Centralized Versus Decentralized Control - A Solvable Stylized Model in Transportation Logistics

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Highly complex decision issues $\Rightarrow$ tendency to decentralize the management

- Huge number of control parameters
- Feedback (*i.e.* non-linearity) in the underlying dynamics
- Ubiquitous presence of randomness in the dynamics
- ...

$\downarrow$

Decisions based on *limited rationality* $\Rightarrow$ Rigid pre-planning offers poor performance

- Mutual interactions $\downarrow$ self-organization

Autonomous agents *might better perform* than an effective central controller

$\downarrow$

*goal of today’s presentation*

**Exhibit a solvable model showing performance of decentralized control**
A Simple Model for Local Imitation Dynamics

\[ \dot{X}_k(t) = v_k(t) + \gamma_k \mathbb{I}_k(\vec{X}(t), X_k(t)) + q_k(v_k(t))dB_{k,t}, \quad k = 1, 2, \ldots, N. \]

Multi-agent interactions:

\[ \mathbb{I}_k(\vec{X}(t), X_k(t)) = \frac{1}{\mathcal{N}_k} \sum_{j \neq k} \mathcal{I}_k(X_j(t)), \quad \mathcal{N}_k := \text{neighbourhood of agent } k, \]

\[ \mathcal{I}_k(X_j(t)) = \begin{cases} 
0 & \text{if } 0 \leq X_j(t) < X_k(t), \quad \text{(velocity unchanged)}, \\
1 & \text{if } X_k(t) \leq X_j(t) < X_k(t) + U, \quad (U > 0), \quad \text{(accelerate)}, \\
0 & \text{if } X_j(t) > X_k(t) + U, \quad \text{(velocity unchanged)}. 
\end{cases} \]

\((U := "\text{mutual influence}" \text{ interval})\)
A Simple Model for Imitation Dynamics - Applications

Logistics

Economy

Human Mimetism

...
Homogeneous Population of Agents

\[ dX_k(t) = \left[ v(t) + \gamma \Pi(\bar{X}(t), X_k(t)) \right] dt + q dB_{k,t}. \]

\( \Pi \) := drift field \( D_{k,v}(x,t) \)

\( dB \) := indep. White Gaussian Noise

\[ \downarrow \text{diffusion process} \]

**Fokker - Planck diffusion equation:**

\[ \frac{\partial}{\partial t} P(\bar{x}, t) = -\sum_k \frac{\partial}{\partial x_k} \left[ D_{k,v(\bar{x}, t)} P(\bar{x}, t) \right] + \frac{1}{2} q^2 \sum_k \frac{\partial^2}{\partial x_k^2} \left[ P(\bar{x}, t) \right], \]

\( P(\bar{x}, t) := \text{conditional probability density} \)
Mean-Field Dynamics for Homogeneous Agents

\[ \mathcal{N}_k \equiv \mathcal{N} \to \infty \quad \Rightarrow \quad \text{Mean-Field Dynamics (MFD)} \]

\[ \Downarrow \quad \text{dynamics for a representative effective agent} \]

**trajectories point of view**

\[ \frac{1}{N} \sum_{j \neq k} \mathcal{I}(X_j(t)) \]

proportion of velocity-active agents acting on \( k \)

\[ \approx \int_x^{x+U} P(x, t) \, dx \]

proportion of representative agents located in \([x, x+U]\)

\[ \Downarrow \]

**Effective Fokker-Planck equation:**

\[
\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \left\{ \left[ v(t) + \gamma \left( \int_x^{x+U} P(x, t) \, dx \right) \right] P(x, t) \right\} + \frac{1}{2} q^2 \frac{\partial^2}{\partial x^2} \left[ P(x, t) \right],
\]

*non-linear and non-local field equation*
Stylized Model for Smart Parts Dynamics

Small Influence Region - Burgers’ Equation Dynamics

Small values of $U \Rightarrow$ Taylor expand up to 1$^{st}$ order in $U$

\[ \int_{x}^{x+U} P(x, t) dx \simeq U P(x, t) \]

\[
\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} \left\{ [v(t) + \gamma U P(x, t)] P(x, t) \right\} + \frac{1}{2} q^2 \frac{\partial^2}{\partial x^2} [P(x, t)]
\]

non-linear but local drift field

\[ t \mapsto \tau = \gamma t \quad \downarrow \quad x \mapsto z = \frac{x - \int_{0}^{t} v(s) \, ds}{2U} \]

Burgers’ Equation (to be solved with initial condition $P(z, t) = \delta(z) \Theta(z)$)

\[
\dot{P}(z, t) = \frac{1}{2} \frac{\partial}{\partial z} \left[ P(z, t)^2 \right] + \left[ \frac{q^2}{8U^2\gamma} \right] \frac{\partial^2}{\partial z^2} [P(z, t)]
\]
Burgers’ Eq. $\iff$ logarithmic transformation (Hopf - Cole) $\implies$ Heat Eq.

$\downarrow$ exact integration

\[
P(y, t) = -\frac{q^2}{4\gamma U^2} \frac{\partial}{\partial y} \ln \left[ 1 + \frac{(e^R - 1)}{2} \text{Erfc} \left( \frac{y}{q\sqrt{t}} \right) \right] = \]

\[
= \frac{1}{R} \left[ (e^R - 1) \frac{1}{\sqrt{\pi q^2 t}} e^{-\frac{y^2}{q^2 t}} \right] := \frac{1}{R} \frac{(e^R - 1) \mathcal{G}(y, t)}{\mathcal{E}(y, t)}
\]

Typical shape of $P(y, t)$ for various $R := \frac{4U^2\gamma}{q^2}$ factors

(viewed from the relative moving frame)

Normalization and positivity are visually manifest !!
Benefit of Competition - **Noise Induced Transport Enhancement**

Position probability distribution: without interaction, with interactions

- **Additional traveled distance when** $R = \frac{4\gamma U^2}{q^2} \to \infty$: $\langle X(t) \rangle_{t \to \infty} \simeq \frac{4U}{3} \sqrt{\gamma t}$,

- **Additional traveled distance when** $R = \frac{4\gamma U^2}{q^2} \to 0$: $\langle X(t) \rangle_{t \to \infty} \simeq 0$. 
Optimal Effective Centralized Control

Controlled diffusion process:

\[ dY_t = c(Y, t) \, dt \quad + \quad q \, dB_t, \]

\[ Y_0 = 0 \quad , \quad (0 \leq t \leq T) , \]

\[ \text{effective central controller} \quad \text{initial condition} \]

\[ \downarrow \quad \text{(Fokker-Planck equation)} \]

\[ \partial_t P_c(y, t) = -\partial_y [c(y, t)P_c(y, t)] + \frac{q^2}{2} \partial^2_{y^2} P_c(y, t) \]

Construct a drift controller \( c(Y, t) \) which, for time \( T \), fulfills

\[ P_c(y, T) = P(y, T) \]

\[ \text{Prob. density with central controller} \quad \text{Prob. density due to agent interactions} \]

Burgers’ exact solution
Optimal Effective Centralized Control (continued)

Introduce a utility function $J_{central,T}[c(y, t; T)]$ defined as:

$$J_{central,T}[c(y, t; T)] = \left\langle \int_0^T \frac{c^2(y, s; T)}{2q^2} ds \right\rangle,$$

where

$$\left\langle \cdot \right\rangle := \text{average over the realization of underlying stochastic process}$$

Optimal Control Problem

Construct an optimal drift $c^*(y, t; T)$ such that:

$$J_{central,T}[c^*(y, t; T)] \leq J_{central,T}[c(y, t; T)]$$

i.e. yielding minimal cost
The **Dai Pra** Solution of the Optimal Control Problem

**Optimal drift controller:**

\[
c^*(y, t; T) = \frac{\partial}{\partial y} \ln [h(y, t)],
\]

\[
h(y, t) = \int_{\mathbb{R}} G[(z - y), (T - t)] \frac{P(z, T)}{G(z, t)} dz.
\]


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**Minimal cost:**

\[
J_{central, T} [c^*(y, t; T)] = \frac{N}{\# \text{ population}} \cdot D(P|G) = \begin{cases} 
0 & \text{for } t = 0, \\
N \frac{R}{2} + N \ln \left( \frac{e^R - 1}{R} \right) & \text{for } t > 0.
\end{cases}
\]
Decentralized Agent Control - Cost Estimation

Cost $J_{\text{agents},T}$ for decentralized evolution during time horizon $T$:

$$J_{\text{agents},T} := N \cdot \rho \cdot \int_0^T ds \Phi(s),$$

- $\rho = \frac{\gamma^2 U^2/2}{q^2}$ := individual cost rate function,
- $\Phi(t) \in [0, 1]$ := proportion of interacting agents at time $t$.

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Cost upper-bound, reached when $\Phi(t) \equiv 1$

$$J_{\text{agents},T} \leq N\rho T$$
Costs Comparison - Centralized vs Decentralized

![Diagram showing cumulative costs, upper-bounded decentralized costs, actual decentralized costs, and centralized costs. The diagram illustrates time horizons for which agent interactions beat the optimal effective centralized controller.](image)
Conclusion and Perspectives

The stylized model exemplifies basic and somehow "universal" features:

- **Agents’ mimetic interactions produce an emergent structure** - (here a "shock"-like wave),
- **Competition enhances global transport flow** - (here a $\sqrt{t}$-increase of the traveled distance),
- **Self-organization via autonomous agents interactions can reduce costs.**

On a Generalized Innovation Diffusion Model

- **Bass’ diffusion model**: describes how a new product gets adopted.
- 2 populations of agents (2 possible states): \( \{ \text{non-adopters}, \text{adopters} \} \)
- Two ways for product adoption: \( \{ \text{spontaneous adoption}, \text{imitation} \} \)
- Output: temporal evolution of the overall adoption rate
- **Aggregated model**, no spatial dimension
On a Generalized Innovation Diffusion Model (continued)

- Introduced a **spatial dimension** into the original Bass’ model ⇒ Agents now described by state and **location**
- Imitation between spatially close neighbors

![Graph showing spatial diffusion models](image)