Lecture Notes on Stochastic Calculus (Part II)

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1 Stochastic integrals

Let $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be a filtration and $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, that is :

- $B_0 = 0$ *a.s.*

- B_t is $\mathcal{F}_t$-measurable $\forall t \in \mathbb{R}_+$ (i.e., B is adapted to $(\mathcal{F}_t, t \in \mathbb{R}_+)$)

- $B_t - B_s \perp\!\!\!\perp \mathcal{F}_s$ $\forall t > s \geq 0$ (independent increments)

- $B_t - B_s \sim B_{t-s} - B_0$ $\forall t > s \geq 0$ (stationary increments)

- $B_t \sim \mathcal{N}(0, t)$ $\forall t \in \mathbb{R}_+$

- B has continuous trajectories *a.s.*

Reminder. In addition, B has the following properties :

- $(B_t,\, t \in \mathbb{R}_+)$ is a continuous ans square-integrable martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, with quadratic variation $\langle B \rangle_t = t$ a.s. (i.e. $B_t^2 - t$ is a martingale).

- $(B_t,\, t \in \mathbb{R}_+)$ is Gaussian process with mean $\mathbb{E}(B_t) = 0$ and covariance $\text{Cov}(B_t, B_s) = t \wedge s := \min(t, s)$.

- $(B_t,\, t \in \mathbb{R}_+)$ is a Markov process with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, that is, $\mathbb{E}(g(B_t)|\mathcal{F}_s) = \mathbb{E}(g(B_t)|B_s)$ a.s., $\forall t > s \geq 0$ and $g : \mathbb{R} \to \mathbb{R}$ continuous and bounded.

1.1 Ito's integral with respect to the standard Brownian motion

Let $(H_t,\, t \in \mathbb{R}_+)$ be a process with continuous trajectories adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and such that

$$\mathbb{E}\left(\int_0^t H_s^2\, ds\right) < \infty, \quad \forall t \in \mathbb{R}_+.$$

It is then possible to define a process $((H \cdot B)_t \equiv \int_0^t H_s\, dB_s,\, t \in \mathbb{R}_+)$ which satisfies the following properties (see lecture notes of the fall semester):

- $\mathbb{E}((H \cdot B)_t) = 0, \quad \mathbb{E}((H \cdot B)_t^2) = \mathbb{E}\left(\int_0^t H_s^2\, ds\right).$

- $\text{Cov}((H \cdot B)_t, (H \cdot B)_s) = \mathbb{E}\left(\int_0^{t \wedge s} H_r^2\, dr\right).$

- $((H \cdot B)_t,\, t \in \mathbb{R}_+)$ is a continuous square-integrable martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, with quadratic variation

$$\langle (H \cdot B) \rangle_t = \int_0^t H_s^2\, ds.$$

- Let

$$(H^{(n)} \cdot B)_t = \sum_{i=1}^n H(t_{i-1}^{(n)}) \left(B(t_i^{(n)}) - B(t_{i-1}^{(n)})\right),$$

where $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = t$ is a sequence of partitions of $[0, t]$ such that

$$\max_{1 \leq i \leq n} |t_i^{(n)} - t_{i-1}^{(n)}| \underset{n \to \infty}{\to} 0.$$

Then $(H^{(n)} \cdot B)_t \xrightarrow{\mathbb{P}} (H \cdot B)_t$ as $n \to \infty$, that is, $\forall \varepsilon > 0$,

$$\mathbb{P}\left(\left|(H^{(n)} \cdot B)_t - (H \cdot B)_t\right| > \varepsilon\right) \underset{n \to \infty}{\to} 0$$

Remark. *In general*, $(H \cdot B)$ is not a Gaussian process; it does not have neither independent increments, nor stationary increments. Moreover, $\langle (H \cdot B) \rangle_t = \int_0^t H_s^2 \, ds$ is not deterministic.

Remark. Processes such as $\int_0^t H(t,s)\, dB_s$ are not martingales in general: at each time t, the integrand H changes. Nevertheless, the above isometry properties remains valid:

$$\mathbb{E}\left(\int_0^t H(t,s)\, dB_s\right) = 0, \quad \mathbb{E}\left(\left(\int_0^t H(t,s)\, dB_s\right)^2\right) = \mathbb{E}\left(\int_0^t H(t,s)^2\, ds\right)$$

and

$$\mathrm{Cov}\left(\int_0^t H(t,s)\, dB_r, \int_0^s H(s,r)\, dB_r\right) = \mathbb{E}\left(\int_0^{t\wedge s} H(t,r)\, H(s,r)\, dr\right).$$

1.2 Wiener's integral

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a deterministic continuous function (so $\int_0^t f(s)^2\, ds < \infty$, $\forall t \in \mathbb{R}_+$). Then the process $((f \cdot B)_t, t \in \mathbb{R}_+)$, in addition of all the above properties (f is a particular case of H), satisfies also:

- $(f \cdot B)$ is a Gaussian process, with mean and covariance:

$$\mathbb{E}((f \cdot B)_t) = 0, \quad \mathrm{Cov}((f \cdot B)_t, (f \cdot B)_s) = \int_0^{t\wedge s} f(r)^2\, dr.$$

- $(f \cdot B)$ has independent increments.

- $\langle (f \cdot B) \rangle_t = \int_0^t f(s)^2\, ds$ is deterministic.

Remark. In general, $(f \cdot B)$ does not have stationary increments and processes such as $\int_0^t f(t,s)\, dB_s$ do not have independent increments.

1.3 Ito's integral with respect to a martingale

Let $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be a filtration and $(M_t,\, t \in \mathbb{R}_+)$ be a continuous square-integrable martingale with respect to $(\mathcal{F}_t, t \in \mathbb{R}_+)$.

Reminder. The quadratic variation of M is the unique process $(\langle M \rangle_t,\, t \in \mathbb{R}_+)$ which is increasing, continuous and adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, such that $\langle M \rangle_0 = 0$ a.s. and $(M_t^2 - \langle M \rangle_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$.

Lemma 1.1. For all $t > s \geq 0$,

$$\mathbb{E}\left((M_t - M_s)^2 \,|\, \mathcal{F}_s\right) = \mathbb{E}(\langle M \rangle_t - \langle M \rangle_s \,|\, \mathcal{F}_s).$$

Proof.

$$\begin{aligned}
\mathbb{E}\left((M_t - M_s)^2 \,|\, \mathcal{F}_s\right) &= \mathbb{E}(M_t^2 - 2M_tM_s + M_s^2 \,|\, \mathcal{F}_s^2) = \mathbb{E}(M_t^2 \,|\, \mathcal{F}_s) - 2\mathbb{E}(M_t \,|\, \mathcal{F}_s)M_s + M_s^2 \\
&= \mathbb{E}(M_t^2 - \langle M \rangle_t + \langle M \rangle_t \,|\, \mathcal{F}_s) - 2M_s^2 + M_s^2 = M_s^2 - \langle M \rangle_s + \mathbb{E}(\langle M \rangle_t \,|\, \mathcal{F}_s) - M_s^2 \\
&= \mathbb{E}(\langle M \rangle_t - \langle M \rangle_s \,|\, \mathcal{F}_s).
\end{aligned}$$

□

Remarks. - Since $t \mapsto \langle M \rangle_t$ is increasing, it is a process with bounded variation, so $\int_0^t H_s\, d\langle M \rangle_s$ is a well-defined Riemann-Stieltjes integral, as long as H has continuous trajectories.

- In general, $\langle M \rangle_t$ is not deterministic, but when M has independent increments, then $\langle M \rangle_t = \mathbb{E}(M_t^2) - \mathbb{E}(M_0^2)$ (and is therefore deterministic).

Let $(H_t,\, t \in \mathbb{R}_+)$ be a continuous process adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ such that

$$\mathbb{E}\left(\int_0^t H_s^2\, d\langle M\rangle_s\right) < \infty, \quad \forall t \in \mathbb{R}_+.$$

It is then possible to define a process $((H \cdot M)_t \equiv \int_0^t H_s\, dM_s,\, t \in \mathbb{R}_+)$ which satisfies the following properties:

- $\mathbb{E}((H \cdot M)_t) = 0, \quad \mathbb{E}((H \cdot B)_t^2) = \mathbb{E}\left(\int_0^t H_s^2\, d\langle M\rangle_s\right).$

- $\mathrm{Cov}((H \cdot M)_t, (H \cdot M)_s) = \mathbb{E}\left(\int_0^{t\wedge s} H_r^2\, d\langle M\rangle_s\right).$

- $((H \cdot M)_t,\, t \in \mathbb{R}_+)$ is a continuous square-integrable martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, with quadratic variation

$$\langle (H \cdot M)\rangle_t = \int_0^t H_s^2\, d\langle M\rangle_s.$$

- Let

$$(H^{(n)} \cdot M)_t = \sum_{i=1}^n H(t_{i-1}^{(n)}) \left(M(t_i^{(n)}) - M(t_{i-1}^{(n)})\right),$$

where $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = t$ is a sequence of partitions of $[0,t]$ such that

$$\max_{1\le i\le n} |t_i^{(n)} - t_{i-1}^{(n)}| \overset{n\to\infty}{\to} 0.$$

Then $\left(H^{(n)} \cdot M\right)_t \overset{\mathbb{P}}{\longrightarrow} (H \cdot M)_t$ as $n \to \infty$, that is, $\forall \varepsilon > 0$,

$$\mathbb{P}\left(\left|(H^{(n)} \cdot M)_t - (H \cdot M)_t\right| > \varepsilon\right) \underset{n\to\infty}{\to} 0.$$

Let us give here a short explanation regarding the construction of the integral in this case and the isometry property. For a simple predictable process of the form

$$H_s(\omega) = \sum_{i=1}^n X_i(\omega)\, 1_{]t_{i-1},t_i]}(s), \quad s \in [0,t],$$

where $0 = t_0 < t_1 < \ldots < t_n = t$ is a partition of $[0,t]$ and X_i is $\mathcal{F}_{t_{i-1}}$-measurable and bounded, the stochastic integral $H \cdot M$ is defined as

$$(H \cdot M)_t = \sum_{i=1}^n X_i\, (M(t_i) - M(t_{i-1})).$$

Let us then compute

$$\begin{aligned}
\mathbb{E}\left((H \cdot M)_t^2\right) &= \sum_{i,j=1}^n \mathbb{E}(X_i\, X_j\, (M(t_i) - M(t_{i-1}))\, (M(t_j) - M(t_{j-1}))) \\
&= \sum_{i=1}^n \mathbb{E}(\mathbb{E}(X_i^2\, (M(t_i) - M(t_{i-1})^2 \,|\, \mathcal{F}_{t_{i-1}})) \\
&\quad +2\sum_{i<j} \mathbb{E}(\mathbb{E}(X_i\, X_j\, (M(t_i) - M(t_{i-1}))\, (M(t_j) - M(t_{j-1}))\,|\, \mathcal{F}_{t_{j-1}})).
\end{aligned}$$

Since X_i is $\mathcal{F}_{t_{j-1}}$-measurable and X_i, X_j and $M(t_i) - M(t_{i-1})$ are $\mathcal{F}_{t_{j-1}}$-measurable for $i < j$, we have

$$\begin{aligned}
\mathbb{E}\left((H \cdot M)_t^2\right) &= \sum_{i=1}^n \mathbb{E}(X_i^2\, \mathbb{E}((M(t_i) - M(t_{i-1}))^2 \,|\, \mathcal{F}_{t_{i-1}})) \\
&\quad +2\sum_{i<j} \mathbb{E}(X_i\, X_j\, (M(t_i) - M(t_{i-1}))\, \mathbb{E}(M(t_j) - M(t_{j-1}) \,|\, \mathcal{F}_{t_{j-1}})).
\end{aligned}$$

Since M is a martingale, $\mathbb{E}(M(t_j) - M(t_{j-1}) \,|\, \mathcal{F}_{t_{j-1}}) = 0$, so the second term on the right-hand side drops. For the first term, Lemma 1.1 tells us that

$$\mathbb{E}((M(t_i) - M(t_{i-1}))^2 \,|\, \mathcal{F}_{t_{i-1}}) = \mathbb{E}(\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}} \,|\, \mathcal{F}_{t_{i-1}}),$$

so

$$\mathbb{E}\left((H \cdot M)_t^2\right) = \sum_{i=1}^n \mathbb{E}(X_i^2\, \mathbb{E}(\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}} \,|\, \mathcal{F}_{t_{i-1}})) = \sum_{i=1}^n \mathbb{E}(X_i^2\, (\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}})).$$

On the other hand,

$$\mathbb{E}\left(\int_0^t H_s^2\, d\langle M \rangle_s\right) = \mathbb{E}\left(\sum_{i=1}^n X_i^2 \int_{t_{i-1}}^{t_i} d\langle M \rangle_s\right) = \sum_{i=1}^n \mathbb{E}(X_i^2\, (\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}})),$$

which shows the above isometry property.

Remark. We now have a map

$$(H, M) \mapsto (H \cdot M)$$

where H is a continuous and adapted process, M is a continuous square-integrable martingale and $(H \cdot M)$ is a continuous square-integrable martingale. So we can iterate the procedure:

- $(H, B) \mapsto M_t = (H \cdot B)_t = \int_0^t H_s\, dB_s$.

- $(K, M) \mapsto N_t = (K \cdot M)_t = \int_0^t K_s\, dM_s$ $(= \int_0^t K_s\, H_s\, dB_s)$ etc.

We will also use in the sequel the following (formal) differential notations:

- $M_t = \int_0^t H_s\, dB_s$ reads $dM_t = H_t\, dB_t$.

- $\langle M \rangle_t = \int_0^t H_s^2\, ds$ reads $d\langle M \rangle_t = H_t^2\, dt$, etc.

Reminder. The quadratic covariation of two continuous square-integrable martingales M and N with respect to the same filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ is the unique process $(\langle M, N \rangle_t,\, t \in \mathbb{R}_+)$ which is continuous, has bounded variation and is adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, such that $\langle M, N \rangle_0 = 0$ a.s. and $(M_t\, N_t - \langle M, N \rangle_t,\, t \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Remember also the following properties:

- $\langle M, M \rangle_t = \langle M \rangle_t$.

- $\langle M, N \rangle_t = 0$ if M and N are independent.

- $\langle M, N \rangle_t = \frac{1}{4}\left(\langle M + N \rangle_t - \langle M - N \rangle_t\right)$ *(polarization identity)*.

Remark. If M, N are continuous square-integrable martingales (with respect to the same filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$) and H, K are continuous adapted processes such that

$$\mathbb{E}\left(\int_0^t H_s^2\, d\langle M \rangle_s\right) < \infty \text{ and } \mathbb{E}\left(\int_0^t K_s^2\, d\langle N \rangle_s\right) < \infty, \quad \forall t \in \mathbb{R}_+,$$

then we have

$$\mathrm{Cov}((H \cdot M)_t, (K \cdot N)_s) = \mathbb{E}\left(\int_0^{t \wedge s} H_r\, K_r\, d\langle M, N \rangle_r\right)$$

and

$$\langle (H \cdot M), (K \cdot N) \rangle_t = \int_0^t H_s\, K_s\, d\langle M, N \rangle_s.$$

2 Ito-Doeblin's formula(s)

2.1 First formulations

(I) Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and $f \in \mathcal{C}^2(\mathbb{R})$ be such that $\mathbb{E}\left(\int_0^t f'(B_s)^2\, ds\right) < \infty$, $\forall t \in \mathbb{R}_+$. Then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

(II) Let now $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ be such that $\mathbb{E}\left(\int_0^t f'_x(s, B_s)^2\, ds\right) < \infty$, $\forall t \in \mathbb{R}_+$. Then

$$f(t, B_t) - f(0, B_0) = \int_0^t f'_t(s, B_s)\, ds + \int_0^t f'_x(s, B_s)\, dB_s + \frac{1}{2}\int_0^t f''_{xx}(s, B_s)\, ds \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

In particular, if f is such that

$$f'_t(t, x) + \frac{1}{2}\, f''_{xx}(t, x) = 0, \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R},$$

then

$$f(t, B_t) = f(0, B_0) + \int_0^t f'_x(s, B_s)\, dB_s$$

is a martingale. Particular examples of such functions are $f(t, x) = x^2 - t$ and $f(t, x) = e^{x - \frac{t}{2}}$.

For an idea of the proofs, see lecture notes of the fall semester.

2.2 Generalizations

(III) Let $(M_t,\, t \in \mathbb{R}_+)$ be a continuous square-integrable martingale with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and $f \in \mathcal{C}^2(\mathbb{R})$ be such that $\mathbb{E}\left(\int_0^t f'(M_s)^2\, d\langle M\rangle_s\right) < \infty$, $\forall t \in \mathbb{R}_+$. Then

$$f(M_t) - f(M_0) = \int_0^t f'(M_s)\, dM_s + \frac{1}{2}\int_0^t f''(M_s)\, d\langle M\rangle_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

(IV) Let also $(V_t,\, t \in \mathbb{R}_+)$ be a continuous process with bounded variation, adapted to the same filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ as $(M_t,\, t \in \mathbb{R}_+)$, and $f \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R})$ be such that

$$\mathbb{E}\left(\int_0^t f'_x(V_s, M_s)^2\, d\langle M\rangle_s\right) < \infty, \quad \forall t \in \mathbb{R}_+. \tag{1}$$

Then

$$f(V_t, M_t) - f(V_0, M_0) = \int_0^t f'_t(V_s, M_s)\, dV_s + \int_0^t f'_x(V_s, M_s)\, dM_s + \frac{1}{2}\int_0^t f''_{xx}(V_s, M_s)\, d\langle M\rangle_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

In particular, if $V_t = \langle M\rangle_t$ and f is again such that $f'_t(t, x) + \frac{1}{2} f''_{xx}(t, x) = 0$, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}$, then

$$f(\langle M\rangle_t, M_t) = f(\langle M\rangle_0, M_0) + \int_0^t f'_x(\langle M\rangle_s, M_s)\, dM_s$$

is a martingale. Again, examples are $f(t, x) = x^2 - t$ (we already knew that $M_t^2 - \langle M\rangle_t$ is a martingale) and $f(t, x) = \exp\left(x - \frac{t}{2}\right)$: $\exp\left(M_t - \frac{\langle M\rangle_t}{2}\right)$ is a martingale, called the exponential martingale associated to M, *provided that condition* (1) *is satisfied!*

Remark. The above integrals may seem to be quite abstract ones. Remember nevertheless that in most cases, we will consider processes such as $V_t = \int_0^t K_s\, ds$ (so $dV_s = K_s\, ds$) and $M_t = \int_0^t H_s\, dB_s$, (so $dM_s = H_s\, dB_s$, and $\langle M\rangle_t = \int_0^t H_s^2\, ds$, so $d\langle M\rangle_s = H_s^2\, ds$).

2.3 Continuous semi-martingales

Definition 2.1. A *continuous semi-martingale* is a process $(X_t,\, t \in \mathbb{R}_+)$ that can be written as $X_t = M_t + V_t$, where

- $(M_t,\, t \in \mathbb{R}_+)$ is a continuous square-integrable martingale with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$,

- $(V_t,\, t \in \mathbb{R}_+)$ is a continuous process with bounded variation, adapted to the same filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and such that $V_0 = 0$ a.s.

Remarks. - The above terminology is non standard. There are variations in the definition.

- From the above definition, it is tempting to deduce that "basically any continuous process is a semi-martingale". This is however far from being true!

Example 2.2. - By Doob's decomposition theorem, every continuous (square-integrable) submartingale may be written as the sum of a martingale and an increasing process : it is therefore a semi-martingale.

- Let H and K be adapted and continuous processes such that $\mathbb{E}\left(\int_0^t H_s^2\, ds\right) < \infty$, $\forall t \in \mathbb{R}_+$. Then the process $(X_t,\, t \in \mathbb{R}_+)$ defined as

$$X_t = \underbrace{X_0 + \int_0^t H_s\, dB_s}_{M_t} + \underbrace{\int_0^t K_s\, ds}_{V_t}.$$

is a continuous semi-martingale (in the literature, this particular type of semi-martingales are called Ito processes).

- Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $f \in \mathcal{C}^2(\mathbb{R})$ be such that $\mathbb{E}\left(\int_0^t f'(B_s)^2\, ds\right) < \infty$, $\forall t \in \mathbb{R}_+$. The process $(X_t = f(B_t),\, t \in \mathbb{R}_+)$ is a continuous semi-martingale, since by Ito-Doeblin's formula (I),

$$f(B_t) = \underbrace{f(B_0) + \int_0^t f'(B_s)\, dB_s}_{M_t} + \underbrace{\frac{1}{2}\int_0^t f''(B_s)\, ds}_{V_t}.$$

Definition 2.3. - Let $(X_t = M_t + V_t,\, t \in \mathbb{R}_+)$ be a continuous semi-martingale. Its *quadratic variation* is defined as

$$\langle X \rangle_t = \langle M \rangle_t, \quad t \in \mathbb{R}_+.$$

- Let $(Y_t = N_t + U_t,\, t \in \mathbb{R}_+)$ be another continuous semi-martingale. The *quadratic covariation* of X and Y is defined as

$$\langle X, Y \rangle_t = \langle M, N \rangle_t, \quad t \in \mathbb{R}_+.$$

Properties.

- $\langle X, X \rangle_t = \langle X \rangle_t$, $\langle Y, X \rangle_t = \langle X, Y \rangle_t$.

- If X has bounded variation, then $\langle X \rangle_t = 0$ and $\langle X, Y \rangle_t = 0$, whatever Y is.

- If X and Y are independent, then $\langle X, Y \rangle_t = 0$.

Remark. Pay attention that the process $(X_t^2 - \langle X \rangle_t,\, t \in \mathbb{R}_+)$ is not a martingale in general. It is actually a martingale only if X is. Likewise, $(X_t Y_t - \langle X, Y \rangle_t,\, t \in \mathbb{R}_+)$ is not a martingale in general. Nevertheless, the *polarization identity*, which was established previously for martingales, still holds:

$$\langle X, Y \rangle_t = \frac{1}{4}\left(\langle X + Y \rangle_t - \langle X - Y \rangle_t\right).$$

Definition 2.4. Let $(X_t = M_t + V_t,\, t \in \mathbb{R}_+)$ be a continuous semi-martingale. Let $(H_t,\, t \in \mathbb{R}_+)$ be a continuous process, adapted to the same filtration as $(X_t,\, t \in \mathbb{R}_+)$ and such that

$$\mathbb{E}\left(\int_0^t H_s^2\, d\langle X\rangle_s\right) \equiv \mathbb{E}\left(\int_0^t H_s^2\, d\langle M\rangle_s\right) < \infty, \quad \forall t \in \mathbb{R}_+.$$

Then the *stochastic integral of H with respect to X* is defined as

$$(H \cdot X)_t \equiv \int_0^t H_s dX_s = \underbrace{\int_0^t H_s dM_s}_{\text{Ito's integral}} \qquad + \qquad \underbrace{\int_0^t H_s dV_s}_{\text{Riemann-Stieltjes integral}}$$

(V) Ito-Doeblin's formula for a semi-martingale. Let $(X_t = M_t + V_t,\, t \in \mathbb{R}_+)$ be a continuous semi-martingale. Let $g \in \mathcal{C}^2(\mathbb{R})$ be such that $\mathbb{E}\left(\int_0^t g'(X_s)^2\, d\langle X\rangle_s\right) < \infty$, $\forall t \in \mathbb{R}_+$. Then

$$g(X_t) - g(X_0) = \int_0^t g'(X_s)\, dX_s + \frac{1}{2}\int_0^t g''(X_s)\, d\langle X\rangle_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

Notice that the first integral on the right-hand side is now the sum of two integrals of different kind.

Proof. We use the version (IV) of Ito-Doeblin's formula with $f(t,x) = g(t+x)$: $f'_t = f'_x = g'$ and $f''_{xx} = g''$, so

$$\begin{aligned} g(X_t) - g(X_0) &= f(V_t, M_t) - f(V_0, M_0) \\ &= \int_0^t f'_t(V_s, M_s)\, dV_s + \int_0^t f'_x(V_s, M_s)\, dM_s + \frac{1}{2}\int_0^t f''_{xx}(V_s, M_s)\, d\langle M\rangle_s \\ &= \int_0^t g'(X_s)\, dX_s + \frac{1}{2}\int_0^t g''(X_s)\, d\langle X\rangle_s. \end{aligned}$$

□

2.4 Integration by parts formula

This is still another variation on the theme of Ito-Doeblin's formulas. Let X, Y be two continuous semi-martingales such that $\mathbb{E}\left(\int_0^t X_s^2\, d\langle Y\rangle_s\right) < \infty$ and $\mathbb{E}\left(\int_0^t Y_s^2\, d\langle X\rangle_s\right) < \infty$, $\forall t \in \mathbb{R}$. Then

$$X_tY_t - X_0Y_0 = \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

In differential form, the above formula reads:

$$d(X_tY_t) = X_t\, dY_t + Y_t\, dX_t + d\langle X, Y\rangle_t$$

Proof. By using Ito-Doeblin's formula (V), we have:

- $(X_t + Y_t)^2 - (X_0 + Y_0)^2 = 2\int_0^t (X_s + Y_s)\, d(X_s + Y_s) + \langle X + Y\rangle_t$

- $(X_t - Y_t)^2 - (X_0 - Y_0)^2 = 2\int_0^t (X_s - Y_s)\, d(X_s - Y_s) + \langle X - Y\rangle_t$

Subtracting these two formulas gives:

$$4X_tY_t - 4X_0Y_0 \;=\; 4\int_0^t X_s\, dY_s + 4\int_0^t Y_s\, dX_s + \underbrace{(\langle X + Y\rangle_t - \langle X - Y\rangle_t)}_{=4\langle X,Y\rangle_t}$$

which completes the proof. □

Remark. We are back to the classical integration by parts formula if $\langle X, Y\rangle = 0$, i.e., if either X or Y has bounded variation (or if they are independent).

2.5 Back to Fisk-Stratonovič's integral

Definition 2.5. Let B be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and H be a continuous semi-martingale adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ such that $\mathbb{E}\left(\int_0^t H_s^2\, ds\right) < \infty\ \forall t \in \mathbb{R}$. Then the Fisk-Stratonovič integral of H with respect to B is defined as

$$(H \circ B)_t \equiv \int_0^t H_s \circ dB_s = \int_0^t H_s dB_s + \frac{1}{2}\langle H, B\rangle_t, \quad t \in \mathbb{R}_+.$$

Remarks. - $H \circ B$ is in general not a martingale.

- The second term on the right-hand side is equal to zero if H has bounded variation (in which case $H \circ B$ becomes a martingale).

Fisk-Stratonovič's formula. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Let $f \in \mathcal{C}^3(\mathbb{R})$ be such that $\mathbb{E}\left(\int_0^t f'(B_s)^2\, ds\right) < \infty$ and $\mathbb{E}\left(\int_0^t f''(B_s)^2\, ds\right) < \infty$, $\forall t \in \mathbb{R}_+$. Then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) \circ dB_s \quad a.s., \quad \forall t \in \mathbb{R}_+,$$

which is the rule of classical calculus.

Proof. Let $g = f'$. Then

$$\int_0^t g(B_s) \circ dB_s = \int_0^t g(B_s)\, dB_s + \frac{1}{2}\langle g(B), B\rangle_t.$$

By applying Ito-Doeblin's formula (I),

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)\, dB_s + \frac{1}{2}\int_0^t g''(B_s)\, ds = M_t + V_t.$$

Notice that all the integrals are well defined, because $\mathbb{E}\left(\int_0^t (g'(B_s))^2 ds\right) < \infty$ and $f \in \mathcal{C}^3(\mathbb{R})$. So

$$\langle g(B), B\rangle_t = \langle M, B\rangle_t = \int_0^t g'(B_s)\, ds.$$

Therefore,

$$\int_0^t g(B_s) \circ dB_s = \int_0^t g(B_s)\, dB_s + \frac{1}{2}\int_0^t g'(B_s)\, ds$$

i.e.,

$$\int_0^t f'(B_s) \circ dB_s = \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds = f(B_t) - f(B_0),$$

by a second application of Ito-Doeblin's formula (I). This completes the proof. □

3 Stochastic differential equations (SDE's)

3.1 Reminder on ordinary differential equations (ODE's)

A (time-homogeneous) ODE is of the form

$$X'(t) = \frac{dX}{dt}(t) = f(X(t)), \quad t \in \mathbb{R}_+, \quad X(0) = x_0, \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$.

Solving method. - Write $\frac{dX}{f(X)} = dt \to \displaystyle\int \frac{dX}{f(X)} = \int dt.$

- Let G be a primitive of $\frac{1}{f}$ (i.e., $G' = \frac{1}{f}$). Then we have $G(X(t)) = G(x_0) + t$.

- When possible, invert G to obtain $X(t) = G^{-1}\left(G(x_0) + t\right)$.

Examples.

1) $X'(t) = c\,X(t)$: $\frac{dX}{X} = c\,dt \to \ln X - \ln x_0 = c\,t \to X(t) = x_0\,e^{ct}$.

2) $X'(t) = X^2(t)$: $\frac{dX}{X^2} = dt \to -\frac{1}{X} + \frac{1}{x_0} = t \to X(t) = \frac{1}{\frac{1}{x_0} - t}$. Notice that the solution explodes in $t = \frac{1}{x_0}$.

3) $X'(t) = \sqrt{X(t)}$: $\frac{dX}{\sqrt{X}} = dt \to 2(\sqrt{X} - \sqrt{x_0}) = t \to X(t) = \left(\frac{t}{2} + \sqrt{x_0}\right)^2$. Notice that if $x_0 = 0$ in this last example, then $X(t) = \frac{t^2}{4}$ is a solution, but $X(t) \equiv 0$ is also a solution, as well as

$$X(t) = \begin{cases} \frac{(t-c)^2}{4}, & t > c, \\ 0, & t \leq c, \end{cases}$$

for any $c \geq 0$. So the solution is not unique in this case.

Definition 3.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitz* if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L\,|x - y|, \quad \forall x, y \in \mathbb{R}.$$

Remark. - If f is Lipschitz, then it is continuous (clear).

- If f is continuously differentiable and its derivative is bounded, then it is Lipschitz. Indeed,

$$|f(x) - f(y)| = \left| \int_y^x f'(z)\,dz \right| \leq \underbrace{\sup_{z \in \mathbb{R}} |f'(z)|}_{<\infty}\, |x - y|.$$

Theorem 3.2. If f is Lipschitz, then (2) admits a unique solution $(X(t),\, t \in \mathbb{R}_+)$.

Back to the examples.

1) $f(x) = c\,x$ is Lipschitz $(L = |c|)$: $X(t) = x_0\,e^{ct}$ exists $\forall t \in \mathbb{R}_+$.

2) $f(x) = x^2$ is not Lipschitz (it is actually "locally Lipschitz", but the constant L explodes at infinity): $X(t) = \frac{1}{\frac{1}{x_0} - t}$ exists only up to $t = \frac{1}{x_0}$, where it explodes.

3) $f(x) = \sqrt{x}$ is not Lipschitz at $x = 0$ (the function has infinite slope): the solution $X(t)$ is not unique if one starts from $x_0 = 0$.

3.2 Time-homogeneous SDE's

While seeing applications of Ito-Doeblin's formula, we have already seen an instance of an SDE. Indeed, if $X(t) = e^{B_t}$, where B is a standard Brownian motion, then applying the above mentioned formula leads to the conclusion that X satisfies the following SDE (in integral form):

$$X_t = 1 + \int_0^t X_s\, dB_s + \frac{1}{2}\int_0^t X_s\, ds \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

This SDE may be rewritten in differential form as

$$dX_t = X_t\, dB_t + \frac{1}{2} X_t\, dt, \quad X_0 = 1.$$

(in order to avoid writing $\frac{dB_t}{dt}$, which does not exist).

A generic time-homogeneous SDE is of the form

$$dX_t = \underbrace{f(X_t)\, dt}_{\text{drift term}} + \underbrace{g(X_t)\, dB_t}_{\text{diffusion term}}, \quad X_0 = x_0,$$

where $x_0 \in \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ and $(B_t,\, t \in \mathbb{R}_+)$ is a standard Brownian motion.

Preliminary remarks. - Solving an SDE is in general much harder than solving an ODE. There are many functions f and g for which we do not know an analytic expression for the solution $(X_t,\, t \in \mathbb{R}_+)$ (or do not even know whether such a solution exists).

- But the good news is that sometimes, only knowing that a process X is solution of an SDE provides already lots of information on X.

Theorem 3.3. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, $x_0 \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be Lipschitz functions. Then there exists a unique continuous process $(X_t,\, t \in \mathbb{R}_+)$, adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and such that

$$X_t = x_0 + \int_0^t f(X_s)\, ds + \int_0^t g(X_s)\, dB_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

$(X_t,\, t \in \mathbb{R}_+)$ is called a *strong solution* of the above equation.

Remark. It can also be proven that $\mathbb{E}(X_t^2) < \infty$, $\forall t \in \mathbb{R}_+$, so the process X is a continuous semi-martingale. Indeed, $X_t = M_t + V_t$, where

$$M_t = x_0 + \int_0^t g(X_s)\, dB_s \quad \text{and} \quad V_t = \int_0^t f(X_s)\, ds.$$

Besides, the quadratic variation of X is given by

$$\langle X \rangle_t = \langle M \rangle_t = \int_0^t g(X_s)^2\, ds.$$

Example: Ornstein-Uhlenbeck process. Let us consider the SDE

$$dX_t = -a\, X_t\, dt + \sigma\, dB_t, \quad X_0 = x_0,$$

where $a, \sigma > 0$ and $x_0 \in \mathbb{R}$. Here, $f(x) = -ax$ and $g(x) = \sigma$ are Lipschitz, so there exists a unique strong solution $(X_t,\, t \in \mathbb{R}_+)$ to the above equation.

Solving method. - Let ϕ be the (deterministic) process solution of

$$d\phi_t = -a\, \phi_t\, dt, \quad \phi_0 = 1.$$

i.e., $\phi'_t = -a\, \phi_t$, so $\phi_t = e^{-at}$.

- Let us write $X_t = \phi_t Y_t$ and search for an equation for Y_t. By the integration by parts formula (in differential form), we have

$$\begin{aligned} dX_t &= d(\phi_t Y_t) = \phi_t \, dY_t + Y_t \, d\phi_t + d\langle \phi, Y \rangle_t \\ &= \phi_t \, dY_t - a\phi_t Y_t \, dt + 0, \end{aligned}$$

since ϕ has bounded variation. On the other hand :

$$dX_t = -a \, X_t \, dt + \sigma \, dB_t = -a\phi_t Y_t \, dt + \sigma \, dB_t,$$

so

$$\sigma \, dB_t = \phi_t \, dY_t, \quad \text{i.e.,} \quad dY_t = \frac{\sigma}{\phi_t} \, dB_t$$

with $Y_0 = X_0/\phi_0 = x_0 1 = x_0$. This implies that

$$Y_t = x_0 + \sigma \int_0^t \frac{1}{\phi_s} \, dB_s = x_0 + \sigma \int_0^t e^{-as} \, dB_s$$

and

$$X_t = \phi_t \, Y_t = e^{-at} \, x_0 + \sigma \int_0^t e^{-a(t-s)} \, dB_s.$$

Example: Black & Scholes equation. Let us consider the SDE

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t, \quad X_0 = x_0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $x_0 > 0$. Here, $f(x) = \mu x$ and $g(x) = \sigma x$ are Lipschitz, so there exists a unique strong solution $(X_t, \, t \in \mathbb{R}_+)$ to the above equation. The solving of this equation is left as an (important) exercise.

3.3 Time-inhomogeneous SDE's

A generic time-inhomogeneous SDE is of the form

$$dX_t = f(t, X_t) \, dt + g(t, X_t) \, dB_t, \quad X_0 = x_0,$$

where $x_0 \in \mathbb{R}$, $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and B is a standard Brownian motion.

Theorem 3.4. If f, g are jointly continuous in (t, x) and Lipschitz in x (i.e., there exists a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L \, |x - y|$, $\forall t \in \mathbb{R}_+ \; x, y \in \mathbb{R}$), then there exists a unique strong solution $(X_t, \, t \in \mathbb{R}_+)$ to the above equation, that is,

$$X_t = x_0 + \int_0^t f(s, X_s) \, ds + \int_0^t g(s, X_s) \, dB_s, \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

Again, X is a continuous semi-martingale.

A particular subclass: linear SDE's. Let us consider the SDE

$$dX_t = a(t) \, X_t \, dt + \sigma(t) \, dB_t, \quad X_0 = x_0,$$

where $x_0 \in \mathbb{R}$, $a, \sigma : \mathbb{R}_+ \to \mathbb{R}$ are continuous and bounded. Here, $f(t, x) = a(t) \, x$ and $g(t, x) = \sigma(t)$ are jointly continuous in (t, x) and Lipschitz in x, so by the above theorem, there exists a unique strong solution $(X_t, \, t \in \mathbb{R}_+)$ to this equation.

Solving method. It is a slight generalization of that used for the Ornstein-Uhlenbeck process.

- Let $(\phi_t, \, t \in \mathbb{R}_+)$ be the (deterministic) process solution of

$$d\phi_t = a(t) \, \phi_t \, dt, \quad \phi_0 = 1.$$

It turns out that $\phi_t = \exp\left(\int_0^t a(s)\,ds\right)$.

- Let $X_t = \phi_t\,Y_t$. By the integration by parts formula, we have

$$\begin{aligned} dX_t &= \phi_t\,dY_t + Y_t\,d\phi_t + 0 = \phi_t\,dY_t + a(t)\,\phi_t\,Y_t\,dt \\ &= a(t)\,X(t)\,dt + \sigma(t)\,dB_t \end{aligned}$$

so

$$\phi_t\,dY_t = \sigma(t)\,dB_t, \quad \text{i.e.,} \quad Y_t = x_0 + \int_0^t \frac{\sigma(s)}{\phi_s}\,dB_s,$$

and

$$X_t = \phi_t\,x_0 + \int_0^t \frac{\sigma(s)\,\phi_t}{\phi_s}\,dB_s$$

Remark. Being a Wiener integral, X is a Gaussian process (but it is not a martingale, nor a process with independent increments, because of the presence of ϕ_t in the integral).

3.4 Weak solutions

Let $(X_t,\, t \in \mathbb{R}_+)$ be the strong solution (assuming it exists) of the SDE

$$dX_t = f(X_t)\,dt + g(X_t)\,dB_t, \quad X_0 = x_0.$$

Then notice that

- $M_t = X_t - \int_0^t f(X_s)\,ds = x_0 + \int_0^t g(X_s)\,dB_s$ is a martingale.

- $\langle X\rangle_t = \int_0^t g(X_s)^2\,ds$, so $N_t = M_t^2 - \int_0^t g(X_s)^2\,ds$ is also a martingale.

This gives rise to the following definition.

Definition 3.5. A *weak solution* to the above equation is a continuous process $(X_t,\, t \in \mathbb{R}_+)$ such that

- $M_t = X_t - \int_0^t f(X_s)\,ds$ is a martingale.

- $N_t = M_t^2 - \int_0^t g(X_s)^2\,ds$ is also a martingale.

Remark. There is no more B in this definition! The weak solution X of an equation need therefore not to be related to it; in particular, it need not be adapted to the same filtration. The weak solution of an SDE can actually be seen as the *distribution* of the process X satisfying the above two properties. Notice also that if X is a strong solution, then it is a weak solution, by what has been said above.

Examples. - A weak solution of $dX_t = a\,X_t\,dt + \sqrt{X_t}\,dB_t$ is a continuous process X such that the processes

$$M_t = X_t - a\int_0^t X_s\,ds \quad \text{and} \quad N_t = M_t^2 - \int_0^t |X_s|\,ds$$

are martingales.

- A weak solution of $dX_t = \text{sgn}(X_t)\,dB_t$ is a continuous process X such that both

$$M_t = X_t \quad \text{and} \quad N_t = M_t^2 - \int_0^t \text{sgn}(X_s)^2\,ds = X_t^2 - t$$

are martingales. Therefore, X is a standard Brownian motion (by Lévy's theorem). But notice that X cannot be equal (nor adapted) to B!

4 Change of probability measure

Let us first give a brief motivation. Let X be the solution of the Black-Scholes SDE :

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t, \quad X_0 = x_0,$$

representing the evolution over time of a stock price with drift μ and volatility σ. Let now ϕ be an investment strategy on X (i.e., ϕ_t represents the number of shares of X owned at time t). Then the process G defined as

$$G_t = \int_0^t \phi_s\, dX_s$$

represents the gain made over the period $[0, t]$ by investing with the strategy ϕ on the stock X. If X is a martingale, then the process G is also a martingale. For computational reasons (see next section), G being a martingale is very useful. The problem is that when $\mu \neq 0$, the process X (and therefore also the process G) is a semi-martingale, but not a martingale. So the question is: can one change the underlying probability measure $\mathbb{P}$ so as to transform X into a martingale? The answer is yes: this is Girsanov's theorem.

4.1 Exponential martingale

Let M be a continuous square-integrable martingale and let Y be the process defined as

$$Y_t = \exp\left(M_t - \frac{\langle M \rangle_t}{2}\right), \quad t \in \mathbb{R}_+,$$

Notice that Y is not necessarily a martingale, a priori.

Fact. (to be proven later)
If there exists a constant $K > 0$ such that

$$\langle M \rangle_t \leq K\, t \quad a.s., \quad \forall t \in \mathbb{R}_+, \tag{3}$$

then Y is a continuous square-integrable martingale. Y is said to be the *exponential martingale* associated to M.

Example. If $M_t = B_t$, then $\langle B \rangle_t = t$ and $Y_t = \exp\left(B_t - \frac{t}{2}\right)$ is indeed a martingale.

Remarks. - There exists a more general condition than (3) which ensures that the process Y is a martingale up to a finite time horizon $T > 0$. This more general condition, called Novikov's condition, reads:

$$\mathbb{E}\left(\exp\left(\frac{\langle M \rangle_T}{2}\right)\right) < \infty.$$

- Under condition (3), one can apply Ito-Doeblin's formula to conclude that

$$Y_t = 1 + \int_0^t Y_s\, dM_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

i.e., Y is solution of the SDE:

$$dY_t = Y_t\, dM_t, \quad Y_0 = 1.$$

4.2 Change of probability measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be a filtration defined on this probability space. Let also M be a square-integrable martingale M with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, satisfying condition (3) and let Y be the exponential martingale associated to M.

Let now $T > 0$ be a given (fixed) horizon in time. We then define a new probability measure $\widetilde{\mathbb{P}}_T$ on $(\Omega, \mathcal{F})$:

$$\widetilde{\mathbb{P}}_T = \mathbb{E}(1_A Y_T), \quad A \in \mathcal{F}.$$

Let us first check that $\widetilde{\mathbb{P}}_T$ is indeed a probability measure :

- $\widetilde{\mathbb{P}}_T(A) \geq 0$, $\forall A \in \mathcal{F}$, since $Y_T = e^{M_T - \langle M \rangle_T/2} > 0$.

- $\widetilde{\mathbb{P}}_T(\emptyset) = 0$ (clear) and $\widetilde{\mathbb{P}}_T(\Omega) = \mathbb{E}(Y_T) = \mathbb{E}(Y_0) = 1$, since Y is a martingale.

- Let $(A_n)_{n=1}^\infty$ be such that $A_n \cap A_m = \emptyset$, $\forall n \neq m$. Then

$$\widetilde{\mathbb{P}}_T\left(\bigcup_{n=1}^\infty A_n\right) = \mathbb{E}\left(1_{\cup_{n=1}^\infty A_n} Y_T\right) = \mathbb{E}\left(\sum_{n=1}^\infty 1_{A_n} Y_T\right) = \sum_{n=1}^\infty \mathbb{E}\left(1_{A_n} Y_T\right) = \sum_{n=1}^\infty \widetilde{\mathbb{P}}_T(A_n),$$

where the third equality follows from the dominated convergence theorem.

Next, notice that $\mathbb{P}$ and $\widetilde{\mathbb{P}}_T$ are *equivalent*, which means

$$\mathbb{P}(A) = 0 \quad \text{if and only if} \quad \widetilde{\mathbb{P}}_T(A) = 0.$$

Indeed, if $\mathbb{P}(A) = 0$, then $\mathbb{E}(1_A Y_T) = 0$, since $1_A = 0$ a.s. On the other hand, if $\mathbb{E}(1_A Y_T) = 0$, then the fact that $Y_T > 0$ implies that $1_A = 0$ a.s., i.e., $\mathbb{P}(A) = 0$.

Notice also that alternate definitions of the two probability measures being equivalent are:

$$\begin{aligned}
\mathbb{P}(A) > 0 \quad &\text{if and only if} \quad \widetilde{\mathbb{P}}_T(A) > 0,\\
\mathbb{P}(A) = 1 \quad &\text{if and only if} \quad \widetilde{\mathbb{P}}_T(A) = 1,\\
\mathbb{P}(A) < 1 \quad &\text{if and only if} \quad \widetilde{\mathbb{P}}_T(A) < 1.
\end{aligned}$$

Finally, it can be shown that the expectation with respect to the new probability measure $\widetilde{\mathbb{P}}_T$ of a random variable X such that $\mathbb{E}(|XY_T|) < \infty$ is given by

$$\widetilde{\mathbb{E}}_T(X) = \mathbb{E}(X\, Y_T).$$

Remark. In the literature, Y_T is called the *Radon-Nikodym derivative* of $\widetilde{\mathbb{P}}_T$ with respect to $\mathbb{P}$.

4.3 Martingales under $\mathbb{P}$ and martingales under $\widetilde{\mathbb{P}}_T$

Lemma 4.1. Let $t \in [0, T]$ and Z be an $\mathcal{F}_t$-measurable random variable such that $\mathbb{E}(|Z\, Y_T|) < \infty$. Then $\widetilde{\mathbb{E}}_T(Z) = \mathbb{E}(ZY_t)$

Proof. By definition, we have

$$\widetilde{\mathbb{E}}_T(Z) = \mathbb{E}(Z\, Y_T) = \mathbb{E}(\mathbb{E}(Z\, Y_T|\mathcal{F}_t)) = \mathbb{E}(Z\, \mathbb{E}(Y_T|\mathcal{F}_t)) = \mathbb{E}(Z\, Y_t),$$

where the third equality holds since Z is $\mathcal{F}_t$-measurable and the last equality holds since Y is a martingale. □

Lemma 4.2. Let $t \in [0, T]$ and Z be an $\mathcal{F}_t$-measurable random variable such that $\mathbb{E}(|Z\, Y_T|) < \infty$. Then

$$\widetilde{\mathbb{E}}_T(Z \,|\, \mathcal{F}_s) = \mathbb{E}\left(\frac{ZY_t}{Y_s} \,\middle|\, \mathcal{F}_s\right) \quad a.s., \quad \forall 0 \leq s \leq t.$$

Proof. Set $W = \mathbb{E}\left(\frac{ZY_t}{Y_s} \,\middle|\, \mathcal{F}_s\right)$. We need to check that $\widetilde{\mathbb{E}}_T(Z\,|\,\mathcal{F}_s) = W$, that is:

i) W is $\mathcal{F}_s$-measurable: this holds since W is by definition a conditional expectation with respect to $\mathcal{F}_s$.

ii) $\widetilde{\mathbb{E}}_T(ZU) = \widetilde{\mathbb{E}}_T(WU)$, for any random variable U $\mathcal{F}_s$-measurable and bounded: indeed, since WU is $\mathcal{F}_s$-measurable, we can use Lemma 4.1 to obtain

$$\widetilde{\mathbb{E}}_T(WU) = \mathbb{E}(WUY_s) = \mathbb{E}\left(\mathbb{E}\left(\frac{ZY_t}{Y_s} \,\middle|\, \mathcal{F}_s\right) UY_s\right) = \mathbb{E}\left(\mathbb{E}\left(\frac{ZY_t}{Y_s} UY_s \,\middle|\, \mathcal{F}_s\right)\right) = \mathbb{E}(ZY_tU) = \widetilde{\mathbb{E}}_T(ZU),$$

by Lemma 4.1 again and the fact that ZU is $\mathcal{F}_t$-measurable. □

With these lemmas in hand, we can now establish the following relation between martingales under $\mathbb{P}$ and martingales under $\widetilde{\mathbb{P}}_T$.

Proposition 4.3. Let $(X_t,\, t \in [0.T])$ be a continuous and adapted process such that $\mathbb{E}(|X_t Y_T|) < \infty$, $\forall t \in [0,T]$. Then $(X_t,\, t \in [0,T])$ is a martingale under $\widetilde{\mathbb{P}}_T$ if and only if $(X_t\, Y_t,\, t \in [0,T])$ is a martingale under $\mathbb{P}$.

Proof. Assume that $\widetilde{\mathbb{E}}_T(X_t\,|\,\mathcal{F}_s) = X_s$, $\forall 0 \le s \le t \le T$. Then, by Lemma 4.2, we have

$$\mathbb{E}(X_t\, Y_t\,|\,\mathcal{F}_s) = \mathbb{E}\left(\frac{X_t\, Y_t}{Y_s} \,\middle|\, \mathcal{F}_s\right) Y_s = \tilde{\mathbb{E}}_T(X_t\,|\,\mathcal{F}_s)\, Y_s = X_s\, Y_s,$$

given the assumption made. The reciprocal statement follows the same logic. □

The above proposition establishes a correspondence between martingales under $\mathbb{P}$ and martingales under $\widetilde{\mathbb{P}}_T$. This is nevertheless not sufficient for our purpose, which is to "transform" martingales under $\mathbb{P}$ into martingales under $\widetilde{\mathbb{P}}_T$.

4.4 Girsanov's theorem

Theorem 4.4. Let $(Z_t,\, t \in [0,T])$ be a continuous square-integrable martingale under $\mathbb{P}$. Then the process $(Z_t - \langle M, Z\rangle_t,\, t \in [0,T])$ is a continuous square-integrable martingale under $\widetilde{\mathbb{P}}_T$.

Proof. Let $A_t = \langle M, Z\rangle_t$. In order to show that $(Z_t - A_t)$ is a martingale under $\widetilde{\mathbb{P}}_T$, it suffices to show, by Proposition 4.3, that $((Z_t - A_t)\, Y_t)$ is a martingale under $\mathbb{P}$. By the integration by parts formula, we have :

$$\begin{aligned}(Z_t - A_t)Y_t - (Z_0 - A_0)Y_0 &= \int_0^t (Z_s - A_s)\, dY_s + \int_0^t Y_s\, d(Z_s - A_s) + \langle Y, Z - A\rangle_t \\ &= \int_0^t (Z_s - A_s)\, dY_s + \int_0^t Y_s\, dZ_s - \int_0^t Y_s\, dA_s + \langle Y, Z\rangle_t,\end{aligned}$$

since A has bounded variation. Moreover, since Y and Z are martingales under $\mathbb{P}$, the first two terms are also martingales under $\mathbb{P}$. In order to conclude, we therefore need to show that

$$\int_0^t Y_s\, dA_s = \langle Y, Z\rangle_t.$$

Remember that $dY_t = Y_t\, dM_t$, so $dM_t = \frac{1}{Y_t}\, dY_t$ and $dA_t = d\langle M, Z\rangle_t = \frac{1}{Y_t}\, d\langle Y, Z\rangle_t$. Therefore,

$$\int_0^t Y_s\, dA_s = \int_0^t Y_s\, \frac{1}{Y_s}\, d\langle Y, Z\rangle_s = \langle Y, Z\rangle_t,$$

which concludes the proof. □

4.5 First application to SDE's

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion, $x_0 \in \mathbb{R}$ and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t, x), Lipschitz in x and bounded (i.e., $\exists K_1 < \infty$ such that $|f(t, x)| \leq K_1$, $\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$). Let also $(X_t,\, t \in \mathbb{R}_+)$ be the strong solution of the SDE

$$dX_t = f(t, X_t)\, dt + dB_t, \quad X_0 = x_0,$$

i.e.,

$$X_t - x_0 = \int_0^t f(s, X_s)\, ds + B_t \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

Under which probability measure $\widetilde{\mathbb{P}}_T$ is the process $(X_t,\, t \in [0, T])$ a martingale?

In order to use Girsanov's theorem, we need to start from a martingale Z under $\mathbb{P}$ and find another martingale M under $\mathbb{P}$ such that

$$X_t - x_0 = Z_t - \langle M, Z \rangle_t.$$

A priori, the only martingale Z under $\mathbb{P}$ that is present in the above equation is B. Let us now assume that the martingale M is of the form

$$M_t = \int_0^t H_s\, dB_s$$

for some continuous adapted process H (we will see later that this is not such a restriction), and deduce what the process H should be. We want

$$X_t - x_0 = Z_t - \langle M, Z \rangle_t = B_t - \int_0^t H_s\, ds.$$

As X satisfies the SDE

$$X_t - x_0 = \int_0^t f(s, X_s)\, ds + B_t,$$

we deduce that $H_s = -f(s, X_s)$. Indeed, the martingale

$$M_t = -\int_0^t f(s, X_s)\, dB_s$$

is a continuous square-integrable martingale, that moreover satisfies condition (3):

$$\langle M \rangle_t = \int_0^t f(s, X_s)^2\, ds \leq \int_0^t K_1^2\, ds = K_1^2\, t.$$

Let then Y be the exponential martingale associated to M and $\widetilde{\mathbb{P}}_T$ be the probability measure defined as $\widetilde{\mathbb{P}}(A) = \mathbb{E}(1_A Y_T)$.

Proposition 4.5. i) $(X_t,\, t \in [0, T])$ is a continuous square-integrable martingale under $\widetilde{\mathbb{P}}_T$.
ii) $(X_t,\, t \in [0, T])$ is even a standard Brownian motion under $\widetilde{\mathbb{P}}_T$!

Proof. Part (i) follows from what has been said above. For part (ii), we need the following fact, given here without proof.

Fact. The quadratic variation of a semi-martingale is invariant under a change of probability measure. Notice however that $(X_t^2 - \langle X \rangle_t)$ is a martingale only under the probability measure under which X is a martingale.

Here, we have $\langle X \rangle_t = \langle B \rangle_t = t$ under $\mathbb{P}$. So by the above fact, it also holds that $\langle X \rangle_t = t$ under $\widetilde{\mathbb{P}}_T$. Since we just proved that X is a continuous square-integrable martingale under $\widetilde{\mathbb{P}}_T$, we obtain by Lévy's theorem that X is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$. □

4.6 Second application to SDE's

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion, $x_0 \in \mathbb{R}$ and $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t,x), Lipschitz in x, and such that $|f(t,x)| \leq K_1 < \infty$ and $|g(t,x)| \geq K_2 > 0$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Let also $(X_t,\, t \in \mathbb{R}_+)$ be the strong solution of the SDE

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dB_t, \quad X_0 = x_0,$$

i.e.,

$$X_t - x_0 = \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

Under which probability measure $\widetilde{\mathbb{P}}_T$ is the process $(X_t,\, t \in [0,T])$ a martingale?

As above, we are looking for martingales Z and M under $\mathbb{P}$ such that

$$X_t - x_0 = Z_t - \langle M, Z \rangle_t.$$

In the above equation, there are two martingales under $\mathbb{P}$:

$$Z_t = B_t \quad \text{and} \quad Z_t = \int_0^t g(s, X_s)\, dB_s.$$

We will see that both choices lead to interesting conclusions, but let us start with the second one for now. Let us again assume that the martingale M is of the form

$$M_t = \int_0^t H_s\, dB_s$$

for some continuous adapted process H. This gives

$$X_t - x_0 = Z_t - \langle M, Z \rangle_t = \int_0^t g(s, X_s)\, dB_s - \int_0^t H_s\, g(s, X_s)\, ds.$$

As X satisfies the SDE

$$X_t - x_0 = \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s,$$

we obtain that $H_s = -\dfrac{f(s, X_s)}{g(s, X_s)}$. Indeed, the martingale M defined as

$$M_t = -\int_0^t \frac{f(s, X_s)}{g(s, X_s)}\, dB_s$$

is a continuous square-integrable martingale satisfying condition (3):

$$\langle M \rangle_t = \int_0^t \left(\frac{f(s, X_s)}{g(s, X_s)} \right)^2 ds \leq \int_0^t \frac{K_1^2}{K_2^2}\, ds = \frac{K_1^2}{K_2^2}\, t.$$

Let then Y be the exponential martingale associated to M and $\widetilde{\mathbb{P}}_T$ be the probability measure defined as $\widetilde{\mathbb{P}}(A) = \mathbb{E}(1_A Y_T)$. From what has been said above, we deduce the following proposition.

Proposition 4.6. $(X_t,\, t \in [0,T])$ is a continuous square-integrable martingale under $\widetilde{\mathbb{P}}_T$.

Remarks. - $\langle X \rangle_t = \int_0^t g(s, X_s)^2\, ds \neq t$ in general, so the process X cannot be transformed into a standard Brownian motion under any probability measure equivalent to $\mathbb{P}$.
- The condition that $|g(t,x)| \geq K_2 > 0$ is a *non-degeneracy* condition: it implies that the process X always has Brownian fluctuations. If this were not the case, then X would be the solution of a classical ODE in some interval, i.e., a deterministic function. And deterministic functions are not martingales, except if

they are constant. Moreover, changes of probability measure are obviously inoperative on deterministic functions!

Choosing now the first option $Z_t = B_t$ above, let us see what martingale under $\widetilde{\mathbb{P}}_T$ do we obtain (keeping the same choice for M, that is, the same change of probability measure). In this case,

$$Z_t - \langle M, Z\rangle_t = B_t + \int_0^t \frac{f(s,X_s)}{g(s,X_s)}\, ds$$

is a continuous square-integrable martingale under $\widetilde{\mathbb{P}}_T$, whose quadratic variation is equal to $\langle B\rangle_t = t$. By Lévy's theorem, we therefore obtain that

$$\widetilde{B}_t = B_t + \int_0^t \frac{f(s,X_s)}{g(s,X_s)}\, ds$$

is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$. In addition, notice that $d\widetilde{B}_t = dB_t + \dfrac{f(s,X_s)}{g(s,X_s)}\, dt$, so that

$$dX_t = g(t,X_t)\, d\widetilde{B}_t, \quad X_0 = x_0,$$

i.e., we have obtained here an alternate proof that the process X is a martingale under $\widetilde{\mathbb{P}}_T$ (since $\widetilde{B}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$).

Remark. There is no process $\widetilde{X}$!

4.7 A particular case: the Black-Scholes model

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion with respect to its natural filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Let $x_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and let us consider the SDE

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t, \quad X_0 = x_0. \tag{4}$$

The strong solution of this SDE represents the evolution in time of a stock price with drift μ and volatility σ, starting at x_0. As seen in the exercises, the solution is given by

$$X_t = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right), \quad t \in \mathbb{R}_+.$$

Under what probability measure $\widetilde{\mathbb{P}}_T$ is X a martingale? Does such a probability measure even exist? Here, $f(x) = \mu x$ is unbounded and $g(x) = \sigma x$ equals zero when $x = 0$, so there might be a problem. Nevertheless, let us try defining

$$M_t = -\int_0^t \frac{f(s,X_s)}{g(s,X_s)}\, dB_s = -\int_0^t \frac{\mu X_s}{\sigma X_s}\, dB_s = -\frac{\mu}{\sigma}\, B_t.$$

We see that $\langle M\rangle_t = \dfrac{\mu^2}{\sigma^2}\, t \le K\, t$; it is therefore possible to define a probability measure $\widetilde{\mathbb{P}}_T$ starting from this martingale M. The reason why there is no problem here is that f and g compensate each other exactly in this model. In addition, the degeneracy of g at $x = 0$ does not create a problem, as the process X remains always strictly positive.

Let now $Z_t = \displaystyle\int_0^t \sigma X_s\, dB_s$; Z is a martingale under $\mathbb{P}$ and

$$Z_t - \langle M, Z\rangle_t = \int_0^t \sigma X_s\, dB_s + \int_0^t \frac{\mu}{\sigma}\, \sigma X_s\, ds = X_t - x_0,$$

so X is a continuous square-integrable martingale under $\widetilde{\mathbb{P}}_T$, according to Girsanov's theorem.

Let also $Z_t = B_t$ be the standard Brownian motion under $\mathbb{P}$. Then

$$Z_t - \langle M, Z \rangle_t = B_t + \int_0^t \frac{\mu}{\sigma} \, ds = B_t + \frac{\mu}{\sigma} \, t = \widetilde{B}_t,$$

is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$.

Finally, observe that $dX_t = \sigma X_t \, d\widetilde{B}_t$, so

$$X_t = x_0 \, \exp\left(\sigma \widetilde{B}_t - \frac{\sigma^2 \, t}{2} \right).$$

4.8 Application : pricing of a European call option (Black-Scholes formula)

Remark. In the sequel, we assume for simplicity that the risk-free interest rate $r = 0$.

Definition 4.7. A *European call option* is the right to buy a stock X at a future time T (called the *maturity*), at a given price K (called the *strike*). The payoff of such an option at time T is therefore given by $C_T = \max(X_T - K, 0)$.

Let us assume that the time evolution of the stock X is given by the Black-Scholes equation of the previous section, with initial price x_0, drift μ and volatility σ. X being the strong solution of the SDE (4), it is adapted to the natural filtration of the standard Brownian motion B.

Question 1. What premium c_0 should the seller of such an option ask at time $t = 0$ in order to be ensured to recover the wealth C_T at time $t = T$?

From what has been done above, we know that there exists a probability measure $\widetilde{\mathbb{P}}_T$ under which the process $(X_t, \, t \in [0, T])$ is a martingale with respect to the Brownian filtration $(\mathcal{F}_t, \, t \in [0, T])$ and we also know that

$$dX_t = \sigma \, X_t \, d\widetilde{B}_t, \quad \forall t \in [0, T],$$

where $\widetilde{B}_t = B_t + \dfrac{\mu}{\sigma} \, t$ is the standard Brownian motion under $\widetilde{\mathbb{P}}_T$. Notice moreover that B and $\widetilde{B}$ are adapted to the same Brownian filtration.

In order to answer the above question, we need now the following *martingale representation theorem*, given here without proof.

Theorem 4.8. i) Let $(M_t, \, t \in [0, T])$ be a continuous square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t, \, t \in [0, T])$. There exists then a (unique) continuous process $(\psi_t, \, t \in [0, T])$ adapted to $(\mathcal{F}_t, \, t \in [0, T])$ such that

$$M_t = M_0 + \int_0^t \psi_s \, d\widetilde{B}_s \quad a.s., \quad \forall t \in [0, T].$$

ii) In particular, every square-integrable $\mathcal{F}_T$-measurable random variable M_T admits the (unique) following representation:

$$M_T = M_0 + \int_0^T \psi_t \, d\widetilde{B}_t \quad a.s.,$$

where M_0 is an $\mathcal{F}_0$-measurable random variable and $(\psi_t, \, t \in [0, T])$ is continuous and adapted to $(\mathcal{F}_t, \, t \in [0, T])$.

In the sequel, we will assume for simplicity that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, the trivial σ-field, so that every $\mathcal{F}_0$-measurable random variable is a constant. From part (ii) of the above theorem, we know therefore that there exist a constant $c_0 > 0$ and a continuous and adapted process $(\psi_t, \, t \in [0, T])$ such that

$$C_T = \max(X_T - K, 0) = c_0 + \int_0^T \psi_t \, d\widetilde{B}_t \quad a.s.$$

As $dX_t = \sigma\, X_t\, d\widetilde{B}_t$, this can be rewritten as

$$C_T = c_0 + \int_0^T \phi_t\, dX_t, \quad \text{where } \phi_t = \frac{\psi_t}{\sigma\, X_t}.$$

The above theorem does not tell us what the value of c_0 is. Nevertheless, it tells us that the c_0 defined above is the right value for the premium, as starting from an initial wealth c_0 and investing on the stock X with the strategy ϕ allows the seller to reach the wealth C_T at time T.

Question 2. How to compute c_0?

As $(X_t,\, t \in [0,T])$ is a martingale under $\widetilde{\mathbb{P}}_T$, the same is true for the process of gains $G_t = \int_0^t \phi_s\, dX_s$. So

$$\widetilde{\mathbb{E}}_T(C_T) = \underbrace{\widetilde{\mathbb{E}}_T(c_0)}_{=c_0 \text{ (constant)}} + \underbrace{\widetilde{\mathbb{E}}_T\left(\int_0^T \phi_t\, dX_t\right)}_{=0}, \quad \text{i.e.,} \quad c_0 = \widetilde{\mathbb{E}}_T(C_T) = \widetilde{\mathbb{E}}_T(\max(X_T - K, 0)).$$

Now, remember that

$$X_T = x_0 \, \exp\left(\sigma \widetilde{B}_T - \frac{\sigma^2 T}{2}\right),$$

and that $\widetilde{B}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$, i.e., $\widetilde{B}_T \sim \mathcal{N}(0,T)$ under $\widetilde{\mathbb{P}}_T$. Therefore,

$$\begin{aligned} c_0 &= \widetilde{\mathbb{E}}_T\left(\max\left(x_0\, \exp\left(\sigma \widetilde{B}_T - \frac{\sigma^2 T}{2}\right) - K, 0\right)\right) \\ &= \int_{\mathbb{R}} dy\, p_T(y)\, \max\left(x_0 \exp\left(\sigma y - \frac{\sigma^2 T}{2}\right) - K, 0\right), \end{aligned}$$

where

$$p_T(y) = \frac{1}{\sqrt{2\pi T}}\, \exp\left(-\frac{y^2}{2T}\right), \quad y \in \mathbb{R}.$$

This is the Black-Scholes formula. Notice that μ has disappeared from the formula, i.e., the drift of the stock price does not enter into the computation of the premium!

Remark. This does not solve yet the problem of deciding which strategy ϕ to apply in order to hedge the option C_T with the initial wealth c_0.

Let us now push the computation further: let y_0 be such that $x_0\, \exp\left(\sigma y_0 - \frac{\sigma^2 T}{2}\right) - K = 0$, i.e., $y_0 = \frac{1}{\sigma}\left(\log\left(\frac{K}{x_0}\right) + \frac{\sigma^2\, T}{2}\right)$. Then

$$\begin{aligned} c_0 &= \int_{y_0}^\infty dy\, \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{y^2}{2T}} \left(x_0\, e^{\sigma y - \frac{\sigma^2 T}{2}} - K\right) = x_0 \int_{y_0}^\infty dy\, \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{(y-\sigma T)^2}{2T}} - K \int_{y_0}^\infty dy\, \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} \\ &= x_0 \int_{y_0 - \sigma T}^\infty dz\, \frac{1}{\sqrt{2\pi T}}\, e^{-\frac{z^2}{2T}} - K \int_{y_0}^\infty dy\, \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} \\ &= x_0 \left(1 - N\left(\frac{y_0 - \sigma T}{\sqrt{T}}\right)\right) - K\left(1 - N\left(\frac{y_0}{\sqrt{T}}\right)\right) = x_0\, N(d_1) - K\, N(d_2), \end{aligned}$$

where $N(x) = \int_{-\infty}^x dz\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}$ and

$$\begin{aligned} d_1 &= \frac{\sigma T - y_0}{\sqrt{T}} = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{x_0}{K}\right) + \frac{\sigma^2 T}{2}\right), \\ d_2 &= -\frac{y_0}{\sqrt{T}} = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{x_0}{K}\right) - \frac{\sigma^2 T}{2}\right). \end{aligned}$$

5 Relation between SDE's and PDE's

The goal of this section is to show that solutions of classical (parabolic) partial differential equations (PDE's) can be represented by means of stochastic processes which are solutions of SDE's.

5.1 Forward PDE

Let $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t, x) and Lipschitz in x. Let us also assume that $|g(t, x)| \geq K > 0$, $\forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$ (NB: we have already encountered this non-degeneracy condition before), and let $u_0 : \mathbb{R} \to \mathbb{R}$ be continuous. We consider the following second order parabolic PDE:

$$\begin{cases} u_t'(t,x) = f(t,x)\, u_x'(t,x) + \dfrac{1}{2}\, g(t,x)^2\, u_{xx}''(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{5}$$

where the second line is called the *initial condition* for the above equation.

Fact. It can be shown that under the above assumptions, there exists a solution $u(t, x)$ to the above equation, which is moreover unique if we impose an additional (weak) technical condition on the growth of u in x.

Our aim in the following is to find a probabilistic representation of this solution. Let $T > 0$ and $x \in \mathbb{R}$ be fixed, and let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion. Let also $(X_t^x,\, t \in [0, T])$ be the (unique) strong solution of the following SDE:

$$\begin{cases} dX_t^x = f(T-t, X_t^x)\, dt + g(T-t, X_t^x)\, dB_t, & t \in [0,T], \\ X_0^x = x. \end{cases} \tag{6}$$

Lemma 5.1. Let u be the solution of (5) and X^x be the solution of (6).
Then the process $(u(T-t, X_t^x),\, t \in [0, T])$ is a martingale.

Proof. Applying Ito-Doeblin's formula to $u(T-t, X_t^x)$, we obtain

$$\begin{aligned} u(T-t, X_t^x) &= u(T-0, X_0^x) - \int_0^t u_t'(T-s, X_s^x)\, ds + \int_0^t u_x'(T-s, X_s^x)\, dX_s^x \\ &\quad + \frac{1}{2} \int_0^t u_{xx}''(T-s, X_s^x)\, d\langle X^x \rangle_s \\ &= u(T,x) + \int_0^t \Big(- u_t'(T-s, X_s^x) + f(T-s, X_s^x)\, u_x'(T-s, X_s^x) \\ &\quad + \frac{1}{2}\, g(T-s, X_s^x)^2\, u_{xx}''(T-s, X_s^x) \Big)\, ds + \int_0^t g(T-s, X_s^x)\, u_x'(T-s, X_s^x)\, dB_s, \end{aligned}$$

since

$$dX_s^x = f(T-s, X_s^x)\, ds + g(T-t, X_s^x)\, dB_s$$

and

$$d\langle X^x \rangle_s = g(T-s, X_s^x)^2\, ds.$$

As u satisfies (5), the integrand in the above Riemann integral is equal to zero, so we obtain that

$$u(T-t, X_t^x) = u(T,x) + \int_0^t g(T-s, X_s^x)\, u_x'(T-s, X_s^x)\, dB_s$$

is a martingale (remember that $T > 0$ is fixed). □

Corollary 5.2. Let u be the solution of (5) and X^x be the solution of (6).
Then for all $(T,x) \in \mathbb{R}_+ \times \mathbb{R}$, we have $u(T,x) = \mathbb{E}(u_0(X_T^x))$.

Proof. Indeed, by the above lemma,

$$\mathbb{E}(u(T-t, X_t^x)) = \mathbb{E}(u(T,x)) = u(T,x), \quad \forall t \in [0,T].$$

So choosing $t = T$ gives

$$u(T,x) = \mathbb{E}(u(0, X_T^x)) = \mathbb{E}(u_0(X_T^x)),$$

where the last equality is obtained using the initial condition of (5). □

The above formula is one of the many instances of the celebrated Feynman-Kac formula.

Particular case. If $f(t,x) \equiv 0$ and $g(t,x) \equiv 1$, then the PDE (5) becomes

$$\begin{cases} u_t'(t,x) = \dfrac{1}{2}\, u_{xx}''(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

and is called the *heat equation*. The SDE (6) becomes

$$\begin{cases} dX_t^x = dB_t, & t \in [0,T], \\ X_0^x = x. \end{cases}$$

whose solution $X_t^x = B_t^x$ is a Brownian motion starting at point $x \in \mathbb{R}$ at time $t = 0$. Notice that $B_t^x = B_t + x$, where B is a standard Brownian motion. The above corollary then says that

$$u(T,x) = \mathbb{E}(u_0(B_T^x)).$$

Since $B_T^x \sim \mathcal{N}(x,T)$, we further obtain that

$$u(T,x) = \int_{\mathbb{R}} dy\, p_T(x-y)\, u_0(y),$$

where

$$p_T(x-y) = \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{(x-y)^2}{2T}\right).$$

This is indeed the solution of the heat equation known from analysis. The term $p_T(x-y)$ is called the *Green Kernel* of equation (5); it is actually the solution of (5) if one replaces the initial condition u_0 with the Dirac measure δ_y.

5.2 Backward PDE

Let $T > 0$ be fixed and let $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t,x) and Lipschitz in x. Let us also assume that $|g(t,x)| \geq K > 0$, $\forall (t,x) \in [0,T] \times \mathbb{R}$, and let $h : \mathbb{R} \to \mathbb{R}$ be continuous. We consider the following second order parabolic PDE:

$$\begin{cases} u_t'(t,x) + f(t,x)\, u_x'(t,x) + \dfrac{1}{2}\, g(t,x)^2\, u_{xx}''(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ u(T,x) = h(x), & x \in \mathbb{R}, \end{cases} \tag{7}$$

where the second line is called the *terminal condition* for the above equation. This type of condition is of course more relevant to option pricing, where the option payoff is known at maturity. Notice also the sign difference in front of the term $u_t'(t,x)$ in the two types of equations.

Fact. Under the above assumptions, there exists a solution to the above equation, which is again unique under a technical condition on the growth of u is x.

Let now $t_0 \in [0,T[$ and $x_0 \in \mathbb{R}$ be fixed. Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and let $(X_t^{t_0,x_0},\, t \in [t_0,T])$ be the (unique) strong solution of the following SDE:

$$\begin{cases} dX_t = f(t,X_t)\,dt + g(t,X_t)\,dB_t, \quad t \in [t_0,T], \\ \\ X_{t_0} = x_0, \end{cases} \tag{8}$$

where we have not kept the superscripts t_0, x_0 in order to lighten the notation.

Lemma 5.3. Let u be the solution of (7) and X^{t_0,x_0} be the solution of (8).
Then the process $(u(t,X_t^{t_0,x_0}),\, t \in [t_0,T])$ is a martingale.

Proof. Applying Ito-Doeblin's formula, we obtain (dropping again the superscripts t_0, x_0):

$$\begin{aligned} u(t,X_t) &= u(t_0,X_{t_0}) + \int_{t_0}^t u_t'(s,X_s)\,ds + \int_{t_0}^t u_x'(s,X_s)\,dX_s \\ &\quad + \frac{1}{2}\int_{t_0}^t u_{xx}''(s,X_s)\,d\langle X\rangle_s \\ &= u(t_0,x_0) + \int_{t_0}^t \Big(u_t'(s,X_s) + f(s,X_s)\,u_x'(s,X_s) + \frac{1}{2}\,g(s,X_s)^2\,u_{xx}''(s,X_s)\Big)\,ds \\ &\quad + \int_{t_0}^t g(s,X_s)\,u_x'(s,X_s)\,dB_s, \end{aligned}$$

since

$$dX_s = f(s,X_s)\,ds + g(s,X_s)\,dB_s$$

and

$$d\langle X\rangle_s = g(s,X_s)^2\,ds.$$

As u satisfies (7), the integrand in the above Riemann integral is equal to zero and therefore,

$$u(t,X_t) = u(t_0,x_0) + \int_{t_0}^t g(s,X_s)\,u_x'(s,X_s)\,dB_s$$

is a martingale. □

Corollary 5.4. Let u be the solution of (7) and X^{t_0,x_0} be the solution of (8).
Then for all $(t_0,x_0) \in [0,T] \times \mathbb{R}$, we have $u(t_0,x_0) = \mathbb{E}(h(X_T^{t_0,x_0}))$.

Proof. Indeed,

$$\mathbb{E}(u(t,X_t^{t_0,x_0})) = \mathbb{E}(u(t_0,x_0)) = u(t_0,x_0), \quad \forall t \in [t_0,T].$$

So choosing $t = T$ and using the terminal condition of (7) gives

$$u(t_0,x_0) = \mathbb{E}(u(T,X_T^{t_0,x_0})) = \mathbb{E}(h(X_T^{t_0,x_0})).$$

□

Particular case. If $f(t,x) \equiv 0$ and $g(t,x) \equiv 1$, then the PDE (7) becomes

$$\begin{cases} u_t'(t,x) + \frac{1}{2}\,u_{xx}''(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ \\ u(T,x) = h(x), & x \in \mathbb{R}, \end{cases}$$

and the SDE (8) becomes

$$\begin{cases} dX_t = dB_t, & t \in [t_0, T], \\ X_{t_0} = x_0. \end{cases}$$

That is, $X_t = B_t^{t_0,x_0}$ is a Brownian motion starting at point $x_0 \in \mathbb{R}$ at time $t_0 \in [0,T]$. Notice that $B_t^{t_0,x_0} = B_t - B_{t_0} + x_0$, where B is a standard Brownian motion. Since $B_T^{t_0,x_0} \sim \mathcal{N}(x_0, T-t_0)$, we obtain from the above corollary that

$$u(t_0, x_0) = \mathbb{E}(h(B_T^{t_0,x_0})) = \int_{\mathbb{R}} dy\, p_{T-t_0}(x_0 - y)\, h(y),$$

where

$$p_{T-t_0}(x_0 - y) = \frac{1}{\sqrt{2\pi(T-t_0)}} \exp\left(-\frac{(x_0-y)^2}{2(T-t_0)}\right)$$

is the Green Kernel of the PDE (7).

Remarks. - The non-degeneracy condition $|g(t,x)| \geq K > 0$ is crucial to all this. Otherwise, the process X solution of the SDE may stop having fluctuations on some interval; for the corresponding PDE, this means that the term in u''_{xx} drops, which changes drastically the nature of the PDE (from second order to first order).

- It is possible to show directly that the function $u(t,x)$ *defined as* $\mathbb{E}(h(X_T^{t,x}))$ is solution of (7), but the proof is much more cumbersome!

5.3 Generator of a diffusion

Let $x_0 \in \mathbb{R}$, $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t,x), Lipschitz in x and such that $|g(t,x)| \geq K > 0$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Let also $(X_t,\, t \in \mathbb{R}_+)$ be the strong solution of the SDE

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dB_t, \quad X_0 = x_0.$$

X is sometimes called a diffusion process, or more simply a diffusion. In the previous section, we have seen that if $u(t,x)$ is a solution of the following PDE :

$$u'_t(t,x) + f(t,x)\, u'_x(t,x) + \frac{1}{2}\, g(t,x)^2\, u''_{xx}(t,x) = 0,$$

then the process $(u(t, X_t),\, t \in \mathbb{R}_+)$ is a martingale (notice that this holds without specifying a terminal condition for the PDE; it follows from a direct application of Ito-Doeblin's formula). Let us then define the linear differential operator $A_t : \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}) \to \mathcal{C}(\mathbb{R}_+ \times \mathbb{R})$ as

$$A_t u(t,x) = f(t,x)\, u'_x(t,x) + \frac{1}{2}\, g(t,x)^2\, u''_{xx}(t,x),$$

A_t is called the *(infinitesimal) generator of the diffusion X*. A reformulation of the above statement gives: if $u'_t(t,x) + A_t u(t,x) = 0$, then the process $(u(t, X_t),\, t \in \mathbb{R}_+)$ is a martingale.

More generally, the following statement holds: for any $u \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, the process

$$\left(u(t, X_t) - \int_0^t (u'_s(s, X_s) + A_s u(s, X_s))\, ds,\; t \in \mathbb{R}_+\right)$$

is a martingale. Again, this is a direct consequence of Ito-Doeblin's formula. Likewise, if $v \in \mathcal{C}^2(\mathbb{R})$, then the process

$$\left(v(X_t) - \int_0^t A_s v(X_s)\, ds,\; t \in \mathbb{R}_+\right)$$

is a martingale, where $A_t : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ is now defined as

$$A_t v(x) = f(t,x)\, v'(x) + \frac{1}{2}\, g(t,x)^2\, v''(x).$$

In the particular case where the diffusion X is time-homogeneous, i.e., where

$$dX_t = f(X_t)\, dt + g(X_t)\, dB_t, \quad X_0 = x_0,$$

with $f, g : \mathbb{R} \to \mathbb{R}$ Lipschitz and such that $|g(x)| \geq K > 0$, $\forall x \in \mathbb{R}$, we obtain that for any $v \in \mathcal{C}^2(\mathbb{R})$, the process

$$\left(v(X_t) - \int_0^t Av(X_s)\, ds,\; t \in \mathbb{R}_+ \right)$$

is a martingale, where $A : \mathcal{C}^2(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ is defined as

$$Av(x) = f(x)\, v'(x) + \frac{1}{2}\, g(x)^2\, v''(x).$$

Moreover, the following properties hold (the proof is left as an exercise):

$$\lim_{t\downarrow 0} \mathbb{E}\left(\frac{v(X_t) - v(x_0)}{t} \right) = Av(x_0), \quad \lim_{t\downarrow 0} \mathbb{E}\left(\frac{X_t - x_0}{t} \right) = f(x_0), \quad \lim_{t\downarrow 0} \mathbb{E}\left(\frac{(X_t - x_0)^2}{t} \right) = g(x_0)^2.$$

5.4 Markov property

Let $T > t > 0$ and $h : \mathbb{R} \to \mathbb{R}$ be continuous and bounded.

- Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be its natural filtration. We have already seen that

$$\mathbb{E}(h(B_T)|\mathcal{F}_t) = \mathbb{E}(h(B_T - B_t + B_t)|\mathcal{F}_t) = \varphi(B_t),$$

where $\varphi(x) = \mathbb{E}(h(B_T - B_t + x))$, as B_t is $\mathcal{F}_t$-measurable and $B_T - B_t$ is independent of $\mathcal{F}_t$. A similar reasoning gives that $\mathbb{E}(h(B_T)|B_t) = \varphi(B_t)$ also, so the Markov property holds for the standard Brownian motion B.

- Let $\mu \in \mathbb{R}$, $\sigma > 0$, $x_0 > 0$ and let us consider the SDE

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t, \quad X_0 = x_0.$$

We have already seen that the strong solution of this SDE is given by

$$X_t = x_0 \,\exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right)$$

and is adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$. We therefore also have

$$X_T = x_0 \,\exp\left(\left(\mu - \frac{\sigma^2}{2} \right) T + \sigma B_T \right) = X_t \,\exp\left(\left(\mu - \frac{\sigma^2}{2} \right) (T - t) + \sigma (B_T - B_t) \right).$$

This implies that

$$\mathbb{E}(h(X_T)|\mathcal{F}_t) = \mathbb{E}\left(h\left(X_t \,\exp\left(\left(\mu - \frac{\sigma^2}{2} \right) (T - t) + \sigma (B_T - B_t) \right) \right) \Big| \mathcal{F}_t \right) = \varphi(X_t)$$

where

$$\varphi(x) = \mathbb{E}\left(h\left(x \,\exp\left(\left(\mu - \frac{\sigma^2}{2} \right) (T - t) + \sigma (B_T - B_t) \right) \right) \right),$$

as X_t is $\mathcal{F}_t$-measurable and $B_T - B_t$ is independent of $\mathcal{F}_t$. Likewise, $\mathbb{E}(h(X_T)|\mathcal{F}_t) = \varphi(X_t)$, so the Markov property also holds for the process X.

- More generally, let $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t, x) and Lipschitz in x, and let X be the strong solution of the SDE

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dB_t, \quad X_0 = x_0.$$

It can then be shown in this more general case that

$$\mathbb{E}(h(X_T)|\mathcal{F}_t) = \varphi(X_t) = \mathbb{E}(h(X_T)|X_t),$$

where $\varphi(x) = \mathbb{E}(h(X_T^{t,x}))$, with $X^{t,x}$ the strong solution of

$$\begin{cases} dX_s = f(s, X_s)\, ds + g(s, X_s)\, dB_s, \quad s \in [t, T], \\ X_t = x. \end{cases}$$

5.5 Application: option pricing and hedging

Let $(B_t,\, t \in \mathbb{R}_+)$ be a standard Brownian motion and $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ be its natural filtration. Let $x_0 \in \mathbb{R}$, $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be jointly continuous in (t, x), Lipschitz in x and such that $|f(t, x)| \leq K_1 < \infty$ and $|g(t, x)| \geq K_2 > 0, \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Let X be the strong solution of the SDE

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dB_t, \quad X_0 = 0,$$

and assume that this SDE describes the time evolution of a stock price X, in a market where the risk-free interest rate $r = 0$.

Definition 5.5. A *European option on the stock X with maturity T* is an option whose payoff Z_T at time T only depends on the final value of the stock X_T, i.e., $Z_T = h(X_T)$, for some function $h \in \mathcal{C}(\mathbb{R})$.

Question 1. What premium Z_t should the seller of such an option ask at time $t < T$?

Question 2. What strategy $(\phi_s,\, s \in [t, T])$ should the seller use during the time interval $[t, T]$ in order to hedge the option?

As already seen, there exists a (unique) probability measure $\widetilde{\mathbb{P}}_T$ under which the process $(X_t,\, t \in [0, T])$ is a martingale. Moreover,

$$dX_t = g(t, X_t)\, d\widetilde{B}_t,$$

where $\widetilde{B}$ is a standard Brownian motion under $\widetilde{\mathbb{P}}_T$.

Using again part (ii) of Theorem 4.8, we know that there exist a constant $z_0 \in \mathbb{R}$ and a continuous and adapted process $(\psi_s,\, s \in [0, T])$ such that the payoff $Z_T = h(X_T)$ of the above option may be written as

$$Z_T = z_0 + \int_0^T \psi_t\, d\widetilde{B}_t.$$

As $d\widetilde{B}_t = \frac{dX_t}{g(t, X_t)}$, this says that

$$Z_T = z_0 + \int_0^T \phi_t\, dX_t, \tag{9}$$

where $\phi_t = \frac{\psi_t}{g(t, X_t)}$ is also a continuous and adapted process (remember that by assumption, $|g(t, x)| \geq K > 0$). Let us now *define* Z_t as

$$Z_t = z_0 + \int_0^t \phi_s\, dX_s.$$

It is then clear that

$$Z_T = Z_t + \int_t^T \phi_s\, dX_s.$$

At time $t < T$, Z_t is therefore the right price for the premium, as it allows to reach a wealth Z_T at time T with the strategy ϕ on $[t,T]$.

Defining now $G_t = \int_0^t \phi_s \, dX_s$, we obtain

$$\widetilde{\mathbb{E}}_T(Z_T|\mathcal{F}_t) = \widetilde{\mathbb{E}}_T(Z_t|\mathcal{F}_t) + \widetilde{\mathbb{E}}_T\left(\int_t^T \phi_s \, dX_s \middle| \mathcal{F}_t\right) = Z_t + \widetilde{E}_T(G_T - G_t|\mathcal{F}_t) = Z_t,$$

as Z_t is $\mathcal{F}_t$-measurable and G is a martingale under $\widetilde{\mathbb{P}}_T$. So

$$Z_t = \widetilde{\mathbb{E}}_T(Z_T|\mathcal{F}_t) = \widetilde{\mathbb{E}}_T(h(X_T)|\mathcal{F}_t) = \widetilde{\mathbb{E}}_T(h(X_T)|X_t) = z(t, X_t),$$

as X satisfies the Markov property. Moreover, we have seen above that

$$z(t,x) = \widetilde{\mathbb{E}}_T(h(X_T^{t,x})),$$

where $X^{t,x}$ is the strong solution of

$$\begin{cases} dX_s = g(s, X_s) \, d\widetilde{B}_s, \quad s \in [t,T], \\ X_t = x. \end{cases}$$

Remembering now the link established previously between SDE's and PDE's, we see that $z(t,x)$ is the solution of the following PDE:

$$\begin{cases} z'_t(t,x) + \dfrac{1}{2} g(t,x)^2 \, z''_{xx}(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}, \\ z(T,x) = h(x), & x \in \mathbb{R}. \end{cases}$$

So the premium at time t is given by $Z_t = z(t, X_t)$, where $z(t,x)$ is the solution of the above PDE. This solves Question 1.

Now, what about Question 2, i.e, the hedging strategy ϕ? Using Ito-Doeblin's formula, we see that

$$\begin{aligned} Z_T - z_0 &= z(T, X_T) - z(0, x_0) \\ &= \int_0^T z'_t(t, X_t) \, dt + \int_0^T z'_x(t, X_t) \, dX_t + \frac{1}{2} \int_0^T z''_{xx}(t, X_t) \, d\langle X \rangle_t. \end{aligned}$$

As $d\langle X \rangle_t = g(t, X_t)^2 \, dt$, we obtain

$$\begin{aligned} Z_T - z_0 &= \int_0^T \left(z'_t(t, X_t) + \frac{1}{2} g(t, X_t)^2 \, z''_{xx}(t, X_t) \right) dt + \int_0^T z'_x(t, X_t) \, dX_t \\ &= \int_0^T z'_x(t, X_t) \, dX_t, \end{aligned}$$

since $z(t,x)$ satisfies the above PDE. Comparing now this formula with (9), we deduce that $\phi_t = z'_x(t, X_t)$, for $t \in [0,T]$. This strategy is called the *delta-hedging strategy* (where *delta* actually stands for *derivative*).

6 Multidimensional processes

In this chapter, we quickly review all the notions of the class in the multidimensional context. The aim here is to point towards concepts whose generalization to the multidimensional case is not immediate. Let us start with some basic definitions.

- A *standard n-dimensional Brownian motion* is a vector $\underline{B} = (B^{(1)}, \ldots, B^{(n)})$ of n standard (one-dimensional) Brownian motions with respect to a given filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, which are moreover assumed to be *independent*.

- An *n-dimensional martingale* is a vector $\underline{M} = (M^{(1)}, \ldots, M^{(n)})$ of n martingales with respect to a given filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ (these need not be independent).

- An *n-dimensional continuous semi-martingale* is a vector $\underline{X} = (X^{(1)}, \ldots, X^{(n)})$ of n continuous semi-martingales with respect to a given filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ (again, these need not be independent).

Multidimensional stochastic integral.
- Let $\underline{B} = (B^{(1)}, \ldots, B^{(m)})$ be a standard m-dimensional Brownian motion with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$.

- Let $H = (H^{(i,j)})_{i,j=1}^{n,m}$ be an $n \times m$ matrix of continuous processes adapted to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and such that

$$\mathbb{E}\left(\int_0^t (H_s^{(i,j)})^2 \, ds\right) < \infty, \quad \forall t \in \mathbb{R}_+,\, 1 \le i \le n,\, 1 \le j \le m.$$

- Let us then define the processes

$$M_t^{(i)} = \sum_{j=1}^m \int_0^t H_s^{(i,j)} \, dB_s^{(j)} = \sum_{j=1}^m (H^{(i,j)} \cdot B^{(j)})_t, \quad 1 \le i \le n.$$

The process $\underline{M} = (M^{(1)}, \ldots, M^{(n)})$ is an n-dimensional continuous square-integrable martingale. We may use the following *vector notation:* $\underline{M}_t = \int_0^t H_s \, d\underline{B}_s$.

Interpretation. $\underline{M}$ describes the fluctuations of n processes generated by m independent sources of noise $B^{(1)}, \ldots, B^{(m)}$. The quadratic covariations of these processes can be computed as follows :

$$\begin{aligned} \langle M^{(i)}, M^{(k)} \rangle_t &= \sum_{j,l=1}^m \langle (H^{(i,j)} \cdot B^{(j)}), (H^{(k,l)} \cdot B^{(l)}) \rangle_t \\ &= \sum_{j,l=1}^m \int_0^t H_s^{(i,j)} \, H_s^{(k,l)} \, d\langle B^{(j)}, B^{(l)} \rangle_s = \sum_{j=1}^m \int_0^t H_s^{(i,j)} \, H_s^{(k,j)} \, ds, \end{aligned}$$

since

$$\langle B^{(j)}, B^{(l)} \rangle_t = \begin{cases} t, & \text{if } j = l, \\ 0, & \text{otherwise, since } B^{(j)} \perp\!\!\!\perp B^{(l)} \text{ for } j \neq l. \end{cases}$$

6.1 Multidimensional Ito-Doeblin's formula

Let $\underline{X} = (X^{(1)}, \ldots, X^{(n)})$ be an n-dimensional continuous semi-martingale and let $f \in \mathcal{C}^2(\mathbb{R}^n)$ (with values in $\mathbb{R}$) be such that

$$\mathbb{E}\left(\int_0^t (f'_{x_i}(\underline{X}_s))^2 \, d\langle X^{(i)} \rangle_s\right) < \infty, \quad \forall t \in \mathbb{R}_+,\, 1 \le i \le n.$$

Then

$$f(\underline{X}_t) - f(\underline{X}_0) = \sum_{i=1}^n \int_0^t f'_{x_i}(\underline{X}_s) \, dX_s^{(i)} + \frac{1}{2} \sum_{i,k=1}^n \int_0^t f''_{x_i,x_k}(\underline{X}_s) \, d\langle X^{(i)}, X^{(k)} \rangle_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

Remark. This formula contains all the previous versions that we have seen.

Important example. If $\underline{X} = \underline{B}$ is a standard n-dimensional Brownian motion, then as seen above,

$$\langle B^{(i)}, B^{(k)} \rangle_t = \begin{cases} t, & \text{if } i = k, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, if

$$\mathbb{E}\left(\int_0^t (f'_{x_i}(\underline{B}_s))^2 \, ds \right) < \infty, \quad \forall t \in \mathbb{R}_+, \, 1 \leq i \leq n, \tag{10}$$

then

$$f(\underline{B}_t) - f(\underline{B}_0) = \sum_{i=1}^n \int_0^t f'_{x_i}(\underline{B}_s) \, dB_s^{(i)} + \frac{1}{2} \int_0^t \Delta f(\underline{B}_s) \, ds \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

where

$$\Delta f(\underline{x}) = \sum_{i=1}^n f''_{x_i, x_i}(\underline{x})$$

is the Laplacian of the function f. Therefore, if condition (10) is satisfied, then the process

$$\left(f(\underline{B}_t) - f(\underline{B}_0) - \frac{1}{2} \int_0^t \Delta f(\underline{B}_s) \, ds, \, t \in \mathbb{R}_+ \right)$$

is a martingale (*NB:* this says that $A = \frac{1}{2} \Delta$ is the generator of the n-dimensional Brownian motion $\underline{B}$). In particular:

- If f is harmonic (i.e., $\Delta f(\underline{x}) = 0$, $\forall \underline{x} \in \mathbb{R}^n$), then the process $(f(\underline{B}_t), \, t \in \mathbb{R}_+)$ is a martingale.

- If f is superharmonic (i.e., $\Delta f(\underline{x}) \leq 0$, $\forall \underline{x} \in \mathbb{R}^n$), then the process $(f(\underline{B}_t), \, t \in \mathbb{R}_+)$ is a supermartingale.

- If f is subharmonic (i.e., $\Delta f(\underline{x}) \geq 0$, $\forall \underline{x} \in \mathbb{R}^n$), then the process $(f(\underline{B}_t), \, t \in \mathbb{R}_+)$ is a submartingale.

This is the explanation for the counter-intuitive terminology adopted for sub- and supermartingales.

6.2 Multidimensional SDE's

Let $\underline{x}_0 \in \mathbb{R}^n$ and $\underline{B}$ be a standard m-dimensional Brownian motion. We need now an extension of the notion of Lipschitz function to the multidimensional case.

Definition 6.1. A function $\underline{f} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, \underline{x}) \mapsto \underline{f}(t, \underline{x})$ is *Lipschitz in* $\underline{x}$ if there exists a constant $L > 0$ such that

$$\|\underline{f}(t, \underline{x}) - \underline{f}(t, \underline{y})\| \leq L \|\underline{x} - \underline{y}\|, \quad \forall t \in \mathbb{R}_+, \, \underline{x}, \underline{y} \in \mathbb{R}^n,$$

where $\|\underline{x}\|^2 = \sum_{i=1}^n x_i^2$ is the Euclidean norm in $\mathbb{R}^n$.

Let then $\underline{f}, \underline{g}^{(1)}, \ldots, \underline{g}^{(m)} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be jointly continuous in $(t, \underline{x})$ and Lipschitz in $\underline{x}$. Let us consider the multidimensional SDE

$$d\underline{X}_t = \underline{f}(t, \underline{X}_t) \, dt + \sum_{j=1}^m \underline{g}^{(j)}(t, \underline{X}_t) \, dB_t^{(j)}, \quad \underline{X}_0 = \underline{x}_0,$$

which reads, component by component, as

$$dX_t^{(i)} = f^{(i)}(t, \underline{X}_t) \, dt + \sum_{j=1}^m g^{(i,j)}(t, \underline{X}_t) \, dB_t^{(j)}, \quad X_0^{(i)} = x_0^{(i)}, \quad 1 \leq i \leq n,$$

where $g^{(i,j)}$ stands for the i^{th}-component of the function $\underline{g}^{(j)}$.

Remark. In general, these n equations are coupled, which makes their resolution much harder than in the one-dimensional case.

Theorem 6.2. Under the above assumptions, there exists a unique strong solution $\underline{X}$ to the above SDE.

Example 1.

$$\begin{cases} dX_t = -\frac{1}{2} X_t\, dt - Y_t\, dB_t, & X_0 = 1, \\ dY_t = -\frac{1}{2} Y_t\, dt + X_t\, dB_t, & Y_0 = 0. \end{cases}$$

Here, $n = 2$ and $m = 1$, $\underline{f}(t,x,y) = (-\frac{1}{2}x, -\frac{1}{2}y)$ and $\underline{g}^{(1)}(t,x,y) = (-y, x)$ are jointly continuous in (t,x,y) and Lipschitz in (x,y) (since linear). There exists therefore a unique strong solution (X_t, Y_t). We have seen in the exercices that $X_t = \cos(B_t)$ and $Y_t = \sin(B_t)$. That is, the solution "lives" on the circle of radius 1, i.e., a submanifold of dimension 1 in $\mathbb{R}^2$. This is related to the fact the two-dimensional SDE is "driven" by a single one-dimensional Brownian motion.

Example 2.

$$\begin{cases} dX_t = Y_t\, dt, & X_0 = 1, \\ dY_t = -X_t\, dt + X_t\, dB_t, & Y_0 = 0. \end{cases}$$

Here again, there exists a unique strong solution. Nevertheless, as simple as this equation may look, its solution does not have a simple analytical form!

Notice also that if we allow ourselves to write $\frac{dB_t}{dt}$ (the *white noise*), we can rewrite the above equation as a second order SDE:

$$\frac{d^2 X_t}{dt^2} = -X_t + X_t \frac{dB_t}{dt}.$$

Example 3. Multidimensional Black-Scholes equation.

$$dX_t^{(i)} = \mu_i X_t^{(i)}\, dt + \sum_{j=1}^m \sigma_{ij} X_t^{(i)}\, dB_t^{(j)}, \quad X_0^{(i)} = x_0^{(i)}, \quad 1 \le i \le n.$$

Notice that these equations are decoupled and therefore much easier to solve than the previous ones. The solution reads

$$X_t^{(i)} = x_0^{(i)} \exp\left(\left(\mu_i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij}^2\right) t + \sum_{j=1}^m \sigma_{ij} B_t^{(j)}\right), \quad 1 \le i \le n.$$

Whether the n-dimensional process $\underline{X} = (X^{(1)}, \ldots, X^{(n)})$ fills the whole space $\mathbb{R}^n$ or not depends on the number of Brownian motions (or sources of noise) generating it, as well as the volatilities σ_{ij}. We will come back to this when talking about the existence of a martingale measure.

Linear multidimensional SDE's (or multidimensional Ornstein-Uhlenbeck process).
Let $\underline{x}_0 \in \mathbb{R}^n$, $\underline{B}$ be a standard m-dimensional Brownian motion, $A = (a_{ik})$ be an $n \times n$ matrix and $\Sigma = (\sigma_{ij})$ be an $n \times m$ matrix. We consider the SDE

$$d\underline{X}_t = A\underline{X}_t\, dt + \Sigma\, d\underline{B}_t, \quad \underline{X}_0 = \underline{x}_0.$$

Here, $f^{(i)}(t, \underline{x}) = \sum_{k=1}^n a_{ik} x_k$ and $g^{(i,j)}(t, \underline{x}) = \sigma_{ij}$. There exists therefore a unique strong solution $\underline{X}$ to the above equation.

Solving method.
- Let Φ be the $n \times n$ matrix-valued (and deterministic) process solution of

$$d\Phi_t = A\Phi_t\, dt, \quad \Phi_0 = I.$$

The solution of this equation reads $\Phi_t = \exp(tA) = \sum_{k \ge 0} \frac{(tA)^k}{k!}$, $t \in \mathbb{R}$. It can moreover be shown that

$$(\Phi_t)^{-1} = \Phi_{-t} \quad \text{and} \quad \Phi_t \Phi_s = \Phi_s \Phi_t = \Phi_{t+s}. \tag{11}$$

- Let us then write $\underline{X}_t = \Phi_t \, \underline{Y}_t$. Since Φ has bounded variation, we obtain, using the multidimensional integration by parts formula, that

$$d\underline{X}_t = (d\Phi_t) \, \underline{Y}_t + \Phi_t \, d\underline{Y}_t + 0 = A\Phi_t \underline{Y}_t \, dt + \Phi_t \, d\underline{Y}_t.$$

Since $d\underline{X}_t = A\underline{X}_t \, dt + \Sigma \, d\underline{B}_t$ also, we obtain that $\Phi_t \, d\underline{Y}_t = \Sigma \, d\underline{B}_t$, i.e.,

$$d\underline{Y}_t = \Phi_t^{-1} \Sigma \, d\underline{B}_t, \quad \underline{Y}_0 = \underline{x}_0,$$

leading to

$$\underline{Y}_t = \underline{x}_0 + \int_0^t \Phi_s^{-1} \, \Sigma \, d\underline{B}_s.$$

Finally,

$$\underline{X}_t = \Phi_t \, \underline{x}_0 + \int_0^t \Phi_t \, \Phi_s^{-1} \, \Sigma \, d\underline{B}_s = \Phi_t \, \underline{x}_0 + \int_0^t \Phi_{t-s} \, \Sigma \, d\underline{B}_s.$$

using the above properties (11) of the process Φ.

6.3 Drift vector, diffusion matrix and weak solution

Let $\underline{X} = \left(X^{(1)}, \ldots, X^{(n)}\right)$ be the strong solution of the multidimensional SDE

$$dX_t^{(i)} = f^{(i)}(t, X_t) \, dt + \sum_{j=1}^m g^{(i,j)}(t, X_t) \, dB_t^{(j)}, \quad X_0^{(i)} = x_0^{(i)}, \quad 1 \leq i \leq n, \tag{12}$$

where $\underline{B} = \left(B^{(1)}, \ldots, B^{(m)}\right)$ is a standard m-dimensional Brownian motion and $\underline{f}$ and $\underline{g}^{(1)}, \ldots, \underline{g}^{(m)}$ are jointly continuous in $(t, \underline{x})$ and Lipschitz in $\underline{x}$. In order to define what is a weak solution of equation (12), we first observe that

(i) The process $M_t^{(i)} = X_t^{(i)} - \int_0^t f^{(i)}(s, \underline{X}_s) \, ds$ is a martingale $\forall 1 \leq i \leq n$.

The vector $\underline{f}(t, \underline{x})$ is called the *drift vector* of the process $\underline{X}$.

Next, let us compute the quadratic covariation of the martingales $M^{(i)}$ and $M^{(k)}$:

$$\begin{aligned} \langle M^{(i)}, M^{(k)} \rangle_t &= \sum_{j,l=1}^m \int_0^t g^{(i,j)}(s, \underline{X}_s) \, g^{(k,l)}(s, \underline{X}_s) \, d\langle B^{(j)}, B^{(k)} \rangle_s \\ &= \sum_{j=1}^m \int_0^t g^{(i,j)}(s, \underline{X}_s) g^{(k,j)}(s, \underline{X}_s) \, ds. \end{aligned}$$

We therefore obtain that

(ii) The process $N_t^{(i,k)} = M_t^{(i)} M_t^{(k)} - \sum_{j=1}^m \int_0^t g^{(i,j)}(s, \underline{X}_s) \, g^{(k,j)}(s, \underline{X}_s) \, ds$ is a martingale $\forall 1 \leq i, \, k \leq n$.

The matrix $G(t, \underline{x})$ whose entries are defined as

$$G^{(i,k)}(t, \underline{x}) = \sum_{j=1}^m g^{(i,j)}(t, \underline{x}) g^{(k,j)}(t, \underline{x})$$

is called the *diffusion matrix* of the process $\underline{X}$. In matrix notation, this gives $G(t, \underline{x}) = g(t, \underline{x}) g(t, \underline{x})^T$. This leads finally to the following definition.

Definition 6.3. An n-dimensional process $\underline{X}$ such that the processes $M^{(i)}$ and $N^{(i,k)}$ defined in (i) and (ii) respectively are martingales $\forall 1 \leq i, \, k \leq n$ is called a weak solution of the SDE (12).

The process $\underline{X}$ is called a *diffusion*; its statistical properties are entirely characterized by the drift vector $\underline{f}(t, \underline{x})$ and the diffusion matrix $G(t, \underline{x})$. Notice that there are different matrices $g(t, \underline{x})$ leading to the same diffusion matrix $G(t, \underline{x})$ (like in the one-dimensional case, where $g(t, x)$ and $-g(t, x)$ give rise to the same diffusion coefficient $g(t, x)^2$). Finally, notice that the process $\underline{X}$ satisfies the Markov property.

6.4 Existence of a martingale measure

Question. Under which conditions on $\underline{f}$ and G does there exist a probability measure $\widetilde{\mathbb{P}}_T$ under which the process $\underline{X}$ solution of (12) is an n-dimensional martingale up to time T (i.e., $X^{(1)}, \ldots, X^{(n)}$ are simultaneously martingales under $\widetilde{\mathbb{P}}_T$)?

Remark. The question of whether the measure $\widetilde{\mathbb{P}}_T$ is unique is not addressed here (and was not addressed before either).

Let us make the following additional assumptions on $\underline{f}$ and G:

(i) $\exists K_1 < \infty$ such that $\|\underline{f}(t, \underline{x})\| \leq K_1$, $\forall (t, \underline{x}) \in \mathbb{R}_+ \times \mathbb{R}^n$.

(ii) $\exists K_2 > 0$ such that $\sum_{i,j=1}^n G^{(i,j)}(t, \underline{x})\, \xi_i\, \xi_k \geq K_2\, \|\underline{\xi}\|^2$, $\forall (t, \underline{x}) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\forall \underline{\xi} \in \mathbb{R}^n$.

If assumption (ii) is satisfied, the diffusion $\underline{X}$ is said to be *non-degenerate.* Let us make here some remarks about this non-degeneracy condition, starting with a slightly more general one: we say that the diffusion $\underline{X}$ is *non-degenerate in the domain* $D \subset \mathbb{R}^n$ if

$$\sum_{i,k=1}^n G^{(i,k)}(t, \underline{x})\, \xi_i\, \xi_k > 0 \quad \forall (t, \underline{x}) \in \mathbb{R}_+ \times D, \forall \underline{\xi} \in \mathbb{R}^n \text{ such that } \xi \neq 0. \tag{13}$$

Notice that since $G(t, \underline{x}) = g(t, \underline{x})\, g(t, \underline{x})^T$, it always holds that $\forall (t, \underline{x}) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n G^{(i,k)}(t, \underline{x})\, \xi_i\, \xi_k = \underline{\xi}^T\, G(t, \underline{x})\, \underline{\xi} = \|g(t, \underline{x})^T\, \underline{\xi}\|^2 \geq 0,$$

i.e., that the matrix $G(t, \underline{x})$ is positive semi-definite. Condition (13) with the strict inequality imposes moreover that $G(t\underline{x})$ is positive definite on the domain D. Conditions equivalent to (13) are :

- all eigenvalues of $G(t, \underline{x})$ are strictly positive, $\forall (t, \underline{x}) \in \mathbb{R}_+ \times D$.
- $\det G(t, \underline{x}) > 0$, $\forall (t, \underline{x}) \in \mathbb{R}_+ \times D$.
- $G(t, \underline{x})$ is invertible, $\forall (t, \underline{x}) \in \mathbb{R}_+ \times D$.
- $\text{rank}(G(t, \underline{x})) = n$, $\forall (t, \underline{x}) \in \mathbb{R}_+ \times D$.

Notice that as $G(t, \underline{x}) = g(t, \underline{x})\, g(t, \underline{x})^T$ and $g(t, \underline{x})$ is an $n \times m$ matrix, $\text{rank}(G(t, \underline{x}) \leq \min(n, m)$. So if $m < n$ (i.e., if the number of Brownian motions is less than the number of processes X in (12)), then the above condition cannot be satisfied, so the diffusion $\underline{X}$ is degenerate.

Proposition 6.4. Under the above assumptions (i) and (ii), there exists a probability measure $\widetilde{\mathbb{P}}_T$ under which the process $\underline{X}$ solution of (12) is a multidimensional martingale.

Proof. For the proof, we follow the strategy used in the one-dimensional case, i.e., we search for a martingale M under $\mathbb{P}$ such that $\langle M \rangle_t \leq Kt$ and n martingales $Z^{(1)}, \ldots, Z^{(n)}$ under $\mathbb{P}$ such that

$$Z_t^{(i)} - \langle M, Z^{(i)} \rangle_t = X_t^{(i)} - X_0^{(i)}, \quad \forall 1 \leq i \leq n, \tag{14}$$

as we know from Girsanov's theorem that if this is the case, then the processes $X^{(i)}$ are all martingales under the probability measure $\widetilde{\mathbb{P}}_T$ defined from the martingale M (remember that $\widetilde{\mathbb{P}}_T$ is defined as $\widetilde{\mathbb{P}}_T(A) = \mathbb{E}(1_A\, Y_T)$, where $Y_T = \exp(M_T - \langle M \rangle_T / 2)$).

Natural candidates for the processes $Z^{(i)}$ are the martingale parts of the processes $X^{(i)}$, i.e.,

$$Z_t^{(i)} = \sum_{j=1}^m \int_0^t g^{(i,j)}(s, \underline{X}_s)\, dB_s^{(j)}, \quad 1 \leq i \leq n.$$

Let us then define

$$h^{(l)}(t, \underline{x}) = \sum_{k=1}^{n} f^{(k)}(t, \underline{x})\,(G^{-1})^{(k,l)}(t, \underline{x}), \quad 1 \le l \le n,$$

and

$$M_t = -\sum_{l=1}^{n} \int_0^t h^{(l)}(s, \underline{X}_s)\, dZ_s^{(l)}.$$

Notice first that the functions $h^{(l)}(t, \underline{x})$ are well defined, as $G(t, \underline{x})$ is invertible by assumption (ii), and that the process M is a martingale (under $\mathbb{P}$), being a multiple sum of stochastic integrals (leaving aside the usual technical condition). Let us then check that $\langle M \rangle_t \le Kt$ for some $K > 0$: M can be rewritten as

$$\begin{aligned} M_t &= -\sum_{k,l=1}^{n} \sum_{j=1}^{m} \int_0^t f^{(k)}(s, \underline{X}_s)\,(G^{-1})^{(k,l)}(s, \underline{X}_s)\, g^{(l,j)}(s, \underline{X}_s)\, dB_s^{(j)} \\ &= -\sum_{j=1}^{m} \int_0^t \underline{f}(s, \underline{X}_s)^T\, G^{-1}(s, \underline{X}_s)\, \underline{g}^{(j)}(s, \underline{X}_s)\, dB_s^{(j)}, \end{aligned}$$

in matrix form. So

$$\begin{aligned} \langle M \rangle_t &= \sum_{j=1}^{m} \int_0^t \Big(\underline{f}(s, \underline{X}_s)^T\, G^{-1}(s, \underline{X}_s)\, \underline{g}^{(j)}(s, \underline{X}_s) \Big)^2\, ds \\ &= \sum_{j=1}^{m} \int_0^t \underline{f}(s, \underline{X}_s)^T\, G^{-1}(s, \underline{X}_s)\, \underline{g}^{(j)}(s, \underline{X}_s)\, \underline{g}^{(j)}(s, \underline{X}_s)^T\, G^{-1}(s, \underline{X}_s)\, \underline{f}(s, \underline{X}_s)\, ds \\ &= \int_0^t \underline{f}(s, \underline{X}_s)^T\, G^{-1}(s, \underline{X}_s)\, G(s, \underline{X}_s)\, G^{-1}(s, \underline{X}_s)\, \underline{f}(s, \underline{X}_s)\, ds \\ &= \int_0^t \underline{f}(s, \underline{X}_s)^T\, G^{-1}(s, \underline{X}_s)\, \underline{f}(s, \underline{X}_s)\, ds. \end{aligned}$$

We now use assumptions (i) and (ii). Assumption (ii) actually says that $\exists K_2 > 0$ such that

$$\underline{\xi}^T\, G^{-1}(t, \underline{x})\, \underline{\xi} \le \frac{1}{K_2}\, \|\underline{\xi}\|^2, \quad \forall (t, \underline{x}) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad \forall \underline{\xi} \in \mathbb{R}^n.$$

This implies that

$$\langle M \rangle_t \le \frac{1}{K_2} \int_0^t \|\underline{f}(s, \underline{X}_s)\|^2\, ds \le \frac{K_1^2}{K_2}\, t,$$

where we have used assumption (i) for the second inequality. So by what was said in the one-dimensional case, we know that $\widetilde{\mathbb{P}}_T$ defined above is a valid probability measure. Let us then check equation (14), computing first $\langle M, Z^{(i)} \rangle_t$ for $1 \le i \le n$:

$$\begin{aligned} \langle M, Z^{(i)} \rangle_t &= -\sum_{k,l=1}^{n} \sum_{j=1}^{m} \int_0^t f^{(k)}(s, \underline{X}_s)\,(G^{-1})^{(k,l)}(s, \underline{X}_s)\, g^{(l,j)}(s, \underline{X}_s)\, g^{(i,j)}(s, \underline{X}_s)\, ds \\ &= -\sum_{l,k=1}^{n} \int_0^t f^{(k)}(s, \underline{X}_s)\,(G^{-1})^{(k,l)}(s, \underline{X}_s)\, G^{(l,i)}(s, \underline{X}_s)\, ds \\ &= -\sum_{k=1}^{n} \int_0^t f^{(k)}(s, \underline{X}_s)\, \delta_{ki}\, ds = -\int_0^t f^{(i)}(s, \underline{X}_s)\, ds. \end{aligned}$$

Therefore, using the definition of $Z^{(i)}$ and the fact that $X^{(i)}$ is solution of (12), we obtain that

$$Z_t^{(i)} - \langle M, Z^{(i)} \rangle_t = X_t^{(i)} - X_0^{(i)}, \quad \forall 1 \le i \le n,$$

and these process are all martingales under $\widetilde{\mathbb{P}}_T$ by Girsanov's theorem, so the proposition is proved. □

6.5 Relation between SDE's and PDE's in the multidimensional case

The stochastic representation of solutions of second order parabolic PDE's can be generalized to the multidimensional case in a relatively straightforward manner. Namely, if $\underline{X}^{t_0,\underline{x}_0}$ is the solution of the SDE (12) with the initial condition replaced by $\underline{X}_{t_0} = \underline{x}_0$, if the above assumptions (i) and (ii) are satisfied and if $u(t,\underline{x})$ is the solution of the PDE

$$\begin{cases} u'_t(t,x) + \sum_{i=1}^n f^{(i)}(t,\underline{x})\, u'_{x_i}(t,\underline{x}) + \frac{1}{2} \sum_{i,k=1}^n G^{(i,k)}(t,x)\, u''_{x_i,x_k}(t,\underline{x}) = 0, & (t,\underline{x}) \in [0,T] \times \mathbb{R}^n, \\ u(T,\underline{x}) = h(\underline{x}), & \underline{x} \in \mathbb{R}^n, \end{cases} \tag{15}$$

then the process $(u(t,\underline{X}_s^{t_0,\underline{x}_0}), s \in [t_0,T])$ is a martingale and hence,

$$u(t_0,\underline{x}_0) = \mathbb{E}(h(\underline{X}_T^{t_0,\underline{x}_0})).$$

Below, we study the stochastic representation of the solution of a multidimensional elliptic PDE.

Fact. Let D be an open and bounded domain in $\mathbb{R}^n$ and ∂D be its (smooth) boundary. Let $h \in \mathcal{C}(\partial D)$. Then there exists a unique function $u \in \mathcal{C}^2(D)$ such that

$$\begin{cases} \Delta u(\underline{x}) = 0, & \forall \underline{x} \in D, \\ u(\underline{x}) = h(\underline{x}), & \forall \underline{x} \in \partial D. \end{cases} \tag{16}$$

Remark. The function u takes values in $\mathbb{R}$ (not in $\mathbb{R}^n$).

Stochastic representation of the solution.
Let $\underline{B}^{\underline{x}}$ be an n-dimensional Brownian motion starting at point $\underline{x} \in D$ at time $t = 0$ (i.e., $\underline{B}_t^{\underline{x}} = \underline{x} + \underline{B}_t$, where $\underline{B}$ is a standard n-dimensional Brownian motion). Let also

$$\tau = \inf\{t > 0 : \underline{B}_t^{\underline{x}} \notin D\}$$

be the first exit time of $\underline{B}^{\underline{x}}$ from the domain D; τ is a stopping time. Notice also that $\underline{B}_\tau^{\underline{x}} \in \partial D$.

Proposition 6.5. The solution of (16) reads

$$u(\underline{x}) = \mathbb{E}(h(\underline{B}_\tau^{\underline{x}})), \quad \forall \underline{x} \in D.$$

Proof. - Let us first show that the process $\left(u(\underline{B}_t^{\underline{x}}),\, 0 \le t \le \tau\right)$ is a martingale. By the multidimensional Ito-Doeblin formula, we have

$$\begin{aligned} u(\underline{B}_t^{\underline{x}}) - u(\underline{B}_0^{\underline{x}}) &= \sum_{i=1}^n \int_0^t u'_{x_i}(\underline{B}_s^{\underline{x}})\, dB_s^{(i)} + \frac{1}{2} \int_0^t \Delta u(\underline{B}_s^{\underline{x}})\, ds \\ &= \sum_{i=1}^n \int_0^t u'_{x_i}(\underline{B}_s^{\underline{x}})\, dB_s^{(i)}, \end{aligned}$$

since $\Delta u(\underline{x}) = 0$, $\forall \underline{x} \in D$ and $\underline{B}_s^{\underline{x}} \in D$, $\forall s \le \tau$.

- Applying therefore the optional stopping theorem, we obtain

$$u(\underline{x}) = \mathbb{E}(u(\underline{B}_0^{\underline{x}})) = \mathbb{E}(u(\underline{B}_\tau^{\underline{x}})) = \mathbb{E}(h(\underline{B}_\tau^{\underline{x}})),$$

where the third equality holds since $u(\underline{x}) = h(\underline{x})$ on ∂D and $\underline{B}_\tau^{\underline{x}} \in \partial D$. □

7 Local martingales

7.1 Preliminary: unbounded stopping times

Let us first recall a result mentioned previously.

Optional stopping theorem (version 1). Let M be a continuous martingale with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$, and let τ_1, τ_2 be two stopping times with respect to this filtration such that

$$0 \le \tau_1 \le \tau_2 \le K < \infty \quad a.s.$$

(i.e. τ_1, τ_2 are bounded stopping times). Then

$$\mathbb{E}(M_{\tau_2}|\mathcal{F}_{\tau_1}) = M_{\tau_1} \quad a.s., \quad \text{so} \quad \mathbb{E}(M_{\tau_2}) = \mathbb{E}(M_{\tau_1}).$$

The following proposition is a variation of the above theorem, and is given here without proof.

Proposition 7.1. Let M be a continuous martingale with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ and let τ be a stopping time with respect to this filtration. Then the stopped process $M^\tau = (M_{t\wedge\tau},\, t \in \mathbb{R}_+)$ is also a martingale, i.e.

- (i) $\mathbb{E}(|M_{t\wedge\tau}|) < \infty$, $\forall t \in \mathbb{R}_+$.

- (ii) $\mathbb{E}(M_{t\wedge\tau}|\mathcal{F}_s) = M_{s\wedge\tau}$ a.s., $\forall t > s \ge 0$.

In order to deal with unbounded stopping times, we first need to make sure that the processes evaluated at these stopping times converge to some limit as time goes to infinity. We give below a sufficient condition ensuring that a martingale converges to some limit M_∞ as $t \to \infty$.

Proposition 7.2. Let M be a continuous square integrable martingale such that

$$\mathbb{E}\left(\sup_{t\ge 0} |M_t|^2\right) < \infty. \tag{17}$$

Then there exists a square-integrable random variable M_∞ such that

$$\lim_{t\to\infty} M_t = M_\infty \quad a.s. \quad \text{and} \quad M_t = \mathbb{E}(M_\infty|\mathcal{F}_t) \quad a.s. \quad \forall t \in \mathbb{R}_+.$$

Terminology. In this case, the martingale M is said to be *closed at infinity.*

Optional stopping theorem (version 2). Let M be a continuous square-integrable martingale satisfying condition (17) and let τ_1, τ_2 be two stopping times such that

$$0 \le \tau_1 \le \tau_2 \le \infty \quad a.s.$$

Then

$$\mathbb{E}(M_{\tau_2}|\mathcal{F}_{\tau_1}) = M_{\tau_1} \quad a.s., \quad \text{so} \quad \mathbb{E}(M_{\tau_2}) = \mathbb{E}(M_{\tau_1}).$$

Application. Let M be a continuous square-integrable martingale such that $M_0 = 0$; let then $a > 0$ and $\tau_a = \inf\{t > 0 \,|\, |M_t| \ge a\}$; τ_a is a stopping time. So by Proposition 7.1, M^{τ_a} is a martingale; it moreover satisfies condition (17), since

$$\mathbb{E}(\sup_{t\ge 0} |M_{t\wedge\tau_a}|^2) \le a^2 < \infty.$$

Choose then $\tau_1 = 0$, $\tau_2 = \infty$ and apply the above theorem:

$$\mathbb{E}(M_{\infty\wedge\tau_a}) = \mathbb{E}(M_{0\wedge\tau_a}) \quad \text{i.e.,} \quad \mathbb{E}(M_{\tau_a}) = \mathbb{E}(M_0) = 0,$$

even though τ_a is an unbounded stopping time and M does not satisfy condition (17).

Remark. The same reasoning cannot be made with $\tau_a' = \inf\{t > 0 \,:\, M_t \ge a\}$. This would indeed lead to the following contradiction:

$$0 < a = \mathbb{E}(M_{\tau_a'}) = \mathbb{E}(M_0) = 0.$$

7.2 Local martingales

Definition 7.3. A *local martingale* with respect to a filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ is a process M such that there exists an increasing sequence $(\tau_n,\, n \geq 1)$ of stopping times with respect to $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ such that $\tau_n \underset{n\to\infty}{\to} \infty$ a.s. and M^{τ_n} is a martingale, $\forall n \geq 1$. That is, if M is a local martingale, then

(i) $\mathbb{E}(|M_{t\wedge\tau_n}|) < \infty$, $\forall t \geq 0$, $\forall n \geq 1$.

(ii) $\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) = M_{s\wedge\tau_n}$, $\forall t > s \geq 0$, $\forall n \geq 1$.

Facts.

- From Proposition 7.1, we see that if M is a martingale, then it is also a local martingale.

- If M is a local martingale such that $\mathbb{E}(\sup_{0\leq s\leq t} |M_s|) < \infty$, $\forall t \geq 0$, then M is a martingale (proof to come in the next section).

- If M is a local martingale such that $\mathbb{E}(|M_t|) < \infty$, $\forall t \geq 0$, then this does not necessarily imply that M is a martingale.

- If M is a continuous local martingale, then it is always possible to replace the sequence $(\tau_n,\, n \geq 1)$ in the definition by

$$\tau'_n = \inf\{t > 0 \,:\, |M_t| \geq n\}, \quad n \geq 1.$$

Why should one be interested in local martingales?

- Local martingales allow to get rid of the integrability condition $\mathbb{E}(|M_t|) < \infty$ and many other technical conditions, as we will see below.

- The above definition also allows to deal easily with processes defined only up to a stopping time τ (replacing the condition $\tau_n \underset{n\to\infty}{\to} \infty$ by $\tau_n \underset{n\to\infty}{\to} \tau$).

Quadratic variation.

Theorem 7.4. Let M be a continuous local martingale. Then there exists a unique process A which is increasing, continuous, adapted and such that $A_0 = 0$ and $(M_t^2 - A_t,\, t \geq 0)$ is a continuous local martingale.

Terminology. A is called the quadratic variation of M and is denoted as $A_t = \langle M \rangle_t$.

Continuous semi-martingale.

Definition 7.5. A continuous semi-martingale is a process X that can be expressed as the sum of a continuous local martingale M and a continuous process V with bounded variation adapted to the same filtration as M such that $V_0 = 0$, i.e., $X_t = M_t + V_t$, $t \in \mathbb{R}_+$. The quadratic variation of X is defined as $\langle X \rangle_t = \langle M \rangle_t$, $t \in \mathbb{R}_+$.

Remark. The above definition is the standard one found in textbooks. The previous definition of semi-martingale given in this class is a non-standard one.

Stochastic integral.

Let M be a continuous local martingale and let H be a continuous and adapted process. Let also $n \geq 1$ and

$$\tau_n = \inf\left\{t > 0 \,:\, |M_t| \geq n \quad \text{or} \quad \int_0^t H_s^2\, d\langle M \rangle_s \geq n\right\}.$$

Up to time τ_n, M^{τ_n} is a continuous square-integrable martingale and the technical condition

$$\mathbb{E}\left(\int_0^{t\wedge\tau_n} H_s^2\, d\langle M \rangle_s\right) < \infty$$

is satisfied. It is therefore possible to define a process ($N_t = \int_0^t H_s\, dM_s$, $t \in \mathbb{R}_+$) such that N^{τ_n} matches with the previous definition of stochastic integral, $\forall n \geq 1$. In particular, N^{τ_n} is a continuous martingale $\forall n \geq 1$, so N is a continuous local martingale.

Ito-Doeblin's formula.

Let us consider three particular instances of the formula.

- Let M be a continuous local martingale and $f \in \mathcal{C}^2(\mathbb{R})$. Then

$$f(M_t) - f(M_0) = \underbrace{\int_0^t f'(M_s)\, dM_s}_{\text{continuous local martingale}} + \underbrace{\frac{1}{2}\int_0^t f''(M_s)\, d\langle M\rangle_s}_{\text{process with bounded variation}} \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

The formula holds now without additional technical condition, and says that the process ($f(M_t)$, $t \in \mathbb{R}_+$) is a continuous semi-martingale (in the sense defined above).

- Let X be a continuous semi-martingale and $f \in \mathcal{C}^2(\mathbb{R})$. Then

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

Again, the process ($f(X_t)$, $t \in \mathbb{R}_+$) is a continuous semi-martingale.

- Let $\underline{B}$ be a standard n-dimensional Brownian motion and $f \in \mathcal{C}^2(\mathbb{R}^n)$. Then

$$f(\underline{B}_t) - f(\underline{B}_0) = \sum_{i=1}^n \int_0^t f'_{x_i}(\underline{B}_s)\, dB_s^{(i)} + \frac{1}{2}\int_0^t \Delta f(\underline{B}_s)\, ds \quad a.s., \quad \forall t \in \mathbb{R}_+.$$

Again, the process ($f(\underline{B}_t)$, $t \in \mathbb{R}_+$) is a continuous semi-martingale.

In particular:

- if $\Delta f(\underline{x}) = 0$, $\forall \underline{x} \in \mathbb{R}^n$, then $f(\underline{B})$ is a continuous local martingale.

- if $\Delta f(\underline{x}) \geq 0$, $\forall \underline{x} \in \mathbb{R}^n$, then $f(\underline{B})$ is a continuous local submartingale.

- if $\Delta f(\underline{x}) \leq 0$, $\forall \underline{x} \in \mathbb{R}^n$, then $f(\underline{B})$ is a continuous local supermartingale.

Whether the word "local" can be removed or not in the above sentences depends now on technical conditions. From what we have already seen, we know that if $\Delta f(\underline{x}) = 0$, $\forall \underline{x} \in \mathbb{R}^n$ and

$$\mathbb{E}\left(\int_0^t (f'_{x_i}(\underline{B}_s))^2\, ds\right) < \infty, \quad \forall t \in \mathbb{R}_+, \quad \forall 1 \leq i \leq n, \tag{18}$$

then $f(\underline{B})$ is a continuous square-integrable martingale. Since

$$\langle f(\underline{B})\rangle_t = \sum_{i=1}^n \int_0^t (f'_{x_i}(\underline{B}_s))^2\, ds$$

(notice that this process is always well defined, even in the case where $f(\underline{B})$ is not a martingale), we see that condition (18) is equivalent to

$$\mathbb{E}(\langle f(\underline{B})\rangle_t) < \infty, \quad \forall t \in \mathbb{R}_+.$$

More generally, if M is a continuous local martingale such that $\mathbb{E}(\langle M\rangle_t) < \infty$, $\forall t \in \mathbb{R}_+$, then M is a continuous square-integrable martingale. Such a condition therefore guarantees that M is a continuous square-integrable martingale, but it is not a necessary condition for M being a martingale. The following section addresses this issue more closely.

7.3 When is a local martingale also a martingale?

Let us first recall three important theorems from measure theory.

Reminder. Let $(a_n, n \geq 1)$ be a sequence of real numbers:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{k\geq n} a_k = \sup_{n\geq 1} \inf_{k\geq n} a_k$$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k = \inf_{n\geq 1} \sup_{k\geq n} a_k$$

$$\lim_{n\to\infty} a_n \text{ exists if and only if } \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

Monotone convergence theorem.

Let $(X_n,\, n \geq 1)$ be a sequence of non-negative random variables such that

$$X_n \leq X_{n+1} \quad \forall n \geq 1 \quad \text{and} \quad \lim_{n\to\infty} X_n = X \quad a.s., \quad \text{with} \quad \mathbb{E}(X) < \infty.$$

Then $\mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}(X_n)$.

Fatou's lemma.

Let $(X_n,\, n \geq 1)$ be a sequence of non-negative random variables. Then

$$\mathbb{E}\left(\liminf_{n\to\infty} X_n\right) \leq \liminf_{n\to\infty} \mathbb{E}(X_n).$$

Dominated convergence theorem.

Let $(X_n,\, n \geq 1)$ be a sequence of random variables such that

$$\lim_{n\to\infty} X_n = X \quad a.s. \quad \text{and} \quad |X_n| \leq Y \quad \forall n \geq 1, \quad \text{with} \quad \mathbb{E}(Y) < \infty.$$

Then $\mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}(X_n)$.

In addition, the above three theorems continue to hold if we replace expectations with conditional expectations (with respect to an arbitrary σ-field $\mathcal{G}$).

With these theorems in hand, we are now ready to prove the following propositions.

Proposition 7.6. Let M be a local martingale such that $\mathbb{E}(\sup_{s\in[0,t]} |M_s|) < \infty$, $\forall t \in \mathbb{R}_+$. Then M is a martingale.

Proof. (i) By assumption, $\mathbb{E}(|M_t|) < \infty$, $\forall t \in \mathbb{R}_+$.

(ii) Since M is a local martingale, there exists an increasing sequence of stopping times τ_n such that $\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) = M_{s\wedge\tau_n}$ for all $n \geq 1$. Since $\tau_n \underset{n\to\infty}{\to} \infty$ a.s.,

$$M_{s\wedge\tau_n} \underset{n\to\infty}{\to} M_s \quad a.s.$$

Likewise, $M_{t\wedge\tau_n} \underset{n\to\infty}{\to} M_t$ a.s. and for all $n \geq 1$, $|M_{t\wedge\tau_n}| \leq \sup_{s\in[0,t]} |M_s| = Y$, with $\mathbb{E}(Y) < \infty$ by assumption. So by the dominated convergence theorem,

$$\mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) \underset{n\to\infty}{\to} \mathbb{E}(M_t|\mathcal{F}_s) \quad a.s.$$

i.e., $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ a.s. $\forall t \geq s \geq 0$. □

Proposition 7.7. Let M be a non-negative local martingale such that $\mathbb{E}(M_0) < \infty$. Then M is a supermartingale.

Proof. (i) Notice that

$$\liminf_{n\to\infty} M_{t\wedge\tau_n} = \lim_{n\to\infty} M_{t\wedge\tau_n} = M_t \quad a.s.,$$

so by Fatou's lemma,

$$\mathbb{E}(M_t) = \mathbb{E}\left(\liminf_{n\to\infty} M_{t\wedge\tau_n}\right) \le \liminf_{n\to\infty} \mathbb{E}(M_{t\wedge\tau_n})$$

and $\mathbb{E}(M_{t\wedge\tau_n}) = \mathbb{E}(M_{0\wedge\tau_n}) = \mathbb{E}(M_0) < \infty$ by assumption, so $\mathbb{E}(M_t) \le \mathbb{E}(M_0) < \infty$, $\forall t \in \mathbb{R}_+$.

(ii) By Fatou's lemma again, we have

$$\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}\left(\liminf_{n\to\infty} M_{t\wedge\tau_n}\middle|\mathcal{F}_s\right) \le \liminf_{n\to\infty}\ \mathbb{E}(M_{t\wedge\tau_n}|\mathcal{F}_s) = \liminf_{n\to\infty} M_{s\wedge\tau_n} = M_s \quad a.s.$$

□

Proposition 7.8. Let $M = (M_t,\, t \in [0,T])$ be a supermartingale. Then M is a martingale if and only if $\mathbb{E}(M_T) = \mathbb{E}(M_0)$.

Proof. Only the "$\Leftarrow$" requires a proof. By assumption, we know that $\mathbb{E}(M_t|\mathcal{F}_s) \le M_s$ a.s. We need to prove that actually, $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ a.s. Assume by contradiction that $\mathbb{P}(\mathbb{E}(M_t|\mathcal{F}_s) < M_s) > 0$. This would imply that $\mathbb{E}(M_t) = \mathbb{E}(\mathbb{E}(M_t|\mathcal{F}_s)) < \mathbb{E}(M_s)$, i.e, $\mathbb{E}(M_T) \le \mathbb{E}(M_t) < \mathbb{E}(M_s) \le \mathbb{E}(M_0)$, which is in contradiction with the assumption. □

Corollary 7.9. Let M be a non-negative local martingale such that $\mathbb{E}(M_0) < \infty$. Then M is a martingale if and only if $\mathbb{E}(M_T) = \mathbb{E}(M_0)$.

Exponential martingale.

Let M be a continuous local martingale such that $M_0 = 0$. Then the process Y defined as

$$Y_t = \exp\left(M_t - \frac{\langle M\rangle_t}{2}\right), \quad t \in \mathbb{R}_+,$$

is also a continuous local martingale. Indeed, by Ito-Doeblin's formula, $Y_t = 1 + \int_0^t Y_s\, dM_s$, as already seen (but now, we know that the stochastic integral does not require an additional technical condition in order to be well defined).

We are now in position to prove the following statement, which was already used in the section on Girsanov's theorem.

Theorem 7.10. Let M be a continuous local martingale such that $M_0 = 0$ a.s. and $\exists K > 0$ with $\langle M\rangle_t \le Kt$, $\forall t \in \mathbb{R}_+$. Then the process Y defined as

$$Y_t = \exp\left(M_t - \frac{\langle M\rangle_t}{2}\right), \quad t \in \mathbb{R}_+,$$

is a martingale.

Terminology. In this case, the process Y is called the *exponential martingale associated to M*.

Remark. The condition $\langle M\rangle_t \le Kt$, $\forall t \in \mathbb{R}_+$, can be replaced by the weaker condition:

$$\mathbb{E}\left(\exp\left(\frac{\langle M\rangle_t}{2}\right)\right) < \infty.$$

This condition is called *Novikov's condition.*

Proof. As already mentioned, $Y_t = \exp(M_t - \frac{\langle M \rangle_t}{2})$ is a local martingale. Likewise, the process $Z_t = \exp(2M_t - \frac{\langle 2M \rangle_t}{2})$ is a local martingale. Notice that

$$Y_t^2 = \exp(2M_t - \langle M \rangle_t) = Z_t \, \exp(\langle M \rangle_t).$$

Let now $\tau_n = \inf\{t > 0 \, : \, |Y_t| \geq n \text{ or } |Z_t| \geq n\}$. By Doob's inequality applied to the martingale Y^{τ_n} (notice that Y^{τ_n} is continuous and square-integrable) and the assumption, we have

$$\begin{aligned} \mathbb{E}\left(\sup_{s\in[0,t]} Y_{s\wedge\tau_n}^2\right) &\leq 4\mathbb{E}(Y_{t\wedge\tau_n}^2) = 4\mathbb{E}(Z_{t\wedge\tau_n} \, \exp(\langle M \rangle_{t\wedge\tau_n})) \\ &\leq 4e^{Kt}\,\mathbb{E}(Z_{t\wedge\tau_n}) = 4e^{Kt}\,\mathbb{E}(Z_0) = 4e^{Kt}. \end{aligned}$$

So applying successively Cauchy-Schwarz' inequality and Fatou's lemma, we obtain

$$\begin{aligned} \left(\mathbb{E}\left(\sup_{s\in[0,t]} Y_s\right)\right)^2 &\leq \mathbb{E}\left(\sup_{s\in[0,t]} Y_s^2\right) = \mathbb{E}\left(\liminf_{n\to\infty} \sup_{s\in[0,t]} Y_{s\wedge\tau_n}^2\right) \\ &\leq \liminf_{n\to\infty} \mathbb{E}\left(\sup_{s\in[0,t]} Y_{s\wedge\tau_n}^2\right) \leq 4e^{Kt} < \infty, \quad \forall t \in \mathbb{R}_+. \end{aligned}$$

Proposition 7.6 then implies that Y is a martingale. □

Therefore, under the condition that $\langle M \rangle_t \leq Kt$, $\forall t \in \mathbb{R}_+$, it is possible to define a new probability measure $\widetilde{\mathbb{P}}_T$ as $\widetilde{\mathbb{P}}_T(A) = \mathbb{E}(1_A Y_T)$.

Remark. Notice that Y is a non-negative local martingale, but Corollary 7.9 giving the simple condition $\mathbb{E}(Y_T) = \mathbb{E}(Y_0)$ for testing whether Y is a martingale or not is useless in the present context. Indeed, this simple condition is exactly the thing we want in order to be able to define the new probability measure $\widetilde{\mathbb{P}}_T$ (it ensures that $\widetilde{\mathbb{P}}_T(\Omega) = 1$). Whether Y is a martingale or not is actually not our concern here.

We are now in position to restate Girsanov's theorem in its full version.

Girsanov's theorem. Let M be a continuous local martingale (under $\mathbb{P}$) such that $\langle M \rangle_t \leq Kt$, $\forall t \in \mathbb{R}_+$, and $\widetilde{\mathbb{P}}_T$ be the above defined probability measure. If Z is a continuous local martingale under $\mathbb{P}$, then $(Z_t - \langle M, Z \rangle_t, \, t \in [0,T])$ is a continuous local martingale under $\widetilde{\mathbb{P}}_T$.

Likewise, the full version of Lévy's theorem is given below, along with its proof.

Lévy's theorem. Let X be a continuous local martingale such that $X_0 = 0$ a.s. and $\langle X \rangle_t = t$ a.s., $\forall t \in \mathbb{R}_+$. Then X is a standard Brownian motion.

Proof. $\forall c \in \mathbb{R}$, cX is a continuous local martingale such that $\langle cX \rangle_t = c^2 t$. Therefore, by Theorem 7.10, the process $(Y_t = \exp(cX_t - \frac{c^2 t}{2}), \, t \in \mathbb{R}_+)$ is a martingale, i.e.,

$$\mathbb{E}\left(\exp\left(cX_t - \frac{c^2 t}{2}\right) \,\middle|\, \mathcal{F}_s\right) = \exp\left(cX_s - \frac{c^2 s}{2}\right), \quad \forall c \in \mathbb{R},$$

or

$$\mathbb{E}(\exp(c(X_t - X_s))|\mathcal{F}_s) = \exp\left(\frac{c^2(t-s)}{2}\right), \quad \forall c \in \mathbb{R}.$$

Fact 1: since the right-hand side is deterministic, $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$. Moreover, by taking expectations, we obtain

$$\mathbb{E}(\exp(c(X_t - X_s))) = \exp\left(\frac{c^2(t-s)}{2}\right), \quad \forall c \in \mathbb{R}.$$

Fact 2: this implies that $X_t - X_s \sim \mathcal{N}(0, t-s)$. Therefore, X is a standard Brownian motion. □

Remark. The "innocent" condition that the (local) martingale X should be continuous is actually crucial. Here is an important counter-example. Let N be the classical Poisson process with intensity 1, that is,

$$N_t = \sup \left\{ k \geq 0 \; : \; \sum_{j=1}^{k} \tau_j \leq t \right\},$$

where $(\tau_j, j \geq 1)$ is a sequence of i.i.d. exponential random variables with parameter 1. Then it can be shown that the discounted Poisson process X defined as

$$X_t = N_t - t, \quad t \in \mathbb{R}_+,$$

is a martingale. Moreover, the process Y defined as

$$Y_t = X_t^2 - t = (N_t - t)^2 - t, \quad t \in \mathbb{R}_+,$$

is also a martingale, which is saying that $\langle X \rangle_t = t$ (even though we have not formally defined the quadratic variation of a process with jumps). Nevertheless, X is far from being a standard Brownian motion!

Finally, we obtain the following corollary (which was already known to us before, by the way).

Corollary 7.11. Let Z be a continuous local martingale under $\mathbb{P}$ such that $Z_0 = 0$ and $\langle Z \rangle_t = t$ a.s., $\forall t \in \mathbb{R}_+$ (i.e., Z is a standard Brownian motion under $\mathbb{P}$ by Lévy's theorem). Then the process $(Z_t - \langle M, Z \rangle_t, \, t \in [0, T])$ is a standard Brownian motion under the probability measure $\widetilde{\mathbb{P}}_T$ defined above.

7.4 Change of time

A nice consequence of Lévy's theorem is the following proposition, saying that basically every local martingale is a time change of a Brownian motion (see corollary below).

Proposition 7.12. Let M be a continuous local martingale with respect to a filtration $(\mathcal{F}_t, \, t \in \mathbb{R}_+)$ such that

$$M_0 = 0 \quad a.s \quad \text{and} \quad \lim_{t \to \infty} \langle M \rangle_t = \infty \quad a.s. \tag{19}$$

Let us also define

$$\tau(s) = \inf\{t > 0 \, : \, \langle M \rangle_t \geq s\}$$

and $B_s = M_{\tau(s)}$, $\mathcal{G}_s = \mathcal{F}_{\tau(s)}$. Then B is a standard Brownian motion with respect to $(\mathcal{G}_s, \, s \in \mathbb{R}_+)$.

Proof. As already mentioned, the idea is to use Lévy's theorem, i.e., to show that

(i) B has continuous trajectories.

(ii) B is a local martingale with respect to $(\mathcal{G}_s, \, s \in \mathbb{R}_+)$.

(iii) $\langle B \rangle_s = s$, i.e., $(B_s^2 - s, \, s \in \mathbb{R}_+)$ is a local martingale with respect to $(\mathcal{G}_s, \, s \in \mathbb{R}_+)$.

Let us verify these three statements.

(i) As M is continuous, $t \to \langle M \rangle_t$ is also continuous. Moreover, if $\langle M \rangle$ is constant on some interval, then M also is, so the function $s \mapsto B_s = M_{\tau(s)}$ is continuous.

(ii) Let $\tau_n = \inf\{t > 0 \, : \, |M_t| \geq n\}$, $n \geq 1$. For each n, M^{τ_n} is a martingale such that

$$\mathbb{E}\left(\sup_{t \in [0,T]} |M_{t \wedge \tau_n}|^2 \right) < \infty, \quad \forall T > 0,$$

so by the optional stopping theorem (version 2), we have

$$\mathbb{E}(M_{\tau(s_2)\wedge\tau_n}|\mathcal{F}_{\tau(s_1)}) = M_{\tau(s_1)\wedge\tau_n} \quad a.s., \quad \forall s_2 > s_1 \geq 0.$$

By the dominated convergence theorem (and some details), this implies that

$$\mathbb{E}(M_{\tau(s_2)}|\mathcal{F}_{\tau(s_1)}) = M_{\tau(s_1)} \quad a.s.$$

i.e.,

$$\mathbb{E}(B_{s_2}|\mathcal{G}_{s_1}) = B_{s_1} \quad a.s.$$

i.e., B is a martingale with respect to $(\mathcal{G}_s,\, s \in \mathbb{R}_+)$.

(iii) Let $X_t = M_t^2 - \langle M\rangle_t$. By assumption, X^{τ_n} is a martingale $\forall n$, so

$$\mathbb{E}(X_{\tau(s_2)\wedge\tau_n}|\mathcal{F}_{\tau(s_1)}) = X_{\tau(s_1)\wedge\tau_n} \quad a.s., \quad \forall s_2 > s_1 \geq 0.$$

Then again by the dominated convergence theorem (and some details), we obtain that

$$\mathbb{E}(X_{\tau(s_2)}|\mathcal{F}_{\tau(s_1)}) = X_{\tau(s_1)} \quad a.s.$$

i.e.,

$$\mathbb{E}(M_{\tau(s_2)}^2 - \langle M\rangle_{\tau(s_2)}|\mathcal{F}_{\tau(s_1)}) = M_{\tau(s_1)}^2 - \langle M\rangle_{\tau(s_1)} \quad a.s.$$

As $\langle M\rangle_{\tau(s)} = s$ by definition, we obtain:

$$\mathbb{E}(B_{s_2}^2 - s_2|\mathcal{G}_{s_1}) = B_{s_1}^2 - s_1 \quad a.s., \quad \forall s_2 > s_1 \geq 0.$$

i.e., $(B_s^2 - s,\, s \in \mathbb{R}_+)$ is a martingale with respect to $(\mathcal{G}_s,\, s \in \mathbb{R}_+)$. □

Remark. Even though it is somehow hidden in the proof given above, condition (19) is needed to ensure two facts: first, that the process B is defined for all times s up to infinity; second, that the process M actually takes all possible values in $\mathbb{R}$, as the Brownian motion does. One can for example show that if the process M is bounded above or below by some constant, then condition (19) cannot be satisfied.

Corollary 7.13. Any continuous local martingale M satisfying (19) may be written as $M_t = B(\langle M\rangle_t)$, where B is a standard Brownian motion.

Remark. This does not say in general that any continuous local martingale satisfying (19) is Gaussian! If $\langle M\rangle_t$ is random, this is not the case (but whenever $\langle M\rangle_t$ is deterministic, then M is Gaussian; this holds in particular for Wiener integrals).

7.5 Local time

Although there is the word "local" in the above title, local times are not directly related to local martingales.

Let B be a standard (one-dimensional) Brownian motion and $f(x) = |x|$, $x \in \mathbb{R}$. Applying naively Ito-Doeblin's formula to $f(B_t)$ gives

$$|B_t| - \underbrace{|B_0|}_{=0\ a.s.} = \int_0^t \mathrm{sgn}(B_s)\, dB_s + 0 \quad ?$$

as

$$f'(x) = \mathrm{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and $f''(x) = 0$, $\forall x \neq 0$. Neglecting the fact that f is twice continuously differentiable in $x = 0$ works in higher dimensions, but not here. The explanation is simple: the one-dimensional Brownian motion comes back much more often to zero! So "something" happens in $x = 0$ that cannot be neglected.

Another direct explanation as to why the above formula cannot possibly hold is that the process $W_t = \int_0^t \operatorname{sgn}(B_s)\, dB_s$ is a continuous local martingale with quadratic variation $\langle W \rangle_t = t$, so W is a standard Brownian motion by Lévy's theorem. Therefore, W_t cannot be equal to $|B_t|$, which is a non-negative random variable (certainly not Gaussian).

It actually turns out that the following formula holds:

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)\, dB_s + L_t \quad a.s., \quad \forall t \in \mathbb{R}_+,$$

where

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|B_s| \leq \varepsilon\}}\, ds$$

Remark. L_t is surprisingly non-zero. The surprise comes from the fact that the trajectories of the Brownian motion may be seen as having infinite derivative (either $+\infty$ or $-\infty$), so it seems that the time spent by this process close to zero should be negligible.

Notice that if we allow ourselves to write

$$\text{“}\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} 1_{\{|x| \leq \varepsilon\}},\text{”}$$

the Dirac mass at $x = 0$, then writing further that "$L_t = \int_0^t \delta(B_s)\, ds$" and "$\operatorname{sgn}'(x) = 2\delta(x)$" (in a distributional sense), we recover a generalized Ito-Doeblin formula :

$$\text{“} \underbrace{|B_t|}_{=f(B_t)} = \int_0^t \underbrace{\operatorname{sgn}(B_s)}_{=f'(B_s)}\, dB_s + \frac{1}{2} \int_0^t \underbrace{2\delta(B_s)}_{=f''(B_s)}\, ds \quad a.s.\text{”}$$

More generally, it holds for any $a \in \mathbb{R}$ and $t \in \mathbb{R}_+$ that

$$|B_t - a| - |B_0 - a| = \int_0^t \operatorname{sgn}(B_s - a)\, dB_s + L_t(a) \quad a.s.,$$

where

$$L_t(a) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{|B_s - a| \leq \varepsilon\}}\, ds.$$

Formally, let us again write "$L_t(a) = \int_0^t \delta(B_s - a)\, ds$". This leads to the formula, valid $\forall g \in \mathcal{C}(\mathbb{R})$ and $t \in \mathbb{R}_+$ (provided one does not pay attention to the invalid interchange of integrals in the middle):

$$\int_0^t g(B_s)\, ds = \int_0^t \int_{\mathbb{R}} g(a)\, \delta(B_s - a)\, da\, ds = \int_{\mathbb{R}} g(a) \int_0^t \delta(B_s - a)\, ds\, da = \int_{\mathbb{R}} g(a)\, L_t(a)\, da.$$

Equivalently, this means that for all $a < b$,

$$\int_0^t 1_{\{a \leq B_s \leq b\}}\, ds = \int_a^b L_t(x)\, dx.$$

So $L_t(a)$ is the density of the occupation measure of the process B over the period $[0, t]$. More naively, $L_t(a)$ can be thought of as the *time spent by the process B in $x = a$ over the period* $[0, t]$.