Highly complex decision issues $\Rightarrow$ tendency to decentralized the management

- Huge number of control parameters
- Feedback (*i.e.* non-linearity) in the underlying dynamics
- Ubiquitous presence of randomness in the dynamics
- ...

$\Downarrow$

Decisions based on **limited rationality** $\Rightarrow$ Rigid pre-planning offers poor performance

- mutual interactions $\Downarrow$ self-organization

Autonomous agents **might better perform** than an effective central controller

$\Downarrow$ **goal of today’s presentation**

**Exhibit a solvable model showing performance of decentralized control**
A Simple Model for Competitive Dynamics

$$\dot{X}_k(t) = v_k(t) + \gamma_k \left[ \mathbb{1}_k(\bar{X}(t), X_k(t)) \right] + q_k(v_k(t)) dB_{k,t}, \quad k = 1, 2, \ldots, N.$$  

Multi-agent interactions:

$$\mathbb{1}_k(\bar{X}(t), X_k(t)) = \frac{1}{\mathcal{N}_k} \sum_{j \neq k} \mathcal{I}_k(X_j(t)), \quad \mathcal{N}_k := \text{neighbourhood of agent } k,$$

$$\mathcal{I}_k(X_j(t)) = \begin{cases} 0 & \text{if } 0 \leq X_j(t) < X_k(t), \quad \text{(velocity unchanged)}, \\ 1 & \text{if } X_k(t) \leq X_j(t) < X_k(t) + U, \quad \text{(accelerate)}, \\ 0 & \text{if } X_j(t) > X_k(t) + U, \quad \text{(velocity unchanged}). \end{cases}$$

(U := "mutual influence" interval)
A Simple Model for Competitive Dynamics - Applications

Logistics

Economy

Human Mimetism

...
Homogeneous Population of Agents

\[ dX_k(t) = \left[ v(t) + \gamma \Pi(\tilde{X}(t), X_k(t)) \right] dt + q dB_{k,t}. \]

\[ \text{diffusion process} \]

Fokker - Planck diffusion equation:

\[ \frac{\partial}{\partial t} P(\vec{x}, t) = -\sum_k \frac{\partial}{\partial x_k} \left[ D_{k,v}(\vec{x}, t) P(\vec{x}, t) \right] + \frac{1}{2} q^2 \sum_k \frac{\partial^2}{\partial x_k^2} \left[ P(\vec{x}, t) \right], \]

\[ P(\vec{x}, t) := \text{conditional probability density} \]
Mean-Field Dynamics for Homogeneous Agents

\[ \mathcal{N}_k \equiv \mathcal{N} \to \infty \Rightarrow \text{Mean-Field Dynamics (MFD)} \]

\[ \downarrow \quad \text{dynamics for a representative effective agent} \]

trajectories point of view

\[ \frac{1}{\mathcal{N}} \sum_{j \neq k}^{\mathcal{N}} \mathcal{T}(X_j(t)) \]

\[ \approx \int_x^{x+U} P(x, t) \, dx \]

proportion of velocity-active agents acting on \( k \)

proportion of representative agents located in \([x, x+U]\)

\[ \downarrow \]

Effective Fokker-Planck equation:

\[ \frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \left\{ \left[ v(t) + \gamma \left( \int_x^{x+U} P(x, t) \, dx \right) \right] P(x, t) \right\} + \frac{1}{2} q^2 \frac{\partial^2}{\partial x^2} [P(x, t)], \]

non-linear and non-local field equation
Small Influence Region - Burgers’ Equation Dynamics

Small values of $U$ $\Rightarrow$ Taylor expand up to 1st order in $U$

$$\downarrow \quad \int_{x}^{x+U} P(x, t) dx \simeq U P(x, t)$$

$$\frac{\partial}{\partial t} P(x, t) = - \frac{\partial}{\partial x} \left\{ [v(t) + \gamma U P(x, t)] P(x, t) \right\} + \frac{1}{2} q^2 \frac{\partial^2}{\partial x^2} [P(x, t)]$$

\textit{non-linear but local drift field}

$$t \mapsto \tau = \gamma t \quad \downarrow \quad x \mapsto z = \frac{x - \int_{0}^{t} v(s) \, ds}{2U}$$

Burgers’ Equation (to be solved with initial condition $P(z, t) = \delta(z) \Theta(z)$)

$$\dot{P}(z, t) = \frac{1}{2} \frac{\partial}{\partial z} \left[ P(z, t)^2 \right] + \left[ \frac{q^2}{8U^2 \gamma} \right] \frac{\partial^2}{\partial z^2} [P(z, t)]$$
Stylized Model for Smart Parts Dynamics

Burgers’ Eq. \iff logarithmic transformation (Hopf - Cole) \implies Heat Eq.

\[ P(y, t) = -\frac{q^2}{4\gamma U^2} \frac{\partial}{\partial y} \ln \left[ 1 + \frac{(e^R - 1)}{2} \text{Erfc} \left( \frac{y}{q\sqrt{t}} \right) \right] = \]

\[ = \frac{1}{R} \left[ \frac{(e^R - 1)}{\sqrt{\pi q^2 t}} e^{-\frac{y^2}{q^2 t}} \right] \cdot \frac{1 + \frac{(e^R - 1)}{2} \text{Erfc} \left( \frac{y}{q\sqrt{t}} \right)}{1 + \frac{(e^R - 1)}{2} \text{Erfc} \left( \frac{y}{q\sqrt{t}} \right)} := \frac{1}{R} \left( e^R - 1 \right) \mathcal{G}(y, t) \]

Typical shape of $P(y, t)$ for various $R := \frac{4U^2\gamma}{q^2}$ factors (viewed from the relative moving frame)

Normalization and positivity are visually manifest !!
Stylized Model for Smart Parts Dynamics

Benefit of Competition - Noise Induced Transport Enhancement

Position probability distribution: without interaction, with interactions

- Additional traveled distance when \( R = \frac{4\gamma U^2}{q^2} \to \infty \): \( \langle X(t) \rangle_{t \to \infty} \simeq \frac{4U}{3} \sqrt{\gamma t} \),

- Additional traveled distance when \( R = \frac{4\gamma U^2}{q^2} \to 0 \): \( \langle X(t) \rangle_{t \to \infty} \simeq 0 \).
Optimal Effective Centralized Control

Controlled diffusion process:

\[ dY_t = c(Y, t) \, dt + q \, dB_t, \quad Y_0 = 0, \quad (0 \leq t \leq T), \]

\[ \text{effective central controller} \]

\[ \text{initial condition} \]

\[ \Downarrow \quad \text{(Fokker-Planck equation)} \]

\[ \frac{\partial}{\partial t} P_c(y, t) = -\frac{\partial}{\partial y} [c(y, t) P_c(y, t)] + \frac{q^2}{2} \frac{\partial^2}{\partial y^2} P_c(y, t) \]

Construct a drift controller \( c(Y, t) \) which, for time \( T \), fulfills

\[ P_c(y, T) = P(y, T) \]

\[ \text{Prob. density with central controller} \]

\[ \text{Prob. density due to agent interactions} \]

Burgers’ exact solution
Optimal Effective Centralized Control (continued)

Introduce a utility function $J_{\text{central}, T} [c(y, t; T)]$ defined as:

$$J_{\text{central}, T} [c(y, t; T)] = \langle \int_0^T \frac{c^2(y, s; T)}{2q^2} ds \rangle,$$

where $c(y, s; T)$ is the cost rate and $\rho(y, s)$ is the underlying stochastic process.

(\langle \cdot \rangle := \text{average over the realization of underlying stochastic process})

Optimal Control Problem

Construct an optimal drift $c^*(y, t; T)$ such that:

$$J_{\text{central}, T} [c^*(y, t; T)] \leq J_{\text{central}, T} [c(y, t; T)].$$
The **Dai Pra** Solution of the Optimal Control Problem

**Optimal drift controller:**

\[
c^* (y, t; T) = \frac{\partial}{\partial y} \ln [h(y, t)],
\]

\[
h(y, t) = \int_{\mathbb{R}} \mathbb{G} [(z - y), (T - t)] \frac{P(z, T)}{G(z, t)} dz.
\]


**Minimal cost:**

\[
J_{\text{central}, T} [c^* (y, t; T)] = \frac{N}{\# \text{ population}} \cdot \mathcal{D}(P \| \mathcal{G}) = \begin{cases} 
0 & \text{for } t = 0, \\
N \frac{R^2}{2} + N \ln \left[ \frac{(e^R-1)}{R} \right] & \text{for } t > 0.
\end{cases}
\]
Decentralized Agent Control - Cost Estimation

Cost $J_{\text{agents},T}$ for decentralized evolution during time horizon $T$:

$$J_{\text{agents},T} := N \cdot \rho \cdot \int_0^T ds \Phi(s)$$

- $\rho = \frac{\gamma^2 U^2}{2q^2}$ := individual cost rate function,
- $\Phi(t) \in [0, 1]$ := proportion of interacting agents at time $t$.

Cost upper-bound, reached when $\Phi(t) \equiv 1$

$$J_{\text{agents},T} \leq N\rho T$$
Costs Comparison - Centralized vs Decentralized

- Average Costs Estimation
- Costs Comparison - Centralized vs Decentralized
- Time horizon $T$
- Cumulative costs
- Actual decentralized costs
- Upper-bounded decentralized costs
- Centralized costs
- Time horizons for which agent interactions beat the optimal effective centralized controller
- Kullback-Leibler entropy

Olivier Gallay (EPFL)
To Summarize and to Somehow "Philosophically" Conclude

The stylized model *cartoons basic and somehow "universal" features:*

- **Agents' mimetic interactions produce an emergent structure** - (here a "shock"-like wave),
- **Competition enhances global transport flow** - (here a $\sqrt{t}$-increase of the traveled distance),
- **Self-organization via autonomous agents interactions can reduce costs.**