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Sufficient conditions for the existence of Q-balls in gauge theories

Alexander Kusenko,[1*] Mikhail Shaposhnikov,[1†] and P. G. Tinyakov[2‡]

[1]Theory Division, CERN, CH-1211 Geneva 23, Switzerland

[2]Institute for Nuclear Research, 60th October Anniversary Prospect 7a, Moscow, Russia 117312

Abstract

We formulate a set of simple sufficient conditions for the existence of Q-balls in gauge theories.

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* email address: Alexander.Kusenko@cern.ch
† email address: mshaposh@nxth04.cern.ch
‡email address: peter@flint.inr.ac.ru

Abelian Q-balls are non-topological solitons that accommodate some conserved global charge at a lesser energetic toll than a collection of free scalar particles [1, 2]. They exist in theories that preserve some global U(1) symmetry[1] and whose scalar potential satisfies certain dynamical constraints.

Q-balls arise naturally in theories with supersymmetry. Supersymmetric extensions of the Standard Model, *e. g.*, MSSM, predict the existence of new scalar baryons and leptons that have the requisite interactions that allow for Q-balls [5]. Baryonic Q-balls that form along a flat direction in the potential [6, 7] can be entirely stable [8].

In addition to scalar interactions, the scalar fields may have gauge interactions as well. This is the case in the MSSM, where the only scalar fields that carry a baryon number, squarks, transform non-trivially under the color SU(3) gauge group. If the effect of the gauge fields cannot be eliminated, the semiclassical description of the solitons may be hampered by the complications related to confinement and other aspects of gauge dynamics. It is important, therefore, to design a proper description of non-topological solitons in the presence of gauge interactions. Previous treatment of supersymmetric Q-balls ignored the effects of the gauge fields because in many cases of interest it is sufficient to deal with the gauge-invariant scalar degrees of freedom.

A straightforward approach to Q-balls in gauge theories would be to find a solution to the equations of motion with a fixed global charge. With the use of Hamiltonian formalism, the problem may be formulated as follows. Let the scalar fields ϕ be a (reducible, in general) representation of some semi-simple (unbroken) gauge group G spanned by the generators T^k. And let the scalar potential $U(\phi)$ preserve a global U(1) symmetry $\phi \to e^{iB\theta}\phi$, where B is the U(1) generator that is assumed to commute with T^a. To construct a Q-ball solution, one can find a minimum of the energy functional

$$E_{total} = \int d^3x \, \left[\frac{1}{2}(E^a)^2 + \frac{1}{2}(H^a)^2 + p^\dagger p + |D_i\phi|^2 + U(\phi) \right] \tag{1}$$

[1]The conservation of a local U(1) charge [3] or a global non-abelian charge [4] can also lead to the appearance of non-topological solitons.

with an additional condition

$$\int d^3x \hat{B} \equiv \int d^3x \, \frac{1}{i}(p^\dagger B\phi - \phi^\dagger B p) = Q. \tag{2}$$

In addition, the Gauss constraint must also be satisfied,

$$D_i E_i^a - \hat{T}^a = 0. \tag{3}$$

Here E^a (H^a) is a generic notation for the non-abelian electric (magnetic) field, D_μ is a covariant derivative, p's are the canonical momenta of the scalar fields, $p = \delta\mathcal{L}/\delta(D_0\phi)^\dagger = D_0\phi$, and $\hat{T}^a$ are the non-abelian charge densities,

$$\hat{T}^a \equiv \frac{1}{i}(p^\dagger T^a \phi - \phi^\dagger T^a p). \tag{4}$$

In general, such a solution can have a non-zero non-abelian charge, with the gauge fields dying away slowly at infinity. It is unclear how to interpret such a solution in a theory with confinement of non-abelian charge. It is also difficult to find such solutions by solving a complicated system of coupled non-linear field equations.

In this note we formulate a set of simple sufficient conditions for the existence of Q-ball solutions that do not carry any overall non-abelian charge (even though the charge densities may not vanish locally). For this type of non-topological solitons, the issues of confinement are not essential and the semiclassical description is valid.

Let us look for a minimum of functional (1), where all gauge fields are taken to be zero, with additional conditions (2) and (3). If the energy of a configuration found this way is less than the energy of a collection of free scalar particles with the same charge Q, then a Q-ball does exist. A field configuration that minimizes the energy over a subspace of classical trajectories with zero gauge fields may not, of course, be the global minimum of energy, nor is it necessarily a solution of the equations of motion. Clearly, a conditional minimum of energy E over a subset of configurations is greater or equal to the global minimum over the whole functional space. If the former is less that the energy of a free-particle state, then so is the latter. By construction, Q-balls of this type have zero gauge charges.

In order to formulate the sufficient conditions, we introduce the Lagrange multipliers λ and ξ^a that correspond to the constraints (2) and (3) (the latter is simply $\hat{T}^a(x) = 0$ now), respectively, and reduce the problem to that of finding an extremum of

$$\mathcal{E}_{\lambda,\xi} = \int d^3x\,[p^\dagger p + |\partial_i\phi|^2 + U(\phi)] - \lambda[\int d^3x \hat{B}(x) - Q] - \int d^3x \xi^a(x)\hat{T}^a(x). \tag{5}$$

The equation of motion for p gives

$$p(x) = -i\lambda B\phi - i\xi^a(x)T^a\phi. \tag{6}$$

The equations for λ and ξ are

$$\lambda\phi^\dagger BT^a\phi + \xi^b\phi^\dagger\,\frac{1}{2}\{T^a, T^b\}\phi \;=\; 0, \quad a = 1, ..., dim(G) \tag{7}$$

$$\int d^3x[\lambda\phi^\dagger B^2\phi + \xi^b(x)\phi^\dagger BT^b\phi] \;=\; Q. \tag{8}$$

A Q-ball exists if the system of equations (7) and (8) has a solution, and if the corresponding extremal value of $\mathcal{E}_{\lambda,\xi}$ is less than the energy of any free-particle state with the same charge:

$$\mathcal{E}_{\lambda,\xi} < Q \min_i \{m_i/b_i\}, \tag{9}$$

where m_i is the mass of the i's particle, which has the global charge b_i.

It is easy to see that the energy of a soliton can be found from the minimisation of a functional without the conjugate momenta,

$$E_\lambda = \int d^3x\,[|\partial_i\phi|^2 + \hat{U}_\lambda(\phi)], \tag{10}$$

where

$$\hat{U}_\lambda(\phi) = U(\phi) - \lambda[\lambda\phi^\dagger B^2\phi + \xi^a(\lambda,\phi)\phi^\dagger BT^a\phi], \tag{11}$$

and $\xi^a(\lambda,\phi)$ are found from the system of equations (7). As in Refs. [2, 9], one can use the correspondence between a Q-ball in the potential $U(\phi)$ and a bounce in $d = 3$ Euclidean dimensions in the potential $\hat{U}_\lambda(\phi)$.

These conditions simplify in the thin-wall limit, where one can approximate the Q-ball solution by a field configuration that vanishes outside a sphere with radius R, and is $\phi(x) = e^{-i(\lambda B+\xi T)t}\phi_0$ for $\vec{x}^2 \equiv r < R$. If one defines $\bar{\lambda} = 2V\lambda/Q$ and $\bar{\xi}^a = 2V\xi^a/Q$, where $V = 4\pi R^3/3$, equations (7) and (8) become a system of linear equations for $\bar{\lambda}$ and $\bar{\xi}$. It has a solution if there exists a gauge-invariant polynomial of ϕ and $\phi^\dagger$ with a non-zero baryon number (*cf.* Ref. [10]). The condition of stability of a Q-ball with respect to its decay into the free scalar particles becomes

$$\min_{\phi_0} \sqrt{U(\phi_0)\bar{\lambda}(\phi_0)} \leq \min_i \{m_i/b_i\}. \tag{12}$$

For illustration, let us consider a scalar condensate associated with a *udd* flat direction in the MSSM [11], where the squarks $q_a^{(j)}$ have non-zero VEV's. A color-singlet condensate that satisfies equations (7) at $\xi^a = 0$ can have the form $q_a^{(j)} = e^{i\lambda t/3}\varphi^{(j)}(x)\delta_a^j$. The constraints (7) are automatically satisfied for the off-diagonal generators of color SU(3) (in the Gell-Mann basis). The remaining two equations for T^3 and T^8 demand that $\varphi^{(1)}(x) = \varphi^{(2)}(x) = \varphi^{(3)}(x) \equiv \phi(x)$. At the same time, the global $\mathrm{U}_{B}(1)$ current $j_B^\mu(x) \equiv \frac{1}{3} q_a^{(j)\dagger} \overleftrightarrow{\partial}^\mu q_a^{(j)} = \frac{1}{3}\lambda\phi^2(x) \neq 0$. Of course, the vanishing of the gauge charge is automatic for every flat direction of the MSSM and need not be verified explicitly thanks to the general theorems [10]. The remaining condition (9) is also satisfied as long as the scalar potential grows slower than the second power of the scalar VEV along the flat direction.

We have formulated the sufficient conditions for the existence of Q-balls in a class of gauge theories. Although the true ground state in the sector of fixed charge may have non-vanishing gauge fields, its energy is less than that of the configuration we have constructed. The latter, in turn, is less than the energy of any free-particle state with the same global charge, which ensures the existence of a soliton.

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