The Vehicle Routing Problem with Discrete Split Deliveries and Time Windows

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Outline

- Problem description and applications
- MILP formulation and Column generation
- Branch-and-price algorithm
- Computational results
- Conclusions and further research
The Vehicle Routing Problem (VRP)

Given

- set of customers with demands to be served (within time windows: VRPTW);
- set of capacitated vehicles available at a single depot;
The Vehicle Routing Problem (VRP)

Objective

- design the optimal minimum-cost routes for vehicles,
- such that every customer is visited exactly once.
VRP with Split Deliveries (SDVRP)

Splittable demand

- demand can be split and thus served by more than one vehicle;
- customers can be visited more than once.
VRP with Discrete Split Deliveries (DSDVRP)

Demand is discretized in items

- variant of VRP with (continuous) split deliveries;
- the demand of each customer is represented by a set of items;
- demand can be split but items cannot.

Demand is delivered in combinations of items

- items are grouped and delivered in combinations (subsets of items);
- for every customer, feasible combinations of items are predefined and known;
- some combinations of items are not allowed.

Sierksma & Tijssen (1998); Nakao & Nagamochi (2007); Ceselli et al. (2009).
DSDVRP with Time Windows (DSDVRPTW)

Service time is quantity-dependent

- we define *service time* the time needed to perform the delivery once the vehicle is arrived at customer’s location;
- in our model, service time depends on the quantity delivered;
- in particular, a specific service time is associated to every feasible combination of items;
- we relax the usual assumption of constant service times.
Known properties

**Property 1.** The SDVRPTW is NP-Hard.

- Property 1 holds for DSDVRPTW.

**Property 2.** There exists an optimal solution of the SDVRPTW in which no two routes have more than one split demand in common.

- Property 2 doesn’t hold for DSDVRPTW (because of discrete demand and quantity-dependant service times).
Field Technician Scheduling Problem

Xu & Chiu (2001)

Problem description

- different types of jobs which require different skills;
- each technician is specialized in a field with certain skills;
- time windows on job starting and completion;
- assignment problem (jobs to technicians) + scheduling problem, where the duration of a job depends on the assignment.

Objective

- maximize the number of jobs completed within a time frame.
TBAP with QC assignment in container terminals

Giallombardo, Moccia, Salani & Vacca (2010)

Problem description

- Tactical Berth Allocation Plan (TBAP): assignment and scheduling of ships to berths;
- Quay-Cranes (QC) assignment: a QC profile (number of QCs per shift) is assigned to each ship;
- feasible profiles can vary in length (number of shifts dedicated to the ship) and in size (number of QCs dedicated to the ship in each active shift);
- time windows on ship arrival and on berth availabilities.

Objective

- maximize the value of chosen profiles.
Modeling the Discrete Split Delivery VRPTW

- $G = (V, E)$ complete graph with $V = \{0\} \cup N$;
- $(c_{ij}, t_{ij})$: cost and travel time of arc $(i, j) \in E$;
- $N$: set of customers $\{1, ..., n\}$; node $\{0\}$ represents the depot;
- $K$: set of vehicles with identical capacity $Q$;
- $R$: set of items; $R = \bigcup_{i \in N} R_i$, $R_i \cap R_j = \emptyset \ \forall i \neq j, \ i, j \in N$;
- $C$: set of combinations of items; $C = \bigcup_{i \in N} C_i$, $C_i \cap C_j = \emptyset \ \forall i \neq j, \ i, j \in N$;
- $e^r_c$: 1 if item $r \in R$ belongs to combination $c \in C$;
- $t_c$: service time of combination $c \in C$ such that $\max_{r \in R} e^r_c t_r \leq t_c \leq \sum_{r \in R} e^r_c t_r$;
- $q_c$: size of combination $c \in C$, $q_c = \sum_{r \in R} e^r_c t_r$;
- $[a_i, b_i]$: time window for customer $i \in N$. 

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Arc-flow formulation

Decision variables

- $x_{ij}^k$ binary: 1 if arc $(i, j) \in E$ is used by vehicle $k \in K$, 0 otherwise;
- $y_c^k$ binary: 1 if vehicle $k \in K$ delivers combination $c \in C$, 0 otherwise;
- $T_{ki}^k \geq 0$: time when vehicle $k \in K$ arrives at customer $i \in N$.

Objective function

- minimize the total traveling costs:
  $$z^* = \min \sum_{k \in K} \sum_{(i,j) \in E} c_{ij} x_{ij}^k$$

Constraints

- flow and precedence constraints;
- demand-satisfaction constraints;
- time-windows constraints;
- capacity constraints.
Arc-flow formulation

Flow and linking constraints

\[ \sum_{j \in V} x_{0j}^k = 1 \quad \forall k \in K, \quad (1) \]

\[ \sum_{j \in V} x_{ij}^k - \sum_{j \in V} x_{ji}^k = 0 \quad \forall k \in K, \forall i \in V, \quad (2) \]

\[ \sum_{j \in V} x_{ij}^k = \sum_{c \in C_i} y_c^k \quad \forall k \in K, \forall i \in N, \quad (3) \]

Covering constraints

\[ \sum_{k \in K} \sum_{c \in C} e_r^c y_c^k = 1 \quad \forall r \in R, \quad (4) \]

\[ \sum_{c \in C_i} y_c^k \leq 1 \quad \forall k \in K, \forall i \in N, \quad (5) \]
Arc-flow formulation

**Precedence constraints**

\[
T^k_i + \sum_{c \in C_i} t_c y^k_c + t_{ij} - T^k_j \leq (1 - x^k_{ij})M \quad \forall k \in K, \forall i \in N, \forall j \in V, \tag{6}
\]

\[
T^k_i - t_{0i} \geq (1 - x^k_{0i})M \quad \forall k \in K, \forall i \in N, \tag{7}
\]

**Time windows**

\[
T^k_i \geq a_i \sum_{j \in V} x^k_{ij} \quad \forall k \in K, \forall i \in N, \tag{8}
\]

\[
T^k_i + \sum_{c \in C_i} t_c y^k_c \leq b_i \sum_{j \in V} x^k_{ij} \quad \forall k \in K, \forall i \in N, \tag{9}
\]

**Capacity constraints**

\[
\sum_{c \in C} q_c y^k_c \leq Q \quad \forall k \in K. \tag{10}
\]
Dantzig-Wolfe reformulation

- $P$: set of feasible routes (constraints (1)-(3) + (5)-(10));
- $c_p$: cost of route $p \in P$;
- $e^r_p$: binary parameter equal to 1 if item $r \in R$ is delivered in route $p \in P$;
- $\lambda_p$: binary decision variable equal to 1 if route $p \in P$ is chosen;

Master problem (path formulation)

$$\min \sum_{p \in P} c_p \lambda_p$$  \hspace{1cm} (11)

$$\sum_{p \in P} e^r_p \lambda_p = 1 \quad \forall r \in R,$$  \hspace{1cm} (12)

$$\sum_{p \in P} \lambda_p \leq |K|,$$  \hspace{1cm} (13)

$$\lambda_p \geq 0 \quad \forall p \in P.$$  \hspace{1cm} (14)
Column generation scheme

The so-called Restricted Master Problem (RMP) is repeatedly solved on a subset of variables $\lambda$, which otherwise would be an exponential number.

At each iteration of column generation we add profitable variables not yet in the formulation by solving the pricing subproblem.

**Pricing subproblem**

Find the route (column) $p$ with the minimum reduced cost $\tilde{c}_p$:

$$p^* = \arg \min_{p \in P} \{\tilde{c}_p\} = \arg \min_{p \in P} \{c_p - \sum_{r \in R} \pi_r e^r_p - \pi_0\}$$ (15)

where $\pi_r$ are the dual variables associated to constraints (12) and $\pi_0$ is the dual variable associated to constraint (13).
Column generation scheme

Generation of columns

- solve the pricing subproblem (identify the min-red-cost column);
- if $\tilde{c}_{p^*} < 0$, then column $p^*$ is added to the (restricted) master problem;
- otherwise, the current master problem solution is proven to be optimal.

Remarks

- the pricing subproblem is an Elementary Resource Constrained Shortest Path Problem solved via dynamic programming;
- the underlying network has one node for each combination and transit time equal to ($t_{ij} + t_c$).
Branch & Price for the DSDVRPTW

- exact algorithm based on branch-and-bound;
- column generation at each node of the search tree;
- pricing solved using bi-directional dynamic programming;
- branching rules:
  1. total number of vehicles;
  2. number of vehicles visiting a customer;
  3. flow on one arc;
  4. flow on two consecutive arcs.
- cutting: 2-path inequalities at the root node.
Instances

- derived from Solomon's data sets R1, C1 and RC1 for the VRPTW;
- \( N = 25, 50 \) customers;
- \( Q = 30, 50, 100; \)
- the demand of each customer is discretized in 12 items;
- we generated 3 scenarios:

<table>
<thead>
<tr>
<th>scenario</th>
<th>combinations</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>full, 50-50%;</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>full, 50-50%, 75-25%;</td>
</tr>
<tr>
<td>C</td>
<td>7</td>
<td>full, 50-50%, 75-25%, 90-10%;</td>
</tr>
<tr>
<td>O</td>
<td>1</td>
<td>full (unsplit case).</td>
</tr>
</tbody>
</table>
Computational results: $n = 25$ customers

<table>
<thead>
<tr>
<th>class</th>
<th>nb_inst</th>
<th>$Q$</th>
<th>A</th>
<th></th>
<th>B</th>
<th></th>
<th>C</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>nb_solved</td>
<td>t</td>
<td>nb_solved</td>
<td>t</td>
<td>nb_solved</td>
<td>t</td>
</tr>
<tr>
<td>R1</td>
<td>12</td>
<td>30</td>
<td>12</td>
<td>7</td>
<td>12</td>
<td>75</td>
<td>8</td>
<td>466</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>12</td>
<td>6</td>
<td>12</td>
<td>60</td>
<td>12</td>
<td>430</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>12</td>
<td>9</td>
<td>12</td>
<td>41</td>
<td>12</td>
<td>113</td>
</tr>
<tr>
<td>C1</td>
<td>9</td>
<td>30</td>
<td>4</td>
<td>1108</td>
<td>0</td>
<td>x</td>
<td>0</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>9</td>
<td>37</td>
<td>4</td>
<td>2137</td>
<td>0</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>7</td>
<td>706</td>
<td>4</td>
<td>705</td>
<td>2</td>
<td>1876</td>
</tr>
<tr>
<td>RC1</td>
<td>8</td>
<td>30</td>
<td>2</td>
<td>1988</td>
<td>0</td>
<td>x</td>
<td>0</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>8</td>
<td>3</td>
<td>8</td>
<td>11</td>
<td>8</td>
<td>35</td>
</tr>
</tbody>
</table>

Time limit: 1 hour.
Computational results: $n = 50$ customers

<table>
<thead>
<tr>
<th>class</th>
<th>nb_inst</th>
<th>$Q$</th>
<th>A \nb_solved</th>
<th>t</th>
<th>B \nb_solved</th>
<th>t</th>
<th>C \nb_solved</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>12</td>
<td>30</td>
<td>1</td>
<td>1010</td>
<td>0</td>
<td>x</td>
<td>0</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>50</td>
<td>3</td>
<td>1572</td>
<td>1</td>
<td>385</td>
<td>0</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>3</td>
<td>1035</td>
<td>2</td>
<td>167</td>
<td>2</td>
<td>535</td>
</tr>
<tr>
<td>RC1</td>
<td>8</td>
<td>50</td>
<td>7</td>
<td>54</td>
<td>6</td>
<td>902</td>
<td>0</td>
<td>x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td>8</td>
<td>529</td>
<td>6</td>
<td>809</td>
<td>3</td>
<td>2832</td>
</tr>
</tbody>
</table>

Time limit: 1 hour.
Optimal solutions for class RC1, $n = 50, Q = 50$

<table>
<thead>
<tr>
<th>id</th>
<th>O $z_{IP}$ veh t</th>
<th>A $z_{IP}$ veh t</th>
<th>B $z_{IP}$ veh t</th>
<th>C $z_{IP}$ veh t</th>
</tr>
</thead>
<tbody>
<tr>
<td>rc101</td>
<td>1713.2 20 0</td>
<td>1708.9 20 13</td>
<td>1708.3 20 594</td>
<td>x</td>
</tr>
<tr>
<td>rc102</td>
<td>1704.3 20 0</td>
<td>1700.5 20 62</td>
<td>1700.5 20 1938</td>
<td>x</td>
</tr>
<tr>
<td>rc103</td>
<td>1703.4 20 1</td>
<td>1696.8 20 37</td>
<td>1696.8 20 427</td>
<td>x</td>
</tr>
<tr>
<td>rc104</td>
<td>1702.2 20 1</td>
<td>1696.7 20 54</td>
<td>1696.7 20 677</td>
<td>x</td>
</tr>
<tr>
<td>rc105</td>
<td>1703.9 20 0</td>
<td>1700.1 20 73</td>
<td>1700.1 20 1132</td>
<td>x</td>
</tr>
<tr>
<td>rc107</td>
<td>1704.1 20 1</td>
<td>1698.6 20 58</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>rc108</td>
<td>1702.2 20 2</td>
<td>1696.7 20 83</td>
<td>1696.7 20 645</td>
<td>x</td>
</tr>
</tbody>
</table>
Summing up

- 76% (A), 60% (B) and 48% (C) instances solved for 25 customers;
- 25% (A), 17% (B) and 6% (C) instances solved for 50 customers;
- 96% (R1), 37% (C1) and 36% (RC1) instances solved for 25 customers;
- 11% (R1), 0% (C1) and 42% (RC1) instances solved for 50 customers;
- difficulty increases with the number of customers and combinations;
- split deliveries are more frequent with small values of $Q$;
- in some cases, split deliveries not only decrease the total traveling costs but also allow to save one vehicle.
Conclusions

- finding optimal solutions is difficult, already with a few combinations;

- only a limited number of instances with 50 customers could be solved;

- the bottleneck is the pricing problem:
  - the underlying network is huge (one node per each combination!)
  - how to efficiently handle this feature of the problem?
Ongoing work

New framework: two-stage column generation

- methodology to accelerate an overall B&P algorithm via generation of compact formulation variables;
- useful when the compact formulation exhibits a large number of variables;
- useful to detect sub-optimal compact formulation variables.

Application to the DSDVRPTW

- we start with a subset of combinations (corresponding to a subset of $y_c$ variables);
- we compute a bound of the reduced cost for variables $y_c$ not yet considered;
- we add the most profitable variable and we iterate.
Thanks for your attention!

More details on transp-or.epfl.ch: