

## Fermion zero-modes on brane-worlds

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### Abstract

We study localization of bulk fermions on a brane with inclusion of Yang-Mills and scalar backgrounds in higher dimensions and give the conditions under which localized chiral fermions can be obtained.

**Introduction.** Suggestions that extra dimensions may not be compact [1]-[5] or large [6, 7] can provide new insights for a solution of gauge hierarchy problem [7], of cosmological constant problem [2, 4], and give new possibilities for model building. One of the interesting questions, related to these ideas, is localization of different fields on a brane. It has been shown that the graviton [5] and the massless scalar field [8] have normalizable zero modes on branes of different types, that the abelian vector fields are not localized in the Randall-Sundrum (RS) model in five dimensions but can be localized in some higher-dimensional generalizations of it [9]. In contrast, in [8] it was shown that fermions do not have normalizable zero modes in five dimensions, while in [9] the same result was derived for a compactification on a string [10, 11] in six dimensions<sup>1</sup>. It is known, though, that fermion interaction with a scalar domain wall in five dimensions can lead to localization of chiral fermions [1]. Gauge field localization by confinement effects were discussed in [13], bulk fields in a slice of AdS in [14].

In this note we shall prove that under quite general assumptions about the geometry and topology of the internal manifold of the higher-dimensional warp factor compactification there exist massless Dirac fermions. However, these fermionic modes are generically non-normalizable. On the other hand if we include a Yukawa-type coupling to a scalar field of a domain-wall type we can ensure chirality as well as localization of the fermions. To generate chiral fermions by this mechanism the topology of the internal Kaluza-Klein manifold and the gauge field defined on it should be such that the index of the Dirac operator defined on this manifold is non-zero. At the end of this note we shall mention the example of the  $K_3$  surface and  $S^4$  with a background instanton configuration defined on it.

**Branes with gauge and gravity backgrounds.** We shall consider  $D = D_1 + D_2 + 1$  - dimensional manifolds with the geometry

$$ds^2 = e^{A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{B(r)}g_{mn}(y)dy^m dy^n + dr^2, \quad (1)$$

where  $\mu, \nu = 0, 1, \dots, D_1 - 1$ ,  $m, n = 1, \dots, D_2$ . The coordinates  $y^m$  cover an internal manifold  $K$  with the metric  $g_{mn}(y)$ . The  $D$ -dimensional Dirac equation is

$$\Gamma^A E_A^M (\partial_M - \Omega_M + A_M) \Psi(x, y, r) = 0, \quad (2)$$

where  $E_A^M$  is the vielbein,  $\Omega_M = \frac{1}{2}\Omega_{M[AB]}\Sigma^{AB}$  is the spin connection,  $\Sigma_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B]$ , and  $A_M$  is a Yang-Mills field in the algebra of some gauge group  $G$ . The RS model is the special case with  $D_2 = 0$  and  $A_M = 0$ .

The non-vanishing components of  $\Omega_M$  are

$$\Omega_\mu = \frac{1}{4}A'e^{\frac{A}{2}}\delta_\mu^a \Gamma_r \Gamma_a, \quad (3)$$

$$\Omega_m = \frac{1}{4}B'e^{\frac{B}{2}}e_m^a \Gamma_r \Gamma_a + \omega_m, \quad (4)$$

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<sup>1</sup>Bulk fermions were localized on brane world studied some years ago in [12] by the use of a bulk magnetic field, which falls out of the framework of our ansatz. However, these solutions generally have problems with normalizability of the graviton modes.

where  $\Gamma_r, \Gamma_a$ ,  $a = 0, 1, \dots, D_1 - 1$  and  $\Gamma_{\underline{a}}$ ,  $\underline{a} = 1, \dots, D_2$  are the constant Dirac matrices and  $\omega_m = \frac{1}{8}\omega_{m[\underline{a}, \underline{b}]}[\Gamma_{\underline{a}}, \Gamma_{\underline{b}}]$  is the spin connection derived from the metric  $g_{mn}(y) = e^{\underline{a}}_m e^{\underline{b}}_n \delta_{\underline{a}\underline{b}}$ .

Assume  $A_\mu = A_r = 0$ . The Dirac equation then becomes

$$\left\{ e^{-\frac{A}{2}} \not{\partial}_x + \Gamma^r \left( \partial_r + \frac{D_1}{4} A' + \frac{D_2}{4} B' \right) + e^{-\frac{B}{2}} \not{\Delta}_y \right\} \Psi = 0, \quad (5)$$

where  $\not{\Delta}_y = \Gamma^{\underline{a}} e^{\underline{m}}_{\underline{a}} (\partial_m - \omega_m + A_m)$  is the Dirac operator on the internal manifold  $K$  and in the background of the gauge field  $A_m$ . With an appropriate choice of  $K$  and  $A_m$  this operator will have zero modes [15]. Denote these modes by  $\psi(y)$ . We can then write

$$\Psi(x, y, r) = \psi(y) f(r) \phi(x), \quad (6)$$

where  $f$  and  $\phi$  satisfy

$$\not{\partial}_x \phi(x) = 0, \quad (7)$$

$$f' + \left( \frac{D_1}{4} A' + \frac{D_2}{4} B' \right) f = 0, \quad (8)$$

or

$$f(r) = \text{const.} \cdot e^{-\left(\frac{D_1}{4} A + \frac{D_2}{4} B\right)}. \quad (9)$$

The effective Lagrangian for  $\phi$  then becomes

$$\int dr dy \sqrt{-G} \bar{\Psi} \Gamma^A E_A^M (\partial_M - \Omega_M + A_M) \Psi = \bar{\phi}(x) \not{\partial}_x \phi(x) \times \int e^{-\frac{A}{2}} dr dy \sqrt{g} \psi^\dagger(y) \psi(y). \quad (10)$$

This should be compared with the expression of  $D_1$ -dimensional Newton constant  $G_{D_1}$  in terms of the  $D$ -dimensional one,

$$G_{D_1}^{-1} = G_D^{-1} V_{D_2} \int dr \exp \left( \left( \frac{D_1}{2} - 1 \right) A + \frac{B}{2} \right), \quad (11)$$

where  $V_{D_2}$  is the volume of the manifold  $K$ . Thus, to have the localization of gravity and finite kinetic energy for  $\phi$ , both integrals (10,11) must be simultaneously finite. This is not the case for the exponential warp-factor  $A \propto -|r|$  considered in the literature so far. In fact, for such  $A$  and  $B$ , the function  $f$  in (9) diverges as  $r \rightarrow \infty$ .

So, for presently known solutions, the bulk fermions cannot be localized on a brane with the use of gravity and gauge fields only.

**Yukawa Coupling and Chiral Fermions.** Now let us include a real scalar field  $\chi$  in our problem. The modification of the Dirac equation will be through some Yukawa term, with the coupling  $\lambda$ , viz.

$$\left\{ e^{-\frac{A}{2}} \not{\partial}_x + \Gamma^r \left( \partial_r + \frac{D_1}{4} A' + \frac{D_2}{4} B' \right) + \lambda \chi + e^{-\frac{B}{2}} \not{\Delta}_y \right\} \Psi = 0. \quad (12)$$

The details of the  $\chi$ -field dynamics will not be important for our discussion. We shall only assume that its equation of motion admits a localized  $r$ -dependent solution such that  $\chi(r) \rightarrow |v|\epsilon(r)$  as  $|r| \rightarrow \infty$ , where  $v = \langle \chi \rangle$ , and  $\epsilon(r)$  is the sign function. With this assumption and imposing the chirality condition  $\Gamma^r \Psi = +\Psi$ , far away from the core region we need to solve

$$\begin{aligned} \not{\partial}_x \phi &= 0, \\ \left( \partial_r + \frac{D_1}{4} A' + \frac{D_2}{4} B' + \lambda |v| \epsilon(r) \right) f &= 0, \\ \not{\Delta}_y \psi &= 0. \end{aligned} \tag{13}$$

The solution of the above equation can be written as

$$\psi(x, y, r) = e^{-\left(\frac{D_1}{4} A + \frac{D_2}{4} B\right) - \lambda |v| r \epsilon(r)} \cdot \psi(y) \phi(x), \tag{14}$$

where  $\not{\Delta}_y \psi(y) = 0$ .

Thus, to have localized  $\Psi(x, y, r)$  it is sufficient that

$$-\frac{A}{2} - 2\lambda |v| r \epsilon(r) < 0. \tag{15}$$

This can be achieved for large enough values of  $\lambda |v|$ . For example, for solutions of Einstein equations with  $A = B = cr\epsilon(r)$ , where  $c < 0$ , that can be obtained for a string in 6 dimensions [10] or on  $K = K3$  in higher dimensions [16], it is sufficient to have  $\lambda |v| > -c/4$ .

Now we come to the issue of chirality of the normalizable zero modes. First we note that for an even  $D_2$  normalizable solutions of  $\not{\Delta}_y \psi(y) = 0$  have definite chirality. The index theorem gives the difference  $n_+ - n_-$ , where  $n_+$  and  $n_-$  are respectively the number of positive and negative chirality zero modes of the operator  $\not{\Delta}_y$ . Since we have imposed  $\Gamma^r = 1$ , the chiralities of  $\psi(y)$  and  $\phi(x)$  will be identical<sup>2</sup>. Thus the number of chiral families will be equal to  $n_+ - n_-$ . This mechanism is identical to the one which generates chiral fermions in the standard Kaluza-Klein compactification [17].

On the example of a  $K = K_3$  compactification we obtain two chiral families, while for  $K = S^4$  with an  $SU(2)$  instanton on it there will be  $\frac{2}{3}t(t+1)(2t+1)$  chiral localized families in 4 dimensions, where  $t$  is the spin of the fermion representations. For  $D = 7$  we can take  $K = S^2$  with a  $U(1)$  magnetic monopole field of charge  $n$  on it. The number of chiral families will then be equal to  $n$  [18].

**Conclusions.** We defined conditions under which bulk fermions can be localized on a brane in models with localized gravity in higher dimensional generalizations of the RS model if only gauge and gravitational backgrounds are considered. We show how the domain-wall scalar field structures can insure localization and chirality at the same time. The number of chiral fermions is related to the topology of the manifold  $K$  and the gauge field background.

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<sup>2</sup>This is due to the fact that for an odd  $D$   $\Gamma^r = \Gamma_{D_1} \cdot \Gamma_{D_2}$  where  $\Gamma_{D_1}$  and  $\Gamma_{D_2}$  are the chirality matrices in  $D_1$  and  $D_2$  dimensions.

It remains to be seen if one can find solutions which incorporate all the required features, namely, localized fields of various spins with the correct standard model quantum numbers in a non-singular background geometry. The non-singularity of the localized geometry seems to be rather difficult to achieve, at least without presence of a brane. It has been shown in [4] that for the metrics of the type given in eq.(1) which are regular at  $r = 0$  the vacuum Einstein equations for  $D_2 > 1$  produce generally singular solutions, although with a finite volume in the  $y, r$  subspace. It has been recently argued by Witten [19] that such naked singularities make the physical interpretation of these solutions problematic.

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