PROPOSAL STRATEGIES FOR JOINT STATE-SPACE TRACKING WITH PARTICLE FILTERS

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ABSTRACT

A proposal function determines the random particle support of a particle filter. When this support is distributed close to the true target density, filter's estimation performance increases for a given number of particles. In this paper, a proposal strategy for joint state-space tracking using particle filters is given. The state-spaces are assumed Markovian and not-exact; however, each state-space is assumed to sufficiently describe the underlying phenomenon. The joint tracking is achieved by carefully placing the random support of the joint filter to where the final posterior is likely to lie. Computer simulations demonstrate improved performance and robustness of the joint state-space through the proposed strategy.

1. INTRODUCTION

State-space models are mathematical relations used for describing a system's evolution and have extensive applications in many practical problems in control theory, signal processing, and telecommunications. Since exact state-space models of real systems are extremely rare, approximate models are used. Hence, during analytical modelling of some natural phenomena, the emphasis is usually placed on choosing a minimum set of variables that completely describe a system's internal status relevant to the problem at hand. In this way, satisfactory results can still be achieved despite incomplete modelling of a system due to ignorance or lack of knowledge [1, 2].

Once a system's state-space is described in a probabilistic fashion, sequential Monté-Carlo methods, also known as *particle filters*, can be used to track the state vector as the observations arrive in sequence. In the filter mechanics, posteriors describing the state vector are represented by randomly distributed discrete state realizations, called particles, along with associated weights. A *proposal function* determines the internal distribution of the particles and directly affects the efficiency of the filter. Given the random particle support, a particle filter can estimate any statistics of the posterior by proper weighting, and the estimation accuracy can be improved up to the theoretical bounds by increasing the number of particles [3, 4].

Recently, there is much interest in combining multiple tracking algorithms described by different state-spaces with overlapping state parameters. The motivation for joint estimation is basically two-fold: (i) to improve the performance of the estimates by merging different information streams and (ii) to maintain adequate robustness of the estimates in the face of unexpected model variations or noisy data. For joint state-space estimation problems, the particle filter is a natural choice because it propagates the probability density function (pdf) of the state vectors. Hence, it allows for heuristic combination methods, which may be problem specific [5], or a general probabilistic framework for combining information.

In this paper, a general framework is described for tracking a single joint state vector by merging two overlapping state-space models using the particle filter. It should be stressed that combining two particle filters for different state spaces is different from formulating one filter that will track them jointly [6]. A proposal strategy is described that carefully combines the proposal strategies optimal for the individual state-spaces such that the random support of the particle filter is concentrated where the final posterior of the joint state-space lies. The resulting filter can have better estimation accuracy with the same number of particles as the individual filters.

The joint proposal strategy assumes that the state-space formulations are not exact. If there is an exact relation governing the system parameters, then a single consistent state space can be formulated easily from the individual state-spaces. Loosely speaking, each state space locally explains the underlying phenomenon and the different independent observations can be used to assist the estimation in the individual state-spaces. For example, an acoustic tracker with a constant velocity motion model can assist a visual tracker using a random-walk model through an occlusion scenario if the acoustic propagation path from the target is not blocked.

The paper is organized as follows. Section 2 sets up the state spaces and describes the assumptions in mathematical terms. Section 3 provides the derivation of the joint proposal strategy for particle filters. Computer simulations are given in Sect. 4.

2. STATE-SPACE ASSUMPTIONS

Two state-spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ described below are used to demonstrate the framework. The state update and observation functions of $\mathcal{S}_1$ and $\mathcal{S}_2$ are assumed to be time-invariant; however, the results can be generalized to time-varying systems including nuisance parameters. It is also assumed that the state dimensions are constant even if the system is time-varying. Define

$$\begin{aligned} \mathcal{S}_\mathrm{i}: \qquad & x_{i,t} = \begin{bmatrix} \chi_t \\ \psi_{i,t} \end{bmatrix} \sim q_i(x_{i,t}|x_{i,t-1}) \\ & y_{i,t} \sim f_i(y_{i,t}|x_{i,t}) \end{aligned} \tag{1}$$

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where the observed data in each space is represented by $\{y_{i,t}, i = 1,2\}$ and the overlapping (possibly multi-dimensional) state parameters are represented by χ_t. The state transition density functions $q_i(\cdot|-)$ are assumed known or, in the general time-varying case, they can be determined dynamically. The observations are explained through the density functions $f_i(\cdot|-)$. The observation sets y_i are modelled as statistically independent given the state through conditionally independent observation densities, e.g., one of them might be the acoustic observations and the other one video. This assumption is justified in many cases but may be hard to verify mathematically for a specific problem at hand [6, 7].

To track the joint state vector $x_t = [\chi_t, \psi_{1,t}, \psi_{2,t}]$ with a particle filter, the following target posterior should be determined:

$$p(x_t|x_{t-1}, y_{1,t}, y_{2,t}) \propto \pi_t(y_{1,t}, y_{2,t})\pi_{t-1}(x_t), \tag{2}$$

where $\pi_s(\cdot) = p(\cdot|x_s)$. In (2), the Markovian property is implicitly assumed on the state-spaces. That is, given the previous state and the current data observation, the current state distribution does not depend on the previous state track and the previous observations.

Equation (2) can calculate the target posterior up to a proportionality constant, where the proportionality is independent from the current state x_t. The first pdf on the right hand side of (2) is called the joint data-likelihood and can be simplified using the conditional independence assumption on the observations:

$$\pi_t(y_{1,t}, y_{2,t}) = f_1(y_{1,t}|x_{1,t})f_2(y_{2,t}|x_{2,t}). \tag{3}$$

The last pdf in (2), corresponding to a joint state update, requires a little bit more finesse. State-spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ may have different updates for the common parameter set since they are not exact. This poses a challenge in terms of formulating the common state update for x_t. Instead of assuming a given analytical form for the joint state update as in [6], we combine the individual state update marginal pdfs for the common state parameter by, in effect, convolving the models as follows:

$$\pi_{t-1}(\chi_t) = cp_1(\chi_t)^{o_1}p_2(\chi_t)^{o_2}r(\chi_t)^{o_3} \tag{4}$$

where $c \geq 1$ is a constant, $p_i(\chi_t) \triangleq p(\chi_t|x_{i,t-1})$ is the marginal density and the probabilities o_i for $i = 1,2$ ($\sum_i o_i = 1$) define an ownership of the underlying phenomenon by the state models; and $r(\chi_t)$ is a (uniform/reference) prior in the natural space of parameter χ_t [8] to account for unexplained observations by the state models. If we define D as the Kullback-Leibler distance then

$$D(\alpha(\chi_t)||\pi_{t-1}(\chi_t)) = -\log c + \sum_i o_i D(\alpha(\chi_t)||p_i(\chi_t)) \tag{5}$$

where α is the unknown true χ_t distribution. Hence, $D(\alpha||\pi_{t-1}) \leq \max_i\{D(\alpha||p_i)\}$. This implies that (4) minimizes the worst case divergence from the true distribution [9]. This way, one of the trackers can assist the other one in this framework.

The ownership probabilities, o_i's, can be determined by using an error criteria. For example, one way is to monitor how well each each partition $x_{i,t}$ in x_t explains the information streams $y_{i,t}$ through their state-observation equation pair defined by $\mathcal{S}_i$, (1). Then, the respective likelihood functions can be aggregated with an exponential envelope to solve for the o_i's recursively. In this case, the target posterior will be dynamically shifting towards the better self-consistent model while still taking into account the information coming from the other model, which might be unable temporarily to explain the temporary data stream.

If one believes that both models explain the underlying process equally likely regardless of their self-consistency, one can set $o_1 = o_2 = 1/2$ to have the marginal distribution of χ_t resemble the product of the marginal distributions imposed by both state spaces. The proposal strategy in this chapter is derived with this assumption on the ownership probabilities, because, interestingly, it is possible to show that assuming equal ownership probabilities along with (4) leads to the following conditional independence relation on the state spaces:

$$\pi_{t-1}(x_{1,t})\pi_{t-1}(x_{2,t}) = q_1(x_{1,t}|x_{1,t-1})q_2(x_{2,t}|x_{2,t-1}) \tag{6}$$

Equation (6) decouples the partition $x_{i,t}$ distributions in the joint update for x_t, and results in the following update equation:

$$\begin{aligned}\pi_{t-1}(x_t) &= \pi_{t-1}(\psi_{1,t}, \psi_{2,t}|\chi_t)\pi_{t-1}(\chi_t) \\ &= \pi_{t-1}(\psi_{1,t}|\chi_t)\pi_{t-1}(\psi_{2,t}|\chi_t)\pi_{t-1}(\chi_t) \\ &= \frac{\pi_{t-1}(x_{1,t})\pi_{t-1}(x_{2,t})}{\pi_{t-1}(\chi_t)} \\ \Rightarrow \pi_{t-1}(x_t) &= \frac{q_1(x_{1,t}|x_{1,t-1})q_2(x_{2,t}|x_{2,t-1})}{\pi_{t-1}(\chi_t)}, \quad \text{where}\end{aligned} \tag{7}$$

$$\pi_{t-1}(\chi_t) \propto \left[\iint q_1(x_{1,t}|x_{1,t-1})d\psi_{1,t}q_2(x_{2,t}|x_{2,t-1})d\psi_{2,t}\right]^{1/2}. \tag{8}$$

3. PROPOSAL STRATEGY

A proposal function, denoted as $g(x_t|x_{t-1}, y_t)$, determines the random support for particle candidates to be weighted. Two very popular choices are (i) the state update $g \propto q_i(x_t|x_{t-1})$, and (ii) the full posterior $g \propto f_i(y_t|x_t)q_i(x_t|x_{t-1})$. The first one is attractive because of its tractability. The second one performs better because it incorporates the latest data while proposing particles, thereby decreasing the variance of the importance weights (since, in effect, it directly samples the posterior) [3, 4]. Moreover, it can be approximated for faster particle generation by using local linearization techniques (see [4]), where the posterior is approximated by a Gaussian. An example proposal function obtained by local linearization of the posterior is

$$g(x_t|x_{t-1}, y_t) \approx \mathcal{N}(\mu(z) + z, \Sigma(z)), \quad \text{where} \tag{9}$$

$$\Sigma(z) = -\left[l_f''(z) + l_q''(z)\right]^{-1}, \quad \mu(z) = \Sigma(z)\left[l_f'(z) + l_q'(z)\right],$$

and where l' and l'' are the gradients and the Hessians of the respective log-likelihood functions. Also, z is judiciously chosen as the mode of the posterior, which can be calculated. Hence, by either way of proposing particles, one can assume that an analytical relation for g_i, defining the support of the actual posterior for each state space, can be obtained.

Figure 1 describes the proposal strategy used for the joint state space. It is assumed that each state space has a proposal strategy described by the analytical functions $\{g_i, i = 1,2\}$ defined over the whole state-spaces. Then, the proposal functions of the each state g_i are used to propose particles for the joint space by carefully combining the supports of the individual posteriors. First, marginalize out the parameters $\psi_{i,t}$:

$$\hat{g}_i(\chi_t|x_{i,t-1}, y_{i,t}) = \int g_i(x_{i,t}|x_{i,t-1}, y_{i,t})d\psi_{i,t}. \tag{10}$$

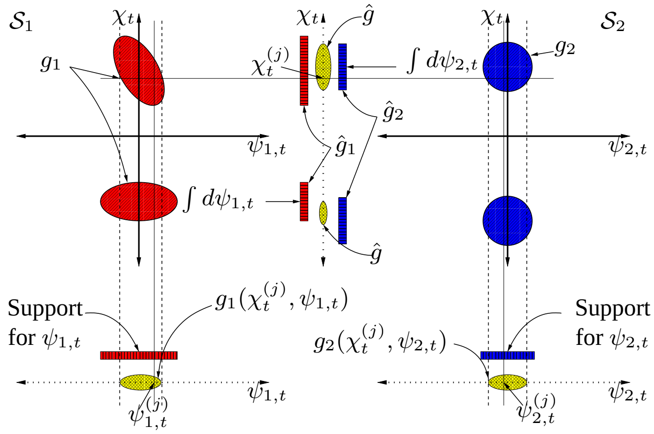

Fig. 1. The supports g_i for the posterior distribution in each state space are shown on the axes χ_t vs. $\psi_{i,t}$. Particles for the joint state are generated by first generating χ_t's from the combined supports of the marginal distributions of χ_t. Parameters $\psi_{i,t}$ are then sampled from g_i, as constrained by χ_t.

The functions $\hat{g}_i$ describe the random support the for the common state parameter χ_t, and can be combined in the same way as the joint state update (4). Hence, the following function

$$\hat{g}(\chi_t|x_{t-1}, y_{1,t}, y_{2,t}) \propto [\hat{g}_1(\chi_t|x_{1,t-1}, y_{1,t})\hat{g}_2(\chi_t|x_{2,t-1}, y_{2,t})]^{1/2} \tag{11}$$

can be used to generate the candidates $\chi_t^{(j)}$ for the overlapping state parameters. Then using $\chi_t^{(j)}$, one can generate $\psi_{i,t}^{(j)}$ from $g_i(\chi_t^{(j)}, \psi_{i,t}|x_{i,t-1}, y_{i,t})$, and form $x_t^{(j)} = [\chi_t^{(j)}, \psi_t^{(j)}, \varphi_t^{(j)}]$.

In general, Monté-Carlo simulation methods can be used to simulate the marginal integrals in this paper [10]. Here, we show how to calculate the marginal integrals of the state models. Simulation of the other integrals are quite similar. Given $\chi_t^{(j)}$, draw M samples using $\psi_{i,t}^{(m)} \sim g_i(\chi_t^{(j)}, \psi_{i,t}|x_{i,t-1}, y_{i,t})$[1]. Then,

$$\int q_1(\chi_t^{(j)}, \psi_{i,t}|x_{1,t-1})d\psi_{i,t} \approx \frac{1}{M}\sum_{m=1}^{M} \frac{q_1(\chi_t^{(j)}, \psi_{i,t}^{(m)}|x_{1,t-1})}{g_1(\chi_t^{(j)}, \psi_{i,t}^{(m)}|x_{1,t-1}, y_{1,t})} \tag{12}$$

Pseudo-code for the joint strategy is given in Table 1. Finally, the importance weights for the particles generated by the joint strategy described in this section can be calculated as follows:

$$w^{(j)} \propto \frac{p(x_t^{(j)}|x_{t-1}, y_{1,t}, y_{2,t})\hat{g}(\chi_t^{(j)}|x_{t-1}, y_{1,t}, y_{2,t})}{g_1(\chi_t^{(j)}, \psi_{1,t}^{(j)}|x_{1,t-1}, y_{1,t})g_2(\chi_t^{(j)}, \psi_{2,t}^{(j)}|x_{2,t-1}, y_{2,t})}. \tag{13}$$

4. EXAMPLES

This section demonstrates the proposal strategy with an analytical example, emphasizing that the joint tracker (called $\mathcal{J}$) has less RMS error in tracking when compared to the trackers (also called $\mathcal{S}_i$) formulated using the individual state spaces, and that

[1]It is actually not necessary to draw the samples directly from $g_i(\chi_t^{(j)}, \psi_{i,t}|-)$. An easier distribution function approximating only q_i can be used for simulating the marginalization integral (12).

Table 1. **Pseudo Code for Joint Proposal Strategy**

1. Given the state update q_i and observation relations f_i, determine analytical relations for the proposal functions g_i. It is important to approximate the true posterior as close as possible because these approximations are used to define the random support for the final joint posterior. For this purpose, Gaussian approximation of the posterior (9) or linearization of the state equations can be used [4].
2. Determine the support for the common state parameter χ_t using (11). The expression for $\hat{g}$ should be approximated or simulated to generate candidates $\chi_t^{(j)}$, $j = 1, 2, \ldots, N$ where N is the number of particles.
3. Given $\chi_t^{(j)}$, (i). calculate the marginal integrals by using (12), (ii). generate $\psi_{i,t}^{(j)} \sim g_i(\chi_t^{(j)}, \psi_{i,t}|x_{i,t-1}, y_{i,t})$, (iii). form $x_t^{(j)} = [\chi_t^{(j)}, \psi_{1,t}^{(j)}, \psi_{2,t}^{(j)}]$, and (iv). calculate the importance weights, $w^{(j)}$'s, using (13).

the joint tracker manifest robustness in the common parameter estimates even if one of the models diverges. For the analytic example, the ownership parameters for the state spaces are fixed to $o_i = 1/2, i = 1, 2$ and the prior component $r(\cdot)$ in (4) is not used.

Consider the following state-space descriptions

$$\mathcal{S}: \quad \begin{bmatrix} \chi_t \\ \psi_t \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \chi_{t-1} + \cos(\omega t)\psi_t \\ \psi_{t-1} \end{bmatrix}, \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}\right) \tag{14}$$

$$\begin{aligned} \mathcal{S}_1: \quad & \begin{bmatrix} \chi_t \\ \psi_t \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \chi_{t-1} + \psi_t \\ \psi_{t-1} \end{bmatrix}, \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}\right) \\ & y_t \sim \mathcal{N}\left(\begin{bmatrix} \chi_t \\ \psi_t \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right) \end{aligned} \tag{15}$$

$$\begin{aligned} \mathcal{S}_2: \quad & \chi_t \sim \mathcal{N}(\chi_{t-1}, \lambda_3^2) \\ & \theta_t \sim \mathcal{N}\left(\tan^{-1}\left(\frac{\chi_t}{W}\right), \sigma_3^2\right) \end{aligned} \tag{16}$$

where $\mathcal{S}$ is the true state model with a deterministic parameter ω. $\mathcal{S}_1$ and $\mathcal{S}_2$ are the image and acoustic tracker formulations of $\mathcal{S}$. In this case, the joint state space is included in the state space of $\mathcal{S}_1$. Hence, the objective is to do a better job in tracking χ_t given the aggregate observations.

Since the data-likelihood and the state-likelihood functions of $\mathcal{S}_1$ are linear Gaussian, the full posterior can be analytically determined. In this case, the proposal function g_1 will be the actual posterior for $\mathcal{S}_1$:

$$g_1(\chi_t, \psi_t|\chi_{t-1}, \psi_{t-1}, y_t) \sim \mathcal{N}(\mu_1, \Sigma_1), \tag{17}$$

where

$$\Sigma_1 = \begin{bmatrix} \frac{1}{\sigma_1^2} + \frac{1}{\lambda_1^2} & -\frac{1}{\lambda_1^2} \\ -\frac{1}{\lambda_1^2} & \frac{1}{\sigma_2^2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \end{bmatrix}^{-1}, \tag{18}$$

$$\mu_1 = \Sigma_1 \left\{ \begin{bmatrix} y_{t,1}/\sigma_1^2 \\ y_{t,2}/\sigma_2^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{\lambda_1^2} & -\frac{1}{\lambda_1^2} \\ -\frac{1}{\lambda_1^2} & \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \end{bmatrix} \begin{bmatrix} \chi_{t-1} + \psi_{t-1} \\ \psi_{t-1} \end{bmatrix} \right\}. \tag{19}$$

The marginal distribution of χ_t from $\mathcal{S}_1$ is calculated using its state update. The resulting marginal distribution is

$$p(\chi_t|\chi_{t-1}, \psi_{t-1}) \sim \mathcal{N}\left(\chi_{t-1} + \psi_{t-1}, \lambda_1^2 + \lambda_2^2\right). \tag{20}$$

Table 2. RMS errors for χ_t and ψ_t. Large numbers in the table are caused by tracking divergence. The underlying model $\mathcal{S}$ is closer to $\mathcal{S}_1$ when ω is small, whereas it is closer to $\mathcal{S}_2$ when ω is large.

ω	$\mathcal{J}(\chi_t)$	$\mathcal{S}_1(\chi_t)$	$\mathcal{S}_2(\chi_t)$	$\mathcal{J}(\psi_t)$	$\mathcal{S}_1(\psi_t)$
1/500	44.38	80.24	6669.5	21.43	24.83
1/50	42.99	101.05	1511.7	26.55	27.68
1/5	235.7	1134.7	48.96	222.59	134.21

Moreover, (10), and (17) lead to

$$\hat{g}_1(\chi_t|\chi_{t-1}, \psi_{t-1}, y_t) \sim \mathcal{N}\left(\mu_1(1), \Sigma_1(1,1)\right). \qquad (21)$$

For $\mathcal{S}_2$, the data-likelihood is given by

$$L_\theta \doteq -\frac{1}{2\sigma_3^2}\left[\theta_t - \tan^{-1}\left(\frac{\chi_t}{W}\right)\right]^2. \qquad (22)$$

The proposal function for $\mathcal{S}_2$, which approximates the full posterior is obtained by the local linearization of the model as follows:

$$g_2(\chi_t|\chi_{t-1}, \theta_t) = \hat{g}_2(\chi_t|\chi_{t-1}, \theta_t) \sim \mathcal{N}\left(\mu_2, \Sigma_2\right), \qquad (23)$$

$$\Sigma_2 = \left[\frac{1}{\sigma_3^2} + \left(\frac{W}{W^2+\chi_t^2}\right)^2 \frac{1}{\lambda_3^2}\right]^{-1}, \qquad (24)$$

$$\mu_2 = \Sigma_2\left[\frac{\chi_{t-1}}{\sigma_3^2} + \frac{W}{W^2+\chi_t^2}\frac{1}{\lambda_3^2}\left(y_{t,2} - \tan^{-1}\left(\frac{\chi_{t-1}}{W}\right) + \frac{W\chi_{t-1}}{W^2+\chi_t^2}\right)\right]. \qquad (25)$$

Hence, to generate χ_t, substitute (21) and (23) into (11) to obtain

$$\hat{g}(\chi_t|\chi_{t-1}, \psi_{t-1}, y_t, \theta_t) \sim \mathcal{N}\left(\mu, \Sigma\right) \qquad (26)$$

$$\Sigma = 2\left(\frac{1}{\Sigma_1(1,1)} + \frac{1}{\Sigma_2}\right)^{-1}, \qquad (27)$$

$$\mu = \left(\frac{1}{\Sigma_1(1,1)} + \frac{1}{\Sigma_2}\right)^{-1}\left(\frac{\mu_1(1)}{\Sigma_1(1,1)} + \frac{\mu_2}{\Sigma_2}\right). \qquad (28)$$

Given $\chi_t^{(j)}$, (17) can be used to generate ψ_t as follows

$$g_1(\chi_t^{(j)}, \psi_t|\chi_{t-1}, \theta_t) \sim \mathcal{N}\left(\mu_1(2) + \frac{\rho_1\left(\chi_t^{(j)} - \mu_1(1)\right)}{\Sigma_1(1,1)}, \Sigma_1(2,2)\right) \qquad (29)$$

where ρ_1 is the correlation coefficient of Σ_2.

Three particle filters ($\mathcal{J}$,$\mathcal{S}_1$,$\mathcal{S}_2$) are implemented each with 100 particles. $\mathcal{J}$ uses the joint proposal strategy, whereas $\mathcal{S}_1$ and $\mathcal{S}_2$ use (17) and (26) as proposal functions. Each filter also has a resampling stage, where particles are resampled with replacement according to their probabilities. Simulation parameters are $\lambda_1^2 = 1$, $\lambda_2^2 = .5$, $\sigma_1^2 = 2$, $\sigma_2^2 = 1$, $\lambda_3^2 = 9$, $\sigma_3^2 = (1^\circ)^2$, $W = 50$, and various ω's. The simulation is done for 50 iterations total. Table 2 demonstrates the RMS error averages of the Monté-Carlo simulation, where model $\mathcal{S}$ is simulated 100 times with different ω parameters.

It should be noted that when neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ diverges, their performances are comparable. In those cases, the joint filter has less than one-fourth of their combined RMS error, because the particles are not wasted in redundant areas. Table 2 also shows that the joint filter demonstrates improved performance as well as robustness. Even though one of the models diverges, the joint filter still does a good job, because it is, in effect, assisted by the presence of the other built-in model, which can still track the phenomenon. The filter's performance may be further improved by implementing the adaptive ownership probabilities. Lastly, the joint tracker's robustness vs. time is shown in Fig. 2.

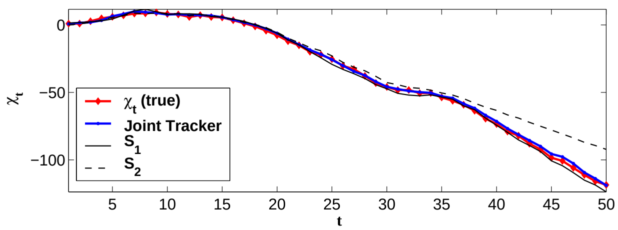

Fig. 2. Example tracking realization with $\omega = 1/50$: Even though the particle filter designed using $\mathcal{S}_2$ is unable to track after $t = 35$s, the joint tracker still does a good job since it uses the information from coming both state-space models.

5. CONCLUSIONS

In this paper, a general proposal strategy is demonstrated for joint state-space tracking with particle filters. The framework is quite general and it allows different state-spaces to assist each other in tracking a common parameter. The framework is demonstrated using an analytical example that mimics a joint acoustic image tracker.

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