Geodesic Active Fields - A Geometric Framework for Image Registration

Dominique Zosso*, Xavier Bresson† and Jean-Philippe Thiran*

*Signal Processing Laboratory (LTS5), Ecole Polytechnique Fédérale de Lausanne (EPFL), Station 11, CH-1015 Lausanne, Switzerland. {dominique.zosso,jp.thiran}@epfl.ch
†Department of Mathematics, University of California (UCLA), Los Angeles, CA 90095-1555, USA. xbresson@math.ucla.edu

Abstract. In this paper we present a novel geometric framework called geodesic active fields for general image registration. In image registration, one looks for the underlying deformation field that best maps one image onto another. This is a classic ill-posed inverse problem, which is usually solved by adding a regularization term. Here, we propose to embed the deformation field in a weighted minimal surface problem. The energy of the deformation field is measured with the Polyakov energy weighted by a suitable image distance, borrowed from standard registration models. Minimizing this weighted Polyakov energy drives the deformation field toward a minimal surface, while being attracted by the solution of the registration problem. Our geometric framework involves two important contributions. Firstly, our general formulation for registration works on any parametrizable, smooth and differentiable surface, including non-flat and multiscale images. Secondly, to the best of our knowledge, this method is the first re-parametrization invariant registration method introduced in the literature.

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INTRODUCTION

Image registration is the concept of mapping homologous points of different images, representing a same object. In practice, however, it is often difficult to establish homology in images based on this definition. For automatic image registration, it is commonplace to substitute homology by a measurable criterion of image dissimilarity, which is to be minimized by an unknown deformation field $\mathbf{u}$. The determination of this deformation field is an ill-posed inverse problem, requiring regularization. Here we propose to use a weighted version of the Beltrami framework to perform regularized registration.

The Beltrami Framework

In their seminal work, Sochen, Kimmel and Malladi introduced in [1] and [2] a general geometrical framework for low-level vision, based on an energy functional defined by Polyakov in [3]. In this framework, images are seen as surfaces or hypersurfaces embedded in higher dimensional spaces.

An n-dimensional manifold Σ with coordinates $\sigma^{1\dots n}$ is embedded in an m-dimensional manifold M with coordinates $X^{1\dots m}$, with $m > n$. The embedding map $X : \Sigma \mapsto M$ is given by m functions of n variables. For example, a 2D grey-level image can be seen as a surface embedded in 3D: $X : (x,y) \to (x,y,I)$. A Riemannian structure can be introduced: the metric $g_{\mu\nu}$ locally measures the distances on Σ, whereas on M distances are measured using h_{ij}.

To measure the weight of the mapping $X : \Sigma \mapsto M$, Sochen *et al.* [2] use the Polyakov energy, known from high energy physics [3]:

$$S[X^i, g_{\mu\nu}, h_{ij}] = \int \sqrt{g} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j h_{ij} \mathrm{d}^n\sigma, \tag{1}$$

where the Einstein summation convention is used, g is the determinant of the image metric, and $g^{\mu\nu}$ is its inverse, such that $g^{\mu\nu} g_{\nu\gamma} = \delta^\mu_\gamma$ (δ^μ_γ is the Kronecker delta). Naturally, the metric g is chosen as the induced metric, obtained by the

pullback-relation: $g_{\mu\nu} = h_{ij}\partial_\mu X^i \partial_\nu X^j$. Under such a metric, the Polyakov energy shortens to:

$$S = \int \sqrt{g}\mathrm{d}^n\sigma, \tag{2}$$

which represents the area of the embedded image surface. Assuming the embedding is in a Euclidean space with Cartesian coordinates, the corresponding gradient descent equation is

$$\partial_t X^i = -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta X^i} = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu X^i) \equiv H^i, \tag{3}$$

known as the Beltrami flow, where H^i denotes the i-th component of the mean curvature vector of the manifold.

With geodesic active fields (GAF), we propose to regularize a deformation field using a weighted Beltrami embedding, where the weighting attracts the deformation field toward a solution of the registration problem.

GEODESIC ACTIVE FIELDS

In this section we define the GAF framework for image registration. Now, the deformation field is embedded as a mapping between the n-dimensional image domain and a m-dimensional space, where $m > n$. This is achieved by letting the components of the deformation field become additional dimensions of the embedding space. The embedded manifold then evolves toward a weighted minimal surface, while being attracted by a deformation field that brings the two images into registration. The main strengths of this framework are twofold: The freedom to register images on any Riemannian manifold, i.e., on any smooth and parametrized surface, and its invariance under re-parametrization of the proposed energy, like the GAC energy [4] for the segmentation problem.

The General Case

In the general form, we register a pair of n-dimensional images defined on a Riemannian domain Ω with coordinates $\mathbf{x} = (x_1, \ldots, x_n)$. The deformation field acts along $p \leq n$ dimensions, i.e., $\mathbf{u} : \Omega \mapsto \mathbb{R}^p, \mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \ldots, u_p(\mathbf{x}))$. At the very core of GAF, the deformation field is seen as a surface or hypersurface embedded in a higher dimensional space, much like images embedded with the Beltrami framework [2]. On these embeddings, a Riemannian structure can be introduced: the metric $g_{\mu\nu}$ locally measures the distances on the embedded deformation field, whereas in the higher dimensional embedding space distances are measured using h_{ij}.

The embedding X and the metric tensors h_{ij} and $g_{\mu\nu}$ are chosen as follows:

$$\begin{cases} X : (x_1, \ldots, x_n) \to (x_1, \ldots, x_n, u_1, \ldots, u_p) \\ h_{ij} \text{ is arbitrary} \\ g_{\mu\nu} = \partial_\mu X^i \partial_\nu X^j h_{ij}, \end{cases} \tag{4}$$

where $x_1, \ldots, x_n$ denote the spatial components of the image and $u_1, \ldots, u_p$ are the components of the dense deformation field.

If the image domain is Euclidean, we may choose the metric tensor

$$h_{ij} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{\beta^2, \ldots, \beta^2}_{p}), \tag{5}$$

where β defines the aspect ratio between spatial and feature dimensions.

Based on this choice, we define the following general registration energy functional, which is a weighted Polyakov energy [5, 3], and the corresponding direct minimizing flow for the geodesic active fields (GAF):

$$\begin{cases} E_{GAF} = \int f\sqrt{g}\,\mathrm{d}\mathbf{x} \\ \partial_t u_i = fH^{n+i} + \partial_k f g^{\mu\nu}\partial_\mu X^k \partial_\nu u_i - \frac{m \cdot n}{2}\partial_k f h^{k(n+i)}, \quad 1 \leq i \leq p, \end{cases} \tag{6}$$

where the weighting function $f = f(\mathbf{x}, \mathbf{u})$ is arbitrary, as detailed below.

Weighting Function

The purpose of the weighting function f is to drive the deformation field toward minimal surfaces that bring the two images into registration. As such, the flow must stop when the deformed image perfectly matches the target image. Hence, the weighting function is naturally chosen to be an image distance metric, which approaches zero when the two images locally match. An intuitive primer for monomodal image registration is the squared error metric [6], leading to:

$$f(\mathbf{x},\mathbf{u}) = 1 + \alpha \cdot (\mathcal{M}(\mathbf{x}+\mathbf{u}) - \mathcal{F}(\mathbf{x}))^2, \tag{7}$$

where $\mathcal{F}$ and $\mathcal{M}$ refer to the fix and moving images, respectively. In other cases, e.g., for stereo vision, the absolute error norm can be more appropriate:

$$f(\mathbf{x},\mathbf{u}) = 1 + \alpha \cdot \left|\mathcal{M}(\mathbf{x}+\mathbf{u}) - \mathcal{F}(\mathbf{x})\right|, \tag{8}$$

For other examples, e.g., suitable for multimodal image registration, or more complicated deformation models, the reader is referred to [7].

The Stereo Vision Example

Let us consider a case of stereo vision disparity recovery of 2D images. Without loss of generality, this choice reduces the co-dimension of the deformation field and heavily simplifies the notation. The stereo embedding is defined as:

$$\begin{cases} X : (x,y) \to (x,y,u) \\ h_{ij} = \mathrm{diag}(1,1,\beta^2) \\ g_{\mu\nu} = \begin{bmatrix} 1+\beta^2 u_x^2 & \beta^2 u_x u_y \\ \beta^2 u_x u_y & 1+\beta^2 u_y^2 \end{bmatrix} \\ g = 1+\beta^2|\nabla u|^2. \end{cases} \tag{9}$$

Put into the general equations we get the following energy functional and minimizing flow:

$$\begin{cases} E_{GAF} = \int f\sqrt{1+\beta^2|\nabla u|^2}\mathrm{d}x\mathrm{d}y \\ \partial_t u = fH^u + \partial_k f g^{\mu\nu}\partial_\mu X^k \partial_\nu u - \frac{3}{\beta^2} f_u. \end{cases} \tag{10}$$

If $\beta \to \infty$, the 1 in the GAF energy becomes negligible, and the energy approaches the TV-norm, well-known in image denoising [8, 9]. If, however, $\beta \to 0$, then the minimizing flow reproduces the isotropic heat diffusion [2].

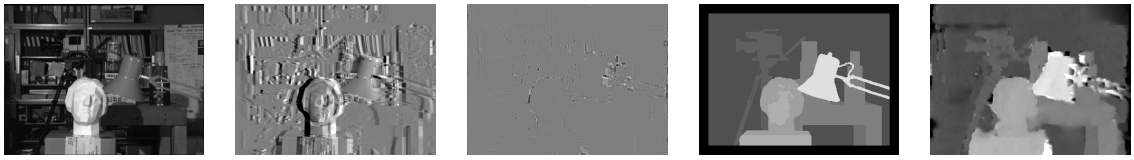
A Splitting Scheme

Since we are dealing with discretely sampled images, we may rewrite the GAF energy in terms of a standard vectorial inner product:

$$E_{GAF} = \langle F, G\rangle = \sum_{i=1}^{N} F_i G_i, \quad F, G \in \mathbb{R}^N, \tag{11}$$

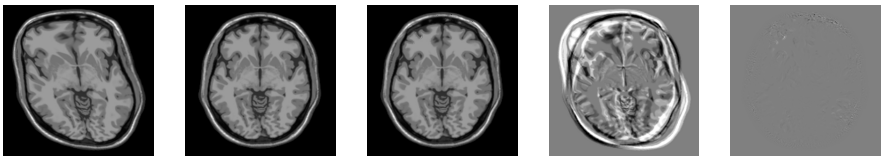
where F_i and G_i are the N samples of the weighting function and the square root of the metric tensor determinant, respectively.

Now, instead of a direct implementation of the flow (6) using a simple forward Euler scheme, we suggest to use a splitting scheme to speed up the optimization process. This scheme minimizes the weighting function term and the metric tensor term of the GAF energy separately, but tightly coupled through an augmented Lagrangian s.a. [10, 11].



(a) $\mathcal{F}$ (b) $\mathcal{M} - \mathcal{F}$ (c) $\mathcal{M}' - \mathcal{F}$ (d) (e)

FIGURE 1. (a) `tsukuba` test image (b)–(c) Mismatch before/after registration. (d) Ground truth (e) GAF result.



(a) $\mathcal{M}$ (b) $\mathcal{F}$ (c) $\mathcal{M}'$ (d) $\mathcal{M} - \mathcal{F}$ (e) $\mathcal{M}' - \mathcal{F}$

FIGURE 2. (a)–(b) Fix/moving image. (c) Warped moving image. (d)–(e) Mismatch before/after registration.

RESULTS

An example of stereo vision depth recovery problem is shown in Fig. 1. The image pair `tsukuba` is a well known test image, taken from the middlebury benchmark set for stereo vision. The registration is set up using the absolute error weighting function (8).

The second case deals with 2D registration of a highly misaligned monomodal medical image pair. An axial slice through a T1 MRI volume is heavily deformed by a given 2D deformation field. The images have a resolution of 317×317 pixels. Registration is set up with the squared error weighting function (7). The image pair and initial error are illustrated along with the registration results in Fig. 2.

CONCLUSIONS

Geodesic Active Fields represent a novel, geometric framework for image registration. It can be considered as a generalization of the popular demons algorithm [12] in various respects. First, it is directly applicable to non-Euclidean and multiscale images, and not restricted to Cartesian images only. Also, the GAF energy can be shown to be parametrization invariant, which is a rare but important feature in image registration. Further, the anisotropy of the geometric regularization can be tuned between TV-like and Gaussian diffusion. Moreover, there is a wide choice of weighting functions which measure the image mismatch locally, including well-known metrics such as L_1, L_2 or local entropy. Finally, the use of a splitting scheme allows to optimize the GAF energy more efficiently.

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