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The Stationary Behaviour of Fluid Limits of Reversible Processes is Concentrated on Stationary Points

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Abstract

Assume that a stochastic processes can be approximated, when some scale parameter gets large, by a fluid limit (also called "mean field limit", or "hydrodynamic limit"). A common practice, often called the "fixed point approximation" consists in approximating the stationary behaviour of the stochastic process by the stationary points of the fluid limit. It is known that this may be incorrect in general, as the stationary behaviour of the fluid limit may not be described by its stationary points. We show however that, if the stochastic process is reversible, the fixed point approximation is indeed valid. More precisely, we assume that the stochastic process converges to the fluid limit in distribution (hence in probability) at every fixed point in time. This assumption is very weak and holds for a large family of processes, among which many mean field and other interaction models. We show that the reversibility of the stochastic process implies that any limit point of its stationary distribution is concentrated on stationary points of the fluid limit. If the fluid limit has a unique stationary point, it is an approximation of the stationary distribution of the stochastic process.

1. Introduction

This paper is motivated by the use of fluid limits in models of interacting objects or particles, in contexts such as communication and computer system modelling [7], biology [8] or game theory [4]. Typically, one has a stochastic process Y^N, indexed by a size parameter N; under fairly general assumptions, one can show that the stochastic process Y^N converges to a deterministic fluid limit φ [17]. We are interested in the stationary distribution of Y^N, assumed to exist and be unique, but which may be too complicated to be computed explicitly. The "fixed point assumption" is then sometimes invoked [15, 6, 19, 14]: it consists in approximating the stationary distribution of Y^N by a stationary point of the deterministic fluid limit φ. In the frequent case where the fluid limit φ is described by an Ordinary Differential Equation (ODE), say of the form $\dot{y} = F(y)$, the stationary points are obtained by solving $F(y) = 0$. If Y^N is an empirical measure, convergence to a deterministic limit implies propagation of chaos, i.e. the states of different objects are asymptotically independent, and the distribution of any particular object at any time is obtained from the fluid limit. Under the fixed point assumption, the stationary distribution of one object is approximated by a stationary point of the fluid limit.

A critique of the fixed point approximation method is formulated in [3], which observes that one may only say, in general, that the stationary distribution of Y^N converges to a stationary distribution of the fluid limit. For a deterministic fluid limit, a stationary distribution is supported by the Birkhoff center of the fluid limit, which may be larger than the set of stationary points. An example is given where the fluid limit has a unique stationary point, but the stationary distribution of Y^N does not converge to the Dirac mass at this stationary point; in contrast, it converges to a distribution supported by a limit cycle of the ODE. If the fluid limit has a unique limit point, say y^*, to which all trajectories converge, then this unique limit point is also the unique stationary point and the stationary distribution of Y^N does converge to the Dirac mass at y^* (i.e. the fixed point approximation is then valid). However, as illustrated in [3], this assumption may be difficult to verify, as it often does not hold, and when it does, it may be difficult to establish. For example, in [9] it is shown that the fixed point assumption does not hold for some parameter settings of a wireless system analyzed in [6], due to limit cycles in the fluid limit.

In this paper we show that there is a class of systems for which such complications may not arise, namely the class of reversible stochastic processes. Reversibility is classically defined as a property of time reversibility in stationary regime [13]. For example, the stochastic process Y^N of [14], which describes the occupancy of inter-city telecommunication links, is reversible. In such cases, we show that the fluid limit must have stationary points, and any limit point of the stationary distribution of Y^N must be supported by the set of stationary points. Thus, for reversible processes that have a fluid limit, the fixed point approximation is justified.

2. Assumptions and Notation

2.1. A Collection of Reversible Random Processes

Let E be a Polish space and let d be a measure that metrizes E. Let $\mathcal{P}(E)$ be the set of probability measures on E, endowed with the topology of weak convergence. Let $\mathcal{C}_b(E)$ be the set of bounded continuous functions from E to $\mathbb{R}$, and similarly $\mathcal{C}_b(E \times E)$ is the set of bounded continuous functions from $E \times E$ to $\mathbb{R}$.

We are given a collection of probability spaces $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ indexed by $N = 1, 2, 3, ...$ and for every N we have a process Y^N defined on $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$. Time is continuous. Let $D_E[0, \infty)$ be the set of cádlág functions $[0, \infty) \to E$; Y^N is then a stochastic process with sample paths in $D_E[0, \infty)$.

We denote by $Y^N(t)$ the random value of Y^N at time $t \geq 0$. Let $E^N \subset E$ be the support of $Y^N(0)$, so that $\mathbb{P}^N(Y^N(0) \in E^N) = 1$.

We assume that, for every N, the process Y^N is Feller, in the sense that for every $t \geq 0$ and $h \in \mathcal{C}_b(E)$, $\mathbb{E}^N \left[h(Y^N(t)) \middle| Y^N(0) = y_0 \right]$ is a continuous function of $y_0 \in E$. Examples of such processes are continuous time Markov chains as in [16], or linear interpolations of discrete time Markov chains as in [5], or the projections of a Markov process as in [12]. Note that apart from the first example, these are not Markov.

Definition 1. A probability $\Pi^N \in \mathcal{P}(E)$ is *invariant* for Y^N if $\Pi^N(E^N) = 1$ and for every $h \in \mathcal{C}_b(E)$ and every $t \geq 0$:

$$\int_E \mathbb{E}^N \left[h\left(Y^N(t)\right) \middle| Y^N(0) = y \right] \Pi^N(dy) = \int_E h(y)\Pi^N(dy)$$

We are interested in reversible processes, i.e. processes that keep the same stationary law under time reversal. A weak form of such a property is defined as follows

Definition 2. Assume Π^N is a probability on E such that $\Pi^N(E^N) = 1$, for some N. We say that Y^N is *reversible* under Π^N if for every time $t \geq 0$ and any $h \in \mathcal{C}_b(E \times E)$:

$$\int_E \mathbb{E}^N \left[h\left(y, Y^N(t)\right) \middle| Y^N(0) = y \right] \Pi^N(dy) = \int_E \mathbb{E}^N \left[h\left(Y^N(t), y\right) \middle| Y^N(0) = y \right] \Pi^N(dy)$$

Note that, necessarily, Π^N is an invariant probability for Y^N. If Y^N is a Markov process, then Definition 2 coincides with the classical definition of reversibility as in [13]. Similarly, if Y^N is a projection of a reversible Markov process X^N, as in [10], then Y^N is reversible under the projection of the stationary probability of X^N; note that in such a case, Y^N is not Markov.

2.2. A Limiting, Continuous Semi-Flow

Further, let φ be a deterministic process, i.e. a mapping

$$\begin{array}{rccl} \varphi : & [0, \infty) \times E & \rightarrow & E \\ & t, y_0 & \mapsto & \varphi_t(y_0) \end{array}$$

We assume that φ_t is a semi-flow, i.e.

1. $\varphi_0(y) = y$,
2. $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s \geq 0$ and $t \geq 0$,

and we say that φ is "space continuous" if for every $t \geq 0$, $\varphi_t(y)$ is continuous in y.

Definition 3. We say that $y \in E$ is a *stationary point* of φ if $\varphi_t(y) = y$ for all $t \geq 0$

In cases where E is a subset of $\mathbb{R}^d$ for some integer d, the semi-flow φ may be an autonomous ODE, of the form $\dot{y} = F(y)$; here the stationary points are the solutions of $F(y) = 0$.

Definition 4. We say that the semi-flow φ is *reversible* under the probability $\Pi \in \mathcal{P}(E)$ if for every time $t \geq 0$ and any $h \in \mathcal{C}_b(E \times E)$:

$$\int_E h(y, \varphi_t(y))\Pi(dy) = \int_E h(\varphi_t(y), y)\Pi(dy) \tag{1}$$

As we show in the next section, reversible semi-flows must concentrate on stationary points.

2.3. Convergence Hypothesis

We assume that, for every fixed t the processes Y^N converge in distribution to the deterministic process φ as $N \to \infty$ for every collection of converging initial conditions. More precisely:

Hypothesis 1. *For every y_0 in E, every sequence $(y_0^N)_{N=1,2,...}$ such that $y_0^N \in E^N$ and $\lim_{N\to\infty} y_0^N = y_0$, and every $t \geq 0$, the conditional law of $Y^N(t)$ given $Y^N(0) = y_0^N$ converges weakly to the Dirac mass at $\varphi_t(y_0)$. That is*

$$\lim_{N\to\infty} \mathbb{E}^N \left[h(Y^N(t)) \middle| Y^N(0) = y_0^N \right] = h \circ \varphi_t(y_0)$$

for all $h \in \mathcal{C}_b(E)$ and any fixed $t \geq 0$.

Hypothesis 1 holds in [18, 16, 20, 7, 3] as a consequence of stronger convergence results; for example in [16] there is almost sure, uniform convergence for all $t \in [0, T]$, for any $T \geq 0$. In [12] the convergence is on the set of trajectories and is thus stronger than what we require.

Under Hypothesis 1, φ is called the *hydrodynamic limit* [1], or simply *fluid limit* of Y^N.

3. Reversible Semi-Flows Concentrate on Stationary Points

Theorem 1. *Let φ be a space continuous semi-flow, reversible under Π. Let S be the set of stationary points of φ. Then Π is concentrated on S, i.e. $\Pi(S) = 1$.*

Proof.

Step 1. Denote with $\bar{S}$ the complement of the set of stationary points. Take some fixed but arbitrary $y_0 \in \bar{S}$. By definition of S, there exists some $\tau > 0$ such that

$$\varphi_{2\tau}(y_0) \neq y_0 \tag{2}$$

Define $\varphi_\tau(y_0) = y_1$, $\varphi_\tau(y_1) = y_2$, so that $y_2 \neq y_0$.

For $y \in E$ and $\epsilon > 0$ we denote with $B(y, \epsilon)$ the open ball $= \{x \in E, d(x, y) < \epsilon\}$. Let $\epsilon = d(y_0, y_2) > 0$ and let $B_2 = B(y_2, \epsilon/2)$. Since the semi-flow is continuous in space, there is some $\alpha_1 > 0$ such that $B_1 = B(y_1, \alpha_1)$ and $\varphi_\tau(B_1) \subset B_2$. Also let $B_1' = B(y_1, \alpha_1/2)$. By the same argument, there exists some $\alpha_0 > 0$ such that $\alpha_0 < \epsilon/2$, $B_0 = B(y, \alpha_0)$ and $\varphi_\tau(B_0) \subset B_1'$. We have thus:

$$\begin{aligned} \varphi_\tau(B_0) &\subset B_1' \subset B_1 \\ \varphi_\tau(B_1) &\subset B_2 \\ B_0 \cap B_2 &= \emptyset \end{aligned}$$

Let ξ be some continuous function $[0, +\infty) \to [0, 1]$ such that $\xi(u) = 1$ whenever $0 \leq u \leq 1/2$ and $\xi(u) = 0$ whenever $u \geq 1$ (for example take a linear interpolation). Now take

$$h(y, z) \stackrel{\text{def}}{=} \xi\left(\frac{d(y_0, y)}{\alpha_0}\right) \xi\left(\frac{d(y_1, y)}{\alpha_1}\right) \tag{3}$$

so that $h \in \mathcal{C}_b(E \times E)$ and

$$\begin{aligned} h(y,z) &= 0 \qquad \text{whenever } y \notin B_0 \text{ or } z \notin B_1 \\ h(y,z) &= 1 \qquad \text{whenever } d(y_0,y) < \alpha_0/2 \text{ and } z \in B_1' \end{aligned}$$

It follows that $h(\varphi_\tau(z), z) = 0$ for every $z \in E$ and

$$\int_E h(y, \varphi_\tau(y))\Pi(dy) \geq \Pi(B(y_0, \alpha_0/2)) \tag{4}$$

Apply Definition 4, it comes $\Pi\left(B(y_0, \alpha_0/2)\right) = 0$; thus, for any non stationary point y_0 there is some $\alpha > 0$ such that

$$\Pi\left(B(y_0, \alpha)\right) = 0 \tag{5}$$

Step 2. The space is polish thus also separable, i.e. has a dense enumerable set, say Q.

For every $y \in \bar{S}$ let α be as in Eq.(5) and pick some $q(y) \in Q$ and $n(y) \in \mathbb{N}$ s.t. $d(y, q(y)) < \frac{1}{n(y)} < \alpha$. Thus $y \in B(q(y), \frac{1}{n(y)})$ and $\Pi\left(B(q(y), \frac{1}{n(y)})\right) = 0$.

Let $F = \bigcup_{y \in \bar{S}} (q(y), n(y))$. $F \subset Q \times \mathbb{N}$ thus F is enumerable and

$$\bar{S} \subset \bigcup_{(q,n)\in F} B\left(q, \frac{1}{n}\right)$$

Thus

$$0 \leq \Pi(\bar{S}) \leq \sum_{(q,n)\in F} \Pi\left(B\left(q, \frac{1}{n}\right)\right) = 0 \tag{6}$$

□

Note that it follows that a semi-flow that does not have any stationary point cannot be reversible under any probability.

4. Stationary Behaviour of Fluid Limits of Reversible Processes

Theorem 2. *Assume the processes Y^N are reversible under some probabilities Π^N. Assume the convergence Hypothesis 1 holds and that $\Pi \in \mathcal{P}(E)$ is a limit point of the sequence Π^N. Then the fluid limit is reversible under Π. In particular, it follows from Theorem 1 that Π is concentrated on the set of stationary points S of the fluid limit φ.*

Proof. All we need to show is that Π verifies Definition 4. Let N_k be a subsequence such that $\lim_{k\to\infty} \Pi^{N_k} = \Pi$ in the weak topology on $\mathcal{P}(E)$. By Skorohod's representation theorem for Polish spaces [11, Thm 1.8], there exists a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which some random variables X^k for $k \in \mathbb{N}$ and X are defined such that

$$\begin{cases} \text{law of } X^k = \Pi^{N_k} \\ \text{law of } X = \Pi \\ X^k \to X \ \mathbb{P} - \text{a.s.} \end{cases}$$

Fix some $t \geq 0$ and $h \in \mathcal{C}_b(E \times E)$, and define, for $k \in \mathbb{N}$ and $y \in E$

$$
\begin{aligned}
a^k(y) &\stackrel{\text{def}}{=} \mathbb{E}\left(h\left(y, Y^{N_k}(t)\right) \middle| Y^{N_k}(0) = y\right) \\
b^k(y) &\stackrel{\text{def}}{=} \mathbb{E}\left(h\left(Y^{N_k}(t), y\right) \middle| Y^{N_k}(0) = y\right)
\end{aligned}
$$

Since Y^N is reversible under Π^{N_k}:

$$\int_E a^k(y)\Pi^{N_k}(dy) = \int_E b^k(y)\Pi^{N_k}(dy) \tag{7}$$

Hypothesis 1 implies that $\lim_{k\to\infty} a^k(x^k) = h(x, \varphi_t(x))$ for every sequence x^k such that $x^k \in E^{N_k}$ and $\lim_{k\to\infty} x^k = x \in E$. Now $X^k \in E^{N_k}$ $\mathbb{P}-$ almost surely, since the law of X^k is Π^{N_k} and Y^{N_k} is reversible under Π^{N_k} . Further, $X^k \to X$ $\mathbb{P}-$ almost surely; thus

$$\lim_{k\to\infty} a^k(X^k) = h(X, \varphi_t(X)) \quad \mathbb{P}- \text{ almost surely} \tag{8}$$

Now $a^k(X^k) \leq \|h\|_\infty$ and, thus, by dominated convergence:

$$\lim_{k\to\infty} \mathbb{E}\left(a^k(X^k)\right) = \mathbb{E}\left(h(X, \varphi_t(X))\right) \tag{9}$$

and similarly for b^k. Thus

$$\int_E h(y, \varphi_t(y))\Pi(dy) = \int_E h(\varphi_t(y), y)\Pi(dy) \tag{10}$$

□

In particular, if the semi-flow has a unique stationary point, we have:

Corollary 1. *Assume the processes Y^N are reversible under some probabilities Π^N. Assume Hypothesis 1 holds and:*

1. *the sequence $(\Pi^N)_{N=1,2,\dots}$ is tight;*
2. *the semi-flow φ has a unique stationary point y^*.*

It follows that the sequence Π^N converges weakly to the Dirac mass at y^*.

Recall that tightness means that for every $\epsilon > 0$ there is some compact set $K \subset E$ such that $\Pi^N(K) \geq 1 - \epsilon$ for all N. If E is compact then $(\Pi^N)_{N=1,2,\dots}$ is necessarily tight.

Compare Corollary 1 to known results for the non reversible case [2]: there we need that the fluid limit φ has a unique limit point to which all trajectories converge. In contrast, here, we need a much weaker assumption, which bears only on stationary points. It is possible for a semi-flow to have a unique stationary point, without this stationary point being a limit of all trajectories (for example because it is unstable, or because there are stable limit cycles as in [3]). In Corollary 1, we do not need to show stability of the unique stationary point y^*.

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