) = 0$ and, since P, Γ_1, Γ_2, Γ_3, Γ_r, and $B_s \Lambda$ are positive definite, $V(e, e_u, \Theta_1, \Theta_2, \Theta_3, \Theta_r) > 0$ for all $(e, e_u, \Theta_1, \Theta_2, \Theta_3, \Theta_r) \neq (0, 0, \Theta_1^*, \Theta_2^*, \Theta_3^*, \Theta_r^*)$. Now, using (7) and (42)–(45), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
&\dot{V}(e(t), e_u(t), \Theta_1(t), \Theta_2(t), \Theta_3(t), \Theta_r(t)) \\
&= e^{\mathrm{T}}(t) P \dot{e}(t) + \dot{e}^{\mathrm{T}}(t) P e(t) + e_u(t)^{\mathrm{T}} \dot{e}_u(t) + 2\,\mathrm{tr}[B_s \Lambda (\Theta_1(t) - \Theta_1^*) \Gamma_1^{-1} \dot{\Theta}_1^{\mathrm{T}}(t)] \\
&\quad + 2\,\mathrm{tr}[B_s \Lambda (\Theta_r(t) - \Theta_r^*) \Gamma_r^{-1} \dot{\Theta}_r^{\mathrm{T}}(t)] + 2\,\mathrm{tr}[(\Theta_2(t) - \Theta_2^*) \Gamma_2^{-1} \dot{\Theta}_2^{\mathrm{T}}(t)] \\
&\quad + 2\,\mathrm{tr}[(\Theta_3(t) - \Theta_3^*) \Gamma_3^{-1} \dot{\Theta}_3^{\mathrm{T}}(t)] \\
&= e^{\mathrm{T}}(t) P (A_r + B_r K) e(t) + e^{\mathrm{T}}(t) (A_r + B_r K)^{\mathrm{T}} P e(t) - e^{\mathrm{T}}(t) K_e e(t) - e_u^{\mathrm{T}}(t) K_u e_u(t) \\
&\quad + 2\,\mathrm{tr}[B_s \Lambda (\Theta_1(t) - \Theta_1^*)(\Gamma_1^{-1} \dot{\Theta}_1^{\mathrm{T}}(t) + \varphi_1(t) e^{\mathrm{T}}(t) P_2)] \\
&\quad + 2\,\mathrm{tr}[B_s \Lambda (\Theta_r(t) - \Theta_r^*)(\Gamma_r^{-1} \dot{\Theta}_r^{\mathrm{T}}(t) + x_{r2}(t) e^{\mathrm{T}}(t) P_2)] \\
&\quad + 2\,\mathrm{tr}[(\Theta_2(t) - \Theta_2^*)(\Gamma_2^{-1} \dot{\Theta}_2^{\mathrm{T}}(t) + \varphi_2(t) e_u^{\mathrm{T}}(t) h(t))] \\
&\quad + 2\,\mathrm{tr}[(\Theta_3(t) - \Theta_3^*)(\Gamma_3^{-1} \dot{\Theta}_3^{\mathrm{T}}(t) - P e(t) e_u^{\mathrm{T}}(t))] \\
&= -e^{\mathrm{T}}(t)(R_1 + K^{\mathrm{T}} R_2 K + K_e) e(t) - e_u^{\mathrm{T}}(t) K_u e_u(t) \\
&\leqslant 0, \quad t \geqslant 0
\end{aligned} \tag{51}$$

Hence, the closed-loop system given by (3), (39), (42)–(45) is Lyapunov stable. Furthermore, it follows from the LaSalle–Yoshizawa theorem [14] that $\lim_{t\to\infty} e^{\mathrm{T}}(t)(R_1 + K^{\mathrm{T}} R_2 K + K_e) e(t) + e_u^{\mathrm{T}}(t) K_u e_u(t) = 0$ and hence $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} e_u(t) = 0$. □

Remark 3.1

Note that a parallel can be drawn between (29) and the actuator dynamics of a physical system. In particular, the form of (29) was chosen to be an integrator for simplicity; however, (29) can be modified to represent the actuator dynamics of a particular system. Hence, the presented approach can account for actuator dynamics in the control framework.

The expression for $f_{\mathrm{c}}(\cdot)$ given by (41) implicitly depends upon the form of $\dot{u}_{\mathrm{d}}(t)$, $t \geqslant 0$. The control algorithm can be significantly simplified by using an estimate of $\dot{u}_{\mathrm{d}}(t)$, $t \geqslant 0$, as detailed in the following corollary. The resulting control algorithm then guarantees ultimate boundedness of $e(t)$ and $e_u(t)$, $t \geqslant 0$; that is, convergence of $e(t)$ and $e_u(t)$, $t \geqslant 0$, to a neighborhood of the origin.

Corollary 3.1

Consider the controlled nonlinear system $\mathcal{G}$ given by (1) and reference system (31), and assume that the hypothesis of Theorem 3.1 holds with $k_\lambda > \lambda_{\min}(\Lambda\Lambda^{\mathrm{T}})$. In addition, define $u_{\mathrm{d}1}(t) \triangleq \Lambda\Theta_1(t)\varphi_1(t),\ t \geqslant 0$, and assume that $\dot{u}_{\mathrm{d}1}(t),\ t\geqslant 0$, can be approximated by $\dot{v}(t),\ t\geqslant 0$, such that $\|u_{\mathrm{d}1}(t)-v(t)\| \leqslant \varepsilon \in \mathbb{R},\ t\geqslant 0$. Finally, define $\hat{u}_{\mathrm{d}}(t) \triangleq v(t)+\Lambda\Theta_r(t)x_{r2}(t)$ and $\hat{e}_u(t)\triangleq u(t)-\hat{u}_{\mathrm{d}}(t)$. Then, the control input generated by (29) and (30) with

$$f_{\mathrm{c}}(t) = f_{\hat{u}_{\mathrm{d}}}(t) - 2\Theta_3(t)Pe(t) - K_u\hat{e}_u(t) \tag{52}$$

where

$$f_{\hat{u}_{\mathrm{d}}}(t) \triangleq \dot{v}(t) + \Lambda(-P_2^{\mathrm{T}}e(t)x_{r2}^{\mathrm{T}}(t)\Gamma_r x_{r2}(t) + \Theta_r(t)T_r^{-1}(r(t)-x_{r2}(t))), \quad t\geqslant 0 \tag{53}$$

along with update laws (42), (43), and

$$\dot{\Theta}_3(t) = \hat{e}_u(t)e^{\mathrm{T}}(t)P\Gamma_3, \quad \Theta_3(0)=\Theta_{30},\ \ t\geqslant 0 \tag{54}$$

guarantees that the tracking errors $e(t)$ and $\hat{e}_u(t),\ t\geqslant 0$, converge to a neighborhood of the origin given by $\mathscr{D}_e \triangleq \{e, \hat{e}_u : e^{\mathrm{T}}(R_1+K^{\mathrm{T}}R_2K)e + \hat{e}_u^{\mathrm{T}}K_u\hat{e}_u \leqslant \varepsilon^2\}$.

Proof

From the definitions of $u_{\mathrm{d}1}(t)$, $\hat{u}_{\mathrm{d}}(t)$, and $\hat{e}_u(t),\ t\geqslant 0$, it follows that

$$u(t) = \Lambda(\Theta_1(t)\varphi_1(t) + \Theta_r(t)x_{r2}(t)) + v(t) - u_{\mathrm{d}1}(t) + \hat{e}_u(t), \quad t\geqslant 0 \tag{55}$$

or, equivalently, using (10),

$$\begin{aligned} u(t) = {} & \Lambda(\Theta^* F(x(t)) + \Theta_r^*(x_{r2}(t)+Ke(t))) - \tfrac{1}{2}k_\lambda \Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}Pe(t) + \Lambda(\Theta_1(t)-\Theta_1^*)\varphi_1(t) \\ & + \Lambda(\Theta_r(t)-\Theta_r^*)x_{r2}(t) + v(t) - u_{\mathrm{d}1}(t) + \hat{e}_u(t) \end{aligned} \tag{56}$$

Substituting (4), (5), and (56) into (3), we obtain

$$\begin{aligned} \dot{e}(t) = {} & (A_r + B_rK - \tfrac{1}{2}k_\lambda B\Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}P)e(t) + B\Lambda(\Theta_1(t)-\Theta_1^*)\varphi_1(t) + B\Lambda(\Theta_r(t)-\Theta_r^*)x_{r2}(t) \\ & + B(v(t)-u_{\mathrm{d}1}(t)) + B\hat{e}_u(t), \quad e(0)=e_0,\ \ t\geqslant 0 \end{aligned} \tag{57}$$

Similarly, from (29), (30), (53), and the definition of $\hat{u}_{\mathrm{d}}(t)$ and $\hat{e}_u(t),\ t\geqslant 0$, we obtain

$$\dot{\hat{e}}_u(t) = f_{\mathrm{c}}(t) - f_{\hat{u}_{\mathrm{d}}}(t), \quad \hat{e}_u(0) = \hat{e}_{u0},\ \ t\geqslant 0 \tag{58}$$

which, using (52) and $\Theta_3^* = B^{\mathrm{T}}$, can be rewritten as

$$\dot{\hat{e}}_u(t) = -2B^{\mathrm{T}}Pe(t) - K_u\hat{e}_u(t) + 2(\Theta_3^* - \Theta_3(t))Pe(t), \quad \hat{e}_u(0)=\hat{e}_{u0},\ \ t\geqslant 0 \tag{59}$$

Now, consider the Lyapunov function candidate

$$\begin{aligned} V_s(e,\hat{e}_u,\Theta_1,\Theta_3,\Theta_r) = {} & e^{\mathrm{T}}Pe + \tfrac{1}{2}\hat{e}_u^{\mathrm{T}}\hat{e}_u + \mathrm{tr}[B_s\Lambda(\Theta_1-\Theta_1^*)\Gamma_1^{-1}(\Theta_1^{\mathrm{T}}-\Theta_1^{*\mathrm{T}})] \\ & + \mathrm{tr}[(\Theta_3-\Theta_3^*)\Gamma_3^{-1}(\Theta_3^{\mathrm{T}}-\Theta_3^{*\mathrm{T}})] \\ & + \mathrm{tr}[B_s\Lambda(\Theta_r-\Theta_r^*)\Gamma_r^{-1}(\Theta_r^{\mathrm{T}}-\Theta_r^{*\mathrm{T}})] \end{aligned} \tag{60}$$

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Note that $V_s(0,0,\Theta_1^*,\Theta_3^*,\Theta_r^*)=0$ and, since P, Γ_1, Γ_3, Γ_r, and $B_s\Lambda$ are positive definite, $V_s(e,\hat{e}_u,\Theta_1,\Theta_3,\Theta_r)>0$ for all $(e,\hat{e}_u,\Theta_1,\Theta_3,\Theta_r)\neq(0,0,\Theta_1^*,\Theta_3^*,\Theta_r^*)$. Now, using (7), (42), (43), and (54), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
&\dot{V}_s(e(t),\hat{e}_u(t),\Theta_1(t),\Theta_3(t),\Theta_r(t),t)\\
&\quad=e^{\mathrm{T}}(t)P(A_r+B_rK)e(t)+e^{\mathrm{T}}(t)(A_r+B_rK)^{\mathrm{T}}Pe(t)\\
&\qquad-k_\lambda e^{\mathrm{T}}(t)PB\Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}Pe(t)-\hat{e}_u^{\mathrm{T}}(t)K_u\hat{e}_u(t)\\
&\qquad+2\,\mathrm{tr}[B_s\Lambda(\Theta_1(t)-\Theta_1^*)(\Gamma_1^{-1}\dot{\Theta}_1^{\mathrm{T}}(t)+\varphi_1(t)e^{\mathrm{T}}(t)P_2)]\\
&\qquad+2\,\mathrm{tr}[B_s\Lambda(\Theta_r(t)-\Theta_r^*)(\Gamma_r^{-1}\dot{\Theta}_r^{\mathrm{T}}(t)+x_{r2}(t)e^{\mathrm{T}}(t)P_2)]\\
&\qquad+2\,\mathrm{tr}[(\Theta_3(t)-\Theta_3^*)(\Gamma_3^{-1}\dot{\Theta}_3^{\mathrm{T}}(t)-Pe(t)\hat{e}_u^{\mathrm{T}}(t))]-2e^{\mathrm{T}}(t)PB(u_{\mathrm{d}1}(t)-v(t))\\
&\quad=-e^{\mathrm{T}}(t)(R_1+K^{\mathrm{T}}R_2K)e(t)-\hat{e}_u^{\mathrm{T}}(t)K_u\hat{e}_u(t)\\
&\qquad-2e^{\mathrm{T}}(t)PB(u_{\mathrm{d}1}(t)-v(t))-k_\lambda e^{\mathrm{T}}(t)PB\Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}Pe(t),\quad t\geqslant 0
\end{aligned}\tag{61}$$

which, by completing the square, gives

$$\begin{aligned}
\dot{V}_s(e(t),\hat{e}_u(t),\Theta_1(t),\Theta_3(t),\Theta_r(t),t)&=-e^{\mathrm{T}}(t)(R_1+K^{\mathrm{T}}R_2K)e(t)-\hat{e}_u^{\mathrm{T}}(t)K_u\hat{e}_u(t)\\
&\quad-\|B^{\mathrm{T}}Pe(t)+(u_{\mathrm{d}1}(t)-v(t))\|^2+\|u_{\mathrm{d}1}(t)-v(t)\|^2\\
&\quad-e^{\mathrm{T}}(t)PB(k_\lambda\Lambda\Lambda^{\mathrm{T}}-I_m)B^{\mathrm{T}}Pe(t),\quad t\geqslant 0
\end{aligned}\tag{62}$$

where I_m is the $m\times m$ identity matrix. Since, by assumption, $\|u_{\mathrm{d}1}(t)-v(t)\|\leqslant\varepsilon,\ t\geqslant 0$, we obtain

$$\begin{aligned}
&\dot{V}_s(e(t),\hat{e}_u(t),\Theta_1(t),\Theta_3(t),\Theta_r(t),t)\\
&\quad\leqslant-e^{\mathrm{T}}(t)(R_1+K^{\mathrm{T}}R_2K)e(t)-\hat{e}_u^{\mathrm{T}}(t)K_u\hat{e}_u(t)+\varepsilon^2,\quad t\geqslant 0
\end{aligned}\tag{63}$$

Now, it follows from (63) that the Lyapunov derivative is strictly negative outside $\mathscr{D}_e$, which guarantees convergence of $(e(t),\hat{e}_u(t)),\ t\geqslant 0$, to $\mathscr{D}_e$ [17, 18]. □

One technique that allows one to estimate the derivative of $u_{\mathrm{d}1}(t),\ t\geqslant 0$, can be found in [19]. In particular, provided there exists $c>0$ such that $\dot{u}_{\mathrm{d}1}(t)\leqslant c,\ t\geqslant 0$, the technique in [19] leads to an ε-estimate of $u_{\mathrm{d}1}(t),\ t\geqslant 0$, and allows for the determination of an approximate value of c. More specifically, the algorithm in [19] can be used to approximate a signal $\upsilon(t)$ by $v(t)$, with bounded estimation error $s(t)\triangleq\upsilon(t)-v(t),\ t\geqslant 0$, where $v(t),\ t\geqslant 0$, is given by

$$\dot{v}(t)=k_0\,\mathrm{sgn}(s(t))+k_1(s(t)-\varepsilon\,\mathrm{sat}(s(t)/\varepsilon))+k_2(t)\,\mathrm{sat}(s(t)/\varepsilon),\quad v(0)=v_0,\ t\geqslant 0\tag{64}$$

where the constants k_0, k_1, and ε are positive, $\mathrm{sgn}(s)\triangleq|s|/s$, for $s\neq 0$, and $\mathrm{sgn}(0)\triangleq 0$. In addition,

$$\mathrm{sat}(s)\triangleq\begin{cases}\mathrm{sgn}(s) & \text{for } \|s\|>1\\ s & \text{for } \|s\|\leqslant 1\end{cases}\tag{65}$$

and the gain $k_2(t),\ t\geqslant 0$, is given by the update law

$$\dot{k}_2(t)=\begin{cases}\gamma\|s(t)\| & \text{for } \|s\|>\varepsilon,\\ 0 & \text{for } \|s\|\leqslant\varepsilon,\end{cases}\quad k_2(0)=k_{20},\quad t\geqslant 0 \tag{66}$$

with $\gamma>0$. It is shown in [19] that, for sufficiently small $\eta>0$, there exist $\varepsilon<\eta$ and $t^*\geqslant 0$ such that $\|s(t)\|<\eta,\ t\geqslant t^*$.

4. ADAPTIVE TRACKING WITH ACTUATOR AMPLITUDE AND RATE SATURATION CONSTRAINTS

In this section, we extend the adaptive control framework presented in Section 3 to account for actuator amplitude and rate saturation constraints. Recall that Theorem 2.2 guarantees asymptotic convergence of the tracking error $e(t),\ t\geqslant 0$, to zero; that is, the state vector $x(t),\ t\geqslant 0$, converges asymptotically to the reference state vector $x_{r1}(t),\ t\geqslant 0$. Furthermore, it is important to note that the compensator dynamics $f_{\mathrm{c}}(\cdot)$ given by (41) depend on the *reference input* $r(t),\ t\geqslant 0$, through $\dot{x}_{r2}(t),\ t\geqslant 0$. Since for a fixed set of initial conditions there exists a one-to-one mapping between the reference input $r(t),\ t\geqslant 0$, and the reference state $x_{r1}(t),\ t\geqslant 0$, it follows that the control signal in (16) guarantees asymptotic convergence of the state $x(t),\ t\geqslant 0$, to the reference state $x_{r1}(t),\ t\geqslant 0$, corresponding to the specified reference input $r(t),\ t\geqslant 0$. Of course, the reference input $r(t),\ t\geqslant 0$, should be chosen so as to guarantee asymptotic convergence to a *desired* state vector $x_{\mathrm{d}}(t),\ t\geqslant 0$. However, the choice of such a reference input $r(t),\ t\geqslant 0$, is not unique since the reference state vector $x_{r1}(t),\ t\geqslant 0$, can converge to the desired state vector $x_{\mathrm{d}}(t),\ t\geqslant 0$, without matching its transient behavior.

Next, we provide a framework wherein we construct a family of reference inputs $r(t),\ t\geqslant 0$, with associated reference state vectors $x_{r1}(t),\ t\geqslant 0$, which guarantee that a given reference state vector within this family converges to a desired state vector $x_{\mathrm{d}}(t),\ t\geqslant 0$, in the face of actuator amplitude and rate saturation constraints.

From (29) and (30), it is clear that $\dot{u}(t),\ t\geqslant 0$, is explicitly dependent on $f_{\mathrm{c}}(t),\ t\geqslant 0$, which in turn depends upon the reference signal $r(t),\ t\geqslant 0$. More specifically, from (29), (30), (38), and (41) we have

$$\begin{aligned}\dot{u}(t)&=H(s(t),r(t))\\&=g_1(t)+h(t)\Theta_2(t)\varphi_2(t)-2\Theta_3(t)Pe(t)-K_u e_u(t)-\Lambda\Theta_r(t)T_r^{-1}r(t),\quad t\geqslant 0\end{aligned} \tag{67}$$

where $s(t)\triangleq(x(t),x_r(t),\Theta_r(t),\Theta_2(t),\Theta_3(t),e(t),e_u(t))$ and

$$\begin{aligned}g_1(t)\triangleq{}&\Lambda(-P_2^{\mathrm{T}}e(t)(\varphi_1^{\mathrm{T}}(t)\Gamma_1\varphi_1(t)+x_{r2}^{\mathrm{T}}(t)\Gamma_r x_{r2}(t))-\Theta_1(t)[0_{n\times s}\ \ K^{\mathrm{T}}\ \ -\tfrac{1}{2}k_\lambda P]^{\mathrm{T}}\dot{x}_{r1}(t)\\&+\Theta_r(t)T_r^{-1}x_{r2}(t))+h(t)A_r x(t)\end{aligned} \tag{68}$$

Using (67), the reference input $r(t),\ t\geqslant 0$, can be expressed as

$$\begin{aligned}r(t)&=H^{-1}(s(t),\dot{u}(t))\\&=T_r\Theta_r^{-1}\Lambda^{-1}(g_1(t)+h(t)\Theta_2(t)\varphi_2(t)-2\Theta_3(t)Pe(t)-K_u e_u(t)-\dot{u}(t))\end{aligned} \tag{69}$$

The above expression relates the reference input to the time rate of change of the control input.

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Next, we assume that the control signal is amplitude and rate limited so that $|u_i(t)|\leqslant u_{\max}$ and $|\dot{u}_i(t)|\leqslant \dot{u}_{\max}$, $t\geqslant 0$, $i=1,\ldots,m$, where $u_i(t)$ and $\dot{u}_i(t)$ denote the ith component of $u(t)$ and $\dot{u}(t)$, respectively, and $u_{\max}>0$ and $\dot{u}_{\max}>0$ are given. We will enforce amplitude saturation of the command $u(t)$, $t\geqslant 0$, by adjusting the rate of change of $u(t)$, $t\geqslant 0$, to zero. For the statement of our main result the following definitions are needed. For $i\in\{1,\ldots,m\}$ define

$$\sigma(u_i(t),\dot{u}_i(t))\triangleq\begin{cases}0 & \text{if } |u_i(t)|=u_{\max} \text{ and } u_i(t)\dot{u}_i(t)>0, \quad t\geqslant 0\\ 1 & \text{otherwise}\end{cases} \tag{70}$$

$$\sigma^*(u_i(t),\dot{u}_i(t))\triangleq\min\left\{\sigma(u_i(t),\dot{u}_i(t)),\frac{\dot{u}_{\max}}{|\dot{u}_i(t)|}\right\},\quad t\geqslant 0 \tag{71}$$

Note that for $i\in\{1,\ldots,m\}$ and $t=t_1>0$, the function $\sigma^*(\cdot,\cdot)$ is such that the following properties hold:

(i) If $|u_i(t_1)|=u_{\max}$ and $u_i(t_1)\dot{u}_i(t_1)>0$, then $\dot{u}_i(t_1)\sigma^*(u_i(t_1),\dot{u}_i(t_1))=0$.
(ii) If $|\dot{u}_i(t_1)|>\dot{u}_{\max}$ and $|u_i(t_1)|<u_{\max}$ or if $|\dot{u}_i(t_1)|>\dot{u}_{\max}$ and $|u_i(t_1)|=u_{\max}$ and $u_i(t_1)\dot{u}_i(t_1)\leqslant 0$, then $\dot{u}_i(t_1)\sigma^*(u_i(t_1),\dot{u}_i(t_1))=\dot{u}_{\max}\operatorname{sgn}(\dot{u}_i(t_1))$.
(iii) If no constraint is violated, then $\dot{u}_i(t_1)\sigma^*(u_i(t_1),\dot{u}_i(t_1))=\dot{u}_i(t_1)$.

Finally, define the component decoupled diagonal nonlinearity $\Sigma(u,\dot{u})$ by

$$\Sigma(u(t),\dot{u}(t))\triangleq\operatorname{diag}[\sigma^*(u_1(t),\dot{u}_1(t)),\sigma^*(u_2(t),\dot{u}_2(t)),\ldots,\sigma^*(u_m(t),\dot{u}_m(t))] \tag{72}$$

Theorem 4.1
Consider the controlled nonlinear system $\mathscr{G}$ given by (1) and reference system (31). Assume that there exist gain matrices $\Theta^*\in\mathbb{R}^{m\times s}$ and $\Theta_r^*\in\mathbb{R}^{m\times m}$ and a continuously differentiable function $F:\mathbb{R}^n\to\mathbb{R}^s$ such that (4) and (5) hold. Furthermore, let $K\in\mathbb{R}^{m\times n}$ be given by (6), where $P>0$ satisfies (7). In addition, for a given desired reference input $\mathrm{d}(t)$, $t\geqslant 0$, let the reference input $r(t)$, $t\geqslant 0$, be given by

$$r(t)=H^{-1}(s(t),\Sigma(u(t),\dot{u}^*(t))\dot{u}^*(t)),\quad t\geqslant 0 \tag{73}$$

where $s(t)=(x(t),x_r(t),\Theta_r(t),\Theta_2(t),e(t),e_u(t))$ and $\dot{u}^*(t)\triangleq H(s(t),r_{\mathrm{d}}(t))$. Then the adaptive feedback control law (41), with update laws (42)–(45) and reference input $r(t)$, $t\geqslant 0$, given by (73), guarantees that the following statements hold:

(i) The zero solution $(e(t),e_u(t))\equiv(0,0)$ to (3) and (39) is asymptotically stable.
(ii) $|u_i(t)|\leqslant u_{\max}$ for all $t\geqslant 0$ and $i=1,\ldots,m$.
(iii) $|\dot{u}_i(t)|\leqslant\dot{u}_{\max}$ for all $t\geqslant 0$ and $i=1,\ldots,m$.

Proof
Statement (i) is a direct consequence of Theorem 3.1 with $r(t)$, $t\geqslant 0$, given by (73). To prove (ii) and (iii) note that it follows from (67), (69), and (73) that

$$\begin{aligned}\dot{u}(t)&=H(s(t),r(t))=H(s(t),H^{-1}(s(t),\Sigma(u(t),\dot{u}^*(t))\dot{u}^*(t)))\\&=\Sigma(u(t),\dot{u}^*(t))\dot{u}^*(t),\quad t\geqslant 0\end{aligned} \tag{74}$$

which implies that $\dot{u}_i(t)=\sigma^*(u_i(t),\dot{u}_i^*(t))\dot{u}_i^*(t)$, $i=1,\ldots,m$, $t\geqslant 0$. Hence, if the control input $u_i(t)$, $t\geqslant 0$, with a rate of change $\dot{u}_i^*(t)$, $i=1,\ldots,m$, $t\geqslant 0$, does not violate the amplitude and

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rate saturation constraints, then it follows from (71) that $\sigma^*(u_i(t), \dot{u}_i^*(t))=1$ and $\dot{u}_i(t)=\dot{u}_i^*(t)$, $i=1,\ldots,m,\ t\geqslant 0$. Alternatively, if the pair $(u_i(t), \dot{u}_i^*(t))$, $i=1,\ldots,m,\ t\geqslant 0$, violates one or more of the input amplitude and/or rate constraints, then (70), (71), and (74) imply:

(i) $\dot{u}_i(t)=0$ for all $t\geqslant 0$ if $|u_i(t)|=u_{\max}$ and $u_i(t)\dot{u}_i^*(t)>0$ and
(ii) $\dot{u}_i(t)=\dot{u}_{\max}\,\mathrm{sgn}(\dot{u}_i^*(t))$ for all $t\geqslant 0$ if $|\dot{u}_i^*(t)|>\dot{u}_{\max}$ and $|u_i(t)|<u_{\max}$ or if $|\dot{u}_i^*(t)|>\dot{u}_{\max}$ and $|u_i(t)|=u_{\max}$ and $u_i(t)\dot{u}_i(t)\leqslant 0$,

which, for $u_i(0)\leqslant u_{\max}$, guarantee that $|u_i(t)|\leqslant u_{\max}$ and $|\dot{u}_i(t)|\leqslant \dot{u}_{\max}$ for all $t\geqslant 0$ and $i=1,\ldots,m$. □

Note that it follows from Theorem 4.1 that if the desired reference input $\mathrm{d}(t),\ t\geqslant 0$, is such that the actuator amplitude and/or rate saturation constraints are not violated, then $r(t)=\mathrm{d}(t),\ t\geqslant 0$, and hence $x(t),\ t\geqslant 0$, converges to $x_{\mathrm{d}}(t),\ t\geqslant 0$. Alternatively, if there exists $t=t^*>0$ such that the desired reference input drives one or more of the control inputs to the saturation boundary, then $r(t)\neq r_{\mathrm{d}}(t),\ t>t^*$. However, as long as the time interval over which the control input remains saturated is finite, the reference signal ultimately reverts to its desired value, and the tracking properties are preserved. Of course, if there exists a solution to the tracking problem wherein the input amplitude and rate saturation constraints are not violated when the tracking error is within certain bounds, then our approach is guaranteed to work.

5. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, we present three numerical examples to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization and tracking in the face of actuator amplitude and rate saturation constraints.

Example 5.1
Consider the uncertain controlled Liénard system given by

$$\ddot{z}(t)+\mu(z^4(t)-\alpha)\dot{z}(t)+\beta z(t)+\gamma\tanh(z(t))=bu(t),\quad z(0)=z_0,\ \ \dot{z}(0)=\dot{z}_0,\ \ t\geqslant 0 \tag{75}$$

where $\mu,\alpha,\beta,\gamma,b\in\mathbb{R}$ are unknown. Note that with $x_1=z$ and $x_2=\dot{z}$, (75) can be written in state space form (1) with $x=[x_1,x_2]^{\mathrm{T}}$, $f(x)=[x_2,-\beta x_1-\gamma\tanh x_1-\mu(x_1^4-\alpha)x_2]^{\mathrm{T}}$, and $B=[0,b]^{\mathrm{T}}$. Here, we assume that $f(x)$ and B are unknown and can be parametrized as $f(x)=[x_2,\theta_\ell x+\theta_{n\ell 1}\tanh x_1+\theta_{n\ell 2}x_1^4x_2]^{\mathrm{T}}$ and $B=b[0,1]^{\mathrm{T}}$, where $\theta_\ell\in\mathbb{R}^2$, $\theta_{n\ell 1}\in\mathbb{R}$, and $\theta_{n\ell 2}\in\mathbb{R}$ are unknown. Next, let $F(x)=[x^{\mathrm{T}},\tanh(x_1),x_1^4x_2]^{\mathrm{T}}$, $A_r=[A_0^{\mathrm{T}},\theta_n^{\mathrm{T}}]^{\mathrm{T}}$, $B_r=[0,b_r]^{\mathrm{T}}$, $b_r\in\mathbb{R}$, $\Lambda=1$, $\Theta_r^*=b_r/b$, and $\Theta^*=[\theta_n-\theta_\ell,-\theta_{n\ell 1},-\theta_{n\ell 2}]/b$, where $A_0=[0,1]$ and θ_n is an arbitrary vector, so that

$$B\Lambda\Theta_r^*=\begin{bmatrix}0\\ b\end{bmatrix}\cdot 1\cdot\frac{b_r}{b}=\begin{bmatrix}0\\ b_r\end{bmatrix}=B_r$$

$$\begin{aligned}B\Lambda\Theta^*F(x)&=\begin{bmatrix}0\\ b\end{bmatrix}\frac{1}{b}[\theta_n-\theta_\ell,-\theta_{n\ell 1},-\theta_{n\ell 2}]F(x)\\ &=A_rx-f(x)\end{aligned}$$

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and, hence, (4) and (5) hold. Now, it follows from Theorem 3.1 that the adaptive feedback controller (41) guarantees that $e(t) \to 0$ as $t \to \infty$ in the face of input amplitude and rate saturation constraints. Specifically, here we choose $\theta_n = [-4, -1.6]$, $R_1 = I_2$, and $R_2 = 1$ so that K and P satisfying (6) and (7) are given by

$$P = \begin{bmatrix} 1.2434 & 0.1036 \\ 0.1036 & 0.1891 \end{bmatrix}, \quad K = [-0.4142 \;\; -0.7562] \tag{76}$$

In order to analyze this design we assume that $\mu = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $b = 3$, with initial condition $x(0) = [1, 1]^{\mathrm{T}}$, $u(0) = 0$, $x_{r1}(0) = [0, 0]^{\mathrm{T}}$, $x_{r2}(0) = 0$. First, we consider a regulation problem, that is, stabilization to the origin. The initial parameter estimates are chosen as $\Theta_{10} = [-1, -1, 0, 1, 1, 0, 4]$, $\Theta_{r0} = 2$, $\Theta_{20} = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & -1 & -1 & 3 \end{smallmatrix}\right]$, and $\Theta_{30} = [0, 4]$. Figure 1 shows the case where no input saturation constraints are considered and Figure 2 shows the case where $u_{\max} = 1$ and $\dot{u}_{\max} = 2$.

Next, we consider the case where we seek to track $z_{\mathrm{d}}(t) = \sin(\pi/5t)$. Figure 3 shows the case where no input saturation constraints are considered, while Figure 4 shows the case where $u_{\max} = 0.6$ and $\dot{u}_{\max} = 2$. As seen in Figure 4, the control algorithm is able to achieve perfect tracking, in spite of the actuation constraint. Should the constraint prove too restrictive to physically allow the system to track the given desired trajectory, while our formulation still guarantees that $x(t) \to x_{r1}(t)$ as $t \to \infty$, we cannot guarantee that $x_{r1}(t) \to x_{\mathrm{d}}(t)$ as $t \to \infty$. However, our approach provides a 'close' agreement between the desired signal to be tracked and the achieved tracked signal for the given saturation levels, as illustrated by Figure 5. The amplitude saturation constraint is chosen as $u_{\max} = 0.53$, which, with $\dot{u}_{\max} = 2$, is too restrictive to allow perfect tracking of the given desired trajectory. At these amplitude and rate saturation levels, the control signal remains periodically saturated, and $x_{r1}(t)$, $t \geqslant 0$, is unable to perfectly track the desired trajectory. However, Figure 5 shows that the control algorithm still provides as close an agreement between the trajectory $x(t)$, $t \geqslant 0$, and $x_{\mathrm{d}}(t)$, $t \geqslant 0$, as made possible by the saturation constraint. Finally, we consider a case where $\mu = -0.1$, $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $b = 3$, with $u_{\max} = 4$ and $\dot{u}_{\max} = 1$. Figure 6 shows the results of a 15-s numerical simulation (left), obtained with the adaptive controller of Theorem 3.1 and the reference input as described in Theorem 4.1. The resulting trajectory, represented by a solid line, converges smoothly towards the desired trajectory. However, if the reference input is not modified as described in Theorem 4.1, which implies that the adaptation mechanism in Theorem 3.1 is not aware of the saturation, then the closed-loop system's trajectory diverges. This is shown in Figure 6 by a dashed line, and is particularly clear when focusing on the first 5 s of the simulation (right).

Example 5.2

Consider the nonlinear dynamical system representing a controlled rigid spacecraft given by

$$\dot{x}(t) = -I_{\mathrm{b}}^{-1} X I_{\mathrm{b}} x(t) + I_{\mathrm{b}}^{-1} u(t), \quad x(0) = x_0, \quad t \geqslant 0 \tag{77}$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}}$ represents the angular velocities of the spacecraft with respect to the body-fixed frame, $I_{\mathrm{b}} \in \mathbb{R}^{3\times 3}$ is an unknown positive-definite inertia matrix of the spacecraft, $u(t) = [u_1, u_2, u_3]^{\mathrm{T}}$ is a control vector with control inputs providing body-fixed torques about three

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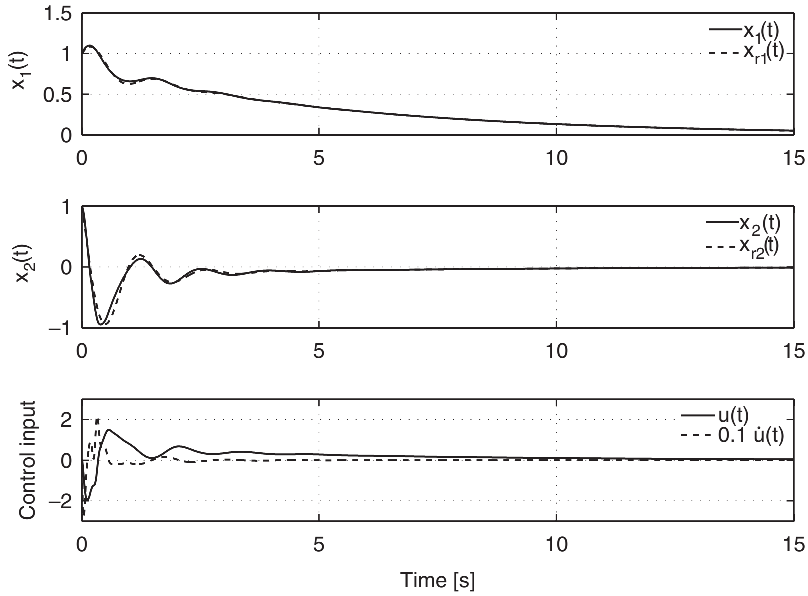

Figure 1. Stabilization of the Liénard system with no saturation constraints.

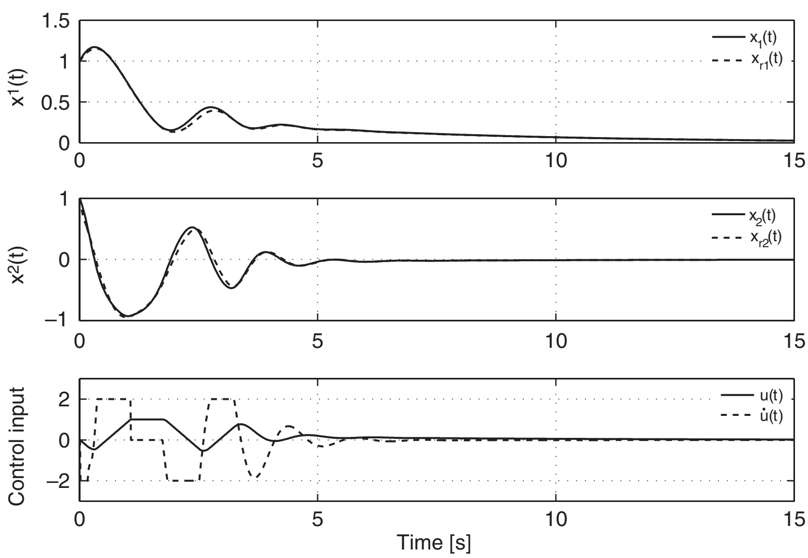

Figure 2. Stabilization of the Liénard system with amplitude and rate saturation constraints.

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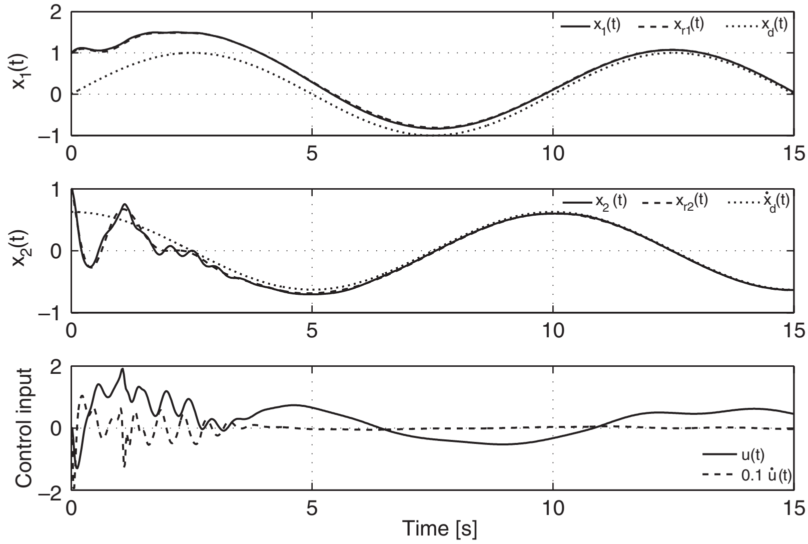

Figure 3. Tracking of the Liénard system with no saturation constraints.

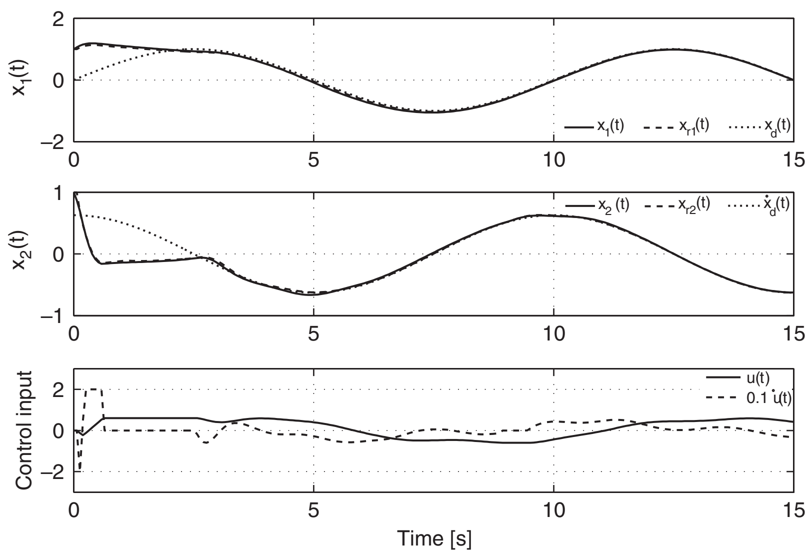

Figure 4. Tracking of the Liénard system with amplitude and rate saturation constraints.

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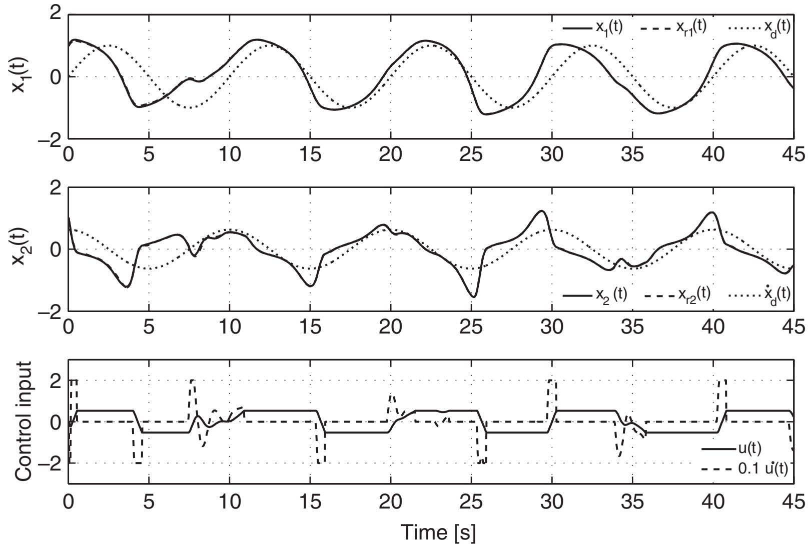

Figure 5. Tracking of the Liénard system with excessive amplitude and rate saturation constraints.

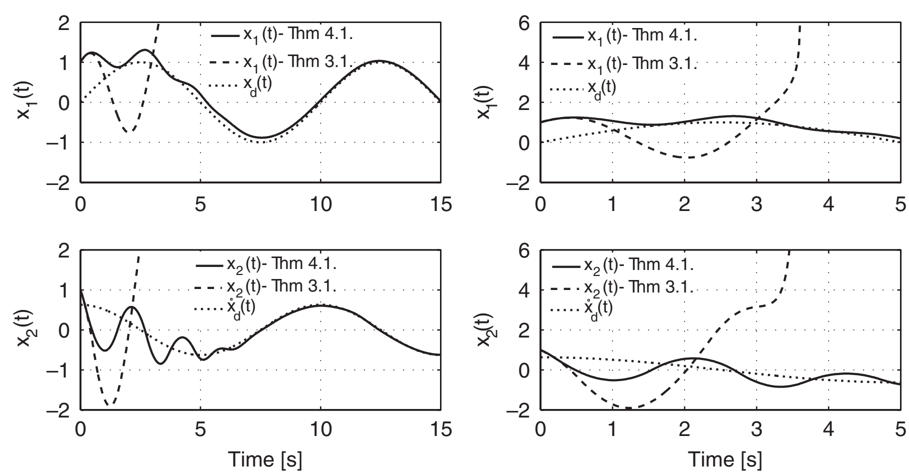

Figure 6. Tracking of the Liénard system with amplitude and rate saturation constraints.

mutually perpendicular axes defining the body-fixed frame of the spacecraft, and X denotes the skew-symmetric matrix

$$X \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{78}$$

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Note that (77) can be written in state space form (1) with $f(x) = -I_b^{-1} X I_b x$ and $B = I_b^{-1}$. Since $f(x)$ is a quadratic function, we parametrize $f(x)$ as $f(x) = \Theta_{n\ell} f_{n\ell}(x)$, where $\Theta_{n\ell} \in \mathbb{R}^{3\times 6}$ is an unknown matrix and $f_{n\ell}(x) = [x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1]^{\mathrm{T}}$.

Next, let $F(x) = [x^{\mathrm{T}}, f_{n\ell}(x)^{\mathrm{T}}]^{\mathrm{T}}$, $B_r = I_3$, $\Lambda = I_3$, $\Theta_r^* = I_b$, and $\Theta^* = I_b[A_r, \Theta_{n\ell}]$ so that

$$B\Lambda\Theta_r^* = I_b^{-1} I_3 I_b = I_3 = B_r$$

$$f(x) - B\Lambda\Theta^* F(x) = f(x) - I_b^{-1} I_b [A_r, \Theta_{n\ell}] F(x) = -A_r x$$

and, hence, (4) and (5) hold. Now, it follows from Theorem 4.1 that the dynamic adaptive controller (29)–(30), (41), guarantees that $e(t) \to 0$ as $t \to \infty$ when considering input amplitude and rate saturation constraints. Specifically, here we choose

$$A_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix}$$

$R_1 = I_3$, and $R_2 = 0.1 I_3$ so that K and P satisfying (6) are given by

$$P = \begin{bmatrix} 0.3738 & 0.1340 & -0.0369 \\ 0.1340 & 0.4401 & -0.0359 \\ -0.0369 & -0.0359 & 0.0709 \end{bmatrix}, \quad K = -10P$$

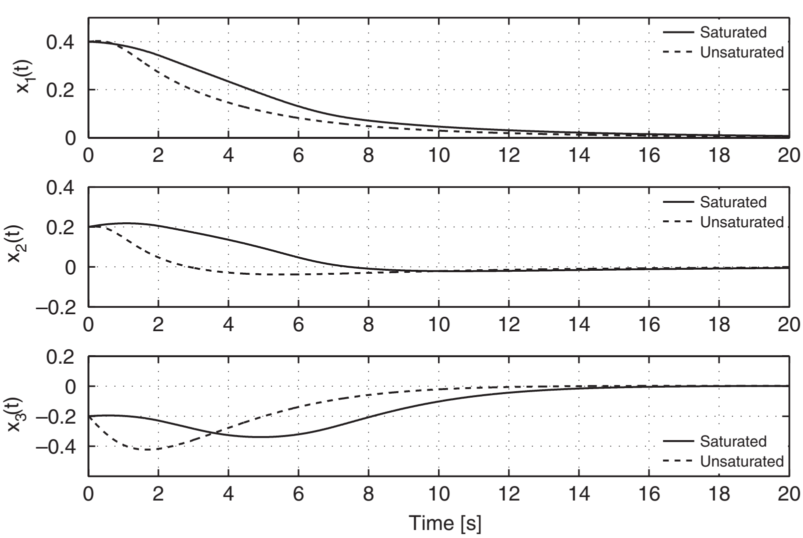

Figure 7. Angular velocities versus time.

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In order to analyze this design we assume that

$$I_b = \begin{bmatrix} 20 & 0 & 0.9 \\ 0 & 17 & 0 \\ 0.9 & 0 & 15 \end{bmatrix}, \quad Q_1 = Q_2 = I_3$$

with initial condition $x(0) = [0.4, 0.2, -0.2]^T$, $x_{r1}(0) = x(0)$, $\Theta_{10} = [I_3 \ 0_{3\times6} \ I_3 \ 0_{3\times3}]$, $\Theta_{20} = [-I_3 \ 0_{3\times9}]$, $\Theta_{30} = 0_{3\times3}$, and $\Theta_{r0} = I_3$. Figure 7 shows the angular velocities versus time for the case where no saturation constraints are enforced and the case where $u_{\max} = 1$ and $\dot{u}_{\max} = 0.5$. The corresponding control inputs and their time rate of change are shown in Figures 8 and 9. The control algorithm successfully regulates the system in spite of the actuator saturation.

Example 5.3

In this example, we compare our approach with the μ-modification approach developed in [8]. Consider the system

$$\dot{x}(t) = \theta_1 x(t) + \theta_2 x^3(t) - \theta_3 e^{-10(x(t)+1/2)^2} - \theta_4 e^{-10(x(t)-1/2)^2} + \theta_5 \sin(2x(t)) + \theta_6 u(t), \quad x(0) = x_0, \quad t \geqslant 0 \tag{79}$$

where θ_i, $i = 1, \ldots, 6$, are unknown parameters. For the simulation, we assume that $\Theta \triangleq [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4 \ \theta_5 \ \theta_6] = [\frac{1}{5} \ \frac{1}{100} \ 1 \ 1 \ \frac{1}{2} \ 2]$. The system given by (79) is considered in [8]. Defining $f(x) \triangleq \frac{1}{5}x + \frac{1}{100}x^3 - e^{-10(x+1/2)^2} - e^{-10(x-1/2)^2} + \frac{1}{2}\sin(2x)$, $B = 2$, and choosing $A_r = -6$ and $B_r = 6$, we obtain

$$f(x) - A_r x = [\tfrac{31}{5} \ \tfrac{1}{100} \ 1 \ 1 \ \tfrac{1}{2}][x \ x^3 \ -e^{-10(x+1/2)^2} \ -e^{-10(x-1/2)^2} \ \sin(2x)]^T \tag{80}$$

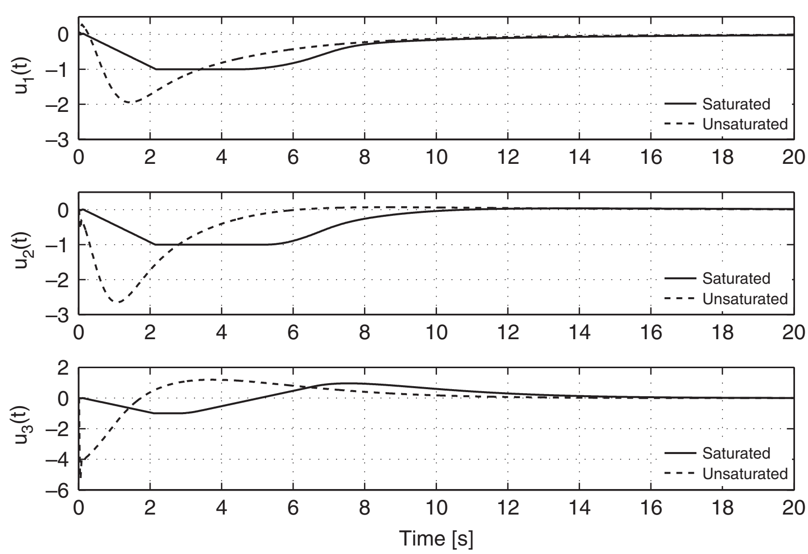

Figure 8. Control command, saturated and unsaturated.

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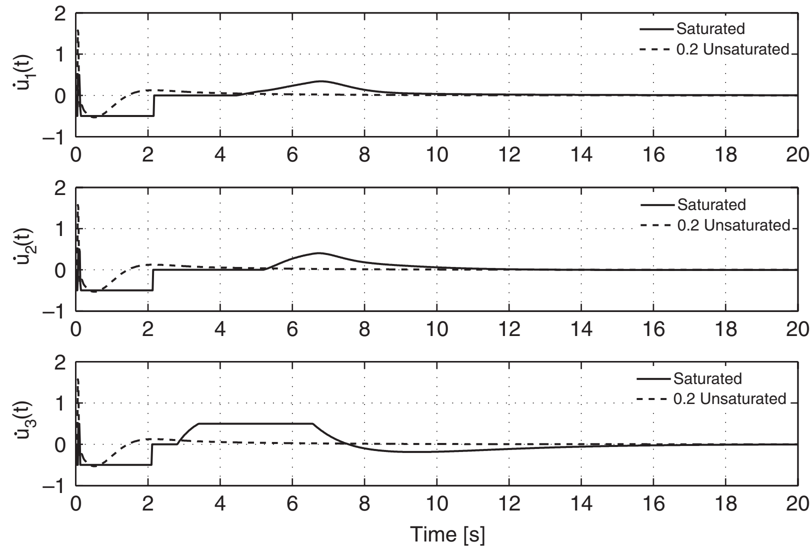

Figure 9. Control command rate, saturated and unsaturated.

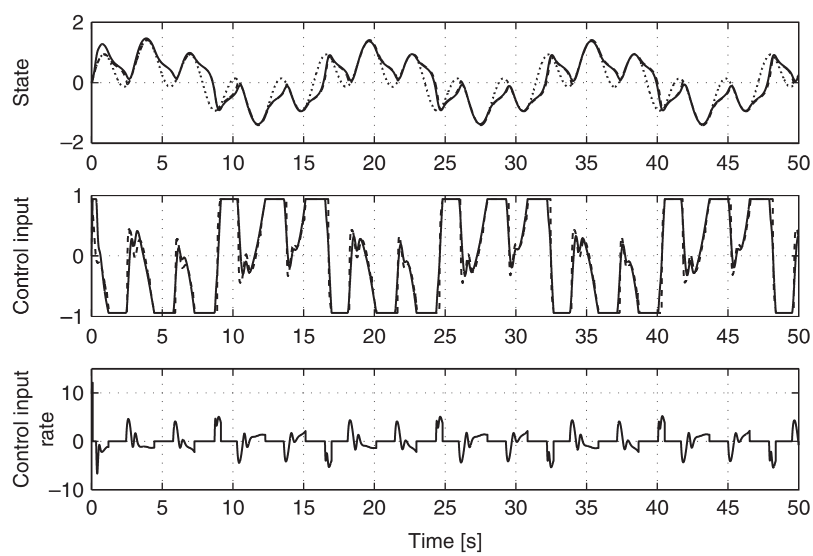

Figure 10. State and control input with $u_{\max}=0.94$ and no rate saturation ($\cdots$ desired trajectory, $--$ μ-modification [8], — Theorem 3.1).

Now, with

$$\Theta^* = \tfrac{1}{2}\left[\tfrac{31}{5} \;\; \tfrac{1}{100} \;\; 1 \;\; 1 \;\; \tfrac{1}{2}\right], \quad F(x) = \left[x \;\; x^3 \;\; -\mathrm{e}^{-10(x+1/2)^2} \;\; -\mathrm{e}^{-10(x-1/2)^2} \;\; \sin(2x)\right]^{\mathrm{T}} \tag{81}$$

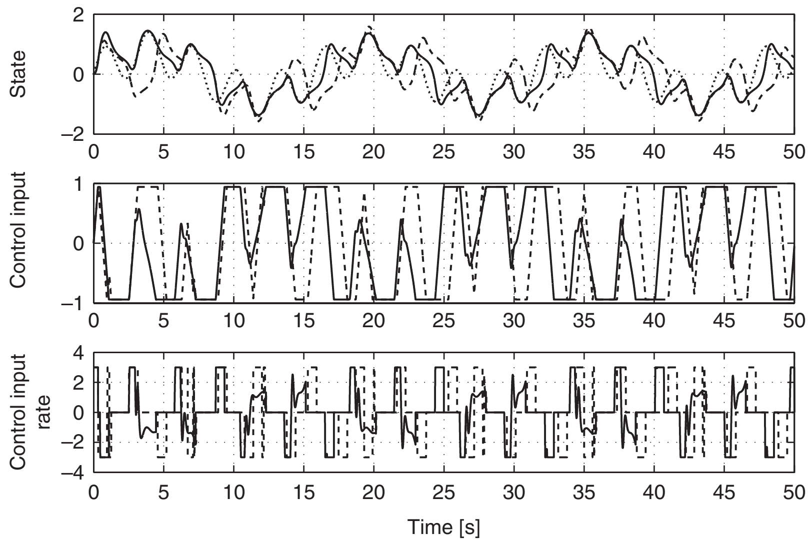

Figure 11. State and control input with $u_{\max}=0.94$ and $\dot{u}_{\max}=3$ ($\cdots$ desired trajectory, $--$ μ-modification [8], — Theorem 3.1).

and $\Lambda=1$, (4) holds. In addition, (5) holds with $\Theta_r^*=3$. Choosing $R_1=5$ and $R_2=0.1$, we obtain $P=0.1024$ and $K=-6.1414$. In addition, we set $K_u=400$, $\Gamma_1=15.5I_7$, $\Gamma_2=15.5I_6$, and $\Gamma_3=\Gamma_r=15.5$. The amplitude saturation constraint is chosen at $u_{\max}=0.94$. The initial conditions are $x_0=u_0=0$ and $\Theta_r(0)=5$, while all other parameter estimates are initially set to zero. The desired trajectory is defined as $x_d(t)\triangleq 0.7(\sin(2t)+\sin(0.4t))$, $t\geqslant 0$.

When no rate saturation is enforced, the control law from Theorem 3.1 yields results very similar to those obtained by the μ-modification algorithm presented in [8]; see Figure 10. The choice of design parameters for the μ-algorithm is identical to those given in [8], and the obtained results replicate those shown in Figure 1(a) of [8]. When enforcing amplitude saturation only, the two algorithms perform comparably. However, in addition to amplitude saturation, our framework allows for rate saturation, whereas the μ-modification framework of [8] does not account for rate saturation. Figure 11 shows the results for $u_{\max}=0.94$ and $\dot{u}_{\max}=3$. The trajectory and the corresponding control effort are again compared with those obtained from the μ-modification approach. As seen in Figure 11, the performance of the μ-modification algorithm is significantly degraded by the presence of the rate saturation constraint, whereas the performance degradation using our controller is marginal.

6. CONCLUSION

A direct adaptive nonlinear tracking control framework for multivariable nonlinear uncertain systems with actuator amplitude and rate saturation constraints was developed. By appropriately modifying the adaptive control signal to the reference system dynamics, the proposed approach guarantees asymptotic stability of the error system dynamics in the face of actuator amplitude and rate limitation constraints. Finally, three numerical examples were presented to show the utility of the proposed adaptive tracking scheme.

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