Brane-worlds and theta-vacua

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Reductions from odd to even dimensionalities ($5 \to 4$ or $3 \to 2$), for which the effective low-energy theory contains chiral fermions, present us with a mismatch between ultraviolet and infrared anomalies. This applies to both local (gauge) and global currents; here we consider the latter case. We show that the mismatch can be explained by taking into account a change in the spectral asymmetry of the massive modes—an odd-dimensional analog of the phenomenon described by the Atiyah-Patodi-Singer theorem in even dimensionalities. The result has phenomenological implications: we present a scenario in which a QCD-like θ-angle relaxes to zero on a certain (possibly, cosmological) timescale, despite the absence of any light axion-like particle.

I. INTRODUCTION

The nontrivial vacuum structure of non-Abelian gauge theories [1, 2, 3, 4, 5] plays an important role in particle theory. It underlies baryon-number non-conservation in electroweak theory and the existence of θ-vacua in QCD. These θ-vacua present a problem, since $\theta \neq 0$ leads to CP violation in strong interactions, which is severely constrained by experiment. Because the vacuum structure depends on topology of the gauge fields, it is sensitive to the dimensionality of space-time. So, one may wonder what happens to electroweak instantons and the θ-vacua in scenarios where the number of space-time dimensions is extended beyond the usual four, and if perhaps a solution to the strong-CP problem can be achieved along these lines.

Topology of gauge fields is best discussed when the space is compact. So, in what follows, we consider only space-times of the form

$$\mathrm{spacetime}_d = \mathcal{S}_{d-1} \times R_1 \ , \tag{1}$$

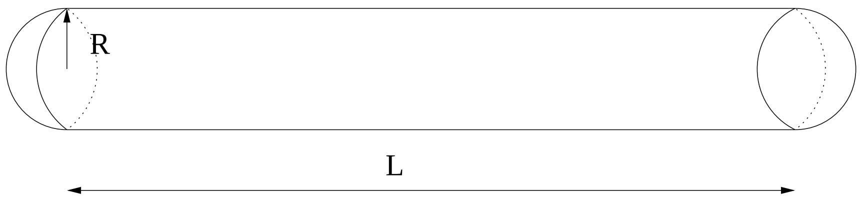

FIG. 1: A sausage-like manifold leading to topologically trivial vacuum and to the absence of θ problem.

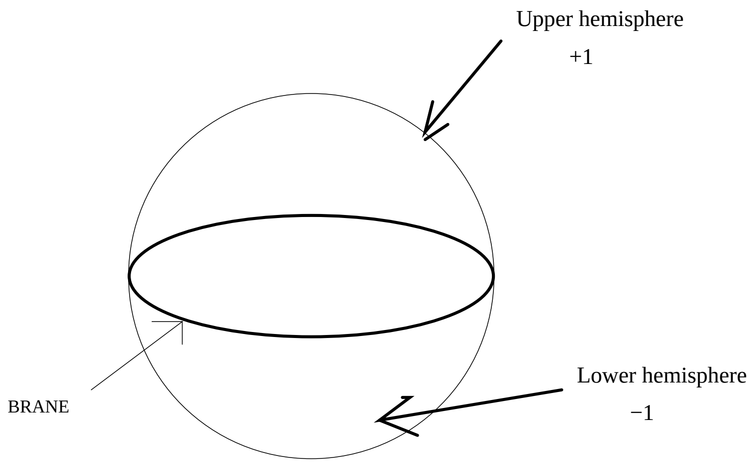

FIG. 2: A manifold with the topology of a sphere and a domain wall along the equator.

where $\mathcal{S}_{d-1}$ is a compact space, and R_1 corresponds to time. The case $d = 5$ is a situation that can be of phenomenological interest, but we also consider Abelian theories in $d = 3$, which are useful models.

Several kinds of such higher-dimensional scenarios can be considered. The simplest one (and, as far as we know, the first invoked in connection with the strong-CP problem [6]) is when $\mathcal{S}_{d-1}$ has the topology of a 4-sphere but the geometry of a 4-dimensional sausage: three dimensions large, and one small, see Fig. 1.

Another possibility is a brane-world: let the geometry of $\mathcal{S}_{d-1}$ be more or less arbitrary—take a round 4-sphere, for example—but suppose that we live on a domain wall along the equator see Fig.2 . Brane-world scenarios have been quite popular recently, but not exactly the kind we envision here—those where $\mathcal{S}_{d-1}$ is compact. Recently, a solution to Einstein equations with this topology was found in [7].

Finally, one can consider $\mathcal{S}_{d-1} = O \times \mathcal{S}_{d-2}$, where the extra dimension is an interval O, see Fig. 3. In what follows, we will often call such an interval an orbifold; these two terms

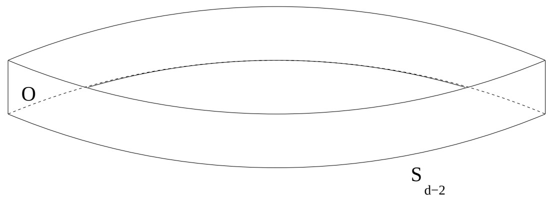

FIG. 3: A space $\mathcal{S}_{d-1} = O \times S_{d-2}$, where O is an orbifold (interval).

will be used interchangeably.

A question closely related to topology of gauge fields is the existence of chiral fermions and of anomalies in the corresponding currents. Indeed, by using an anomalous chiral transformation, we can rotate the θ angle out of the vacuum wavefunction and into the mass matrix of the fermions. This can be sometimes a very convenient way to represent the θ angle, since the θ-dependence can now be picked up by a calculation of the fermionic determinant. Yet, when we try to embed this picture into a higher-dimensional scenario, we encounter a paradox.

The lore holds that there is no anomaly in $d = 3, 5$ (or any other odd dimension; in orbifold scenarios this applies in the bulk of the orbifold, but not necessarily at the boundary [8]). This means that the chiral transformation, which we—from our 4-dimensional perspective—decided was anomalous, is in fact anomaly-free. Does that mean that it can be used to safely rotate the phase of the mass matrix to zero, without any extra terms appearing in the effective action? If that were true, it would imply, among other things, that any odd-dimensional theory solves the θ-problem *automatically*, i.e., without any reference to the theory's specific dynamics. On the other hand, if we recall that at low energies our odd-dimensional theory reduces to a 4-dimensional one, and so must share its properties, this kind of automatic solution looks exceedingly formal and suspect.

The present paper grew out of an attempt to resolve this paradox. The solution we are going to describe reminds us of the Atiyah-Patodi-Singer theorem [9], in that it emphasizes the role of high-frequency fermion modes. Although at low energies these modes are not observable directly, changes in their spectral asymmetry can lead to interesting low-energy consequences. At this point, though, the similarity with the APS theorem remains largely qualitative; in particular, they consider an even-dimensional Dirac operator, for which there is an anomaly, while we consider an odd-dimensional one, for which there is none. There is also some connection between our solution and the Callan-Harvey mechanism [10], which relates the anomaly in, say, four dimensions to a variation of a Chern-Simons term in five. However, the Callan-Harvey mechanism reproduces a gauge anomaly, while we are interested in a global (i.e., non-gauge) chiral transformation. The Chern-Simons term is immune to global transformations and therefore by itself will not do the job for us.

It is clear from the preceding that the paradox we are facing does not depend very sensitively on whether we are considering a non-Abelian gauge theory in five dimensions, or an Abelian theory in three. So, in most of the paper we concentrate on the second case as technically the simpler. With regard to the three types of extra-dimensional models listed above, we observe that, to our knowledge, chiral fermions have not been obtained for sausage-like compactifications. So, in what follows, we confine ourselves to brane-worlds and orbifolds. These two cases have many similarities and can be treated in parallel.

For the type of questions that we address here, the global topology of space is essential. We consider a brane-world for which the space is a two-sphere (will be a four-sphere in the 5d version), with the domain wall positioned along the equator.

The paper is organized as follows. In Sect. II, we describe how chiral fermions appear. In Sect. III, we consider the ultraviolet anomaly, as given by nonconservation of the current in the odd-dimensional theory. This anomaly is zero on a sphere, while on an orbifold it is concentrated at the endpoints [8]. We show that it can be alternatively interpreted as a flow of charge through the endpoints. In Sect. IV, we compute the infrared anomaly—the dependence of the fermion determinant on the phase of the fermion mass. We find that it is nonzero and coincides, in the low-energy limit, with the dependence computed using the effective low-energy theory from the start. The mismatch between the two anomalies is explained in Sect. V by considering the change in the spectral asymmetry of massive fermion modes. Gauge-field dynamics, responsible for existence (or non-existence) of θ-vacua, is considered in Sect. VI. There, we find that even though there is no true θ-vacuum on a sphere (in agreement with topological considerations), one can have an effective, time-dependent θ-angle. On the one hand, this suggests a solution to the strong-CP problem; on the other, it can have interesting cosmological consequences, if the relaxation of $\theta_{\rm eff}$ occurs on the cosmological timescale. In Sect. VII, we briefly discuss the case when the space is a disk, which turns out to be similar to the case of a sphere. Sect. VIII is a conclusion.

II. CHIRAL FERMIONS FROM COMPACT EXTRA DIMENSIONS

For most of this section, we consider $d = 3$ (corresponding to two-dimensional"observable" space-time). Generalization to the realistic case $d = 5$ is straightforward in the case of orbifold and is expected to present only technical difficulties in the case of a domain-wall on a sphere. Emergence of chiral fermions on an orbifold is well-known in the literature, see [11] (and also [8] and references therein), but we nevertheless describe it here for completeness and to fix the notations. Domain-wall fermions are well-known for the case when the extra dimension is a line [12]. Here, we are interested in the case when the higher-dimensional space is a sphere, with the domain wall positioned along the equator. Our analysis of this case is, as far as we know, new.

A. Chiral fermion on orbifold

Perhaps the simplest type of compactification leading to existence of chiral fermions is related to orbifolds. Consider a 3d space-time of the form $S_1 \times O \times R_1$, where R_1 is (non-compact) time, S_1 corresponds to large observable dimension with size L ($0 < x \leq L$), and O is a (short) interval corresponding to extra dimension ($-R/2 \leq z \leq R/2$), $L \gg R$. The Dirac equation for the 3-dimensional two-component fermion

$$\Psi(t, x, z) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2}$$

has the form

$$i\gamma^A \partial_A \Psi + m(z)\Psi = 0 \ , \tag{3}$$

where $m(z)$ is a mass term which in general depends on the extra coordinate z. We will keep $m(z)$ arbitrary as much as we can but occasionally, for the sake of simplicity, will specialize to $m(z) = 0$. Uppercase Latin indices scan all the coordinates, while the Greek ones "our" space-time. For this subsection it is convenient to choose the γ matrices as follows: $\gamma_0 = \tau_1$, $\gamma_1 = i\tau_2$ and $\gamma_2 = i\tau_3 \equiv i\gamma_5$, $\gamma_5 = \mathrm{diag}(1, -1)$, where τ_i are the Pauli matrices. The signature of the metric is $(1, -1, -1)$.

The boundary condition leading to existence of a left chiral fermion is [8]

$$(1 - \gamma_5)\Psi(\pm R/2) = 0 \ , \quad \text{or} \quad \psi_2(\pm R/2) = 0 \ . \tag{4}$$

Note that since eq. (3) is a system of two differential equations of the first order, one needs exactly two boundary conditions to specify the spectrum, as in (4). The wave-function of the chiral fermion with momentum k is simply

$$\Psi^{(0)} \propto e^{-ik(t+x)} \begin{pmatrix} \chi_0 \\ 0 \end{pmatrix} , \tag{5}$$

where $k = 2\pi l/L$ with integer l because of the periodic boundary condition $\Psi(t,0,z) = \Psi(t,L,z)$, and the zero mode is

$$\chi_0 = \frac{1}{N} \exp\left(- \int_0^z m(z')dz' \right) , \quad N^2 = \int_{-R/2}^{R/2} dz \exp\left(-2 \int_0^z m(z')dz' \right) . \tag{6}$$

The wave functions of the Kaluza-Klein tower of massive Dirac fermions with masses M_n ($n = 1, 2, \ldots$) are

$$\Psi^{(n)}(z) = \begin{pmatrix} \chi_n(z) \\ \psi_n(z) \end{pmatrix} , \tag{7}$$

where $\psi_n(z)$ and $\chi_n(z)$ are two sets of orthogonal normalized functions which satisfy the equations

$$\begin{aligned} \left[-\frac{d^2}{dz^2} + m^2(z) + \frac{dm}{dz} \right] \psi_n(z) &= M_n^2 \psi_n(z) , \\ \left[-\frac{d^2}{dz^2} + m^2(z) - \frac{dm}{dz} \right] \chi_n(z) &= M_n^2 \chi_n(z) \end{aligned} \tag{8}$$

with boundary conditions

$$\psi_n(\pm R/2) = 0 \quad \text{and} \quad \left(\frac{d\chi_n}{dz} + m(z)\chi_n \right) |_{\pm R/2} = 0 . \tag{9}$$

The spectrum of both operators is the same, except for the zero mode (6), as they are partner Hamiltonians from the point of view of supersymmetric quantum mechanics [13]. The relation between ψ and χ is given by $\chi_n = \frac{1}{M_n}(\partial_z - m(z))\psi_n$, $\psi_n = \frac{1}{M_n}(-\partial_z - m(z))\chi_n$.

For the case $m(z) = 0$, the wave functions have a simple form:

$$\psi_n(z) = \sqrt{\frac{2}{R}} \sin \pi n (\frac{z}{R} - \frac{1}{2}), \quad \chi_n(z) = \sqrt{\frac{2}{R}} \cos \pi n (\frac{z}{R} - \frac{1}{2}) , \tag{10}$$

for $n = 1, 2, 3 \ldots$ and $\chi_0(z) = \frac{1}{\sqrt{R}}$. The fermion masses are given by

$$M_n^2 = \left(\frac{\pi n}{R} \right)^2 . \tag{11}$$

The low-energy effective theory consists of a massless left chiral fermion described by a one component spinor in $1+1$ dimensions.

Similar considerations apply to a theory defined in the five-dimensional space-time $S_3 \times O \times R_1$. The only difference is that the 5d fermion has four components, while the low-energy (4d) massless chiral fermion now has two components.

B. Chiral fermions on S_2

An alternative way to obtain chiral fermions from extra dimensions is to consider a (2+1)-dimensional theory for which the space is a 2d sphere (of unit radius). The action of a single fermionic species is

$$\mathcal{A} = \int dt \sin\theta d\theta d\phi L \; , \quad L = i\bar{\Psi}\hat{\partial}\Psi - \Phi(\theta)\bar{\Psi}\Psi \; , \tag{12}$$

where

$$\hat{\partial} = \gamma^0\partial_0 + \gamma^1\partial_\theta + \gamma^2\frac{1}{\sin\theta}(\partial_\phi + \frac{1}{2}\gamma^1\gamma^2\cos\theta) \; . \tag{13}$$

Here θ and ϕ are the usual polar coordinates on the sphere, and Φ is a scalar field, whose dependence on θ is for a moment arbitrary, although later we will specify it to be a domain wall localized on the equator (i.e., at $\theta = \pi/2$). The field Ψ is a two-component spinor: $\Psi = (\psi_1, \psi_2)^T$. A convenient choice of γ-matrices for this subsection is $\gamma^0 = \tau_3$, $\gamma^1 = i\tau_1$, and $\gamma^2 = i\tau_2$. (Note that this is different from the choice we made in the case of orbifold.)

The problem has translational symmetry with respect to time and the azimuthal angle ϕ, so we can take the spinor to depend on these as $\exp[-iEt + im\phi]$, where $m = \pm\frac{1}{2}, \ldots$ is a half-integer (which should not be confused with the mass $m(z)$ we use for the orbifold theory). We then obtain the following equations for the components:

$$\left[\partial_\theta + \frac{1}{2}\mathrm{ctg}\theta + \frac{m}{\sin\theta}\right]\psi_2 + \Phi\psi_1 \;=\; E\psi_1 \; , \tag{14}$$

$$\left[\partial_\theta + \frac{1}{2}\mathrm{ctg}\theta - \frac{m}{\sin\theta}\right]\psi_1 + \Phi\psi_2 \;=\; -E\psi_2 \; . \tag{15}$$

These equations form the eigenvalue problem for the operator

$$\mathcal{O} = \begin{pmatrix} \Phi & \partial_\theta + \frac{1}{2}\mathrm{ctg}\theta + \frac{m}{\sin\theta} \\ -[\partial_\theta + \frac{1}{2}\mathrm{ctg}\theta - \frac{m}{\sin\theta}] & -\Phi \end{pmatrix} , \tag{16}$$

whose square is

$$\mathcal{O}^2 = \begin{pmatrix} \Phi^2 - [\partial_\theta^2 + \mathrm{ctg}\theta\partial_\theta - \frac{m^2 - m\cos\theta + \frac{1}{4}}{\sin^2\theta} - \frac{1}{4}] & -\partial_\theta\Phi \\ -\partial_\theta\Phi & \Phi^2 - [\partial_\theta^2 + \mathrm{ctg}\theta\partial_\theta - \frac{m^2 + m\cos\theta + \frac{1}{4}}{\sin^2\theta} - \frac{1}{4}] \end{pmatrix} . \tag{17}$$

We now see that the problem becomes particularly simple for Φ of the form of a step-function ($\Phi_0 > 0$):

$$\Phi(\theta) = \begin{cases} \Phi_0 \ , & \theta < \pi/2 \ , \\ -\Phi_0 \ , & \theta > \pi/2 \ . \end{cases} \tag{18}$$

This corresponds to the limit of an infinitely thin domain wall. In this case, the eigenvalue equation for $\mathcal{O}^2$ becomes diagonal everywhere outside the equator, while at the equator the off-diagonal terms in $\mathcal{O}^2$ produce δ-function "potentials". We adopt this choice of the scalar-field profile in what follows. We can then use solutions for constant fermion mass Φ_0 and match them at the equator.

Solutions for constant mass can be expressed through hypergeometric functions, using transformations described in ref. [14]. In what follows, we assume that $m > 0$. Solutions for $m < 0$ can be obtained by reflection about the equator. Define a new coordinate variable $z = \cos^2\frac{\theta}{2}$, and a new pair of functions $\xi(z)$ and $\eta(z)$:

$$\psi_1 = (1-x)^{\frac{m}{2} - \frac{1}{4}}(1+x)^{\frac{m}{2} + \frac{1}{4}}\xi \ , \tag{19}$$

$$\psi_2 = (1-x)^{\frac{m}{2} + \frac{1}{4}}(1+x)^{\frac{m}{2} - \frac{1}{4}}\eta \ , \tag{20}$$

where $x = \cos\theta = 2z - 1$. Then, the problem reduces to the eigenvalue problem for the operator

$$\begin{pmatrix} z(1-z)\frac{d^2}{dz^2} + [m + \frac{3}{2} - (2m+2)z]\frac{d}{dz} - ab & -(1-z)\Phi_{,z} \\ -z\Phi_{,z} & z(1-z)\frac{d^2}{dz^2} + [m + \frac{1}{2} - (2m+2)z]\frac{d}{dz} - ab \end{pmatrix} , \tag{21}$$

where

$$a = m + \frac{1}{2} + \sqrt{E^2 - \Phi^2} \ , \tag{22}$$

$$b = m + \frac{1}{2} - \sqrt{E^2 - \Phi^2} \ . \tag{23}$$

For the scalar field (18), we can construct the eigenfunctions (ξ, η) at $z \le \frac{1}{2}$ and $z \ge \frac{1}{2}$ from solutions to the hypergeometric equation that are regular at the north pole and the south pole, respectively. We obtain

$$\xi = \begin{cases} \mathcal{F}(a,b,m+\frac{3}{2};z) \ , & z \le \frac{1}{2} \ , \\ \nu\mathcal{F}(a,b,m+\frac{1}{2};1-z) \ , & z \ge \frac{1}{2} \ , \end{cases} \tag{24}$$

and

$$\eta = \begin{cases} -\sigma\nu\mathcal{F}(a,b,m+\frac{1}{2};z) \ , & z \le \frac{1}{2} \ , \\ -\sigma\mathcal{F}(a,b,m+\frac{3}{2};1-z) \ , & z \ge \frac{1}{2} \ . \end{cases} \tag{25}$$

where $\mathcal{F} \equiv {}_2F_1$. From continuity,

$$\nu = \frac{\mathcal{F}(a,b,m+\frac{3}{2};\frac{1}{2})}{\mathcal{F}(a,b,m+\frac{1}{2};\frac{1}{2})} \ . \tag{26}$$

From the jump of the derivatives on the equator, we obtain $\sigma = \pm 1$ and the eigenvalue equation

$$\frac{\nu\mathcal{F}'(a,b,m+\frac{1}{2};\frac{1}{2}) + \mathcal{F}'(a,b,m+\frac{3}{2};\frac{1}{2})}{4\Phi_0\mathcal{F}(a,b,m+\frac{3}{2};\frac{1}{2})} = \sigma = \pm 1 \ , \tag{27}$$

which determines the allowed energies E.

Using the differentiation formula

$$\mathcal{F}'(a,b,m+\frac{1}{2};\frac{1}{2}) = \frac{ab}{m+\frac{1}{2}}\mathcal{F}(a+1,b+1,m+\frac{3}{2};\frac{1}{2}) \tag{28}$$

and these formulas for special values of $\mathcal{F}$ [15]:

$$\begin{aligned} \mathcal{F}(a,b,\frac{1}{2}a+\frac{1}{2}b+1;\frac{1}{2}) &= 2\sqrt{\pi}\frac{\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)}{b-a}\left[X(a,b)-X(b,a)\right] \ , \\ \mathcal{F}(a,b,\frac{1}{2}a+\frac{1}{2}b;\frac{1}{2}) &= \sqrt{\pi}\Gamma(\frac{1}{2}a+\frac{1}{2}b)\left[X(a,b)+X(b,a)\right] \ , \end{aligned}$$

where

$$X(a,b) = \frac{1}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})} \ , \tag{29}$$

and Γ is Euler's Γ-function, we take the eigenvalue equation (27) to the form

$$b-a = \sigma\Phi_0\left[\frac{X(a,b)}{X(b,a)} - \frac{X(b,a)}{X(a,b)}\right] \ . \tag{30}$$

Eq. (30) can be explored in considerable detail in the limit $|a|$, $|b| \gg 1$, when we can use the expansion

$$\frac{\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(\frac{1}{2}a)} = \sqrt{\frac{a}{2}}\left[1 - \frac{1}{4a} + \frac{1}{32a^2} + O(a^{-3})\right] \ . \tag{31}$$

In particular, this limit applies in the case of main interest to us: $\Phi_0 \gg 1$ and $E \ll \Phi_0$, corresponding to a light bound state on the domain wall. This state has $\sigma = 1$, and for its energy we obtain

$$E^2 = m^2 + O(m^2/\Phi_0^2) \ . \tag{32}$$

We recall that m is a half-integer. In units where the radius of the sphere is $\mathcal{R}$ (rather than 1), eq. (32) gives $E^2 \approx m^2/\mathcal{R}^2$, which is the dispersion law of a massless fermion propagating along the equator.

The sign of E can be found by returning to eqs. (14), (15). We find $E \approx -m/\mathcal{R}$, which corresponds to a left-moving, i.e., chiral fermion in (1+1) dimensions.

The transition to the effective (1+1) theory is achieved by projecting the field Ψ onto the massless mode, i.e., by writing

$$\Psi(z, \phi; t) = \frac{1}{\sqrt{2\pi}} \sum_m e^{im\phi} \begin{pmatrix} \xi_m(z) \\ \eta_m(z) \end{pmatrix} A_m(t) \ , \tag{33}$$

where A_m is the amplitude of a single-component (chiral) 2d fermion. Note that we have indicated explicitly the dependence of ξ and η on m, which was implicit before. Also, we now assume that the basis spinor is normalized by the condition (no sum over m)

$$\int (\xi_m^* \xi_m + \eta_m^* \eta_m) \sin\theta d\theta = 1 \ . \tag{34}$$

This condition makes A_m canonically normalized. Note that the fermionic mode of an opposite chirality is singular at the poles of a sphere and is not normalizable.

In what follows, we will consider theory with two such chiral fermions, produced by two fields Ψ_1 and Ψ_2, whose interactions with the domain-wall field Φ have opposite signs. If $\Psi_1 = \Psi$ and is given by (33), then

$$\Psi_2(z, \phi; t) = \frac{1}{\sqrt{2\pi}} \sum_m e^{im\phi} \begin{pmatrix} \xi_m(z) \\ -\eta_m(z) \end{pmatrix} B_m(t) \ . \tag{35}$$

The presence of two fields makes possible a mass term $\mu \bar{\Psi}_1 \Psi_2$ with a complex μ. Let us see what becomes of this mass term upon the reduction to 2d. We have

$$\int \bar{\Psi}_1 \Psi_2 \sin\theta d\theta d\phi = \sum_m A_m^\dagger B_m \int (\xi_m^*, \ \eta_m^*) \gamma^0 \begin{pmatrix} \xi_m \\ -\eta_m \end{pmatrix} \sin\theta d\theta \ . \tag{36}$$

Recalling that $\gamma^0 = \tau_3$ and using the normalization condition (34), we see that the result is the canonical mass term connecting two chiral 2d fermions.

III. FERMION CURRENT ON AN ORBIFOLD

As discussed in the introduction, one of the ingredients of the paradox that motivated the present study is the popular assertion of the absence of anomalies in odd dimensions (d = odd). While for the case when the space is a $(d-1)$-dimensional sphere we have no reason to doubt that assertion, for the case of an orbifold the precise statement requires some care. Namely, it is known that in that case anomalies are absent in the bulk of the orbifold but may exist on its boundary [8]. We pause here to review this boundary anomaly and to show that it can be interpreted as the flow of the corresponding current through the boundary of the orbifold.

Consider the theory of just one 3d fermion Ψ with coupling e to a gauge field A_B, $B = 0, 1, 2$. The interpretation that we are going to derive will apply also to global currents in theories with more than one fermion species.

On the orbifold $-R/2 \leq z \leq R/2$, the single-fermion theory, according to the calculation in ref. [8], is inconsistent as it contains a gauge anomaly concentrated at the end points $z = \pm R/2$. It is customary to write

$$\partial_A J^A = \frac{e}{4\pi} \epsilon_{\mu\nu} F^{\mu\nu} \left[\delta(z - R/2) + \delta(z + R/2)\right] \ , \tag{37}$$

where $J_A = \bar{\Psi}\gamma_A \Psi$. We will show here that the mathematical inconsistency of this theory has a simple physical interpretation. Namely, we will demonstrate that the limit

$$\lim_{\epsilon \to +0} \Big(J^2(R/2 - \epsilon) - J^2(-R/2 + \epsilon) \Big) = \frac{e}{8\pi} \epsilon_{\mu\nu} \left[F^{\mu\nu}(-R/2) + F^{\mu\nu}(+R/2)\right] \ . \tag{38}$$

is non-zero in the presence of background gauge field. Note that the values $J^2(\pm(R/2))$ *are equal to zero,* as this is enforced by the boundary conditions (4). In other words, one may either insist that the generator of the global gauge transformation is given by

$$Q = \int_{-R/2}^{+R/2} dz d^2x J^0 \tag{39}$$

and is not conserved because of the anomaly (37), while the flux through the endpoints of the orbifold is zero, or one may define the charge as a limit

$$Q = \lim_{\epsilon \to +0} \int_{-R/2+\epsilon}^{+R/2-\epsilon} dz d^2x J^0 \tag{40}$$

and relate its non-conservation to non-zero charge flux through the endpoints.

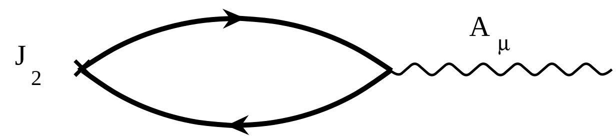

FIG. 4: The Feynman that gives a non-zero contribution to the current through the endpoints of the interval.

Let us now derive eq. (38). Formulas for fermionic propagators on the orbifold are collected in Appendix A. For the present theory, the fermionic propagator is given by S_{11} with $\mu = 0$, where S_{11} is defined in (A9). The value of J^2 in a background gauge field can be found from the diagram in Fig. 4 and is given by

$$J^2(x^\mu, z) = e \int d^2x' dz' \epsilon^{\mu\nu} A_\mu(x', z') \times \\ [(\partial_z - m) G_D(x - x'; z, z') \partial'_\nu G_D(x' - x; z', z) - \\ (\partial_z + m) G_N(x - x'; z, z') \partial'_\nu G_N(x' - x; z', z) + \\ \partial_\nu G_D(x - x'; z, z')(\partial'_z + m) G_N(x' - x; z', z) - \\ \partial_\nu G_N(x - x'; z, z')(\partial'_z - m) G_D(x' - x; z', z)] \quad , \tag{41}$$

where G_N and G_D are the Green functions defined by (A1) with $\mu = 0$. With the use of (A5) this can be simplified further to give

$$J^2(x^\mu, z) = 2e \int d^2x' dz' \epsilon^{\mu\nu} A_\mu(x', z') \times \\ [\partial_\nu G_D(x - x'; z, z')(\partial'_z + m) G_N(x' - x; z', z) - \\ \partial_\nu G_N(x - x'; z, z')(\partial'_z - m) G_D(x' - x; z', z)] \quad . \tag{42}$$

In this section we will compute the current for a background gauge field that is independent of z; a more general situation—a field slowly varying with z—is considered in Appendix B.

If the background field does not depend on z, the integration in (42) over z can be performed with the help of (A3) and orthogonality of the functions ψ_n and χ_n. The result is

$$J^2(x^\mu, z) = -\frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}\lambda(z) \tag{43}$$

where

$$\lambda(z) = \sum_{n=1}^{\infty}\frac{\psi_n(z)\chi_n(z)}{M_n} = \left[\frac{1}{2}\partial_z - m(z)\right]G_D(z) = \frac{1}{2}\left(\rho(z, R/2) - \rho(-R/2, z)\right) \ , \tag{44}$$

and ρ is defined in eq. (A11). Formally, $\lambda(\pm R/2) = 0$; however, $\lim_{\epsilon\to+0}\lambda(\pm(R/2-\epsilon)) \neq 0$. The flux of the charge through the interval end points is

$$\lim_{\epsilon\to+0}(J^2(R/2-\epsilon) - J^2(-R/2+\epsilon)) = \frac{e}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ . \tag{45}$$

This does not depend on the function $m(z)$ and exactly reproduces the anomaly in eq. (37).

Eq. (45) applies when the gauge field is independent of z. In Appendix B, we consider vector potentials slowly changing with z and show that in that case the flux depends on the value of the gauge field at the endpoints only.

IV. COMPUTATION OF THE FERMION DETERMINANT

Now, consider two species of fermions in (2+1) dimensions coupled to an external Abelian gauge field. Dynamics of the gauge field, and in particular the question of existence of a θ-structure, will be the subject of the subsequent sections. Here, we consider the determinant of fermions in an external field and, specifically, its dependence on the phase of the fermion mass. Thus, our starting point is the following (2+1)-dimensional Lagrangian

$$L_\Psi = i\bar{\Psi}_1\hat{D}\Psi_1 + m(z)\bar{\Psi}_1\Psi_1 + i\bar{\Psi}_2\hat{D}\Psi_2 - m(z)\bar{\Psi}_2\Psi_2 - \mu\bar{\Psi}_1\Psi_2 - \mu^*\bar{\Psi}_2\Psi_1 \ , \tag{46}$$

where $\hat{D} = \gamma^B D_B$, D_B is the covariant derivative, and $\mu = |\mu|e^{i\theta_M}$ is a small (complex) mass. For simplicity, we take μ to be independent of coordinates, while the 3d mass $m(z)$ can, at the moment, depend arbitrarily on the extra coordinate z. We will be interested in the dependence of the fermion determinant on the phase of μ.

We consider situations when the (2+1) theory (46) reduces at low energies to a (1+1)-dimensional theory. In other words, there will be a quasi-zero fermionic mode, while all other modes will be separated from it by a large gap. This can be arranged in both of the extra-dimensional scenarios considered in sect. II.

The resulting (1+1)-dimensional theory has a pair of chiral fermions, forming a two-component Dirac spinor of mass μ. In fact, this theory is nothing but the fermionic sector of a massive Schwinger model in $1+1$ dimensions, with the Lagrangian

$$L_{2d} = \bar{\psi} i \gamma^\mu D_\mu \psi - \mu \bar{\psi}_L \psi_R - \mu^* \bar{\psi}_R \psi_L \; . \tag{47}$$

If we start from five dimensions and a non-Abelian gauge group (say, SU(3)) we will similarly get the massive fermions of quantum chromodynamics.

The global current of the theory (46)

$$J_G^A = \bar{\Psi}_1 \gamma^A \Psi_1 - \bar{\Psi}_2 \gamma^A \Psi_2 \tag{48}$$

becomes, at low energies, the chiral current of the effective (1+1) theory:

$$J_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi \; . \tag{49}$$

In (1+1), the chiral symmetry, in addition to being broken by the mass μ, is also broken by the anomaly, which manifests itself in a dependence of the (1+1) determinant on $\theta_M = \arg \mu$, a dependence that does not disappear in the limit $\mu \to 0$. A naive expectation would be that this θ_M-dependence carries over to the full (2+1)-dimensional theory. Indeed, the masses of the heavy modes depend only weakly on μ, and therefore any contribution they make to the determinant should be regular in μ, i.e., independent of θ_M at $\mu \to 0$.

In this section, we show that the naive reasoning is in fact entirely correct (in particular, it is not affected by ultraviolet divergences). Thus, even though the complete (2+1) theory has no anomaly in the (bulk) chiral current, the low energy manifestations of the phase θ_M are the same as in the (1+1) theory, which has such an anomaly. On the one hand, this resolves the paradox formulated in the Introduction, but on the other, indicates that in brane-world scenarios the breakdown of chiral symmetry is realized rather non-trivially. Namely, non-conservation of the fermion number is not simply counted by the anomaly: we are pointed towards an additional effect, having to do with the spectral asymmetry.

A. Fermion determinant on an orbifold

To obtain an effective (1+1) theory that is free of a gauge anomaly, from an orbifold compactification, we use different boundary conditions for Ψ_1 and Ψ_2: for Ψ_1, they are those of eq. (4) whereas for Ψ_2 they single out the right-handed fermion,

$$(1+\gamma_5)\Psi_2(\pm R/2) = 0 \ . \tag{50}$$

The mixing of the left-handed fermion ψ_L in Ψ_1 and right-handed fermion ψ_R in Ψ_2 produces a Dirac fermion ψ with mass μ. Note that the 3d masses of Ψ_1 and Ψ_2 in (46) are the same (up to a sign), which allows one to use the eigenfunctions defined in eq. (8).

Both the complete 3d and effective 2d theories are free from gauge anomalies. However, the global currents (48) and (49) are anomalous: at $\mu = 0$,

$$\partial_A J^A_G = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}\left[\delta(z-R/2)+\delta(z+R/2)\right] \ , \tag{51}$$

$$\partial_\mu J^\mu_5 = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ , \tag{52}$$

where the δ-function at the boundary is defined so that its integral over z is equal to 1/2. The covariant derivative in this subsection is $D_B = \partial_B - ieA_B$.

As discussed in Sect. III, the 3d anomaly (51) is concentrated at the boundary of the orbifold. This anomaly, which determines non-conservation of the current, will be referred to as the ultraviolet anomaly. We now wish to see if it matches the “infrared” anomaly, which comes from the dependence of the fermionic determinant on the phase θ_M.

Consider the variation of the vacuum energy with respect to θ_M in a slowly-varying gauge field background in three dimensions. It is given by the diagram in Fig. 4, which can be immediately computed with the result

$$\frac{\partial\Omega}{\partial\theta_M}|_{\theta_M=0} = e\int d^2x dz\ \epsilon_{\mu\nu}F^{\mu\nu}(x,z)\kappa(z) \ , \tag{53}$$

where

$$\kappa(z) = |\mu|^2 \int d^2x' dz' \left[G_N(x';z,z')^2 - G_D(x';z,z')^2\right] \ . \tag{54}$$

This result is valid for arbitrary $m(z) \neq 0$. (Definitions of various Green functions are given in Appendix A.)

With the use of the mode expansion this can be rewritten as

$$\kappa(z) = |\mu|^2 \int \frac{d^2p}{(2\pi)^2}\sum_{m=0}\frac{\chi_m(z)^2-\psi_m(z)^2}{(p^2-M_m^2-|\mu|^2)^2} = \frac{|\mu|^2}{4\pi}\left[\tilde{G}_N(0,z,z)-\tilde{G}_D(0,z,z)\right] \ . \tag{55}$$

The function $\kappa(z)$ has the following important property

$$\int dz\kappa(z) = \frac{1}{4\pi} \ , \tag{56}$$

which shows that for z-independent field strengths the θ_M dependence of the vacuum energy is given entirely by the "ultraviolet" anomaly. Indeed, in this case, we can pull $F_{\mu\nu}$ out of the integral over z in (53) and use (56) to obtain

$$\frac{\partial\Omega}{\partial\theta_M}|_{\theta_M=0} = \frac{e}{4\pi}\int d^2x\ \epsilon_{\mu\nu}F^{\mu\nu}\ . \tag{57}$$

However, for arbitrary z-dependent background fields, that is no longer true. In this case, the θ_M dependence is more complicated. For example, for a theory with $m(z) = 0$ one finds, with the help of equations from Appendix A, that

$$\kappa(z) = \frac{1}{4\pi}\frac{\mu}{\sinh\mu R}\cosh 2\mu z\ , \tag{58}$$

so that

$$\frac{\partial\Omega}{\partial\theta_M}|_{\theta_M=0} = \frac{e}{8\pi}\left(\int d^2x\epsilon_{\mu\nu}\left[F^{\mu\nu}(x,-R/2) + F^{\mu\nu}(x,R/2)\right] - \int d^2x dz\frac{\sinh 2\mu z}{\sinh\mu R}\epsilon_{\mu\nu}\partial_z F^{\mu\nu}(x,z)\right)\ . \tag{59}$$

The first term is a boundary contribution that can be seen to match the "ultraviolet" anomaly (51). However, the second—bulk—term is new. It represents a mismatch between the "ultraviolet" and "infrared" anomalies for the case of orbifold. For example, if $F_{\mu\nu}$ vanishes at the endpoints, the "ultraviolet" anomaly is zero, but the bulk contribution still persists.

B. Determinant of domain-wall fermions in infinite flat space

Before we consider a domain wall on the equator of a sphere, let us look at a simpler case that has all the relevant features—a domain wall in flat space with an infinite extra dimension. In other words, instead of $\mathcal{S}_{d-1} = S_2$, we consider

$$\mathcal{S}_{d-1} = R_1 \times S_1\ . \tag{60}$$

The line R_1 is the extra dimension. Such a theory holds no promise for solving the strong-CP problem, but the structure of the fermion determinant is very similar to that on a sphere. In fact, after we handle the case (60), transition to a sphere will be relatively straightforward.

In this subsection, we absorb charge $(-e)$ into the field A_B, so that the covariant derivative is $D_B = \partial_B + iA_B$. Also, the 3d mass in (46) is assumed to be entirely due to the coupling with the domain-wall field Φ:

$$m(z) = -\Phi(z)\ . \tag{61}$$

Fermion determinant produces the following contribution to the effective action of the gauge field:

$$\Delta\mathcal{A} = -i\mathrm{Tr}\ln\begin{pmatrix} i\hat{D}-\Phi & -\mu \\ -\mu^* & i\hat{D}+\Phi \end{pmatrix} \equiv -i\mathrm{Tr}\ln\mathcal{M} \ . \tag{62}$$

We are interested in the derivatives of this action with respect to real and imaginary parts of $\mu = \mu_R + i\mu_I$ or, more precisely, in the dependence of these derivatives on the gauge field A, for example,

$$\frac{\partial}{\partial\mu_R}\Delta\mathcal{A} - [\ldots]_{A=0} = i\mathrm{Tr}\left\{\mathcal{M}^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - [\ldots]_{A=0}\right\} \ . \tag{63}$$

We use notation $-[\ldots]_{A=0}$ to denote a subtraction at zero A_B.

To invert the operator $\mathcal{M}$, we write

$$\begin{pmatrix} i\hat{D}+\Phi & \mu \\ \mu^* & i\hat{D}-\Phi \end{pmatrix}\mathcal{M} = \begin{pmatrix} (i\hat{D}+\Phi)(i\hat{D}-\Phi)-|\mu|^2 & 0 \\ 0 & (i\hat{D}-\Phi)(i\hat{D}+\Phi)-|\mu|^2 \end{pmatrix} \tag{64}$$

and then compute

$$(i\hat{D}+\Phi)(i\hat{D}-\Phi) \ = \ -D^B D_B - \frac{1}{2}\epsilon^{ABC}\gamma_A F_{BC} + \gamma^5\Phi' - \Phi^2 \ , \tag{65}$$

$$(i\hat{D}-\Phi)(i\hat{D}+\Phi) \ = \ -D^B D_B - \frac{1}{2}\epsilon^{ABC}\gamma_A F_{BC} - \gamma^5\Phi' - \Phi^2 \tag{66}$$

($\epsilon^{012} = 1$). We number coordinates in the way consistent with Sect. II: the extra coordinate z corresponds to $B = 1$, while "our" coordinate x to $B = 2$. In this computation we have assumed that the scalar field Φ depends only on z, so that $\Phi' = \partial_z\Phi$. The choice of γ-matrices is the same as in Sect. II: $\gamma^0 = \tau_3$, $\gamma^1 = i\tau_1$, and $\gamma^2 = i\tau_2$. In addition, we have introduced $\gamma_5 = -i\gamma^1$; this will be the γ_5 matrix of the effective (1+1)-dimensional theory.

We can now use (64) to express $\mathcal{M}^{-1}$ through the inverse of the operator on the right-hand side. This operator is diagonal in "isospin", i.e., the index distinguishing the two species, Ψ_1 and Ψ_2. The isospin trace can then be found explicitly; we continue to denote the remaining spin and coordinate trace by Tr.

We will only need the derivative (63) to the first order in F_{BC}. We find that to this order it can be written as a sum of two pieces:

$$\frac{\partial}{\partial\mu_R}\Delta\mathcal{A} - [\ldots]_{A=0} = I_1 + I_2 \ , \tag{67}$$

where

$$I_1 = i\mathrm{Tr}\left\{\frac{\mu^*}{-D^2-\gamma^5\Phi'-M^2}+\frac{\mu}{-D^2+\gamma^5\Phi'-M^2}\right\}-[\ldots]_{A=0}\ , \tag{68}$$

$$I_2 = \frac{i}{2}\mathrm{Tr}\epsilon^{ABC}\gamma_A F_{BC}\left\{\frac{\mu^*}{(\partial^2+\gamma^5\Phi'+M^2)^2}+\frac{\mu}{(\partial^2-\gamma^5\Phi'+M^2)^2}\right\}\ . \tag{69}$$

with $M^2=\Phi^2+|\mu|^2$. We now consider these two pieces in turn.

Calculation of I_1. This term reflects the coupling of the gauge field to the translational motion of fermions. As expected, no anomaly comes from this coupling; nevertheless, for completeness, we describe the calculation in some detail.

Define $P_B = iD_B$ and consider traces of various powers of the operator

$$\mathcal{O} = P^2-\gamma_5\Phi'-M^2\ . \tag{70}$$

In eq. (68) we need $\mathrm{Tr}\mathcal{O}^{-1}-[\ldots]_{A=0}$ (and an analogous trace with $\Phi'\to-\Phi'$). An anomaly in I_1 would correspond to a non-analytic behavior in the limit $\mu\to 0$: for a slowly varying $F=F_{0x}$, we would have

$$\mathrm{Tr}\mathcal{O}^{-1}-[\ldots]_{A=0}\approx\frac{\text{const.}}{|\mu|^2}\int dxdtF_{0x}(0,x,t) \tag{71}$$

(assuming that the domain wall is at $z=0$). Since traces of higher powers of $\mathcal{O}^{-1}$ can be obtained by differentiating with respect to $|\mu|^2$, they would have similar singular limits, for example,

$$\mathrm{Tr}\mathcal{O}^{-2}-[\ldots]_{A=0}\approx-\frac{\text{const.}}{|\mu|^4}\int dxdtF_{0x}(0,x,t)\ . \tag{72}$$

If we find that at least one of these traces does not have the requisite behavior, that means that the constant in (71) is in fact zero.

To verify the presence (or, rather, the absence) of these singular contributions, we use the "shift" method described in ref. [16]. In the presence of a domain wall, this method needs to be slightly generalized. In particular, we use only shift vectors q that lie within "our" (1+1)-dimensional subspace.

Consider

$$\mathrm{Tr}h\ln[(P-q)^2-\gamma_5\Phi'-M^2]-[\ldots]_{A=0}=\mathrm{Tr}h\ln[\mathcal{O}-2Pq+q^2]-[\ldots]_{A=0}\ , \tag{73}$$

where q is an arbitrary constant (1+1) vector, and h is an operator that depends only on the z component of P and so is immune to the shift. Expression (73), as well as the above expressions (71) and (72), is assumed to be properly regularized in the ultraviolet. For example, we can use a set of Pauli-Villars regulators. Such a regularization will be assumed in what follows, but it will not be indicated explicitly. The final result will be ultraviolet-finite.

Expanding in q to the second order, we obtain

$$\ln[\mathcal{O} - 2Pq + q^2] = \ln \mathcal{O} + \mathcal{O}^{-1}(-2Pq + q^2) - \frac{1}{2}\mathcal{O}^{-1}(-2Pq)\mathcal{O}^{-1}(-2Pq) + \ldots . \tag{74}$$

The idea of the "shift" method [16] is that, since the regularized trace is independent of q, traces of the order q^2 terms in eq. (74) should add up to zero. Averaging over directions of q, we see that this leads to

$$\mathrm{Tr} h\mathcal{O}^{-1} - [\ldots]_{A=0} = \mathrm{Tr} h\mathcal{O}^{-1}P_\mu \mathcal{O}^{-1}P^\mu - [\ldots]_{A=0} , \tag{75}$$

where μ takes values 0 and 2. Using the commutators

$$[P_\mu, \mathcal{O}^{-1}] = -\mathcal{O}^{-1}[P_\mu, \mathcal{O}]\mathcal{O}^{-1} , \tag{76}$$

and

$$[P_\mu, \mathcal{O}] = [P_\mu, P^2] = -i\{F_{\mu B}, P^B\} , \tag{77}$$

where the braces denote an anti-commutator, we can rewrite eq. (75) as

$$[\mathrm{Tr} h\mathcal{O}^{-1} - \mathrm{Tr} h\mathcal{O}^{-2}P^\mu P_\mu] - [\ldots]_{A=0} = i\mathrm{Tr} h\mathcal{O}^{-2}\{F_{\mu B}, P^B\}\mathcal{O}^{-1}P^\mu . \tag{78}$$

The difference of the traces on the left-hand side can be rewritten as

$$\mathrm{Tr} h\mathcal{O}^{-1} - \mathrm{Tr} h\mathcal{O}^{-2}P^\mu P_\mu = \mathrm{Tr} h\mathcal{O}^{-2}(-P_z^2 - \gamma_5\Phi' - M^2) , \tag{79}$$

so if we choose

$$h = (-P_z^2 - \gamma_5\Phi' - M^2)^{-1} , \tag{80}$$

eq. (78) becomes

$$\mathrm{Tr}\mathcal{O}^{-2} - [\ldots]_{A=0} = i\mathrm{Tr} h\mathcal{O}^{-2}\{F_{\mu B}, P^B\}\mathcal{O}^{-1}P^\mu . \tag{81}$$

This is to be compared to the would-be anomalous behavior, eq. (72). By inspection of the right-hand side of (81), we find that the anomalous term is absent. We conclude that there is no anomaly in I_1.

Calculation of I_2. This term reflect the coupling of the gauge field to the spin of fermions, which is the coupling that usually leads to an anomaly. In our case, the calculation of (69) in the limit of a slowly varying F amounts to a study of the spectra of two effective one-dimensional Hamiltonians: $H_1 = -\partial_z^2 + \Phi^2 - \gamma_5 \Phi'$ and $H_2 = -\partial_z^2 + \Phi^2 + \gamma_5 \Phi'$. These Hamiltonians are supersymmetric partners, and in addition both commute with γ_5. So, their spectra can be analyzed in some detail. However, for our present purposes, we only need the infrared parts of the spectra. In the presence of a domain wall of Φ, H_1 and H_2 each have a zero mode, with opposite chiralities. These are the only modes that give a singular contribution in the limit $\mu \to 0$. Therefore, in this limit, for slowly-varying (in comparison with $|\mu|$) fields, we obtain

$$I_2 \approx 2i\mu_I \int dxdt F_{0x}(0,x,t) \int \frac{d\omega dk_x}{(2\pi)^2} \frac{1}{(\omega^2 - k_x^2 - |\mu|^2 + i\epsilon)^2} \;, \tag{82}$$

which is the anomaly.

Combining the above results for I_1 and I_2, we find that the anomalous term in the effective action is

$$(\Delta\mathcal{A})_{\rm anom} = \frac{\theta_M}{2\pi} \int dxdt F_{0x}(0,x,t) \;, \tag{83}$$

where $\theta_M = \arg \mu$. This is precisely the same anomaly that would obtained in the effective (1+1) theory describing chiral fermions on the wall:

$$L_{2d} = i\bar{\psi}\gamma^\mu D_\mu \psi - \mu_R \bar{\psi}\psi - i\mu_I \bar{\psi}\gamma_5 \psi \;. \tag{84}$$

C. Determinant of domain-wall fermions on a sphere

On a sphere, the covariant derivative is

$$\hat{D} = \gamma^0(\partial_0 + iA_0) + \gamma^1(\partial_\theta + iA_\theta) + \gamma^2 \frac{1}{\sin\theta}(\partial_\phi + \frac{1}{2}\gamma^1\gamma^2 \cos\theta + iA_\phi) \;. \tag{85}$$

The relevant infrared limit now is

$$\mathcal{R}^{-1} \ll |\mu| \ll \Phi_0 \;, \tag{86}$$

where $\mathcal{R} = 1$ is the radius of the sphere, and Φ_0 the magnitude of the scalar field away from the equator. Because of the explicit dependence of $\hat{D}$ on the polar angle θ, various additional terms appear in the calculation of the determinant. Nevertheless, in the limit (86), the final answer is the natural adaptation of eq. (83):

$$(\Delta\mathcal{A})_{\rm anom} \approx \frac{\theta_M}{2\pi} \int d\phi dt F_{0\phi}(\frac{\pi}{2}, \phi, t) \;. \tag{87}$$

D. Limit of a thin orbifold

The orbifold and domain-wall results are related to each other. To see that, consider the limit when the orbifold becomes thin: $|\mu|R \ll 1$. Restricting ourselves to the case $m(z) = 0$, for which the explicit formula (59) was obtained, we see that in the limit $|\mu|R \ll 1$ we can approximate the sinh functions in (59) by their arguments and then integrate over z by parts. The boundary terms cancel, and we obtain

$$\frac{\partial \Omega}{\partial \theta_M}|_{\theta_M=0} = \frac{e}{4\pi R}\int d^2x dz \epsilon_{\mu\nu} F^{\mu\nu}(x,z) \; . \tag{88}$$

This agrees with the effective action (83) of the domain-wall scenario, with the role of $F_{0x}(0,x,t)$ now being played by the average of F_{0x} over the extra dimension. Thus, in a sense, in the thin-orbifold limit, the entire orbifold plays the role of a domain wall.

V. SPECTRAL ASYMMETRY

We have seen that, in all of our examples, the θ-dependence of the d-dimensional theory agrees with that calculated using the low-energy $(d-1)$-dimensional fields alone, and disagrees with what one might expect from the anomaly equation for the d-dimensional current. In other words, there is a mismatch between the "ultraviolet" and "infrared" anomalies.

This mismatch implies that the anomalous production of fermions is not counted correctly by the d-dimensional anomaly. The situation is analogous to that described by the Atiyah-Patodi-Singer theorem for a Dirac operator in even dimensions [9], see also ref. [17]. There, the index of the Dirac operator is not given simply by the anomaly equation, but includes an additional term (the η-invariant) having to do with the change in spectral asymmetry. In this section, we show that a similar mechanism is at work in our odd-dimensional theories.

The argument is the simplest when the space is a two-sphere (the total dimensionality of space-time is $d = 3$). As seen from eq. (87), in this case, the θ-dependence is activated by fluctuations that change the integral of A_ϕ around the equator:

$$\int d\phi dt F_{0\phi} = \int d\phi A_\phi(t_2) - \int d\phi A_\phi(t_1) \neq 0 \; , \tag{89}$$

where t_1 and t_2 are some initial and final times. For brevity, we will refer to such fluctuations as "instantons", even though they do not have to be associated with tunneling and may as well take place in real time.

Now, on a sphere, the integral of A_ϕ along the equator equals the magnetic flux through the northern hemisphere:

$$\int d\phi A_\phi = \int d\phi \int_0^{\pi/2} b \sin\theta d\theta \ , \tag{90}$$

where

$$b = \frac{1}{\sin\theta} F_{\theta\phi} = \frac{1}{\sin\theta}(\partial_\theta A_\phi - \partial_\phi A_\theta) \ . \tag{91}$$

It will be convenient to visualize the transport of flux as motion of particle-like flux quanta—vortices. Flux can be localized into vortices, for instance, by introduction of a suitable Higgs field.

We will be interested in scattering of vortices off the domain wall (positioned along the equator). Consider the process when a vortex-antivortex pair is created from vacuum in the southern hemisphere, and then the vortex is transported across the equator to the northern hemisphere, while the antivortex remains where it was. This changes $\frac{1}{2\pi}\int d\phi A_\phi$ by one. The energetics of this process does not concern us at present; it will be the subject of the next section. Here we simply assume that the vortices are light enough to be a part of our low-energy theory.

Consider first the case when the small mass in eq. (46) is zero, $\mu = 0$. Then, the anomaly in the $(d-1)$ current (49) tells us that the scattering process should produce two massless fermions on the equator, with the total of 2 units of chirality. On the other hand, the corresponding current of the d-dimensional theory, eq. (48), is conserved exactly, so there should be an additional contribution to the charge balance.

To see where this additional contribution comes from, recall that in $d = 3$ a vortex, in the presence of a single massive fermion with mass M, acquires half a unit of the fermion number [18, 19]:

$$\langle J^A \rangle = \frac{M}{8\pi |M|} \epsilon^{ABC} F_{BC} \ , \tag{92}$$

where J^A is the current of that single species. This effect occurs in the bulk of the $d = 3$ spacetime, where the vortex is initially positioned, and can be regarded as a result of the polarization of the massive Dirac sea by the field F_{BC}.

In our Lagrangian (46), there are two species of fermions, with opposite signs of the mass. As a result, the gauge charge of the vortex is now zero (so that in contrast to the Callan-Harvey mechanism [10], there is no net Chern-Simons action), but the global charge, corresponding to the current (48), is doubled. In addition, in the presence of a domain wall, the mass M for each fermionic species has opposite signs in the two hemispheres. Thus, the global charge of the vortex is now equal to ± 1, depending on the hemisphere. So, as the vortex crosses the equator, it produces two units of chirality in the form of fermions bound to the wall, but its own charge also changes, precisely by the opposite amount. In this way, the exact conservation of the d-dimensional current is reconciled with the anomalous production of fermions on the equator.

Next, let us restore the small mass in eq. (46), $\mu \neq 0$. In this case, the conservation of the current (48) is no longer exact. Instead, we have

$$\partial_A J_G^A = 2i\mu^* \bar{\Psi}_2 \Psi_1 - 2i\mu \bar{\Psi}_1 \Psi_2 \ . \tag{93}$$

The vortex states are no longer exact eigenstates of the corresponding global charge,

$$Q_G = \int J_G^0 \sin\theta d\theta d\phi \ , \tag{94}$$

but we can still consider averages of Q_G in these states. Specifically, let us consider the adiabatic limit, when the vortex crosses the equator very slowly—the timescale of its motion is much larger than μ^{-1}. In this limit, fermions remain, to a good accuracy, in adiabatic vacuum. Averaging eq. (93) over this state and integrating over the sphere and over an interval of time, we find

$$\langle Q_G(t_2)\rangle - \langle Q_G(t_1)\rangle = \int [2i\mu^* \langle \bar{\Psi}_2 \Psi_1\rangle - 2i\mu \langle \bar{\Psi}_1 \Psi_2\rangle] \sin\theta d\theta d\phi dt \ . \tag{95}$$

The averages on the right-hand side can be obtained through the derivatives of the anomalous action (87) with respect to μ and μ^*. In this way, we find

$$\langle Q_G(t_2)\rangle - \langle Q_G(t_1)\rangle \approx \frac{1}{\pi} \int d\phi dt F_{0\phi}(0, \phi, t) \ . \tag{96}$$

The approximation sign reminds us that in the action (87) we have neglected terms of higher orders in μ. To the same accuracy, the average charges of the vortex before and after the equator crossing are still determined by eq. (92): $\langle Q_G(t_1)\rangle \approx -1$, $\langle Q_G(t_2)\rangle \approx 1$. We see that the change in $\langle Q_G\rangle$, due to restructuring of the Dirac sea of the massive modes, is precisely as required for eq. (96) to hold. In fact, the flow of information could have been reverted: we could have used the simple counting of charges to restore the coefficient in the effective action (87).

The fact that for $\mu \neq 0$ the current J_G^A is not exactly conserved, and therefore vortex states are not eigenstates of the charge, has important consequences for the realization of the global $U(1)$ symmetry. In (2+1), this $U(1)$ is anomaly-free, and we can use it to rotate the phase $\theta_M = \arg\mu$ to zero. However, in quantum theory, this will also transform the state vector. If we think of the vortex-antivortex state as a superposition of components with different values of Q_G, the transformation will change the relative phases of the components. These relative phases then become a counterpart, in the (2+1) theory with an non-anomalous J_G^A, to the vacuum θ-angle in the (1+1) theory with an anomalous J_5^μ.

To summarize, for the case of a domain wall on a sphere, the motion of vortices provides a very visual way to understand the effect of spectral asymmetry. We have not developed a corresponding visual tool for the case of an orbifold, but we expect that in that case the mismatch between the "ultraviolet" and "infrared" anomalies can similarly be attributed to restructuring of the massive part of the fermionic spectrum in a z-dependent field $F_{\mu\nu}$.

VI. GAUGE DYNAMICS AND THETA-VACUA

The presence, for $\mu \neq 0$, of instanton processes that do not produce any fermions implies a possibility to have an observable θ angle. (This is similar to how in QCD, to have a θ-vacuum, all quarks should be massive.) However, the presence of such processes is only one necessary condition for the existence of θ. The other condition is that these processes connect states that are degenerate, or nearly degenerate, in energy. If instantons have to climb a high potential ladder, they will be blocked at low energies.

Note that a description of gauge dynamics requires that we construct scenarios where the gauge field splits into a low-energy mode, corresponding to a $(d-1)$-dimensional gauge field, and high-energy modes separated from the low-energy mode by a large gap. This is relatively straightforward to achieve on an orbifold, and somewhat less straightforward on a sphere. We now consider these cases in turn.

A. Theta-vacua on an orbifold

For our toy model in $2+1$ dimensions we take an Abelian gauge field, as it is for this theory that the vacuum in $1+1$ dimensions has complicated structure. We consider a massless gauge field with the Lagrangian

$$L = -\frac{1}{4} F_{AB} F^{AB} \; , \tag{97}$$

where $F_{AB} = \partial_A A_B - \partial_B A_A$. This massless case is somewhat degenerate, since it has no physical propagating mode in $1+1$ dimensions: the only physical mode in a massless $1+1$ gauge theory is a uniform electric field and its canonically conjugate coordinate given by the Wilson line $\int dx A_1(x,z)$. However, it is precisely the dynamics of this mode that is of interest to us here. Indeed, a constant (in time) uniform electric field plays the role of a θ-angle in (1+1) [20].

We consider this theory on a orbifold $O \times S_1$, where O is an interval of length R, and S_1 is a large circle of length L. First, we consider the free gauge theory, defined by the bilinear Lagrangian (97), and then add interaction with fermions.

Variation of the action, besides the ordinary Maxwell equations

$$\partial_A F^{AB} = 0 \tag{98}$$

valid in the bulk, gives now the extra boundary terms,

$$\int d^2x \left(\delta A^\mu(R/2,x) F_{2\mu}(R/2,x) - \delta A^\mu(-R/2,x) F_{2\mu}(-R/2,x)\right) = 0 \ , \tag{99}$$

which lead to boundary conditions [21]

$$F_{\mu 2}|_{\pm R/2} = 0 \ . \tag{100}$$

assuming arbitrary variations of A_μ at the boundaries. Note that since A_0 plays the role of a gauge function, gauge transformations with arbitrary continuous gauge functions are admitted (see below for more detail).

The general solution to the free Maxwell equations (98) with boundary conditions (100) has the form:

$$A_\mu = \frac{\partial \alpha(x^A)}{\partial x^\mu} + \sum_{n=0}^{\infty} A^n_\mu(x^\nu)\chi_n(z) \ , \tag{101}$$

$$A_z = \frac{\partial \alpha(x^A)}{\partial z} \ , \tag{102}$$

where α is an arbitrary function reflecting the gauge freedom, fields $A^n_\mu(x^\nu)$ satisfy the vector field equation

$$\partial^\mu F^n_{\mu\nu} + M_n^2 A^n_\nu = 0 \ , \tag{103}$$

where $F^n_{\mu\nu} = \partial_\mu A^n_\nu - \partial_\nu A^n_\mu$, the mass M_n is given by eq. (11), and χ_n are defined in (10). This decomposition is valid for the five-dimensional case as well. The low-energy theory is just the electrodynamics in $1+1$ or $3+1$ dimensions.

In 3d, the field strengths for the solution (101), (102) are

$$\begin{aligned} E_1 &= E_1^0 - \sum_{n=1} \frac{\partial a_n(x,t)}{\partial t} M_n \chi_n(z) \ , \\ E_2 &= \sum_{n=1} \frac{\partial^2 a_n(x,t)}{\partial t \partial x} \psi_n(z) \ , \\ B &= \sum_{n=1} \left(\frac{\partial^2 a_n(x,t)}{\partial x^2} - a_n(x,t) M_n^2 \right) \psi_n(z) \ , \end{aligned} \tag{104}$$

where $a_n(x,t)$ satisfy the Klein-Gordon equation $(\partial_\mu \partial^\mu + M_n^2) a_n(x,t) = 0$. Thus, E_1 includes a constant electric field E_1^0 in x direction, which is a Lorentz scalar in $1+1$ dimensions. The fact that this constant electric field is allowed by the 3d equations of motion and by the boundary conditions is essential for discussion of θ vacua on orbifold. Note that in $1+1$ dimensional electrodynamics the constant electric field is also a solution of equations of motion and plays the role of a θ angle [20].

We now want to address the following questions: (i) Does the complete (2+1)-dimensional theory have a complicated vacuum structure characterized by a vacuum angle θ? (ii) Does the phase θ_M of the fermion mass μ contribute to an observable θ-angle? As we will see, the answers to both of these questions are affirmative.

We begin with constructing classical vacua, i.e., states of minimal classical energy. Let us choose the gauge $A_0 = 0$ and consider time-independent gauge transformations. The group of all such transformations consists of functions α for which $\exp[i\alpha(z,x)]$ is continuous on $O \times S_1$. In addition to "local" or "small" gauge transformations, for which functions $\alpha(z,x)$ themselves are continuous, this condition allows for functions $\alpha(z,x)$ that have a jump $2\pi n$ with integer n along a line in the (x,z) plane. Such a line can either form a closed loop or connect the opposite points of the interval O. In the first case, the loop is contractible, and the transformation can be continuously deformed to a small gauge transformation, but the second case is a non-contractible, "large" transformation. For $n=1$, these large gauge transformations can be reduced to the form

$$\alpha(z,x) = \begin{cases} \frac{2\pi}{L}[x - x_0(z)] \ , & 0 \le x \le x_0(z) \ , \\ \frac{2\pi}{L}[x - x_0(z) - L] \ , & x_0(z) < x \le L \ , \end{cases} \tag{105}$$

where the function $x_0(z)$ defines a line (without intersections) on which α jumps by 2π. So, the N-vacuum is the gauge-field configuration

$$A_1^{(N)} = \frac{2\pi N}{eL}, \quad A_2^{(N)} = -\frac{2\pi N}{eL} \frac{\partial x_0}{\partial z} \ , \tag{106}$$

which is characterized by an integer Chern-Simons number

$$N_{CS} = \frac{e}{2\pi}\int dx A_1^{(N)}(x,z) = N \ . \tag{107}$$

Note that for the vacuum configurations N_{CS} does not depend on z (for an arbitrary gauge background that is not so) and that it is invariant under small gauge transformations for any $A_1(x,z)$. Under the large gauge transformation (105), it changes by one.

This construction of the classical N-vacua on the orbifold is almost identical to the similar construction in (1+1) dimensions. As we now go to quantum theory, we construct a θ-vacuum as a linear superposition of states built near these classical vacua [4, 5]. The functional integral for vacuum-vacuum transitions can be written as

$$\int \mathcal{D}A_B \exp\left(i\mathcal{A}_3 - i\frac{e\theta_{\rm vac}}{4\pi}\int d^2x \epsilon_{\mu\nu}F^{\mu\nu}(z,x)\right) \ , \tag{108}$$

where $\mathcal{A}_3$ is the complete three-dimensional action together with the necessary gauge fixing and ghost terms. Note that

$$\int d^2x\epsilon_{\mu\nu}F^{\mu\nu}(z,x) \tag{109}$$

is z-independent for the vacuum-vacuum transitions. Thus, the vacuum of a $U(1)$ gauge theory on orbifold is characterized by an angle θ, exactly in the same way as in the effective (1+1)-dimensional theory. Clearly, this is related to the topology of the orbifold: the mapping $O\times S_1 \to U(1)$ is non-trivial and characterized by an integer Z, just as the mapping $S_1 \to U(1)$ is in the low-energy theory. A similar argument applies to 5d non-Abelian theories, for which the space $O \times S_3$ has a non-trivial mapping to the group SU(2).

To detect a θ-angle, we need to have charged particles. So, let us include interaction of the gauge field with two species of fermions, such as those described in Sect. IV. The fermions can be integrated out, and for small and slowly varying F_{AB}, their main contribution to the effective action is given by eq. (57). So, in this approximation, the effective action still has the form (108) but with $\theta_{\rm vac}$ replaced by

$$\theta_{\rm tot} = \theta_{\rm vac} + \theta_M \ , \tag{110}$$

where θ_M is the phase of the fermion mass μ.

B. Vacuum structure on a sphere

We have seen that in this case instanton fluctuations, which activate the θ_M dependence, correspond to scattering of vortices on the domain wall. The essential difference with the case of the orbifold is that such a fluctuation now does not connect two vacuum states. Rather, it connects the vacuum to a state with a vortex in the northern hemisphere and an antivortex in southern (or vice versa). This is consistent with topological considerations: the mapping from S_2 to $U(1)$ is trivial, so there are no "large" gauge transformations and no degenerate N-vacua.

The question we want to address in this subsection is if there can nevertheless be an effective θ-angle, due to existence of vortex states that are *nearly* degenerate with the vacuum. This question needs to be answered within a scenario where the effective low-energy theory is that of a (1+1)-dimensional gauge field, while all other gauge modes have a large gap.

To construct such a scenario, we consider a family of Abelian theories with a coupling constant dependent on the spherical angle θ:

$$L = -\frac{1}{4}\sqrt{g}\frac{1}{h(\theta)}g^{AB}g^{CD}F_{AC}F_{BD} = \frac{\sin\theta}{2h}(F_{0\theta}^2 + \frac{1}{\sin^2\theta}F_{0\phi}^2 - \frac{1}{\sin^2\theta}F_{\theta\phi}^2) \ , \tag{111}$$

where $g_{AB} = \mathrm{diag}(1, -1, -\sin^2\theta)$ is the metric. The θ-dependent coupling $h(\theta) > 0$ will be referred to as the warp factor. Such space-dependent couplings arise naturally in brane-world scenarios [22, 23, 24].

Magnetic field b has been defined in eq. (91): $b = F_{\theta\phi}/\sin\theta$. The time derivative of this expression gives Faraday's law on the sphere:

$$\dot{b} = \frac{1}{\sin\theta}(\partial_\theta \dot{A}_\phi - \partial_\phi \dot{A}_\theta) = \frac{1}{\sin\theta}(\partial_\theta F_{0\phi} - \partial_\phi F_{0\theta}) \ . \tag{112}$$

Eq. (112) shows that the total magnetic flux through the sphere is conserved:

$$\mathcal{B} = \int d\phi \int_0^\pi b \sin\theta d\theta = \text{const.} \tag{113}$$

In what follows, we restrict ourselves to the sector with zero flux,

$$\mathcal{B} = 0 \ , \tag{114}$$

i.e., we assume that there is no monopole inside the sphere.

Equations of motion following from (111) are

$$\partial_0(\frac{\sin\theta}{h}F_{0\theta}) - \partial_\phi(\frac{1}{h\sin\theta}F_{\phi\theta}) \;=\; 0\;, \tag{115}$$

$$\partial_0(\frac{1}{h\sin\theta}F_{0\phi}) - \partial_\theta(\frac{1}{h\sin\theta}F_{\theta\phi}) \;=\; 0\;, \tag{116}$$

$$\partial_\theta(\frac{\sin\theta}{h}F_{\theta 0}) + \partial_\phi(\frac{1}{h\sin\theta}F_{\phi 0}) \;=\; 0\;. \tag{117}$$

Consider first solutions for which all F_{AB} are time-independent. Then, the first two of the equations of motion reduce to $\partial_\phi b = 0$ and $\partial_\theta(b/h) = 0$, which are solved by

$$b = c_1 h\;, \tag{118}$$

where c_1 is an arbitrary space- and time-independent coefficient. This is the monopole solution characterized, for $c_1 \neq 0$, by a non-zero total flux. We have projected it away by imposing the zero-flux condition (114).

Next, consider solutions for which all F_{AB} depend on time as $e^{-i\omega t}$ with $\omega \neq 0$. Then, the electric fields are

$$F_{0\theta} \;=\; \frac{\partial_\phi b}{i\omega\sin\theta}\;, \tag{119}$$

$$F_{0\phi} \;=\; -\frac{h\sin\theta}{i\omega}\partial_\theta(b/h)\;. \tag{120}$$

Substituting these expressions into eq. (112) and expanding in the eigenstates of the angular momentum, we obtain a closed equation for component of b with angular momentum m (m =integer):

$$-\frac{1}{\sin\theta}\partial_\theta[h\sin\theta\partial_\theta(b/h)] + \frac{m^2}{\sin^2\theta}b = \omega^2 b\;. \tag{121}$$

Defining $B = b/h$ and $H = h\sin\theta$, we can rewrite this equation as

$$\partial_\theta(H\partial_\theta B) = -(\omega^2 - m^2/\sin^2\theta)HB\;. \tag{122}$$

Setting $B = \chi/\sqrt{H}$ and $H = e^f$, we rewrite it further as a Schrödinger equation

$$\chi'' - [\frac{1}{2}f'' + \frac{1}{4}(f')^2]\chi = -(\omega^2 - m^2/\sin^2\theta)\chi\;. \tag{123}$$

Primes denote derivatives with respect to θ. The ground state of this Schrödinger problem is $\chi \propto \sqrt{H}$. This coincides with the monopole solution (118), which we have projected out. We are interested in the lowest-energy mode satisfying the condition (114).

Let us consider the case when all modes are concentrated mostly in small regions near the poles. A simple choice of the warp factor that leads to such an arrangement is

$$H = C \exp(-\frac{1}{2}\kappa^2 \sin^2\theta) , \tag{124}$$

where C is a constant. Taking $\kappa \gg 1$ and considering only a vicinity of the north pole, $\theta \ll 1$, we see that in this case the potential in (123) is approximately that of a harmonic oscillator:

$$V(\theta) = \frac{1}{2}f'' + \frac{1}{4}(f')^2 \approx \frac{1}{4}\kappa^4\theta^2 - \frac{1}{2}\kappa^2 . \tag{125}$$

Upon replacement $\theta \to \pi - \theta$, we obtain a corresponding expression near the south pole. The ground states of these oscillators comprise the low-energy subspace of our system. Due to the (exponentially small) overlap at the equator, these ground states form symmetric and antisymmetric linear combinations. The symmetric combination, which is the true ground state of the system, is once again the monopole solution (118). The antisymmetric combination is the state we are interested in: it has zero total flux and, if suitably populated, corresponds to a vortex at the north pole and an antivortex at the south pole. The fields in this state oscillate at exponentially small frequency

$$\omega_1 \sim \kappa \exp(-\mathrm{const.} \times \kappa^2\mathcal{R}^2) , \tag{126}$$

where we have restored the radius $\mathcal{R}$ of the sphere. All other modes are separated from this one by the gap $\omega_2^2 \approx \kappa^2$.

The above spectrum is reminiscent of the one that occurs in models that use a warped gauge coupling as a means to obtain light vector bosons [25]. The crucial difference is that in our case the exponentially light mode occurs only for angular momentum $m = 0$. So, it does not correspond to a vector particle propagating along the domain wall. Rather, it is the counterpart of the "topological" mode, for which A_ϕ and $F_{0\phi}$ are constant along "our" dimension. If that mode were constant in time, it would correspond to a conventional θ-angle, just as in (1+1) dimensions or in the case of orbifold. We see, however, that on the sphere this mode acquires a small but nonzero frequency, resulting in a variation of the effective θ-angle with time.

In a static universe, the case for which the above results have been obtained, the dynamics of the "topological" mode is oscillatory. In an expanding universe, we expect this mode to be damped by the expansion. Furthermore, if ω_1 is not particularly small, and the gauge field interacts with light matter, the oscillations of the "topological" mode can decay into matter particles. For QCD in 4d, either of these scenarios constitutes a solution to the strong-CP problem.

If ω_1 is, in fact, small (as in the above example, where it is suppressed exponentially by the size of "our" dimensions), ω_1^{-1} may well be a cosmological timescale, so the relaxation of the effective θ-angle and of the associated vacuum energy will occur relatively late in the cosmological history.

C. Effective Lagrangian for a time-dependent θ

As follows from eq. (120), the "topological" mode, oscillating at frequency ω_1, corresponds to an oscillating (in time) and uniform (in ϕ) electric field on the equator. We know that in (1+1) dimensions or in the case of orbifold a constant electric field is the classical counterpart of a θ-angle [20]. In quantum theory on the orbifold, the θ dependence can be described by the effective Lagrangian appearing in eq. (108). One may expect that a similar description, but with an effective, time-dependent θ-angle, exists in the case of a sphere.

Such a description is provided by the following dimensionally reduced action for the fields on the equator:

$$\mathcal{A} = \mathcal{A}_2 + \frac{1}{2\omega_1^2}\int dt \left(\frac{da}{dt}\right)^2 + \frac{1}{\sqrt{2\pi}}\int dt d\phi a(t) F_{0\phi}^{(2)} \;, \tag{127}$$

where $\mathcal{A}_2$ is the action for the theory on a circle, and $F_{0\phi}^{(2)}$ is the canonically normalized field strength of that theory. The quantum-mechanical variable $a(t)$ depends only on time but not on space: it is a dual representation of the "topological" mode of the gauge field; ω_1 is the frequency of that mode, eq. (126).

If the action $\mathcal{A}_2$ contains fermions, we integrate them out and, for small, slowly-changing field strengths, obtain as the leading terms the anomalous action (87), proportional to the phase θ_M of the fermion mass. We see, however, that in the present case this anomalous action can be absorbed by a shift in the variable a. In other words, it only changes the initial conditions for $a(t)$. The same applies to any term of the form

$$\mathcal{A}_3' = \int dt d\theta d\phi u(\theta) F_{0\phi} \;, \tag{128}$$

which we might have added by hand to the original 3d action (u is some function). This is because at low energies $F_{0\phi}$ projects onto the "topological" mode, so (128) becomes of the same form as the last term in (127). So, in what follows we assume that a in (127) already includes the effect of θ_M and of any term such as (128), i.e., we set $\theta_M = 0$ and $u = 0$. (Strictly speaking, this requires that the combined $a(t)$ is sufficiently small to prevent the decay of the uniform electric field, eq. (129) below, into fermion pairs, cf. ref. [20].)

Now, integrating out the uniform component of $A_\phi^{(2)}$, we obtain

$$\langle F_{0\phi}^{(2)} \rangle = -\frac{1}{\sqrt{2\pi}} a(t) \tag{129}$$

and a simple oscillatory Lagrangian for $a(t)$:

$$L_a = \frac{1}{2\omega_1^2}\left(\frac{da}{dt}\right)^2 - \frac{1}{2}a^2 \; . \tag{130}$$

We see that in the limit $\omega_1 \to 0$, the inertia of a grows indefinitely. Formally setting $\omega_1 = 0$ would convert a into a conventional time-independent θ-angle.

The condition that $F_{0\phi}^{(2)}$ is slowly-varying implies that $\omega_1 \ll |\mu|$, where μ is the fermion mass. For large enough ω_1, the oscillating a can efficiently decay into fermions, and eq. (130) is no longer applicable. (A single quantum of a can decay into fermions when $\omega_1 > 2|\mu|$, two such quanta will be required when $|\mu| < \omega_1 \leq 2|\mu|$, etc.)

The variable $a(t)$ can be viewed as a "global axion", in the sense that it couples to the topological density in a way similar to how the usual axion [26, 27] does. However, since a only depends on time, and not on space, it does not correspond to a new particle. In fact, it is not even an additional degree of freedom, external to the original 3d theory: as eq. (129) shows, it is simply a different representation of the time-dependent uniform electric field.

Clearly, existence of such a variable would not be possible in a perfectly Lorentz-invariant theory but, of course, the 2d Lorentz invariance is not exact in our brane-world scenario.

Finally, we note that although eq. (129) is specific to a 3d $U(1)$ gauge theory, the general structure of eq. (127) is not. For a 5d non-Abelian theory (with $\mathcal{S}_{d-1} = S_4$), we would replace (127) with

$$\int \mathcal{D}A_B \exp\left\{ i\mathcal{A}_4 + i\frac{1}{2\omega_1^2}\int dt \left(\frac{da}{dt}\right)^2 + iv\int d^4x a(t) F_{\mu\nu}\tilde{F}^{\mu\nu} \right\} \; , \tag{131}$$

where $\mathcal{A}_4$ is the conventional 4d action, and v is a suitably chosen constant.

VII. THEORY ON A DISK

Our results on chiral fermions and the vacuum structure on a sphere are related to the topology, rather than geometry, of the spatial manifold. Similar results are valid for a simpler, flat geometry. Simply cut a sphere along the equator, choose a hemisphere, and make it flat by replacing it with a disk. Then, substitute the domain wall by a suitable boundary condition. In this subsection we present the corresponding equations. We will call the boundary of the disk the brane and its interior the bulk.

A. Fermions on a disk

Introduce Cartesian coordinates x and y with origin at the center of the disk. Three-dimensional γ-matrices used in this subsection are $\gamma^0 = \tau_3,\ \gamma^1 = i\tau_1,\ \gamma^2 = i\tau_2$. Note that these matrices are associated with the Cartesian coordinates. Then, the Dirac equation $i\gamma^\mu\partial_\mu\Psi - M\Psi = 0$, where $M > 0$ is a constant fermion mass in the bulk, can be written in polar coordinates ($x = r\cos\phi,\ y = r\sin\phi$) in the form of a Schrödinger equation $i\frac{\partial\Psi}{\partial t} = H\Psi$ with the Hamiltonian

$$H = \begin{pmatrix} M & e^{-i\phi}\left(-\partial_r + \frac{i}{r}\partial_\phi\right) \\ e^{i\phi}\left(\partial_r + \frac{i}{r}\partial_\phi\right) & -M \end{pmatrix} , \tag{132}$$

leading to the energy eigenvalue problem $H\Psi = E\Psi$. The regular at $r = 0$ solutions are:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{in\phi} J_n(kr) E_+ \\ -e^{i(n+1)\phi} J_{n+1}(kr) E_- \end{pmatrix} , \tag{133}$$

for $E^2 > M^2$ and

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{in\phi} I_n(kr) E_+ \\ e^{i(n+1)\phi} I_{n+1}(kr) E_- \end{pmatrix} , \tag{134}$$

for $E^2 < M^2$. Here $E_+ = \sqrt{|E+M|}$, $\ E_- = \sqrt{|E-M|}$, $\ k = \sqrt{|E^2 - M^2|}$, J_n and I_n are the Bessel and modified Bessel functions, n is an integer.

A boundary condition that produces a left chiral fermion on the brane and is consistent with the hermiticity of the Hamiltonian is

$$(1 - \gamma_5(\phi))\Psi|_{r=R} = 0, \quad \text{or} \quad \psi_2 = e^{i\phi}\psi_1 , \tag{135}$$

where $\gamma_5(\phi) = -i(\gamma^1 \cos\phi + \gamma^2 \sin\phi)$ is a chirality matrix in polar coordinates. Eq. (135) leads to the eigenvalue equations

$$I_n(kR)E_+ = I_{n+1}(kR)E_-, \quad E^2 < M^2 , \tag{136}$$

$$J_n(kR)E_+ = -J_{n+1}(kR)E_-, \quad E^2 > M^2 , \tag{137}$$

where R is the radius of the disk. Solution to eq. (136), at $MR \gg 1$ gives a chiral mode with dispersion relation $E \approx -(n + \frac{1}{2})/R$, exactly as we have obtained for a sphere. This mode is localized at the boundary of the disk, with an exponential wave function $\sim e^{-M(R-r)}$ for $(R-r)/R \ll 1$. Solutions to eq. (137) lead to massive bulk modes with energies greater than M.

A right-handed fermion can be derived in a similar manner, by choosing the negative mass parameter $M < 0$ and by changing the boundary condition (135) to $(1 + \gamma_5(\phi))\Psi|_{r=R} = 0$.

A massive (with mass μ), Dirac fermion living on the boundary of the disk can be introduced exactly in the way it has been done for the orbifold or a sphere, namely by including two fermions, the first one (Ψ_1) producing the left one and the second (Ψ_2) producing the right fermion, with the mixing mass term $\mu\bar{\Psi}_1\Psi_2$.

B. Gauge fields on a disk

Similarly to the case of a sphere, a gauge field that has a (1+1)-like low-energy mode, while other modes are separated by a large gap, can be introduced through Lagrangian

$$L = -\frac{1}{4}\Delta(r)F_{AB}F^{AB} , \tag{138}$$

where the warp factor $\Delta(r)$ is of order one in a small vicinity of the disk boundary and goes to zero at $r \to 0$. A typical model for $\Delta(r)$ could be

$$\Delta(r) = \left(\frac{r}{R}\right)^2 e^{-M(R-r)} . \tag{139}$$

As for the orbifold case, the boundary condition to the gauge field is

$$F_{r\phi}|_{r=R} = F_{r0}|_{r=R} = 0 . \tag{140}$$

However, in contrast to the orbifold case, the vacuum is topologically trivial, as follows from the fact that the boundary of the disk is a simply connected manifold. Namely, allowed gauge functions may contain a $2\pi n$ jump along a closed loop on the disk or along a line connecting two points at the boundary. All these transformations can be continuously transformed into trivial gauge transformations.

The absence of a conventional θ-angle on a disk still leaves us with the possibility to have an effective θ-angle, due to transitions that connect the vacuum to a vortex state. The only difference with the sphere in this regard is that the total magnetic flux through the disk is not conserved. But this is in fact necessary for a candidate vortex state to be connected to the vacuum: the disk is analogous to a hemisphere, rather than the entire sphere in our previous example. For a suitable warp factor in (138), the lowest-energy vortex state can be light enough to produce a slowly-changing $\theta_{\rm eff}(t)$.

C. Scalar fields on a disk

Scalar fields can be localized on the boundary of a disk similarly to fermions. In this subsection, we consider a real scalar field as a prototype for a (complex) field that could give rise to the Higgs mechanism.

We start from the standard Lagrangian

$$L = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{M^2}{2}\varphi^2. \tag{141}$$

The regular at $r = 0$ (in polar coordinates) solutions to the equations of motion are

$$\varphi = e^{-iEt-in\phi} \begin{cases} I_n(kr) \ , & E^2 < M^2 \ , \\ J_n(kr) \ , & E^2 > M^2 \ . \end{cases} \tag{142}$$

The boundary condition

$$\left(\frac{\partial \varphi}{\partial r} - \sqrt{M^2 - m^2}\varphi\right)|_R = 0 \ , \tag{143}$$

where $m^2 \ll M^2$, leads to the following dispersion relations:

$$\frac{1}{2}(I_{n+1}(kR) + I_{n-1}(kR)) = \sqrt{\frac{M^2 - m^2}{M^2 - E^2}} I_n(kR), \qquad E^2 < M^2 \ , \tag{144}$$

$$\frac{1}{2}(J_{n-1}(kR) - J_{n+1}(kR)) = \sqrt{\frac{M^2 - m^2}{E^2 - M^2}} J_n(kR), \qquad E^2 > M^2 \ , \tag{145}$$

which single out a light mode with dispersion relation (in the physically interesting limit $R \to \infty, \ n \to \infty, \ n/R = \text{const.}$, with m and M fixed)

$$E^2 \approx \left(\frac{n}{R}\right)^2 + m^2 \tag{146}$$

and a wave-function localized on the brane. Other, bulk modes, have large masses and are non-observable at small energies.

VIII. DISCUSSION AND CONCLUSIONS

Results of this paper are two-fold. First, we have shown how conservation of a global current in odd dimensionalities can be reconciled with the presence of an anomaly in the reduced, even-dimensional theory. The central observation here is the presence of an additional contribution to the charge balance, due to restructuring of the massive fermion modes.

Second, we have presented a brane-world scenario that leads to a time-dependent effective θ-angle. In this scenario, the space is a sphere, and a domain wall is positioned along the equator. Since the mapping from the sphere to the gauge group is trivial, the usual, time-independent θ-angle is absent. However, the requirement that the low-energy limit is a dimensionally reduced gauge theory automatically brings in an effective θ-angle.

We have discussed this scenario in detail for the case of a $U(1)$ gauge field in a $(2+1) \to (1+1)$ compactification, see Fig. 2. In this case, a simple picture of the effective θ-angle can be obtained, based on tracking the motion of flux between the two hemispheres. For a free gauge field (but with a warped action), when exact results can be obtained, the dynamics of θ_{eff} turns out to be oscillatory. We expect that when the gauge field interacts with light matter or in a non-static universe these oscillations will be damped, so that θ_{eff} relaxes to zero. We also expect that a similar relaxation dynamics will obtain for the $(4+1) \to (3+1)$ compactification of a non-Abelian theory, thus providing a solution to the strong-CP problem.

The idea that the vacuum structure of a gauge theory can be modified in the presence of extra dimensions is by itself not new. Indeed, already quite a while ago [6] we pointed out that if the higher-dimensional theory is defined on a space manifold $\mathcal{S}_{d-1}$, which is compact and obeys the property $\pi_3(\mathcal{S}_{d-1}) = 1$, the vacuum is topologically trivial and that this can be a basis for a solution to the strong-CP problem.

Let us compare the structure of the manifolds of ref. [6] and of the present work. We start from the $2+1 \to 1+1$ compactification. In both cases the topology of the space is that of a 2-sphere. In ref. [6], we proposed that the low-energy theory is $1+1$ dimensional because the manifold has the form of a sausage, with $L \gg R$, see Fig. 1.

In this setup, there is no complete translational invariance along "our" dimensions because of the presence of two highly curved regions, where our space "ends". Nevertheless, if an observer resides far from these regions, the low-energy physics looks $1+1$ dimensional, as the size of extra dimension R is assumed to be small. This setup solves the θ problem in the following way. First, the topology of space is such that no non-trivial gauge transformations exist. Second, the determinant of the fermionic mass matrix is always real, because the fermions are vectorlike. The generalization of this picture to $4+1$ dimensions has qualitatively the same features.

Note that compactness of the higher-dimensional space is essential: only in this case the topological argument is unambiguous. (Thus, for instance, a recent proposal [28] for solving the strong-CP problem with a non-compact manifold will not work.) The easiest way to see the role of compactness is to step back to the 3d $\to$ 2d case. In this case, the θ-angle corresponds to a time-independent electric field [20]. On a non-compact manifold, there is always a choice of boundary conditions at infinity for which such a time-independent solution can be found. As long as no a priori way to reject these boundary conditions is proposed, the θ problem is not solved.

Disadvantages of a sausage-like manifold are quite obvious: it breaks the translational invariance in a very peculiar way, and it is far from being obvious that a structure like this can arise as a solution of the Einstein equations when gravity is incorporated. Moreover, one cannot include chiral fermions, and therefore possibility of construction of a phenomenologically acceptable electroweak theory is doubtful.

In the present paper, we have proposed another structure, which solves the above-mentioned problems. First, the manifold of the type shown in Fig. 2, where the standard-model fields are localized on a brane, leads to physics that is translationally invariant in "our" dimensions (i.e., along the equator). There is a trivial breaking of the Lorentz invariance, since our space is compact, but this is suppressed by the size of our dimensions. Moreover, a similar geometry can be obtained as a solution to the Einstein equations, as was demonstrated in ref. [7]. In that solution, two slices of AdS space are glued together along a three-dimensional sphere representing the observable space. Finally, the presence of a domain wall leads naturally to chiral fermions and thus to possibility to construct a realistic theory.

A convenient way to visualize the dynamics of the low-energy mode that plays the role of an effective θ-angle in this setup is through its dual—the “global” axion introduced in eqs. (127) and (131). This “global axion” is very different form the usual axion in that it does not depend on space and therefore does not correspond to a new particle. Such a global axion is not subject to any astrophysical constraints, as it cannot be excited in stars, whereas the cosmological constrains for it may remain in force.

The timescale of changes in $\theta_{\rm eff}$ is controlled by the size of extra dimensions and can easily be very much larger than the inverse of the QCD mass scale $\Lambda_{\rm QCD}$. In this case, all the standard QCD dynamics—except for the strong-CP problem—remains intact. In particular, the mechanism that gives mass to the η' meson is unaffected by the presence of the global axion, regardless of whether one associated this mechanism with instantons or any other non-perturbative fluctuations in the QCD vacuum.

Such a global axion may look bizarre from the point of view of relativistic field theory, but as we have shown in this paper it may be quite natural in higher-dimensional theories. Thus, the absence of strong CP violation may indicate that the number of spatial dimensions in our world is greater than three, and moreover that the space has certain topological properties and is compact.

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APPENDIX A: GREEN FUNCTIONS ON THE ORBIFOLD

The computation of anomalies requires computation of several Feynman diagrams. In this appendix we construct the relevant fermionic propagators for the theory defined by Lagrangian (46).

Let us define for this end two scalar propagators, $G_D(x^\mu; z, z')$ and $G_N(x^\mu; z, z')$ which satisfy the equations

$$\begin{aligned}\left[\partial_\nu\partial^\nu - \frac{d^2}{dz^2} + m^2(z) + \frac{dm}{dz} + |\mu|^2\right] G_D(x^\mu; z, z') &= \delta^2(x)\delta(z - z') \ , \\ \left[\partial_\nu\partial^\nu - \frac{d^2}{dz^2} + m^2(z) - \frac{dm}{dz} + |\mu|^2\right] G_N(x^\mu; z, z') &= \delta^2(x)\delta(z - z')\end{aligned} \tag{A1}$$

and boundary conditions

$$G_D(x^\mu; \pm R/2, z') = 0, \quad \left(\frac{d}{dz} + m(z)\right) G_N(x^\mu; z, z')|_{\pm R/2} = 0 \ . \tag{A2}$$

They can be expressed through the orthogonal sets of functions ψ_n and χ_n defined in (8) as follows:

$$\begin{aligned}G_D(x^\mu; z, z') &= \int \frac{d^2k}{(2\pi)^2} e^{ikx} \tilde{G}_D(k; z, z'), \quad \tilde{G}_D(k; z, z') = \sum_{m=1}^{\infty} \frac{\psi_m(z)\psi_m(z')}{-k^2 + M_n^2 + |\mu|^2} \ , \\ G_N(x^\mu; z, z') &= \int \frac{d^2k}{(2\pi)^2} e^{ikx} \tilde{G}_N(k; z, z'), \quad \tilde{G}_N(k; z, z') = \sum_{m=0}^{\infty} \frac{\chi_m(z)\chi_m(z')}{-k^2 + M_n^2 + |\mu|^2} \ .\end{aligned} \tag{A3}$$

The Green functions in Fourier space $G_D(k; z, z')$ and $G_N(k; z, z')$ satisfy the equation

$$\left[-\frac{d^2}{dz^2} + m^2(z) + \frac{dm}{dz} + |\mu|^2 - k^2\right] G_{D,N}(x^\mu; z, z') = \delta(z - z') \tag{A4}$$

and boundary conditions following from (A2).

A helpful relation between the two functions is

$$(\partial_z - m) G_D(x^\mu; z, z') = (-\partial'_z - m) G_N(x^\mu; z, z') \ . \tag{A5}$$

In five-dimensional theory one simply replaces $\frac{d^2k}{(2\pi)^2} \to \frac{d^4k}{(2\pi)^4}$. The integral over spatial components of momentum should be understood as a sum over Fourier harmonics since we work on a compact torus.

The free fermion propagator

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \ , \tag{A6}$$

obeys the equation

$$\begin{pmatrix} i\gamma^A\partial_A + m(z) & -\mu \\ -\mu^* & i\gamma^A\partial_A - m(z) \end{pmatrix} S = \delta^2(x)\delta(z-z') \tag{A7}$$

and the boundary conditions

$$P_L S_{1i}|_{z=\pm R/2} = 0 \ , \quad P_R S_{2i}|_{z=\pm R/2} = 0 \ , \quad i = 1,2 \ . \tag{A8}$$

It is easy to check that it can be expressed through scalar propagators G_D and G_N as

$$\begin{aligned} S_{11} &= -(i\gamma^A\partial_A - m(z))\left[P_L G_D + P_R G_N\right], \quad S_{21} = \mu^* \left[P_L G_D + P_R G_N\right] \ , \\ S_{22} &= -(i\gamma^A\partial_A + m(z))\left[P_L G_N + P_R G_D\right], \quad S_{12} = \mu \ \left[P_L G_N + P_R G_D\right] \ , \end{aligned} \tag{A9}$$

where $P_L = \frac{1}{2}(1+\gamma_5)$ and $P_R = \frac{1}{2}(1-\gamma_5)$ are the chirality projectors.

Now we construct explicitly the Green function $\tilde{G}_D(k; z, z')|_{k=0,\mu=0} \equiv G_D(z, z')$ needed for a number of applications. It satisfies the equation

$$\left[-\frac{d^2}{dz^2} + m^2(z) + \frac{dm}{dz}\right] G_D(z, z') = \delta(z - z') \tag{A10}$$

and the boundary conditions $G_D(\pm R/2, z') = 0$.

Let us define the function

$$\rho(a, b) = \int_a^b \chi_0(z)^2 \ dz \ , \tag{A11}$$

where $\chi_0(z)$ is the zero mode defined in (6). The obvious properties of the function $\rho(a, b)$ are: $\rho(-R/2, R/2) = 1$, $\rho(a, a) = 0$. Then one can easily check that the function

$$G_D(z, z') = \frac{1}{2\chi_0(z)\chi_0(z')}\left[\rho(-R/2, z)\rho(z', R/2)\theta(z'-z) + \rho(-R/2, z')\rho(z, R/2)\theta(z-z')\right] \tag{A12}$$

satisfies eq. (A10) and the boundary conditions.

Another function we will need is

$$G_D(z) \equiv G_D(z, z) = \frac{\rho(-R/2, z)\rho(z, R/2)}{\chi_0(z)^2} \ . \tag{A13}$$

Now, we give the explicit expressions for the Green functions $\tilde{G}_D$ and $\tilde{G}_N$ in a theory with $m(z) = 0$:

$$\begin{aligned} \tilde{G}_D(0, z, z') = \frac{1}{\mu \sinh \mu R}\big[&\sinh \mu(z + R/2) \sinh \mu(z' - R/2)\theta(z' - z) + \\ &\sinh \mu(z - R/2) \sinh \mu(z' + R/2)\theta(z - z')\big] \end{aligned} \tag{A14}$$

$$\tilde{G}_N(0,z,z') = -\frac{1}{\mu\sinh\mu R}\left[\cosh\mu(z+R/2)\cosh\mu(z'-R/2)\theta(z'-z)+ \right.$$
$$\left. \cosh\mu(z-R/2)\cosh\mu(z'+R/2)\theta(z-z')\right] \quad . \tag{A15}$$

Finally, we construct the Green function $G_D(k;z,z')$ for a sharp domain wall residing in an infinite space-time. For this we put in (A4)

$$m(z) = m_0\epsilon(z) \tag{A16}$$

and define a function

$$\Phi(z) = \theta(z)e^{-E(k)z} + \theta(-z)\left(e^{-E(k)z} - \frac{m_0}{m_0+E(k)}e^{E(k)z}\right) , \tag{A17}$$

where

$$E(k) = \sqrt{m_0^2+\mu^2-k^2} \ . \tag{A18}$$

Then

$$G_D(k;z,z') = \theta(z-z')\Phi(z)\Phi(-z') + \theta(z'-z)\Phi(z')\Phi(-z) \ . \tag{A19}$$

APPENDIX B: DERIVATION OF EQ. (38)

In this appendix we will show that the flux of the gauge current depends on the value of the gauge field at the end points of interval only. Note that in 3d no regularization is needed as all integrals are convergent.

For slowly varying in x^μ gauge fields the expression for the current can be written as

$$J^2(x^\mu,z) = \frac{e}{2}\int d^2x'dz'\epsilon_{\mu\nu}F^{\mu\nu}(x',z') \times \tag{B1}$$
$$x'^\alpha\left[\partial_\alpha G_D(x';z,z')(\partial_z'+m)G_N(x';z',z) - \right.$$
$$\left. \partial_\alpha G_N(x';z,z')(\partial_z'-m)G_D(x';z',z)\right] \quad .$$

The integral over x' can be performed (going first to momentum space) to give

$$J^2(x^\mu,z) = \frac{e}{2\pi}\int dz' \sum_{m=0}\sum_{n=1} G_{mn}(z,z')M_nF(M_m,M_n)\epsilon_{\mu\nu}F^{\mu\nu}(x,z') , \tag{B2}$$

where

$$G_{mn}(z,z') = [\chi_m(z)\chi_m(z')\chi_n(z')\psi_n(z) + \psi_m(z)\psi_m(z')\psi_n(z')\chi_n(z)] \tag{B3}$$

and

$$F(M_m,M_n) = \frac{1}{(M_m^2-M_n^2)^2}\left[M_n^2 - M_m^2 + M_m^2\log\left(\frac{M_m^2}{M_n^2}\right)\right] \ . \tag{B4}$$

Now, if one puts $z = \pm R/2$ directly in (B1) one gets zero because the expression for the current contains $\psi_n(z)$ which is zero at the end points because of the boundary conditions. Nevertheless, the limit $\lim_{\epsilon\to+0} J^2(\pm(R/2-\epsilon))$ is not equal to zero. The reason is that in spite of the fact that the sum in (B2) converges for any z, z', its derivative with respect to z does not converge at $z = \pm R/2$. As this is an ultraviolet effect which involves infinite sums, the limit of small ϵ can be computed in a theory without mass term $m(z)$: the wave function ψ_n and χ_n approach their limit (10) for large n.

With this in mind we have:

$$\Delta J \equiv J^2(x^\mu, R/2-\epsilon) - J^2(x^\mu, -R/2+\epsilon) = \frac{e}{2\pi}\int dz'\epsilon_{\mu\nu}F^{\mu\nu}(x,z')\frac{4}{R^2}\times \tag{B5}$$
$$\sum_{m+n=even} \sin\frac{\pi n\epsilon}{R}\left[\cos\pi(n-m)(\frac{z'}{R}-\frac{1}{2})F_s(M_m,M_n) + \cos\pi(n+m)(\frac{z'}{R}-\frac{1}{2})F_a(M_m,M_n)\right] ,$$

where

$$F_{s,a}(M_m,M_n) = \frac{1}{M_n \pm M_m}\left[1 \pm \frac{M_m M_n}{M_n^2 - M_m^2}\log\frac{M_n^2}{M_m^2}\right] \tag{B6}$$

are symmetric (s) and antisymmetric (a) functions respectively.

The gauge field A_μ can be expanded over a complete set of orthogonal functions on the interval as follows:

$$\frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu}(x,z') = \sum_{k=0} B_k(x)\cos 2\pi k(\frac{z'}{R}-\frac{1}{2}) + \sum_{k=1} C_k(x)\sin 2\pi k(\frac{z'}{R}-\frac{1}{2}) \tag{B7}$$

so that

$$\Delta J = \frac{2}{R}\sum_{k=0} B_k(x)\left[\sum_{n=2k}^{\infty}\sin\frac{\pi n\epsilon}{R}F_s(M_{n-2k},M_n) + \sum_{n=0}^{2k}\sin\frac{\pi n\epsilon}{R}F_a(M_{2k-n},M_n)\right] . \tag{B8}$$

For any fixed finite k the limit of the second term is equal to zero as the sum over n contains a finite number of terms. On the contrary,

$$\lim_{\epsilon\to 0}\sum_{n=2k}^{\infty}\sin\frac{\pi n\epsilon}{R}F_s(M_{n-2k},M_n) = \lim_{\epsilon\to 0}\sum_{n=2k}^{\infty}\sin\frac{\pi n\epsilon}{R}\frac{1}{M_n} = \frac{R}{2} \tag{B9}$$

since the sum over n can be replaced by an integral for small ϵ. Finally,

$$\Delta J = \sum_{k=0}^{\infty} B_k(x) = \frac{e}{4\pi}\epsilon_{\mu\nu}\left[F^{\mu\nu}(x,R/2) + F^{\mu\nu}(x,-R/2)\right] . \tag{B10}$$

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beyond the scope of this paper.

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