On the Decoupling of Heavy Modes in Kaluza-Klein Theories

S. Randjbar-Daemi[a1], A. Salvio[b2] and M. Shaposhnikov[c3]

[a] *International Center for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy*

[b] *International School for Advanced Studies, Via Beirut 2-4, 34014 Trieste, Italy*

[c] *Institut de Théorie des Phénomènes Physiques, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland*

Abstract

In this paper we examine the 4-dimensional effective theory for the light Kaluza-Klein (KK) modes. Our main interest is in the interaction terms. We point out that the contribution of the heavy KK modes is generally needed in order to reproduce the correct predictions for the observable quantities involving the light modes. As an example we study in some detail a 6-dimensional Einstein-Maxwell theory coupled to a charged scalar and fermions. In this case the contribution of the heavy KK modes are geometrically interpreted as the deformation of the internal space.

[1]Email: seif@ictp.trieste.it
[2]Email: salvio@sissa.it
[3]Email: Mikhail.Shaposhnikov@epfl.ch

Contents

1 Introduction

In studying the low energy physics of the light modes of a 4+d dimensional theory the attention is usually paid only to the spectral aspects. After determining the quantum numbers of the light modes the nature and the form of the interaction terms are often assumed to be dictated by symmetry arguments. Such arguments fix the general form of all the renormalilzable terms and if the effective theory is supersymmetric certain relationship between the couplings can also be established by supersymmetry. The masses are derived from the bilinear part of the effective action and the role of the heavy modes in the actual values of the masses and the couplings of the effective theory for the light modes are seldom taken into account. It is, however, well known from the study of the GUT's in 4-dimensions that the heavy modes have an important role to play even at low energies [1]. This happens through their contributions to the couplings entering into the effective Lagrangians describing the low energy physics of the light modes. According to Wilsonian approach, in order to obtain an effective theory applicable in large distances, the heavy modes should be integrated out [2, 3]. The processes of "integrating out" has the effect of modifying the couplings of the light modes or introducing additional terms which are suppressed by inverse powers of the heavy masses [4].

The aim of the present paper is to examine the role of the heavy modes in the low energy description of a higher dimensional theory. To this end we shall basically perform two complementary calculations. The first one will start from a solution of a higher dimensional theory with a 4-dimensional Poincaré invariance and develop an action functional for the light modes of the effective 4-dimensional theory. This effective action generally has a local symmetry which should be broken by Higgs mechanism. Our interest is in the spectrum of the broken theory. The procedure is essentially what is adopted in the effective description of higher dimensional theories including superstring and M-theory compactifications. In this construction the heavy KK modes are generally ignored simply by reasoning that their masses are of the order of the compactification mass and this can be as heavy as the Planck mass. Therefore they can not affect the low energy physics of the light modes.

In the second approach which we shall call the *geometrical approach* we shall find a solution of the higher dimensional equations with the same symmetry group as the one of the broken phase of the effective 4-dimensional theory for the light modes. We shall then study the physics of the 4-dimensional light modes around this solution. The result for the effective 4-dimensional theory will turn out to be different from the first approach. The aim of this paper is to show that the difference is precisely due to the fact that in constructing the effective theory along the lines of the first approach the contribution of the heavy KK modes have been ignored. Indeed it will be argued - and demonstrated by working out some explicit examples - that taking due care of the role of the heavy modes a complete equivalence is established between the two approaches.

To motivate the discussion in a simpler context, in section 2 we shall work out a simple model of two coupled scalar fields in 4-dimensions which will be generalized to a multiplet of scalar fields in arbitrary dimensions in section 3. The examples in sections 2 and 3 will clarify the relevance of the heavy modes in the low energy description of the light modes. In sections 4 and 5 we shall study a higher dimensional (in this case six dimensional) theory of Einstein-Maxwell system [5] coupled to a charged scalar and eventually also to charged fermions. Such a model can arise in the compactification of string or M-theory to lower dimensions. The system has enough number of adjustable parameters to allow us to go to various limits in order to establish the main point of our paper. The result will of course confirm the above mentioned expectation that in order to obtain a correct 4-dimensional description of the physics of the light modes the contribution of the heavy modes should be duly taken into account[4]. This example is particularly interesting because the first kind of solution will produce an effective 4-dimensional gauge theory with a $SU(2)\times U(1)$ symmetry which will be broken to $U(1)$ by a complex triplet of Higgs fields. The geometrical approach, on the other hand, will take us directly to the unbroken $U(1)$ phase by deforming a round sphere into an ellipsoid[5]. In the geometrical approach the W and the Z masses originate from the deformation of the internal space. In this sense the standard Higgs mechanism acquires a geometrical origin[6]. We elaborate a little more on this point in section 6 which summarizes our results. Some technical aspects of various derivations have been detailed in the appendices.

2 A Simple 4D Theory

Let's consider a 4-dimensional theory, which contains two real scalar fields φ and χ and with the lagrangian

$$\mathcal{L}=-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}\partial_\mu\chi\partial^\mu\chi-\frac{1}{2}m_\varphi^2\varphi^2-\frac{1}{2}m^2\chi^2-\frac{1}{4}\lambda_\varphi\varphi^4-\frac{1}{4}\lambda_\chi\chi^4-a\varphi^2\chi^2,$$

where m_φ^2, m^2, λ_φ, λ_χ and a are real parameters[7]. Here we have the symmetry:

$$\begin{aligned} Z_2 &: \varphi\to\pm\varphi,\\ Z_2' &: \chi\to\pm\chi. \end{aligned} \tag{2.1}$$

This is a very particular example and of course we don't want to present any general result in this section, we just want to provide a framework in which the general equivalence that we spoke about in the introduction emerges in a simple way and is not obscured by technical difficulties.

For $m_\varphi^2<0$ we have the following solution of the equations of motion (EOM):

$$\chi=0,\qquad \varphi=\sqrt{\frac{-m_\varphi^2}{\lambda_\varphi}}\equiv\varphi_{eff}, \tag{2.2}$$

[4]Of course this doesn't prove that the heavy modes contribution never vanishes: for instance [6] proves the decoupling of the heavy modes in the $(Minkowski)_4\times S^2$ compactification of the 6-dimensional chiral supergravity [7], which is basically the supersymmetric version of our 6-dimensional theory.

[5]This corresponds to the magnetic monopole charge of 2 as explained in section 5. A monopole charge of unity will produce a Higgs doublet of SU(2).

[6]It should be mentioned that all of our discussion is (semi-) classical. To include quantum and renormalization effects is beyond the scope of the present paper.

[7]Of course we consider only the values of these parameters such that the scalar potential is bounded from below.

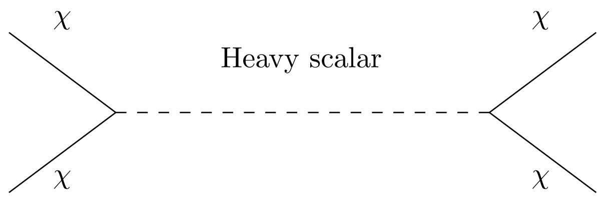

Figure 1: A tree diagram which describes the scattering of two light χ, through the exchange of an heavy scalar. This kind of diagram gives a contribution to the quartic term in the effective theory potential.

which breaks Z_2 but preserves Z_2'. We can express the lagrangian in terms of the fluctuation $\delta\varphi$ and χ around this background:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\nu\delta\varphi\partial^\nu\delta\varphi - \frac{1}{2}\partial_\nu\chi\partial^\nu\chi + m_\varphi^2\left(\delta\varphi\right)^2 - \frac{1}{2}\mu^2\chi^2 \\ &\quad -\sqrt{-m_\varphi^2\lambda_\varphi}\left(\delta\varphi\right)^3 - \frac{1}{4}\lambda_\varphi\left(\delta\varphi\right)^4 - \frac{1}{4}\lambda_\chi\chi^4 - 2a\sqrt{\frac{-m_\varphi^2}{\lambda_\varphi}}\delta\varphi\chi^2 \\ &\quad -a\left(\delta\varphi\right)^2\chi^2 + constants, \end{aligned} \tag{2.3}$$

where

$$\mu^2 \equiv m^2 - 2a\frac{m_\varphi^2}{\lambda_\varphi}. \tag{2.4}$$

If $|\mu^2| \ll |m_\varphi^2|$, we expect that the heavy mode $\delta\varphi$ can be integrated out and an effective theory for χ can be constructed for both the signs of μ^2. However it's important to note that $\delta\varphi$ cannot be simply neglected because it gives a contribution, because of the trilinear[8] coupling $\delta\varphi\chi^2$ in (2.3), to the operator χ^4 in the effective theory, through the diagram 1. This is similar to what is usually done in GUT theories [1], where, for instance, four fermions effective interactions emerge by integrating out the heavy gauge fields [8]. At the classical level the effective lagrangian for χ is

$$\mathcal{L}_{eff} = -\frac{1}{2}\partial_\nu\chi\partial^\nu\chi - \frac{1}{2}\mu^2\chi^2 - \frac{1}{4}\left(\lambda_\chi - \frac{4a^2}{\lambda_\varphi}\right)\chi^4 + \ldots, \tag{2.5}$$

where the dots represent higher dimensional operators. The term $a^2\chi^4/\lambda_\varphi$ is the contribution of the heavy mode. The result (2.5) was originally derived in [9], but here we want also to study the effective theory with spontaneous symmetry breaking and we want to compare it with the low energy limit of the fundamental theory.

[8] Also the quartic coupling $\left(\delta\varphi\right)^2\chi^2$ gives a contribution to the operator χ^4, but this is negligible in the classical limit.

For $\mu^2 > 0$, the minimum of the effective theory potential is for $\chi = 0$. Instead for $\mu^2 < 0$ we have

$$\chi = \sqrt{\frac{-\mu^2}{\lambda_\chi - \frac{4a^2}{\lambda_\varphi}}} \tag{2.6}$$

and the fluctuation $\delta\chi$ over this background has the following mass squared:

$$M^2(\delta\chi) = -2\mu^2. \tag{2.7}$$

This results will be not modified by the higher dimensional operator at the leading order[9] in μ. The equations (2.6) and (2.7) represent the effective theory prediction for the VEV and the spectrum in the phase where Z_2' is broken.

On the other hand, a solution of the fundamental EOM, namely the EOM derived from the fundamental lagrangian $\mathcal{L}$, is

$$\begin{aligned} \chi^2 &= \frac{-\mu^2}{\lambda_\chi - \frac{4a^2}{\lambda_\varphi}} + O(\mu^3), \\ \varphi^2 &= -\frac{m_\varphi^2}{\lambda_\varphi} + \frac{2a\mu^2}{\lambda_\varphi\lambda_\chi - 4a^2} + O(\mu^3) \end{aligned} \tag{2.8}$$

which is a small deformation of (2.2) at the leading non trivial order in μ and breaks the Z_2' symmetry. Moreover the light mode which corresponds to this solution has a mass squared $-2\mu^2$.

Therefore the effective theory prediction for the light mode VEV and spectrum is correct, at the order μ, in this simple framework, but the heavy mode contribution is necessary in order the effective theory prediction to be correct.

3 A More General Case

Now we want to extend the result of section 2 and ref [9] to a more general class of theories. We consider a set of real D-dimensional scalars Φ_i with a general potential V: the lagrangian is

$$\mathcal{L} = -\frac{1}{2}\partial_M\Phi_i\partial^M\Phi_i - V(\Phi), \tag{3.1}$$

where $M, N, ...$ run over all the space-time dimensions, while $\mu, \nu, ...$ and $m, n, ...$ are respectively the 4-dimensional and the internal coordinates indices. The EOM are

$$\partial_M\partial^M\Phi_i - \frac{\partial V}{\partial\Phi_i}(\Phi) = 0. \tag{3.2}$$

We consider now a solution Φ_{eff} of (3.2) which preserves the 4-dimensional Poincaré invariance and some internal symmetry group $\mathcal{G}$; the corresponding mass squared eigenvalue

[9] The mass μ is small in the sense $|\mu| \ll |m_\varphi|$.

problem for the 4-dimensional states is

$$-\partial_m \partial_m \delta\Phi_i + \frac{\partial^2 V}{\partial\Phi_i \partial\Phi_j}(\Phi_{eff})\delta\Phi_j = M^2 \delta\Phi_i, \tag{3.3}$$

where $\delta\Phi$ is the fluctuation around Φ_{eff}. We assume that there are n normalizable solutions $\mathcal{D}_l$ with small eigenvalues ($M^2 \sim \mu^2$), other, in principle infinite, solutions[10] $\tilde{\mathcal{D}}_h$with large eigenvalues ($M^2 \gg |\mu^2|$) and nothing else. These hypothesis are needed in order to define the concept of light KK modes.

We can expand the scalars Φ_i as follows

$$\Phi_i = (\Phi_{eff})_i + \chi_l(x)\mathcal{D}_{li}(y) + \tilde{\chi}_h(x)\tilde{\mathcal{D}}_{hi}(y), \tag{3.4}$$

where χ_l and $\tilde{\chi}_h$ are respectively the light and heavy KK modes. We choose the $\mathcal{D}_l$ and $\tilde{\mathcal{D}}_h$ in order that they form an orthonormal basis for the functions over the internal space:

$$\begin{aligned}
\langle \mathcal{D}_l | \mathcal{D}_{l'} \rangle &\equiv \int d^{D-4}y \mathcal{D}_{li}(y)\mathcal{D}_{l'i}(y) = \delta_{ll'}, \\
\left\langle \tilde{\mathcal{D}}_h | \tilde{\mathcal{D}}_{h'} \right\rangle &\equiv \int d^{D-4}y \tilde{\mathcal{D}}_{hi}(y)\tilde{\mathcal{D}}_{h'i}(y) = \delta_{hh'}, \\
\left\langle \mathcal{D}_l | \tilde{\mathcal{D}}_h \right\rangle &\equiv \int d^{D-4}y \mathcal{D}_{li}(y)\tilde{\mathcal{D}}_{hi}(y) = 0.
\end{aligned} \tag{3.5}$$

We note that χ_l and $\tilde{\chi}_h$ could both belong to some non trivial representation of the internal symmetry group $\mathcal{G}$.

3.1 The Effective Theory Method

We construct now some relevant terms in the effective theory for the light KK modes χ_l. Here "relevant terms" mean relevant terms in the classical limit and in case we have a small point of minimum of the order μ of the effective theory potential: we want to compare the results of the effective theory for the light KK modes with the low energy limit of the fundamental theory expanded around a vacuum which is a small perturbation of Φ_{eff}. Further we calculate everything at leading non trivial order[11] in μ. The relevant terms can be computed by putting just the light KK modes in the action and performing the integration over the extra dimensions and then by taking into account the effect of heavy KK modes through the diagrams like figure 1. In order to calculate those diagrams, we give the interactions between two light modes χ_l and one heavy mode $\tilde{\chi}_h$:

$$-\frac{1}{2}\left(\int d^{D-4}y V_{ijk}\mathcal{D}_{li}\mathcal{D}_{mj}\tilde{\mathcal{D}}_{hk}\right)\chi_l \chi_m \tilde{\chi}_h, \tag{3.6}$$

where we have used the notation

$$V_{i_1 \ldots i_N} \equiv \frac{\partial^N V}{\partial\Phi_{i_1} \ldots \partial\Phi_{i_N}}(\Phi_{eff}). \tag{3.7}$$

[10] In principle h can be a discrete or a continuous variable.
[11] The μ mass scale is small in the sense $|\mu|$ is much smaller than the heavy masses.

We get the following relevant terms in the effective theory potential $\mathcal{U}$:

$$\mathcal{U}(\chi) = \frac{1}{2}c_l\mu^2\chi_l\chi_l + \frac{1}{3}\lambda^{(3)}_{lmp}\chi_l\chi_m\chi_p + \frac{1}{4}\lambda^{(4)}_{lmpq}\chi_l\chi_m\chi_p\chi_q + \dots, \tag{3.8}$$

where the dots represent non relevant terms, c_l are dimensionless numbers and

$$\lambda^{(3)}_{lmp} \equiv \frac{1}{2}\int d^{D-4}y V_{ijk}\mathcal{D}_{li}\mathcal{D}_{mj}\mathcal{D}_{pk}, \tag{3.9}$$

$$\lambda^{(4)}_{lmpq} \equiv \frac{1}{3!}\left(\int d^{D-4}y V_{ijkk'}\mathcal{D}_{li}\mathcal{D}_{mj}\mathcal{D}_{pk}\mathcal{D}_{qk'}\right) + a_{lmpq}, \tag{3.10}$$

where the quantities a_{lmpq} represent the heavy modes contribution and they are given by

$$a_{lmpq} = c_{lmpq} + c_{lpmq} + c_{lqpm} \tag{3.11}$$

and

$$c_{lmpq} \equiv -\frac{1}{6}\int d^{D-4}yd^{D-4}y' V_{ijk}(y)\mathcal{D}_{li}(y)\mathcal{D}_{mj}(y)G_{kk'}(y,y')V_{i'j'k'}(y')\mathcal{D}_{pi'}(y')\mathcal{D}_{qj'}(y'). \tag{3.12}$$

The object $G_{kk'}$ is the Green's function for the mass squared operator at the left hand side of (3.3) and it's explicitly given by

$$G_{kk'}(y,y') = \sum_h \frac{1}{m_h^2}\tilde{\mathcal{D}}_{hk}(y)\tilde{\mathcal{D}}_{hk'}(y'), \tag{3.13}$$

where m_h^2 is the eigenvalue associated to the eigenfunction $\tilde{\mathcal{D}}_h$.

In the rest of this section we consider the predictions of the effective theory with spontaneous symmetry breaking. The potential (3.8) has to be considered as a generalization of (2.5), which was originally derived in [9]. A non vanishing VEV breaks in general $\mathcal{G}$ to some subgroup and it must satisfies

$$\frac{\partial\mathcal{U}}{\partial\chi_l} = c_l\mu^2\chi_l + \lambda^{(3)}_{lmp}\chi_m\chi_p + \lambda^{(4)}_{lmpq}\chi_m\chi_p\chi_q = 0. \tag{3.14}$$

Since we require that χ_l goes to zero as μ goes to zero we have

$$\chi_l = \chi_{l1} + \chi_{l2} + \dots \tag{3.15}$$

where χ_{l1} is proportional to μ, χ_{l2} is proportional to μ^2 and so on. At the order μ^2 the equations (3.14) reduce to

$$\lambda^{(3)}_{lmp}\chi_{m1}\chi_{p1} = 0 \tag{3.16}$$

which implies

$$\lambda^{(3)}_{lmp}\chi_{p1} = 0. \tag{3.17}$$

While, at the order μ^3, the equations (3.14) reduce to

$$c_l\mu^2\chi_{l1} + \lambda^{(4)}_{lmpq}\chi_{m1}\chi_{p1}\chi_{q1} = 0, \tag{3.18}$$

where we have used the equations (3.17).

Finally the mass spectrum corresponding to a solution of (3.14) is given by the eigenvalues of the hessian matrix of $\mathcal{U}$ in that solution:

$$\frac{\partial^2 \mathcal{U}}{\partial \chi_l \partial \chi_{l'}} = c_l \mu^2 \delta_{ll'} + 2\lambda^{(3)}_{ll'm}\chi_m + 3\lambda^{(4)}_{ll'mq}\chi_m\chi_q. \tag{3.19}$$

If we assume, for simplicity, $\lambda^{(3)}_{ll'm} = 0$, which corresponds to the absence of cubic terms in $\mathcal{U}$, the leading order approximation of the hessian is simply given by

$$\frac{\partial^2 \mathcal{U}}{\partial \chi_l \partial \chi_{l'}} = c_l \mu^2 \delta_{ll'} + 3\lambda^{(4)}_{ll'mq}\chi_{m1}\chi_{q1} + O(\mu^3). \tag{3.20}$$

In subsection 3.2 we show that this matrix, which represents the mass spectrum for the light KK modes, and the equations (3.17) and (3.18) for the light modes VEVs are exactly reproduced by a D-dimensional analysis.

3.2 D-dimensional analysis

Now we want to find a solution of (3.2) which is a small perturbation, of the order μ, of Φ_{eff} and then we want to find the low energy mass spectrum of the fluctuations around this solution. In general this solution will break $\mathcal{G}$ to some subgroup like a solution of (3.14) does in the effective theory method. The explicit form of such solution in the simple case of section 2 is given by (2.8) and the low energy mass spectrum in that simple case is represented by the squared mass $-2\mu^2$; now we want to generalize these results.

Let's consider the expansion (3.4); we observe that the statement that the solution is a small perturbation of Φ_{eff} means

$$\begin{aligned} \chi_l &= \chi_{l1} + \chi_{l2} + \dots, \\ \tilde{\chi}_h &= \tilde{\chi}_{h1} + \tilde{\chi}_{h2} + \dots, \end{aligned} \tag{3.21}$$

that is there are no big $\mu-$independent terms in χ_l and $\tilde{\chi}_h$. We consider now a Taylor expansion of the equations (3.2) around Φ_{eff}:

$$\begin{aligned} &\partial_m\partial_m \left(\Phi_i - (\Phi_{eff})_i\right) \\ &-\sum_{k=1}^{N} \frac{1}{k!} V_{ii_1\dots i_k} \left(\Phi_{i_1} - (\Phi_{eff})_{i_1}\right) \cdot \dots \cdot \left(\Phi_{i_k} - (\Phi_{eff})_{i_k}\right) \\ &+O(\mu^{N+1}) = 0. \end{aligned} \tag{3.22}$$

At the order μ the equations (3.22) reduce to

$$\left(\partial_m\partial_m\delta_{ij} - V_{ij}\right)\left(\Phi_j - (\Phi_{eff})_j\right) + O(\mu^2) = 0, \tag{3.23}$$

which simply states

$$\tilde{\chi}_{h1} = 0. \tag{3.24}$$

Moreover at the order μ^2 the equations (3.22) imply

$$\tilde{\chi}_{h2}\left(\partial_m\partial_m\delta_{ij} - V_{ij}\right)\tilde{\mathcal{D}}_{hj} = \frac{1}{2}V_{ijk}\mathcal{D}_{lj}\mathcal{D}_{mk}\chi_{l1}\chi_{m1}, \tag{3.25}$$

which has two consequences: the first one is

$$\lambda^{(3)}_{lmp}\chi_{p1} = 0, \tag{3.26}$$

which can be derived from (3.25) by projecting over $\mathcal{D}_l$ and it exactly reproduces (3.17) of the effective theory method; the second consequence is

$$\tilde{\chi}_{h2}\tilde{\mathcal{D}}_{hi'}(y) = -\frac{1}{2}\chi_{l1}\chi_{m1}\int d^{D-4}y' G_{i'i}(y,y')V_{ijk}(y')\mathcal{D}_{lj}(y')\mathcal{D}_{mk}(y'), \tag{3.27}$$

where G still represents the Green's function for the operator at the left hand side of (3.3). Now we can write the μ^3 part of the equation (3.22) as follows

$$\begin{aligned}
&-c_l\mu^2\chi_{l1}\mathcal{D}_{li} - m_h^2\tilde{\chi}_{h3}\tilde{\mathcal{D}}_{hi}\\
&-\frac{1}{2}V_{ijk}\chi_{l1}\mathcal{D}_{lj}\left(\tilde{\chi}_{h2}\tilde{\mathcal{D}}_{hk} + \chi_{m2}\mathcal{D}_{mk}\right)\\
&-\frac{1}{2}V_{ijk}\left(\chi_{l2}\mathcal{D}_{lj} + \tilde{\chi}_{h2}\tilde{\mathcal{D}}_{hj}\right)\chi_{m1}\mathcal{D}_{mk}\\
&-\frac{1}{3!}V_{ijkk'}\mathcal{D}_{lj}\mathcal{D}_{mk}\mathcal{D}_{pk'}\chi_{l1}\chi_{m1}\chi_{p1} = 0.
\end{aligned} \tag{3.28}$$

If one projects this equation over $\mathcal{D}_l$ and uses the equations (3.26) and (3.27) one gets exactly the equations (3.18). Therefore, at the order μ, all the solutions of (3.14) are reproduced by the D-dimensional analysis and viceversa. Moreover we observe that these light KK modes VEVs, predicted by the effective theory, constitute approximate solutions of the fundamental D-dimensional EOM at leading non trivial order because of the equation (3.24), which states that the heavy KK modes VEVs are higher order quantity with respect to the light KK modes VEVs.

Now we consider the mass squared eigenvalue problem which corresponds to a solution Φ; moreover we assume for simplicity $\lambda^{(3)}_{lmp} = 0$, like in the effective theory method. This eigenvalue problem is

$$\mathcal{O}_{ij}\delta\Phi_j \equiv -\partial_m\partial_m\delta\Phi_i + \frac{\partial^2 V}{\partial\Phi_i\partial\Phi_j}(\Phi)\delta\Phi_j = M^2\delta\Phi_i, \tag{3.29}$$

where $\delta\Phi_i$ represents the fluctuations of the scalars around the solution Φ. We observe now that the equation (3.29) can be considered a time-independent Schrodinger equation: $\mathcal{O}$ is the hamiltonian and M^2 the generic energy level. Moreover we can perform a Taylor expansion of $\mathcal{O}$ around $\mu = 0$:

$$\mathcal{O} = \mathcal{O}_0 + \mathcal{O}_1 + \mathcal{O}_2 + \ldots. \tag{3.30}$$

The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ can be easily expressed just in terms of χ_{l1} and χ_{l2} by using (3.4), (3.21) and the constraints (3.27) and (3.24) which come from the EOM. From the perturbation theory of quantum mechanics we know that the leading value of the low energy mass spectrum is given by the eigenvalues of the following mass squared matrix:

$$M^2_{ll'} \equiv A_{ll'} + B_{ll'}, \tag{3.31}$$

where

$$A_{ll'} \equiv < \mathcal{D}_l|\mathcal{O}_2|\mathcal{D}_{l'} > \tag{3.32}$$

and

$$B_{ll'} \equiv -\sum_h \frac{1}{m_h^2} < \mathcal{D}_l|\mathcal{O}_1|\tilde{\mathcal{D}}_h >< \tilde{\mathcal{D}}_h|\mathcal{O}_1|\mathcal{D}_{l'} > . \tag{3.33}$$

If one express the matrices A and B in terms[12] of χ_{l1} , one finds exactly the corresponding result (3.20) predicted by the effective theory.

So we have two equivalent (at least at the leading non trivial order in μ) approaches to study the breaking of $\mathcal{G}$: the spontaneous symmetry breaking in the 4-dimensional effective theory and the D-dimensional analysis. We stress that, like in the simple model of section 2, also in this more general case the heavy KK modes contribution in the effective theory can't be neglected if one wants to reproduce the D-dimensional result, even at the classical level. In general this is true not only in scalar theories but also in theories which involve gauge and gravitational interactions, as we illustrate in sections 4 and 5.

4 A 6D Gauge and Gravitational Theory

Now we consider a 6-dimensional field theory of gravity with a $U(1)$ gauge invariance, including a charged scalar field ϕ and eventually fermions. The bosonic action is[13]

$$S_B = \int d^6X\sqrt{-G}\left[\frac{1}{\kappa^2}R - \frac{1}{4}F_{MN}F^{MN} - (\nabla_M\phi)^*\nabla^M\phi - V(\phi)\right], \tag{4.1}$$

where R is the Ricci scalar, κ represents the 6-dimensional Planck scale, F_{MN} is the field strength of the $U(1)$ gauge field A_M, defined by

$$F_{MN} = \partial_M A_N - \partial_N A_M \tag{4.2}$$

and

$$\nabla_M\phi = \partial_M\phi + ieA_M\phi, \tag{4.3}$$

where e is the $U(1)$ gauge coupling. Moreover V is a scalar potential and we choose

$$V(\phi) = m^2\phi^*\phi + \xi(\phi^*\phi)^2 + \lambda, \tag{4.4}$$

where m^2 and ξ are generical real constants, with the constraint $\xi > 0$ and λ represents the 6-dimensional cosmological constant.

[12] The dependence on χ_{l2} disappears because we assume $\lambda^{(3)}_{lmp} = 0$, as one can easily check.

[13] Some conventions are fixed in appendix A.

From the action (4.1) we can derive the general bosonic EOM. However we focus on the following class of backgrounds, which are invariant under the 4-dimensional Poincaré group:

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + g_{mn}(y)dy^m dy^n. && (4.5)\\
A &= A_m(y)dy^m, && (4.6)\\
\phi &= \phi(y), && (4.7)
\end{aligned}
$$

where g_{mn} is the metric of a 2-dimensional compact internal manifold K_2; so the 6-dimensional space-time manifold is $(Minkowski)_4 \times K_2$. By using (4.5), (4.6) and (4.7), we can write the bosonic EOM in the following form:

$$
\begin{aligned}
&\nabla^2\phi - m^2\phi - 2\xi(\phi^*\phi)\phi = 0,\\
&\nabla_m F^{mn} + ie\left[\phi^*\nabla^n\phi - (\nabla^n\phi)^*\phi\right] = 0,\\
&\frac{1}{\kappa^2}R_{mn} - \frac{1}{2}F_{mp}F_n^{\ p} - \frac{1}{2}(\nabla_m\phi)^*\nabla_n\phi - \frac{1}{2}(\nabla_n\phi)^*\nabla_m\phi = 0,\\
&\frac{1}{4}F^2 - \lambda - m^2\phi^*\phi - \xi(\phi^*\phi)^2 = 0, \qquad (4.8)
\end{aligned}
$$

where $\nabla^2 \equiv \nabla_m\nabla^m$ is the covariant laplacian over the internal manifold. The equations (4.8) must be satisfied by the bosonic VEV.

We introduce also fermions and gauge invariant coupling with the scalar ϕ. In order to do that it's necessary to introduce at least a pair of 6-dimensional Weyl spinors ψ_+ and ψ_-, where $+$ and $-$ refer here to the 6-dimensional chirality. We consider the following fermionic action:

$$
S_F = \int d^6X\sqrt{-G}\left(\overline{\psi_+}\Gamma^M\nabla_M\psi_+ + \overline{\psi_-}\Gamma^M\nabla_M\psi_- + g_Y\phi^*\overline{\psi_+}\psi_- + g_Y\phi\overline{\psi_-}\psi_+\right), \qquad (4.9)
$$

where g_Y is a real Yukawa coupling constant. In (4.9) ∇_M represents the covariant derivative acting on spinor, which includes the gauge and the spin connection; moreover our conventions for Γ^M are given in appendix A. The $U(1)$ charge e_+ and e_- of ψ_+ and ψ_- have to satisfy the condition $e_- = e_+ + e$ coming from the gauge invariance of the Yukawa terms. In the following we consider the choice $e_+ = e/2$ and $e_- = 3e/2$, corresponding to a simple harmonic expansion for the compactification over $(Minkowski)_4 \times S^2$. From (4.9) we get the following EOM:

$$
\Gamma^M\nabla_M\psi_+ + g_Y\phi^*\psi_- = 0, \quad \Gamma^M\nabla_M\psi_- + g_Y\phi\psi_+ = 0. \qquad (4.10)
$$

Now we define the following 4-dimensional Weyl spinors:

$$
\psi_{\pm L} = \frac{1-\gamma^5}{2}\psi_\pm, \quad \psi_{\pm R} = \frac{1+\gamma^5}{2}\psi_\pm, \qquad (4.11)
$$

where γ^5 is the 4-dimensional chirality matrix. In terms of $\psi_{\pm L}$ and $\psi_{\pm R}$ the EOM, for a $(Minkowski)_4 \times K_2$ background space-time, are[14]

[14] We rearrange the equations in a way that the left handed and right handed sector are split.

$$
\begin{aligned}
\left(\partial^2 + 2\nabla_+\nabla_- - g_Y^2|\phi|^2\right)\psi_{+L} - \sqrt{2}g_Y\left(\nabla_+\phi^*\right)\psi_{-L} = 0,\\
\left(\partial^2 + 2\nabla_-\nabla_+ - g_Y^2|\phi|^2\right)\psi_{-L} - \sqrt{2}g_Y\left(\nabla_-\phi\right)\psi_{+L} = 0,\\
\left(\partial^2 + 2\nabla_-\nabla_+ - g_Y^2|\phi|^2\right)\psi_{+R} + \sqrt{2}g_Y\left(\nabla_-\phi^*\right)\psi_{-R} = 0,\\
\left(\partial^2 + 2\nabla_+\nabla_- - g_Y^2|\phi|^2\right)\psi_{-R} + \sqrt{2}g_Y\left(\nabla_+\phi\right)\psi_{+R} = 0,
\end{aligned}
\tag{4.12}
$$

where $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$,

$$
\nabla_\pm = \frac{1}{\sqrt{2}}(\nabla_5 \pm i\nabla_6) \tag{4.13}
$$

and $\nabla_{5,6}$ are the covariant derivative components in an orthonormal basis. The equations (4.12) will be used in order to compute the fermionic spectrum.

4.1 The $SU(2)\times U(1)$ Background Solution

An $SU(2)\times U(1)$-invariant solution of (4.8) is [5]

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + a^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), && (4.14)\\
A &= \frac{n}{2e}(\cos\theta - 1)d\varphi \equiv -\frac{n}{2e}e^3(y), && (4.15)\\
\phi &= 0, && (4.16)
\end{aligned}
$$

subject to the constraints

$$
\lambda = \frac{n^2}{8e^2a^4} = \frac{1}{\kappa^2 a^2}, \tag{4.17}
$$

where n is a (integer) monopole number. So here we have $K_2 = S^2$, and a is the radius of S^2. We introduce also an orthonormal basis in the internal cotangent space [5]:

$$
e^\pm(y) = \pm\frac{i}{\sqrt{2}}e^{\pm i\varphi}\left(d\theta \pm i\sin\theta d\varphi\right). \tag{4.18}
$$

In the following we consider, just for simplicity, the case

$$
n = 2. \tag{4.19}
$$

In fact for this value of the monopole charge we can find a very simple solution of the fundamental 6-dimensional EOMs (4.8) which is invariant under a $U(1)$ subgroup of $SU(2)\times U(1)$; this solution is discussed in section 5. Like in section 3 our purpose is in fact to construct the 4-dimensional $SU(2)\times U(1)$-invariant effective theory, study the spontaneous symmetry breaking $SU(2)\times U(1) \to U(1)$ and the Higgs mechanism in the effective theory and then compare the results with the corresponding quantities predicted by the 6-dimensional theory; therefore, in order to do that, one has to find a 6-dimensional U(1)-invariant solution of the EOMs.

If Φ_λ is a field with an integer or half-integer iso-helicity λ, we can perform an harmonic expansion [5]:

$$\Phi_\lambda(x,\theta,\phi) = \sum_{l\geq|\lambda|}\sum_{|m|\leq l} \Phi^l_m(x)\sqrt{\frac{2l+1}{4\pi}}\mathcal{D}^{(l)\lambda}_m(\theta,\varphi), \tag{4.20}$$

where, for a given l, $\mathcal{D}^{(l)\lambda}_m$ is a $(2l+1)\times(2l+1)$ unitary matrix. The $\mathcal{D}^{(l)\lambda}_m$ were originally introduced in [10] and our conventions are given in appendix A. For example ϕ has an expansion like (4.20) with $\lambda = 1$.

The low energy 4-dimensional spectrum coming from the background (4.14), (4.15) and (4.16) is given in the reference [5] for the spin-1 and spin-2 sectors. The massless sector is the following: there are a graviton (helicities ± 2, $l = 0$), a $U(1)$ gauge field (helicities ± 1, $l = 0$) coming from $\mathcal{V}_\mu$ and a Yang-Mills $SU(2)$ triplet (helicities ± 1, $l = 1$) coming from $h_{\mu\alpha}$ and $\mathcal{V}_\mu$, where $\mathcal{V}_M$ and h_{MN} are the fluctuations of the gauge field and the metric around the solution (4.14), (4.15) and (4.16). Regarding the scalar spectrum all the scalars from G_{MN} and A_M have very large masses, of the order $1/a$, and we can get only an $SU(2)$-triplet from ϕ in the low energy spectrum if we choose m^2 such that

$$|\mu^2| \ll \frac{1}{a^2}, \tag{4.21}$$

where

$$\mu^2 \equiv -\frac{1}{a^2}\eta \equiv m^2 + \frac{1}{a^2}. \tag{4.22}$$

In fact $-1/a^2$ is the eigenvalue of the laplacian operator acting on the harmonic with $l = 1$ and $\lambda = 1$, as one can check using the related formula of [5]. The parameter μ^2 is in fact the squared mass of the triplet from ϕ, and it can be in principle either positive or negative. If (4.21) holds all the remaining scalars have masses at least of the order $1/a$ and they don't appear in the low energy theory. So we assume that (4.21) holds. Finally in order to find the low energy fermionic spectrum we have to calculate the associated iso-helicities by using the explicit expression for the background covariant derivative of $\psi_\pm$ along the internal space:

$$\nabla_m\psi_\pm = \left(\partial_m \pm \omega_m\frac{1}{2}\gamma^5 + ie_\pm A_m\right)\psi_\pm, \tag{4.23}$$

where $\omega_\theta = 0$, $\omega_\varphi = \frac{i}{a}(\cos\theta - 1)$, $e_+ = e/2$ and $e_- = 3e/2$. We get

$$\lambda_{+L} = 0, \quad \lambda_{+R} = 1, \quad \lambda_{-L} = 2, \quad \lambda_{-R} = 1 \tag{4.24}$$

and the corresponding expansions are given by (4.20). So the equations (4.12) tell us that there are 4 zero-modes: the $l = 0$, $m = 0$ mode in ψ_{+L} and the $l = 1$, $m = +1, -1, 0$ in ψ_{-R}. So we have a massless $SU(2)$ singlet from ψ_{+L} and a massless $SU(2)$ triplet from ψ_{-R}.

4.2 The 4D $SU(2)\times U(1)$ Effective Lagrangian and the Higgs Mechanism

Now we want to study the 4D effective theory: which is the 4-dimensional theory obtained from the background (4.14), (4.15) and (4.16) retaining only the low energy spectrum we discussed at the end of subsection 4.1, that is the particles with masses much smaller than $1/a$, and integrating out all the heavy modes, namely those with mass at least of the order $1/a$. This is an $SU(2)\times U(1)$-invariant theory, which includes a charged scalar, that we call χ, in the 3-dimensional representation of $SU(2)$, and, if we want, two Weyl spinors in the $1_{1/2}$ and $3_{3/2}$ of $SU(2)\times U(1)$. The background (4.14), (4.15) and (4.16) is the analogous of what we called Φ_{eff} in section 3. In this section we give only some relevant terms[15] appearing in the lagrangian of this theory. In particular we calculate the scalar potential, we study the Higgs mechanism, which is active only for $\mu^2<0$, and we give in this case the masses of the spin-1, spin-0 and spin-1/2 particles.

Like in the general scalar theory of section 3, in the following we perform all the calculations at the order η. If we use the information regarding the low-energy spectrum which we discussed at the end of subsection 4.1, we can construct some relevant terms of the 4D effective theory through the following ansatz[16]

$$\begin{aligned}
E^a(x) &= E^a_\mu(x)dx^\mu,\\
E^\alpha(x,y) &= e^\alpha(y)-\frac{\kappa}{a\sqrt{4\pi}}W^{\hat{\alpha}}_\mu(x)dx^\mu\mathcal{D}^\alpha_{\hat{\alpha}}(y),\\
A(x,y) &= -\frac{n}{2ea}e^3(y)\\
&\quad +\frac{1}{a\sqrt{4\pi}}V_\mu(x)dx^\mu-\frac{n\kappa}{2ea^2\sqrt{4\pi}}U^{\hat{\alpha}}_\mu(x)dx^\mu\mathcal{D}^3_{\hat{\alpha}}(y),\\
\phi(x,y) &= \frac{1}{a}\sqrt{\frac{3}{4\pi}}\chi^{\hat{\alpha}}(x)\mathcal{D}_{-,\hat{\alpha}}(y),\\
\psi_{+R} &= \psi_{-L}=0,\\
\psi_{-R} &= \frac{1}{a}\sqrt{\frac{3}{4\pi}}\psi^{\hat{\alpha}}_R(x)\mathcal{D}_{-,\hat{\alpha}}(y),\\
\psi_{+L} &= \frac{1}{a\sqrt{4\pi}}\psi_L(x),
\end{aligned}\tag{4.25}$$

where E^A, $A=0,1,2,3,+,-$, are the 6-dimensional orthonormal 1-form basis, E^a_μ is the 4-dimensional vielbein, V_μ is the 4-dimensional $U(1)$ gauge field coming from $\mathcal{V}_\mu$, a linear combination[17] of W_μ and U_μ is the Yang-Mills $SU(2)$ triplet [5] coming from $h_{\mu\alpha}$ and $\mathcal{V}_\mu$; finally ψ_L and ψ_R are the $SU(2)$ fermion singlet and fermion triplet, respectively. Actually the ansatz (4.25) is the background (4.14), (4.15) and (4.16) plus some fluctuations, which include all the light KK states.

Now we want to write some relevant terms of the effective lagrangian for χ by using the light-mode ansatz (4.25) and by taking into account the heavy modes contribution.

[15] Here "relevant terms" has the same meaning as in the subsection 3.1.
[16] The ansatz (4.25) is a generalization of the zero-mode ansatz of [5], which doesn't include scalar fields.
[17] The orthogonal linear combination has a large mass; we show this in appendix B.3.

The scalar potential $\mathcal{U}$ in the 4D effective theory, including the bilinears and the quartic interactions, is

$$\mathcal{U}(\chi) = \mu^2 \chi^\dagger \chi + (\lambda_H + c_1 \lambda_G) \left(\chi^\dagger \chi\right)^2 - \frac{\lambda_H + c_2 \lambda_G}{3} \left|\chi^{\hat{\alpha}} g_{\hat{\alpha}\hat{\beta}} \chi^{\hat{\beta}}\right|^2 + ..., \tag{4.26}$$

where c_1 and c_2 are dimensionless parameters,

$$\lambda_H \equiv \frac{9}{20\pi a^2}\xi, \qquad \lambda_G \equiv \frac{9\kappa^2}{80\pi a^4} \tag{4.27}$$

and the dots represent higher order non relevant terms, for example terms with a product of 6 χ or 8 χ. These terms don't contribute to the VEV of χ as we want this VEV to be of the order[18] $\eta^{1/2}$. In (4.26) we took into account that the quartic part of $\mathcal{U}$ comes from the quartic term in the 6-dimensional potential V in (4.4) and from the contribution of the heavy scalars, namely $h_{\alpha\beta}$ and $\mathcal{V}_\alpha$, through diagrams like figure 1. The latter contribution is represented by $c_1\lambda_G$ and $c_2\lambda_G$, the analogous of a_{lmpq} in the equation (3.10). Moreover we give also the expression for the gauge covariant derivative of χ:

$$D_\mu \chi^{\hat{\alpha}} = \partial_\mu \chi^{\hat{\alpha}} + i g_1 V_\mu \chi^{\hat{\alpha}} + g_2 \mathcal{A}_\mu^{\hat{\beta}} \epsilon_{\hat{\beta}\hat{\gamma}}^{\ \ \hat{\alpha}} \chi^{\hat{\gamma}}, \tag{4.28}$$

where $\mathcal{A}_\mu$ is defined in appendix B.3 and it represents the $SU(2)$ Yang-Mills field, $\epsilon_{\hat{\gamma}\hat{\beta}\hat{\alpha}}$ is a totally antisymmetric symbol with $\epsilon_{+-3} = i$, and

$$g_1 = \frac{e}{\sqrt{4\pi}a}, \quad g_2 = \sqrt{\frac{3}{16\pi}}\frac{\kappa}{a^2}, \tag{4.29}$$

are the 4-dimensional $U(1)$ and $SU(2)$ gauge couplings.

Therefore the complete lagrangian for χ is

$$\mathcal{L}_{\chi eff} = -\left(D_\mu \chi\right)^\dagger D^\mu \chi - \mathcal{U}(\chi). \tag{4.30}$$

Let's look for the points of minimum of the order $\eta^{1/2}$ of the potential $\mathcal{U}$ in (4.26). We have a minimum, in the case $\mu^2 < 0$, for

$$\chi_1 = \chi_2 = 0, \quad \chi_3 = v \equiv \sqrt{\frac{-3\mu^2}{4\left[\lambda_H + \frac{1}{2}\left(3c_1 - c_2\right)\lambda_G\right]}}, \tag{4.31}$$

which corresponds to the global minimum

$$\mathcal{U}_0 = 0 \tag{4.32}$$

at the order η. This fact states that, at leading order, the 4-dimensional flatness condition in the background is compatible with the procedure of the 4D effective theory. In fact $\mathcal{U}_0$ can be interpreted as a 4-dimensional cosmological constant and the flatness implies $\mathcal{U}_0 = 0$.

[18]The order $\eta^{1/2}$ corresponds to the order μ because of equation (4.22).

Instead for $\mu^2 > 0$ we don't have any order parameter because the global minimum $\mathcal{U}_0 = 0$ corresponds to $\chi = 0$.

If we take, for $\mu^2 < 0$, the vacuum (4.31), $SU(2) \times U(1)$ breaks to $U(1)_3$, where $U(1)_3$ is the $U(1)$-subgroup of $SU(2)$ generated by its third generator. The gauge field of $U(1)$ and $SU(2)$ are respectively V_μ and $\mathcal{A}_\mu$; before Higgs mechanism these gauge field are of course massless as one can see by looking at their bilinear lagrangian given in appendix B.3. From (4.30) and (4.28) we can calculate the masses of these vector fields in the 4D effective theory after the Higgs mechanism. We get a massless vector field $\mathcal{A}^3_\mu$, which corresponds to the unbroken $U(1)_3$ gauge symmetry. Instead V_μ and $\mathcal{A}^\pm_\mu$ acquire respectively the following squared masses

$$M_V^2 = \frac{3e^2}{8\pi a^2} \frac{-\mu^2}{\lambda_H + \frac{1}{2}(3c_1 - c_2)\lambda_G}, \tag{4.33}$$

$$M_{V\pm}^2 = \frac{9e^2}{16\pi a^2} \frac{-\mu^2}{\lambda_H + \frac{1}{2}(3c_1 - c_2)\lambda_G}, \tag{4.34}$$

where the subscript V indicates that we're dealing with vector particles. Moreover, in the spin-0 sector, we have two physical scalar fields: a real scalar and a complex one, which is charged under the residual $U(1)_3$ symmetry. Their squared masses are respectively

$$M_S^2 = -2\mu^2, \tag{4.35}$$

$$M_{S\pm}^2 = -\mu^2 \frac{\lambda_H + c_2\lambda_G}{\lambda_H + \frac{1}{2}(3c_1 - c_2)\lambda_G}. \tag{4.36}$$

Finally we can determine the fermionic spectrum by examining the fermionic lagrangian in the effective theory:

$$\mathcal{L}_{Feff} = \overline{\psi_L}\gamma^\mu D_\mu \psi_L + \overline{\psi_R}\gamma^\mu D_\mu \psi_R + g_4 \overline{\psi_L}\chi^\dagger \psi_R + g_4 \overline{\psi_R}\chi \psi_L, \tag{4.37}$$

where

$$g_4 = \frac{g_Y}{a\sqrt{4\pi}}. \tag{4.38}$$

The result is a neutral Dirac fermion, with squared mass

$$M_F^2 = \frac{3g_Y^2}{16\pi a^2} \frac{-\mu^2}{\lambda_H + \frac{1}{2}(3c_1 - c_2)\lambda_G}, \tag{4.39}$$

and a pair of massless right-handed Weyl fermions. We observe that the mass spectrum that we gave here is parametrized by the c_i. Of course these constants are not free parameters but they can be in principle computed by evaluating explicitly the heavy modes contribution. In the rest of this paper we don't compute the c_i but we prove that the 4D effective theory without heavy modes contribution, that is $c_i = 0$, is not correct because it predicts a wrong VEV of the light KK scalars and a wrong mass spectrum.

5 Symmetry Breaking in the 6D Theory

Now we perform a 6-dimensional (or geometrical) analysis of spontaneous symmetry breaking: this method corresponds to the contents of section 3 for scalar theories. Of course we perform all the calculations at the order η, as in the effective theory method.

The most simple solution, up to higher order terms in η, that we find is similar to the background which appears in the reference [11] [19] :

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + a^2\left[(1+|\eta|\beta\sin^2\theta)d\theta^2 + \sin^2\theta d\varphi^2\right], \\
A &= -\frac{1}{e}e^3, \\
\phi &= \eta^{1/2}\alpha\exp(i\varphi)\sin\theta,
\end{aligned}
\tag{5.1}
$$

where $\beta \equiv \kappa^2|\alpha|^2$. As required, for $\eta = 0$ this background reduces to the background of subsection 4.1. The value of ϕ in (5.1) is proportional to the harmonic $\mathcal{D}_{-,0}$, that is the harmonic with $l = 1$, $\lambda = 1$ and $m = 0$. In order that (5.1) is a solution, up to $O(\eta^{3/2})$, it is necessary that (4.17) holds and $|\alpha|^2$ is given by the following equation:

$$
\frac{1}{a^2}\eta\int D^*\phi + \int D^*L_2\phi = 2\xi\int D^*|\phi|^2\phi, \tag{5.2}
$$

where D is $\mathcal{D}_{-,0}$ and $L_2\phi$ is the function proportional to $\eta^{3/2}$ in the expansion of $\nabla^2\phi$ in powers of $\eta^{1/2}$. Further in (5.2) the integrals are performed with the round S^2 measure. The equation (5.2) can be derived by putting (5.1) in the Klein-Gordon equation. For $\mu^2 < 0$ the equation (5.2) has a solution for

$$
\lambda_H > \lambda_G, \tag{5.3}
$$

where λ_H and λ_G are defined by (4.27), while, for $\mu^2 > 0$, we have a solution for

$$
\lambda_H < \lambda_G. \tag{5.4}
$$

Whether $\mu^2 > 0$ or $\mu^2 < 0$, the solution of (5.2) is

$$
|\alpha|^2 = \frac{5}{|8\xi a^2 - 2\kappa^2|} = \frac{9}{32\pi a^4}\frac{1}{|\lambda_H - \lambda_G|}. \tag{5.5}
$$

Note that here we have symmetry breaking for both signs of μ^2. This is not so interesting because the solution with $\mu^2 > 0$ is unstable, as it is discussed in subsection 5.2. We want to stress that the value of $|\alpha|^2$ predicted by the 4D effective theory is not equal to (5.5) if we neglect the heavy modes contribution to the effective theory, namely for $c_i = 0$: indeed in this case the effective theory predicts a value of $|\alpha|^2$ equal to

$$
|\alpha|^2_{eff} = \frac{9}{32\pi a^4}\frac{1}{\lambda_H}, \tag{5.6}
$$

[19] This solution was discussed in reference [11], but incorrectly.

which is equal to (5.5) only for $\lambda_G = 0$. However from (4.27) it's clear that λ_G cannot be taken equal to zero. Therefore we have already proved that the heavy modes contribution is needed at least for the light mode VEV. We shall prove that this is the case also for the mass spectrum.

As required the background (5.1) has the symmetry

$$U(1)_3 \subset SU(2). \tag{5.7}$$

So the 4-dimensional effective low energy theory, which follows from this background, is $U(1)_3$-invariant and comparing these results with the effective theory predictions makes sense.

We note that the symmetry breaking (5.7) is associated, in the 6-dimensional theory, to a geometrical deformation of the internal space. Further we observe that (5.1) tell us the heavy modes VEVs are higher order corrections with respect to the light modes VEVs like in the scalar theories of section 3.

Now we calculate the low energy vector, scalar and fermion spectrum by analyzing the 4-dimensional bilinear lagrangian for the fluctuations around the solution (5.1).

5.1 Spin-1 Spectrum

The spin-1 spectrum can be calculated in a way similar to the light mode ansatz (4.25). However it must be noted that the sectors with different l no longer decouple for $\eta \neq 0$, but the mixing terms are of the order η and they give negligible corrections of the order η^2 to the vector boson masses. These facts are evident from the general formula of [12]. So we can neglect the modes with $l > 1$ in the calculation of spin-1 spectrum. Therefore we can compute the vector boson masses by putting the following ansatz in the action and integrating over the extra dimensions:

$$\begin{aligned}
E^a(x) &= E^a_\mu(x)dx^\mu, \\
E^\alpha(x,y) &= e^\alpha(y,\eta) - \frac{\kappa}{a\sqrt{4\pi}} W^{\hat\alpha}_\mu(x)dx^\mu \mathcal{D}^\alpha_{\hat\alpha}(y), \\
A(x,y) &= -\frac{1}{ea}e^3(y) \\
&\quad + \frac{1}{a\sqrt{4\pi}} V_\mu(x)dx^\mu - \frac{\kappa}{ea^2\sqrt{4\pi}} U^{\hat\alpha}_\mu(x)dx^\mu \mathcal{D}^3_{\hat\alpha}(y), \\
\phi(x,y) &= \eta^{1/2}\alpha \exp(i\varphi)\sin\theta,
\end{aligned} \tag{5.8}$$

where $e^\alpha(y,\eta)$ is the orthonormal basis for the 2-dimensional metric in (5.1): :

$$e^{\pm}(y,\eta) = \pm\frac{i}{\sqrt{2}}e^{\pm i\varphi}\left[\left(1 + |\eta|\frac{\beta}{2}\sin^2\theta\right)d\theta \pm i\sin\theta d\varphi\right]. \tag{5.9}$$

In (5.8) we consider the spin-1 fluctuations but we don't consider the spin-0 fluctuations, because they are not necessary for the calculation of vector boson masses. It's important to note that in (5.8) the VEV of E^α is $e^\alpha(y,\eta)$, it's not $e^\alpha(y)$ as in (4.25).

From (5.8) it follows that some of the previous ($\eta = 0$) massless states acquire masses for $\eta \neq 0$. Up to $O(\eta^{3/2})$, the $U(1)$ gauge boson ($l = 0$) has the mass squared

$$M_V^2 = \eta \frac{20}{3} \frac{e^2}{8\xi a^2 - 2\kappa^2} = \frac{3e^2}{8\pi a^2} \frac{-\mu^2}{\lambda_H - \lambda_G}, \tag{5.10}$$

while the Yang-Mills triplet $\mathcal{A}$ ($l = 1$) is separated in a massless gauge boson, which is associated to $U(1)_3$ gauge invariance, and a couple of massive vector fields with the same mass squared

$$M_{V\pm}^2 = \eta \frac{10e^2}{8\xi a^2 - 2\kappa^2} = \frac{9e^2}{16\pi a^2} \frac{-\mu^2}{\lambda_H - \lambda_G}. \tag{5.11}$$

By comparing (5.10) and (5.11) with (4.33) and (4.34), we get that the heavy modes contribution is needed in the effective theory. However we observe that the ratio $M_V^2/M_{V\pm}^2$ is correctly predicted by the 4D effective theory for every c_i.

Since the computation of vector bosons masses is complicated we present it explicitly. In order to prove (5.10) and (5.11) it's useful to split the action in four terms:

$$S_B = S_R + S_F + S_\lambda + S_\phi, \tag{5.12}$$

where

$$\begin{aligned} S_R &= \int d^6X \sqrt{-G} \frac{1}{\kappa^2} R, &(5.13)\\ S_F &= -\frac{1}{4} \int d^6X \sqrt{-G} F^2, &(5.14)\\ S_\lambda &= \int d^6X \sqrt{-G}\,(-\lambda)\,, &(5.15)\\ S_\phi &= \int d^6X \sqrt{-G} \left[-\left(\nabla_M \phi\right)^* \nabla^M \phi - V(\phi) \right]. &(5.16) \end{aligned}$$

In appendix B we prove that the contributions coming from S_R and S_F vanish, so only S_ϕ contributes to the spin-1 masses up to $O(\eta^{3/2})$. The same low-energy spin-1 masses in (5.10) and (5.11) can be obtained also by using the general formula of [12], which contains all the bilinear terms in the light cone gauge. The light cone gauge advantage is that the sectors with different spin decouple. However the derivation that we presented here shows that the unique contribution (at the leading order) to the spin-1 masses comes from S_ϕ, like in the effective theory approach. This explains why the ratio $M_V^2/M_{V\pm}^2$ is correctly predicted by the 4D effective theory for every values of c_i.

5.2 Spin-0 Spectrum

We choose the light cone gauge [12, 13] in order to evaluate the spin-0 spectrum. In this gauge we have just two independent values for the indexes $\mu, \nu, ...$ which label the 4-dimensional coordinates. The bilinears for the fluctuations over the solution (5.1) can be simply computed with the general formula of [12]. For our model the helicity-0 $\mathcal{L}_0$ part is given by

$$\mathcal{L}_0 = \mathcal{L}_0(\phi,\phi) + \mathcal{L}_0(h,h) + \mathcal{L}_0(\mathcal{V},\mathcal{V}) + \mathcal{L}_0(\phi,h) + \mathcal{L}_0(\phi,\mathcal{V}) + \mathcal{L}_0(h,\mathcal{V}), \tag{5.17}$$

where

$$\begin{aligned}
\mathcal{L}_0(\phi,\phi) &= \phi^*\partial^2\phi + \phi^*\nabla^2\phi - \Big[m^2 + (4\xi+e^2)|\Phi|^2 + \kappa^2\left(\nabla_m\Phi\right)^*\nabla^m\Phi\Big]\,|\phi|^2 \\
&\quad -\frac{1}{2}\left\{\Big[(2\xi-e^2)\left(\Phi^*\right)^2 + \kappa^2\left(\nabla_m\Phi\nabla^m\Phi\right)^*\Big]\,\phi^2 + c.\,c.\right\}, \qquad (5.18)\\
\mathcal{L}_0(h,h) &= \frac{1}{4\kappa^2}\left\{h_{mn}\partial^2 h^{mn} + h_{mn}\nabla^2 h^{mn} + 2R_{mn}{}^{kl}h^m_l h^n_k\right. \\
&\quad +\kappa^2 h_{ks}h_{mn}F^{km}F^{sn} - 2\kappa^2 h^l_m h_{ln}\left[\frac{1}{2}F^m{}_k F^{nk} + \left(\nabla^m\Phi\right)^*\nabla^n\Phi\right] \\
&\quad \left.+\frac{1}{2}h^i_i\partial^2 h^j_j + \frac{1}{2}h^i_i\nabla^2 h^j_j\right\}, \qquad (5.19)\\
\mathcal{L}_0(\mathcal{V},\mathcal{V}) &= \frac{1}{2}\left\{\mathcal{V}_m\partial^2\mathcal{V}^m + \mathcal{V}_m\nabla^2\mathcal{V}^m - R_{mn}\mathcal{V}^m\mathcal{V}^n\right. \\
&\quad \left.-2e^2|\Phi|^2\mathcal{V}^m\mathcal{V}_m - \kappa^2\left(F_{ml}\mathcal{V}^l\right)^2\right\}, \qquad (5.20)\\
\mathcal{L}_0(\phi,h) &= \nabla_l h^{lm}\phi^*\nabla_m\Phi + h^{mn}\left(\nabla_m\phi\right)^*\nabla_n\Phi + c.\,c.\,, \qquad (5.21)\\
\mathcal{L}_0(\phi,\mathcal{V}) &= 2ie\mathcal{V}^m\phi^*\nabla_m\Phi - \kappa^2 F^{lm}\mathcal{V}_m\phi^*\nabla_l\Phi + c.\,c.\,, \qquad (5.22)\\
\mathcal{L}_0(h,\mathcal{V}) &= \mathcal{V}^n\left(\nabla_m h_{ln}F^{lm} - h^m_l\nabla_m F^l{}_n\right), \qquad (5.23)
\end{aligned}$$

where Φ and ϕ are the background and the fluctuation of the 6-dimensional scalar. In this vanishing-helicity sector, it turns out that we have not only mixing terms of the order η but also mixing terms of the order $\eta^{1/2}$, coming from $\mathcal{L}_0(\phi,h)$ and $\mathcal{L}_0(\phi,\mathcal{V})$. So now we can't neglect the mixing between the sectors with different values of l, as we did in the helicity ±1 sector. If we integrate these bilinears over the extra-dimensions we get an infinite dimensional squared mass matrix. However we are interested only in the light masses, therefore we can use the perturbation theory of quantum mechanics in order to extract the correction of the order η to the masses of the 6 real scalars which are massless for $\eta = 0$. We already used this method for the computation of the mass spectrum in the scalar theories of section 3. We explain now how to use it in this framework.

Formally we can write the bilinears $\mathcal{L}_0$ of the scalar fields in this way

$$\mathcal{L}_0 = \frac{1}{2}S^\dagger\partial^2 S - \frac{1}{2}S^\dagger\mathcal{O}S, \qquad (5.24)$$

where S is an array which includes all the scalar fluctuations; we choose

$$S = \begin{pmatrix} \phi \\ \phi^* \\ h_{++} \\ h_{--} \\ h_{+-} \\ \mathcal{V}_+ \\ \mathcal{V}_- \end{pmatrix}. \qquad (5.25)$$

We have just to solve a 2-dimensional eigenvalue problem for the squared mass operator[20] $\mathcal{O}$:

$$\mathcal{O}S = M^2 S. \tag{5.26}$$

In particular we want to find the 6 values of M^2 which go to zero as η goes to zero. Since we're working at the order η we decompose $\mathcal{O}$ as follows

$$\mathcal{O} = \mathcal{O}_0 + \mathcal{O}_1 + \mathcal{O}_2, \tag{5.27}$$

where $\mathcal{O}_0$ doesn't depend on η, $\mathcal{O}_1$ is proportional to $\eta^{1/2}$ and $\mathcal{O}_2$ is proportional to η. From the perturbation theory of quantum mechanics in the degenerate case we know that the 6 values of M^2 we are interested in are the eigenvalues of the following 6×6 matrix[21]:

$$M^2_{ij} = -\sum_{\tilde{i}} \frac{< i|\mathcal{O}_1|\tilde{i} >< \tilde{i}|\mathcal{O}_1|j >}{M^2_{\tilde{i}}} + < i|\mathcal{O}_2|j >, \tag{5.28}$$

where $|i >$, $i = 1, ...6$ represent the 6 orthonormal eigenfunctions of $\mathcal{O}_0$ with vanishing eigenvalue and they have the form

$$|i >= \begin{pmatrix} \phi \\ \phi^* \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}. \tag{5.29}$$

Moreover $|\tilde{i} >$ are all the remaining orthonormal eigenfunctions of $\mathcal{O}_0$ and $M^2_{\tilde{i}}$ the corresponding eigenvalues. We note that the matrix elements $< i|\mathcal{O}_1|\tilde{i} >$ are non vanishing for

$$|\tilde{i} >= \begin{pmatrix} 0 \\ 0 \\ h_{++} \\ h_{--} \\ h_{+-} \\ \mathcal{V}_+ \\ \mathcal{V}_- \end{pmatrix}. \tag{5.30}$$

Further the operator $\mathcal{O}_1$ modifies the integration measure just by a factor proportional to the harmonics $\mathcal{D}^{(1)}$, therefore we need just a finite subset of $|\tilde{i} >$ for the evaluation of M^2_{ij}, namely those constructed through the harmonics with $l = 0, 1, 2$, which are given in appendix A. An explicit form for $|i >$ and $|\tilde{i} >$, and the preliminary computations of the 6 eigenvalues we are interested in, are given in appendix C.

[20]The matrix elements of $\mathcal{O}$ can be computed by comparing (5.24) with the explicit expression of $\mathcal{L}_0$.

[21]Like in section 3 we use the Dirac notation; for two states $|S_1 >$ and $|S_2 >$ and for an operator A, $< S_1|A|S_2 >$ represents $\int S_1^\dagger A S_2$, where the integral is performed with the round S^2 metric.

We give here just the final result: we have two unphysical scalar fields (a real and a complex one) which form the helicity-0 component of the massive vector fields; they have in fact the same squared masses given in (5.10) and (5.11), as it's required by Lorentz invariance, which is not manifest in the light cone gauge. Then we have a physical real scalar and a physical complex scalar, charged under the residual U(1) symmetry, with squared masses given respectively by (for $\mu^2 < 0$)

$$\begin{aligned} M_S^2 &= -2\mu^2, \\ M_{S\pm}^2 &= -\mu^2 \frac{\lambda_H + \lambda_G}{\lambda_H - \lambda_G}. \end{aligned} \tag{5.31}$$

For $\mu^2 > 0$, we get a negative value for M_S^2, therefore the corresponding solution is unstable. Note that the squared mass M_S^2 has exactly the same expression as in the 4D effective theory, for every c_i. But for $c_i = 0$, which corresponds to neglecting the heavy modes contribution, the effective theory prediction for $M_{S\pm}^2$ in (4.36) is not equal to the correct value (5.31). We note that this is a physical inequivalence because the ratio $M_S^2/M_{S\pm}^2$, which is in principle a measurable quantity, is not correctly predicted by the 4D effective theory without the heavy modes contribution. More precisely the effective theory prediction for $M_S^2/M_{S\pm}^2$, in the case $c_i = 0$, is always greater than the correct value.

5.3 Spin-1/2 Spectrum

The spin-1/2 spectrum can be calculated by linearizing the equation of motion (4.12): for $n = 2$ we get

$$\begin{aligned} &\left(\partial^2 + 2\nabla_+\nabla_- - g_Y^2|\Phi|^2\right)\psi_{+L} = 0, \\ &\left(\partial^2 + 2\nabla_-\nabla_+ - g_Y^2|\Phi|^2\right)\psi_{-L} = 0, \\ &\left(\partial^2 + 2\nabla_-\nabla_+ - g_Y^2|\Phi|^2\right)\psi_{+R} + \sqrt{2}g_Y\left(\nabla_+\Phi\right)^*\psi_{-R} = 0, \\ &\left(\partial^2 + 2\nabla_+\nabla_- - g_Y^2|\Phi|^2\right)\psi_{-R} + \sqrt{2}g_Y\nabla_+\Phi\,\psi_{+R} = 0, \end{aligned} \tag{5.32}$$

where Φ represents again the background of the 6-dimensional scalar, namely the third line of (5.1), and the covariant derivatives are evaluated with the background metric and background gauge field given by the first and the second line of (5.1). These covariant derivatives are in the $\pm$ basis defined by (5.9) and it includes the modified spin connection when it acts on spinors:

$$\nabla_\alpha\psi_{\pm R} = e_\alpha^m(y,\eta)\left(\partial_m \pm \omega_m\frac{1}{2} + ie_\pm A_m\right)\psi_{\pm R}, \tag{5.33}$$

$$\nabla_\alpha\psi_{\pm L} = e_\alpha^m(y,\eta)\left(\partial_m \mp \omega_m\frac{1}{2} + ie_\pm A_m\right)\psi_{\pm L}, \tag{5.34}$$

where $\omega_\theta = 0$, $\omega_\varphi \equiv \omega_\varphi{}^+{}_+$ is given in equation (B.13) and the value of the charges $e_\pm$ and the iso-helicities[22] of the fermions are given at the end of subsection 4.1. There we give

[22] For $\eta \neq 0$ we adopt the same harmonic expansion as in the $\eta = 0$ case; this gives the correct result for the fermionic masses squared at the order η.

also the fermionic massless spectrum for $\eta = 0$: an $SU(2)$ singlet from ψ_{+L} and an $SU(2)$ triplet from ψ_{-R}.

From (5.32) it's clear that the left handed sector doesn't present mixing terms of the order $\eta^{1/2}$ but only of the order η. Therefore the calculation of the squared mass M_F^2 of the light fermion coming from ψ_{+L} is quite easy. The result is

$$M_F^2 = \frac{3g_Y^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H - \lambda_G}. \tag{5.35}$$

Instead the evaluation of the right-handed spectrum is complicated by the presence of mixing terms of the order $\eta^{1/2}$, as in the scalar sector. Therefore we use the perturbation theory of quantum mechanics also in the fermion right-handed sector. Formally we can write the eigenvalue equation for the mass squared operator $\mathcal{O}$ acting in the right-handed sector as follows

$$\mathcal{O}F_R = M^2 F_R, \tag{5.36}$$

where F_R is an array which includes both the right-handed fermions; we choose

$$F_R = \begin{pmatrix} \psi_{+R} \\ \psi_{-R} \end{pmatrix}. \tag{5.37}$$

One can easily compute $\mathcal{O}$ acting on F_R by performing the substitution $\partial^2 \to M^2$ in the last two equations of (5.32). Then we can proceed as in the scalar spectrum, performing the decomposition (5.27). However in this case the matrix $M_{\tilde{i}\tilde{j}}^2$ in (5.28) is a 3×3 matrix as the number of zero modes for $\eta = 0$ in the right-handed sector is 3. Like in the scalar spectrum we need only those $|\tilde{i}>$ vectors made of harmonics with $l \leq 2$, because the operator $\mathcal{O}_1$ modifies the integration measure just by a factor proportional to the harmonics $\mathcal{D}^{(1)}$. In appendix D we give an expression for the $|i>, i = 1, -1, 0$, vectors, for the $|\tilde{i}>$ vectors and the $M_{\tilde{i}}^2$ eigenvalues for the relevant values of l: $l = 1, 2$. Here we give the final result: the right-handed low energy spectrum has a pair of massless right-handed fermions as in the 4D effective theory, which have opposite charge under the residual $U(1)$ symmetry, and a massive right-handed fermion with the same squared mass given in (5.35). This right-handed fermion together with the massive left-handed fermion form a massive Dirac spinor with mass M_F.

Also in the fermionic sector we note that the heavy modes contribution is needed in order that the effective theory reproduces the correct 6-dimensional result; this sentence is evident if one compares the effective theory prediction (4.39) with the correct result (5.35).

6 Summary and Conclusions

The principal result of this paper is that the contribution of the heavy KK modes to the effective 4-dimensional action is necessary in order to reproduce the correct D-dimensional predictions concerning the light KK modes. We have calculated such a contribution for a class of scalar theories. However this result holds in a more general framework. In order to show this, we have studied a 6-dimensional gauge and gravitational theory which involves a

Squared Mass	4D Effective Theory	6D Theory
M_V^2	$\frac{3e^2}{8\pi a^2}\frac{-\mu^2}{\lambda_H}$	$\frac{3e^2}{8\pi a^2}\frac{-\mu^2}{\lambda_H-\lambda_G}$
$M_{V\pm}^2$	$\frac{9e^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H}$	$\frac{9e^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H-\lambda_G}$
M_S^2	$-2\mu^2$	$-2\mu^2$
$M_{S\pm}^2$	$-\mu^2$	$-\mu^2\frac{\lambda_H+\lambda_G}{\lambda_H-\lambda_G}$
M_F^2	$\frac{3g_Y^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H}$	$\frac{3g_Y^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H-\lambda_G}$
$M_{F\pm}^2$	0	0

Table 1: The spectra predicted by the 4D effective theory without heavy modes contribution ($c_i = 0$) and by the 6D theory.

complex scalar and, possibly, fermions. In particular we have considered the compactification over S^2, for a particular value of the monopole number ($n = 2$), and the construction of a 4D $SU(2) \times U(1)$ effective theory. The latter contains a scalar triplet of $SU(2)$ which, through an Higgs mechanism, gives masses to the vector, scalar and fermion fields. An explicit expressions for these masses and for the VEV of the scalar triplet was found at the leading order in the small mass ratio μ/M, where M is the lightest heavy mass. On the other hand, for $n = 2$, we found a simple perturbative solution of the fundamental 6-dimensional EOMs with the same symmetry of the 4D effective theory in the broken phase. This solution presents a deformation of the internal space S^2 to an ellipsoid, which has isometry group $U(1)$ instead of $SU(2)$. Moreover we computed the corresponding vector, scalar and fermion spectrum with quantum mechanics perturbation theory technique. We have demonstrated by direct calculation that these quantities, computed in the 6-dimensional approach, are equal to the corresponding predictions of the 4D effective theory only if the contribution of the heavy KK modes are taken into account. In table 1 we give the spectrum predicted by the 4D effective theory for $c_i = 0$, namely, without heavy KK modes contribution, and the low energy spectrum predicted by the 6-dimensional theory for the stable ($\mu^2 < 0$) solution, that we gave in the text. We observe that ratios of masses which involve only vector and fermion excitations are correctly predicted by the 4D effective theory even without the heavy KK modes contribution. But the ratios of masses which involve at least one scalar mode are not correctly predicted and the error is measured by λ_G/λ_H, where λ_G and λ_H are defined in equations (4.27). We can roughly estimate the magnitude of this disagreement: if we require g_1 and g_2 in (4.29) to be of the order of 1 and we consider also the relation between κ and the 4-dimensional Planck length κ_4

$$\frac{4\pi a^2}{\kappa^2} = \frac{1}{\kappa_4^2}, \tag{6.1}$$

we get that $\sqrt{\kappa}$, e and a are all of the order of κ_4. So roughly speaking the condition $\lambda_G/\lambda_H \ll 1$ becomes $\lambda_H \gg 1$, which is a strong coupling regime. Therefore we can't probably neglect the heavy KK modes contribution and believe in the perturbation theory of quantum field theory at the same time.

Finally we note that there's a value of c_1 and c_2 ($c_1 = -1/3$, $c_2 = 1$) such that the effective theory VEV and vector, scalar and fermion spectrum turn out to be correct, namely, they are equal to the corresponding quantities given in section 5. This is a sign of the equivalence between the geometrical approach, which involves the deformed internal space geometry, to the spontaneous symmetry breaking and the Higgs mechanism in the 4D effective theory. In particular the heavy KK modes contribution can be interpreted in a geometrical way as the internal space deformation of the 6-dimensional solution: in fact if we put $\beta = 0$ but we keep $\alpha \neq 0$ in (5.1), which corresponds to neglecting the S^2 deformation, we get exactly the VEV and the spectrum predicted by the 4D effective theory without heavy KK modes contribution.

Possible uses of our work can be its extension to the case which resembles more the standard electro-weak theory. The latter could be for instance the 6D gauge and gravitational theory of this paper, compactified over S^2 but with monopole number $n = 1$; in this case we have in fact an Higgs doublet in the 4D effective theory. Other interesting applications could be models without fundamental scalars, which, in some sense, geometrize the Higgs mechanism or the context of supersymmetric version of 6D gauge and gravitational theories. Such supersymmetric theories have been recently investigated in connection with attempts to find a solution to the cosmological dark energy problem, a summary of which can be found in [14].

Acknowledgments. This work was supported in part by the Swiss Science Foundation. A.S. is appreciative of the hospitality at the IPT of Lausanne and the support by INFN.

Appendix

A Conventions and Notations

We choose the signature $-,+,+,+,...$ for the metric G_{MN}. The Riemann tensor is defined as follows

$$R^R_{MNS} = \partial_M \Gamma^R_{NS} - \partial_N \Gamma^R_{MS} + \Gamma^R_{MP}\Gamma^P_{NS} - \Gamma^R_{NP}\Gamma^P_{MS}, \tag{A.1}$$

where the $\Gamma' s$ are the Levi-Civita connection. While the Ricci tensor and the Ricci scalar

$$R_{MN} = R^P_{PMN}, \qquad R = G^{MN} R_{MN}. \tag{A.2}$$

Our choice for the 6-dimensional gamma matrices is

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix}, \quad \Gamma^6 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{A.3}$$

where the γ^μ are the 4-dimensional gamma matrices and γ^5 the 4-dimensional chirality matrix.

We define the harmonics $\mathcal{D}^{(l)\lambda}_m$ as proportional to the matrix element

$$\langle l, \lambda| \, e^{i\varphi Q_3} e^{i(\pi-\theta)Q_2} e^{i\varphi Q_3} \, |l, m\rangle \,, \tag{A.4}$$

where the Q_j, $j = 1, 2, 3$, are the generators of $SU(2)$:

$$[Q_j, Q_k] = i\epsilon_{jkl} Q_l, \tag{A.5}$$

where ϵ_{jkl} is the totally antisymmetric Levi-Civita symbol with $\epsilon_{123} = 1$. Moreover $|l, m\rangle$ is the eigenvector of $\sum_j Q_j^2$ with eigenvalue $l(l+1)$ and the eigenvector of Q_3 with eigenvalue m.

We introduce also $\mathcal{D}^{(l)}_{\lambda,m} \equiv \mathcal{D}^{(l)-\lambda}_m$ and, for $l = 1$, $\mathcal{D}_{\lambda,m} \equiv \mathcal{D}^{(1)}_{\lambda,m}$; our choice is

$$\mathcal{D}_{\hat\alpha,\hat\beta}(\theta,\varphi) = \begin{pmatrix} \frac{1}{2}(\cos\theta + 1) & \frac{1}{2}(\cos\theta - 1)e^{-2i\varphi} & -\frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi} \\ \frac{1}{2}(\cos\theta - 1)e^{2i\varphi} & \frac{1}{2}(\cos\theta + 1) & -\frac{1}{\sqrt{2}}\sin\theta e^{i\varphi} \\ \frac{1}{\sqrt{2}}\sin\theta e^{i\varphi} & \frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi} & \cos\theta \end{pmatrix}. \tag{A.6}$$

In (A.6) the first, second and third rows correspond to $\hat\alpha = +, -, 3$, the first, second and third columns to $\hat\beta = +, -, 3$.

While our choice for $\mathcal{D}^{(2)}_{\lambda,m}$ is

$$\mathcal{D}^{(2)}_{\lambda,2}(\theta,\varphi) = \begin{pmatrix} \frac{1}{4}\left(1+\cos\theta\right)^2 \\ -\frac{1}{2}\sin\theta(1+\cos\theta)e^{i\varphi} \\ \sqrt{\frac{3}{8}}\sin^2\theta e^{2i\varphi} \\ -\frac{1}{2}\sin\theta(1-\cos\theta)e^{3i\varphi} \\ \frac{1}{4}\left(1-\cos\theta\right)^2 e^{4i\varphi} \end{pmatrix}, \; \mathcal{D}^{(2)}_{\lambda,1}(\theta,\varphi) = \begin{pmatrix} -\frac{1}{2}\sin\theta(1+\cos\theta)e^{-i\varphi} \\ \frac{1}{2}(1-\cos\theta - 2\cos^2\theta) \\ \sqrt{\frac{3}{2}}\sin\theta\cos\theta e^{i\varphi} \\ \frac{1}{4}(4\cos^2\theta - 2\cos\theta - 2)e^{2i\varphi} \\ \frac{1}{2}\sin\theta(1-\cos\theta)e^{3i\varphi} \end{pmatrix},$$

$$\mathcal{D}^{(2)}_{\lambda,0}(\theta,\varphi)=\begin{pmatrix}\sqrt{\frac{3}{8}}\sin^2\theta e^{-2i\varphi}\\ \sqrt{\frac{3}{2}}\sin\theta\cos\theta e^{-i\varphi}\\ \frac{1}{2}(3\cos^2\theta-1)\\ -\sqrt{\frac{3}{2}}\sin\theta\cos\theta e^{i\varphi}\\ \sqrt{\frac{3}{8}}\sin^2\theta e^{2i\varphi}\end{pmatrix},\ \mathcal{D}^{(2)}_{\lambda,-1}(\theta,\varphi)=\begin{pmatrix}-\frac{1}{2}\sin\theta(1-\cos\theta)e^{-3i\varphi}\\ \frac{1}{4}(4\cos^2\theta-2\cos\theta-2)e^{-2i\varphi}\\ -\sqrt{\frac{3}{2}}\sin\theta\cos\theta e^{-i\varphi}\\ \frac{1}{2}(1-\cos\theta-2\cos^2\theta)\\ \frac{1}{2}\sin\theta(1+\cos\theta)e^{i\varphi}\end{pmatrix},$$

$$\mathcal{D}^{(2)}_{\lambda,-2}(\theta,\varphi)=\begin{pmatrix}\frac{1}{4}\left(1-\cos\theta\right)^2 e^{-4i\varphi}\\ \frac{1}{2}\sin\theta(1-\cos\theta)e^{-3i\varphi}\\ \sqrt{\frac{3}{8}}\sin^2\theta e^{-2i\varphi}\\ \frac{1}{2}\sin\theta(1+\cos\theta)e^{-i\varphi}\\ \frac{1}{4}\left(1+\cos\theta\right)^2\end{pmatrix},$$

where λ is the row index.

B Spin-1 Mass Terms from S_F and S_R

B.1 S_F Contribution

In this subsection we write the contribution of

$$S_F\equiv-\frac{1}{4}\int d^6X\sqrt{-G}F^2 \tag{B.1}$$

to the bilinear terms of V, U and W. By direct computation we get kinetic terms for V and U and some mass terms for U and W:

$$\begin{aligned}-\frac{1}{4}\int d^2y\det\left(e_m^\alpha\right)F^2 &= -\frac{1}{4}V_{\mu\nu}V^{\mu\nu}K-\frac{1}{6}U^{\hat\alpha}_{\mu\nu}U^{\mu\nu\hat\beta}K_{\hat\alpha\hat\beta}\\ &-\frac{2}{3}U^{\hat\alpha}_{\mu}U^{\mu\hat\beta}M^{(1)}_{\hat\alpha\hat\beta}+\frac{4}{3}U^{\hat\alpha}_{\mu}W^{\mu\hat\beta}M^{(2)}_{\hat\alpha\hat\beta}-\frac{2}{3}W^{\hat\alpha}_{\mu}W^{\mu\hat\beta}M^{(3)}_{\hat\alpha\hat\beta}+...,\end{aligned} \tag{B.2}$$

where the 4-dimensional curved indices μ and ν are contracted with the 4-dimensional metric $g_{\mu\nu}$, the dots are constant terms and interaction terms, moreover

$$V_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu,\ \ U^{\hat\alpha}_{\mu\nu}=\partial_\mu U^{\hat\alpha}_\nu-\partial_\nu U^{\hat\alpha}_\mu \tag{B.3}$$

and

$$\begin{aligned}K &= \frac{1}{4\pi a^2}\int d^2y\det\left(e_m^\alpha\right),\quad K_{\hat\alpha\hat\beta}=\frac{3}{4\pi}\left(\frac{\kappa}{\sqrt{2}ea^2}\right)^2\int d^2y\det\left(e_m^\alpha\right)\ \mathcal{D}^3_{\hat\alpha}\mathcal{D}^3_{\hat\beta},\\ M^{(1)}_{\hat\alpha\hat\beta} &= \frac{3}{8\pi}\left(\frac{\kappa}{\sqrt{2}ea^2}\right)^2\int d^2y\det\left(e_m^\alpha\right)\ g^{mn}\partial_m\mathcal{D}^3_{\hat\alpha}\partial_n\mathcal{D}^3_{\hat\beta}\\ M^{(2)}_{\hat\alpha\hat\beta} &= -\frac{3\kappa^2}{16\pi ea^3}\int d^2y\det\left(e_m^\alpha\right)\ \partial_m\mathcal{D}^3_{\hat\alpha}\mathcal{D}^\alpha_{\hat\beta}e^n_\alpha g^{mq}F_{nq}\\ M^{(3)}_{\hat\alpha\hat\beta} &= \frac{3\kappa^2}{16\pi a^2}\int d^2y\det\left(e_m^\alpha\right)\ \mathcal{D}^\alpha_{\hat\alpha}e^m_\alpha\mathcal{D}^\beta_{\hat\beta}e^p_\beta F_p{}^nF_{mn}.\end{aligned} \tag{B.4}$$

The results (B.4) are valid for all background e^α and e^3. We use the $SU(2)\times U(1)$ background in the subsection B.3, the $U(1)_3$ background in the subsection B.4.

B.2 S_R Contribution

In this subsection we write the contribution of

$$S_R \equiv \int d^6X\sqrt{-G}\frac{1}{\kappa^2}R \tag{B.5}$$

to the bilinear terms of W. The complete contribution of S_R to the 4-dimensional action is given in [15] in the case of non deformed background solutions. Here we need explicit expressions, at least for the bilinears, which are also valid for deformed solutions. We get a kinetic term and a mass term of W: up to a total derivative we have

$$\begin{aligned}&\int d^2y\frac{1}{\kappa^2}\det\left(e_m^\alpha\right)\,R = -\frac{1}{6}W^{\hat\alpha}_{\mu\nu}W^{\mu\nu\hat\beta}K'_{\hat\alpha\hat\beta}\\ &+W^{\hat\alpha}_{\mu}W^{\mu\hat\beta}M^{(4)}_{\hat\alpha\hat\beta}+...,\end{aligned}\tag{B.6}$$

where the dots include constant and interaction terms; moreover

$$W^{\hat\alpha}_{\mu\nu} = \partial_\mu W^{\hat\alpha}_\nu - \partial_\nu W^{\hat\alpha}_\mu, \tag{B.7}$$

and

$$\begin{aligned}K'_{\hat\alpha\hat\beta} &= \frac{3}{8\pi a^2}\int d^2y\det\left(e_m^\alpha\right)\,\mathcal{D}^\alpha_{\hat\alpha}\mathcal{D}^\beta_{\hat\beta}g_{\alpha\beta},\\ M^{(4)}_{\hat\alpha\hat\beta} &= \frac{1}{4\pi a^2}\int d^2y\det\left(e_m^\alpha\right)\Big[\partial_n\mathcal{D}^\alpha_{\hat\alpha}\mathcal{D}^\beta_{\hat\beta}\left(-e^m_\alpha\omega_m{}^\gamma{}_\beta e^n_\gamma - g_{\alpha\delta}g^{nm}\omega_m{}^\delta{}_\beta + 2e^n_\alpha e^m_\gamma\omega_m{}^\gamma{}_\beta\right)+\\ &\quad +D^\alpha_{\hat\alpha}\mathcal{D}^\beta_{\hat\beta}\left(-\frac{1}{2}\omega_n{}^\gamma{}_\alpha e^m_\gamma\omega_m{}^\delta{}_\beta e^n_\delta - \frac{1}{2}\omega_n{}^\delta{}_\alpha g^{nm}\omega_{m\delta\beta} + \omega_n{}^\delta{}_\alpha e^n_\delta\omega_m{}^\gamma{}_\beta e^m_\gamma\right)\\ &\quad +\partial_n\mathcal{D}^\alpha_{\hat\alpha}\partial_m\mathcal{D}^\beta_{\hat\beta}\left(-\frac{1}{2}e^m_\alpha e^n_\beta + e^n_\alpha e^m_\beta - \frac{1}{2}g_{\alpha\beta}g^{nm}\right)\Big],\end{aligned}\tag{B.8}$$

where $\omega_n{}^\alpha{}_\beta$ is the 2-dimensional spin connection for e^α_n. The results (B.8) are also valid for every background e^α and e^3. We use the $SU(2)\times U(1)$ background in the subsection B.3, the $U(1)_3$ background in the subsection B.4.

B.3 The case of $SU(2)\times U(1)$ background

We use now the $SU(2)\times U(1)$ background, that is $\eta = 0$. This computation is performed in [5]. We have the following bilinear terms for V, U and W:

$$\begin{aligned}&-\frac{1}{4}V_{\mu\nu}V^{\mu\nu} - \frac{1}{6}U^{\hat\alpha}_{\mu\nu}U^{\mu\nu}_{\hat\alpha} - \frac{1}{6}W^{\hat\alpha}_{\mu\nu}W^{\mu\nu}_{\hat\alpha}\\ &-\frac{2}{3a^2}\left(U_{\mu\hat\alpha} - W_{\mu\hat\alpha}\right)\left(U^{\mu\hat\alpha} - W^{\mu\hat\alpha}\right).\end{aligned}\tag{B.9}$$

If we define

$$\begin{aligned}\mathcal{A} &= \sqrt{\frac{1}{3}}(W+U),\\ X &= \sqrt{\frac{1}{3}}(W-U),\end{aligned} \tag{B.10}$$

we can write (B.9) as follows

$$\begin{aligned}&-\frac{1}{4}V_{\mu\nu}V^{\mu\nu} - \frac{1}{4}\mathcal{A}^{\hat{\alpha}}_{\mu\nu}\mathcal{A}^{\mu\nu}_{\hat{\alpha}}\\ &-\frac{1}{4}X^{\hat{\alpha}}_{\mu\nu}X^{\mu\nu}_{\hat{\alpha}} - \frac{2}{a^2}X_{\mu\hat{\alpha}}X^{\mu\hat{\alpha}},\end{aligned} \tag{B.11}$$

So $\mathcal{A}$ is a massless field, in fact it's the $SU(2)$ Yang-Mills field [5], while X is a massive field which can be neglected in the low-energy limit.

B.4 The Case of $U(1)_3$ Background

Let's consider now the solution (5.1). First we note that S_R and S_F don't give mass terms for V; so the only source for the mass of V is S_ϕ.

We want to prove now that also the $SU(2)$ Yang-Mills fields masses don't receive contributions from S_R and S_F. First we give the bilinears for U and W, which come from S_R and S_F:

$$\begin{aligned}&-\frac{1}{6}U^{\hat{\alpha}}_{\mu\nu}U^{\mu\nu\hat{\beta}}g_{\hat{\alpha}\hat{\beta}}\left(1+|\eta|\beta k_{\hat{\alpha}}\right) - \frac{1}{6}W^{\hat{\alpha}}_{\mu\nu}W^{\mu\nu\hat{\beta}}g_{\hat{\alpha}\hat{\beta}}\left(1+|\eta|\beta k'_{\hat{\alpha}}\right)\\ &-\frac{2}{3}U^{\hat{\alpha}}_{\mu}U^{\mu\hat{\beta}}g_{\hat{\alpha}\hat{\beta}}\left(1+|\eta|\beta m^{(1)}_{\hat{\alpha}}\right) + \frac{4}{3}U^{\hat{\alpha}}_{\mu}W^{\mu\hat{\beta}}g_{\hat{\alpha}\hat{\beta}}\left(1+|\eta|\beta m^{(2)}_{\hat{\alpha}}\right)\\ &-\frac{2}{3}W^{\hat{\alpha}}_{\mu}W^{\mu\hat{\beta}}g_{\hat{\alpha}\hat{\beta}}\left(1+|\eta|\beta m^{(3)}_{\hat{\alpha}}\right),\end{aligned} \tag{B.12}$$

where

$$k_+ = k_- = \frac{2}{5},\ \ k_3 = \frac{1}{5},\ \ k'_+ = k'_- = \frac{3}{10},\ \ k'_3 = \frac{2}{5},$$

$$m^{(1)}_+ = m^{(1)}_- = \frac{1}{5},\ \ m^{(1)}_3 = -\frac{2}{5},$$

$$m^{(2)}_+ = m^{(2)}_- = -\frac{1}{20},\ \ m^{(2)}_3 = -\frac{2}{5},\ \ m^{(3)}_+ = m^{(3)}_- = -\frac{3}{10},\ \ m^{(3)}_3 = -\frac{2}{5}.$$

In order to prove (B.12) it's useful to use the following formula for the background spin connection:

$$\omega_{\varphi}{}^{+}{}_{+} = -\omega_{\varphi}{}^{-}{}_{-} = \frac{i}{a}(\cos\theta - 1 - \frac{1}{2}|\eta|\beta\cos\theta\sin^2\theta) \tag{B.13}$$

and $\omega_{\varphi}{}^{-}{}_{+} = \omega_{\varphi}{}^{+}{}_{-} = 0$.

Now we define X and A as follows

$$\left(1+\frac{|\eta|\beta}{2}k'_{\hat{\alpha}}\right)W^{\hat{\alpha}}=\sqrt{\frac{3}{2}}\left(\cos\theta_\eta^{\hat{\alpha}}X^{\hat{\alpha}}+\sin\theta_\eta^{\hat{\alpha}}\mathcal{A}^{\hat{\alpha}}\right),$$
$$\left(1+\frac{|\eta|\beta}{2}k_{\hat{\alpha}}\right)U^{\hat{\alpha}}=\sqrt{\frac{3}{2}}\left(-\sin\theta_\eta^{\hat{\alpha}}X^{\hat{\alpha}}+\cos\theta_\eta^{\hat{\alpha}}\mathcal{A}^{\hat{\alpha}}\right), \tag{B.14}$$

where the angle $\theta_\eta^{\hat{\alpha}}$ is defined by

$$\cos\theta_\eta^{\hat{\alpha}}=\frac{1+|\eta|\beta\delta^{\hat{\alpha}}}{\sqrt{2}},\ \ \sin\theta_\eta^{\hat{\alpha}}=\frac{1-|\eta|\beta\delta^{\hat{\alpha}}}{\sqrt{2}}, \tag{B.15}$$

and the quantities $\delta^{\hat{\alpha}}$ are not still fixed. It's simple to check that the kinetic terms for X and $\mathcal{A}$ are in the standard form for every $\delta^{\hat{\alpha}}$ up to $O(\eta^{3/2})$. The definition (B.14) reduce to (B.10) for $\eta=0$.

If we choose

$$\delta^{\hat{\alpha}}=\frac{1}{8}\left(m_{\hat{\alpha}}^{(3)}-k'_{\hat{\alpha}}-m_{\hat{\alpha}}^{(1)}+k_{\hat{\alpha}}\right) \tag{B.16}$$

we have no mass terms for $\mathcal{A}$ coming from S_R+S_F.

So the only source for the spin-1 low energy spectrum is S_ϕ and the result is given in equations (5.10) and (5.11).

C Explicit Calculation of Spin-0 Spectrum for the 6D Theory

As we pointed out in the text, in order to find the spin-0 spectrum the expression of the $|i>$, $i=1,...,6$, vectors is needed; these are defined by $\mathcal{O}_0|i>=0$, which is equivalent to $\nabla^2\phi+\phi/a^2=0$, where $\nabla^2\phi$ is the laplacian over the charged scalar ϕ, calculated with the round S^2 metric. Our choice for the orthonormal vectors[23] $|i>$ is

$$|1>=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,1}\\ \sqrt{\frac{3}{4\pi}}\left(\mathcal{D}^{(1)}_{-1,1}\right)^*\\ 0\\ .\\ .\\ .\\ 0\end{pmatrix},\ |2>=\frac{1}{\sqrt{2}}\begin{pmatrix}i\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,1}\\ -i\sqrt{\frac{3}{4\pi}}\left(\mathcal{D}^{(1)}_{-1,1}\right)^*\\ 0\\ .\\ .\\ .\\ 0\end{pmatrix},$$

$$|3>=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,0}\\ \sqrt{\frac{3}{4\pi}}\left(\mathcal{D}^{(1)}_{-1,0}\right)^*\\ 0\\ .\\ .\\ .\\ 0\end{pmatrix},\ |4>=\frac{1}{\sqrt{2}}\begin{pmatrix}i\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,0}\\ -i\sqrt{\frac{3}{4\pi}}\left(\mathcal{D}^{(1)}_{-1,0}\right)^*\\ 0\\ .\\ .\\ .\\ 0\end{pmatrix},$$

[23] We express a generic vector as in (5.25).

$$|5>=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,-1}\\ \sqrt{\frac{3}{4\pi}}\left(\mathcal{D}^{(1)}_{-1,-1}\right)^*\\ 0\\ .\\ .\\ .\\ 0\end{pmatrix},\ |6>=\frac{1}{\sqrt{2}}\begin{pmatrix}i\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,-1}\\ -i\sqrt{\frac{3}{4\pi}}\left(\mathcal{D}^{(1)}_{-1,-1}\right)^*\\ 0\\ .\\ .\\ .\\ 0\end{pmatrix}.$$

Another ingredient for the calculation of the spin-0 spectrum is an explicit expression of the vectors $|\tilde{i}>$ and of the eigenvalues $M^2_{\tilde{i}}$. As we explained in the text, only the $|\tilde{i}>$ like (5.30) and made of $l=0,1,2$ harmonics are needed. The $|\tilde{i}>$ vectors must satisfy the following eigenvalue equations[24]:

$$\begin{aligned}
&-\nabla^2 h_{++}+2R_{+-+-}h_{++}-2\kappa^2F^2_{+-}h_{++}-\sqrt{2}\kappa\nabla_+\mathcal{V}_+F_{-+}=M^2h_{++},\\
&-\nabla^2 h_{--}+2R_{+-+-}h_{--}-2\kappa^2F^2_{+-}h_{--}+\sqrt{2}\kappa\nabla_-\mathcal{V}_-F_{-+}=M^2h_{--},\\
&-\nabla^2 h_{+-}-R_{+-+-}h_{+-}-\frac{\kappa}{\sqrt{2}}\nabla_+\mathcal{V}_-F_{-+}+\frac{\kappa}{\sqrt{2}}\nabla_-\mathcal{V}_+F_{-+}=M^2h_{+-},\\
&-\nabla^2\mathcal{V}_++R_{+-}\mathcal{V}_+-\kappa^2\mathcal{V}_+F^2_{+-}+\frac{\kappa}{\sqrt{2}}\nabla_+h_{+-}F_{-+}-\sqrt{2}\kappa\nabla_-h_{++}F_{-+}=M^2\mathcal{V}_+,\\
&-\nabla^2\mathcal{V}_-+R_{+-}\mathcal{V}_--\kappa^2\mathcal{V}_-F^2_{+-}-\frac{\kappa}{\sqrt{2}}\nabla_-h_{+-}F_{-+}+\sqrt{2}\kappa\nabla_+h_{--}F_{-+}=M^2\mathcal{V}_-,
\end{aligned}\tag{C.1}$$

where the background objects (∇^2, R_{+-+-},...) correspond to the background (4.14), (4.15) and (4.16). We can transform the differential problem (C.1) into an algebraic one by using the expansion (4.20). We get an eigenvalue problem for every value of l and we give now an explicit expression for the $|\tilde{i}>$ vectors for the relevant value of l, namely $l=0,1,2$. For $l=0$ we get just one eigenvector $|\tilde{1}>$ with $M^2=1/a^2$:

$$|\tilde{1}>=\begin{pmatrix}0\\0\\0\\0\\1/\sqrt{4\pi}\\0\\0\end{pmatrix}.\tag{C.2}$$

For $l=1$ we get three different eigenvalues: $M^2=2/a^2,4/a^2,5/a^2$. The eigenvectors which correspond to $M^2=2/a^2$ are

$$|\tilde{2}_0>,\quad \frac{1}{\sqrt{2}}\left(|\tilde{2}_1>+|\tilde{2}_{-1}>\right),\quad \frac{1}{\sqrt{2}i}\left(|\tilde{2}_1>-|\tilde{2}_{-1}>\right),\tag{C.3}$$

[24]We derive (C.1) evaluating (5.19), (5.20) and (5.23) in the basis (4.18) and performing the redefinition $h_{\pm\pm}\to\sqrt{2}\kappa h_{\pm\pm}$ and $h_{+-}\to h_{+-}\kappa/\sqrt{2}$, which normalizes the kinetic terms in the standard way.

where

$$|\tilde{2}_m> \equiv \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{0,m} \\ -\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{1,m} \\ -\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,m} \end{pmatrix}. \tag{C.4}$$

Instead the eigenvectors which correspond to $M^2 = 4/a^2$ are

$$i|\tilde{3}_0>, \quad \frac{1}{\sqrt{2}i}\left(|\tilde{3}_1> + |\tilde{3}_{-1}>\right), \quad \frac{1}{\sqrt{2}}\left(|\tilde{3}_1> - |\tilde{3}_{-1}>\right), \tag{C.5}$$

where

$$|\tilde{3}_m> \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{1,m} \\ \sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,m} \end{pmatrix}. \tag{C.6}$$

Moreover the eigenvectors which correspond to $M^2 = 5/a^2$ are

$$|\tilde{4}_0>, \quad \frac{1}{\sqrt{2}}\left(|\tilde{4}_1> + |\tilde{4}_{-1}>\right), \quad \frac{1}{\sqrt{2}i}\left(|\tilde{4}_1> - |\tilde{4}_{-1}>\right), \tag{C.7}$$

where

$$|\tilde{4}_m> \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{0,m} \\ \sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{1,m} \\ \sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,m} \end{pmatrix}. \tag{C.8}$$

Finally, for $l = 2$ the values of M^2 are given by

$$a^2M^2 = 6,\ 2(3-\sqrt{3}),\ 2(3+\sqrt{3}),\ \frac{1}{2}(13-\sqrt{73}),\ \frac{1}{2}(13+\sqrt{73}). \tag{C.9}$$

The eigenvectors with $a^2M^2 = 6$ are

$$\begin{aligned} &|\tilde{5}_0>, \quad \frac{1}{\sqrt{2}}\left(|\tilde{5}_1> - |\tilde{5}_{-1}>\right), \quad \frac{1}{\sqrt{2}i}\left(|\tilde{5}_1> + |\tilde{5}_{-1}>\right), \\ &\frac{1}{\sqrt{2}}\left(|\tilde{5}_2> + |\tilde{5}_{-2}>\right), \quad \frac{1}{\sqrt{2}i}\left(|\tilde{5}_2> - |\tilde{5}_{-2}>\right), \end{aligned} \tag{C.10}$$

where

$$|\tilde{5}_m> \equiv \frac{1}{3\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{2,m} \\ -\sqrt{2}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-2,m} \\ -2\sqrt{3}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{0,m} \\ -\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{1,m} \\ \sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-1,m} \end{pmatrix}. \tag{C.11}$$

For $a^2M^2 = 2(3-\sqrt{3})$ we have the eigenvectors

$$\begin{aligned} &i|\tilde{6}_0>, \quad \frac{1}{\sqrt{2}}\Big(|\tilde{6}_1> + |\tilde{6}_{-1}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{6}_1> - |\tilde{6}_{-1}>\Big), \\ &\frac{1}{\sqrt{2}}\Big(|\tilde{6}_2> - |\tilde{6}_{-2}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{6}_2> + |\tilde{6}_{-2}>\Big), \end{aligned} \tag{C.12}$$

where

$$|\tilde{6}_m> \equiv \frac{1}{\sqrt{2(3+\sqrt{3})}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1+\sqrt{3}}{\sqrt{2}}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{2,m} \\ \frac{1+\sqrt{3}}{\sqrt{2}}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-2,m} \\ 0 \\ \sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{1,m} \\ \sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-1,m} \end{pmatrix}. \tag{C.13}$$

For $a^2M^2 = 2(3+\sqrt{3})$ we have the eigenvectors

$$\begin{aligned} &i|\tilde{7}_0>, \quad \frac{1}{\sqrt{2}}\Big(|\tilde{7}_1> + |\tilde{7}_{-1}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{7}_1> - |\tilde{7}_{-1}>\Big), \\ &\frac{1}{\sqrt{2}}\Big(|\tilde{7}_2> - |\tilde{7}_{-2}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{7}_2> + |\tilde{7}_{-2}>\Big), \end{aligned} \tag{C.14}$$

where

$$|\tilde{7}_m> \equiv \frac{1}{\sqrt{2(3-\sqrt{3})}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1-\sqrt{3}}{\sqrt{2}}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{2,m} \\ \frac{1-\sqrt{3}}{\sqrt{2}}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-2,m} \\ 0 \\ \sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{1,m} \\ \sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-1,m} \end{pmatrix}. \tag{C.15}$$

Then for $a^2M^2 = (13-\sqrt{73})/2$:

$$
\begin{aligned}
&|\tilde{8}_0>, \quad \frac{1}{\sqrt{2}}\Big(|\tilde{8}_1> - |\tilde{8}_{-1}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{8}_1> + |\tilde{8}_{-1}>\Big), \\
&\frac{1}{\sqrt{2}}\Big(|\tilde{8}_2> + |\tilde{8}_{-2}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{8}_2> - |\tilde{8}_{-2}>\Big),
\end{aligned}
\tag{C.16}
$$

where

$$
|\tilde{8}_m> \equiv \frac{1+\sqrt{73}}{\sqrt{438+30\sqrt{73}}}
\begin{pmatrix}
0 \\
0 \\
\frac{13\sqrt{2}+\sqrt{146}}{2(1+\sqrt{73})}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{2,m} \\
\frac{13\sqrt{2}+\sqrt{146}}{2(1+\sqrt{73})}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-2,m} \\
-\frac{4\sqrt{3}}{1+\sqrt{73}}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{0,m} \\
-\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{1,m} \\
\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-1,m}
\end{pmatrix}.
\tag{C.17}
$$

Finally for $a^2M^2 = (13+\sqrt{73})/2$:

$$
\begin{aligned}
&|\tilde{9}_0>, \quad \frac{1}{\sqrt{2}}\Big(|\tilde{9}_1> - |\tilde{9}_{-1}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{9}_1> + |\tilde{9}_{-1}>\Big), \\
&\frac{1}{\sqrt{2}}\Big(|\tilde{9}_2> + |\tilde{9}_{-2}>\Big), \quad \frac{1}{\sqrt{2}i}\Big(|\tilde{9}_2> - |\tilde{9}_{-2}>\Big),
\end{aligned}
\tag{C.18}
$$

where

$$
|\tilde{9}_m> \equiv \frac{1-\sqrt{73}}{\sqrt{438-30\sqrt{73}}}
\begin{pmatrix}
0 \\
0 \\
\frac{13\sqrt{2}-\sqrt{146}}{2(1-\sqrt{73})}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{2,m} \\
\frac{13\sqrt{2}-\sqrt{146}}{2(1-\sqrt{73})}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-2,m} \\
-\frac{4\sqrt{3}}{1-\sqrt{73}}\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{0,m} \\
-\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{1,m} \\
\sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-1,m}
\end{pmatrix}.
\tag{C.19}
$$

We can now calculate the 6×6 matrix M^2_{ij} given in (5.28). In order to do that we need just the matrix elements $< i|\mathcal{O}_1|\tilde{i} >$ and $< i|\mathcal{O}_2|j >$, which can be computed by evaluating[25] $\mathcal{L}_0(\phi,h)$ and $\mathcal{L}_0(\phi,\phi)$, which appears in (5.18) and (5.21), in the $\pm$ basis given in (4.18). After the redefinitions $h_{\pm\pm}\to\sqrt{2}kh_{\pm\pm}$ and $h_{+-}\to h_{+-}k/\sqrt{2}$, which normalize the kinetic terms in the standard way, we get (for $n=2$)

$$
\begin{aligned}
\mathcal{L}_0(\phi,h) \;=\;& \sqrt{2}\kappa\nabla_+\Phi\nabla_+h_{--}\phi^* + \frac{\kappa}{\sqrt{2}}\nabla_+\Phi\nabla_-h_{+-}\phi^* \\
&+\sqrt{2}\kappa\nabla_+\Phi h_{--}\left(\nabla_-\phi\right)^* + \frac{\kappa}{\sqrt{2}}\nabla_+\Phi h_{+-}\left(\nabla_+\phi\right)^* + c.c.\,,
\end{aligned}
$$

[25] For the background solution (4.14), (4.15) and (4.16) we have $\mathcal{L}_0(\phi,\mathcal{V})=0$.

$$\mathcal{L}_0(\phi,\phi) = \phi^*\partial^2\phi - \phi^*\left[-\nabla^2 + m^2 + (e^2+4\xi)\Phi^*\Phi + \kappa^2\nabla_+\Phi\,(\nabla_+\Phi)^*\right]\phi$$
$$-\frac{1}{2}\left[\phi(2\xi - e^2)\,(\Phi^*)^2\,\phi + c.c.\right]. \qquad \text{(C.20)}$$

By using these expressions and the values of $|i>$ and $|\tilde{i}>$ given before, we find the following expression for M_{ij}^2:

$$\{M_{ij}^2\} = \begin{pmatrix} a_1 & 0 & 0 & 0 & a_4 & 0 \\ 0 & a_1 & 0 & 0 & 0 & -a_4 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & 0 & a_1 & 0 \\ 0 & -a_4 & 0 & 0 & 0 & a_1 \end{pmatrix}, \qquad \text{(C.21)}$$

where

$$a_1 = \frac{|\eta|}{a^2}\left(-sign(\eta) + \frac{3}{10}\beta + \frac{12}{5}\frac{\beta\xi a^2}{\kappa^2}\right),$$
$$a_2 = \frac{|\eta|}{a^2}\left(-sign(\eta) - \frac{6}{5}\beta + \frac{24}{5}\frac{\beta\xi a^2}{\kappa^2}\right),$$
$$a_3 = \frac{|\eta|}{a^2}\left(-sign(\eta) + \frac{4}{15}\beta + \frac{8}{5}\frac{\beta\xi a^2}{\kappa^2}\right),$$
$$a_4 = \frac{|\eta|}{a^2}\beta\left(\frac{3}{10} - \frac{4}{5}\frac{\xi a^2}{\kappa^2}\right). \qquad \text{(C.22)}$$

By diagonalizing M_{ij}^2, we found exactly the spectrum that we discussed in the subsection 5.2: the squared masses of the vector particles are reproduced[26], as required by the light cone gauge; moreover we get the two masses squared given in (5.31).

D Explicit Calculation of Spin-1/2 Spectrum for the 6D Theory

Here we concentrate on the right-handed sector, which is the non trivial one because it presents $\eta^{1/2}$ mixing terms.

The eigenvalue equations for the unperturbed ($\eta = 0$) mass squared operator $\mathcal{O}_0$, acting on the right-handed sector, are

$$-2\nabla_-\nabla_+\psi_{+R} = M^2\psi_{+R},$$
$$-2\nabla_+\nabla_-\psi_{-R} = M^2\psi_{-R}, \qquad \text{(D.1)}$$

The differential equation (D.1) can be transformed in an algebraic one through the harmonic expansion, remembering the iso-helicities of ψ_{+R} and ψ_{-R}: $\lambda_{+R} = \lambda_{-R} = 1$. Therefore an

[26] In order to see that we use the background constraints (4.17).

explicit expression for the vectors $|i>$, which satisfies by definition $\mathcal{O}_0|i>=0$, is given by

$$|i>=\left(\begin{array}{c} 0 \\ \sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,i} \end{array}\right), \qquad i=1,-1,0. \tag{D.2}$$

We give also an expression for the vectors $|\tilde{i}>$ and the corresponding non vanishing eigenvalues $M^2_{\tilde{i}}$. For $l=1$, we have just one eigenvalue $M^2=2/a^2$ and the corresponding eigenvectors are

$$|\tilde{1}_m>=\left(\begin{array}{c} \sqrt{\frac{3}{4\pi}}\mathcal{D}^{(1)}_{-1,m} \\ 0 \end{array}\right). \tag{D.3}$$

For $l=2$ we have an eigenvalue $M^2=6/a^2$, which corresponds to the eigenvectors

$$|\tilde{2}_m>=\left(\begin{array}{c} \sqrt{\frac{5}{4\pi}}\mathcal{D}^{(2)}_{-1,m} \\ 0 \end{array}\right), \tag{D.4}$$

and an eigenvalue $M^2=4/a^2$, which corresponds to the eigenvectors

$$|\tilde{3}_m>=\left(\begin{array}{c} 0 \\ \sqrt{\frac{5}{4\pi}}\mathcal{D}^{(1)}_{-1,m} \end{array}\right). \tag{D.5}$$

By inserting these eigenvectors and eigenvalues in the expression (5.28) we get

$$M^2_{ij}=diag\left(0,\ 0,\ \frac{2}{3}|\eta|g_Y^2\frac{\beta}{\kappa^2}\right), \tag{D.6}$$

which corresponds to the spectrum we discussed at the end of section 5.3.

References

[1] J. C. Pati and A. Salam, "Unified Lepton - Hadron Symmetry, And A Gauge Theory Of The Basic Interactions," Phys. Rev. D **8** (1973) 1240; H. Georgi and S. L. Glashow, "Unity Of All Elementary Particle Forces," Phys. Rev. Lett. **32** (1974) 438;

[2] K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975).

[3] S. Weinberg, "Effective Gauge Theories," Phys. Lett. B **91** (1980) 51.

[4] T. Appelquist and J. Carazzone, "Infrared Singularities And Massive Fields," Phys. Rev. D **11** (1975) 2856.

[5] S. Randjbar-Daemi, A. Salam and J. A. Strathdee, "Spontaneous Compactification In Six-Dimensional Einstein-Maxwell Theory," Nucl. Phys. B **214** (1983) 491.

[6] G. W. Gibbons and C. N. Pope, "Consistent S^2 Pauli reduction of six-dimensional chiral gauged Einstein-Maxwell supergravity," Nucl. Phys. B **697** (2004) 225 [arXiv:hep-th/0307052].

[7] A. Salam and E. Sezgin, “Chiral Compactification On $(Minkowski) \times S^2$ Of N=2 Einstein-Maxwell Supergravity In Six-Dimensions,” Phys. Lett. B **147** (1984) 47.

[8] A. J. Buras, J. R. Ellis, M. K. Gaillard and D. V. Nanopoulos, “Aspects Of The Grand Unification Of Strong, Weak And Electromagnetic Interactions,” Nucl. Phys. B **135** (1978) 66.

[9] L. H. Chan, T. Hagiwara and B. A. Ovrut, “The Effect Of Heavy Particles In Low-Energy Light Particle Processes,” Phys. Rev. D **20** (1979) 1982.

[10] E. P. Wigner, “Group Theory and its Application to the Quantum Mechanics of Atomics Spectra,” New York: Academic Press, 1959.

[11] J. Sobczyk, “Symmetry Breaking In Kaluza-Klein Theories,” Class. Quant. Grav. **4** (1987) 37.

[12] S. Randjbar-Daemi and M. Shaposhnikov, “A formalism to analyze the spectrum of brane world scenarios,” Nucl. Phys. B **645** (2002) 188 [arXiv:hep-th/0206016].

[13] S. Randjbar-Daemi, A. Salam and J. A. Strathdee, “Towards A Selfconsistent Computation Of Vacuum Energy In Eleven-Dimensional Supergravity,” Nuovo Cim. B **84** (1984) 167.

[14] C. P. Burgess, AIP Conf. Proc. **743** (2005) 417 [arXiv:hep-th/0411140]; C. P. Burgess, “Supersymmetric large extra dimensions and the cosmological constant problem,” arXiv:hep-th/0510123.

[15] A. Salam and J. A. Strathdee, “On Kaluza-Klein Theory,” Annals Phys. **141** (1982) 316.