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Anomalies and inflow on D-branes and O-planes

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Abstract

We derive the general form of the anomaly for chiral spinors and self-dual antisymmetric tensors living on D-brane and O-plane intersections, using both path-integral and index theorem methods. We then show that the anomalous couplings to RR forms of D-branes and O-planes in a general background are precisely those required to cancel these anomalies through the inflow mechanism. This allows, for instance, for local anomaly cancellation in generic orientifold models, the relevant Green-Schwarz term being given by the sum of the anomalous couplings of all the D-branes and O-planes in the model.

1. Introduction

One of the most important discoveries in the last few years of intense developments in string theory is that Dp-branes and Op-planes carry the elementary RR p-form charges μ_p and $\mu'_p = -2^{p-4}\mu_p$ [1]. It has also become clear that in a generic string background new charges with respect to lower RR forms are induced. For example, a topologically non-trivial gauge bundle induces charges with respect to (p-2n)-forms [2, 3], whereas the curvature of the tangent and normal bundles induces charges with respect to (p-4n)-forms [4, 5, 6, 7, 8]. All these induced couplings are anomalous with respect to gauge transformations of the background, and are expected to cancel possible anomalies on the defects through the inflow mechanism [9]. This fact was indeed exploited to derive the complete anomalous couplings of a Dp-brane by requiring them to cancel the anomaly arising in the generically chiral world-volume theory on the intersection of two or more D-branes [5, 6]. The presence of anomalous couplings for O-planes was instead predicted by string dualities [8], whereas their relevance for anomaly cancellation has been argued in particular situations [10].

In [11], a direct string computation of the complete anomalous couplings for Dp-branes and Op-planes has been given by factorizing magnetic interactions in a generic string background, confirming the indirect predictions for Dp-branes and correcting and extending those for Op-planes. The results are

$$S_{D_p} = \frac{\mu_p}{2}\int C \wedge \mathrm{ch}_\lambda(F) \wedge \left.\sqrt{\frac{\widehat{A}(R)}{\widehat{A}(R')}}\right|_{(p+1)-form} \tag{1.1}$$

$$S_{O_p} = \frac{\mu'_p}{2}\int C \wedge \left.\sqrt{\frac{\widehat{L}(R/4)}{\widehat{L}(R'/4)}}\right|_{(p+1)-form} \tag{1.2}$$

and have been further checked through disk computations [12, 13], and indirectly in other contexts (see for instance [14]). In these formulae, $C = \oplus_n C_{(n)}$ is the sum over the pulled-back RR form potentials. F is the field strength of the D-brane gauge field in the Chan-Paton factors representation λ and R, R' are the pulled-back curvature two-forms of the tangent and normal bundles to the world-volume, all in units of $4\pi^2\alpha'$. The overall 1/2 normalization has been chosen in order to use later on standard results for the inflow of anomaly, along the lines of [6]. The symbol ch_ρ indicates the Chern class in the representation ρ,

$$\mathrm{ch}_\rho(F) = \mathrm{Tr}_\rho \exp i\frac{F}{2\pi}\ , \tag{1.3}$$

and $\widehat{A}(R)$ and $\widehat{L}(R)$ are the Roof genus and Hirzebruch polynomials, given by

$$\widehat{A}(R) = \prod_i \frac{\lambda_i/4\pi}{\sinh \lambda_i/4\pi}\ ,\quad \widehat{L}(R) = \prod_i \frac{\lambda_i/2\pi}{\tanh \lambda_i/2\pi}\ , \tag{1.4}$$

in terms of the skew-eigenvalues λ_i of $R_{\mu\nu}$. In considering the inflow mechanism, also the Euler class will appear:

$$e(R) = \prod_i \lambda_i/2\pi\ . \tag{1.5}$$

The aim of this paper is to show that the couplings (1.1) and (1.2), and similar additional model-dependent anomalous couplings to be determined case by case, lead to an anomaly inflow on D-brane and O-plane intersections which precisely cancel the anomalies on the corresponding world-volumes. From a string theory point of view, the only diagrams which can potentially give an anomaly are the divergent ones. At the one-loop level, these are the annulus, Möbius strip and Klein bottle, associated respectively to brane-brane (BB), brane-orientifold (BO) and orientifold-orientifold (OO) intersections. The torus is instead always finite, and therefore can not give any anomaly. This means that even the anomaly of closed string fields living in the bulk of spacetime is located on orientifold planes. Quite in general, it is easy to figure out what kind of anomalous fields can appear on BB, BO and OO intersections. The cylinder and the Möbius strip are surfaces with boundaries, corresponding to loops of arbitrary bosonic and fermionic open string states in the Ramond (R) and Neveu-Schwarz (NS) sectors; the only massless anomalous particle which can arise on BB and BO intersections is therefore a charged chiral R spinor reduced form $D = 10$ to $d = p + 1$. The Klein bottle is instead a surface without any boundary, and corresponds to loops of arbitrary bosonic closed string states in the RR and NSNS sectors; the only massless anomalous particle which can arise on OO intersections is therefore a neutral RR self-dual antisymmetric tensor, again reduced from $D = 10$ to $d = p + 1$.

A very interesting consequence of this observation is that the anomalies arising from neutral closed string states have to combine to reproduce that of a self-dual tensor, whereas the anomalies coming from charged open string states that of chiral spinors in suitable representations of the gauge group, as we will see. A slight clarification is here needed. In string theory, physical states generically arise after suitable truncations of the spectrum, implemented by projections like GSO, Ω or orbifold projections. Correspondingly, physical g-loop amplitudes are obtained by summing the contributions of all possible spin-structures on genus g sufaces. In particular, anomalies appear in the parity-violating part of one-loop amplitudes. In string theory, this means genus one surfaces in the odd spin-structures, and restricting the analysis to these spin-structures one can eventually identify which states, among those propagating in the loop, are responsible for potential anomalies. As we have argued, these are a chiral spinor on BB and BO intersections and a self-dual tensor on OO intersections. The point is that these states *do not need* to appear in the physical string spectrum; they might be projected out in the truncated theory, but neverthless appear in different surfaces contributing to the same amplitudes. On the other hand, the inflow mechanism which will cancel these anomalies, that is nothing but the tree-level closed channel interpretation of the same diagrams, has to involve the exchange of physical RR forms only, appearing indeed in the effective action.

As a concrete example, consider for instance Type IIB orientifolds. Since the Type IIB theory one starts with in $D = 10$ is chiral but anomaly free, any orientifold construction satisfying all consistency requirements of string theory, like in particular tadpole cancellation, has to yield a theory which is automatically free of anomalies. However, it is well known that the massless fields arising in a generic orientifold model will in general give a non-vanishing anomaly. This implies that the D-branes and O-planes present in the model have to contribute an equal and opposite anomaly through the inflow mechanism. This is nothing but a generalization of the Green-Schwarz (GS) mechanism [15], the GS term being given by the sum of the anomalous couplings of all the D-branes and O-planes in the model, including possible additional anomalous couplings to RR forms coming from twisted string sectors.

Having identified the potentially anomalous states living on D-brane/O-plane intersections, we will consider in a more general context the possible anomalies arising from chiral spinors and self-dual antisymmetric tensors propagating in a given submanifold of spacetime, but in interaction with gravity of all spacetime.

The plan of the paper is the following. In Section 2 we compute the general form of the anomaly for chiral spinors and self-dual antisymmetric tensors coupling to both the tangent and the normal bundle curvatures, generalizing some standard results [16] (see also [17] for an extended review of anomalies in field theory). Using Fujikawa's approach [18], we first relate these anomalies to chiral anomalies, which can then be interpreted as indices of classical complexes endowed with an additional group action. We then compute these indices in two different ways. The first more physical approach consists in regularizing the index as the high-temperature limit of the partition function of a suitable supersymmetric theory, as in [19, 16, 20]. The second is more mathematical and relies on the application of the so-called index and G-index theorems [21, 22]. Although the general form of the anomaly for chiral spinors has been already obtained in [6] by using the family index theorem [23], we will re-derive and confirm this result in the next sections in the two different ways mentioned above. In particular, the explicit computation based on a supersymmetric quantum mechanics, obtained as the reduction of the supersymmetric non-linear σ-model [24] from 1+1 to 0+1 dimensions with some constraints on the fields, will turn out to be very instructive due to its close connection to the open string σ-model in a curved D-brane background. In Section 3 we give a detailed description of the anomaly inflow on D-brane/O-plane intersections, showing that it always precisely cancels world-volume field theory anomalies. In Section 4 we apply our results to some Type IIB orientifold models, discuss general features of anomaly cancellation in these models and finally in last section we give some conclusions.

2. Anomalies for spinors and tensors

As explained in the introduction, we need to compute the anomaly for a chiral spinor field and a self-dual antisymmetric tensor field (that is with a self-dual field strength) reduced from some D-dimensional manifold X to a submanifold $M \subset X$ of dimension $d < D$. More precisely, with the term reduced we mean here the generalization to a non-trivial normal bundle of the usual dimensional reduction. Recall that upon dimensional reduction from D to d dimensions, the D-dimensional Lorentz group is broken to the d-dimensional Lorentz group plus an R-symmetry corresponding to rotations in the $(D-d)$-dimensional transverse space: $SO(D-1,1) \rightarrow SO(d-1,1) \times SO(D-d)$. The tangent bundle to X restricted to M decomposes as the Whitney sum of the tangent and normal bundles to M:

$$T(X)|_M = T(M) \oplus N(M) \ . \tag{2.1}$$

Correspondingly, a field in D dimensions in some representation R of $SO(D-1,1)$ decomposes into various multiplets of fields in d dimensions, in representations (R_1^i, R_2^i) of $SO(d-1,1) \times SO(D-d)$. More precisely, a section of $T(X)$ in some representation R will decompose into sections of $T(M) \otimes N(M)$ in representations (R_1^i, R_2^i). For simplicity and motivated by D-brane physics, we will consider fields that couple only to gravity on X, restricting any possible gauge field to connections on bundles over M.

Under the above reduction, local Lorentz symmetry (or general covariance) on X is broken to local Lorentz symmetry on M and local R-symmetry on the tranverse space. The former is just standard gravity seen as a gauge theory whose gauge field is the spin connection on the tangent bundle $T(M)$, the fields transforming in the R_1^i representations. The latter corresponds instead to a gauge symmetry under which the fields transform in the R_2^i representations, the gauge field associated to this gauge symmetry being simply the spin connection on the normal bundle $N(M)$. Any anomalous representation R will decompose into pairs of representations (R_1^i, R_2^i) and $(\bar{R}_1^i, \bar{R}_2^i)$ related by conjugation. If $N(M)$ is trivial, the representations R_2^i and $\bar{R}_2^i$ are equivalent and the two components R_1^i and $\bar{R}_1^i$ will give an equal and opposite contribution to the anomaly, which will therefore vanish. On the other hand, if $N(M)$ is non-trivial, so are the bundles lifted from it in the $R_2^i, \bar{R}_2^i$ representations, and the latter are no longer equivalent; the two components R_1^i and $\bar{R}_1^i$ can then give unbalanced contributions and the anomaly can be non-vanishing.

In the following we use the standard notation for Wess-Zumino descents. The anomaly is encoded in a closed and gauge-invariant sum of forms I, function of the curvature 2-forms of the gauge, tangent and normal bundles. Apart from a possible constant term I_0, this will also be exact, since so are the curvatures: $I - I_0 = dI^{(0)}$. $I^{(0)}$ is not gauge invariant; rather its gauge variation defines the Wess-Zumino descent $I^{(1)}$: $\delta_\eta I^{(0)} = dI^{(1)}$. Anomalies in field theory always have the form $A = 2\pi i \int_M I^{(1)}$; this ensures that the Wess-Zumino consistency condition is automatically satisfied.

2.1. Path-integral computation

As anticipated, we will first use Fujikawa's method [18] to compute the anomaly for a reduced chiral spinor and self-dual antisymmetric tensor. In this approach, the anomaly is attributed to the Jacobian arising from the non-invariant path-integral measure under a generic background gauge transformation δ_η. This method presents the advantage of being very easily generalized to the case in which the field is reduced to a lower dimensional manifold. It will prove very convenient to use the strategy of looking at the reduced case as the unreduced case with a constraint. Correspondingly, the generic gauge variation δ_η contains gauge transformations and reparametrizations of both the tangent and the normal bundle. Since the dependence of the anomaly on the gauge field in the reduced case is the same as in the standard unreduced case, we shall concentrate in the following on the gravitational part.

In the spirit of [16], we regularize the ill-defined traces encoding the anomaly as the high-temperature limits of partition functions of suitable supersymmetric theories. These will turn out to be different versions of the supersymmetric non-linear σ-model [24] reduced from 1+1 down to 0+1 dimensions, whose Lagrangian is

$$L = \frac{1}{2}\left[g_{MN}(x)\dot{x}^M\dot{x}^N + i\psi_{1\underline{M}}D_\tau(x)\psi_1^{\underline{M}} + i\psi_{2\underline{M}}D_\tau(x)\psi_2^{\underline{M}} + \frac{1}{2}R_{\underline{MNPQ}}(x)\psi_1^{\underline{M}}\psi_1^{\underline{N}}\psi_2^{\underline{P}}\psi_2^{\underline{Q}}\right] \tag{2.2}$$

where

$$D_\tau\psi_\alpha^{\underline{M}} = \dot{\psi}_\alpha^{\underline{M}} + \omega_M{}^{\underline{M}}{}_{\underline{N}}(x)\,\psi_\alpha^{\underline{N}}\,\dot{x}^M \;, \quad \alpha = 1,2 \;. \tag{2.3}$$

In the action (2.2), we have introduced the Lorentz frame fermion fields $\psi_\alpha^{\underline{M}} = e^{\underline{M}}{}_M\,\psi_\alpha^M$. Here and in the following capital indices $M, N, \ldots$ run over the total space-time, whereas Greek and Latin indices $\mu, \nu, \ldots,\ i, j, \ldots$ denote respectively coordinates on M and the transverse space to it in X; underlined indices denote instead flat indices, non-underlined ones being curved.

2.1.1 Chiral spinor

Consider first a chiral spinor. In the usual unreduced case, the Jacobian giving the gauge and/or the gravitational anomaly can be written as [16]

$$A_X = \lim_{t\to 0} \mathrm{Tr}\,[\Gamma^{D+1}\delta_\eta e^{-t(i\not{D}_X)^2}] \tag{2.4}$$

where Γ^{D+1} is the chiral matrix in D dimensions, the operator δ_η represents the corresponding gauge and/or gravitational variation and the trace runs over the eigenstates of $H = (i\not{D}_X)^2$, $\not{D}_X$ being the Dirac operator on X. The operator δ_η can be exponentiated, resulting in a shift in the background fields. The final result has then to be restricted to the term which is linear in η. This procedure corresponds to the well known fact that gauge and gravitational anomalies can be obtained from chiral anomalies by taking the Wess-Zumino descent.

In the reduced case, in which the chiral spinor propagates on M but couples through a gauge-like coupling to the normal bundle curvature, the anomaly is still given by (2.4) but with $\not{D}_X$ replaced by the Dirac operator $\not{D}_M$ on M, that now includes the connection on the normal bundle of M. Again, by exponentiating the operator δ_η, we are left with

$$Z_M = \lim_{t\to 0} \mathrm{Tr}\,[\Gamma^{D+1} e^{-t(i\not{D}_M)^2}]\ . \tag{2.5}$$

A useful way to evaluate (2.5) is by looking for a supersymmetric quantum mechanical theory whose supercharge is $Q = i\not{D}_M$. It is not difficult to check that such a theory can be obtained by reducing the supersymmetric non-linear σ-model from $1+1$ to $0+1$ dimensions, with Neumann boundary conditions for the fields in the directions inside M and Dirichlet boundary conditions for the fields in the remaining directions. In terms of the fields appearing in (2.2) this means $x^i = 0$, $\psi_1^{\underline{\mu}} = \psi_2^{\underline{\mu}} \equiv \psi^{\underline{\mu}}/\sqrt{2}$ and $\psi_1^{\underline{i}} = -\psi_2^{\underline{i}} \equiv \psi^{\underline{i}}/\sqrt{2}$. The action (2.2) then leads to the following Lagrangian

$$\begin{aligned} L \;=\; & \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{i}{2}\psi_{\underline{\mu}}\left(\dot{\psi}^{\underline{\mu}} + \frac{1}{2}\omega_\rho{}^{\underline{\mu}}{}_{\underline{\nu}}\,\dot{x}^\rho\,\psi^{\underline{\nu}}\right) + \frac{i}{2}\psi_{\underline{i}}\left(\dot{\psi}^{\underline{i}} + \frac{1}{2}\omega_\rho{}^{\underline{i}}{}_{\underline{j}}\,\dot{x}^\rho\,\psi^{\underline{j}}\right) \\ & + \frac{1}{4} R_{\underline{\mu}\underline{\nu}\underline{i}\underline{j}}\psi^{\underline{\mu}}\psi^{\underline{\nu}}\psi^{\underline{i}}\psi^{\underline{j}}\ . \end{aligned} \tag{2.6}$$

This is still invariant under a combination of the two supersymmetries of (2.2). More precisely, the operator $Q = e^\mu{}_{\underline{\mu}}\,\psi^{\underline{\mu}}\,\dot{x}_\mu$ is still a conserved supercharge.

After canonical quantization, the $\psi^{\underline{\mu}}$'s and $\psi^{\underline{i}}$'s satisfy the anticommutation relations $\{\psi^{\underline{\mu}}, \psi^{\underline{\nu}}\} = \delta^{\underline{\mu}\underline{\nu}}$, $\{\psi^{\underline{i}}, \psi^{\underline{j}}\} = \delta^{\underline{i}\underline{j}}$ and $\{\psi^{\underline{\mu}}, \psi^{\underline{i}}\} = 0$, and generate Clifford algebras on M and the transverse space to it in X. They form therefore bases of forms respectively on M and its transverse space. The canonical momentum π_μ conjugate to x^μ is found to be

$$\pi_\mu = g_{\mu\nu}\dot{x}^\nu + \frac{i}{4}\left(\omega_\mu{}^{\underline{\nu}\underline{\rho}}[\psi_{\underline{\nu}}, \psi_{\underline{\rho}}] + \omega_\mu{}^{\underline{i}\underline{j}}[\psi_{\underline{i}}, \psi_{\underline{j}}]\right)\ . \tag{2.7}$$

Upon canonical quantization, $\pi_\mu \to -i\partial_\mu$ and $\psi_{\underline{M}} \to \Gamma_{\underline{M}}/\sqrt{2}$, and the supercharge becomes

$$Q = -\frac{i}{\sqrt{2}} e^{\mu}_{\ \underline{\mu}} \Gamma^{\underline{\mu}} \left[\partial_\mu + \frac{1}{4}\left(\omega_\mu^{\ \underline{\nu\rho}} \Gamma_{\underline{\nu\rho}} + \omega_\mu^{\ \underline{ij}} \Gamma_{\underline{ij}}\right)\right] \tag{2.8}$$

which is indeed the Dirac operator on M: $Q = -\not{D}_M/\sqrt{2}$. The Lagrangian (2.6) is actually a particular case of the one which was found in [19] to have as supercharge the Dirac operator with an arbitrary gauge connection. Finally, the chiral matrix Γ^{D+1}, which can be interpreted also as $\Gamma^{d+1}\Gamma^{D-d+1}$, is represented by the fermion number operator $(-1)^F$.

The partition function (2.5) giving the anomaly is recognized to be the Witten index [25] for the theory described by the Lagrangian (2.6):

$$Z_M = \mathrm{Tr}\left[(-1)^F e^{-tH}\right] . \tag{2.9}$$

Actually, being a topological quantity, this index does not depend on t. Its functional integral representation is

$$Z_M = \int_P \mathcal{D}x^\mu(\tau) \int_P \mathcal{D}\psi^{\underline{\mu}}(\tau) \int_P \mathcal{D}\psi^{\underline{i}}(\tau) \exp\left\{-\int_0^t d\tau L\left(q^\mu(\tau), \psi^{\underline{\mu}}(\tau), \psi^{\underline{i}}(\tau)\right)\right\} . \tag{2.10}$$

Due to the $(-1)^F$ insertion, all the fields are periodic. In order to evaluate this path-integral, it will be convenient to take the high-temperature limit $t \to 0$. In that limit, (2.10) is dominated by constant paths x_0^μ, $\psi_0^{\underline{\mu}}$, $\psi_0^{\underline{i}}$, with minimal energy $E_0 = -R_{\underline{\mu\nu ij}}(x_0)\psi_0^{\underline{\mu}}\psi_0^{\underline{\nu}}\psi_0^{\underline{i}}\psi_0^{\underline{j}}/4$ and Euclidean action $S_0 = tE_0$. The functional integral is evaluated by expanding the fields as fluctuations around these constant paths, $x^\mu = x_0^\mu + \xi^\mu$, $\psi^{\underline{\mu}} = \psi_0^{\underline{\mu}} + \lambda^{\underline{\mu}}$ and $\psi^{\underline{i}} = \psi_0^{\underline{i}} + \lambda^{\underline{i}}$. In the present case, as we will see, it will be enough to consider quadratic fluctuations and perform a one-loop computation, since higher loop corrections come with additional powers of t and are irrelevant in the limit $t \to 0$ we are considering. Expanding then the Lagrangian (2.6) in normal coordinates [26] around x_0^μ and keeping only terms at most bilinear in the fluctuations, one finds a quadratic action for the fluctuations which depends on the fermionic zero modes $\psi_0^{\underline{\mu}}$ and $\psi_0^{\underline{i}}$. The path integral then reduces to the integral over the bosonic and fermionic zero modes of the determinants arising from the Gaussian integration over the fluctuations. The integral over the x_0^μ's is just the integral over the manifold M, whereas the integrals over the $\psi_0^{\underline{\mu}}$'s and the $\psi_0^{\underline{i}}$'s select the d-form component $\psi_0^{\underline{\mu_0}}...\psi_0^{\underline{\mu_d}}$ on M and the (D-d)-form component $\psi_0^{\underline{i_{d+1}}}...\psi_0^{\underline{i_{D-d}}}$ on the transverse space in X.

The quadratic Lagrangian for the fluctuations contains terms with two or less fermionic zero modes. It is clear that, due to the integrals over fermionic zero modes, only interactions providing the maximal number of them (that is 2) will be relevant; indeed, picking up other interactions would increase the total number of vertices required to provide a sufficient number of fermionic zero modes in order to get a non-vanishing result. Among these, we now argue that the t-independent term of the path integral depends (besides the constant term) only on the terms quadratic in the fluctuations and bilinear in the $\psi_0^{\underline{\mu}}$'s. The reason is the following: the tree level term L_0 contains 4 zero modes and a single power of t, whereas quadratic vertices are also effectively proportional to t, as we will see, but provide only 2 fermionic zero modes. The leading contribution to the path-integral for $t \to 0$ comes therefore from correlations involving a maximum number of tree-level terms. If $d > D/2$, this term saturates alone the integral over the $\psi_0^{\underline{i}}$'s, contributing with a power $t^{(D-d)/2}$ and providing $D-d$ $\psi_0^{\underline{\mu}}$'s. Since the d periodic bosons $\xi^{\underline{\mu}}$ give a normalization factor of order $t^{-d/2}$, the total power of t becomes $t^{(D-2d)/2}$. We still need to soak up the remaining $2d-D$ $\psi_0^{\underline{\mu}}$'s. It is then clear that only the terms bilinears in the $\psi_0^{\underline{\mu}}$'s contribute to the leading t-independent term. Any other contribution vanishes in the limit $t \to 0$. This shows that we do not need to consider higher-loop contributions and that we can safely neglect all terms proportional to $\psi_0^{\underline{i}}$'s in the quadratic Lagrangian. If $d < D/2$, a similar line of arguments shows that the anomaly vanishes.

The effective Lagrangian, quadratic in the fluctuations and in the $\psi_0^{\bar{\mu}}$'s, is found to be

$$L^{eff} = \frac{1}{2}\left(\dot{\xi}_{\underline{\mu}}\dot{\xi}^{\underline{\mu}} + i\lambda_{\underline{\mu}}\dot{\lambda}^{\underline{\mu}} + i\lambda_{\underline{i}}\dot{\lambda}^{\underline{i}} + iR_{\underline{\mu\nu}}\dot{\xi}^{\underline{\mu}}\xi^{\underline{\nu}} + R'_{\underline{ij}}\lambda^{\underline{i}}\lambda^{\underline{j}} + R'_{\underline{ij}}\psi_0^{\underline{i}}\psi_0^{\underline{j}}\right) \tag{2.11}$$

where we have defined the tangent and normal bundle curvature 2-forms as

$$R_{\underline{\mu\nu}} = \frac{1}{2}R_{\underline{\mu\nu}\underline{\rho\sigma}}(x_0)\psi_0^{\underline{\rho}}\psi_0^{\underline{\sigma}}\ ,\ \ R'_{\underline{ij}} = \frac{1}{2}R_{\underline{ij}\underline{\rho\sigma}}(x_0)\psi_0^{\underline{\rho}}\psi_0^{\underline{\sigma}}\ . \tag{2.12}$$

The integral over the constant tree-level part of the action gives

$$\int d\psi_0^{\underline{i}} \exp\left\{t\sum_{i=d/2}^{D/2} R'_{\underline{ij}}\psi_0^{\underline{i}}\psi_0^{\underline{j}}\right\} = \prod_{i=d/2}^{D/2}\lambda'_i t\ . \tag{2.13}$$

The evaluation of the one-loop determinants is straightforward. Using ζ-function regularization to carefully normalize them, one finds

$$\begin{aligned}
&\det{}_P^{-1}(\partial_\tau^2\eta_{\underline{\mu\nu}} + iR_{\underline{\mu\nu}}\partial_\tau) = (2\pi t)^{-\frac{d}{2}}\prod_{i=1}^{d/2}\frac{\lambda_i t/2}{\sinh\lambda_i t/2}\ ,\\
&\det{}_P(i\partial_\tau\eta_{\underline{\mu\nu}}) = 1\ ,\ \ \det{}_P(i\partial_\tau\eta_{\underline{ij}} + R'_{\underline{ij}}) = \prod_{i=d/2}^{D/2}\frac{\sinh\lambda'_i t/2}{\lambda'_i t/2}\ .
\end{aligned} \tag{2.14}$$

Note at this point that the inclusion of an arbitrary gauge bundle presents no difficulties. The supersymmetric quantum mechanical model (2.6) has to be extended in such a way that its supercharge acquires an additional term involving the gauge connection, reproducing therefore the Dirac operator for a charged spinor. This modification is achieved exactly in the same way as in the standard case [16], and results in the additional tree-level factor

$$\mathrm{Tr}_\rho \exp\{itF\} \tag{2.15}$$

where Tr_ρ indicates the trace over the gauge group in the representation ρ in which the spinor transforms and

$$F = \frac{1}{2}F_{\underline{\mu\nu}}(x_0)\psi_0^{\underline{\mu}}\psi_0^{\underline{\nu}} \tag{2.16}$$

is the curvature 2-form of the gauge bundle.

Taking into account also the effect of the gauge field, the result for the path-integral (2.10) is then:

$$\begin{aligned} Z_M &= \int dx_0^\mu \int d\psi_0^\mu \, (2\pi t)^{-\frac{d}{2}} \mathrm{Tr}_\rho \exp\{itF\} \prod_{i=1}^{d/2} \frac{\lambda_i t/2}{\sinh \lambda_i t/2} \prod_{i=d/2}^{D/2} \frac{\sinh \lambda'_i t/2}{\lambda'_i t/2} \prod_{i=d/2}^{D/2} \lambda'_i t \\ &= \int dx_0^\mu \int d\psi_0^\mu \, \mathrm{Tr}_\rho \exp\{iF/2\pi\} \prod_{i=1}^{d/2} \frac{\lambda_i/4\pi}{\sinh \lambda_i/4\pi} \prod_{i=d/2}^{D/2} \frac{\sinh \lambda'_i/4\pi}{\lambda'_i/4\pi} \prod_{i=d/2}^{D/2} \lambda'_i/2\pi \end{aligned} \quad (2.17)$$

where in the last step we have used the fact that only the d-form component of the integrand contributes. This result can be rewritten as

$$Z_M = \int_M \mathrm{ch}_\rho(F) \wedge \frac{\widehat{A}(R)}{\widehat{A}(R')} \wedge e(R') \; . \quad (2.18)$$

This is the chiral anomaly of a spinor propagating on M but section of the spin bundle of $X \supset M$. As last step, we have to consider how the operator δ_η is realized, in order to derive the form of gauge and gravitational anomalies on M. Reparametrizations in X are broken down to reparametrizations of M and rotations in the transverse space. The former correspond to tangent bundle gauge transformations, $\delta_\eta \psi_a = -\eta^\mu D_\mu \psi_a$, and the latter to normal bundle gauge transformations, $\delta_{\eta'} \psi_a = -D_{\underline{i}} \eta'_{\underline{j}} \Gamma^{\underline{ij}}_{ab} \psi_b$. It is not difficult to verify that the operators δ_η and $\delta_{\eta'}$ are represented by

$$\delta_\eta = -\eta_\mu \dot{x}^\mu \;, \quad \delta_{\eta'} = -D_{\underline{i}} \eta'_{\underline{j}} \psi^{\underline{i}} \psi^{\underline{j}}$$

after canonical quantization. It is then easy to show that exponentiating these operators and expanding in normal coordinates, their net effect in (2.11) is to shift $R_{\mu\nu} \to R_{\underline{\mu\nu}} + D_{\underline{\mu}} \eta_{\underline{\nu}} - D_{\underline{\nu}} \eta_{\underline{\mu}}$ and $R_{\underline{ij}} \to R_{\underline{ij}} + D_{\underline{i}} \eta_{\underline{j}} - D_{\underline{j}} \eta_{\underline{i}}$. Taking the terms linear in η, η' corresponds therefore to take the Wess-Zumino descent, and the final result for the complex chiral spinor anomaly turns out to be, as expected:

$$A_M = 2\pi i \int_M \left[\mathrm{ch}_\rho(F) \wedge \frac{\widehat{A}(R)}{\widehat{A}(R')} \wedge e(R') \right]^{(1)} \; . \quad (2.19)$$

2.1.2 Self-dual tensor

Consider now a self-dual tensor interacting with gravity on X. The Jacobian giving the anomaly has been shown to be given by [16]

$$A_X = \lim_{t \to 0} \frac{1}{4} \mathrm{Tr}\left[*_D \delta_\eta e^{-t\Box_X} \right] \quad (2.20)$$

where $*_D$ is the Hodge operator in D dimensions and the trace runs over the eigenstates of $H = \Box_X$, with $\Box_X = (d + d^\dagger)^2$ the Laplace-Beltrami operator on X. We want now to consider gravitational anomalies on $M \subset X$, arising by reducing a chiral tensor from D down to d dimensions. The corresponding expression is obtained from (2.20) by tracing only over the states propagating on M. However, it is not possible to follow the same strategy as in the case of the chiral spinor, essentially because there is no simple theory having the required Hamiltonian. Indeed, it is well known that the supersymmetric action (2.2) has a Hamiltonian which is the Laplacian on forms of the target space [25], but there is no evident deformation of it which could constrain the dynamics to a submanifold of the target space. Fortunately, there is a second way of restricting the trace, which will make possible the evaluation of the anomaly: rather than changing the Hamiltonian, one can insert in the trace a suitable operator having the effect of projecting out the unwanted eigenstates. More precisely, we want to keep only those states with vanishing momentum in the transverse space to M in X. The appropriate operator to insert turns out to be quite simple and given by the reflection operator I in the transverse space. Since I acts on the fields x^M and ψ^M in (2.2) with a $+$ and a $-$ sign in the tangent and normal directions to M, states with non vanishing momentum p^i will be reflected into orthogonal states with opposite momentum $-p^i$, giving so a vanishing contribution to the trace. We can therefore extend the trace on the whole set of states propagating on X

$$A_M = \lim_{t\to 0} \mathrm{Tr}\left[I *_D \delta_\eta e^{-t\Box_X}\right] . \tag{2.21}$$

The trace (2.21) is actually taken, as in the standard case, over all the tensor fields (differential forms) on X. However, it is clear that the only non-vanishing contribution to (2.21) comes from $d/2$-forms on M, arising from the reduction of the self-dual $D/2$-form on X.

As before, the operator δ_η can be exponentiated, resulting in a shift in the background fields. We are then left with the "self-duality anomaly"

$$Z_M = \lim_{t\to 0} \mathrm{Tr}\left[I *_D e^{-t\Box_X}\right] . \tag{2.22}$$

Although the operator I *does not* commute in general with the Hamiltonian $\Box_X$, according to the decomposition (2.1), it commutes with the Laplace-Beltrami operator restricted to M. Using standard arguments [25], it is then clear that the trace (2.22) will be again an index. In order to evaluate (2.22), we regard it as the partition function for the supersymmetric quantum mechanical model (2.2), whose Hamiltonian is $H = \Box_X$. The $*_D$ operator, which can be interpreted now as $*_d *_{D-d}$, is implemented as usual by the discrete symmetry Ω, mapping $\psi_1 \to -\psi_1$ and $\psi_2 \to \psi_2$. Only zero energy states can contribute to (2.22); indeed all the massive ones fall into multiplets with equal number of eigenstates of $I\Omega$ with opposite eigenvalues, and the result is therefore independent of t. The path integral representation of (2.22) is

$$\begin{aligned} Z_M = & \int_P \mathcal{D}x^\mu(\tau) \int_A \mathcal{D}x^i(\tau) \int_P \mathcal{D}\psi_1^\mu(\tau) \int_A \mathcal{D}\psi_1^i(\tau) \int_A \mathcal{D}\psi_2^\mu(\tau) \int_P \mathcal{D}\psi_2^i(\tau) \\ & \exp\left\{-\int_0^t d\tau L\left(x^M(\tau), \psi_{1,2}^\mu(\tau), \psi_{1,2}^i(\tau)\right)\right\} \end{aligned} \tag{2.23}$$

where the Lagrangian is given by (2.2). The periodicities are obtained by noting that ΩI acts with a $+$ sign on x^μ, ψ_1^i, ψ_2^μ and with a $-$ sign on x^i, ψ_1^μ, ψ_2^i. Notice that only x^μ, ψ_1^μ and ψ_2^i have then zero modes. Again, it is convenient to take the high-temperature limit $t \to 0$. In that limit, (2.23) is dominated by constant paths and one can therefore expand the fields as $x^\mu = x_0^\mu + \xi^\mu$, $x^i = \xi^i$, $\psi_1^\mu = \psi_0^\mu + \lambda_1^\mu$, $\psi_2^\mu = \lambda_2^\mu$, $\psi_1^i = \lambda_1^{\bar{i}}$ and $\psi_2^i = \psi_0^i + \lambda_2^i$. Expanding the Lagrangian (2.2) in normal coordinates around x_0^μ, it is evident that the last term in (2.2) will again give the same tree-level term as in the spinor case: $L_0 = tR_{\mu\nu\underline{i}\underline{j}}(x_0)\psi_0^\mu\psi_0^\nu\psi_0^{\underline{i}}\psi_0^{\underline{j}}/4$. Then, by applying precisely the same considerations as in the spinor case, the t-independent term in the path integral (2.23) will receive contributions only from the tree-level term and quadratic interactions bilinear in the ψ_0^μ's. The effective Lagrangian that one obtains is then

$$L^{eff} = \frac{1}{2}\Big[\dot{\xi}_{\underline{\mu}}\dot{\xi}^{\underline{\mu}} + \dot{\xi}_{\underline{i}}\dot{\xi}^{\underline{i}} + i\lambda_{1\underline{\mu}}\dot{\lambda}_1^{\underline{\mu}} + i\lambda_{1\underline{i}}\dot{\lambda}_1^{\underline{i}} + i\lambda_{2\underline{\mu}}\dot{\lambda}_2^{\underline{\mu}} + i\lambda_{2\underline{i}}\dot{\lambda}_2^{\underline{i}} + R_{\underline{\mu}\underline{\nu}}\left(i\,\dot{\xi}^{\underline{\mu}}\xi^{\underline{\nu}} + \lambda_2^{\underline{\mu}}\lambda_2^{\underline{\nu}}\right) + R'_{\underline{i}\underline{j}}\left(i\,\dot{\xi}^{\underline{i}}\xi^{\underline{j}} + \lambda_2^{\underline{i}}\lambda_2^{\underline{j}}\right) + R'_{\underline{i}\underline{j}}\psi_0^{\underline{i}}\psi_0^{\underline{j}}\Big] \qquad (2.24)$$

in terms of the tangent and normal bundle curvature 2-forms (2.12).

The constant tree-level part of the action again contributes as in (2.13). The evaluation of the one-loop determinants presents no difficulties, a part from important normalizations which can be fixed by relying again on ζ-function regularization. One finds in this way the following results for the two bosonic and four fermionic fluctuations:

$$\begin{aligned}
&\det{}_P^{-1}(\partial_\tau^2\eta_{\underline{\mu}\underline{\nu}} + i\,R_{\underline{\mu}\underline{\nu}}\partial_\tau) = (2\pi t)^{-\frac{d}{2}}\prod_{i=1}^{d/2}\frac{\lambda_i t/2}{\sinh\lambda_i t/2}\,,\\
&\det{}_A^{-1}(\partial_\tau^2\eta_{\underline{i}\underline{j}} + i\,R_{\underline{i}\underline{j}}\partial_\tau) = \prod_{i=d/2}^{D/2}\frac{1}{4\cosh\lambda'_i t/2}\,,\\
&\det{}_P(i\partial_\tau\eta_{\underline{\mu}\underline{\nu}}) = 1\,,\quad \det{}_A(i\partial_\tau\eta_{\underline{\mu}\underline{\nu}} + R_{\underline{\mu}\underline{\nu}}) = \prod_{i=1}^{d/2}2\cosh\lambda_i t/2\,,\\
&\det{}_A(i\partial_\tau\eta_{\underline{i}\underline{j}}) = \prod_{i=d/2}^{D/2}2\,,\quad \det{}_P(i\partial_\tau\eta_{\underline{i}\underline{j}} + R'_{\underline{i}\underline{j}}) = \prod_{i=d/2}^{D/2}\frac{\sinh\lambda'_i t/2}{\lambda'_i t/2}\,.
\end{aligned} \qquad (2.25)$$

The result for the path-integral (2.23) is then:

$$\begin{aligned}
Z_M &= \int dx_0^\mu\int d\psi_0^\mu\,(\pi t)^{-\frac{d}{2}}\prod_{i=1}^{d/2}\frac{\lambda_i t/2}{\tanh\lambda_i t/2}\prod_{i=d/2}^{D/2}\frac{\tanh\lambda'_i t/2}{\lambda'_i t/2}\prod_{i=d/2}^{D/2}\lambda'_i t/2\\
&= \int dx_0^\mu\int d\psi_0^\mu\prod_{i=1}^{d/2}\frac{\lambda_i/2\pi}{\tanh\lambda_i/2\pi}\prod_{i=d/2}^{D/2}\frac{\tanh\lambda'_i/2\pi}{\lambda'_i/2\pi}\prod_{i=d/2}^{D/2}\lambda'_i/2\pi
\end{aligned} \qquad (2.26)$$

where the last step is valid for the d-form component. This result can be finally rewritten as

$$Z_M = \int_M \frac{\widehat{L}(R)}{\widehat{L}(R')}\wedge e(R')\,. \qquad (2.27)$$

Again the realization of δ_η does not present particular problems. The form of the gauge transformations can be obtained by thinking of the self-dual tensor as a bispinor [16]. The operators δ_η and $\delta_{\eta'}$ are in this case represented by

$$\delta_\eta = -\eta_\mu\dot{x}^\mu - D_{\underline{\mu}}\eta'_{\underline{\nu}}\psi_2^{\underline{\mu}}\psi_2^{\underline{\nu}}\,,\quad \delta_{\eta'} = -\eta'_i\dot{x}^i - D_{\underline{i}}\eta'_{\underline{j}}\psi_2^{\underline{i}}\psi_2^{\underline{j}}$$

upon canonical quantization. As before, one can exponentiate these operators and expand in normal coordinates. The net effect in (2.24) is again to shift $R_{\mu\nu} \to R_{\underline{\mu}\underline{\nu}} + D_{\underline{\mu}}\eta_{\underline{\nu}} - D_{\underline{\nu}}\eta_{\underline{\mu}}$ and $R_{\underline{i}\underline{j}} \to R_{\underline{i}\underline{j}} + D_{\underline{i}}\eta_{\underline{j}} - D_{\underline{j}}\eta_{\underline{i}}$. Therefore, taking the terms linear in η, η' again corresponds to take the Wess-Zumino descent, and the final result for the real self-dual antisymmetric tensor anomaly is

$$A_M = 2\pi i \int_M \left[-\frac{1}{8} \frac{\widehat{L}(R)}{\widehat{L}(R')} \wedge e(R') \right]^{(1)} . \qquad (2.28)$$

The additional factor of 1/2 arises as in [16]: for the anomaly the relevant component of the integrand is the (d+2)-form, whereas it was the d-form component in (2.27), so the rescaling in the second line of (2.26) produces an extra factor of 1/2.

2.2. Index and G-Index theorems

In last subsection we have computed the anomalies associated to chiral spinors and tensors propagating on a submanifold M of X, in interaction with gravitational fields propagating on X, by working out suitable path integrals. The results are indices encoding topological data of M and/or X. In order to check the path integral computation and to have a better understanding of the mathematical nature of the anomalies we have found, we will also compute directly the indices (2.18),(2.27), using the Atiyah-Singer index theorem and its G-index generalization [21, 22] (see also [19, 20] for related applications of index theorems).

2.3. Chiral spinor

It is well known that the index of the Dirac operator in an even dimensional manifold X gives the chiral anomaly of a Dirac spinor on X. Through the descent procedure, this is also related to gauge and gravitational anomalies of a chiral spinor in two lower dimensions. The close relation between index theory and anomalies can be also used in the case we are interested in, i.e. a chiral spinor propagating on a even dimensional submanifold M of X. Although the chiral anomaly (2.18) has been already obtained in [6] using the family index theorem [23], for completeness we re-derive here that result using the standard index theorem [21].

Given the tangent bundle decomposition of X as the Whitney sum of the tangent and normal bundles of M, the corresponding positive and negative chirality spin bundles $S^{\pm}_{T(X)}$ decompose as follows in terms of the positive and negative chirality spin bundles $S^{\pm}_{T(M)}$ and $S^{\pm}_{N(M)}$ lifted from the tangent and normal bundles to M:

$$S^{\pm}_{T(X)} \to [\, S^{\pm}_{T(M)} \otimes S^{+}_{N(M)} \,] \oplus [\, S^{\mp}_{T(M)} \otimes S^{-}_{N(M)} \,] \, . \qquad (2.29)$$

The Dirac operator for the charged and reduced fermion we are considering acts on sections of the bundles (2.29) tensored with the gauge bundle V_ρ in the representation ρ in which the fermion transforms, interchanging positive and negative chiralities. More precisely, we have the two-term complex

$$i\not{D} \;:\; \Gamma(M, E^+) \to \Gamma(M, E^-) \qquad (2.30)$$

where

$$E^{\pm} = \Big([\, S^{\pm}_{T(M)} \otimes S^{+}_{N(M)} \,] \oplus [\, S^{\mp}_{T(M)} \otimes S^{-}_{N(M)} \,] \Big) \otimes V_\rho \, . \qquad (2.31)$$

It is now straightforward to apply the usual index theorem to the particular case of the two-term complex (2.30). One finds

$$\mathrm{index}(i\not{D}) = (-1)^{\frac{d(d+1)}{2}} \int_M \mathrm{ch}_\rho(V) \frac{\mathrm{ch}(S^{+}_{T(M)} - S^{-}_{T(M)})\, \mathrm{ch}(S^{+}_{N(M)} - S^{-}_{N(M)})}{e(T(M))} \mathrm{Td}(T(M^C))$$

where $\mathrm{Td}(T(M^C))$ is the Todd class of the complexified tangent bundle of M and $d = \dim M$. The Chern characters of the spin bundles are:

$$\mathrm{ch}(S^+_{T(M)} - S^-_{T(M)}) = \prod_{i=1}^{d/2} (e^{x_i/2} - e^{-x_i/2}) ,$$
$$\mathrm{ch}(S^+_{N(M)} - S^-_{N(M)}) = \prod_{j=d/2}^{D/2} (e^{x'_i/2} - e^{-x'_i/2}) ,$$

with $D = \dim X$ and x_i, x'_i respectively the eigenvalues of the curvature two-form on $T(M)$ and $N(M)$ (note that $x = \lambda/2\pi$ defined before). By using the standard expressions for the Euler and Todd classes, one then easily reproduces (2.18):

$$\mathrm{index}(i\not{D}) = \int_M \mathrm{ch}_\rho(V) \wedge \frac{\widehat{A}(R)}{\widehat{A}(R')} \wedge e(R') . \tag{2.32}$$

2.4. Self-dual tensor

Let us now turn our attention to self-dual tensors. Again, through the descent procedure, the gravitational anomaly on a 4n+2 dimensional manifold is related to the index of a classical complex over a 4n+4 dimensional manifold X, the signature complex:

$$\mathcal{D}_+ \ : \ {}^+\!\wedge T^*X \longrightarrow {}^-\!\wedge T^*X . \tag{2.33}$$

$\mathcal{D}_+$ maps self-dual forms to anti self-dual forms on X. In the case we are interested, the tensors propagate on $M \subset X$, but they are sections of $\wedge T^*X$. From the results of last section, we learned that a suitable operator that projects onto states propagating on M only is the $\mathbf{Z}_2$ operator acting on the normal coordinates: if (x^μ, y^i) are respectively local coordinates on M and its transverse space in X, then

$$\mathbf{Z}_2 \ : \ (x^\mu, y^i) \longrightarrow (x^\mu, -y^i) . \tag{2.34}$$

What we have to compute is then the so-called G-index [22] of the signature complex, or simply the G-signature, where $G = \mathbf{Z}_2$. Generically, G can be a compact Lie group acting on X by orientation-preserving transformations[1] (see [27] for a nice introduction and more details on the G-index). The action of G on X can be also extended to vector bundles over X, provided that G acts on the bundle E mapping linearly the fiber on the point x to the fiber on gx, $\forall g \in G$. In this case E is also called a G-bundle. Let then $X_G \subset X$ be the subspace left invariant by G, that is $X_G = \{x \in X : gx = x, \forall g \in G\}$. Let us also denote with T_G and N_G respectively the tangent and normal bundles of X_G in X. The G-signature is then given by (see e.g. [27])

$$\mathrm{index}(\mathcal{D}^G_+) = \int_{X^G} \frac{\mathrm{ch}(T^+_G - T^-_G)\,\mathrm{ch}_G(N^+_G - N^-_G)}{\mathrm{ch}_G(\tilde{N}_G)\, e(T_G)} \mathrm{Td}(T^C_G) \tag{2.35}$$

where $e(T_G)$ and $\mathrm{Td}(T^C_G)$ are the usual Euler and Todd classes, $T^\pm_G = {}^\pm\!\wedge T^*X_G$, $N^\pm_G = {}^\pm\!\wedge N^*X_G$ and $\tilde{N}_G = \oplus_i (-)^i \wedge^i N^*X_G$. If E_G is a G-bundle, in general $E_G = \oplus_i E^{(i)}_G$,

[1] Note that in our cases the transverse space is always even-dimensional, and (2.34) is orientation-preserving.

where G acts with the element g_i in $E_G^{(i)}$. In this case, the Chern character ch_G reads $\mathrm{ch}_G(E_G) = \sum_i \mathrm{Tr}\, g_i \exp\{iF_i/2\pi\}$ where F_i is the curvature 2-form on $E_G^{(i)}$. In our particular case, $G = \mathbf{Z}_2$ and clearly $X_{\mathbf{Z}_2} = M$, $T_{\mathbf{Z}_2} = T(M)$ and $N_{\mathbf{Z}_2} = N(M)$. $T(M)$ and $N(M)$ are G-bundles in which $\mathbf{Z}_2$ acts respectively with the elements I and -I. One then finds

$$\mathrm{ch}(T_G^+ - T_G^-) = \prod_{i=1}^{d/2} (e^{x_i} - e^{-x_i}) \ , \ \ \mathrm{ch}_G(N_G^+ - N_G^-) = \prod_{j=d/2}^{D/2} (e^{x_i'} - e^{-x_i'}) \ ,$$
$$\mathrm{ch}_G(\tilde{N}_G) = \prod_{j=d/2}^{D/2} (1 + e^{x_i'})\,(1 + e^{-x_i'}) \ ,$$

where the eigenvalues x_i, x_i' are defined as previously. Putting all together, one finally reproduces eq.(2.27):

$$\mathrm{index}(\mathcal{D}_+^{\mathbf{Z}_2}) = \int_M \frac{\widehat{L}(R)}{\widehat{L}(R')} \wedge e(R') \ . \tag{2.36}$$

3. Inflow on D-brane and O-plane intersections

In previous sections, we have shown that the anomaly polynomials for the chiral spinors and self-dual tensors living on overlapping D-brane/O-planes are given by

$$I_{1/2}^{\rho}(F,R,R') \ = \ \mathrm{ch}_\rho(F) \wedge \frac{\widehat{A}(R)}{\widehat{A}(R')} \wedge e(R') \ , \tag{3.1}$$
$$I_A(R,R') \ = \ -\frac{1}{8}\frac{\widehat{L}(R)}{\widehat{L}(R')} \wedge e(R') \ . \tag{3.2}$$

In this section we show that these anomalies are exactly cancelled by the inflow of anomaly associated to the anomalous couplings (1.1) and (1.2). To this aim, we shall briefly recall how the inflow mechanism works in the general case, following [6].

Consider a set of defects M_i in spacetime X with anomalous couplings of the form

$$S = -\sum_i \frac{\mu_i}{2} \int_{M_i} C \wedge Y_i \ . \tag{3.3}$$

By integrating by parts, the integrand can be rewritten in terms of the constant parts Y_{i0}, which we set to 1 by suitably normalizing the charges μ_i, and the descents $Y_i^{(0)}$, as $C \mp H \wedge Y_i^{(0)}$, the sign depending on whether C contains even (Type IIB) or odd (Type IIA) forms. The complete action for the RR fields in presence of this sources can then be written as an integral over all of spacetime X by using a current representative τ_{M_i} in the space which is dual in X to the forms on M_i:

$$S = -\frac{1}{4}\int_X H \wedge {}^*H - \sum_i \frac{\mu_i}{2} \int_X \tau_{M_i} \left(C \mp H \wedge Y_i^{(0)}\right) \ . \tag{3.4}$$

τ_{M_i} is itself a form of rank equal to the codimension $D - d_i$ of M_i in X. Locally, it can be represented by a generalization to forms of Dirac's δ-function given by

$\tau_{M_i} \sim \delta(x^{d_i})...\delta(x^D)dx^{d_i} \wedge ... \wedge dx^D$. Globally, it is however a section of the normal bundle $N(M_i)$. The equations of motion and Bianchi identity implied by (3.4) are

$$d^*H = \sum_i \mu_i \tau_{M_i} \wedge Y_i \;, \tag{3.5}$$

$$dH = -\sum_i \mu_i \tau_{M_i} \wedge \bar{Y}_i \;, \tag{3.6}$$

where $\bar{Y}_i$ is obtained from Y_i by complex conjugation of the gauge group representation. Moreover, for consistency one must have vanishing total charge for the top RR form. It is clear form the modified Bianchi identity (3.6) that the field-strength H cannot be identified anymore with dC. Rather, the minimal solution of (3.6) is $H = dC \mp \sum_i \mu_i \tau_{M_i} \wedge \bar{Y}_i^{(0)}$. Since H, being a physical observable, must be gauge invariant, C must acquire an anomalous gauge transformation to compensate the gauge variation of the second term: $\delta_\eta C = \sum_i \mu_i \tau_{M_i} \wedge \bar{Y}_i^{(1)}$. Consequently, under a gauge transformation δ_η the couplings (3.3) present an anomaly given by

$$\delta_\eta(iS) = -\,i\sum_{i,j}\frac{\mu_i\mu_j}{2}\int_X \tau_{M_i} \wedge \tau_{M_j} \left(Y_i \wedge \bar{Y}_j\right)^{(1)} \;. \tag{3.7}$$

All the anomaly is localized on the intersections M_{ij} of pairs of defects M_i and M_j. In order to see this explicitly and correctly, remember that τ_{M_i} are sections of the normal bundles $N(M_i)$, with compact support on it. The current is then well defined on $N(M_i)$ and represented on M_i by taking the zero section of $N(M_i)$. It is now a standard result (see for instance [28]) that in cohomology τ_{M_i} can be identified with the Thom class $\Phi[N(M_i)]$ of $N(M_i)$, whose zero section is the Euler class of $N(M_i)$. This implies the following property for the currents [6]: $\tau_{M_i} \wedge \tau_{M_j} = \tau_{M_{ij}} \wedge e(N(M_{ij}))$. Using the freedom left over in the descent procedure, the inflow can then be rewritten as

$$\delta_\eta(iS) = -\,i\sum_{i,j}\frac{\mu_i\mu_j}{2}\int_{M_{ij}} \left[\left(Y_i \wedge \bar{Y}_j\right) \wedge e(N_{ij})\right]^{(1)} \;. \tag{3.8}$$

It is now straightforward to use this result to show that the anomalies (3.1) and (3.2) are cancelled by the inflows on BB, BO and OO intersections. Specializing to two overlapping defects on the same manifold M, the inflow of anomaly can be written as $A_{ij} = 2\pi i \int_{M_{ij}} I_{ij}^{(1)}$ with

$$I_{ij} = -\frac{\mu_i\mu_j}{4\pi} Y_i \wedge \bar{Y}_j \wedge e(N_{ij}) \;. \tag{3.9}$$

We set $d = p+1$ but keep D generic, and use the couplings (1.1) and (1.2) for a Dp-brane and an Op-plane. It is easily seen that the d-form part in (3.9) has precisely the right powers of $(4\pi^2\alpha')$ to cancel the factor $(4\pi^2\alpha')^{-(p+1)/2}$ in the charges $\mu_{i,j}$. The effective Dp-brane and Op-plane charges are then $\mu_p = \alpha\sqrt{2\pi}$ and $\mu'_p = -2^{p+1-D/2}\alpha\sqrt{2\pi}$. The numerical coefficient α depends on the particular model, and we will derive it case by case. The anomaly inflows on the BB, BO and OO are the following:

BB intersection

Using the property $\mathrm{ch}_{\rho_1}(F) \wedge \mathrm{ch}_{\rho_2}(F) = \mathrm{ch}_{\rho_1\otimes\rho_2}(F)$, one finds

$$I_{BB}(F,R,R') = -\frac{\alpha^2}{2}\mathrm{ch}_{\lambda\otimes\bar{\lambda}}(F) \wedge \frac{\widehat{A}(R)}{\widehat{A}(R')} \wedge e(R') \tag{3.10}$$

that is

$$I_{BB}(F,R,R') = -\frac{\alpha^2}{2} I_{1/2}^{\lambda\otimes\bar\lambda}(F,R,R')\ . \tag{3.11}$$

BO intersection

Using the relations $\mathrm{ch}_{\rho_1}(F) + \mathrm{ch}_{\rho_2}(F) = \mathrm{ch}_{\rho_1\oplus\rho_2}(F)$ and $\widehat{A}(R)\wedge\widehat{L}(R/4) = \widehat{A}(R/2)$, one finds in this case

$$\begin{aligned} I_{BO+OB}(F,R,R') &= 2^{p+1-\frac{D}{2}}\frac{\alpha^2}{2}\mathrm{ch}_{\lambda\oplus\bar\lambda}(F)\wedge\frac{\widehat{A}(R/2)}{\widehat{A}(R'/2)}\wedge e(R') \\ &= \frac{\alpha^2}{4}\mathrm{ch}_{\lambda\oplus\bar\lambda}(2F)\wedge\frac{\widehat{A}(R)}{\widehat{A}(R')}\wedge e(R')\ . \end{aligned} \tag{3.12}$$

The second line is obtained by rescaling the argument of the Euler class by 1/2, producing a factor $2^{\frac{D-p-1}{2}}$, and then rescale all the arguments by factor 2, giving an additional factor $2^{-\frac{p+3}{2}}$ for the relevant (p+3)-form component. Therefore

$$I_{BO+OB}(F,R,R') = \frac{\alpha^2}{4} I_{1/2}^{\lambda\oplus\bar\lambda}(2F,R,R')\ . \tag{3.13}$$

OO intersection

One finds in this case

$$\begin{aligned} I_{OO}(F,R,R') &= -4^{p+1-\frac{D}{2}}\frac{\alpha^2}{2}\frac{\widehat{L}(R/4)}{\widehat{L}(R'/4)}\wedge e(R') \\ &= -\frac{\alpha^2}{8}\frac{\widehat{L}(R)}{\widehat{L}(R')}\wedge e(R') \end{aligned} \tag{3.14}$$

where the second equality follows from manipulations similar to those performed before. Then

$$I_{OO}(R,R') = \alpha^2 I_A(R,R')\ . \tag{3.15}$$

This demonstrate that the anomaly inflow on D-branes and O-planes intersections have the required form to cancel the anomaly of the fields living on them. One has chiral spinors in the representation $\lambda\otimes\bar\lambda$ of the gauge group for BB, essentially a chiral spinor in the representation $\lambda\oplus\bar\lambda$ of the gauge group for BO, and a self-dual antisymmetric tensor, neutral under the gauge group, for OO. The precise coefficients depend on the particular model through the parameter α. We will show that they are indeed correct and discuss them in more detail for some particular Type IIB orientifolds in next section.

4. Anomaly cancellation in Type IIB orientifolds

In this section, we shall discuss some simple examples of Type IIB orientifold models and discuss anomaly cancellation in the light of our results. In the following we will consider two simple examples: type I theory in $D=10$ and the $T^4/\mathbf{Z_2}$ orientifold model in $D=6$ dimensions [29, 30]. Although in both cases the cancellation of spacetime anomalies is well understood [15, 31], our aim will be simply to reinterpret those results as special cases of the inflow mechanism discussed in the previous sections. In particular, we will explicitly show that, as mentioned in the introduction, the anomaly coming from neutral states combines into that of a self-dual antisymmetric tensor (OO), whereas the one from charged states can be recast into that of chiral spinors, essentially in the bifundamental (BB) and fundamental (BO) representations of the gauge group. Moreover, the coefficients will turn out to be precisely those required to cancel these anomalies by the inflow mechanism. We will also briefly discuss the anomaly coming from neutral fields in other $T^4\mathbf{Z}_N$ orientifolds. A detailed and complete account of how this mechanism can be extended to these and more general six-dimensional models [32, 33, 34], for which an analysis of anomaly cancellation is still lacking, will be reported elsewhere.

In the following, we will make use of the following relations between the anomaly polynomial of different fields:

$$D=10: \quad I_{1/2}(R) - I_{3/2}(R) - I_A(R) = 0 \tag{4.1}$$

$$D=6: \quad 21\, I_{1/2}(R) - I_{3/2}(R) + 8\, I_A(R) = 0 \tag{4.2}$$

where $I_{1/2}(R)$, $I_A(R)$ and $I_{3/2}(R)$ are the standard gravitational anomaly polynomials for spinors, antisymmetric tensors and Rarita-Schwinger fields [16].

4.1. Type I theory

Consider first Type I theory in $D=10$ as the simplest Type IIB orientifold [36]. Taking n_9 D9-branes together with 1 O9-plane, the gauge group would be $SO(n_9)$, and consistency of the theory requires $n_9 = 32$ [15].

One-loop anomalies

Keeping n_9 arbitrary, the anomalous fields in the model are the following

- 1 grav. mult.: $-\frac{1}{2} I_{1/2}(R) + \frac{1}{2} I_{3/2}(R)$
- 1 vec. mult.: $\frac{1}{2} I_{1/2}^{\frac{\mathbf{n_9(n_9-1)}}{\mathbf{2}}}(F,R)$

where the factors 1/2 are due to the fact that all fermions are real. Using (4.1), the total anomaly from neutral fields is therefore

$$I_n(R) = -\frac{1}{2} I_A(R) \ . \tag{4.3}$$

For the charged fields, it is convenient to rewrite the trace in the adjoint $\mathbf{n_9(n_9-1)/2}$ in terms of traces in the fundamental $\mathbf{n_9}$. It is not difficult to verify order by order that for $SO(n)$ one has

$$\mathrm{ch}_{\frac{\mathbf{n(n-1)}}{\mathbf{2}}}(F) = \frac{1}{2}\left(\mathrm{ch}_{\mathbf{n}\otimes\mathbf{n}}(F) - \mathrm{ch}_{\mathbf{n}}(2F)\right) \ . \tag{4.4}$$

The anomaly from charged fields can then be written as

$$I_c(F,R) = \frac{1}{4} I_{1/2}^{\mathbf{n_9}\otimes\mathbf{n_9}}(F,R) - \frac{1}{4} I_{1/2}^{\mathbf{n_9}}(2F,R) \ . \tag{4.5}$$

Anomaly inflow

Consider now the inflows on the D9D9, D9O9 and O9O9 intersections. The Chan-Paton representation λ is the fundamental $\mathbf{n_9}$. Using then the formulae derived in last section one finds

$$
\begin{aligned}
I_{BB}(F,R) &= -\frac{\alpha^2}{2} I_{1/2}^{\mathbf{n_9}\otimes\mathbf{n_9}}(F,R) \;, \\
I_{BO+OB}(F,R) &= \frac{\alpha^2}{2} I_{1/2}^{\mathbf{n_9}}(2F,R) \;, \\
I_{OO}(R) &= \alpha^2 I_A(R) \;.
\end{aligned} \tag{4.6}
$$

We see therefore that the inflow cancels precisely the anomalies (4.3) and (4.5) only if $\alpha = 1/\sqrt{2}$. This is indeed the correct value for Type I D-branes and O-planes [35].

We have therefore verified in this simple case that the inflow on BB, BO and OO exactly cancels respectively the anomalies of the charged and neutral fields. Moreover, it is very clear from this way of doing that the requirement $n_9 = 32$ appears exclusively from charge cancellation. Notice also that the requirement of vanishing irreducible terms in the anomaly polynomial $\mathrm{tr}F^6, \mathrm{tr}R^6$ is equivalent to charge cancellation for the 10-form RR potential. Indeed, the inflow necessary to cancel these terms would involve the 10-form and require the presence of a clearly inexistent magnetic dual (-2)-form with anomalous couplings proportional to $C_{(-2)} \wedge \mathrm{tr}F^6$, $C_{(-2)} \wedge \mathrm{tr}R^6$ [37].

4.2. K3 orientifolds

Consider now $T^4/\mathbf{Z}_N$ orientifolds of Type IIB theory, which can be interpreted as generalizations of $K3$ compactifications of Type I theory to $D = 6$. The low energy effective theory has $N = 1$ $D = 6$ supersymmetry, and beside the usual gravitational and tensor multiplets of $N = 1$ $D = 6$ supergravity, it will involve a vector multiplet in the adjoint representation of the gauge group and a certain number of charged hyper and neutral tensor matter multiplets, depending on the model. Indicating with n_H the number of hyper multiplets and n_T the number of additional tensor multiplets, the total anomaly from neutral fields is found to be, using (4.2),

$$
I_n(R) = (n_T - 8) I_A(R) + (n_T + n_H - 20) I_{1/2}(R) \;. \tag{4.7}
$$

As pointed out in [32] all consistent $\mathbf{Z}_N$ orientifold models have $n_T + n_H = 20$. This condition is closely related to the geometric properties of the underlying $K3$ surface. Additional models constructed as open descents of Gepner models [34] also satisfy this condition. It is remarkable that the total neutral field anomaly has the form of the anomaly of a self-dual tensor:

$$
I_n(R) = (n_T - 8) I_A(R) \;. \tag{4.8}
$$

The total anomaly coming from charged fields must be analyzed model by model. The simplest example we shall consider in the following is the $\mathbf{Z}_2$ orientifold with maximally enhanced gauge group [30]. Taking $2n_9$ D9-branes and $2n_5$ D5-branes, together with 1 O9-plane and 16 O5-planes, with all the $2n_5$ D5-branes at a single fixed-point on top of the corresponding O5-plane, the gauge group is $U(n_5) \times U(n_9)$, and consistency of the theory requires $n_5 = n_9 = 16$ [30]. Anomaly cancellation in this model has been studied in [31], where it was shown that the anomaly factorizes and is cancelled by a generalization of the GS mechanism. We will give here an interpretation in terms of inflows on D-brane/O-plane intersections.

One-loop anomalies

Keeping $n_{5,9}$ arbitrary, the anomalous fields in the $\mathbf{Z}_2$ model are the following

- 1 grav. mult.: $-I_{3/2}(R) - I_A(R)$
- 1 tens. mult.: $I_A(R) + I_{1/2}(R)$
- 1 vec. mult. in the $(\mathbf{n_5^2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{n_9^2})$: $-I_{1/2}^{(\mathbf{n_5^2},\mathbf{1})\oplus(\mathbf{1},\mathbf{n_9^2})}(F,R)$
- 2 hyp. mult. in the $(\frac{\mathbf{n_5(n_5-1)}}{\mathbf{2}}, \mathbf{1}) \oplus (\mathbf{1}, \frac{\mathbf{n_9(n_9-1)}}{\mathbf{2}})$: $2\, I_{1/2}^{(\frac{\mathbf{n_5(n_5-1)}}{\mathbf{2}},\mathbf{1})\oplus(\mathbf{1},\frac{\mathbf{n_9(n_9-1)}}{\mathbf{2}})}(F,R)$
- 1 hyp. mult. in the $(\mathbf{n_5}, \mathbf{n_9})$: $I_{1/2}^{(\mathbf{n_5},\mathbf{n_9})}(F,R)$
- 20 hyp. mult. in the $(\mathbf{1}, \mathbf{1})$: $20\, I_{1/2}(R)$

Using (4.2), the total anomaly from neutral fields is therefore

$$I_n(R) = -8 I_A(R) \tag{4.9}$$

which is a particular case of (4.8) with $n_T = 0$. For the charged fields, it is as usual convenient to rewrite the traces in all the representations in terms of traces in the fundamental representations. For $U(n)$ one has obviously

$$\mathrm{ch}_{\mathbf{n^2}}(F) = \mathrm{ch}_{\mathbf{n}\otimes\bar{\mathbf{n}}}(F) \tag{4.10}$$

for the adjoint representation, and one can check order by order that

$$\mathrm{ch}_{\frac{\mathbf{n(n\pm 1)}}{\mathbf{2}}}(F) = \frac{1}{2}\left(\mathrm{ch}_{\mathbf{n}\otimes\mathbf{n}}(F) \pm \mathrm{ch}_{\mathbf{n}}(2F)\right) \tag{4.11}$$

for the symmetric and antisymmetric tensor representations.

Using the fact that only even powers of the field strength F appear in the anomaly polynomial, the total anomaly from charged fields is then found to be (the symbol $\ominus$ means that one has to take the differences of the Chern classes in the two representations):

$$\begin{aligned} I_c(F,R) \;=\; & \frac{1}{4} I_{1/2}^{(\mathbf{n_5}\oplus\bar{\mathbf{n}}_\mathbf{5},\mathbf{n_9}\oplus\bar{\mathbf{n}}_\mathbf{9})}(F,R) - \frac{1}{2}\left(I_{1/2}^{(\mathbf{n_5}\oplus\bar{\mathbf{n}}_\mathbf{5},\mathbf{1})} + I_{1/2}^{(\mathbf{1},\mathbf{n_9}\oplus\bar{\mathbf{n}}_\mathbf{9})}\right)(2F,R) \\ & + \frac{1}{4}\left(I_{1/2}^{(\mathbf{n_5}\ominus\bar{\mathbf{n}}_\mathbf{5},\mathbf{n_9}\ominus\bar{\mathbf{n}}_\mathbf{9})} + 2 I_{1/2}^{((\mathbf{n_5}\ominus\bar{\mathbf{n}}_\mathbf{5})\otimes(\mathbf{n_5}\ominus\bar{\mathbf{n}}_\mathbf{5}),\mathbf{1})} + 2 I_{1/2}^{(\mathbf{1},(\mathbf{n_9}\ominus\bar{\mathbf{n}}_\mathbf{9})\otimes(\mathbf{n_9}\ominus\bar{\mathbf{n}}_\mathbf{9}))}\right)(F,R)\,. \end{aligned} \tag{4.12}$$

The first three terms contain only $\mathrm{tr}F^{2m}$ factors, whereas the last three contain only $\mathrm{tr}F^{2m+1}$ factors. The latter are therefore entirely responsible for pure Abelian and mixed Abelian-non Abelian anomalies.

Anomaly inflow

Consider now the inflows on the DpDq, DpOq and OpOq intersections, with $p,q = 5,9$. Due to the $\mathbf{Z}_2$ projection and the consequent appearance of twisted closed strings, there will be in this case two types of anomaly inflow associated to magnetic interactions of the various D-branes and O-planes. The first one is the usual one, involving the exchange of untwisted RR forms and the anomalous couplings (1.1) and (1.2) to them. The second one involves the exchange of twisted RR-forms and additional anomalous couplings to them, as shown in [31].

Consider first the inflow from the untwisted sector. The Chan-Paton representation λ is in this case $(\mathbf{n_5} \oplus \mathbf{\bar{n}_5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{n_9} \oplus \mathbf{\bar{n}_9})$. Using then the formulae derived in last section, one can then compute the inflow directly in terms of D=10 RR fields and then integrate over the compact space. Since we have already explicitly taken into account the singularities at the 16 orbifold fixed-points through 16 O5-planes in the background, we have to integrate over the flat torus T^4, rather than the orbifold $T^4/\mathbf{Z}_2$; otherwise, one would overcount the singularities[2]. The D9D9, D9O9 and O9O9 inflows vanish when integrated over T^4. The D5D5, D5O5 and O5O5 inflows also vanish, since the normal bundle to them is trivial and the corresponding anomalous couplings vanish as well. The only non-vanishing inflows come therefore from the D5D9, D5O9, O5D9 and O5O9 intersections. The normal bundle is null, and all the corresponding characteristic classes are equal to 1. One then finds in total

$$\begin{aligned} I_{D5D9}^{(un.)}(F,R) &= -\alpha^2 I_{1/2}^{(\mathbf{n_5}\oplus\mathbf{\bar{n}_5},\mathbf{n_9}\oplus\mathbf{\bar{n}_9})}(F,R) \;, \\ I_{D5O9+D9O5}^{(un.)}(F,R) &= 2\alpha^2 \left(I_{1/2}^{(\mathbf{n_5}\oplus\mathbf{\bar{n}_5},\mathbf{1})} + I_{1/2}^{(\mathbf{1},\mathbf{n_9}\oplus\mathbf{\bar{n}_9})}\right)(2F,R) \;, \\ I_{O5O9}^{(un.)}(R) &= 32\alpha^2 I_A(R) \;. \end{aligned} \tag{4.13}$$

Consider now the inflow from the twisted sector. It was shown in [31] that this cancels the Abelian anomaly coming from the charged hyper multiplets in the spectrum. The corresponding anomalous couplings are also responsible for a spontaneous breaking of $U(n_5) \times U(n_9)$ to $SU(n_5) \times SU(n_9)$. The gauge field dependence of these couplings was inferred in [31]. The complete result can be obtained by factorizing twisted sector magnetic interactions in the odd spin-structure, along the lines of [11]. The Möbius strip and Klein bottle amplitudes giving the BO and OO twisted magnetic interaction vanish trivially. One can thus immediatly conclude that O-planes do not have anomalous couplings to twisted RR forms. For the cylinder encoding BB twisted magnetic interactions, the gravitational part is unaltered, whereas the net effect of the twist is a conjugation of Chan-Paton wave-function through the symplectic matrix

$$M = \begin{pmatrix} 0 & I_{n_5} & 0 & 0 \\ -I_{n_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_9} \\ 0 & 0 & -I_{n_9} & 0 \end{pmatrix} \;. \tag{4.14}$$

D5-branes at fixed point I and D9-branes have therefore the following anomalous couplings to twisted RR forms $C_I^{tw.}$:

$$\begin{aligned} S_{D_5}^{tw.} &= \frac{\tilde{\mu}_5}{2} \int C_I^{tw.} \wedge \mathrm{ch}_{(\mathbf{n_5}\oplus\mathbf{\bar{n}_5},\mathbf{1})}(MF) \wedge \sqrt{\widehat{A}(R)}\Big|_{6-form} \\ S_{D_9}^{tw.} &= \frac{\tilde{\mu}_9}{8} \sum_{I=1}^{16} \int C_I^{tw.} \wedge \mathrm{ch}_{(\mathbf{1},\mathbf{n_9}\oplus\mathbf{\bar{n}_9})}(MF) \wedge \sqrt{\widehat{A}(R)}\Big|_{6-form} \end{aligned} \tag{4.15}$$

[2]This can be also seen by integrating the anomalous coupling (1.2) for the O9-plane of Type I theory on $T^4/\mathbf{Z}_2$, where the anomalous couplings for the 16 O5-planes in the $\mathbf{Z}_2$ model appear in the fixed-point contributions to $\int_{T^4/\mathbf{Z}_2} \sqrt{\widehat{L}}$.

where $\tilde{\mu}_{5,9} = \beta\sqrt{2\pi}$ and the integral is over the (5+1)-dimensional non-compact space. β is again a numerical coefficient, which will be fixed in the following. A D5-brane at fixed-point I couples therefore only to $C_I^{tw.}$, whereas a D9-brane wrapped on compact space couples to all the 16 $C_I^{tw.}$'s [31]. The net effect of the matrix M in the Chern character is:

$$\mathrm{ch}_{(\mathbf{n_5}\oplus\bar{\mathbf{n}}_5,\mathbf{1})}(MF) = \mathrm{ch}_{(\mathbf{n_5}\ominus\bar{\mathbf{n}}_5,\mathbf{1})}(F)\ , \quad \mathrm{ch}_{(\mathbf{1},\mathbf{n_9}\oplus\bar{\mathbf{n}}_9)}(MF) = \mathrm{ch}_{(\mathbf{1},\mathbf{n_9}\ominus\bar{\mathbf{n}}_9)}(F)\ . \tag{4.16}$$

Due to this property, it is evident that only odd powers of F appear in (4.15). Correspondingly, the associated twisted RR forms are 0-forms and their magnetic dual 4-forms, responsible for the inflow in this sector. The final result for the twisted inflow on BB intersections is then

$$\begin{aligned} I^{(tw.)}_{(D5+D9)(D5+D9)}(F,R) \;=\; -\frac{\beta^2}{4}\Big[& I_{1/2}^{(\mathbf{n_5}\ominus\bar{\mathbf{n}}_5,\mathbf{n_9}\ominus\bar{\mathbf{n}}_9)}(F,R) \qquad\qquad (4.17)\\ & +2\left(I_{1/2}^{((\mathbf{n_5}\ominus\bar{\mathbf{n}}_5)\otimes(\mathbf{n_5}\ominus\bar{\mathbf{n}}_5),\mathbf{1})} + I_{1/2}^{(\mathbf{1},(\mathbf{n_9}\ominus\bar{\mathbf{n}}_9)\otimes(\mathbf{n_9}\ominus\bar{\mathbf{n}}_9))}\right)(F,R)\Big] \end{aligned}$$

whereas the twisted inflow on BO and OO intersections vanish.

We see that the inflows (4.13) from the untwisted sector cancel precisely the one loop anomaly (4.9) of neutral fields and the non-Abelian one of charged fields in eq.(4.12), if one takes $\alpha = 1/2$. This is indeed the minimal value required by the D1-D5 Dirac quantization condition, since in this model the D5-branes are grouped into sets of 4 to make a dynamical D5-brane, due to the orientifold projection: $4(\alpha\sqrt{2\pi})^2 = 2\pi$. Similarly, the inflow (4.17) from the twisted sector cancels the remaining part of the anomaly in eq.(4.12), if one takes $\beta = 1$.

Summarizing, we have shown that the anomalous couplings required to cancel all the one-loop anomalies in the model do indeed arise. Similarly to the Type I case, the condition $n_5 = n_9 = 16$ comes from the requirement of vanishing irreducible terms in the anomaly polynomial, again because these would require unphysical propagating negative forms. This confirms what found in [31].

5. Conclusions

In this paper we have computed the anomalies for reduced chiral spinors and self-dual antisymmetric tensors living on D-brane/O-plane intersections and showed that these are cancelled through the inflow mechanism induced by the couplings (1.1) and (1.2). The main point is that in any consistent string theory model, the one-loop anomaly can be recast into a very particular form. The part arising from charged fields in the open string sector can be recast into the anomaly of a charged spinor in appropriate representations of the gauge group, whereas the part coming from neutral fields in the closed string sector has to combine into that of neutral self-dual antisymmetric tensors. The inflows on BB, BO and OO intersections then cancel these anomalies. The condition that the irreducible part of the anomaly polynomial cancels is mapped to the absence of inflow involving non-existent negative forms.

The relation between inflow and anomalies is very clear in string theory, where the two are related by the usual open-closed duality in the odd spin-structure of potentially divergent annulus, Möbius strip and Klein bottle diagrams involving D-branes and O-planes. In the open string channel, they are interpreted as anomalous one-loop amplitude on the worlvolumes of the corresponding D-branes and/or O-planes, whereas in the closed string channel they correspond to anomalous magnetic interactions responsible for the inflow mechanism. Extending [15], one could imagine to compute anomalies in string theory, by studying one-loop correlation functions with one unphysical external particle probing the breakdown of gauge-invariance. Presumably, the only effect of the unphyiscal vertex will be to implement the descent on the correlation of the physical vertices. Along the lines of [11], the remaining correlation can be exponentiated, reducing the amplitude to an effective supersymmetric partition function in the odd spin-structure. At that point, due to the topological nature of the amplitude, one can reduce the (1+1)-dimensional σ-model to 0+1, the computation boiling then down to that of Section 2.1.

It will be very interesting to discuss string theory compactifications in which the anomaly associated to the normal bundle is potentially non-vanishing. For instance, this is the case of D-branes, and eventually O-planes, wrapped on supersymmetric cycles of Calabi-Yau manifolds. Even more interestingly, gravity in transverse space seems to induce consistent gauge-like couplings for antisymmetric tensors. In presence of a non-trivial normal bundle, this allows, according to eq.(2.28), mixed anomalies for self-dual tensors in 4n dimensions.

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