Game Theory in Communications: a Study of Two Scenarios

THÈSE N^{O} 4811 (2010)

PRÉSENTÉE LE 17 SEPTEMBRE 2010

À LA FACULTÉ INFORMATIQUE ET COMMUNICATIONS
LABORATOIRE DE COMMUNICATIONS MOBILES
PROGRAMME DOCTORAL EN INFORMATIQUE, COMMUNICATIONS ET INFORMATION

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

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ÉCOLE POLYTECHNIQUE
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Suisse
2010

To my family.

Abstract

Multi-user communication theory typically studies the fundamental limits of communication systems, and considers communication schemes that approach or even achieve these limits. The functioning of many such schemes assumes that users always cooperate, even when it is not in their own best interest. In practice, this assumption need not be fulfilled, as rational communication participants are often only interested in maximizing their own communication experience, and may behave in an undesirable manner from the system's point of view. Thus, communication systems may operate differently than intended if the behavior of individual participants is not taken into account.

In this thesis, we study how users make decisions in wireless settings, by considering their preferences and how they interact with each other. We investigate whether the outcomes of their decisions are desirable, and, if not, what can be done to improve them. In particular, we focus on two related issues. The first is the decision-making of communication users in the absence of any central authority, which we consider in the context of the Gaussian multiple access channel. The second is the pricing of wireless resources, which we consider in the context of the competition of wireless service providers for users who are not contractually tied to any provider, but free to chose the one offering the best tradeoff of parameters.

In the first part of the thesis, we model the interaction of self-interested users in a Gaussian multiple access channel using non-cooperative game theory. We demonstrate that the lack of infrastructure leads to an inefficient outcome for users who interact only once, specifically due to the lack of coordination between users. Using evolutionary game theory, we show that this inefficient outcome would also arise as a result of repeated interaction of many individuals over time. On the other hand, if the users correlate their decoding schedule with the outcome of some publicly observed (pseudo) random variable, the resulting outcome is efficient. This shows that sometimes it takes very little intervention on the part of the system planner to make sure that users choose a desirable operating point.

In the second part of the thesis, we consider the competition of wireless service providers for users who are free to chose their service provider based on their channel parameters and the resource price. We model this situation as a two-stage game where the providers announce unit resource prices in the first stage and the users choose how much resource they want to purchase from each provider in the second stage. Under fairly general conditions, we show that the competitive interaction of users and providers results in socially optimal resource allocation. We also provide a decentralized primal-dual algorithm and prove its convergence to the socially optimal outcome.

Keywords: Game Theory, Information Theory, Wireless Communication, Pricing, Microeconomics, Primal-Dual Algorithm

Résumé

La théorie des communications multi-utilisateur (*multi-user communication theory*) a pour principe d'étudier les limites fondamentales des systèmes de communication, et d'examiner des schémas de transmission approchant ou même atteignant ces limites. Le fonctionnement de nombre de ces schémas suppose que les utilisateurs coopèrent, même si cela contredit leur intérêt propre; pourtant, des participants rationnels s'intéressent généralement davantage à l'optimisation de leurs propres communications, et peuvent se comporter de manière indésirable du point de vue de l'ensemble du système.

Dans cette thèse, qui privilégie des scénarios de communications sans fil, nous étudions la manière dont les utilisateurs prennent leurs décisions, en tenant compte de leurs préférences individuelles et en regardant comment les différents agents interagissent. Nous cherchons à savoir si les conséquences de leurs décisions sont souhaitables, et, si ce n'est pas le cas, nous essayons d'améliorer leur prise de décision. Nous nous intéressons plus particulièrement à deux problèmes intimement liés l'un à l'autre. Le premier se rapporte à la prise de décision d'utilisateurs en l'absence d'autorité centrale, dans le contexte de canaux gaussiens à accès multiple (*Gaussian multiple access channel*). Le second touche à la tarification des ressources, dans le cadre d'une compétition entre les fournisseurs d'accès aux réseaux sans fil. Ces derniers désirent attirer les clients qui ne sont pas contractuellement affiliés à un fournisseur, et qui sont libres de choisir, à tout moment, celui présentant le meilleur compromis entre la qualité de la connexion et le prix.

Dans la première partie, nous modélisons l'interaction entre les utilisateurs égoïstes d'un canal gaussien à accès multiple, à l'aide de la théorie des jeux non-coopératifs. Nous montrons que le manque d'infrastructure mène à une situation peu efficace pour des utilisateurs qui n'interagissent qu'une seule fois, en raison du manque de coordination. Grâce à la théorie des jeux d'évolution (*evolutionary game theory*), nous montrons que cette inefficacité surviendrait encore à la suite d'interactions répétées entre les nombreux utilisateurs. En revanche, si les utilisateurs lient leur agenda de décodage à la réalisation d'une variable (pseudo) aléatoire publiquement observable, la coopération devient efficace. Cela montre que, parfois, il suffit d'une très légère intervention de la part du système pour s'assurer que les utilisateurs choisissent un point de fonctionnement souhaitable.

Dans la deuxième partie, nous nous intéressons à la compétition entre les fournisseurs d'accès aux réseaux sans fil, pour ce qui est de servir les utilisateurs ayant le choix du fournisseur. Ces utilisateurs fondent leur choix sur les paramètres des canaux de transmission et sur le prix des ressources non filaires proposées. Nous modélisons cette situation comme un jeu à deux étapes, où les fournisseurs annoncent d'abord le prix unitaire de la ressource, et où les utilisateurs choisissent ensuite la quantité de ressources qu'ils comptent acheter à chacun des fournisseurs. Sous des conditions assez générales, nous montrons que l'interaction compétitive des utilisateurs et des fournisseurs débouche sur une allocation de ressources socialement optimale. Nous proposons également un algorithme primal-dual décentralisé, et nous prouvons sa convergence vers le point de fonctionnement socialement optimal.

Mots Clés: Théorie des jeux, Théorie de l'information, Communication sans fil, Tarification, Microéconomie, Algorithme primal-dual.

Acknowledgements

First and foremost I would like to thank my thesis advisor Prof. Bixio Rimoldi, who took me under his wing and suggested that I consider the topic of user behavior in communications. His perseverance and attention to detail taught me that work on a problem is not finished when we find the solution, but only once we are able to understand its elegant beauty.

I would like to thank my committee members Prof. Patrick Thiran, Prof. Edmund Yeh, and Prof. Jianwei Huang, for taking the time to read the thesis and for offering their helpful comments; also Prof. Martin Hasler for being the president of the jury. I would particularly like to thank Prof. Jianwei Huang and his NCEL group for inviting me to Hong Kong on two occasions. This collaboration resulted in a big part of my thesis, and in a personal experience that I will never forget.

I would like to thank Professors Emre Telatar, Suhas Diggavi, Rudiger Urbanke and Bixio Rimoldi for making the Information Processing Group such a great place to work in, and to Françoise, Muriel and Yvonne for making sure that the labs function in an incredibly smooth manner. I particularly thank Françoise who always miraculously produced the documents, schedules and itineraries that I desperately needed on short notice. Thanks to Damir for always having the answer for computer questions, even if his first answer was always “NO!”; and to Giovanni for buying a foosball table, a development that might have easily cost me my thesis.

The IPG lab would not be what it is if it weren’t for its members, so I would like to thank everyone I’ve met throughout the years, in particular Olivier, Nicolas, Peter, Jasper, Marius, Mohammed, Shrinivas, Dinkar, Ayfer, Irina, Iryna, Etienne and last but not least my office mates Satish and Christine; for all the fun moments, and all the technical and non-technical discussions we had over the years. I also want to mention Jeremie, who in his brief time with us taught us so many important things.

I would like to thank my other EPFL and Lausanne friends for the wonderful time we spent together. In particular Pierre-Jean, Aurelien, Dorothée, Sebastien, Julien, Çan, Nikola, Aleksandar, Mihailo, Miloš, Nedeljko, as well as the other members of the Serbian clan, the lunch crew, the GSA and the ACIDE members.

My greatest words of thanks go to my family, to whom I dedicate this thesis. I especially thank my mom, dad, and brother whose love and unconditional support were, and always will be, my sources of strength. Finally, my most special thanks go to my loving wife Florence, not only for standing by my side during both the difficult and amazing moments so far, but also for wanting to share with me the rest of the ride.

Contents

Introduction

The central topic of this thesis is the study of decision making in multi-user wireless communications settings, in situations where a choice made by one participant influences those available to others. Namely, multi-user communication theory typically studies the limits of communication systems, as well as how to achieve these limits, while disregarding the human factor - the needs and preferences of individual participants. On the other hand, the functioning of many communication schemes often hinges on the assumption that all participants behave in a certain manner. In systems where users try to obtain the best possible parameters for themselves, they may be reluctant to behave in a desirable way from the system's point of view. This motivates a study of communication systems where the behavior of participants is taken into account.

In this thesis, we specifically study how live users make decisions in wireless situations, whether the outcomes of these decisions are desirable, and, if not, what can be done to improve the outcomes. In particular, we focus on two related issues. The first is the decision-making of communication users in the absence of any central authority, which we consider in the context of the Gaussian Multiple Access Channel. The second is the pricing of wireless resources, which we consider in the context of the competition of wireless service providers for users who are not contractually tied to any provider, but free to connect to that (or those) offering the best tradeoff of parameters.

Motivation

Our study is motivated by two recent advances in wireless communications, one related to engineering, and the other one to business.

On the engineering side, the advent of *Software Defined Radio* (SDR) gives users the possibility to regulate their transmit power, bandwidth, frequency and rate in real-time, based on their current needs and channel conditions. These dynamic capabilities are a great step towards true efficient utilization of wireless resources, in contrast to current wireless devices which use preprogrammed operational parameters that users cannot change.

On the business side, the unprecedented success of WiFi technology that came as a result of the opening of the ISM (Industrial, Scientific and Medical) frequency band to unlicensed use has forced regulators to rethink the current exclusive spectrum allocation paradigm. Traditionally, the wireless spectrum is auctioned off by government regulators to mobile operators who then get exclusive use over it. This leads to inefficiencies as large portions of the spectrum are unused much of the time by the license holders, even though other portions of the spectrum may be crowded. Such stringent allocation policy led to the myth that the spectrum is scarce, while in reality it is only inflexibly allocated. A deregulated approach based on open access to spectrum is currently being discussed to address this issue.

The flexible wireless devices that may freely interact in open spectrum are supposed to improve wireless communication experience, but many aspects of user interactions and tradeoffs that will arise as a consequence of these new possibilities are not well understood. The central question is: How will wireless users make decisions about their communication parameters, especially in situations where their own choices influence the ability of other users to communicate?

State of the field

This question instigated an amazing body of research in the past ten years. Far from yielding a single answer, this research raised many concerns and offered many different solutions, depending on the particular wireless situation and underlying assumptions. Worryingly, it is shown in many different situations that wireless users who interact for a brief period of time will experience inefficiencies as a result of their selfish behavior.[1]

More generally, users that share the same spectrum are subject to the *Tragedy of the commons*, which arises when free public resources are over-exploited in the selfish pursuit of gain by individuals. The famous examples include the unregulated use of forests, pastures, and air, which historically resulted in deforestation, overgrazing and pollution. Similarly, when users share the same spectrum, each user has an incentive to transmit at maximum power/rate, generating too much interference and reducing the ability of other users to communicate in the process.

One of the main positive results of the analysis of user behavior in communications so far is that pricing of communication resources often solves the tragedy of the commons problem, or at least yields a more efficient outcome. When wireless users have to pay for the service they are using, they in fact pay for the damage they are doing to others, which in turn makes them use wireless resources in an more efficient manner. On the other hand, the whole point of the open spectrum is deregulation, so it is not clear which entity should do the pricing and how.

To see that user behavior is not just a problem of future systems, consider the recent proliferation of smartphones in the mobile markets. Thanks to the improvements in the processing power and a simple user interface, smartphones allow users to enjoy access to the multimedia content that was until recently only available through personal computers. Faced with flat-rate fees that include unlimited data use, a small portion of smartphone users (ab)use multimedia services to the point that other users sometimes cannot use the services at all. In other words, something similar to the tragedy of the commons occurs. As a consequence, the mobile providers are considering a shift to a hybrid of flat-rate and usage-based pricing, where in addition to the flat monthly rate, all data above a certain

[1] On the other hand, users who interact repeatedly with each other over a long period of time will tend to cooperate. Although this shows that selfishness is not always a problem, in most wireless situations the users will not interact with the same opponents for too long.

threshold has to be paid proportionally. Hence, the analysis of which pricing mechanism results in the most efficient use of wireless resources is paramount in this setting.

Methodology

The tools being used to analyze the behavior of users in wireless communications stem mostly from game theory, a branch of applied mathematics which give formal mathematical models for the behavior of individuals in situations of conflicting interests. The models of game theory assume intelligent and rational decision makers. An intelligent decision maker is one that understands everything about the structure of the interaction, including the available information, assumptions, but also the fact that other decision makers are intelligent and rational. Rational decision makers always make decisions that are in their own best interest, which typically means maximizing an expected utility function. Game theory started out as a branch of economics, but its potential to model and analyze human behavior in a variety of situations was soon understood and it was applied in diverse disciplines such as international relations, evolutionary biology, political science, psychology, design of auctions and voting systems, and so on.

Some comments on game theory

The game theory itself is built on the choice theory formalism, which explains how users make decisions based on their preferences. Consequently, all of game theoretic results, be it for wireless communication scenarios or otherwise, are subject to the limitations of the choice theory, and should be taken with a grain of salt. Most importantly, the assumption that the behavior of a human individual can be reduced to maximization of his or her utility function should not be taken literally. One should always keep in mind that even if individuals know their utility functions perfectly (a great assumption in itself), their behavior can still be irrational, impulsive, using rules of thumb, and so on[2]. In a sense, any result on the behavior of the users in a given communication setting should be understood as a first-degree approximation, and in no way as an absolute prediction.

It should be noted that the utility maximizing users are often interpreted in a different way in wireless communication settings. Instead of considering behavior of human users, we can think of wireless devices equipped with software agents that come with pre-programmed utility functions. These utility functions are chosen (centrally) by the network so that they implement some resource allocation policy in a decentralized manner, and are not under users' control. While this interpretation of utility maximization is useful and does not face the same criticism as the analysis of human behavior does, we emphasize that it is not the subject of this thesis.

Thesis outline

In **Chapter 1** we give the overview of theory and assumptions that will be underlying the analysis in the remainder of the thesis. We review the key notions related to user choice

[2]This gave rise to the field of Behavioral Economics, whose goal is to explain human behavior without the assumption that people are rational.

theory, starting with the definition of preferences and the rational user assumptions and then move to the single user utility maximization problem which models decision-making of a single individual. We then give a brief introduction to game theory, where the decision making process now has to take into account the decisions of other participants. The section on game theory gives a short overview of the entire field. Our hope is that readers new to game theory will find information that will facilitate the reading of (often highly technical) game theoretic references.

Chapter 2 models the individual decision making of users in a Gaussian multiple access channel (MAC) using non-cooperative game theory. This setting is particularly interesting since adding more users strictly increases the set of available choices, in contrast to most other multi-user systems. For users who choose their communication rate under a fixed power assumption, we characterize the Nash equilibria and argue that the resulting outcome is inefficient in the absence of coordination. We further consider the interaction of a large number of individuals in this setting using evolutionary game theory. Finally, we show that using a correlating device can solve the inefficiency problem using the concept of coordinated equilibrium.

In **Chapter 3** we again consider the Gaussian MAC, this time for users who need to decide on both power and rate in an information limited scenario. We adopt a mechanism design approach where the receiver takes the role of the central authority by gathering preference information from the users, and then choosing an operating point. The problem then becomes one of soliciting preference information from the users, in such a way that they have no incentive to misreport. In this setting, the Vickrey-Clarke-Groves mechanisms are the natural (and only) solution.

Chapter 4 considers the competition of service providers for wireless users with no contractual constraints. Each provider has a fixed amount of wireless resource available for sale, which they are selling to wireless users through linear pricing. The emphasis is on how users choose which provider they purchase the service from, and on the efficiency of such allocations in the presence of pricing. Remarkably, it turns out that the price competition results in the socially optimal allocation of resources. In addition, we introduce a decentralized algorithm and prove that it converges to the socially optimal allocation using only local information.

Chapter 5 again considers the service providers for wireless users. Unlike Chapter 4 where providers are charging for resources used, in this chapter we assume that the providers are pricing the service provided to the users (such as communication rate), which is in general not socially optimal. We compare the benefits of pricing for the wireless resource as opposed to pricing for provided service. Under certain conditions pricing for service, although not socially optimal, earns more profit for the service providers. This shows that the service providers may not always have an incentive to adopt pricing for resources used, despite its desirable properties. The admission control and competition of providers charging for provided services are also considered in this chapter.

We give an overview of related work on the interaction of users in wireless communications in **Chapter 6**. The overview is not limited to work directly related to the contributions of this thesis, and the related work section is not required to understand the thesis. Even though the overview is not exhaustive, its sheer volume would obscure the understanding of the main text if placed at the beginning, so we put it at the end of the thesis. We give particular attention to user interaction in information-theoretic physical layer settings, infrastructure based-networks, ad-hoc networks (single and multi-hop), cognitive radio and provider competition for wireless users. The notable absence is the related work on pricing schemes in wireless networks, which is outside the scope of the thesis.

Contributions of the thesis

- **Roadmap for beginners**

  Our hope is that this thesis will serve as a good starting point to researchers who are interested in modeling the behavior of participants in wireless communications. Our overview of modeling assumptions in Chapter 1 is geared towards explaining the applicability of the game-theoretic framework. We also provide a breakdown of different game-theoretic concepts and identify the situations where they can be used. Finally, the extensive overview of related work in Chapter 6 gives an idea of what was done so far in the field and helps identify interesting areas for future research.

  Key message: Game theory is a rich field whose many diverse tools were used to analyze behavior of participants in wireless communications.

- **Modeling of user interaction in AWGN MAC channel**

  We model the interaction of self-interested users in an AWGN MAC using non-cooperative game theory. We demonstrate that the lack of infrastructure leads to inefficiencies. We propose the use of a coordinating device as a solution, in the form of a random (or pseudorandom) variable whose outcome is publicly observed by all users. Its use results in an efficient outcome, which demonstrates that the original inefficiencies are not due to the selfishness of the users, but rather due to lack of coordination. This is good news, since the coordinating device can be easily implemented, and it does not rely on any central authority.

  Key message: Coordination of users can improve inefficient outcomes.

- **Modeling the competition of wireless service providers**

  We consider the competition of wireless service providers for users who are free to choose their provider based on their channel parameters and the service price. We model this situation as a two stage game of complete information where providers propose prices in the first stage and the users choose their provider and amount of resource in the second stage. For a large class of user utility functions, we arrive at the surprising result that providers who are charging for wireless resources that they use (such as bandwidth), instead for the service they provide (such as rate), assign resources in a socially optimal way. This is an important finding for the design of future wireless pricing schemes.

  Key message: When competing service providers are charging for wireless resources, the resulting resource allocation is socially optimal.

- **Providing an algorithm to reach the provider competition game equilibrium**

The socially desirable equilibrium of the provider competition game can be found if all users and providers have complete knowledge of the system parameters, which is typically not the case. We introduce a decentralized algorithm that uses only local information and prove that it converges to the socially optimal equilibrium of the game. The users update their demand for wireless resources solely based on prices, while providers update their prices solely based on user demand for the resource. In addition to requiring only local information, the algorithm does not assume perfect rationality on behalf of the communication participants, which makes it suitable for implementation.

Key message: The socially optimal resource allocation can be reached in a decentralized manner with only local information.

Frequently used terms, abbreviations, and notation

Terms and abbreviations

NE: Nash equilibrium

strategy set: set of available choices to the user

mixed strategy: a user's choice of action consisting of a probability distribution over the strategy set

pure strategy: an atomic choice from a strategy set (a degenerate mixed strategy)

MAC: Multiple Access Channel

AWGN: Additive White Gaussian Noise (Channel), see Chapter 2

VCG mechanism: Vickrey-Clarke-Groves mechanism. Defined in Section 3.3.3 as Groves mechanism

SPE: Sub-game Perfect (Nash) Equilibrium. Generalization of Nash equilibrium for sequential games

BGR: Bipartite Graph Representation. A visualization of the user-provider association in the provider competition game. See Chapter 4 for more details

resources: denoted q_{ij}, resources that limit the operation of a wireless provider, e.g., bandwidth, transmit power, OFDM tones

service: denoted x_i, where $x_i = \sum_{j=1}^{J} q_{ij} c_{ij}$, service that a wireless user obtains, e.g communication rate

Notation

scalars	are denoted by lowercase letters, e.g., x_i, y, a, b, rarely capital letters, e.g., P_i, R_i, C_i.
vectors	are denoted by boldface letters, e.g., $\boldsymbol{x}, \boldsymbol{q_i}$.
sets	are denoted by calligraphic letters, e.g., $\mathcal{J}, \mathcal{I}$
i, I, $\mathcal{I}$	the index of a user, the total number of users, and the set of all users
j, J, $\mathcal{J}$	the index of a service provider, the total number of provider, and the set of all providers
P_i	power of user i
R_i	rate of user i
a_i	willingness to pay factor of user i
$u_i(\cdot)$	utility function of user i
$\mathcal{U}_i(\cdot)$	expected utility of user i
$v_i(\cdot)$	payoff function of user i (utility minus a payment)
s_i, $\mathcal{S}_i$	strategy of user i and the set of possible strategies
$\boldsymbol{s}$	strategy profile, i.e. a vector of user strategies $\boldsymbol{s} = (s_1, \ldots, s_I)$
δ_{s_i}	pure strategy, i.e. mixed strategy where all the probability mass is on strategy s_i
σ_i	mixed strategy of user i
q_{ij}	demand of user i for the resource of provider j
$\boldsymbol{q}_i$	demand vector of user i, $\boldsymbol{q}_i = (q_{i1}, \ldots, q_{iJ})$
$\boldsymbol{q}$	demand vector of all users
c_{ij}	channel offset parameter, an indication of channel quality
x_i	provided/acquired service in Chapters 4 and 5, i.e. $x_i = \sum_{j=1}^{J} q_{ij} c_{ij}$
p_j	price of provider j
$\boldsymbol{p}$	vector of prices/Lagrange multipliers $\boldsymbol{p} = (p_1, \ldots, p_J)$

1 Background

The lynchpin of microeconomic and game theory is a model which assumes that the decisions of an individual are always oriented towards the maximization of some utility function. In this chapter we give an overview of the theory and assumptions that support and develop this model. The chapter is divided in two parts. The first is an overview of choice theory, a mathematical model of the choice making for individual users. An individual has well-defined preferences over the set of choices. Under certain assumptions, these preferences can be expressed via a utility function. Besides introducing the relevant concepts, we also mention the caveats and assumptions that have to be kept in mind whenever the rational individual assumption is explicitly or implicitly invoked. The second part is an overview of game theory, a mathematical model of decision making in conflicting situations. The game theory is built on choice theory, with additional assumptions. These assumptions involve reasoning about the reasoning of other participants. We introduce relevant concepts for a game such as user strategies, payoffs, solution concepts, and provide examples. The central solution concept is that of Nash equilibrium. Finally, we provide a brief overview of most commonly used game theoretic concepts.

1.1 Modeling how individuals make choices

Our ultimate goal is to model how individuals make choices when the decisions of other participants influence their personal interests. To get there, we start with a basic building block, which is a model of how a single individual makes choices when facing a set of alternatives. The material is loosely based on the theory described in [1], adapted to wireless communications, and in particular to the needs of this thesis.

1.1.1 Preferences of a decision-maker

The basic approach for modeling behavior of an individual, (also, consumer or decision maker), is that of investigating his[1] **preferences** over a set of available alternatives[2]. This approach assumes that an individual has preferences over the set of choices, which we label as X. The consumer could prefer some choices to others, or be indifferent between them. The set of choices is deliberately abstract, in order to model just about anything. For a communication example, consider a consumer who needs to decide between different mobile providers, choose a specific calling plan within a single provider, or decide on the speed of his internet connection.

The differences in desirability between different choices is summarized by a preference relation $\succsim$, a binary operator constructed to reflect the preferences of a consumer. When writing $x \succsim y$, where $x, y \in X$, it is understood that the individual considers choice x to be at least as good as choice y.

Remark 1.1 (**Strict preference and indifference**). *The preference relation $\succsim$ defines two additional relations: strict preference $\succ$ and indifference $\sim$. Strict preference $x \succ y$ implies that choice x is strictly preferred to choice y ($x \succsim y$ but not $y \succsim x$), and $x \sim y$ implies that the consumer is indifferent between the two choices ($x \succsim y$ and $y \succsim x$).*

The two most basic properties that can be fulfilled by a preference relation are **completeness** and **transitivity**:

Definition 1.1 (**Complete preference relation**). *A preference relation $\succsim$ (on X) is complete if for any two choices $x, y \in X$ we have $x \succsim y$, $y \succsim x$, or both.*

Definition 1.2 (**Transitive preference relation**). *A preference relation $\succsim$ (on X) is transitive if for any choices $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim y$.*

The completeness property is fairly natural. It says that an individual should know how to rank any two alternatives from a choice set. Sometimes the choice is clear: I prefer vacationing in Tahiti to vacationing in Iraq. I prefer 2 free movie tickets to 1. But it can be quite difficult as well: "do I prefer these shoes or those?", "should I get a new car or remodel the house?", "should I buy a new lens for my camera or not?". Assuming your spouse does not impose the decision on you, answering these questions may require deep introspection and complicated cost-benefit analysis. Such cost-benefit analysis is usually assumed to take place instantaneously in the mind of the decision maker, which we assume to be infinitely intelligent. Hence even the completeness assumption, the first and most basic assumption in building a theory of decision making, is already putting a light strain on interpretability and applicability of the decision-making theory. Luckily, in this thesis, and in communication literature in general, we are usually dealing with individuals that decide on the amount of a desirable quantity such as communication rate (or undesirable rate such as power used or probability of error), where it is much easier to make comparisons. For example, preferring a $10Mbps$ internet connection to one that

[1] In this thesis, we will use male form without any implications on the gender of the decision maker.

[2] It should be noted that there is another approach to the subject, based on the observed actions of individuals when choosing from a set of alternatives. However, when such observations are absent (which is the case here), only the preference based approach is relevant.

is a measly $1Mbps$ sounds perfectly reasonable. The difficulty arises when the choice between the two comes along with a different price tag, though. Is a 10 Mbps internet connection that costs 100.- per month preferable to a 1 Mbps connection that costs only 20.-? The answer is less clear in this case and most likely depends on the individual in question.

The transitivity property also seems natural (if I prefer oranges to apples, and apples to pears, I may also prefer oranges to pears), but again it is not always fulfilled. Consider an example where Florence needs to get a new laptop, and the choices are Macbook Air (light, expensive, not powerful), Macbook Pro (heavy, expensive, powerful), and Macbook (heavy, cheap, not powerful). She prefers Macbook Air to Macbook Pro, since a Macbook Air is much lighter, and they cost about the same. She prefers Macbook Pro to Macbook since Macbook Pro is much more powerful, and they weigh about the same. Finally, she prefers Macbook to Macbook Air, since Macbook costs considerably less, and they have about the same performance. In this case the preferences do not exhibit transitivity and lead to an indecision that may have lasted a whole week. The problem is that each choice has one good characteristic and two bad ones, and it was not clear which characteristic was the most important one. Eventually it turned out that weight was the most important characteristic, and the Macbook Air was chosen. There are many other real-life examples where transitivity can be violated but we shall not dwell on this.

Even though completeness and transitivity are not always present in practice, for theoretical treatment we assume they are a bare minimum fulfilled by any decision-maker's preference relation:

Definition 1.3 (**Rational preference relation**). *The preference relation $\succsim$ is rational if it is complete and transitive.*

Later on in the text we will invoke a rational user assumption when considering user behavior. A rational user will be defined as a user who chooses the alternative(s) that he prefers the most, and should not be confused with the rationality of a preference relation.

It is not difficult to show that a rational preference relation induces rational strict preference and rational indifference relations as well (see Remark 1.1).

1.1.2 Representing preferences using a utility function

A more convenient way of representing a preference relation of a decision maker is by using a utility function $u : X \to \mathbb{R}$. A utility function assigns numerical values to the choices of a consumer, so that they can be easily compared. Formally:

Definition 1.4 (**Utility function representing preferences**). *A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$ if for all $x, y \in X$:*

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Hence a choice that gives more utility to a user is considered to be more preferred. The utility function most commonly expresses the level of satisfaction (or happiness) that a consumer gets from a certain choice. Alternatively, a utility function may express a disutility (level of dissatisfaction, unhappiness) that a choice brings. This is the case when the choices are undesirable, such as the level of pollution experienced or the probability of error in a given communication scheme.

It should be clear that any strictly increasing transformation of a utility function is again a utility function. The actual numerical value of a utility function is not important if we consider simply the preference of a single user over the set of available choices. For example, if $u(x) = 1$ and $y \succ x$, then any $u(y) > 1$ will express the same information that a user prefers choice y to choice x. This is called the **ordinal** property of a utility function. However, the numerical value (**cardinal** property) is very significant if we give a specific meaning to a utility function. For example, the utility function can signify the maximum amount of money that a consumer is willing to spend on a particular choice. In this thesis, the numerical value of a utility function will be of importance whenever the utility functions of several individuals are compared. In Figure 1.1, two different utility functions are shown. These functions have identical ordinal properties, but their cardinal properties are different.

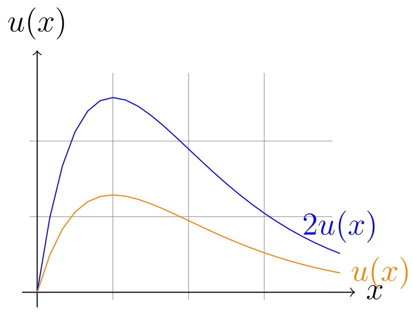

Figure 1.1: Example of a utility function

It is not difficult to verify that a preference relation represented by a utility function is a rational one (i.e. both complete and transitive). On the other hand, not every rational preference relation can be represented by a utility function. This is in particular true of the lexicographic preferences, whose name comes from the way words are arranged in a dictionary. Lexicographic preferences can arise when a user chooses between several options, with the first option being more important than the second one, the second more important than third, and so on. For example, a person may choose a job based on the salary and the distance of the workplace from home. Then, a job offer that comes with the highest salary wins, and only if the two jobs come with the same salary the decision maker looks at how far away the workplace is from home. Even though this example is somewhat artificial, it illustrates how lexicographic preferences work. It turns out that such preferences cannot be represented by a utility function. For example, in communications, this means that we cannot use a utility function to represent preferences of a user who is above all interested in communication rate of a specific operating point, and only if two operating points have the same rate, he looks at how much power needs to be used up, or what is the error rate, etc. So the utility functions may only model preferences that include some tradeoff between different components.

The necessary assumption for the existence of a utility function is that a preference relation be continuous:

Definition 1.5 (**Continuous preference relation**). *The preference relation $\succsim$ on X is continuous if it is preserved under limits. That is, for any sequence of pairs $\{x^n, y^n\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n$ and $y = \lim_{n\to\infty} y^n$, we have $x \succsim y$.*

It turns out that the continuity of a preference relation is enough not only to guarantee the existence of a utility function to represent this relation, but also the existence of a continuous utility function (of course, there are many discontinuous utility functions representing the same relation: just take any increasing, discontinuous transformation of a continuous utility).

We next consider some important preference relation properties, and what these properties imply for the utility function that represents them.

If we are interested in comparing the preferences over different quantities of a commodity, a common assumption for the preferences of a user is that "more of a good thing is better". For example, assuming x and y are scalar quantities of a desirable good, if $x > y$, then it must be that $u(x) \geq u(y)$. Formally:

Definition 1.6 (**Monotone preference relation**). *A preference relation $\succsim$ on X is monotone if for all $x, y \in X$ with $x > y$, we have $x \succsim y$. It is strongly monotone if $x > y$ implies $x \succ y$.*

When x and y are vectors of commodities, the definition is valid when $x - y$ is a non-negative vector. Monotonicity is a property that will often be used when examining behavior of wireless consumers. This is a fairly natural assumption, fulfilled in many situations. For example, even though one may not experience much difference between a connection at 10 Mbps and the one at 10.05 Mbps, given a choice most people would choose the latter[3]. Monotonicity of a preference relation implies an **increasing** utility function.

Another important property that will often be assumed is that of the convexity of the preference relation.

Definition 1.7 (**Convex preference relation**). *Assume that X is a convex choice set. The preference relation $\succsim$ on X is **convex** if for every $x, y, z \in X$ such that $y \succsim x$ and $z \succsim x$, we have $\lambda y + (1 - \lambda) z \succsim x$ for any $\lambda \in [0, 1]$.*

Strict convexity is defined in a similar manner. Convexity can be thought of as an individual's natural inclination towards diversification. For example, if one is indifferent between apples and oranges, then one apple and one orange would be at least as good of a choice as two apples or two oranges. Convexity can also be interpreted in terms of **diminishing marginal rates of substitution**. If we consider some consumption consisting of two goods, then reducing the quantity of one good would require the increase of the other to keep the consumer indifferent. For any successive unit of the first good removed, more and more of the other would be required for compensations under convex preferences.

Contrary to what may seem intuitive, the convexity of a preference relation does not automatically imply the concavity of the utility function being used to represent this relation. It does, however, imply quasiconcavity of any such utility function. In this

[3]Admittedly, some people may say "10 Mbps is more than enough, give the rest to someone who has nothing." These people are the enemy of the rational user assumption.

thesis we will not consider the more general quasi-concave utility functions, but rather focus on the concave ones:

Definition 1.8 (**Concave utility function**). *Assume that X is a convex choice set. A utility function u is concave if $u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$ for any $\lambda \in [0,1]$.*

Indeed, the concavity of a utility function representing a preference relation is a stronger notion and it brings along another important property, namely that of **diminishing marginal utility**. It is somewhat fortunate that concave functions are both convenient to work with, and good at representing a fundamental economic property that the more a consumer has of some desirable good, the less he cares about any additional unit. For example, a man will be a lot more happy about a glass of water if he is thirsty than if he had just drank some. Similarly, if we offer to increase a throughput of a consumer by 1 Mbps, he will thank us a lot more (and pay more) if he has dial-up than if he is already using a broadband connection. Diminishing marginal utility is a common feature in communication; in particular it models well users who are interested in data services.

Another important feature often used in economic analysis is the **numeraire** commodity, which is commonly interpreted as money. The numeraire commodity is one for which the preferences of a consumer are quasilinear:

Definition 1.9 (**Quasilinear preference relation and numeraire commodity**). *The preference relation $\succsim$ is quasilinear with respect to the numeraire commodity if:*

i) $x \sim y$ (x indifferent to y) for some $x \sim y$, implies $x + \alpha \sim y + \alpha$, where $\alpha \in \mathbb{R}$ is an arbitrary quantity of the numeraire commodity.

ii) The numeraire commodity is desirable, in the sense that $x + \alpha \succ x$ for all $\alpha > 0$.

A preference relation that is quasilinear in some commodity admits a utility function that is linear in that commodity, i.e. the utility function is of the form $v(x,\alpha) = u(x) + \alpha$, where α is the amount of the numeraire good. In this thesis, the payoff/happiness/satisfaction of a user will often be in such form, and we interpret the numeraire as **money**. In addition, we will call $v(x,\alpha)$ payoff, and continue to call $u(x)$ utility.

Since $u(x)$ is thought of as the satisfaction that the user is getting from the choice x, the total satisfaction of the user with quasilinear preferences is linearly increasing with the amount of money. There is no a priori reason why user satisfaction should increase linearly in money, as money is similar to other commodities. An individual is likely to experience diminishing marginal rates of substitution for money just as he is for water, bananas, or communication rate. This would imply that money should be represented by a concave function. However, having a quasilinear utility function is convenient, and money is a natural candidate for a numeraire commodity. We can assume that the satisfaction of the consumer with commodity x was transformed to be in line with the numeraire commodity. Now we can also talk about $u(x)$ being interpreted as the **willingness to pay**, meaning that it is the largest amount of money that a user would give in return for good x and still not be dissatisfied with the exchange. Hence, quasilinear utilities are widely used.

In most cases, $-\alpha$ will be the amount of money that a user pays in order to participate in a trade (sometimes by paying for a good, and sometimes by paying a tax). This also implies that a user who does not participate in the exchange would have zero payoff. Then the necessary condition for any consumer to participate in such a scheme is that $v(x, \alpha) > 0$, which is also known as the **individual rationality** condition. This is a very natural condition: unless a consumer stands to make some kind of a gain from a transaction, there is no reason for him to participate in it.

It is worth noting the difference in the quality of certain properties of preference relations and their utility functions. A preference relation that is monotone and convex will be represented by an increasing and quasiconcave function. In this sense, increasingness and quasiconcavity are ordinal properties of a preference relation: any arbitrary increasing transformation of the utility will preserve them. However, the quasilinearity and concavity of utility functions are cardinal properties, and will only be present in some special, convenient-to-work-with utility functions.

1.1.3 Utility maximization as a way to make a choice

From now on we assume that a user has a rational (i.e complete and transitive), continuous, and monotone preference relation, meaning that his preferences can be represented by an increasing and continuous utility function $u(\cdot)$.

The consumer's problem of choosing the preferred consumption choice is traditionally defined in economics literature as:

$$\max_{\boldsymbol{x}>0} u(\boldsymbol{x}) \quad \text{s.t. } \boldsymbol{x}^T\boldsymbol{p} \leq w, \tag{1.1}$$

where $\boldsymbol{x}$ is the vector of commodities, $\boldsymbol{p}$ the vector of prices, $\boldsymbol{x}^T\boldsymbol{p}$ the total amount of money paid, and w the consumer's wealth (i.e. amount of money available). The utility maximizer under the budget constraint, often denoted by $\boldsymbol{x}^*$, is then the consumer's preferred choice. The $\boldsymbol{x}^*$ as a function of prices $\boldsymbol{p}$ and budget w is known in microeconomics literature as the **Walrasian demand function**.

In this thesis, and in the communication economics literature with few exceptions, the budget constraint is not invoked. The reason is that we are usually focusing only on a single good (e.g. communication rate, bandwidth, etc.) which is only a small part of the consumers decision set. In such a case, it is feasible to use the **partial equilibrium analysis**, which is a simplification of the general problem that focuses only on the commodity (or several commodities) that we are interested in.

One of the most interesting features of this simplification is that budget effects disappear, so there is no dependence on w. Another is that the good under study does not influence the prices of other goods in the economy. This is a simplification: the consumers often cannot afford to buy certain products because they have purchased other goods. Nevertheless, the use of no budget constraint is often justified and it reduces complexity. For example, it allows us to gain some insight into the economics of wireless communications. The user's maximization problem then simplifies to the maximization of the user's **payoff** function (see Chapter 10 of [1], in particular Sections 10.C and 10.G):

$$\max_{\boldsymbol{x}>0} v(\boldsymbol{x}) = \max_{\boldsymbol{x}>0} u(\boldsymbol{x}) - \boldsymbol{x}^T\boldsymbol{p}, \tag{1.2}$$

where the utility function $u(\cdot)$ is assumed to be concave, which ensures that a maximizer exists, and the preferences are linear in the numeraire commodity. The demand $\boldsymbol{x}^*(\boldsymbol{p}) = \operatorname{argmax}_{\boldsymbol{x}>0} v(\boldsymbol{x}, \boldsymbol{p})$ is known as Walrasian demand with no wealth effects. We will discuss the consequences of this assumption in Section 1.1.4. For now, notice that the maximizer of (1.2) would change if we consider an increasing transformation of $u(\cdot)$, whereas the maximizer of (1.1) stays invariant under any such transformation.

Hence, a consumer's action is to choose the consumption vector that maximizes his payoff function.

Definition 1.10 (**Rational consumer**). *A rational consumer/user always chooses his most preferred option from the set of alternatives, i.e. he chooses the payoff function maximizer (possibly subject to some constraints).*

A different price vector may result in a different maximizing alternative. The set of maximizing choices defines a demand correspondence: $\boldsymbol{x}^*(\boldsymbol{p}) = \operatorname{argmax}_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{p})$. When the vector $\boldsymbol{x}^*(\boldsymbol{p})$ is unique, then the correspondence is a function, and it becomes feasible to study the changes in the economy as a function of prices.

Definition 1.11 (**Demand function of a consumer**). *Assume that, for any price $\boldsymbol{p}$ there is a unique consumption vector that maximizes the utility function of a user. Then, $\boldsymbol{x}^*(\boldsymbol{p}) = \operatorname{argmax}_{\boldsymbol{x}} v(\boldsymbol{x}, \boldsymbol{p})$ is the demand function of that consumer.*

The total demand of the population is then the aggregate demand of individual consumers. This is not the only way to define demand. A simpler way would be to consider directly a (typically decreasing) function of price and call it demand. Indeed, the aggregate demand of a population is sometimes found in this way (e.g. in Bertrand and Cournot competition [1]). However, we will usually consider demand stemming from a utility function since it clearly shows demand as a result of underlying user preferences.

Before we conclude this section and consider multi-user interaction, we mention some issues that need to be considered while reading the material in the remainder of the thesis.

1.1.4 Issues with assuming a specific utility function

When investigating the behavior of users in wireless communications we often assume a specific user preferences where the user's payoff is linear in the numeraire commodity (i.e. amount of money). The hidden implication of this assumption is that the cardinal properties of users' preference are known, either to the user himself or to the entity in charge for wireless resource allocation. This is a much more involved assumption that the ordinal properties such as completeness and transitivity. For example, the completeness property implies that a consumer can compare two choices and say something about their relative desirability, while the specific utility assumption is implying that a consumer actually knows how much he values every single choice, which is a statement about a choice's absolute desirability. If we mentioned before that the completeness property is not always fulfilled, here we can claim that knowing one's utility function is almost never fulfilled. It is more or less an accepted fact that people are relatively good in making choices when they make comparisons, but they make poor choices when dealing in absolutes [2].

On the other hand, sometimes we may go even further by assuming that the consumers are aware of the utility function of other consumers. In particular, we will see in the next section that this is often the case in game theory, where we talk about players, rather than consumers. We have to be aware that this is a very strong assumption and that the results should be interpreted carefully. Typically this means that the conclusion of the analysis should be of an ordinal rather than cardinal nature. For example, suppose that we do a lengthy calculation and we arrive at a result that $u(x) = u(y) + \epsilon$ for a consumer, where ϵ is some small number. In that case the rationality of a user dictates that he chooses option x. However, unless these two options are easily comparable, the user's introspection reveals that the two options give him approximately the same satisfaction, and he is likely to be indifferent between the two. Similarly, a consumer may not be able to perceive ϵ differences between different options, and may employ rules of thumb to come to a decision.

We make a final note regarding the choices of a consumer. The consumption set is often decided by the producers, an entity that we have so far ignored (and will continue to do so for the most part in this thesis). A firm often incurs some cost in order to produce goods. This, however, is not necessarily true in a wireless setting. A provider may pay some initial fee in order to gain access to the spectrum, but later on this is sunk cost, and is not included in the analysis. The wireless resource can be used over an over. More importantly, in the Chapter 4 when we treat the competition of wireless providers we will simplify things by assuming that the providers have a fixed amount of resource to sell, and can only influence their profits by changing the price of their resource.

1.2 Introduction to game theory

In communications, both wireline and wireless, users are often sharing common communication resources, whether it is optical cables or electromagnetic spectrum. Often, the actions of other consumers adversely impact our own ability to communicate.

Having just introduced the model of how an individual consumer makes choices faced with a consumption set, we now turn to the situation where outcomes depend on the decisions of multiple participants. In such a situation, simply choosing a desired option will not necessarily results in the desired outcome as participants may have conflicting objectives. Then, if an individual is to reach his goal, he needs to take into account the behavior of other participants and act strategically.

Non-cooperative game theory provides a formal mathematical framework for modeling situations where outcomes are decided by independent decisions of many possibly conflicting participants. By contrast, **cooperative game theory** concerns itself with situations where outcomes can be enforced by third party contracts. Although the two are related, we will focus almost exclusively on non-cooperative games.

In the following section, we provide a short introduction to non-cooperative game theory. We consider in detail only the most important concepts and definitions that are consequently used in the remainder of this thesis. In particular, we focus on the **single-shot game** in **normal form**, which is a cornerstone for building the rest of the theory. We will also mention the different sub-fields of game theory, and give their short description. Our hope is to provide a road-map to the world of game theory, by explaining which game theoretic tools are most appropriate for modeling of a specific conflict situation. Additional game theoretic material will be presented when needed in the subsequent chapters.

1.2.1 What is a game

We begin with the most famous example of a game, the prisoner's dilemma:

Example 1.1 (Prisoner's dilemma). *Two prisoners, Chuck and Morgan, are being interrogated in separate cells by the police. The police knows that the suspects are guilty of a small crime, which brings them a sentence of one year in jail each. In addition, the police rightly suspect them to be guilty of a greater crime. During the interrogation, each prisoner is presented with the following choice: confess to the greater crime, or keep quiet. If one prisoner confesses to the greater crime, and the other one keeps quiet, the one that confessed is free to go home, and the one that kept quiet gets 10 years of prison. However, if both prisoners confess, they each get 4 years in prison.*

We give the possible outcomes of this situation for Chuck and Morgan in Table 1.1. There are four table entries, corresponding to four possible outcomes (both confess, both keep quiet, or only one confesses). The first number in the entry is the number of years in prison for Chuck, and the second for Morgan.

		Morgan	
		confess	quiet
Chuck	confess	4,4	0,10
	quiet	10,0	1,1

Table 1.1: The prisoner's dilemma

We now analyze how the prisoners would behave in such a situation. The rational thing to do for the prisoners would be to make choices that minimize their sentences. So, Chuck's reasoning would likely be the following: "If Morgan confesses, I can avoid 10 years of prison and only serve 4 by also confessing. If he doesn't confess, I can confess myself, and walk a free man today! Morgan is not my buddy, this was our first operation together, can I really trust him? Anyway, whatever Morgan does, I am better off confessing." We would expect Morgan to reason in a similar way. Hence, the outcome of this situation would be that both prisoners confess to their greater crime, and serve 4 years of prison each.

For a communication example we will consider the Forwarder's dilemma, introduced in [3].

Example 1.2 (Forwarder's dilemma). *Two wireless users want to send a packet each to their respective destinations. The packet of player 1 needs to be forwarded by player 2 to reach its destination and vice versa. In case his packet is forwarded by the other user, a player receives benefit 1. Each player has a choice to either forward the packet, which costs some power P, or to drop it, which costs him nothing. The payoffs of the players can be expressed by the matrix given in Table 1.2.*

	player 2		
		forward	drop
player 1	forward	1-P,1-P	-P,1
	drop	1,-P	0,0

Table 1.2: The forwarder's dilemma

The forwarder's dilemma has the same form as the prisoner's dilemma, and the same analysis caries over (as well as all the subsequent discussion of the prisoner's dilemma). We expects the wireless users to behave in the same way as the prisoners, resulting in a "drop, drop" outcome.

The reference [3] contains a concise introduction of game theory to communication researchers. In addition to the forwarder's dilemma, it introduces several other communication games. A related, in-depth treatment is given in [4].

We now give a formal definition of game in strategic (or normal) form:

Definition 1.12 (Game in strategic form). *A game in strategic (or normal) form $\mathcal{G} = \{\mathcal{I}, \mathcal{S}, \mathcal{U}\}$ consists of three elements:*

- *the set $\mathcal{I} = \{1, \ldots, I\}$ of I **players**, i.e. the participants in the game.*
- *the set $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_I$ of **player's strategies**, where $\mathcal{S}_i$ contains the strategies available to the player i.*
- *the **player's payoffs** $\mathcal{U} = \{u_1(\cdot), \ldots, u_I(\cdot)\}$ which are functions indicating the values of different outcomes to the players.*

A strategy $s_i \in \mathcal{S}_i$ is an action (choice) that player i can perform. The I-tuple $\boldsymbol{s} = (s_1, \ldots, s_I)$ is called a **strategy profile**, and uniquely defines an outcome, although the same outcome may be the result of different strategy profiles. More importantly, $u_i(\boldsymbol{s})$ is the payoff/utility/satisfaction that player i gets as a result of the outcome that results from a strategy profile $\boldsymbol{s}$. Since what ultimately matters is the payoff of an outcome to a player, many game theory references omit outcomes from the game's definition, as do we. When considering a strategy from the point of view of a single player i, the strategies of other players are often denoted $\boldsymbol{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$. We define $\boldsymbol{\mathcal{S}}_{-i}$ similarly. When considering a single player, we will sometimes say that $\boldsymbol{s}_{-i}$ are the strategies of his opponents. This is a somewhat misleading terminology, as in general players are not true opponents, and the gains to one player do not necessarily entail losses to others[4].

In game theory, the participants are called players in order to emphasize the strategic component of the user's decision making. This is contrary to the previous section, where users only needed to consider the relative attractiveness of different options. These theories should not be thought of as two different theories, rather they complement each other. The players are still considered to be rational, although rationality is less straightforward to define. A player not only has to consider the consequences of his own decisions, but also those of the other players. Furthermore, he also has to consider that the other players know that he is considering their strategies, and so on, ad infinitum. Hence, what is

[4]Games where the players are pure opponents, and sum of their payoffs is zero, such as chess, are called **zero-sum** games.

meant by saying that the players are **rational** is that their actions are consistent with maximizing their payoff considering what they know about the strategies and payoffs of other participants, and what these participants know, and so on.

The prisoner's dilemma contains the three elements of a game in strategic form, which are the players (Chuck and Morgan), their strategies ("confess" or "be quiet") and the values of different outcomes to the participants (the number of years in prison). The outcomes are then that both prisoners confess, only one confesses, or none of them confess. Prisoner's dilemma is a perfect example used to explain a **single-shot** game since it has the following features:

1. **single shot interaction**: The players are playing for the first time (no history of play), and are not going to play this game again.

2. **simultaneous move:** The players have to make their decisions independently of each other. The simultaneity does not necessarily mean that they make their choice at the exact same time, but they do make it at the time when they still do not know the choice of the other player.

3. **complete knowledge:** Each player is perfectly aware of the set of actions and the value of the outcomes for both himself and the other player.

4. **player's rationality:** Each player is trying to maximize his payoff function (or minimize his cost function)

The difference in games will usually come from some variations on these features, but more about this in Section 1.2.6.

The complete knowledge assumption perhaps the most difficult one to justify in most games. In the prisoner's dilemma, this is not a problem, since we can imagine that the policemen explain clearly the possible choices and outcomes to the prisoners. Each prisoner is also perfectly aware of the complete knowledge by his opponent. Most importantly, the prisoners know that not confessing would put them in a difficult position. The actual duration of the sentences in the prisoner's dilemma is of little importance from the analysis point of view. What is important is the relative value compared to the other options. On the other hand, if we replace 4 years by 4 years and 1 day, this is probably no longer the case, and we can expect the prisoners to keep quiet instead of confessing (especially if they make some verbal agreement about keeping quiet before getting arrested). According to the game theoretic analysis, however, the punishments of 10 years and 4 years and 1 day give the same result, namely that both prisoners confess. This is because in theory, two choices that give ϵ difference in utility are distinguishable, whereas real people care less about such small differences.

Another important feature of the prisoner's dilemma game is that the outcome where both players confess is strictly worse than the outcome where both prisoners keep quiet. It is a typical example of how strategic decision-making can lead to socially suboptimal outcomes. This is obviously a negative feature of decentralized decision making. We will formally define notions relative to this issue later.

1.2.2 Solving a game

Once a game is defined, it should be analyzed in order to find out how players would behave when facing a situation described by such a game (as we have done with the prisoner's dilemma prior to formally defining a game). Unlike most engineering disciplines where the analyzed quantity has a well-defined meaning (e.g. communication rate, probability of error), the meaning of game-theoretic solution concepts is a subject of intense and never-ending debates. For example, if we say "this communication rate has probability of error of 10^{-5}", the implications for the communication system employing this rate are clear. On the other hand, saying "this game has a unique Nash equilibrium" does not mean that the behavior of any two people playing this game is precisely determined. As such, game-theoretic solution concepts vary in their applicability and strength, and are usually (at best) approximations to human behavior. We will introduce here only a few such concepts, commenting briefly on their applicability, starting with those that are more easily justifiable. Not surprisingly, we first consider the reasoning used to explain the behavior in the prisoner's dilemma.

Definition 1.13 (**Dominant strategy**). *A strongly dominant strategy s_i^d is a strategy that brings higher payoff to a player than any other strategy, regardless of the strategies of other players: $u_i(s_i^d, \boldsymbol{s}_{-i}) > u_i(s_i, \boldsymbol{s}_{-i})$, for all $s_i \in S_i$, $s_i \neq s_i^d$, for all $\boldsymbol{s}_{-i} \in \mathcal{S}_{-i}$. For a weakly dominant strategy the condition is $u_i(s_i^d, \boldsymbol{s}_{-i}) \geq u_i(s_i, \boldsymbol{s}_{-i})$.*

A strongly dominant strategy is a fairly good prediction of a player's behavior. If such a strategy exists, we can safely assume that a player would choose it. Moreover, we can assume that the other players will make the same conclusion about the actions of that player, which often simplifies the analysis of the game. For example, this is how we concluded the outcome of the prisoner's dilemma game. In particular, the prisoner's dilemma has a characteristic that all players have a strongly dominant strategy, which leads us to our first solution concept.

Definition 1.14 (**Dominant strategy equilibrium**). *A dominant strategy equilibrium is a strategy profile $(s_1^d, \ldots, s_I^d)$ where s_i^d is a strongly dominant strategy for all $i \in \mathcal{I}$.*

It should be clear that there can be only one dominant strategy in any given game. A presence of such an equilibrium is the strongest statement that we can ever make about an outcome of a game. Unfortunately, there are not that many games that have a dominant strategy equilibrium, which means that in most games we need to consider weaker solution concepts.

1.2.3 Mixed strategies

Before we introduce more solution concepts, we give the notion of mixed strategies. The strategies $s_i \in \mathcal{S}_i$ are often called **pure strategies**.

Definition 1.15 (**Mixed strategies**). *A mixed strategy $\sigma_i \in \Delta\mathcal{S}_i$ is a probability distribution on the set of pure strategies $\mathcal{S}_i$, where $\Delta\mathcal{S}_i$ is the set of all probability distributions over the elements of $\mathcal{S}_i$.*

More explicitly, when $\mathcal{S}_i$ is a discrete set, $\sigma_i = \left(\sigma_i(s_i^1), \ldots, \sigma_i(s_i^{|\mathcal{S}_i|})\right)$, where $\sigma_i(s_i^k)$ is the probability of player i choosing pure strategy s_i^k, such that $\sum_{k=1}^{|\mathcal{S}_i|} \sigma_i(s_i^k) = 1$. This definition is changed accordingly when the strategy set $\mathcal{S}_i$ is continuous.

The notion of mixed strategies may seem strange at first in a situation where we are trying to model behavior of consumers, but it turns out that it sometimes the only possible equilibrium consists of mixed strategies. For example, anyone who has ever played rock-paper-scissors knows that playing a deterministic strategy (e.g. choosing a rock) is the fastest way to lose. Hence, a player usually randomizes between the three choices. Another example is a penalty shoot-out in football (soccer), where goalkeepers typically dive left or right before they even see which way the ball is going. In order to be succesful, these goalkeepers need to employ (what appears to be) a randomized strategy.

Finally, notice that pure strategies are nothing more than degenerate mixed strategies, so mixed strategies are a more general concept. It should be said that degenerate mixed strategies (i.e., pure strategies) are often the only considered in order to simplify the analysis. Nevertheless, we consider non-degenerate mixed strategies in Chapter 2, where, using evolutionary game theory, we give an alternate interpretation of mixed strategies as a fraction of a population that plays a certain pure strategy.

Let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_I)$ be a mixed strategy profile. The payoff function has to have the expected utility form (such functions are called von Neumann-Morgenstern utility functions, see Section 6.B of [1] for more details):

$$\mathcal{U}_i = u_i(\boldsymbol{\sigma}) = \sum_{\boldsymbol{s} \in \mathcal{S}} \left[\left(\prod_{j=1}^{I} \sigma_j(s_j) \right) u_i(\boldsymbol{s}) \right]$$

where with some abuse of notation we write $u_i(\boldsymbol{\sigma})$ when the users employ a mixed strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_I)$ over their pure strategies, and label by $\mathcal{U}_i$ the expected payoff. The payoff when employing mixed strategies is calculated as the expected value of the payoffs of different outcomes. For example, if in the Prisoner's dilemma both prisoners play each of their strategies with probability 0.5, then the expected number of years in prison is $\frac{15}{4}$, which is slightly smaller than 4. However, such an outcome is not likely because each of the prisoners would have an incentive to unilaterally deviate from their randomized strategy by always choosing to confess. This reasoning is an introduction to the next equilibrium concept, namely Nash equilibrium.

1.2.4 Nash equilibrium

Arguably the most famous plausible equilibrium strategy profile is the Nash equilibrium.

Definition 1.16 (Pure strategy Nash equilibrium). *A pure strategy profile $\boldsymbol{s}^*$ is a **pure strategy** Nash equilibrium if, for all players $i \in \mathcal{I}$,*

$$u_i(s_i^*, \boldsymbol{s}_{-i}^*) \geq u_i(s_i, \boldsymbol{s}_{-i}^*), \ \forall \ s_i \in S_i,$$

where $\boldsymbol{s}_{-i}^*$ is the tuple of strategies chosen by players other than player i at the equilibrium.

Therefore, a pure strategy Nash Equilibrium is any strategy profile in which it is not beneficial for any user to deviate from his pure strategy. It is typically not difficult to identify Nash equilibria, but sometimes it is not obvious why participants would decide to actually play one. A game may have zero, one, or more than one pure strategy Nash equilibria. The case of multiple equilibria is particularly problematic since it is often not clear which ones the players would end up operating on.

Unsurprisingly, a mixed strategy Nash Equilibrium (or simply Nash Equilibrium) is a mixed strategy profile on which no user can improve unilaterally:

Definition 1.17 (**Nash equilibrium**). *A mixed strategy profile $\boldsymbol{\sigma}^*$ is a Nash Equilibrium if, for all players $i \in \mathcal{I}$,*

$$u_i(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \geq u_i(s_i, \boldsymbol{\sigma}_{-i}^*), \ \forall\, s_i \in S_i.$$

where $\boldsymbol{\sigma}_{-i}^*$ is the tuple of strategies chosen by players other than player i at the equilibrium.

Notice that the mixed strategy Nash equilibrium definition requires that the Nash-equilibrium strategy be weakly better only than any pure strategy of the player. This is since the utility obtained by playing a mixed strategy is a linear combination of the utilities obtained by playing pure strategies. As a consequence, *the expected utility of each strategy that is being played with non-zero probability should be the same*, where the expectation is with respect to other players' strategy probability distribution. Suppose that this was not the case, that a user had a strategy that is giving him more expected utility than the other strategies. Then this user could improve his gain by playing that strategy deterministically.

Mixed strategies are important because of the famous result due to Nash [5]:

Theorem 1.1 (**Nash equilibrium existence in discrete games**). *Every finite strategic-form game (Definition 1.12) has a mixed strategy equilibrium (which may be comprised of degenerate mixed strategies).*

Instead of giving a formal proof (see, e.g. [5]), we give an intuition using the concept of best response.

Definition 1.18 (**Best response**). *The* ***best response correspondence*** $\phi(\boldsymbol{s}_{-i}) = \operatorname{argmax}_{s_i \in \mathcal{S}_i} u_i(s_i, \boldsymbol{s}_{-i})$ *is a set that contains the strategies of user i that give him the highest utility for some fixed strategies of other players. When the best response is a singleton for all $\boldsymbol{s}_{-i}$, we can talk about the* ***best response function.***

For example, a dominant strategy is a best response of a user regardless the strategies of his opponents. A Nash equilibrium achieving strategy of a user is a best response to the strategies of other users in a Nash equilibrium. More formally, $s_i^* = \phi(\boldsymbol{s}_{-i}^*) = \operatorname{argmax}_{s_i \in \mathcal{S}_i} u_i(s_i, \boldsymbol{s}_{-i}^*)$. Let us construct a vector correspondence of users' best responses $\hat{\phi}(\boldsymbol{s}) = [\phi_1(\boldsymbol{s}_{-1})\phi_2(\boldsymbol{s}_{-2}) \cdots \phi_I(\boldsymbol{s}_{-I})]$. Then, at a Nash equilibrium $\hat{\phi}(\boldsymbol{s}^*) = \boldsymbol{s}^*$. In other words, a Nash equilibrium is a fixed point of the correspondence $\hat{\phi}$. Thus, Kakutani's fixed point theorem [6] can be invoked.

Similarly, a more general result holds:

Theorem 1.2 (**Nash equilibrium existence in continuous games**). *Consider a strategic-form game whose strategy spaces $\mathcal{S}_i$ are nonempty compact convex subsets of an Euclidean space. If the payoff functions $u_i(\cdot)$ are continuous in $\boldsymbol{s}$ and quasi-concave in s_i, there exists a pure-strategy Nash equilibrium.*

As we have already mentioned, Nash equilibrium is a weaker solution concept than the dominant strategy equilibrium. In a dominant strategy equilibrium a user plays a strategy that is superior to his other strategies regardless of the strategies of his opponents. In a Nash equilibrium, a user plays a strategy that is weakly superior to the strategies of other players *at the Nash equilibrium only.* So a Nash equilibrium has certain local properties. A dominant strategy equilibrium does not always exist, but when it does, it is unique. A Nash equilibrium always exists, but it need not be unique. It is not difficult to show that every dominant strategy equilibrium is also a Nash equilibrium, while the opposite is not true.

The holly Grail of every game would be to find a dominant strategy equilibrium. Since this is not always possible, a Nash equilibrium is the next best thing. For all its simplicity, it does manage to correctly predict human behavior in a large number of cases. When a single Nash equilibrium exists it is often reasonable to use it as a predictor of the participants' behavior. It is generally assumed that all players can recognize the Nash equilibrium in a game, and assume that their opponents will do the same. So it can be thought of as a reasonable outcome for the system. As the following example demonstrates, multiple Nash equilibria pose problems as it is no longer sure that the players would be able to predict correctly which of the equilibria will be chosen by their opponents.

Example 1.3 (**Coordination game**). *Consider a game defined by the payoff matrix in Table 1.3:*

		her	
		tartar	sushi
him	tartar	1,2	0,0
	sushi	0,0	2,1

Table 1.3: Coordination game

This game goes by the name of "Battle of the sexes," since the background story involves a couple trying to reach a decision about how to spend their evening. In our version the discussion revolves around choosing the restaurant for an evening out. He prefers sushi, she prefers tartar steak. If they disagree on the choice they end up staying at home and eating pasta, which gives them 0 payoff. If they agree on the choice, their preferred choice gives them 2 units of payoff, and 1 unit of payoff if it is not their preferred choice.

This game has three Nash equilibria: two in pure strategies (tartar, tartar) and (sushi, sushi), but the game also admits a mixed strategy equilibrium. It is not difficult to work out that in this mixed strategy equilibrium each person is playing their preferred strategy with probability $\frac{2}{3}$, yielding a payoff of $\frac{2}{3}$. While it is (hopefully) unlikely that couples will pass out on a good time because they cannot reach an agreement (after all, they can agree to alternate to eat tartar one week and sushi the next), the game is good at modeling situations where players who do not know each other need to coordinate their actions. We will see in Chapter 2 that this is the case when we model the behavior of users in an AWGN multiple access channel.

We conclude this chapter with a very intuitive, yet important concept of Pareto optimality:

Definition 1.19 (Pareto optimality). *An outcome $\boldsymbol{s}$ is Pareto optimal if there is no feasible allocation $\boldsymbol{s}'$ such that $u_i(\boldsymbol{s}') \geq u_i(\boldsymbol{s})$ for all $i \in \mathcal{I}$ with strict inequality for at least one i.*

In other words an outcome is Pareto optimal (or Pareto efficient) if we cannot make any player better off without making some other player worse off. The Pareto optimality ensures that an outcome is efficient in the sense that there is no waste. It does not say anything about the *social fairness* of an outcome. In particular, a certain outcome may give all the benefit to one player and none to the others, and still be Pareto optimal. The dominant strategy equilibrium in the prisoner's dilemma is not Pareto optimal, as the "keep quiet, keep quiet" outcome yields strictly better payoffs for both players. The two pure strategy Nash equilibria in the coordination game are Pareto optimal, whereas the mixed strategy equilibrium of the coordination game is Pareto suboptimal.

1.2.5 Some comments on the concept of Nash equilibrium

Since Nash equilibrium is a widely used concept, it is worth making a few comments about its merits as a solution concept. In other words, in a real-world situation that a game is trying to model, why should one expect that the players behave according to the prediction given by a Nash equilibrium? The treatment of this question has been rather informal in the literature until recently, and here we give a brief overview of the discussion.

The main argument is somewhat inverse of what one would expect. Assume for a second that there is an "obvious" way to play the game. A necessary condition for such a way to play it is that it is unique. Then, if such an outcomes exists, rational players will understand it. In particular, no player should have an incentive to change his strategy in such an outcome. So, such an outcome should be a Nash equilibrium.

The key point in the above argument is that the outcome be obvious, and in particular unique. To claim why any given outcome should have such magical properties, different arguments were proposed. Unfortunately, no argument is applicable to all games. Rather, some arguments may work well in some situations, other arguments for others. We present the three main points.

(1) *Focal points* are choices that people make if the outcome seems natural, special, or relevant to them. In other words, in an absence of communication, the players focus on those outcomes that they may be expected to choose, or that may be expected to be expected, and so on. These outcomes are often cultural. For example, if you are informally invited for lunch "tomorrow", the exact time may vary depending on the country, or on the people involved. Another example is determining the location of a meeting when all is known is the meeting time, and a wider area (street, metro

station). In such situations the exact place on the street is usually implied by, for example, some monument or a restaurant or other significant object which is focal.

(2) *Nash equilibrium as a self-enforcing agreement.* Consider a situation where players verbally agree beforehand on the outcome of some game. This outcome is non-binding, and players may not rationally expect the agreement to be fulfilled unless no party has any incentive to deviate from the agreement. Hence, any meaningful agreement should be a Nash equilibrium. In a sense, once the players verbally commit to an outcome, this outcome becomes focal. Meetings typically function in this fashion. All parties agree to a time and a place for a meeting, and get positive payoff for meeting at that place and time. However, meeting at a different place and time would yield similarly positive payoff, but it is the pre-commitment to an outcome that separates it from the other, unlikely, outcomes.

(3) *Nash equilibrium as a stable social convention.* Certain actions repeated over time may yield a particular way for a game to be played, establishing a social convention. For example, party-goers flock to night-clubs to find other people with the same motivation. However, this has to be done at a certain time which may depend on the society in question. Otherwise, a person may find themselves completely alone in an establishment, defying the purpose of their visit.

1.2.6 Other types of games and classification

So far in this chapter we focused on the single shot game of complete information in strategic form. This is this simplest form of a game, and its applicability is limited, often as an approximation to more complicated games. In this section we present other categories of games, most of which build up on the single shot game. The material presented is treated very informally and is not meant to be complete nor rigorous. Our intention is to give as simple as possible of an overview to help facilitate the navigation of an interested reader in this fascinating subject.

We tabulate key concepts for different sub-fields of non-cooperative game theory in Table 1.4, given at the end of this section. We classify games according to two criteria: how many stages they have (duration) and the level of information available to the players about the payoffs of other players (knowledge/information). We caution that the games can also classified according to how much is know about the *observed actions* of other players, and discuss this in more detail later in the section.

Dynamic games of complete information

Games that consist of a single round are called **static** games (also called single shot), whereas any game lasting more than one stage is called a **dynamic** game. We make a distinction between two types of dynamic games with common information: the **repeated games** and what we will call **sequential games** (also known as multi-stage games with observed actions), although their combination is also possible.

In repeated games, same players play the same single-shot game many times over. Their payoff is a discounted function of the payoffs received in different rounds. For example, if there are N rounds, the payoff is $\sum_{k=1}^{N} \delta^k u_i^k$, where u_i^k is the payoff of a user at round k, and $\delta \in (0, 1)$ is the discount factor indicating how important the future is to the player. These games are richer since the strategies of the players may depend on the past actions of their opponents. As we have seen before with the example of prisoner's and forwarder's dilemma, single shot games may be Pareto inefficient, but it turns out that infinitely repeated games often can improve on the single-shot outcome[5]. For example, the (infinitely) repeated prisoner's dilemma game yields a Nash equilibrium which is an improvement on the inefficient dominant strategy equilibrium of the single shot game. The central result is called the Folk theorem. Intuitively, the participants are more likely to make risky non-binding agreements (such as playing "keep quiet") if they know that their good behavior will be rewarded by the other players and bad behavior punished. For example, a shop-keeper may be tempted to rip a customer off if he thinks he will never see that customer again (single-shot game). Many tourists experience this every year. On the other hand if a shopkeeper knows that the customer lives in the neighborhood, it is in the his interest to be fair so that the customer would come back to the store over and over (repeated game). In this case the payoff of both current and future round(s) of the game is important to the player.

In sequential games, the players are not making decisions simultaneously[6]. Rather, a single player makes a choice which is then observed by other players, then the next player makes a move, etc. In such a situation, for example, the player who moves first needs to consider the subsequent behavior of the player who moves second when choosing his strategy. When only one player moves at a time, and the players at any stage are able to observe all of the actions that were chosen beforehand, the game is called one of **perfect information**. For example, the prisoner's dilemma is not a game of perfect information since two players move at the same time, and the action of one player is observed by the other only after they both made their move. The concept of perfect information is not to be confused with the assumption of **complete information**, which supposes that the structure of the game (players, strategies, payoffs) is publicly known to all participants.

Consider the following example of a sequential game:

Example 1.4 (Prize sharing game). *Consider two participants that need to share a* \$10 *prize. For simplicity, assume the players can split the money in* \$1 *increments. It turns out that a fair way to share is to construct a following game: player 1 splits the prize in two, and player 2 then chooses a part of the prize. It is not difficult to see that the equilibrium of this game is: player 1 splits the prize in* (\$5, \$5)*, and player 2 picks one of the two stacks of money.*

Note that, even though both in the prisoner's dilemma and the prize sharing game each player makes a single strategic choice, the former is considered a static game, and the latter a dynamic one. Due to the temporal nature of a sequential game, it is often cumbersome (although possible[7]) to define it in strategic form. Instead, sequential games are often defined in extensive form.

[5]In reality there is no such thing as an infinitely repeated game. However, infinitely repeated games model well situations where participants are not sure after how many rounds of interaction their interaction will end.

[6]Technically speaking, when one player makes a move, the other player can always be thought of as making a "do nothing" move. We make a clear distinction since sequential games lend themselves to somewhat different analysis than the simple repeated games.

[7]For more details, see section 3.4.2 of [5]

Definition 1.20 (Extensive form). *The extensive form of a game contains the following information:*

(1) the set of players

(2) the order of moves

(3) the player's payoffs as a function of the moves that were made

(4) the players' choices at each step

(5) what each players knows when he makes his move

(6) the probability distribution over any moves of nature.

We have seen that the single-shot games are often conveniently represented using matrix form. On the other hand, the sequential games are best represented using a tree. For example, we represent the prize sharing game by a tree in Figure 1.2 (for simplicity only three choices of player 1 are given). The leaves of the tree represent the outcomes for player 1 and player 2, respectively. Since the player 2 always moves after observing the outcome of player 1, he chooses the edges that lead to leaves giving him higher payoff. This is indicated by thick lines. Since player 1 can predict such behavior, he need only consider leaves that emanate from thick edges, where he can obtain payoffs of 5, 4 and 3, respectively. Hence, he chooses a branch that leads to the highest payoff among the thick leaves, which is the leftmost branch.

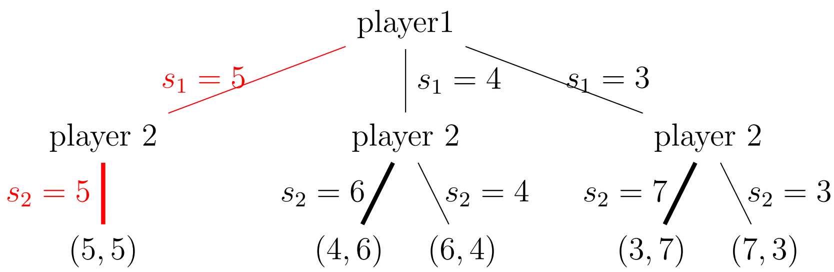

Figure 1.2: Prize sharing game

The reasoning demonstrated here is an example of **backwards induction**, which is a convenient method for solving games of perfect information. In backwards induction, the moves of the final player are first analyzed, then those of the player who moves before him, and so on until reaching the player who moves first.

In some dynamic games there can be a single user moving in one stage, and then several moving in another stage and so on. These are also dynamic games of imperfect information, but they cannot be solved just by using backwards induction. For example, consider a firm that needs to decide whether to build a factory or not in the first stage. In the second stage, this firm and its competitors need to choose production levels. It would be convenient to solve these games using a principle similar to backwards induction. This is the idea behind **subgame-perfect equilibrium**, where every branch of the game should be in equilibrium. For example, if a firm decided to build its factory, the output levels should be in line with Cournot equilibrium. Also, if a firm is to do the opposite, the output levels are chosen by a (different) Cournot equilibrium.

Static games of incomplete information

We now turn to the question of the information available to the players in a game. Both the static and dynamic games considered so far are the games of complete information, in the sense that the players know the strategies and payoffs of all participants. Even though this assumption is sometimes fulfilled, in many situations of strategic interaction such knowledge is absent[8]. When some players do not know the payoffs of others, the game is said to have **incomplete information**. For example, in an auction, the payoff of a player can be his willingness to pay for an object. An auctioneer whose goal is to maximize profit from the sale would gain more money if he was aware of the bidder's (or bidders') willingness to pay. However, an auctioneer almost never has such knowledge.

Before explaining how incomplete information games are dealt with, we first introduce the notion of **randomness due to nature** in games of complete information. For example, consider a modified version of the prisoner's dilemma where the prisoners belong to a criminal organization that has a few judges on its payroll. This organization has no influence on which judge gets the case of the two perpetrators, but if by chance a corrupt judge is appointed, the prisoners would walk free if they do not confess. The prisoners still get long sentences if they confess. Without analyzing the game in detail, it is clear that having more corrupt judges on the payroll implies greater chances that the prisoners would walk free. This means that the "don't confess" strategy may become attractive. The point is that this situation is also considered to be a game of complete but imperfect information in which nature can be thought of as an external player with no payoff. In such games the probability of the nature event is assumed to be known to both players, and the different outcomes of this random event are called **types**. The players then choose their strategy by examining their **expected payoff** averaged over the probability of types.

In games of incomplete information, the players may not have a lot of information about the payoffs of their opponents, but they are always assumed to have some prior **beliefs** about the possible forms that these payoffs may take. As in the case of the nature move, these beliefs are probability distributions over the opponents' types. Then, a game of incomplete information can be transformed into that of imperfect information, where the payoffs of the players occur with some probability.

For example, consider another variation of the prisoner's dilemma where Chuck suspects (with some probability) Morgan to be a police informant, and not really a criminal. If this is the case, Chuck's confession lands him 10 years in prison, and staying quiet the original 1 year sentence (there are no strategies for Morgan). This situation can be modeled as if nature draws a biased coin, and depending on the outcome Chuck either plays the regular prisoner's dilemma, or he plays a smaller game in which there is no danger of a double confession. On the other hand, Morgan chooses his strategy conditioned on his type and there is no uncertainty: if he is a criminal he plays the strategy "confess", and if he is an informant he is not questioned by the police so there is no action to make. From Chuck's point of view, Morgan is playing a mixed strategy. In this case an equilibrium strategy of Chuck would depend on the expected value of his payoff (as in the games of imperfect information where nature moves first). The equilibrium solution concept used

[8] The games of complete information can still be very useful tool to approximate those situations where information is incomplete.

to solve this kind of game is called **Bayesian equilibrium**. Not surprisingly, Bayesian equilibrium is a probabilistic version of Nash equilibrium.

A special version of the games of incomplete information are **mechanism design** games, which consider a central player that invents the rules of some game played by other participants with the intention of maximizing his own utility. Such a central player can be self-interested (a monopolistic firm) or benevolent (trying to maximize social welfare). We will consider such games in more detail in Chapter 3.

Dynamic games of incomplete information

The most general category of games consists then of **multi-stage games of incomplete information**, which are a natural extension of static games of incomplete information. In these games, at the start of a stage, the players observe the actions of players who made decisions in previous stages, but they still do not observe each other's types. The crucial element in these games is then the belief that players have about the distribution of other players' types. The sub-game perfect equilibrium concept that is used for dynamic games with perfect information is no longer applicable, since there is no way for players to know if any outcome is a Nash equilibrium without the complete information. The equilibrium concepts used are then **perfect Bayesian equilibrium**, **sequential equilibrium**, and **trembling hand perfection**. Perfect Bayesian equilibrium, is similar to subgame perfect equilibrium, since each sub-game should result in a Bayesian equilibrium, under a certain assumption on how players update their beliefs about the types of other players. Sequential equilibrium is similar, but the assumptions on how players update their beliefs are more restricted. The third concept, trembling hand equilibrium, assumes that players may "tremble", and play sub-optimal strategies with diminishing probability. For more information on dynamic games of incomplete information, see Chapters 8-10 of [5].

There are many other game theoretic concepts, examples and subtleties that we could talk about, but that would be reaching beyond the scope of this thesis. The key solution concepts, technical terms and examples were given in Table 1.4, but for more information an interested readers is referred to [5] and [7].

<table>
<tr><td colspan="2"></td><td colspan="2">Classification by number of rounds</td></tr>
<tr><td colspan="2"></td><td>static</td><td>dynamic</td></tr>
<tr><td rowspan="2">Classification by available information</td><td>common knowledge</td><td>Single shot game:
e.g. Prisoner's dilemma
Dominant strategy eq.
Nash equilibrium
- iterated dominance</td><td>Sequential games:
e.g. prize sharing
Subgame perfect equilibrium
- extensive form
- backwards induction
Repeated games:
Folk theorem
- discount factor</td></tr>
<tr><td>incomplete information</td><td>Bayesian games:
Bayesian equilibrium
- types
Mechanism Design
e.g. auctions
e.g. optimal taxation
e.g. monopolistic pricing</td><td>Multi-stage games of incomplete information:
Perfect Bayesian equilibrium
Sequential equilibrium
Trembling-hand perfection
Reputation effects
- types
e.g. auctions
e.g. sequential bargaining</td></tr>
</table>

Table 1.4: Non-cooperative game theory breakdown

CHAPTER SUMMARY

- **A utility/payoff function is a convenient way of describing the preferences of a single consumer.**
- **A concave utility function implies:**
  (1) Complete and transitive preferences.
  (2) Convex preferences and diminishing marginal utility.
- **Game theory is used to model interaction of participants in situations where a conflict of interest may exist.**
- **A game is defined by a set of players, strategies, and payoffs.**
- **Nash equilibrium is an outcome where no user has an incentive to unilaterally deviate.**
- **In games of complete information the game structure (players, strategies, payoffs) are publicly known, unlike in games of incomplete information.**
- **In games of perfect information, only one player moves at any time, and his actions are later publicly known**

2 Interaction of Users in an AWGN Multiple Access Channel with a Power Constraint

In this chapter we study the behavior of users in a classical Additive White Gaussian Noise Multiple Access Channel (AWGN MAC) with users who are subject to a maximum average power constraint. This is one of the simplest multiple-user channel, and its capacity region is known and well defined. It also provides a curious example of a situation where having more participants does not necessarily lead to worse performance. Namely, a user can be added to an AWGN MAC with N users in such a way that the rates of the incumbent users stay the same. This is so since the capacity of an AWGN MAC with N users is a subset of the capacity region with $N+1$ users. This situation is quite different than, for example, a highway or a Wi-Fi connection, where every additional user implies a worsening of the situation for the incumbents. Our motivating question is how users in an AWGN MAC choose their operating point in the absence of any regulating authority. This is in contrast to the standard communication setup that implicitly assumes that users already agreed on the individual rates.

Our attention focuses mostly on the two-user case, although a three-user case is briefly considered as well. We model users as rational entities whose only interest is to maximize their own communication rate, and we model their interaction as a noncooperative one-shot game, defined in Section 1.2.1. We first consider a simplified version of the game where the set of users strategies contains only two coding rates, and then generalize the setup to the case where set of strategies contains all achievable rates. For this general game, all of the points on the dominant face of the capacity region are Nash equilibria. Even though all of these equilibria are Pareto optimal, it is not clear which one of them would be played. We then study randomized strategies, and find the probability distributions that constitute the mixed strategy Nash equilibria. As is the case in many games, it turns out that the absence of cooperation and coordination leads to inefficiencies. Indeed, the most interesting mixed strategy Nash equilibrium is not Pareto efficient.

To interpret the mixed strategies in a different way, we turn to evolutionary game theory. We provide a short introduction to the theory, which models the interaction of a population of users over time. The players are not considered rational, but are instead programmed to play a specific strategy. They are drawn two at a time to play a one-shot game, and the resulting payoff is equated with evolutionary fitness. In such a setting, more successful strategies survive and less successful ones disappear. We then apply evolutionary game theory to the AWGN MAC game. A unique evolutionary stable strategy is found for this case, corresponding to the mixed strategy achieving the Nash equilibrium in a simplified one-shot game. This means that a population of users playing this strategy would be getting better rates than a population playing any other strategy. We also find that, when the obtained payoff is interpreted as evolutionary fitness, the user population converges to a state in which the average strategy of the population is the mixed strategy from the simplified version of the one-shot game. The corollary of this result is that the users in this population would, on average, obtain inefficient payoff.

Analyzing the non-cooperative AWGN MAC game both from one-shot and evolutionary perspective leads us to the conclusion that the efficient communication cannot be obtained in a truly decentralized fashion. And while introducing a central planner to rectify the problem is tempting, a semi-decentralized approach also exists. We therefore introduce the concept of correlated equilibrium. Here the players still make independent choices, but they observe the outcome of some auxiliary random variable that can help them coordinate their actions. Hence, their strategies now have an option to be correlated, and not independent, which turns out to be enough to guarantee an efficient equilibrium.

We conclude this chapter with a review of related work, and a partial list of gametheretic work applied to communication settings on the physical layer.

2.1 Two-user interaction in a AWGN MAC channel

We are interested in the behavior of users communicating over an Additive White Gaussian Noise (AWGN) Multiple Access Channel (MAC) in a situation where they make independent decisions about their coding rate, Throughout this chapter, we assume that prior offline agreement between the users is not possible and any central authority is absent.

2.1.1 Communication model

We consider a Gaussian MAC with two senders and one receiver, as indicated in Figure 2.1. The signal at the receiver is

$$Y = X_1 + X_2 + Z, \tag{2.1}$$

where $X_i \in \mathbb{R}$ is the transmitted signal of user i and $Z \in \mathbb{R}$ is zero mean Gaussian noise of unit variance. Each user has an individual average input power constraint $\mathbb{E}[|X_i|^2] \leq P_i$, $i \in \{1, 2\}$. The capacity region $\mathcal{C}$ for this channel [8] is the set of all rate pairs (R_1, R_2)

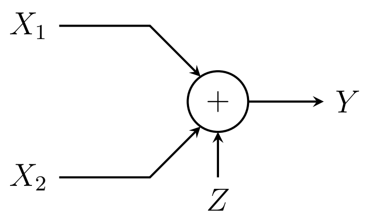

Figure 2.1: Two user AWGN multiple access channel

such that

$$R_i \leq \frac{1}{2}\log(1+P_i),\ i \in \{1,2\}$$
$$R_1 + R_2 \leq \frac{1}{2}\log(1+P_1+P_2).$$

We denote by $C_i = 0.5\log(1+P_i)$ the single-user capacity of user i and by $r_i^0 = 0.5\log(1+\frac{P_i}{1+P_{\bar{\imath}}})$, $i \in \{1,2\}$, the rate of user i when the signal of the other user, $\bar{\imath}$, is treated as Gaussian noise. Note that the signal of user i is decodable when $r_i \leq r_i^0$ even if user $\bar{\imath}$ transmits with $r_{\bar{\imath}} > C_{\bar{\imath}}$, since the receiver can perform the decoding of user i while considering the signal of user $\bar{\imath}$ as noise. Hence, even when the rate pair lies outside the capacity region, the signal of one user may still be decodable. This is illustrated by the shaded regions in Figure 2.2. For obvious reasons we will refer to r_i^0 as the **safe rate**.

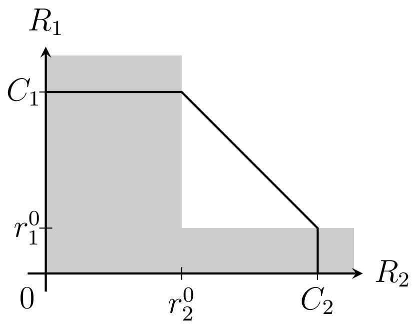

Figure 2.2: AWGN MAC capacity region

The capacity region is the region of all rate pairs that can be achieved by *some* coding strategy of the two users, so that the error probability can be arbitrarily small. We will make a simplifying assumption that all the rates in the capacity region can be obtained if users choose their rates independently.

2.1.2 Two-user AWGN MAC game with two strategies

We assume that the users are only interested in maximizing their own communication rate (as opposed to considering their power, error probability, complexity or delay), and that they are aware that the other user has the same goal. The power constraints are also common knowledge, representing for example, physical limitations of the users' devices.

The users still need to choose their coding rate. Once they do, both of their signals are successfully decoded if they lie inside the capacity region. Otherwise, either the receiver is unable to decode any signal and the observed rates are both zero, or only one of the signals can be decoded (if one of the users is transmitting at or below the safe rate).

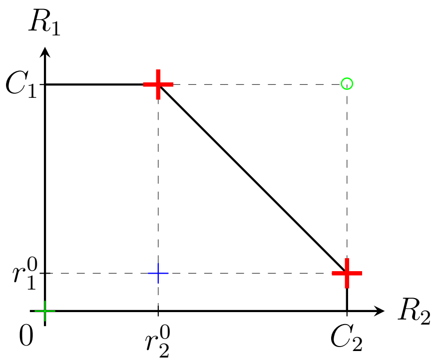

Figure 2.3: Simplified Gaussian MAC game

We begin by assuming that each user has only two strategies: to transmit at rate C_i, or to transmit at the safe rate r_i^0, while treating the other user as noise, as seen in Figure 2.3. Then the strategy sets are $S_1 = \{r_1^0, C_1\}$ and $S_2 = \{r_2^0, C_2\}$ and the payoff matrix is given in Table 2.1. The payoffs are represented simply by the obtained rate, but any increasing transformation of the payoffs in Table 2.1 would give identical results. The precise values do not matter for pure strategy Nash equilibria (as long as the ordering of the values is preserved), but we will see that they do for the mixed strategy Nash equilibria. With these assumptions we define a non-cooperative one-shot game of complete information, which we call the **simplified Gaussian MAC game**.

		rate of player 2	
		safe	capacity
rate of player 1	safe	r_1^0, r_2^0	$\boldsymbol{r_1^0, C_2}$
	capacity	$\boldsymbol{C_1, r_2^0}$	$0, 0$

Table 2.1: Simplified Gaussian MAC game payoffs

The following fact can be verified from the definition of Nash equilibrium.

Fact 2.1. *Strategy profiles* (r_1^0, C_2) *and* (C_1, r_2^0) *are the only two pure strategy Nash equilibria of the simplified Gaussian MAC game.*

If there was only one Nash equilibrium, we could infer that the two users who are in a situation described by this game would likely end up playing that equilibrium. The fact that there are two equilibria is a problem, as it is not clear which one of the two equilibria the users would play. This is similar to the issue discussed for the Coordination game defined in Table 1.3, where the lack of coordination of the users prevents a clear conclusion for the outcome of the game.

In addition, the game admits a mixed strategy Nash equilibrium.

Lemma 2.1. *The mixed strategy profile* (σ_1^*, σ_2^*)*, where* $\sigma_1^* = \frac{r_2^0}{C_2}\delta_{r_1^0} + (1 - \frac{r_2^0}{C_2})\delta_{C_1}$ *and* $\sigma_2^* = \frac{r_1^0}{C_1}\delta_{r_2^0} + (1 - \frac{r_1^0}{C_1})\delta_{C_2}$*, is the mixed strategy Nash equilibrium of the simplified Gaussian MAC game. The corresponding expected utilities are* (r_1^0, r_2^0)*.*

Proof. Suppose that user 1 plays strategy r_1^0 with the probability p, and the strategy C_1 with the probability $\overline{p} = 1 - p$. This means that user 1 is playing safe with probability p, since rate r_1^0 is guaranteed regardless of user 2's actions, and gambles with the remaining probability. Similarly, user 2 plays r_2^0 with probability q and plays C_2 otherwise. The expected payoff for user 1 is then:

$$\begin{aligned}\mathcal{U}_1 = \mathbb{E}[R_1] =& pqr_1^0 + p\overline{q}r_1^0 + \overline{p}qC_1 + \overline{p}\overline{q}0 \\ =& pr_1^0 + \overline{p}qC_1,\end{aligned}$$

where the expectation is taken over both players' mixed strategies. Suppose that $qC_1 > r_1^0$, so there is no incentive to code at the safe rate, then $\mathbb{E}[R_1]$ is maximized if $p = 0$ (in that case $\mathbb{E}[R_1] = qC_1$). On the other hand, if $qC_1 < r_1^0$, then $\mathbb{E}[R_1]$ is maximized if $p = 1$ (in that case $\mathbb{E}[R_1] = r_1^0$, and $r_1^0 > qC_1$). If $q = \frac{r_1^0}{C_1}$, then player 1 should not care about the probability distribution on his strategies, and the expected utility is $\mathbb{E}[R_1] = r_1^0$. In particular, if user 2 plays the mixed strategy σ_2^*, then user 1 cannot improve his payoff by unilaterally deviating from σ_1^* to some other strategy. Similar reasoning holds for user 2 if user 1 plays the mixed strategy σ_2^*. Hence, the mixed strategy profile (σ_1^*, σ_2^*) is the mixed strategy Nash equilibrium of the simplified Gaussian MAC game. □

Some discussion is in order. Note that, in order to arrive to this equilibrium, both users should set their strategy distribution to the probabilities that depend on the other user's parameters. This is a feature of all mixed strategy Nash equilibria. Here, the actual values of the payoffs matter: if we take some increasing transformation of the obtained rates as payoff, this would change the mixed strategy equilibrium as well. So the assumption that players know each other's payoffs (even though perhaps unlikely in practice) is crucial for the mixed strategy Nash equilibrium to be correctly predicted as the outcome. In any case, one can imagine that flipping biased coins is not the most obvious way to go about playing a game and that this requires players with some experience, which is not always the case. We will see in Section 2.4 that there is a different way to interpret mixed strategies, and that mixed strategies arise naturally when the assumption on players' rationality is weakened.

Nevertheless, our analysis so far seems to suggest the following: rational users in a real-world situation, in a AWGN MAC setup with only 2 strategies, without help from some regulating authority and in the absence of cooperation, would likely play conservatively and treat each other as noise, even though this strategy profile is not a Nash equilibrium. Even though the two pure strategy Nash equilibria are Pareto efficient, neither of them is focal, so there is no reason to think that users would naturally recognize one as being more natural than the other. Taking into account the fact that most people are actually risk-averse, rather than perfectly rational, we can conclude that in practice users would rather play the safe rate: people prefer to have some communication rate than risk having none at all.[1] It is a negative result since there are operating points that are strictly better

[1]There may be some users that attempt coding at capacity from time to time, but we will discuss more about this at the end of this section.

than the one that the users will play. Notice that this inefficiency arises not due to the computational complexity that operating at capacity might entail, but simply due to lack of cooperation and coordination among the users. This suggests that we can look into different models that involve cooperation, such as correlated equilibrium and repeated games, in order to make up for the inefficiency. We will do so later in the chapter.

One can also wonder whether the previous argument may be flawed since we allow users to code only at the two extreme strategy points. As we will see, this is not the case and the argument is more general. It is true, however, that restricting the set of strategies of the players can change the way they think about the game. As a degenerate example, consider players whose strategy set is constrained to only one strategy. Then, each player will always play the only available strategy and the only possible strategy profile will be obviously the only equilibrium.

Next, we consider the general case where all the strategies are played with positive probability.

2.2 General two-user AWGN game (continuous strategies)

The set of allowed strategies for user i is now the set of all probability distributions over $[0, C_i]$, and the payoff is the rate at which they communicate. We assume that users have access to codes such that, if the sampled rate tuple lies in the capacity region, users will communicate at that operating point, even if one user is not aware of the code used by the other one. Users have to choose independently the probability distribution over the set of coding rates at the beginning of the communication. With these assumptions we define a non-cooperative one-shot game, which we call the **two-user Gaussian MAC game**.

We denote the (random) attempted rate of user i as R_i, and r_i its sampled value. Notice that r_i and R_i are not the rates at which the users communicate. If the sampled values of the attempted rates are r_1 and r_2, the utility of user i, i.e., the rate at which he communicates, denoted $u_i(r_1, r_2)$, is:

$$u_i(r_1, r_2) = \begin{cases} r_i, & \text{if } (r_1, r_2) \in \mathcal{C} \text{ or } r_i \leq r_i^0 \\ 0, & \text{else.} \end{cases}$$

We denote the probability density function of user i as $\sigma_{R_i}(r_i)$ or simply $\sigma_i(r_i)$. We call $\sigma_i(\cdot)$ the mixed strategy of user i, and the tuple (σ_1, σ_2) a strategy profile. A mixed strategy that chooses some coding rate r_i with probability 1 is called a pure strategy, denoted δ_{r_i}. The quantity of interest for the users is their expected utility $\mathcal{U}_i(\sigma_1, \sigma_2)$, or simply $\mathcal{U}_i$, where $\mathcal{U}_i(\sigma_1, \sigma_2) = \mathbb{E}\left[u_i(R_1, R_2)\right]$. Since the users are making independent decisions, the joint density of (R_1, R_2) is simply the product of the individual densities.

2.2.1 Nash equilibria

We are interested in finding Nash equilibria for this game. We begin with the pure strategy equilibria.

Fact 2.2. *Any pure strategy profile $(\delta_{r_1}, \delta_{C_{1,2}-r_1})$, where $r_1 \in [r_1^0, C_1]$ and $C_{1,2} = 0.5 \log(1+P_1 + P_2)$, is a pure-strategy Nash equilibrium of the Gaussian MAC game. Furthermore, these are the only pure strategy Nash equilibria of the two-user Gaussian MAC game.*

These strategy profiles are exactly those for which (r_1, r_2) is a point on the dominant face of the capacity region. Therefore the Gaussian MAC game has an infinite number of (pure strategy) Nash equilibria. As discussed for the simplified Gaussian MAC game in Section 2.1.2, it is not clear which of these equilibria would be played. In fact, any such equilibrium would require that the users deliberately exclude all but one strategy, and correctly predict that their opponent would do the same.

It may be more reasonable to expect that the users try out different coding rates. In the remainder of this section we study mixed strategies, and in particular we focus on the case where users do not exclude any coding rates from the feasible set of rates.

A mixed strategy Nash equilibrium is a tuple (σ_1^*, σ_2^*) of probability density functions such that $\mathcal{U}_1(\sigma_1^*, \sigma_2^*) \geq \mathcal{U}_1(\sigma_1, \sigma_2^*)$ and $\mathcal{U}_2(\sigma_1^*, \sigma_2^*) \geq \mathcal{U}_2(\sigma_1^*, \sigma_2)$ for all σ_1 and σ_2.

To find mixed strategy Nash equilibria of the Gaussian MAC game, we start with the following observation: a rational user will never transmit at a rate below r_i^0, since that rate can always be improved upon. Therefore, in the mixed strategy equilibrium, user i will have a probability density function that has support on $[r_i^0, C_i]$. We call this interval the set of rational strategies. In Theorem 2.3 we show that there is a unique Nash equilibrium with the entire set of rational strategies of that user as support set.

The following Lemma reveals the crucial property of a mixed strategy Nash equilibrium.

Lemma 2.2. *Let $g_1 : [r_1^0, C_1] \to \mathbb{R}^+$ be defined as*

$$g_1(r) = \mathbb{E}[u_1(r, R_2)] = \int_{r_2^0}^{C_2} \sigma_2(\xi) u_1(r, \xi) d\xi,$$

and, similarly, let $g_2(r) = \mathbb{E}[u_2(R_1, r)]$. Hence, $g_i(r)$ is the conditional expected utility of user i if he plays rate r, where the expectation is over the mixed strategy of his opponent.

For $i \in \{1, 2\}$, let $\bar{\lambda}_i = \max_{r \in [r_i^0, C_i]} g_i(r)$ and let $\bar{\mathcal{J}}_i = g_i^{-1}(\bar{\lambda}_i)$. Here, $\bar{\mathcal{J}}_i$ is the pre-image of $\bar{\lambda}_i$, i.e. the set of rates that give user i the maximum expected utility of $\bar{\lambda}_i$. The mixed strategy profile (σ_1, σ_2) is a Nash equilibrium iff

$$\int_{\bar{\mathcal{J}}_i} \sigma_i(\xi) d\xi = 1, \qquad i = 1, 2. \tag{2.2}$$

Before giving the proof of Lemma 2.2, we give an intuitive explanation. Lemma 2.2 claims that in a Nash equilibrium, the mass of a user's probability density should lie entirely on those strategies that give him maximum expected payoff, where the expectation is over the probability distribution of his opponent. If this were not true, a user could increase his payoff by moving probability mass from strategies that give him suboptimal expected payoff to optimal ones. This is the generalization of a similar argument for discrete strategies.

Proof. By definition, the mixed strategy profile (σ_1, σ_2) is a Nash equilibrium iff, for $i = 1, 2$, the expected utility of user i cannot be increased by unilaterally changing σ_i.

Concerning user 1, observe that

$$\mathbb{E}[u_1(R_1, R_2)] = \int_{r_1^0}^{C_1} g_1(r)\sigma_1(r)dr \leq \bar{\lambda}_1$$

and that the inequality holds with equality iff (2.2) is true for $i = 1$. Similarly, for $i = 2$, $\mathbb{E}[u_2(R_1, R_2)] \leq \bar{\lambda}_2$ holds with equality iff (2.2) holds. □

We begin by finding the unique mixed strategy Nash equilibrium in which users have positive probability density almost everywhere on the set of rational strategies.

Theorem 2.3. *The following probability distributions constitute the unique Nash equilibrium for which the density of R_i, for $i \in \{1, 2\}$, is strictly positive almost everywhere on the interval $[r_i^0, C_i]$:*

$$Pr\{R_1 \leq r\} = \begin{cases} \frac{r_2^0}{C_{1,2}-r}, & r \in [r_1^0, C_1] \\ 0, & else \end{cases} \tag{2.3}$$

$$Pr\{R_2 \leq r\} = \begin{cases} \frac{r_1^0}{C_{1,2}-r}, & r \in [r_2^0, C_2] \\ 0, & else \end{cases} \tag{2.4}$$

Proof. Suppose that there exist σ_1 and σ_2 such that the conditional expected utility of user i, as defined in Lemma 2.2, is equal to some constant λ_i, for all $r_i \in [r_i^0, c_i]$, $i = 1, 2$. Then, as an immediate corollary to Lemma 2.2, the mixed strategy profile (σ_1, σ_2) is a Nash equilibrium. To find such probability distributions, we begin by noting that $g_1(r_1^0) = r_1^0$, regardless of σ_2. Hence, $\lambda_1 = r_1^0$. Then, σ_2 needs to be such that $g_1(r_1) = r_1^0$, for all $r_1 \in [r_1^0, C_1]$:

$$r_1^0 = \lambda_1 = \mathbb{E}[u_1(r_1, R_2)] = r_1 Pr\{R_2 \leq C_{1,2} - r_1\}.$$

By substituting $C_{1,2} - r_1 = r$, $r \in [r_2^0, C_2]$, we can find the probability distribution of user 2 in the mixed strategy Nash equilibrium:

$$Pr\{R_2 \leq r\} = \frac{r_1^0}{C_{1,2} - r}, r \in [r_2^0, C_2].$$

Similarly, we find the probability distribution of user 1. These probability distribution are unique by construction (up to a set of measure zero), hence so is this mixed strategy Nash equilibrium. □

Recall that, due to rationality, user i will not choose any rate below r_i^0. On the other hand, (2.3) evaluated at r_1^0 implies that $Pr\{R_1 \leq r_1^0\} = \frac{r_2^0}{C_2}$. Therefore,

$$Pr\{R_1 = r_1^0\} = \frac{r_2^0}{C_2}.$$

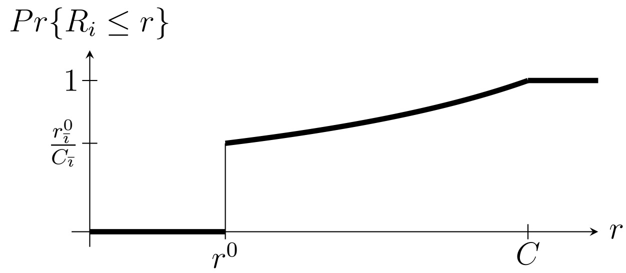

Figure 2.4: Probability distribution achieving Nash equilibrium with support over the set of rational strategies

So, this mixed strategy Nash equilibrium consists of two mixed distributions: each user chooses the safe rate with a positive probability, and the remaining rates in $(r_i^0, C_i]$ by sampling from a continuous probability distribution.

Figure 2.4 shows an example of a probability distribution that achieves the mixed strategy Nash equilibrium where the support is over the set of all rational strategies.

The following result generalizes Theorem 2.3.

Theorem 2.4. *Let $\sigma_i(r_i) > 0$ almost everywhere on $\mathcal{I}^i \subset [r_i^0, C_i]$, $i = 1, 2$, where $\mathcal{I}^i$ is the union of finitely many closed disjoint intervals. Let $a = \min\{r_1 | r_1 \in \mathcal{I}^1\}$, $b = \max\{r_1 | r_1 \in \mathcal{I}^1\}$ and $\mathcal{I}^2 = \{r_2 \text{ s.t. } r_1 + r_2 = C_{1,2},\ r_1 \in \mathcal{I}^1\}$. Then, the following probability distributions constitute a Nash equilibrium*

$$Pr\{R_1 \leq r\} = \begin{cases} 0, & r < a \\ \frac{C_{1,2} - b}{C_{1,2} - \hat{r}_1}, & else \end{cases}$$

$$Pr\{R_2 \leq r\} = \begin{cases} 0, & r < C_{1,2} - b \\ \frac{a}{C_{1,2} - \hat{r}_2}, & else \end{cases}$$

where $\hat{r}_i = \max\{\tilde{r} | \tilde{r} \in \mathcal{I}^i \text{ and } \tilde{r} \leq r\}$, $i = 1, 2$. An example of such a distribution for one of the users can be seen in Figure 2.5, for the case when $\mathcal{I} = [r^0, r^1] \cup [r^2, C]$. The proof of Theorem 2.4 also relies on Lemma 2.2, but is omitted as it is highly technical without providing any insight.

Theorem 2.4 is given for completeness, as its result seems to be relevant only if the sets $\mathcal{I}^1$ and $\mathcal{I}^2$ are fixed ahead of time, which is prevented by the no coordination assumption. One exception where no coordination is assumed is when $\mathcal{I}^1$ and $\mathcal{I}^2$ are $[r_1^0, C_1]$ and $[r_2^0, C_2]$, respectively, in which case Theorem 2.4 reduces to Theorem 2.3. For this reason we consider the mixed strategy Nash equilibrium of Theorem 1 to be the most interesting one.

The expected payoff of a user in the Nash equilibrium of Theorem 2.3 is $\mathcal{U}_i = r_i^0$. One can then wonder why users would not simply transmit at the safe rate with probability 1, resulting in the outcome (r_1^0, r_2^0). But if this were the case, a user would be tempted to unilaterally deviate and transmit at capacity. Therefore, each user has some incentive to code at the safe rate, and some incentive to try out higher rates with positive probability. This is exactly what is predicted by the mixed strategy Nash equilibrium found in

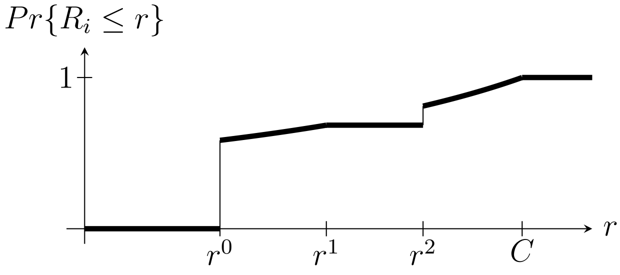

Figure 2.5: An example of a probability distribution achieving Nash equilibrium

Theorem 2.3. The price for the lack of cooperation and coordination is that the mixed strategy Nash equilibrium is not Pareto optimal since any point on the dominant face is component-wise better.

2.3 Short introduction to evolutionary game theory

In this section we give a brief overview of evolutionary game theory, a branch of game theory that will allow us to tackle several issues that arise when using non-cooperative game theory. The exposition is largely based on [9], an excellent source on the subject.

In evolutionary game theory, we consider a population consisting of a large number of individuals. Instead of assuming rational users, we consider individuals who are programmed to play fixed strategies. Typically, the strategies are taken from some symmetric one shot game, where the game is symmetric in the sense that players can switch labels with no change for the game. Then, pairs of players are chosen randomly at some point in time to play the game, and afterwards returned to the population. Then this action is repeated and so on. The payoff obtained by playing these pre-programmed strategies is assumed to be connected in some way, to be defined precisely later, to the evolutionary success of such strategies. Essentially, the strategies that gain more payoff are copied by a greater number of individuals than those that gain less payoff. Such a model makes sense: people tend to copy the good decisions of their peers more than the bad ones.

The evolutionary game theory was initially developed to model the evolution of traits in organisms. The organisms in question being non-sentient, the rationality assumption had to be dropped in favor of assuming pre-programmed strategies. With the advent of experimental economics, it was confirmed that people were not behaving very rationally either, so evolutionary game theory can be applied to human interaction as well.

Evolutionary game theory considers repeated interaction, although this is not to be confused with repeated games. In the repeated games, players are interacting against the same opponent many times, so the history of play matters, but rationality is still assumed on the part of players. In many cases, assuming that users repeatedly interact in a population in a same way but with different opponents is more realistic than considering each of those interactions as a single-shot game. Suppose we play a game we have not played before. In the beginning, we will probably play poorly, but we will soon discover which strategies yield good payoffs and naturally start playing those more often. Hence, what happens in reality is often better modeled as an outcome of a series of trials and errors than as a result of rational decision-making of intelligent players. Of course, how we play will depend on our opponents, as well as what we know about them, and so on.

The evolutionary game theory does away with the rationality assumption, and it models a special form of repeated interaction over time. Another benefit of evolutionary game theory is that multiple equilibria are often taken care of. If a one-shot game predicts multiple equilibria, evolutionary game theory may predict a single one, or it may give conditions on when a certain equilibrium may prevail. For example, if the majority of the population is playing a certain strategy, there may be enough critical mass for that strategy to prevail. Also, the evolutionary game theory gives a different meaning to a mixed strategy. For example, if a fraction of the population is playing some pure strategy, and the remainder a different pure strategy, from the point of view of any given player, his random opponent is playing a mixed strategy.

Consider the following situation: there exist two roads between some town and a nearby city to which people from this town commute. Every morning, each driver has to choose which road to take. Over time, habits form, and each driver may have his preferred road. But if one road is always congested, some drivers will start taking the other one. Notice that not all of the driver will do this, but only some, more entrepreneurial ones. It is reasonable to expect that eventually the two roads will be equally congested (i.e. in equilibrium). Such an outcome is not a result of rational inference: it would be impossible to construct a game of complete information in such a setting, since no driver can possibly know who the other drivers are, what their preferences are and so on. Instead, this outcome comes as a result of local deviations in strategy by some players who adopt strategies that yield higher payoff, and eventually the equilibrium evolves.

In evolutionary game theory there are two points of view: mutation and evolution. The mutation point of view considers a population where all the players are playing the same pure or mixed strategy. Then, a subpopulation of players playing a different strategy is introduced, and the question is if the incumbent users will have higher payoff against the average strategy of the population than the newcomers. The newcomers are often called mutants, since they were introduced by a mutation in the strategy. A strategy which is always stronger than any mutant strategy is called an evolutionary stable strategy, and is one of most important concepts of evolutionary game theory. We first give an informal description.

If σ is the incumbent strategy and $\hat{\sigma}$ a mutant strategy played by an ϵ fraction of the population, then the average strategy of the population is $(1-\epsilon)\sigma + \epsilon\hat{\sigma}$. The definition of the evolutionary stable strategy requires that the expected payoff of an incumbent is strictly greater than the expected payoff of the mutant, both played against the average strategy. This property should hold for any mutant strategy, for a sufficiently small portion of the mutant population. Then, a population of users playing an evolutionary stable strategy will have no incentive to change strategy.

Definition 2.1 (Evolutionary stable strategy). *σ is an evolutionary stable strategy if for every strategy $\hat{\sigma} \neq \sigma$ there exists some $\hat{\epsilon} = \hat{\epsilon}(\hat{\sigma}) \in (0, 1]$ such that the inequality*

$$\mathcal{U}_1(\sigma, (1-\epsilon)\sigma + \epsilon\hat{\sigma}) > \mathcal{U}_1(\hat{\sigma}, (1-\epsilon)\sigma + \epsilon\hat{\sigma})$$

holds for all $\epsilon \in (0, \hat{\epsilon})$.

The concept of evolutionary stable strategy is not concerned with how the population comes to play this strategy, but simply if such a strategy is stable under evolutionary pressures. On the other hand, the concept of replicator dynamics deals with the evolution of strategies.

Replicator dynamics is a system of ordinary differential equations that models how the state of the population is changing, by assuming that the number of offsprings of a user is equal to the payoff of this user in a one-shot game. The assumption is that all the offsprings play the same strategy as their predecessor. Unlike the evolutionary stable strategy, in replicator dynamics the players are only programmed to play pure strategies.

In the following section we apply the evolutionary game theoretic concepts to the simplified Gaussian MAC game.

2.4 Applying evolutionary game theory to Gaussian MAC game

In this section we consider both the mutation and evolution point of view. We begin by asking if there is a strategy that, if adopted by the entire population, will have a greater payoff than a subpopulation of mutants playing some different strategy. In the second part of the section we study how the distribution of strategies in a population evolves. We pose these questions for the population of users who are all playing a **symmetric simplified Gaussian MAC game**, given in Table 2.2, where symmetric refers to $P_1 = P_2 = P$.

	rate of player 2		
	rate	safe	capacity
rate of player 1	safe	r^0, r^0	$\boldsymbol{r^0, C}$
	capacity	$\boldsymbol{C, r^0}$	$0, 0$

Table 2.2: Symmetric simplified Gaussian MAC game payoffs

Formally, we consider a large population of communication users whose power is limited by P. Furthermore, let $r^0 = \log(1 + \frac{P}{1+P})$ and $C = \log(1 + P)$ be the only two allowed coding rates for any user drawn to play the game. Then, the set of allowed distributions is $\Sigma = \{\sigma(\cdot) : \sigma(r) = p\delta_{r^0} + (1 - p)\delta_C,\ p \in [0, 1]\}$. If a fraction $\bar{\epsilon} = 1 - \epsilon$ of the population is programmed to play the incumbent strategy $\sigma_I = p_I\delta_{r^0} + (1 - p_I)\delta_C$, and the remaining fraction ϵ adopts the mutant strategy $\sigma_M = p_M\delta_{r^0} + (1 - p_M)\delta_C$, then the *average strategy* of the population σ_A is defined as

$$\begin{aligned}\sigma_A =& \epsilon\sigma_M + \bar{\epsilon}\sigma_I \\ =& (\epsilon p_M + \bar{\epsilon}p_I)\delta_{r_0} + (\epsilon(1 - p_M) + \bar{\epsilon}(1 - p_I))\delta_C,\end{aligned}$$

Consider first a one-shot symmetric simplified Gaussian MAC game from Table 2.2. This game has three Nash equilibria: two involving pure strategies $((\delta_{r^0}, \delta_C)$ and $(\delta_C, \delta_{r^0}))$, and one involving mixed strategies. The latter is (σ^*, σ^*), where $\sigma^* = \frac{r^0}{C}\delta_{r^0} + (1 - \frac{r^0}{C})\delta_C$, which can be seen by plugging in $P_1 = P_2 = P$ in the mixed strategy equilibrium of the simplified Gaussian MAC game found in Lemma 2.1. Note that, if a user plays against an opponent who is using strategy σ^*, then he will get the same payoff regardless of his own strategy. Indeed: $\mathcal{U}_1(\delta_{r^0}, \sigma^*) = \mathcal{U}_1(\delta_C, \sigma^*) = \mathcal{U}_1(\sigma, \sigma^*) = r^0$, for all $\sigma \in \Sigma$.

This shows that, if the entire population plays σ^*, no single user has an incentive to deviate from this strategy. In the following we prove a stronger property, that no subpopulation of any size has an incentive to deviate from σ^*.

Theorem 2.5. *The mixed strategy $\sigma^* = \frac{r^0}{C}\delta_{r^0} + (1 - \frac{r^0}{C})\delta_C$ is the only evolutionary stable strategy for the simple Gaussian MAC game. Furthermore, $\hat{\epsilon}(\hat{\sigma}) = 1$ for all $\hat{\sigma}$, i.e. users playing σ^* are earning higher average payoff than the mutants regardless of the size of the mutant population.*

Proof. To prove the theorem, we need to show that the strategy σ^* is strictly better than any other mutant strategy. We begin by checking that σ^* is better than any strategy of the type $\hat{\sigma} = (\frac{r^0}{C} - t)\delta_{r^0} + (1 - \frac{r^0}{C} + t)\delta_C$, for $t \in (0, \frac{r^0}{C}]$. We suppose that fraction ϵ of the population plays the mutant strategy $\hat{\sigma}$, and that the remaining $\bar{\epsilon}$ fraction of users stick with the incumbent strategy σ^*. Then,

$$\begin{aligned}\mathcal{U}_1(\sigma^*, \bar{\epsilon}\sigma^* + \epsilon\hat{\sigma}) =& r^0 - \epsilon t C(1 - \frac{r^0}{C}) \text{ and}\\ \mathcal{U}_1(\hat{\sigma}, \bar{\epsilon}\sigma^* + \epsilon\hat{\sigma}) =& r^0 - \epsilon t C(1 - \frac{r^0}{C} + t).\end{aligned}$$

So we see that $\mathcal{U}_1(\sigma^*, \bar{\epsilon}\sigma^* + \epsilon\hat{\sigma}) > \mathcal{U}_1(\hat{\sigma}, \bar{\epsilon}\sigma^* + \epsilon\hat{\sigma})$ for all $t \in (0, \frac{r}{C})$, for all $\epsilon \in (0, 1)$. This means that, whatever the size of the mutant population, the users playing strategy σ^* are strictly better off. Similarly, for all $\hat{\sigma} = (\frac{r^0}{C} + t)\delta_{r^0} + (1 - \frac{r^0}{C} - t)\delta_C$, where $t \in (0, 1 - \frac{r^0}{C}]$ we have:

$$\begin{aligned}\mathcal{U}_1(\sigma^*, \bar{\epsilon}\sigma^* + \epsilon\hat{\sigma}) =& r^0 + \epsilon t C(1 - \frac{r^0}{C}) \text{ and}\\ \mathcal{U}_1(\hat{\sigma}, \bar{\epsilon}\sigma^* + \epsilon\hat{\sigma}) =& r^0 + \epsilon t C(1 - \frac{r^0}{C} - t),\end{aligned}$$

which shows that σ^* is an evolutionary stable strategy.
Since σ^* is an interior point (i.e $\sigma^* \in \Sigma \backslash \{\delta_{r^0}, \delta_C\}$) of the set of probability distributions on r^0 and C, we can invoke Proposition 2.2 of [9], which states that an interior strategy that is evolutionary stable is necessarily the unique evolutionary stable strategy of the game. □

Replicator dynamics for the Gaussian MAC game

Since Theorem 2.5 does not concern itself with how a population gets to playing the evolutionary stable strategy, we now turn to replicator dynamics, which investigates how the strategies played by a population change over time.

We assume that the individuals in a population are programmed to play either pure strategy r^0 or pure strategy C. We also assume that each of these two strategies is present in the population. Without this assumption, a system that starts in a state where all users play a single pure strategy will necessarily stay in the state where all users play that strategy, since mutations are not considered in replicator dynamics.

Let $N_{r^0} > 0$ and $N_C > 0$ denote the number of users playing strategy r^0 and C, respectively, and let $\dot{N}_{r^0}$ and $\dot{N}_C$ be their time derivatives. The key variable in replicator dynamics is the population state, labeled α, where $\alpha = \frac{N_{r^0}}{N_{r^0}+N_C}$ is the fraction of users playing strategy r^0. The fraction of the population playing strategy C is $\bar{\alpha} = 1 - \alpha$ (In a general setup with K strategies played, the state of the system is an N-tuple that indicates the relative frequency of strategies present in the population.). The average strategy of a population with state α is $\sigma = \alpha\delta_{r^0} + (1-\alpha)\delta_C$. Hence, there is a 1-to-1 map between a population state and its average strategy.

We assume that the number of offsprings per unit of time of a user is equal to his expected payoff, and that each offspring inherits its (single) parent's pure strategy[2]:

$$\dot{N}_{r^0} = \mathcal{U}_1(\delta_{r^0}, \sigma)N_{r^0}, \tag{2.5}$$

$$\dot{N}_C = \mathcal{U}_1(\delta_C, \sigma)N_C, \tag{2.6}$$

This model and its analysis are treated in detail in Chapter 3 of [9]. For brevity, we only revisit the key points and equations that are of interest to us. After some manipulations, the fundamental equations of replicator dynamics become:

$$\frac{\partial \alpha}{\partial t} = (\mathcal{U}_1(\delta_{r^0}, \sigma) - \mathcal{U}_1(\sigma, \sigma))\alpha, \tag{2.7}$$

$$\frac{\partial \bar{\alpha}}{\partial t} = (\mathcal{U}_1(\delta_C, \sigma) - \mathcal{U}_1(\sigma, \sigma))\bar{\alpha}. \tag{2.8}$$

Therefore, the growth rate of a population share is equal to the difference between the average payoff that the users playing this strategy are getting, and the average population payoff.

It can be proved that (2.7) and (2.8) define a continuous solution mapping $\xi : \mathbb{R} \times (0,1) \to (0,1)$, where $\xi(t, \alpha^0)$ is the state of the population at time t, if the initial state at $t = 0$ was α^0.

In general, we are interested in the steady state behavior of the system, i.e., in the value of ξ as $t \to \infty$. As it turns out, if the users are playing a game that has an evolutionary stable strategy, the state of the system will converge to one corresponding to this mixed strategy. The statement holds regardless of the initial state. The sufficient condition is that the evolutionary stable strategy is an interior point of the set of mixed strategies (i.e. every pure strategy is played with non-zero probability). This is proved formally in Proposition 3.11 of [9]. The following result is a consequence of this Proposition.

Theorem 2.6. *For any initial state α^0 of the replicator dynamics described by (2.7) and (2.8),*

$$\lim_{t\to\infty} \xi(t, \alpha^0) = \alpha^* = \frac{r^0}{C},$$

where $\alpha^* = \frac{r^0}{C}$ is the state of the system with average strategy $\sigma^* = \frac{r^0}{C}\delta_{r^0} + (1 - \frac{r^0}{C})\delta_C$, the unique evolutionary stable strategy of the simple symmetric Gaussian MAC game.

[2] The model treated in [9] assumes that a population has an underlying birth rate β and death rate γ which do not depend on the strategies being played. We do not consider these since they do not change anything in the analysis. In particular (2.7) and (2.8) hold regardless of the value of β and γ.

As an example, consider a population whose initial state is $\alpha^0 < \frac{r^0}{C}$. This population has too many users that code at C. These users receive a payoff of 0 when playing against each other, and payoff of C when playing against users coding at the safe rate. However, due to the relative frequency of the two strategies, their expected payoff is less than r^0. On the other hand, the users that code at the safe rate will have a payoff of r^0, regardless of the system state. In this case, r^0 is more than the expected average payoff, so α will grow, as predicted by equation (2.7). By equation (2.8), the proportion of users that code at capacity will shrink. The rate of growth of α and the rate of decline of $\bar{\alpha}$ will slow down as the state of the system approaches α^*. The system will eventually converge to the state in which its average strategy is σ^*. Similar reasoning holds if the initial state is $\alpha^0 > \frac{r^0}{C}$. On the other hand, a system with the initial state $\alpha^0 = \alpha^*$ will always stay in that state since $\mathcal{U}_1(\delta_{r^0}, \sigma^*) = \mathcal{U}_1(\delta_C, \sigma^*) = \mathcal{U}_1(\sigma^*, \sigma^*) = r^0$, so $\frac{\partial \alpha}{\partial t} = 0$ and $\frac{\partial \bar{\alpha}}{\partial t} = 0$.

For numerical results, we consider a population of 10000 users who are drawn, two at a time, to interact through the symmetric Gaussian game, and where the payoff gains were interpreted as evolutionary fitness according to replicator dynamics. For $P = 10$, the ratio $\frac{r^0}{C}$ is around 27% which is confirmed in Figure 2.6. The two figures show cases that only differ in the initial proportion of users playing safe rate and those playing capacity. Here, we see that the percentage of users playing the safe rate converges to 27%, regardless of the initial state.

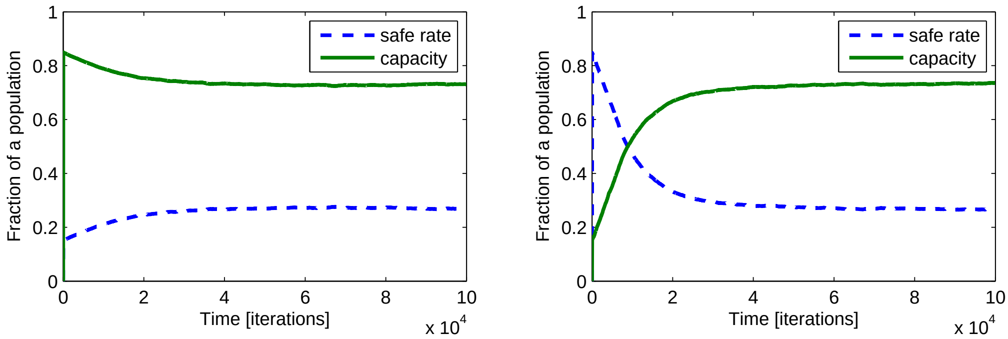

Figure 2.6: Evolution of strategies in a two-user Gaussian MAC game

2.5 Extension to more than two users

In this section, we make an attempt to generalize some of the results from the Gaussian MAC game with two users to the one with an arbitrary number of users. It turns out that the results cannot be extended to the case where more than two users interact.

We begin by treating the case with three users, and assume that all three users have equal power. In particular, we are interested in finding the mixed strategy Nash equilibrium that has support on the set of all rational rates $[r^0, C]$, where now $r^0 = \log(1 + \frac{P}{1+2P})$, and $C = \log(1 + P)$. We also define $r^1 = \log(1 + \frac{P}{1+P})$, which we call the middle rate, that will be useful for the sequel. The three coding rates correspond to the rates that the users would obtain using successive decoding. The safe rate is now the rate obtained by decoding both opponents as noise, and the middle rate is obtained by removing first the signal of the user coding at safe rate. We assume that users can have their signals successfully decoded when choosing the operating point r^1, r^1, r^1, since it lies inside the capacity region of the three-user AWGN MAC.

The **three-user symmetric Gaussian MAC game** is defined in the following way. The users' strategies are the set of all probability distributions over the set of rational rates $[r^0, C]$. The sampled rate tuple r_1, r_2, r_3 is successfully decoded if it lies in the capacity region of the three-user Gaussian MAC: $r_i + r_j \leq \log(1+2P) = C_{12} = C_{23} = C_{31}$ for any $i, j \in \{1, 2, 3\}$ and $r_1 + r_2 + r_3 \leq \log(1+3P) = C_{123}$. In addition, $r_i = r^0$ is always decodable even if both opponents of i try to code at capacity. We begin with the following negative result.

Lemma 2.7. *There is no mixed strategy Nash equilibrium with strategies that have support over the entire set $[r^0, C]$ in a three-user Gaussian MAC game.*

Proof. The application of Lemma 2.2 for the three-user case yields

$$Pr\{R_2 + R_3 \leq C_{123} - r\} = \frac{r^0}{r}, \forall r \in (r^0, C],$$

This means that for r's that are close to r^0 (for example, we take some $r = r^0 + \delta$, $\delta > 0$), we get

$$\begin{aligned} Pr\{R_2 + R_3 \leq C_{123} - r^0 - \delta\} &= \frac{r^0}{r^0 + \delta} \\ Pr\{R_2 + R_3 \leq \log(1+2P) - \delta\} &= \\ 1 - Pr\{R_2 + R_3 > \log(1+2P)\} &- \\ Pr\{R_2 + R_3 \in [\log(1+2P) - \delta, \log(1+2P)]\} &= \frac{r^0}{r^0 + \delta} \end{aligned}$$

Taking the limit of $\delta \to 0$, we get

$$1 - Pr\{R_2 + R_3 > \log(1+2P)\} = 1$$

meaning that

$$Pr\{R_2 + R_3 > \log(1+2P)\} = 0.$$

But, $\log(1+2P) < C_2 + C_3 = \log(1+P) + \log(1+P)$, and $Pr\{R_2 + R_3 \in (\log(1+2P), C_2 + C_3)\} > 0$ since the support of σ_2 and σ_3 is the set of all rational strategies, which is a contradiction. Hence, there exist no mixed strategies that have support on the entire set of rational rates. Notice that this observation holds regardless whether the strategies of the users are symmetric or not. □

This impossibility is the consequence of the continuity of the set of actions. If the user 1 plays at the safe rate, then the receiver will decode his signal, even if the other 2 users are not decodable (i.e. they are transmitting at rates whose sum is larger than $\log(1+2P)$). However, as soon as user 1 is transmitting at slightly more than r^0, then he needs the other two users to be decodable as well for his signal to be decoded with probability very close to 1. However for any probability density that has support on the set of rational strategies, the probability that the other 2 users are not decodable is bounded away from 0.

Since the general mixed-strategy equilibrium does not exist, we now consider the mixed strategy Nash equilibrium on the constrained set of strategies. We consider a **simplified symmetric three-user Gaussian MAC game** where only 3 strategies are played with positive probability: r^0, r^1, C. The payoffs for this game are given in Table 2.3, where the first, second and third table correspond to player 3 choosing strategies r^0, r^1 and C, respectively. We begin by assuming that users have a symmetric probability distribution on these 3 coding rates, and let $Pr\{R_i = r^0\} = x$, $Pr\{R_i = r^1\} = y$, and $Pr\{R_i = C\} = z$. Such an assumption is natural due to the symmetry of the problem. If an equilibrium exists where $\sigma_i = x\delta_{r^0} + y\delta_{r^1} + z\delta_C$, for all $i \in \{1,2,3\}$ then this equilibrium is arguably a good predictor of users' behavior. Other, non-symmetric equilibria exist, but they come in multiplicities. For example, if player 1 always plays r^0, player 2 always plays r^1 and player 3 always plays C, this is a (Pareto optimal) pure strategy Nash equilibrium. However, if we relabel the players by taking any of the remaining 5 permutation on the order of players, the same holds.

r^0		player 2		
		safe	middle	cap.
p.1	safe	r^0, r^0, r^0	r^0, r^1, r^0	r^0, C, r^0
	middle	r^1, r^0, r^0	r^1, r^1, r^0	r^1, C, r^0
	cap.	C, r^0, r^0	C, r^1, r^0	$0, 0, r^0$

r^1		player 2		
		safe	middle	cap.
p.1	safe	r^0, r^0, r^1	r^0, r^1, r^1	r^0, C, r^1
	middle	r^1, r^0, r^1	r^1, r^1, r^1	$0, 0, 0$
	cap.	C, r^0, r^1	$0, 0, 0$	$0, 0, 0$

C		player 2		
		safe	middle	cap.
p.1	safe	r^0, r^0, C	r^0, r^1, C	$r^0, 0, 0$
	middle	r^1, r^0, C	$0, 0, 0$	$0, 0, 0$
	cap.	$0, r^0, 0$	$0, 0, 0$	$0, 0, 0$

Table 2.3: Simplified three-user Gaussian MAC game payoffs

The utility of a player when coding at rate r^0 is r^0, since it does not depend on the choice of rate of the other players. Next, we observe that the expected utility of a player, when coding at the other two rates needs to be also r^0, leading to:

$$\begin{aligned} r^1(x^2 + 2xy + 2xz + y^2) = r^1(1 - z^2 - 2yz) &= r^0 \\ C(x^2 + 2xy) &= r^0, \end{aligned}$$

which, combined with $x + y + z = 1$ gives us a system of 3 equations and 3 unknowns.

The above equations turn out to be difficult to solve in closed form. Alternatively, we perform variable substitution $\alpha = x$, $\beta = x + y$. The governing equations then become:

$$\begin{aligned} \beta^2 + 2\alpha - 2\alpha\beta &= \frac{r^0}{r^1} \triangleq a \\ 2\alpha\beta - \alpha^2 &= \frac{r^0}{C} \triangleq b, \end{aligned}$$

which leads to

$$f(\alpha) \triangleq 2\alpha\sqrt{a + b - 2\alpha + \alpha^2} - \alpha^2 - b = 0. \tag{2.9}$$

For certain values of power P (a and b depend on P) the function $f(\alpha)$ does not have zeros between 0 and 1. For $P \geq 1.56109$, however, there are two values of α such that $f(\alpha) = 0$, leading to two mixed strategy Nash equilibria.

Fact 2.3. *For the three-user symmetric simplified Gaussian MAC game we have the following result concerning the symmetric Nash equilibrium in which users are playing all three strategies r^0, r^1, C with positive probability:*

(1) For $P \leq 1.56108$ *there is no such mixed strategy Nash equilibrium.*

(2) For $1.56109 \leq P \leq 19.3$ *there are 2 such mixed strategy Nash equilibria.*

(3) For $P \geq 19.4$ *there is only one such Nash equilibrium.*

We now turn to replicator dynamics in order to see which mixed strategy Nash equilibrium is likely to arise in the population through repeated interaction.

We consider a population of 10000 users who are drawn, three at a time, to interact through the simplified symmetric three-user Gaussian game, and where the payoff gains are interpreted as evolutionary fitness according to replicator dynamics. We show the results in Figure 2.7. Both figures represent the evolution of strategies over time using replicator dynamics, with $P = 10$, the simulations only differ in the initial distribution of strategies. In the figure on the left the initial proportion of users who play the middle rate is big enough so that the population playing that strategy has an evolutionary advantage. On the other hand, when the safe rate and capacity are initially well represented, the population converges to a state where all three coding strategies are represented, as indicated in the figure on the right. So, depending on the initial state, the system converges either to one of the mixed strategy Nash equilibria states, or to the state where the entire population is playing (r^1, r^1, r^1). This latter result is actually good news, meaning that in some cases all users would play the same strategy, and this strategy would give better expected utility than the other Nash equilibria. However, the simulations also indicate that there is no obvious way to predict how any given population of users would play the game.

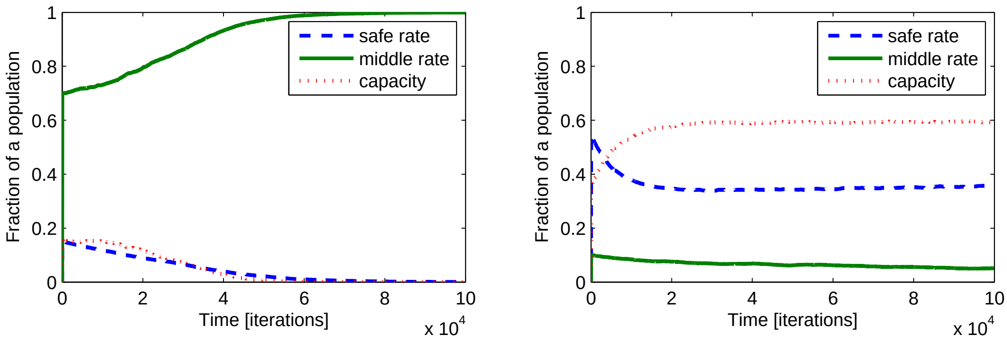

Figure 2.7: Evolution of strategies in a three-user Gaussian MAC game

Running replicator dynamics for $P < 1.56108$ yields an expected result that, regardless the initial state, the population state converges to one where all the users code at middle rate r^1. On the other hand, for $P > 1.56108$ we observe similar behavior as that in Figure 2.7: either only the middle rate survives, or the middle rate is played by a minority. The interesting case is for $P \geq 19.4$ where either the middle rate completely takes over, or it goes extinct. Also curiously, for $1.56109 \leq P \leq 19.3$, where we predict two mixed strategy Nash equilibria where all strategies are played with positive probability, only one of those equilibria is observed as an equilibrium population state.

The above discussion indicates that the three-user Gaussian MAC game is sensitive to the parameters of the problem. Even in the symmetric case with only three strategies, the equilibrium in non-degenerate mixed strategies sometimes does not exist, and sometimes there is a multiplicity of equilibria. The evolutionary game theory does not clarify the issue, as the final state depends also on the initial state. To complete the negative results, the non-trivial mixed strategy Nash equilibria are always Pareto inefficient. So far, the only bright point of the discussion was that any point on the dominant face on the capacity region is a pure strategy Nash equilibrium. Hence, there is hope that users who can agree on a certain efficient point would honor this agreement, as there would be no incentive to deviate. The following section allows for such an agreement to arise even with users who make independent decisions.

2.6 Overcoming the inefficiencies: coordinated equilibrium

Consider now again the two-user simplified Gaussian MAC game from Table 2.1. This game has two pure strategy Nash equilibria, namely (r_1^0, C_2) and (C_1, r_2^0). Playing either of these would be a stable, Pareto efficient solution, although perhaps not an equitable one.

Imagine now that the players can publicly observe an outcome of some binary random variable[3]. This variable could be implemented by means of a pseudorandom binary sequence provided in advance to the users. The variable would be tied to the following convention: when the outcome is 1, the user 1 transmits at capacity and user 2 at his safe rate, and when the outcome is 0, it is the reverse. If the publicly observed random variable is equally likely to be a zero or a one, then the expected payoff by following this strategy is the midpoint of the dominant face region: $\frac{r_1^0+C_1}{2}, \frac{r_2^0+C_2}{2}$. This outcome is desirable since it is both efficient and equitable. Furthermore, it is an equilibrium, since the users have no incentive to deviate from the suggested behavior. This is very important since the convention we introduced is in no ways contractual, it is only added as an optional strategy. But this optional strategy stands out as a desirable solution, which arguably makes it focal (see Section 1.2.5).

The concept we just described is known as the **correlated equilibrium**, first introduced by Aumann [10]. The random variable in question is also known as the correlating device. A correlating device can stem from a natural occurrence: for example everyone can observe that it is a rainy day and opt to have picnic indoors; as opposed to outdoors, which is natural when the weather is nice. We now give an informal definition.

Definition 2.2 (Correlated equilibrium). *Suppose that the players in a game have access to a correlating device (i.e. a random variable) and that the game is expanded by including strategies that are defined relative to the outcome of the correlated device. Then, a correlated equilibrium is a strategy profile which is a Nash equilibrium in the strategies defined relative to the correlating device.*

Fact 2.4 (Correlated equilibrium for AWGN MAC). *Consider the extension of the simplified two-user MAC game by including the correlating device in a form of a random*

[3]We note that more generally the outcome can be partially observed. In some games, a partially observed random variable may lead to a more desirable outcome than a perfectly observed one. For more details, see Section 2.2 of [5].

coin with bias p, where $0 < p < 1$. *The space of strategies is extended by the following correlated strategy pair: when the outcome of the random device is* 1*, user* 1 *transmits at capacity and user 2 at his safe rate, and when the outcome is* 0*, it is the reverse. Then, the correlated strategy pair is a correlated equilibrium of the simple Gaussian MAC game with a correlating device.*

The result follows directly from from Definition 2.2, after noticing that players following a correlated equilibrium strategies have no incentive to deviate. Each player can strongly believe that his opponent will follow the prescribed strategy, so not doing the same can only hurt both players. Hence, it is reasonable to expect that the players will actually play a correlated equilibrium whenever a correlating device can be provided.

We now make several remarks related to the correlated equilibrium.

(1) The use of a correlating device is a form of time-sharing. However, this time-sharing is still decentralized, as it is obtained with no direct communication between the users: it suffices that users obtain the pseudo-random sequence once (e.g., they can be provided by the network manager). Furthermore, the time-sharing is not enforced by a central authority. In particular, the users are still free to ignore the convention and send their information at the rate of their choosing.

(2) If we extend the general two-user Gaussian game (continuous strategies) by including a correlating device, the same way as in the simplified Gaussian MAC game, the same correlated equilibrium exists. Furthermore, we expect the correlated equilibrium to be a reasonable prediction for the game outcome, again using the same argument as for the simplified case. Therefore, by using a correlated device, independent users can operate efficiently and equitably.

(3) The bias of the correlating device in the Gaussian game can be any number p, between 0 and 1, so even less equitable policies, or equitable policies different than proscribing the middle point of the dominant face region, are possible. For example, if there are different classes of users, the users who are from a premium class, who pay more for the communication service, could play in games with a correlated device that proscribes them to play capacity more often than the safe rate. This would have to be accompanied by an opponent who comes from an economy class.

(4) The correlated equilibrium can also be used to solve the coordination issue when a Gaussian game is played by an arbitrary number of users. In each case, the correlating device can ensure that the effective operating point lies somewhere on the dominant face of the capacity region. The main difference is that the correlating device would need to be a discrete random variable that takes N distinct values, where N is the number of players. This would work even in a dynamic setting where the number of users would be constant in one communication session, but could potentially change in the next session.

(5) The labeling of the users makes a difference in the correlated equilibrium, which may not be natural in all games. If both players think they are "player 1", then there is no benefit from the scheme. In the Gaussian game, the players need to exchange some information before the communication session, most importantly in

order to be aware of each-other's power constraints. The players could be assigned labels 1, 2, etc. during this phase, avoiding any potential issues.

(6) In general, a correlating device is useful whenever there is a possibility of a prior arrangement between the players. In the Gaussian MAC game setting, this makes sense. We can imagine that the game is being played by users who subscribed to some service, where the rules of behavior are non-contractual (for example, because they cannot be enforced), but the users are advised to play the game according to the coordinating device.

All in all, the coordinating device solves in a rather elegant manner the issues we faced when considering the Gaussian MAC game. So, why can't it be used to solve any problem arising from lack of cooperation? There are several reasons, but the key one is that not all situations have the special properties that the Gaussian game has. First, a possibility for a prior, non-commital arrangement exists. This is not always the case when players make independent choices. More importantly, the Gaussian game is of the same type as the Coordination game (recall "Battle of the sexes" from Table 1.3). In this type of game, there are two efficient Nash equilibria, so the main issue is how to decide which one of the equilibria to play. The coordinating device takes care of this in an elegant way, and the users have no incentive to deviate regardless the outcome of the correlating device. In other types of games, such as the prisoner's dilemma, the efficient outcome is unstable since the players have an incentive to unilaterally deviate.

2.7 Related work

In this section we briefly consider the work related to ours. For a more detailed treatment, refer to Chapter 6.

In the communication model we considered in this chapter, users are operating under a power constraint over a scalar Gaussian channel. This is in contrast to previous work treating a fading channel model, where users can exploit different fading states to reach a sum-rate optimal operating point [11]. In our work, the optimal power allocation of a user is to transmit at maximum power, so power control is not an issue. The strategies of the users then consist in choosing their communication rates, and the receiver's only role is to decode, if possible. In other works, typically the set of strategies is the set of available powers, not of the available rates. For example, the behavior of users in CDMA was investigated in [12], [13], [14] and [15] just to name a few. In these cases the rate is calculated directly from the SINR, and the strategy of a user consists in choosing the transmit power. In a similar setting [16], the authors describe how users reach the optimal sum-rate power allocation for the limiting case where the number of users goes to infinity and the choice of the decoding order is up to an *arbitrator*. In this case the users' choice is again limited to choosing a power allocation. Similarly, in [11], the receiver imposes a specific decoding order, which leads to users adopting optimal power allocations on different degrees of freedom. Once more the users' rates are uniquely determined from the power allocations.

Evolutionary game theory was already applied in communication settings in [17] and [18]. Recently, our model was extended to communication systems with an arbitrary number of users and continuous evolutionary strategies in [19]. Evolutionary game-theoretic approach was also used in [20] to model the dynamics in a spectrum trading setting involving multiple buyers and sellers. Finally, unlike the work in this chapter which considers non-cooperative game theory, the behavior of users in the AWGN MAC was investigated using cooperative game theory in [21].

CHAPTER SUMMARY

- **In this chapter we model the interaction of users in an AWGN multiple access channel.**
- **We focus on the two-user case, and model it as a non-cooperative single-shot game of complete information.**
- **We observe that there are multiple efficient pure strategy equilibria, indicating that there may be coordination issues in this game.**
- **We characterize the mixed strategy Nash equilibria, and argue that the most interesting one is Pareto inefficient.**
- **We give an evolutionary interpretation of the interaction, which leads to the conclusion that a population of users that participates in this interaction would operate inefficiently.**
- **We suggest a solution in the form of a coordinating device, which is a random variable that can be publicly observed. Such a device would correlate the behavior of users, thus eliminating the issues arising from their independent decision-making.**
- **We provide an overview of the research effort in applications of game theory in a communication setting.**

3 Interaction of Users in an AWGN Multiple Access Channel with no Power Constraint

In this chapter we we consider the behavior of users in the AWGN MAC channel with no power constraint. Our motivation is fueled by the advent of devices equipped with Software Defined Radio (SDR) technology, where users can change the basic operating parameters of their devices, such as power and rate, to match their current needs.

Unlike the discussion in Chapter 2, here we cannot assume that the users know the power constraints of other participants, and hence they are uncertain about the set of achievable rates. In particular, there is no longer a safe rate for a user, since the power of opponents' transmit signal is a priori not limited. We model the users as rational agents maximizing a utility function which is an increasing function of the rate, and a decreasing function of the transmit power, thus ensuring that users do not transmit at infinite power.

The set up of the problem leads naturally to the subject of mechanism design, for which we provide an introduction. In such a setting, the users are revealing their preferences to a benevolent central planner, whose task is to choose a socially optimal point. Then, the main task is to ensure that users truthfully reveal their preferences, so that the outcome assigned by the social planner corresponds to the needs of the users. In particular, we show that the only way to ensure that users in this system behave in line with social goals is to institute a form of Vickrey-Clarke-Groves (VCG) tax.

3.1 System Description

3.1.1 Communication model and user preferences

We consider the multiple access channel $Y = \sum_{i=1}^{I} X_i + Z$, where $Z \sim \mathcal{C}(0,1)$ is a complex zero-mean Gaussian random variable with variance 1, and $X_i \in \mathbb{C}$ is a complex symbol originating from user i, $i \in \mathcal{I} = \{1, \dots, I\}$. We do not impose a power limit on the users, rather we let users decide on which average power they do not want to exceed. We know from information theory [8] that for some vector of power constraints $\boldsymbol{P} = (P_1, \dots, P_I)$ and input distributions such that $\mathbb{E}[|X_i|^2] \leq P_i$ for all $i \in \mathcal{I}$, the rate capacity region $\mathcal{R}(\boldsymbol{P})$ is given by:

$$\sum_{i \in \mathcal{S}} R_i \leq \log(1 + \sum_{i \in \mathcal{S}} P_i)$$

where R_i is the rate of user i, for all subsets $\mathcal{S}$ of the set of users $\mathcal{I}$.

In practice, the power of a user is limited by his communication device. One way to justify the assumption that users do not face a power constraint is to assume that any device power constraint is greater than the one self-imposed by the users. Namely, even with no limits on power, a user would still not transmit at infinite power. This could be, for example, to extend the battery life of the device, or keep the power down due to health concerns.

It is reasonable to assume that the satisfaction of a user decreases as his power use increases, all other factors not considered. On the other hand, the satisfaction of the user increases as his rate increases. In addition, such satisfaction is of diminishing returns. As we already mentioned in Chapter 1, it is natural that an increase in communication rate from, say, 0 to 100 kbps should be worth more to a user than an increase from 2 Mbps to 2.1 Mbs.

For the reasons discussed in the previous two paragraphs, we model the satisfaction of a user with the following utility function[1]:

$$u_i(\mathbf{x}) = w_i(R_i) - g_i(P_i),\ i \in \mathcal{I} \tag{3.1}$$

where $w_i(\cdot)$ is an continuous, increasing, strictly concave function of rate, $g_i(\cdot)$ is a continuous, increasing, strictly convex function of power, and $\mathbf{x} = (\boldsymbol{P}, \boldsymbol{R}) \in X$ is a valid operating point, or *choice* in the space of alternatives, where $\boldsymbol{R} = (R_1, \dots, R_I)$. An operating point is valid if $\boldsymbol{R} \in \mathcal{R}(\boldsymbol{P})$, where $\boldsymbol{P} \in \mathbb{R}_+^I$. We call $\mathbf{x}$ a valid operating point since for all $\mathbf{x} \in X$, there exists a communication scheme that guarantees reliable communication. For all $\mathbf{x} \notin X$, we know that the probability of error is bounded away from zero. Then, u_i is a concave function of (P_i, R_i). We furthermore constrain $w_i(\cdot) = w(\boldsymbol{\theta}_i, \cdot)$ and $g_i(\cdot) = g(\boldsymbol{\theta}_i, \cdot)$ to be functions parametrized by the vector $\boldsymbol{\theta}_i$, which is a preference parameter of a user, otherwise known as the user's **type**. A type is personal to a user, and no other user knows it, while the functions g and w are common knowledge. With this assumption, a user can communicate his preferences simply by sharing his type $\boldsymbol{\theta}_i$, which is simpler than communicating an entire utility function. We also impose $u_i(\mathbf{0}, \boldsymbol{\theta}_i) = 0$, so that a user who is not participating in the scheme obtains no utility nor disutility. The consequence is that a user would never accept an outcome $\boldsymbol{x}$ such that $u_i(\boldsymbol{x}, \boldsymbol{\theta}) < 0$ since he can always obtain more utility by not communicating. We call this **individual rationality**.

[1]There are of course other criteria that could be taken into account for user satisfaction besides rate and power: we treat those we feel are initially the most insightful

As an example, consider the following utility function:

$$u_i(\mathbf{y}, \boldsymbol{\theta}_i) = a_i \log(1 + R_i) - c_i P_i^2;\ i \in \mathcal{I}, \tag{3.2}$$

The utility of user i is completely determined by his type $\boldsymbol{\theta}_i = (a_i, c_i)$. Here, the type of a user consists of a_i, which we will loosely identify as the coefficient of preference for rate R_i, and c_i, which we deem the coefficient of dissatisfaction with power P_i. Such a utility function can model the situation where users can have varying demand for rate and dislike for spending power. For example, smartphone owners need to recharge their phones virtually every night, while some simpler phones can last for two weeks on a simple charge.

The specific form of utility function used in (3.2) possesses the diminishing returns in rate property, and seems simple enough to give us some initial insight into the problem. There are of course numerous other functions that fulfill these properties.

As a negative example, the function $w_i(\cdot)$ cannot be of the form $w_i(R_i) = a_i \log(R_i)$ since $\lim_{R_i \to 0} a_i \log(R_i) = -\infty$.

3.1.2 Single user decision

To begin with, we take a look at what user i would do if there were no other users to share the channel with.

Proposition 3.1 (Single user operating point). *Utility of a single user in an AWGN channel is maximized for a unique and finite power – rate pair.*

Proof. From information theory, we know that $R_i \leq \log(1 + P_i)$. Then, a maximizing point cannot be such that $R_i < \log(1 + P_i)$, since a user can always increase his utility by setting $\hat{R}_i = \log(1 + P_i)$ while keeping the P_i fixed. Hence, the utility $u_i(\cdot)$ is maximized for $R_i = \log(1 + P_i)$, for some P_i, i.e. a single user would always transmit at capacity. The utility function becomes:

$$u_i(P_i) = w_i(\log(1 + P_i)) - g_i(P_i),$$

which is a strictly concave function of P_i, with $\lim_{P_i \to 0} u_i(P_i) = 0$ and $\lim_{P_i \to \infty} u_i(P_i) = -\infty$. Hence, there is a unique value of $P_i^* \geq 0$ and $R_i^* = \log(1 + P_i^*)$ for which u_i is maximized. □

A rational user can find P_i^* that maximizes utility u_i by setting the first derivative to zero:

$$\begin{aligned} \frac{\partial u_i}{\partial P_i} &= \frac{\partial w_i(\log(1 + P_i))}{\partial P_i} - \frac{\partial g_i(P_i)}{\partial P_i} \\ &= \frac{\partial w_i(R_i)}{\partial R_i} \frac{1}{1 + P_i} - \frac{\partial g_i(P_i)}{\partial P_i}. \end{aligned}$$

The marginal satisfaction $\frac{\partial w_i(R_i)}{\partial R_i} \frac{1}{1+P_i}$ due to rate is a strictly decreasing function of the power, with the limit of 0 as $P \to \infty$, and the marginal dissatisfaction $\frac{\partial g_i(P_i)}{\partial P_i}$ due to power is a strictly increasing function. We plot the marginal satisfaction and dissatisfaction with respect to power separately in Figure 3.1. There are two possible cases. Either the marginal dissatisfaction is always greater than the satisfaction, and in that case the utility is maximized by $P_i = R_i = 0$, or the two curves intersect at the point where $\frac{\partial w_i(R_i(P_i))}{\partial P_i} - \frac{\partial g_i(P_i)}{\partial P_i} = 0$, which is the case shown in Figure 3.1. At power P_i^* displayed on the figure, the marginal satisfaction due to rate is perfectly offset by the marginal dissatisfaction due to power and this point maximizes the utility of user i.

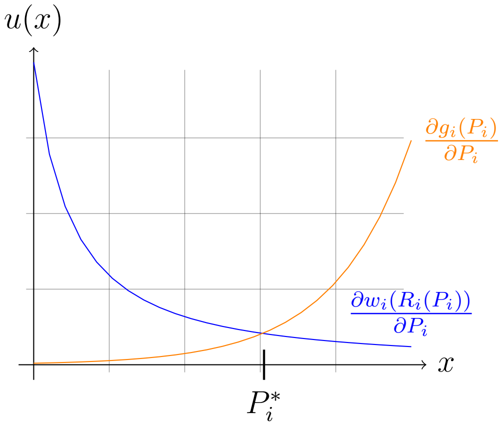

Figure 3.1: Marginal utility with respect to power

So there is no ambiguity as to what a rational user would do if he were alone in the channel. Figure 3.1 can be useful to visualize the change in the best operating power P_i^* for a user if his preferences were to change. For example, an increase in affinity towards rate would increase the optimal operating power.

3.1.3 Multiple user issue

On the other hand, it is not at all clear what the operating point would be if there is more than one user in the channel. Indeed, not all of them can transmit at capacity. Communication theory tells us what the set of feasible rates is as a function of $\boldsymbol{P}$ but it does not concern itself with the question of which operating point will actually be chosen. This is the main issue that we investigate in this section.

We make the assumptions that the users are unable to make coalitions[2]. We are mostly interested in the wireless setting, where users are likely to be geographically separated and unable to directly communicate to one another, which makes coalition creation unlikely. On the other hand, users may need to exchange control messages with the receiver to set up the communication. These messages are exchanged rarely, so we assume that they can be sent reliably, with negligible effects on the communication rate.

In Chapter 2 we considered decentralized decision making of users in a Gaussian MAC. In this chapter, we consider what happens if a central authority is to implement the decision on behalf of the users. In particular, we think of a receiver as an independent entity that is able to enforce a choice on the users, or threaten the users with refusal to

[2]The forming of coalitions in AWGN MAC was investigated in [21].

decode. This is one way of avoiding that users transmit at power and rates that lead to communication breakdown. We assume that users comply with the decision made by the receiver, as long as they get positive utility, since individual rationality prevents a user from participating in the scheme that gives him negative utility. We do not consider the existence of malicious users who try to disrupt other users' communication, since a goal of such participants is not modeled by Equation (3.1).

We now put ourselves in the role of the receiver, with a task to determine which operating point to assign to the users. There is a number of things that a system engineer can try to accomplish in the multiple-user setting. He can try to optimize the sum of rates, assign a single maximum rate that all users can attain, find a point that maximizes the sum of utilities, or in general try to implement any function that yields an operating point based on a certain criteria that is set for the system. Different choices can be justified by different desired properties.

In this work, we consider maximizing the sum of utilities that users get from the system, which has several justifications. To begin with, we can think of the sum of utilities as the overall happiness in the system. Such an allocation criterion is fair, in the sense that it treats everyone's utility equally. Furthermore, resources allocated in this way will go to those that appreciate them the most. For more details on the comparison of different allocation functions and their properties, an interested reader is referred to Chapters 21 and 22 of [1].

The problem can now be stated as follows. We are interested in finding an operating point $\mathbf{x}^* = (\boldsymbol{P}^*, \boldsymbol{R}^*) \in X$ that maximizes the sum of utilities of the users communicating through a multiple access AWGN channel. We call this the Gaussian MAC utility maximization (GUM):

$$\mathbf{x}^*(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_I) = \underset{\boldsymbol{x} \in X}{\operatorname{argmax}} \sum_{i=1}^{I} u_i(\mathbf{x}, \boldsymbol{\theta}_i), \text{ where:}$$
$$X = \{(\boldsymbol{P}, \boldsymbol{R}) : \boldsymbol{R} \in \mathcal{R}(\boldsymbol{P})\}$$

Proposition 3.2 (Uniqueness of the utility maximizer). *The maximization problem GUM is maximized by a unique value of $\boldsymbol{x}^*$.*

Proof. First, note that the sum of utilities is bounded:

$$\underset{x \in X}{\operatorname{argmax}} \sum_{i=1}^{I} u_i(\mathbf{x}, \boldsymbol{\theta}_i) \leq \sum_{i=1}^{I} \underset{x \in X}{\operatorname{argmax}}\, u_i(\mathbf{x}, \boldsymbol{\theta}_i).$$

Recall that any $\boldsymbol{x} \in X$ is subject to AWGN MAC capacity constraints: $\sum_{i \in \mathcal{S}} R_i \leq \sum_{i \in \mathcal{S}} \log(1 + P_i)$ for all $\mathcal{S} \subset \mathcal{I}$. Suppose that there are two maximizing solutions $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, where $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \operatorname{argmax}_{\boldsymbol{x}} \sum_{i=1}^{I} u_i(\boldsymbol{x}, \boldsymbol{\theta}_i)$. Then, both of these fulfill the capacity constraints. It can be verified that any convex combination $\boldsymbol{x}^\lambda = \lambda \boldsymbol{x}^1 + (1 - \lambda)\boldsymbol{x}^2$ of these two maximizing solutions also fulfills the constraints. Since $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are maximizers, $\sum_{i=1}^{I} u_i(\boldsymbol{x}^\lambda, \boldsymbol{\theta}_i) \leq \sum_{i=1}^{I} u_i(\boldsymbol{x}^1, \boldsymbol{\theta}_i) = \sum_{i=1}^{I} u_i(\boldsymbol{x}^2, \boldsymbol{\theta}_i)$. But, since $u_i(\boldsymbol{x}, \boldsymbol{\theta}_i)$ is strictly concave in $\boldsymbol{x}$ for all i, then

$$\begin{aligned}\sum_{i=1}^{I} u_i(\boldsymbol{x}^\lambda) >& \lambda \sum_{i=1}^{I} u_i(\boldsymbol{x}^1) + (1-\lambda) \sum_{i=1}^{I} u_i(\boldsymbol{x}^2) \\ =& \sum_{i=1}^{I} u_i(\boldsymbol{x}^1) = \sum_{i=1}^{I} u_i(\boldsymbol{x}^2),\end{aligned}$$

which is a contradiction. □

We now recap the actual process that takes place. Several users want to communicate to the same receiver. After establishing the connection with the transmitters, the receiver informs each of them that the channel is shared with other users, and that they should disclose their preferences, in the form of types, in order to find the operating point that maximizes the overall utility of the system. Users send their types to the receiver on a separate channel, the receiver finds the operating point which maximizes the sum of utilities, and then informs the users what this operating point is.

3.1.4 Why users might not reveal their true preferences

The procedure we just described maximizes the overall happiness in the system, but it does not maximize the individual user's utilities. This is a problem since rational users do not care that the entire system is better off: they only care about their own utility. Consequently, implementation difficulties may exist. For example, if two users have identical utility functions, each user has an incentive to report a type that implies a higher affinity for rate (higher value of w_i), or conversely, a stronger dissatisfaction with using power (higher value of g_i). This is so since a user can gain considerably by misreporting his type. In that case we say that an outcome is **manipulable**.

> For an example where users misrepresent their preferences, consider a presidential election where there are three candidates. Suppose that 46% prefer candidate A, 48% prefer candidate B, and only 4% prefer of the electorate prefers candidate C. Suppose candidate A is left-wing, and candidate C even more so. Then, rational voters who would normally vote for candidate C will vote for candidate A, since that way a left-wing president can be elected. If they do not do so, a right wing candidate B wins the majority of the vote. In this sense, the election is manipulable by the electorate, since people are not reporting their true preference to the question: "which candidate would you prefer as president".

Hence, in general, we cannot assume that users will report their true types. Instead, it is more reasonable to expect that they will behave strategically, by reporting the value that they think will maximize their own utility. We are interested in designing a system where users will naturally choose to report their true preferences. It is not a priori obvious that something like this is possible, but it should be clear that such a system is desirable. We will investigate when it is possible to do this, for what utility functions, and what are the inherent costs, if any.

In order to do this, we introduce some notions from **Social choice theory** and **Mechanism design**. Social choice theory deals with the *nature* of the choice that can be imposed on multiple persons that have differing utilities and need to choose an alternative. Mechanism Design, is a study of how a certain outcome can be *obtained*, in a situation where the collective decision differs based on the individual strategic behavior. In the following chapter we introduce some notions and results from social choice theory and mechanism design that will give us insight into what is possible and what is not in our communication scenario.

3.2 Introduction to mechanism design

The following definitions are either taken verbatim or paraphrased from [1]. We label the space of alternatives X, and we label the set of users $\mathcal{I}$, where $|\mathcal{I}| = I$ is the number of users.

3.2.1 Definitions

Definition 3.1 (**Social choice function**). *Let $\boldsymbol{\theta}_i \in \Theta_i$ be a type of user i, and X a set of possible outcomes. A social choice function $f : \Theta_1 \times \ldots \times \Theta_I \to X$ is a function that assigns a collective choice $f(\boldsymbol{\theta}) \in X$ for each possible profile $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_I)$ of user types.*

A social choice function is simply a way of choosing an operating point based on the preferences of all the participants. We are interested in those social choice functions that are in the best interest of society. For example, maximizing the sum of utilities in a system is considered a good thing to do, as we have already mentioned in the previous section.

Example 3.1 (**Utility sum maximizing social choice function**). *Suppose that user i has type $\boldsymbol{\theta}_i$. Then f is a social choice function that maximizes the sum of utilities if $f(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_I) = \mathbf{x}^*(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_I) = \mathrm{argmax}_{\mathbf{x} \in \mathbf{X}} \sum_{i=1}^{I} u_i(\mathbf{x}, \boldsymbol{\theta}_i)$.*

Example 3.2 (**Dictatorial social choice function**). *A dictatorial social choice function is such that it chooses an alternative that yields the best possible utility for one user, who is then called the dictator. For example, when $I = 2$, any function that chooses an operating point from the set $X_1 = \mathrm{argmax}_{\mathbf{x}}(u_1(\mathbf{x}, \boldsymbol{\theta}_1))$ is dictatorial with user 1 being the dictator. A special case would be a function that finds the best option for user 2 among the set X_1 of best options for user 1. Formally,*

$$\hat{\mathbf{x}}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \underset{\mathbf{x} \in X_1}{\mathrm{argmax}}\, u_2(\mathbf{x}, \boldsymbol{\theta}_2), \text{ where } X_1 = \underset{\mathbf{x} \in X}{\mathrm{argmax}}(u_1(\mathbf{x}, \boldsymbol{\theta}_1))$$

This social choice function $\hat{x}(\boldsymbol{\theta})$ is obviously dictatorial, however it is the best social choice function for user 2 out of all those in which user 1 is the dictator.

As we mentioned already in Chapter 1, a basic criterion for identifying a good social choice function is Pareto efficiency.

Definition 3.2 (**Pareto efficient social choice function**). *A social choice function f is (Pareto) efficient, if there exists no $x \in X$ such that $u_i(x, \boldsymbol{\theta}_i) \geq u_i(f(\boldsymbol{\theta}), \boldsymbol{\theta}_i)$ for all $i \in \mathcal{I}$, for all $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_I)$, with the inequality being strict for at least one i.*

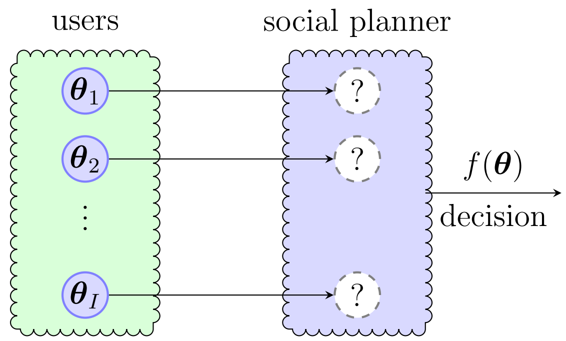

Figure 3.2: Social choice function

So, a social choice function is Paretian, or simply efficient, if no user can be made better off without making another user worse off. It is clear that our social choice function that maximizes the sum of utilities is Pareto efficient. Recall that Pareto efficiency does not imply anything on the fairness of the outcome. For example, even the dictatorial social choice function that we introduced in the previous paragraph is efficient, without necessarily being egalitarian.

The social choice function assumes that the user types $\boldsymbol{\theta}$ are available to the social planner. For example, a city planer that wants to implement a public project would have to know how each person in the city feels about this project in order to make a decision that maximizes the sum of utilities that citizens draw from it. If this were possible (for example, by having an omniscient city planner), there would be no need for designing mechanisms. We depict this in Figure 3.2.

Since this is rarely the case, the city planer can try to solicit information from the citizens about their opinions on the project, and then make a decision based on their input. There are different ways of asking for feedback. The social planner could ask people to fill out questionnaires, write or phone to him with their opinions and even, in theory, ask them how much utility they would receive from this project. In all of these cases the people are required to reveal their preferences by submitting some form of a message. We call S_i the set of allowed messages of user i, and we call this the strategy set of user i. A strategy then for user i is a way of choosing the message to submit to the planner based on his actual preferences, i.e., it is a function $s_i : \Theta_i \to S_i$.

After soliciting information from the citizens, the city planer can implement his project by using an outcome function, which produces a social decision based on submitted user strategies, as depicted in Figure 3.3. We formally call this a mechanism:

Definition 3.3 (**Mechanism**). *A mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ is a collection of I strategy sets $(S_1, \ldots, S_I)$ and an outcome function $g : S_1 \times \ldots \times S_I \to X$.*

So a mechanism is nothing but a decision-making procedure[3]. The outcome function does the same as the social choice function: it assigns an operating point. The social

[3]We remark that the mechanism could be defined simply by specifying the outcome function, since

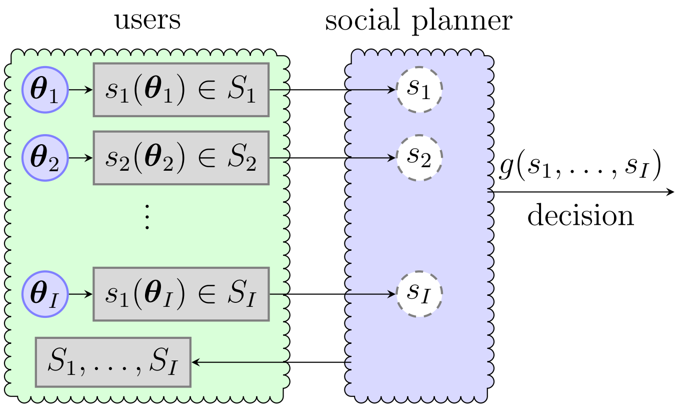

Figure 3.3: Mechanism

choice function gives the *goal* based only on the user types. A mechanism is more general, as it consists of defining both the strategy sets, and the outcome based on the choices from those strategy sets. Usually the goal of a mechanism designer is to construct a mechanism such that the outcome function results in the same operating point as some social choice function.

A special case of a mechanism is a direct revelation mechanism, where the social planner asks the users to reveal their types. This is depicted in Figure 3.4. Notice that users need not reveal their true types. In general user i will report the type $\hat{\boldsymbol{\theta}}_i$ that is strategically chosen to maximize the utility $u_i(\cdot, \boldsymbol{\theta}_i)$ which depends on the real type. The direct revelation mechanism is the most important class of mechanisms.

Definition 3.4 (**Direct revelation mechanism**). *A direct revelation mechanism is a mechanism in which the strategy sets are confined to be the set of types, that is* $S_i = \Theta_i$*, and* $s_i(\boldsymbol{\theta}_i) = \hat{\boldsymbol{\theta}}_i \in \Theta_i$.

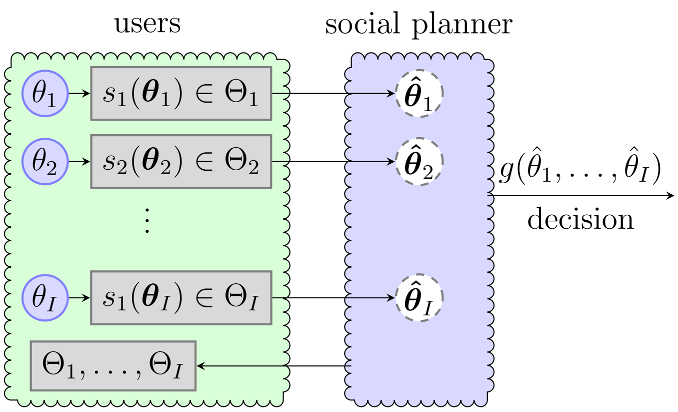

Figure 3.4: Direct revelation mechanism

We now recap the current setup. Each user privately observes their own type $\boldsymbol{\theta}_i$ that other users are unaware of. All users are aware of the allowed strategy sets $(S_1, \ldots, S_I)$, as

the strategy sets are already implied in its definition. Mechanism design theory explicitly mentions the sets in the definition, in order to emphasize their role when designing the mechanism.

well as of the outcome function $g(\cdot)$. Each user then chooses his action $s_i(\boldsymbol{\theta}_i)$ depending on the observed type and sends the message to the receiver. These messages combine to form an I-tuple $s(\boldsymbol{\theta}) = (s_1(\boldsymbol{\theta}_1), \ldots, (s_I(\boldsymbol{\theta}_I))$, which we call a *strategy profile*. For a direct revelation mechanism, users are asked to reveal their type. Based on the received strategy profile, the social planner evaluates the outcome function which produces an operating point.

Notice that the strategy sets defined in the mechanism produce a certain response from the users that can be quite different had the strategy sets been different. For example, a presidential election where voters can rank the candidates and give them points accordingly might give different results than the one where only one candidate can be voted for. We say that a mechanism Γ, consisting of strategy sets and the outcome function, *induces* a game.

> In the language of game theory, the mechanism Γ, the set of types $\Theta = (\Theta_1, \ldots, \Theta_I)$, the probability distribution $\phi(\cdot)$ on the set of types, and the utility functions $\{u_i\}_{i \in I}$ define a Bayesian Game of incomplete information. The fact that we assume some distribution on the user types is important only if the users consider expected utility, which is done in Section 3.4.

The game theoretic terminology just introduced is needed for the following definition:

Definition 3.5. *We say that a mechanism Γ **implements** the social choice function $f(\cdot)$ if there is a strategy profile $(s_1(\cdot), \ldots, s_I(\cdot))$ such that $g(s_1(\boldsymbol{\theta}_1), \ldots, s_I(\boldsymbol{\theta}_I)) = f(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_I)$, for all $\boldsymbol{\theta} \in \Theta$, where $(s_1(\cdot), \ldots, s_I(\cdot))$ is an equilibrium of the game induced by Γ.*

Simply said, a mechanism implements a social choice function if the decision-making procedure chooses the operating point that was the goal to begin with. Notice that we say that the strategy profile in Definition 3.5 is an equilibrium of the game induced by mechanism Γ, but we do not say which equilibrium. Although there are many equilibrium concepts in game theory, we will consider the *dominant strategy equilibrium*. As we already mentioned in Chapter 1, this is the strongest equilibrium concept, but also the one that exists rarely. We will show under which conditions such an equilibrium can be obtained in the AWGN MAC setting.

Definition 3.6 (Dominant strategy equilibrium). *Let $\boldsymbol{s}_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$ be the arbitrary strategies of users other than i. The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_I^*(\cdot))$ is a dominant strategy equilibrium of mechanism Γ if for all $i \in \mathcal{I}$ and for all $\boldsymbol{\theta}_i \in \Theta_i$, $u_i(g(s_i^*(\boldsymbol{\theta}_i), \boldsymbol{s}_{-i}), \boldsymbol{\theta}_i) \geq u_i(g(s_i', \boldsymbol{s}_{-i}), \boldsymbol{\theta}_i)$, for all $s_i' \in S_i$ and all $\boldsymbol{s}_{-i} \in S_{-i}$,*

So for this equilibrium each user, for each type, has a strategy that weakly dominates all of his other strategies, *regardless* of what other users' strategies are. To say that each user has a superior plan of action no matter what his preferences are, and regardless of what other people do, is a rather strong statement. There is no guarantee that an equilibrium like this exists, but if it does, it is natural to assume that users will indeed behave according to it.

> There exist weaker equilibrium notions that are present in a wider class of problems, but there is less reason to believe that the users would actually

converge to those. The dominant strategy equilibrium only assumes that the users are aware of the mechanism Γ, in particular this is why we were able to ignore the probability distribution over the preference types. On the other hand, the strategies comprising the Bayesian Nash Equilibrium are a function of the probability distribution over the set of types.

Definition 3.7 (**Implementation in dominant strategies**). *The mechanism Γ implements the social choice function $f(\cdot)$ in dominant strategies if there exists a dominant strategy equilibrium $s^*(\cdot)$ such that $g(s^*(\boldsymbol{\theta})) = f(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \in \Theta$.*

Definition 3.7 is simply an extension of Definition 3.5, with a well defined notion of an equilibrium. So we say that a mechanism implements a social choice function in dominant strategies if each user has a superior plan of action, and these actions, when evaluated by the outcome function, yield the operating point that was the goal of the social planner from the beginning.

We are interested in characterizing those social choice functions that incite people to tell the truth. We call those functions that have this property truthfully implementable or incentive compatible. Social choice functions that do not belong to this group are called manipulable.

Definition 3.8 (**Incentive compatibility**). *The social function $f(\cdot)$ is incentive compatible if the direct revelation mechanism $\Gamma = (\Theta_1, \dots, \Theta_I, f(\cdot))$ has some equilibrium $s^*(\boldsymbol{\theta})$ in which $s_i^*(\boldsymbol{\theta}_i) = \boldsymbol{\theta}_i$ for all $\boldsymbol{\theta}_i$ and all i.*

In other words, if a mechanism is incentive compatible (we also say that this mechanism is **truthfully implementable**), then the truthful revealing of types by each agent constitutes an equilibrium of the game induced by Γ.

Definition 3.9 (**Strategy proof mechanism**). *We say that a social choice function $f(\cdot)$ is* ***truthfully implementable in dominant strategies*** *(or strategy proof) if truthful revealing of preferences constitutes a dominant strategy equilibrium of the game induced by Γ. That is, for all i and all $\boldsymbol{\theta}_i \in \Theta_i$:*

$$u_i(f(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_i) \geq u_i(f(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_i), \tag{3.3}$$

for all $\hat{\boldsymbol{\theta}}_i \in \Theta_i$ and $\boldsymbol{\theta}_{-i} \in \Theta_{-i}$.

Included in the definition of truthful implementation is that the mechanism Γ is a direct revelation mechanism. But how does this relate to a general mechanism where the set of user strategies is does not consist of simply disclosing one's type?

3.3 Main results

3.3.1 Mechanisms implementable in dominant strategies

It turns out that investigating only the direct revelation mechanisms suffices to determine whether a social choice function is truthfully implementable or not. To see this, consider the following proposition:

Proposition 3.3 (The revelation principle for dominant strategies). *If there exists a mechanism Γ that implements the social choice function $f(\cdot)$ in dominant strategies, then this function is truthfully implementable in dominant strategies.*

In other words, if there exists a mechanism for which truth reporting by the users constitutes a dominant strategy equilibrium, then there exists a direct revelation mechanism where truthful reporting of types constitutes a dominant strategy. We include the proof since it helps to better understand the revelation principle.

Proof. By the premise of the proposition, there exists $s^*(\cdot)$ such that for all i, all $\boldsymbol{\theta}_i$, all $\hat{s}_i \in S_i$ and all $s_{-i} \in S_{-i}$:

$$u_i(g(s_i^*(\boldsymbol{\theta}_i), s_{-i}), \boldsymbol{\theta}_i) \geq u_i(g(\hat{s}_i, s_{-i}), \boldsymbol{\theta}_i).$$

Since this is true for all $\hat{s}_i$, it works for $\hat{s}_i = s_i^*(\hat{\boldsymbol{\theta}}_i)$, and since it is true for all s_{-i}, it works for $s_{-i} = s_{-i}^*(\boldsymbol{\theta}_{-i})$. Substituting, we get:

$$u_i(g(s_i^*(\boldsymbol{\theta}_i), s_{-i}^*(\boldsymbol{\theta}_{-i})), \boldsymbol{\theta}_i) \geq u_i(g(s_i^*(\hat{\boldsymbol{\theta}}_i), s_{-i}^*(\boldsymbol{\theta}_{-i})), \boldsymbol{\theta}_i).$$

But, since the mechanism Γ implements a social choice function $f(\cdot)$, we have $g(s^*(\boldsymbol{\theta})) = f(\boldsymbol{\theta})$ for all $\boldsymbol{\theta}$, then we get:

$$u_i(f(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_i) \geq u_i(f(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_i) \tag{3.4}$$

for all $i \in I$, $\boldsymbol{\theta}_i \in \Theta_i$, $\hat{\boldsymbol{\theta}}_i \in \Theta_i$ and $\boldsymbol{\theta}_{-i} \in \Theta_{-i}$, which is exactly the condition (3.3) required for a function to be truthfully implementable in dominant strategies. □

The revelation principle has two important consequences. First, it allows us to focus our attention on direct revelation mechanisms: if there is no direct revelation mechanism truthfully implementable in dominant strategies, then we can never find any other mechanism that is implementable in dominant strategies. Second, since the incentive compatibility of direct revelation mechanisms depends only on the social choice function and the utility functions of the players, we can conclude that certain social choice functions can be never implemented in dominant strategies. All that needs to be done is to find a counterexample $\boldsymbol{\theta}_i$ and a $\hat{\boldsymbol{\theta}}_i$ for which (3.3) does not hold.

On the other hand, the direct revelation mechanism is the best one can hope for. If a function is truthfully implementable in dominant strategies, in practice we might still need to find a simple non-direct revelation mechanism that implements this function in dominant strategies, since it may not be feasible to ask users directly for their preferences[4]. On the other hand, we will see shortly that for a large class of problems there exists no direct revelation mechanism that implements any social choice function in dominant strategies.

The following theorem[5], introduced and proved in [22], gives us the conditions under which a social choice function is manipulable:

[4]When a type contains a vector, this is not an issue. When types are functions, a practical mechanism should impose a smaller set of possible strategies.

[5]The theorem is similar to the better known Gibbard-Satterwathe Theorem, but the former does not imply the latter, nor vice versa.

Theorem 3.4 (Dictatorial nature of strategy proof social choice functions). *Let X, the set of alternatives, be a space with metric d, let U be the set of continuous real-valued functions on X, and let $\boldsymbol{u} \in U^I$ be the vector of utility functions for I users. A social choice function is a function $f : U^I \to X$, with the range r_f. User i is called a dictator if $u_i(f(\boldsymbol{u})) \geq u_i(x)$ for all $x \in r_f$, in which case $f(\cdot)$ is called dictatorial. Then a social choice function on X that is truthfully implementable in dominant strategies, and whose range contains at least three alternatives is necessarily dictatorial.*

Therefore most interesting social choice functions cannot be implemented in dominant strategies unless they are dictatorial. This may sound bad, but dictatorship does not equal catastrophe. Admittedly, it is different than maximizing the sum of utilities, but since it is the only function that can be truthfully implemented in dominant strategies, let us give it a closer look. For example, if we were to employ successive decoding in the I-user AWGN channel then the user who gets decoded last would be the dictator. Hence, we have a social choice function that is truthfully implementable in dominant strategies, and is furthermore efficient. We could then randomly assign the decoding order in each communication session, for example by using a coordinating device (see Section 2.6), which would make every user a dictator once in a while, and lead to an egalitarian operating point.

Going back to the original goal of maximizing the sum of utilities, the Theorem 3.4 tells us that our goal is impossible to attain. Rather than completely giving up and accepting that dictatorship is the only thing that can be done, we can achieve our goal by solving a slightly different problem. Namely, we give users an incentive to tell the truth by charging them carefully designed taxes [23].

3.3.2 Quasilinear preferences and the Groves mechanism

To account for the tax that would alter users' behavior, we modify their payoffs and allow for the possibility of a monetary transfer (which we will call transfer of "money", although a more correct term would be "numeraire commodity"). We do this by changing the payoff function of the user i to be:

$$v_i(\mathbf{y}, \boldsymbol{\theta}_i) = u_i(\mathbf{x}, \boldsymbol{\theta}_i) + t_i, \; i \in \mathcal{I} \tag{3.5}$$

where $t_i \in \mathbb{R}$ is the monetary transfer of user i (positive if the user is receiving money, negative if the user is giving it away). Now the space of alternatives is changed to include the money transfers: $Y = X \times \mathbb{R}^I$, and a valid alternative is $\mathbf{y} = (\boldsymbol{P}, \boldsymbol{R}, \boldsymbol{t}) = (\mathbf{x}, \boldsymbol{t})$, where $\mathbf{x} \in X$ and $\boldsymbol{t} = (t_1, \ldots, t_I) \in \mathbb{R}^I$. We call u_i the 'pure' utility or simply utility, so it can be distinguished from the total payoff v_i, which is the utility plus the money transfer. As we discussed in Chapter 1, preferences defined in this way are called quasilinear. When dealing with quasilinear utilities we will not worry about how much money is actually at the disposal of the users.

The task of the receiver now is to find the appropriate communication operating point, as well as to mandate the amount of money that the users should receive or give. We impose a constraint that money should not be entering the system: $\sum_{i=1}^{I} t_i \leq 0$, which is called the **budget constraint**. If $\sum_{i=1}^{I} t_i = 0$, then we say that we have budget balance, since no money is leaving the system. On the other hand, $\sum_{i=1}^{I} t_i < 0$ can be justified by inventing another user who gets all the left-over money, but is otherwise excluded from the analysis. For example, the social planner could collect the money, and then distribute it to participants who are not involved in the mechanism, such as communication users in a different geographic area. It is important that the participants in the mechanism do not get the left-over money since that would give them an incentive to misreport their utilities.

The difference between the previous and the current setup is the following: the new social choice function finds an operating point which maximizes the sum of **pure** utilities $\sum_{i=1}^{I} u_i(\cdot)$, and we are trying to find a mechanism that will implement it. Meanwhile, the rational user i is still trying to maximize his/her own payoff $v_i(\cdot)$. Notice that we are no longer trying to maximize the overall payoff in the system (which is equal to $\sum_{i=1}^{I} v_i(\cdot) = \sum_{i=1}^{I} u_i(\cdot) + \sum_{i=1}^{I} t_i$), but simply to make sure that the sum of utilities is maximized. In the case of budget balance, the two goals actually coincide.

A priori, our problem still satisfies the assumptions of Theorem 3.4, so it is not clear why changing the utility functions to quasilinear form would be of any help in designing mechanisms that implement incentive compatible social choice functions in dominant strategies. We will come back to this discussion shortly after introducing the Groves mechanism, which is a direct revelation mechanism that implements a class of social choice functions that are truthfully implementable in dominant strategies.

Proposition 3.5 (Groves mechanism). *Let $\mathbf{x}^*(\hat{\boldsymbol{\theta}}) \in X$ be a choice function that maximizes the sum of pure utilities:*

$$\sum_{i \in \mathcal{I}} u_i \left(\mathbf{x}^*(\hat{\boldsymbol{\theta}}), \hat{\boldsymbol{\theta}}_i \right) \geq \sum_{i \in \mathcal{I}} u_i(\mathbf{x}, \hat{\boldsymbol{\theta}}_i), \text{ for all } x \in X \text{ and all } \hat{\boldsymbol{\theta}} \in \Theta. \tag{3.6}$$

Then the social choice function $f(\cdot) = (\mathbf{x}^*(\cdot), t_1(\cdot), \ldots, t_I(\cdot))$ is truthfully implementable in dominant strategies if, for all $i \in \mathcal{I}$:

$$t_i(\hat{\boldsymbol{\theta}}) = \left[\sum_{j \neq i} u_j(\mathbf{x}^*(\hat{\boldsymbol{\theta}}), \hat{\boldsymbol{\theta}}_j) \right] + h_i(\hat{\boldsymbol{\theta}}_{-i}), \tag{3.7}$$

where $h_i(\hat{\boldsymbol{\theta}}_{-i})$ is an arbitrary function that does not depend on the type of user i, but such that $\sum_i t_i \leq 0$.

Here we offer a slightly different proof than in [1].

Proof. A rational user i will try to maximize his payoff $v_i(\cdot)$, so he needs to decide on the value of $\hat{\boldsymbol{\theta}}_i$ to report to the receiver. Suppose that this user also knows the values $\hat{\boldsymbol{\theta}}_{-i}$ that were reported to the social planner by the other users. Then, armed with all the information needed in the system, user i is in the position to make the best possible choice of strategy. The maximum utility that user i can obtain is:

$$v_i^{max} = \max_{\boldsymbol{\theta}_i' \in \Theta_i} \left[v_i(\mathbf{y}^*(\boldsymbol{\theta}_i', \hat{\boldsymbol{\theta}}_{-i}), \boldsymbol{\theta}_i) \right] = \max_{\boldsymbol{\theta}_i' \in \Theta} \left[u_i(\mathbf{x}^*(\boldsymbol{\theta}_i', \hat{\boldsymbol{\theta}}_{-i}), \boldsymbol{\theta}_i) + t_i(\boldsymbol{\theta}_i', \hat{\boldsymbol{\theta}}_{-i}) \right]$$

If we are using a Groves scheme described in (3.7), then the expression becomes:

$$\begin{aligned}
v_i^{max} &= \max_{\boldsymbol{\theta}_i' \in \Theta_i} \left[u_i(\mathbf{x}^*(\boldsymbol{\theta}_i', \hat{\boldsymbol{\theta}}_{-i}), \boldsymbol{\theta}_i) + \sum_{j \neq i} u_j(\mathbf{x}^*(\boldsymbol{\theta}_i', \hat{\boldsymbol{\theta}}_{-i}), \hat{\boldsymbol{\theta}}_j) + h_i(\hat{\boldsymbol{\theta}}_{-i}) \right] \\
&\overset{(a)}{=} \max_{\boldsymbol{\theta}_i' \in \Theta_i} \left[u_i(\mathbf{x}^*(\boldsymbol{\theta}_i', \hat{\boldsymbol{\theta}}_{-i}), \boldsymbol{\theta}_i) + \sum_{j \neq i} u_j(\mathbf{x}^*(\boldsymbol{\theta}_i', \hat{\boldsymbol{\theta}}_{-i}), \hat{\boldsymbol{\theta}}_j) \right] + h_i(\hat{\boldsymbol{\theta}}_{-i}) \\
&\overset{(b)}{\leq} \max_{\mathbf{x} \in X} \left[u_i(\mathbf{x}, \boldsymbol{\theta}_i) + \sum_{j \neq i} u_j(\mathbf{x}, \hat{\boldsymbol{\theta}}_j) \right] + h_i(\hat{\boldsymbol{\theta}}_{-i}) \\
&\overset{(c)}{=} u_i(\mathbf{x}^*(\boldsymbol{\theta}_i, \hat{\boldsymbol{\theta}}_{-i}), \boldsymbol{\theta}_i) + \sum_{j \neq i} u_j(\mathbf{x}^*(\boldsymbol{\theta}_i, \hat{\boldsymbol{\theta}}_{-i}), \boldsymbol{\theta}_j) + h_i(\hat{\boldsymbol{\theta}}_{-i}) \\
&= u^*(\boldsymbol{\theta}_i, \hat{\boldsymbol{\theta}}_{-i}) + h_i(\hat{\boldsymbol{\theta}}_{-i}).
\end{aligned}$$

Here (a) follows trivially since $h_i(\cdot)$ does not depend on $\boldsymbol{\theta}_i'$ and (b) follows because the maximum over all alternatives of the sum of pure utilities cannot be smaller than sum of utilities evaluated at any point assigned by the mechanism (such as $\mathbf{x}^*(\boldsymbol{\theta}_i, \hat{\boldsymbol{\theta}}_{-i})$). Finally, (c) follows by definition of $\mathbf{x}^*(\cdot)$. So we see that the utility of user i cannot be better than $u^*(\boldsymbol{\theta}_i, \hat{\boldsymbol{\theta}}_{-i}) + h_i(\hat{\boldsymbol{\theta}}_{-i})$ even if he has perfect knowledge of the system, which is the best way to manipulate it. On the other hand, it is not difficult to see that user i can actually attain this maximum by revealing his/her type $\boldsymbol{\theta}_i$ truthfully. Therefore, truth telling constitutes a dominant strategy for user i, and, by the same reasoning, for other users as well. So $f(\cdot)$ is truthfully implementable in dominant strategies. □

On second look, it is almost obvious why users should reveal their preferences truthfully when facing a Groves mechanism. The form of the money transfer makes sure that each user takes into account the utility that other users derive from the channel. This way, any attempt to manipulate the outcome by not telling the truth will actually decrease the utility of a dishonest user.

Now we seem to be facing a contradiction. On one hand, we have Theorem 3.4, that states that the only social choice function that can be truthfully implemented in dominant strategies is a dictatorial one, and on the other hand we have an example of a social choice function that is truthfully implementable in dominant strategies that appears to be defying the theorem. The resolving factor is the specific choice of the money transfer. Namely, once the Groves scheme is announced, the users are facing a maximization problem where the social choice function $f(\cdot)$ is actually **dictatorial**, with all users as dictators.

To see this, recall that a dictatorial social choice function always chooses an alternative that maximizes the utility of user i, in which case user i is a dictator. Consider a mechanism that implements a Groves scheme. This mechanism has already chosen t_i for all $i \in \mathcal{I}$, and now the remaining task is to chose the communication operating point $\mathbf{x}$. The payoffs of the users are then:

$$v_i(\mathbf{y}, \boldsymbol{\theta}_i) = u_i(\mathbf{x}, \boldsymbol{\theta}_i) + \sum_{j \neq i} u_j(\mathbf{x}, \hat{\boldsymbol{\theta}}_j) + h_i(\hat{\boldsymbol{\theta}}_{-i})$$

A dictatorial social choice function with user i as dictator would try to find $v_i^{max} = \max_{\mathbf{y}\in Y} v_i(\mathbf{y}, \boldsymbol{\theta}_i)$:

$$v_i^{max} = \max_{\mathbf{x}\in X} \left[u_i(\mathbf{x}, \boldsymbol{\theta}_i) + \sum_{j\neq i} u_j(\mathbf{x}, \hat{\boldsymbol{\theta}}_{-i}) + h_i(\hat{\boldsymbol{\theta}}_{-i}) \right]$$

Now, we assume that $\boldsymbol{x}^*(\cdot)$ is such that it maximizes the sum of utilities based on reported $\hat{\boldsymbol{\theta}}_i$'s. If user i tells the true $\boldsymbol{\theta}_i$, then $v_i(\mathbf{x}^*, \boldsymbol{\theta}_i) = v_i^{max}$ so this $\boldsymbol{x}^*(\cdot)$ becomes a dictatorial social choice function with user i as a dictator. Since this is true for all $i \in \mathcal{I}$, the $\boldsymbol{x}^*(\cdot)$, combined with the Groves monetary transfers, is then a dictatorial social choice function with all users as dictators. Of course, rational users will always want to be dictators in a society, and hence they choose to tell the truth[6]. So there is no contradiction.

Another way to think about Groves mechanism is that it *internalizes the externality* by the users. Namely, the presence of a user means that other users will receive less than they would receive without him, i.e. that this player presents an externality to others. With Groves mechanism, each player pays for the damage he is doing to others, thereby internalizing his own influence.

However, the economic theory teaches us that "there is no such thing as free lunch [24]" so we suspect that the truth-telling property needs to be paid for somehow. Indeed, we have not yet discussed the budget constraint $\sum_{i=1}^{I} t_i \leq 0$, which will lead to some inefficiencies.

3.3.3 Clarke mechanism

One subset of the Groves mechanism is the **Clarke mechanism**, also called the **pivotal mechanism**. We will see that in this mechanism, each user i pays the amount of money equal to the amount of utility that he deprives others of. We first define a few relevant quantities.

Let $\boldsymbol{P_{-i}} \in \mathcal{R}^{I-1}$ and $\boldsymbol{R_{-i}} \in \mathcal{R}^{I-1}$ be the powers and rates, respectively, of all users except user i, and X_{-i} be the set of all valid choices for an $I-1$ AWGN MAC with all users except user i. Then, $\boldsymbol{x}_{-i} = (\boldsymbol{P_{-i}}, \boldsymbol{R_{-i}})$. Finally, define $\mathbf{x}^*_{-i}(\cdot) \in \mathcal{R}^{2(I-1)}$ to be the maximizer of GUM for a Gaussian MAC without user i:

$$\sum_{j\neq i} u_j(\mathbf{x}^*_{-i}(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_j) \geq \sum_{j\neq i} u_j(\mathbf{x}_{-i}, \boldsymbol{\theta}_j),\ \forall \mathbf{x}_{-i} \in X_{-i} \text{ and all } \boldsymbol{\theta}_{-i} \in \Theta_{-i}. \tag{3.8}$$

We are now ready to define the Clarke mechanism.

Definition 3.10 (Clarke mechanism). *The Clarke mechanism is a Groves mechanism with*

$$\tilde{h}_i(\boldsymbol{\theta}_{-i}) = \sum_{j\neq i} u_j(\mathbf{x}^*_{-i}(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_j),\ \forall \boldsymbol{\theta}_{-i} \in \Theta_{-i}, \tag{3.9}$$

[6]The sentence should be understood in the game theoretic sense of the word "rational."

So $\tilde{h}_i(\cdot)$ is the sum of pure utilities that users other than i would receive had it not been for user i in the channel. In total, the monetary transfer t_i that is assigned to user i in Clarke mechanism is the difference between the pure utility that users other than i get with user i in the system, and the pure utility that they get without user i in the system.

A quick inspection will show that in our case, all users in the system need to pay money, so the budget balance is necessarily negative. We proceed to show that the Clarke mechanism is the best Groves mechanism that we can find for the MAC problem, for a wide class of utility functions. Finally we cite a result which proves that Groves mechanism is the only incentive compatible mechanism implementable in dominant strategies for our case, to conclude that we cannot find a better mechanism than the Clarke mechanism without relaxing some of the problem constraints. We remark that the Clarke mechanism is also the mechanism that maximizes the payments from the users to the mechanism designer among all Groves mechanisms [25].

First of all we show that t_i is not positive in the Clarke mechanism:

$$\begin{aligned} t_i =& \sum_{j \neq i} u_j(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}_j) - \sum_{j \neq i} u_j(\mathbf{x}^*_{-i}(\boldsymbol{\theta}_{-i}), \boldsymbol{\theta}_j) \\ \leq& \sum_{j \neq i} u_j(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}_j) - \sum_{j \neq i} u_j(\mathbf{x}, \boldsymbol{\theta}_j) \text{ for all } \mathbf{x} \in X \\ =& \, 0, \text{ by choosing } \mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta}), \end{aligned}$$

where the inequality follows from (3.8) and the following equality

$$\sum_{j \neq i} u_j(\mathbf{x}_{-i}, \boldsymbol{\theta}_j) = \sum_{j \neq i} u_j(\hat{\mathbf{x}}, \boldsymbol{\theta}_j), \ \forall \mathbf{x}_{-i} \in X_{-i}, \qquad (3.10)$$

where $\hat{\mathbf{x}}$ is obtained by adding an arbitrary rate and power of user i to vector $\mathbf{x}_{-i}$. Notice that any vector $\boldsymbol{x} \in X$ can be obtained as an extension of some vector $\boldsymbol{x}_{-i} \in X_{-i}$ since adding a user in AWGN channel strictly increases the set of possible choices.

Therefore, a user cannot gain money with the Clarke mechanism. Furthermore, since each $t_i \leq 0$, then it must be that $\sum_i t_i \leq 0$, which is exactly the balance condition, and therefore a Clarke mechanism is incentive compatible and satisfies the constraint that no money is entering the system.

Note that the only way that a user can escape paying any money is if the socially optimal point $\mathbf{x}^*(\boldsymbol{\theta})$ is an extension of a socially optimal point without that user in the channel, i.e., $\mathbf{x}^*_{-i}(\boldsymbol{\theta}_{-i})$. This would mean that the presence of this user does not reduce the set of available choices to the others. However, this cannot be true for all users. For example, consider a two user AWGN MAC. Each of these users would transmit at an optimal power and at capacity if they were alone in the channel, but regardless the mechanism in place, the presence of the other user obliges at least one of them to have a worse operating point. This means that at least one of the users would have to pay money to the system when a Clarke tax is in place. Hence, in general $\sum_{i=1}^{I} t_i < 0$.

So now we have a mechanism that has certain desirable properties, but can we do better? First, we note that the Clarke mechanism will inevitably waste some user's money. Is there a mechanism that is more efficient than the Clarke scheme, ideally $\sum_i t_i = 0$? Unfortunately, the answer is no.

Proposition 3.6 (Efficiency of Clarke mechanism). *Clarke mechanism is the most efficient mechanism out of the class of Groves mechanisms applied to the I-user AWGN multiple access channel problem for a parametrized class of utility functions where there exists a sequence of types $\{\boldsymbol{\theta}_2^n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \max_{\mathbf{x}\in X}(u_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) = 0$.*

The proof is given in Appendix 3.A.

The class of utility functions for which the proposition holds is very large, for example any $u_i(\boldsymbol{x}, \boldsymbol{\theta_i}) = \theta_i u(\boldsymbol{x})$, for some strictly concave function $u(\cdot)$ would have this property by choosing a sequence of scalars θ_i^n that tend to zero. On the other hand, it is not difficult to find a counterexample, such as $u_i(\boldsymbol{x}, \boldsymbol{\theta_i}) = u(\boldsymbol{x}) + R_i\theta_i$ with $\max_{x\in X} u(\boldsymbol{x}) > 0$. Such a function would never yield arbitrarily small maximum utility.

Therefore, Clarke mechanism is the most efficient Groves mechanism for our problem, for a large class of utility functions. Next, we will see that the Groves mechanism is the only incentive compatible mechanism that exists for a large class of problems.

The first result stating that the Groves mechanism is the only one that could guarantee truthfully implementable social choice functions was given by Green and Laffont [26]. However, the assumption was that the space of utility functions that the users could have was quite large: it consisted of all possible utility functions on X. In most cases, this assumption is not satisfied as the utility function space is more restricted, but this result initiated a body of work that tried to show this result on more restricted domains. The two theorems that we will quote from [27, 28] show the indispensability of Groves scheme on smoothly connected social choice function domains.

The reason why we are interested in smoothly connected domains is because a large number of functions fall under this domain, including the concave functions that we consider for the AWGN MAC case. More generally, any convex domain is smoothly connected. U is convex function domain if $u, u' \in U$ implies $\lambda u + (1-\lambda)u' \in U$ for all $\lambda \in [0,1]$. This is not to say that the actual functions u, u' need to be concave, but this case is certainly included. The utility functions that we consider constitute a convex domain.

The following definition of the smoothly connected domains is given for completeness:

Definition 3.11 (Smoothly connected). *Let U_i be the set of possible utility functions for user i, and $U = \times_{i\in I} U_i$. Let $\mathbf{x}^* : U \to X$ be a social choice function from the space of users' utility functions to the space of valid alternatives, that maximizes the sum of utility functions. Assume that the set*

$$X^* = \operatorname*{argmax}_{\mathbf{x}\in X} \sum_i u_i(\mathbf{x})$$

is non empty. The domain $U = \times_{i\in I} U_i$ is said to be smoothly connected if for any two functions $u_i,\ u_i' \in U_i$ and any $u_{-i} \in U_{-i}$ there exists a one dimensional parameterized family of functions in U_i:

$$U_i(u_i, u_i') = \{u_i(\boldsymbol{x}; z_i) \in U_i \mid z_i \in [0,1]\},$$

such that for all $\mathbf{x} \in X$

$$\begin{aligned} u_i(\mathbf{x}; 0) =& u_i(\mathbf{x}), \\ u_i(\mathbf{x}; 1) =& u_i'(\mathbf{x}), \\ \frac{\partial u_i(\mathbf{x}; z_i)}{\partial z_i} & \textit{ exists for all } z_i \in [0,1], \end{aligned}$$

and, moreover, for all $z_i \in [0,1]$, $\mathbf{x} \in X^*(u_i, u_i'; u_{-i}) = \{\mathbf{x} \in X \mid \mathbf{x} = \mathbf{x}^*(u_i(\boldsymbol{x}; z_i), u_{-i})$ *for some* $z_i \in [0,1]\}$, *we have*

$$\left| \frac{\partial u_i(\mathbf{x}; z_i)}{\partial z_i} \right| \leq K \textit{ for some } 0 < K < \infty.$$

Now we state two theorems from, the proofs of which can be found in [28].

Theorem 3.7. *If* U*, the domain of the social choice function, is convex, then* U *is smoothly connected.*

Theorem 3.8. *If* U*, the domain of the social choice function, is smoothly connected, then any scheme that is truthfully implementable in dominant strategies is a Groves' scheme.*

The immediate corollary of Theorems 3.7 and 3.8 is that for any social choice function whose domain is convex, any truthfully implementable scheme in dominant strategies is necessarily a Groves scheme.

We see that we have to use the Groves scheme in the I-user AWGN MAC problem in order to incite users to tell the truth and operate at a point that maximizes the sum of utilities. We also found that Clarke mechanism is the most efficient of all Groves mechanisms for this case, for a large class of utility functions.

It is not difficult to show that a non-Clarke mechanism may lead to users getting negative total utility for some values of $\boldsymbol{\theta}$. This violates **individual rationality**, since any user getting negative utility would opt out of the scheme. Therefore, Clarke mechanism is the *only* mechanism that leads to efficient distribution of resources, and assures that users experience non-negative utilities.

3.4 Mechanism implementing Bayesian Nash equilibrium

So far, in order to implement a utility maximizing social choice function, we had to modify the payoffs of the users to include a payment according to the Groves scheme. Furthermore, in order for the mechanism to be individually rational and satisfy budget balance, we considered a special case of the Groves scheme, namely the Clarke scheme, which turns out to be the only scheme with those positive properties. The price to pay for the desirable properties of the Clarke function is literally in the form of a payment from the users to the mechanism designer, since in general budget balance is not satisfied. In this section we show that by relaxing the equilibrium concept for the mechanism, we can have a scheme where budget balance is obtained.

3.4.1 The benefits of considering average utility

From now on, we assume that the types of the players come from some probability distribution, and that all of the players have knowledge of this distribution prior to the game. Then, it makes sense to introduce an equilibrium concept where the users can maximize their expected utility, where the expectation is with respect to the types of the other players.

Definition 3.12 (**Bayesian Nash equilibrium**). *A strategy profile* $\boldsymbol{s} = (s_1(\cdot), \ldots, s_I(\cdot))$ *is a Bayesian Nash equilibrium if for every* $i \in \mathcal{I}$ *and all* $\boldsymbol{\theta}_i \in \boldsymbol{\Theta}$

$$\mathcal{U}_i(s_i(\boldsymbol{\theta}_i), s_{-i}(\cdot), \boldsymbol{\theta}_i) \geq \mathcal{U}_i(s_i'(\boldsymbol{\theta}_i), s_{-i}(\cdot), \boldsymbol{\theta}_i), \text{ for all } s_i'(\cdot) \neq s_i(\cdot),$$

where by $\mathcal{U}_i$ *we designate the expected value of the utility with respect to the distribution of types.*

The key difference between the dominant strategy equilibrium and Bayesian Nash is that a user is now playing a best response strategy to the *distribution* of strategies as opposed to a best response to the *actual* strategies. So a user is not necessarily playing a best-response to the actual strategies of his opponents.

We also remark that in this context, we can make a distinction between Pareto optimality as we described it before (aka **ex post Pareto optimality**), and **ex ante Pareto optimality**, where the optimality is in *expectation.* Similarly, we can distinguish between **ex post budget balance** and **ex ante budget balance**, which permits that players sometimes give and sometimes receive payments, but have no money change in expectation. Finally, we say that a direct revelation mechanism is **Bayesian Nash incentive compatible** if truthful revealing of preferences constitutes a Bayesian Nash equilibrium in a game induced by said mechanism.

The assumption that a user knows the distribution of preferences of other participants is fairly reasonable; more difficult to justify is the assumption of common rationality. In addition, Bayesian Nash equilibria suffer from the problem of multiplicity of equilibria. Here we comment on the inapplicability of Nash equilibrium concept in this setting. Nash equilibrium is useful when the players know the preferences of their opponents, so that they can predict an outcome where no player has an incentive to deviate. In a setting where players do not have that knowledge, it is of little use to talk about Nash equilibria as a predicted equilibrium. In a limited information setting, a Nash equilibrium may be useful only if it arises as a result of some iterative procedure.

Going back to Bayesian Nash equilibrium, we only state one possibility result to give a flavor of what can be achieved by relaxing the equilibrium notion. For more details, an excellent resource is [25] where a breakdown of impossibility and possibility results for mechanism design can be found.

Definition 3.13 (**dAGVA mechanism**). *A dAGVA mechanism is defined in the fol-*

lowing way:

$$\boldsymbol{x}^* = \max_{\boldsymbol{x} \in X} \sum_{i=1}^{I} u_i(\boldsymbol{x}, \hat{\boldsymbol{\theta}}_i)$$

$$t_i(\boldsymbol{\theta}) = h_i(\hat{\boldsymbol{\theta}}_{-i}) - \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left[\sum_{j \neq i} v_j(\boldsymbol{x}^*(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}), \theta_j) \right].$$

A dAGVA mechanism can be thought of as an average version of Groves mechanism. Of special interest is the following function of opponents' types:

$$h_i(\hat{\boldsymbol{\theta}}_{-i}) = \frac{1}{I-1} \sum_{j \neq i} V_{-j}(\hat{\boldsymbol{\theta}}_j).$$

where

$$V_{-i}(\hat{\boldsymbol{\theta}}_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left[\sum_{j \neq i} v_j(\boldsymbol{x}^*(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}), \theta_j) \right]$$

is the averaged expected social welfare given the announced types of agents other than agent i. The following theorem holds:

Theorem 3.9 (dAGVA mechanism properties). *The dAGVA mechanism is Bayesian-Nash incentive compatible, Pareto efficient, and budget balanced when users have quasi-linear preferences.*

Without going into detail, it is good to know that a scheme exists that manages to achieve budget balance, i.e. $\sum_{i=1}^{I} t_i = 0$. Unfortunately, this comes at a cost of violating the individual rationality constraint, meaning that some users may get negative utility by using this scheme, limiting its applicability.

3.5 Discussion

In this chapter we have investigated the optimal resource allocation for an I-user AWGN channel with no power constraint. We have focused on the case where the resources are allocated by a benevolent central authority that collects the preferences of the users, and then imposes a socially optimal outcome. The result is a relatively negative one since it is implied that optimality has to be literally paid for, in the form of a Clarke-Grove tax. In return, the users are offered a scheme that gives them incentive to tell the truth about their preferences, which stimulates cooperation in the system. The results hold for a large class of utility functions, hence the truth tax is inevitable.

The motivation behind searching for the mechanism that implements the maximization of sum of utilities in dominant strategies was that we wanted to find a robust way of utilizing system resources efficiently. Even though we eventually succeed in this goal, this comes at a cost. In particular, some more efficient but less fair schemes can be implemented robustly instead of the Clarke mechanism. For example, using a dictatorial social choice function always yields a Pareto efficient solution. This is of particular interest since an implementation of a dictatorial system involves less overhead; in particular there would be no need for money transfers which usually come with a plethora of security and implementation issues.

3.A Proof of Clarke mechanism's efficiency

Proof. We prove the proposition by means of counterexample. Consider the case with $I = 2$. With the Clarke mechanism, for all $\boldsymbol{\theta}_1 \in \Theta_1$ and $\boldsymbol{\theta}_2 \in \Theta_2$, we have:

$$h_1(\boldsymbol{\theta}_2) = -\max_{\mathbf{x} \in X}(u_2(\mathbf{x}, \boldsymbol{\theta}_2)), \text{ and}$$
$$h_2(\boldsymbol{\theta}_1) = -\max_{\mathbf{x} \in X}(u_1(\mathbf{x}, \boldsymbol{\theta}_1))$$

Suppose that there exist functions $\bar{h}_1(\boldsymbol{\theta}_2)$ and $\bar{h}_2(\boldsymbol{\theta}_1)$ that waste less money than the Clarke mechanism. Then we can write them in the following way:

$$\bar{h}_1(\boldsymbol{\theta}_2) = -\max_{\mathbf{x} \in X}(u_2(\mathbf{x}, \boldsymbol{\theta}_2)) + \hat{h}_1(\boldsymbol{\theta}_2), \text{ and}$$
$$\bar{h}_2(\boldsymbol{\theta}_1) = -\max_{\mathbf{x} \in X}(u_1(\mathbf{x}, \boldsymbol{\theta}_1)) + \hat{h}_2(\boldsymbol{\theta}_1),$$

where $\hat{h}_1(\boldsymbol{\theta}_2) + \hat{h}_2(\boldsymbol{\theta}_1) \geq 0$ for all $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. Without loss of generality, we can take $\hat{h}_1(\boldsymbol{\theta}_2) \geq 0$ and $\hat{h}_2(\boldsymbol{\theta}_1) \geq 0$ for all $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ (see Appendix 3.B).

Now we construct a sequence of types $\{\boldsymbol{\theta}_2^n\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} \max_{\mathbf{x} \in X}(v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) = 0. \tag{3.11}$$

One way to think about the sequence of types $\{\boldsymbol{\theta}_2^n\}_{n=1}^{\infty}$ is that it describes a user 2 who is progressively less and less interested in the resources.

The following inequalities hold since $\max_{\boldsymbol{x} \in X} u_i(\boldsymbol{x}, \cdot) \geq 0$:

$$\max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) \leq \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1) + v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) \leq \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) + \max_{\mathbf{x} \in X}(v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)).$$

Taking the limit $n \to \infty$ on all three terms yields:

$$\max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) \leq \lim_{n \to \infty} \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1) + v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) \leq \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) + \lim_{n \to \infty} \max_{\mathbf{x} \in X}(v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)).$$

which by Equation (3.11) yields

$$\lim_{n \to \infty} \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1) + v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) = \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)). \tag{3.12}$$

Essentially, as $n \to \infty$ the maximum utility obtained by user 1 is the same as if he were alone in the channel. Now we inspect the budget balance for this case:

$$\begin{aligned}
&t_1 + t_2 = \\
&= \left[v_1(\mathbf{x}^*, \boldsymbol{\theta}_1) - \max_{\mathbf{x} \in X}(v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) + \hat{h}_1(\boldsymbol{\theta}_2^n)\right] + \left[v_2(\mathbf{x}^*, \boldsymbol{\theta}_2^n) - \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) + \hat{h}_2(\boldsymbol{\theta}_1)\right] \\
&= [v_1(\mathbf{x}^*, \boldsymbol{\theta}_1) + v_2(\mathbf{x}^*, \boldsymbol{\theta}_2^n)] - \max_{\mathbf{x} \in X}(v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) - \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) + \hat{h}_1(\boldsymbol{\theta}_2^n) + \hat{h}_2(\boldsymbol{\theta}_1) \\
&= \max_{\mathbf{x} \in X}\,(v_1(\mathbf{x}, \boldsymbol{\theta}_1) + v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) - \max_{\mathbf{x} \in X}(v_2(\mathbf{x}, \boldsymbol{\theta}_2^n)) - \max_{\mathbf{x} \in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) + \hat{h}_1(\boldsymbol{\theta}_2^n) + \hat{h}_2(\boldsymbol{\theta}_1),
\end{aligned}$$

by simple rearranging of terms, for all n. After taking the limit as $n \to \infty$, we use Equation (3.12), to get:

$$
\begin{aligned}
0 &\geq \lim_{n\to\infty} t_1 + t_2 = \max_{\mathbf{x}\in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) - \max_{\mathbf{x}\in X}(v_1(\mathbf{x}, \boldsymbol{\theta}_1)) + \lim_{n\to\infty} \hat{h}_1(\boldsymbol{\theta}_2^n) + \hat{h}_2(\boldsymbol{\theta}_1) \\
&= \lim_{n\to\infty} \hat{h}_1(\boldsymbol{\theta}_2^n) + \hat{h}_2(\boldsymbol{\theta}_1) \geq 0,
\end{aligned}
$$

which is only possible if $\hat{h}_2(\boldsymbol{\theta}_1) = 0$ for all $\boldsymbol{\theta}_1$, since we assumed that $\hat{h}_1(\boldsymbol{\theta}_2) \geq 0$ and $\hat{h}_2(\boldsymbol{\theta}_1) \geq 0$ for all $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. Similarly, we obtain that $\hat{h}_1(\boldsymbol{\theta}_2)$. So the Clarke function cannot be improved upon for a large class of functions where types can be found such that Equations (3.11) holds. □

3.B Independence of offset functions

We are interested in finding functions $q(x)$ and $r(y)$ such that $q(x)+r(y) \geq 0$, for all x, y. We want to show that, without loss of generality, we can take $q(x) \geq 0$ and $r(y) \geq 0$, for all x, y.

Proof.

$$
\begin{aligned}
&\text{If } q(x) + r(y) \geq 0.\ \forall x, y \\
&\Rightarrow q(x) \geq \max_y[-r(y)] = -\min_y r(y) = -b.
\end{aligned}
$$

Then we can construct non-negative $q'(x)$ and $r'(y)$:

$$
q'(x) = q(x) + b \geq 0,\ r'(y) = q(y) - b \geq 0,
$$

for all x, y, which completes the proof. □

CHAPTER SUMMARY

- **In this chapter we study one of the ways for users in an AWGN multiple access channel with no power constraint to choose an operating point.**
- **We adopt a centralized approach where users submit their preferences to the receiver, who takes the role of a benevolent social planner. In such a setting, the biggest challenge is to get users to truthfully reveal their preferences.**
- **For a large class of utility functions, the only way to make users behave truthfully is to institute a Clarke-Groves tax.**
- **The cost of using the Clarke Groves tax is that users have to pay in order to ensure correct operation of the mechanism, and money is leaving the system.**
- **When users maximize their expected utility as opposed to single-shot utility, it is possible to have budget balance, but some users may not have an incentive to participate in the mechanism.**

4 Competition of Wireless Resource Providers

In this chapter we study a scenario where service providers compete for wireless users who are not contractually tied to a single provider. Due to the deregulation of the telecommunication industry, in the future wireless users are likely to freely choose a provider (or providers) offering the best tradeoff of parameters in real time. This is already the case with some public Wi-Fi connections. The Wi-Fi users are able to connect to the provider of their choosing with no contractual obligation, with the payments being proportional to the duration of their connection. Despite the common presence of a free public network, many users choose more expensive providers who offer better quality of service. Another example of the deregulation trend is the analog television (UHF) spectrum which was recently open for unregulated use in the US [29].

We consider wireless service providers who sell a limited amount of some wireless resource such as frequency band, time slots, transmission power, etc. We investigate how providers set prices for the resource, and how users choose the amount of resource they purchase and from which providers. The focus of our study is to characterize the outcome of this interaction. We make the assumption that users experience different channel conditions to different providers and we model the users as rational economic agents who have different utility functions.

We model the user-provider interaction as a two-stage extensive game of complete information [5]. The providers announce the wireless resource prices in the first stage, and the users announce how much resource they want and from which provider in the second stage. A user may choose to buy from a provider with an inferior channel if the price of the resource is lower, or conversely choose a more expensive provider if the channel is better. The providers select their prices to maximize their revenues, keeping in mind the impact of their prices on the demand of the users.

The outline of this chapter is as follows. In Section 4.1, we construct a general network model that captures the heterogeneity of wireless users and service providers. The users have different utility functions, the providers have different resource constraints, the channel gains between users and providers are independent and arbitrarily distributed, and the numbers of users and providers can be arbitrary. We model the user-provider interaction as a multi-stage game. We begin the analysis in Section 4.2 by considering a related problem of social welfare maximization. We show that, under fairly broad conditions, there exists a *unique* optimal solution to the problem with probability 1. We return to the provider competition game in Section 4.3, where we prove existence and uniqueness of the subgame perfect Nash equilibrium, under an easily verifiable sufficient condition on the users' utility functions. Moreover, we show that the unique equilibrium maximizes the social welfare, despite the selfish nature of the providers and users. In Section 4.4, we provide a decentralized algorithm that converges to an equilibrium of the provider competition game. The participants only need local information during the execution of this algorithm. Providers only need to know the demand of the users, while users only need to consider the prices given by the providers. We provide some numerical results and dicussion in Section 4.5. We consider the related work in Section 4.6 and conclude in Section 4.7.

4.1 Problem formulation

We consider a set $\mathcal{J} = \{1, \ldots, J\}$ of service providers and a set $\mathcal{I} = \{1, \ldots, I\}$ of users. Provider $j \in \mathcal{J}$ maximizes its revenue by selling up to Q_j amount of resource to the users. A user $i \in \mathcal{I}$ maximizes its payoff by purchasing resources from one or more providers. The communication can be both downlink or uplink, as long as users do not interfere with each other by using orthogonal resources. We model the interaction as a multi-leader-follower game (see [31, 32]), where providers are the leaders and users are the followers. We assume that the users play the game during the coherence time of their channels. This can be reasonable for a relative static network environment (e.g., users are laptops or smart phones in offices or airports). This means that the channel gains remain roughly constant and can be made known to all parties. For example, each provider collects its channel condition information to each user, and then broadcasts this information to all users. This assumption will be relaxed in Section 4.4, where we consider a decentralized algorithm that results in the same outcome as the multi-leader-follower game, also called provider competition game.

Provider competition game

The provider competition game consists of two stages. In the first stage, providers announce prices $\boldsymbol{p} = [p_1 \ \cdots \ p_J]$, where p_j is the unit resource price charged by provider j. In the second stage, each user $i \in \mathcal{I}$ chooses a demand vector $\boldsymbol{q_i} = [q_{i1} \ \cdots \ q_{iJ}]$, where q_{ij} is the demand to provider j. We denote by $\boldsymbol{q} = [\boldsymbol{q}_1 \cdots \boldsymbol{q}_I]$ the demand vector of all users.

In the second stage where prices $\boldsymbol{p}$ are given, a user i's goal is to choose $\boldsymbol{q}_i$ to maximize its payoff, which is utility minus payment:

$$v_i(\boldsymbol{q_i}, \boldsymbol{p}) = u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right) - \sum_{j=1}^{J} p_j q_{ij}, \tag{4.1}$$

where c_{ij} is the *channel quality offset* between user i and the base station of provider j (see Example 4.1 and Assumption 4.2), and u_i is an increasing and concave utility function. The term $\sum_{j=1}^{J} q_{ij}c_{ij}$ that the utility function depends on is the **service** acquired by the user, which is a function of the resources used. In the first stage, a provider j chooses price p_j to maximize its revenue $p_j \sum_{i=1}^{I} q_{ij}$ subject to the resource constraint $\sum_{i=1}^{I} q_{ij} \leq Q_j$, by taking into account the demand of the users in the second stage. We consider linear pricing with no price discrimination across the users.

Under this model, a user is allowed to purchase from several providers at the same time. For this to be feasible, a user's device might need to have several wireless interfaces. Mathematically, the solution of this model gives an upper bound on the best performance of any situation where users are constrained to purchase from one provider alone. Interestingly, our results show that for most users, i.e. no fewer than $I - J$, the optimal strategy is to choose exactly one provider (or none).

Next we give a concrete example of how our model is mapped into a physical wireless system.

Example 4.1 (TDMA). *Consider wireless providers operating on orthogonal frequency bands $W_j, j \in \mathcal{J}$. Let q_{ij} be be the fraction of time that user i is allowed to transmit exclusively on the frequency band of provider j, with the constraint that $\sum_{i \in \mathcal{I}_j} q_{ij} = 1$, $j \in \mathcal{J}$. Furthermore, assume that each user has a peak power constraint P_i. We can then define $c_{ij} = W_j \log(1 + \frac{P_i |h_{ij}|^2}{\sigma_{ij}^2 W_j})$, where h_{ij} is the channel gain and σ_{ij}^2 is the Gaussian noise variance for the channel between user i and network j. In this case, a user's payoff is the difference between its utility obtained from the service (in terms of total rate) and the payments for the resource used, $v_i = u_i(\sum_{j=1}^{J} q_{ij}c_{ij}) - \sum_{j=1}^{J} p_j q_{ij}$.*

Similarly, our model is applicable when providers sell bandwidth or OFDM tones to users who face a maximum power constraint. Although the c_{ij} channel quality offset factor represents channel capacity in Example 4.1, it can be any increasing function of the channel strength depending on the specific application scenario.

Finally, we remark that the problem shares certain similarity with the multi-path routing problem in a generalized network flow setting, where each source corresponds to a user and each link corresponds to a provider. The key difference is that in our model the user-provider connections have different weights, which is not the case for the multipath routing problem.

Model assumptions

We make the following assumptions throughout this chapter:

Assumption 4.1 (Utility functions). *For every user $i \in \mathcal{I}$, $u_i(x)$ is differentiable, increasing and, strictly concave in x. This is a standard way to model elastic data applications in network literature (see, e.g., [33]).*

Assumption 4.2 (Channel quality offsets and channel gains). *Channel quality offsets c_{ij} are drawn independently from continuous, possibly different probability distributions. In particular $Pr(c_{ij} = c_{kl}) = 0$ for any $i, k \in \mathcal{I}$, $j, l \in \mathcal{J}$. The channel quality offset accounts for the effect that buying the same amount of resource from different providers typically has different effects on a user's quality of service. As Example 4.1* *shows, channel quality offset c_{ij} may be a function of the channel gain h_{ij} between user i and provider j. In this case the assumption is fulfilled if channel gains are drawn from independent continuous probability distributions (e.g., Ralyleigh, Rician, distance-based path-loss model).*

Assumption 4.3 (**Atomic and price-taking users**). *The demand for an atomic user is not infinitely small and can have an impact on providers' prices. Precise characterization of this impact is one of the focuses of this chapter. On the other hand, users are price-takers by the assumption of the two-stage game, and do not strategically influence prices.*

To analyze the properties of the provider competition game, in Section 4.2 we study a related socially optimal resource allocation problem and show the uniqueness of its solution in terms of users' demands. Then, in Section 4.3, we come back to the provider competition game. We show that the unique socially optimal solution solution corresponds to the unique equilibrium of the provider competition game, in which case the selfish and strategic behavior of providers and users leads to zero efficiency loss.

4.2 Social optimum

Social welfare maximization

In this section we consider a social welfare problem, which aims at maximizing the sum of payoffs of all participants, (users and providers), and show the uniqueness of its solution in terms of users' demands. The social welfare problem is equivalent to maximizing the sum of users' utility functions since the payments between users and providers cancel out. We will define the social welfare maximization problem as a function of *services acquired* by the users, which is ultimately what the users are interested in.

Definition 4.1 (**Service acquired**). *Let $\boldsymbol{x} = [x_1 \ \cdots \ x_I]$ be the vector of* services acquired*, where the service acquired by user i, $x_i = \sum_{j=1}^{J} q_{ij}c_{ij}$ is a function of user i's demand $\boldsymbol{q}_i = [q_{i1} \ldots q_{iJ}]$ for resource.*

The social welfare optimization problem (SWO) is:

$$\textbf{SWO} : \max \ u(\boldsymbol{x}) = \sum_{i=1}^{I} u_i(x_i) \tag{4.2}$$

$$\text{subject to } \sum_{j=1}^{J} q_{ij}c_{ij} = x_i \ i \in \mathcal{I} \tag{4.3}$$

$$\sum_{i=1}^{I} q_{ij} = Q_j, \ j \in \mathcal{J} \tag{4.4}$$

$$\text{over } q_{ij}, x_i \geq 0 \ \forall i \in \mathcal{I}, j \in \mathcal{J}. \tag{4.5}$$

We expressed the SWO in terms of two different variables: service acquired vector $\boldsymbol{x}$ and demand vector $\boldsymbol{q}$, even though the problem can be expressed entirely in terms of $\boldsymbol{q}$. In particular, a vector $\boldsymbol{q}$ uniquely determines a vector $\boldsymbol{x}$ through equations (4.3), i.e.

we can write $\boldsymbol{x}$ as $\boldsymbol{x}(\boldsymbol{q})$. With some abuse of notation we will write $u(\boldsymbol{q})$ when we mean $u(x(\boldsymbol{q}))$.

Lemma 4.1 (Uniqueness of maximizing service acquired). *The social welfare optimization problem SWO has a unique optimal solution $\boldsymbol{x}^*$.*

Proof. Since $u_i(x_i)$ is strictly concave in x_i, then $u(\boldsymbol{x}) = \sum_{i=1}^{I} u_i(x_i)$ is strictly concave in $\boldsymbol{x}$. The feasible region defined by constraints (4.3)-(4.5) is convex. Assume there are two maximizers of SWO, $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, where $u(\boldsymbol{x}^1) = u(\boldsymbol{x}^2) = u^{max}$. Due to strict concavity of $u(\cdot)$ in $\boldsymbol{x}$, for any $\lambda \in (0,1)$ we have $u(\lambda\boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2) > \lambda u(\boldsymbol{x}^1) + (1-\lambda)u(\boldsymbol{x}^2) = u^{max}$, which is a contradiction. Hence, $u(\boldsymbol{x})$ has a unique optimal solution $\boldsymbol{x}^*$ subject to constraints (4.3)-(4.5). $\square$

Uniqueness of the socially optimal demand vector $\boldsymbol{q}^*$

Even though $u_i(\cdot)$'s are strictly concave in x_i, they are not strictly concave in the demand vector $\boldsymbol{q_i}$. Hence, SWO is non-strictly concave in $\boldsymbol{q}$. It is well-known that a non-strictly concave maximization problem might have several different global optimizers (several different demand vectors $\boldsymbol{q}$ in our case) (see e.g. [34],[35]). In particular, one can choose c_{ij}'s, Q_j's, and $u_i(\cdot)$'s in such a way that a demand maximizing vector $\boldsymbol{q}^*$ of SWO is not unique. For example, taking $c_{ij} = c$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $Q_j = Q$ for all $j \in \mathcal{J}$ results in a non-unique maximizer of SWO. However, we can show that such cases arise with zero probability whenever channel offsets factors c_{ij}'s are independent random variables drawn from continuous distributions (see Assumption 4.2).

In the remainder of this section, we show that SWO has a unique maximizing demand vector with probability 1. We begin by proving Lemma 4.2, which is an intermediate result stating that any two maximizing demand vectors of SWO must have different non-zero components. We then observe that any convex combination of two maximizing demand vectors is also a maximizing demand vector. Finally, we show that all convex combinations of maximizing demand vectors have the same non-zero components, which is a contradiction with Lemma 4.2. This proves the main result of this section (Theorem 4.3).

To make our argument precise, we first define the support set of a demand vector $\boldsymbol{q}_i$ as follows.

Definition 4.2 (Support set). *The support set of a demand vector $\boldsymbol{q}_i$ contains the indices of its non-zero entries, i.e. the set of providers that user i has strictly positive demand from:*

$$\hat{\mathcal{J}}_i(\boldsymbol{q}_i) = \{j \in \mathcal{J} : q_{ij} > 0\}.$$

Given a demand vector $\boldsymbol{q}$, the ordered collection *of support sets $\hat{\mathcal{J}}_1, \hat{\mathcal{J}}_2, \ldots, \hat{\mathcal{J}}_I$ is denoted by $\{\hat{\mathcal{J}}_i\}_{i=1}^I$.*

Lemma 4.2. *Let $\boldsymbol{q}^*$ be an optimal solution of SWO (a maximizing demand vector) and $\{\hat{\mathcal{J}}_i\}_{i=1}^I$ be the corresponding collection of support sets. Then, $\boldsymbol{q}^*$ is almost surely[1] the unique maximizing demand vector corresponding to $\{\hat{\mathcal{J}}_i\}_{i=1}^I$.*

[1] This holds on the probability space defined by the distributions of c_{ij}'s.

Proof. For a maximizing demand vector $\boldsymbol{q}^*$, equations (4.3)-(4.5) hold. To prove the lemma, we will uniquely construct $\boldsymbol{q}^*$ from $\boldsymbol{x}^*$ and $\{\hat{\mathcal{J}}_i\}_{i=1}^{I}$.

We can divide the users into two categories.

Definition 4.3 (Decided and undecided users). *The* decided *users purchase from only one provider ($|\hat{\mathcal{J}}_i| = 1$), and the* undecided *users from several ($|\hat{\mathcal{J}}_i| > 1$).*

It is also possible that some users have zero demand to all providers. Without loss of generality, we treat such users as decided. Recall that for all users we have $x_i^* = \sum_{j=1}^{J} q_{ij}^* c_{ij}$. For a *decided* user i who purchases only from provider $\bar{\jmath}$, this reduces to $x_i^* = q_{i\bar{\jmath}}^* c_{i\bar{\jmath}}$, and the corresponding unique demand vector is $\boldsymbol{q}_i^* = [0 \cdots 0 \frac{x_i^*}{c_{i\bar{\jmath}}} 0 \cdots 0]$.

For undecided users, finding the unique $\boldsymbol{q}_i^*$ is less straightforward as there is more than one $\boldsymbol{q}_i$ such that $\sum_{j \in \hat{\mathcal{J}}_i} q_{ij} c_{ij} = x_i^*$. To show that the demand of undecided users is unique, we construct the bipartite graph representation (BGR) $\mathcal{G}$ of the undecided users' support sets as follows. We represent undecided users by circles, and providers of undecided users as squares. We place an edge (i, j) between a provider node j and a user node i if $j \in \hat{\mathcal{J}}_i$.

We give an example of a BGR in Figure 4.1, where $\hat{\mathcal{J}}_1 = \{a, b, c\}$, $\hat{\mathcal{J}}_2 = \{b, d\}$, $\hat{\mathcal{J}}_3 = \{d, e, f\}$ and $\hat{\mathcal{J}}_4 = \{b, g\}$.

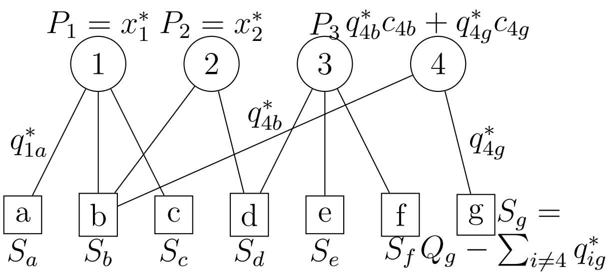

Figure 4.1: Bipartite graph representation

The BGR has the following properties[2] (refer to Figure 4.1):

(1) The sum of service acquired on all the edges connecting user i is the optimal service acquired $x_i^* = \sum_{j \in \hat{\mathcal{J}}_i} q_{ij}^* c_{ij} = P_i$. Borrowing from coding theory and with some abuse of terminology, we call P_i the *check-sum* of user node i.

(2) The sum of all edges connecting to provider node j equals to the difference between the supply Q_j and the demand from decided users who connect to provider j: $\sum_{i:(i,j) \in \mathcal{G}} q_{ij}^* = Q_j - \sum_{i:(i,j) \notin \mathcal{G}} q_{ij}^* = S_j$. We call S_j the *check-sum* of provider j.

(3) With probability 1, the BGR does not contain any loops. This is proved in Appendix 4.A.

As it is the case in Figure 4.1, the number of undecided users is smaller than the number of providers. This is a direct consequence of Property (3).

[2]Figure 4.1 shows a connected graph, but this need not be the case.

We can use the BGR to uniquely determine the demands of undecided users. Here we use Figure 4.1 as an illustrative example, the formal description is given in Appendix 4.A. We call this procedure the BGR decoding algorithm. Consider leaf node (a node with only one edge) g and edge q_{4g}^*. The BGR implies that user 4 is the only undecided customer of provider g. Since the demands of all decided users have been determined, then we know that $q_{4g}^* = S_g = Q_g - \sum_{i \neq 4} q_{ig}^*$. We can then remove edge q_{4g}^* and node g from the BGR, and update the check-sum value of node 4 to $P_4 = x_4^* - q_{4g}^* c_{4g}$. Now consider node 4 and edge q_{4b}^*. Since edge q_{4b}^* is now the only edge connecting with user node 4, we have $q_{4b}^* c_{4g} = P_4$ and hence $q_{4b}^* = \frac{P_4}{c_{4b}}$. Next we can consider node a, e, or f, and so on. Property 3) is crucial in this procedure since it guarantees that we can always find a leaf node in the reduced graph.

In each step of the algorithm, we determine the unique value of q_{ij}^* associated with the edge of one leaf, and this value is independent of the order in which we pick the leaf nodes, as seen from Appendix 4.A. So, we can construct unique demand vectors $\boldsymbol{q}_i^*$ for each undecided user i in the BGR. Together with the unique demand vectors of the decided users, we have found the unique maximizing demand vector of SWO with support sets $\{\hat{\mathcal{J}}_i\}_{i=1}^I$. □

Theorem 4.3 (Uniqueness of maximizing demand vector). *The social welfare optimization problem SWO has a unique maximizing solution $\boldsymbol{q}^*$ with probability 1.*

Proof. The detailed proof is in Appendix 4.A, here we provide an outline. Assume there exist two maximizing demand vectors of SWO which, by Lemma 4.2, have different supports sets. The support set of a convex combination of any two non-negative vectors is the union of support sets of these two vectors. Hence, all convex combinations of two maximizing demand vectors of SWO, which are also maximizing demand vectors, have the same support. This is a contradiction to Lemma 4.2. □

Given an optimal demand vector $\boldsymbol{q}^*$ of the SWO problem, there exists a unique corresponding Lagrange multiplier vector $\boldsymbol{p}^*$, associated with the resource constraints of J providers [36]. In the next section, we show that $(\boldsymbol{q}^*, \boldsymbol{p}^*)$ is the unique equilibrium of the provider competition game defined in Section 4.1.

4.3 Analysis of the two stage game

In this section we show that there exists a unique equilibrium (defined more precisely shortly) of the multi-leader-follower provider competition game. In particular, this equilibrium corresponds to the unique social optimal solution of SWO and the associated Lagrange multipliers. The idea is to interpret the Lagrange multipliers as the prices announced by the providers. Moreover, we show that there are at most $J-1$ undecided users at this equilibrium.

First, we define the equilibrium concept [5]:

Definition 4.4 (Subgame perfect equilibrium (SPE)). *A price demand tuple $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ is a subgame perfect equilibrium for the provider competition game if no player has an incentive to deviate unilaterally at any stage of the game. In particular, each user $i \in \mathcal{I}$*

maximizes its payoff given prices $\boldsymbol{p}^*$. Each provider $j \in \mathcal{J}$ maximizes its revenue given other providers' prices $p^*_{-j} = (p^*_1, \cdots, p^*_{j-1}, p^*_{j+1}, \cdots p^*_J)$ and the users' demand $\boldsymbol{q}^*(\boldsymbol{p}^*)$.

We compute the equilibrium using backward induction. In Stage II, we compute the best response of the users $\boldsymbol{q}^*(\boldsymbol{p})$ as a function of any given vector of prices $\boldsymbol{p}$. Then in Stage I, we compute the equilibrium prices $\boldsymbol{p}^*$. For equilibrium prices $\boldsymbol{p}^*$, the best response of the users $\boldsymbol{q}^*(\boldsymbol{p}^*)$ is uniquely determined via BGR decoding.

Equilibrium strategy of the users in Stage II

Consider users who face prices $\boldsymbol{p}$ in the second stage. Each user solves a user payoff maximization (UPM) problem:

$$\textbf{UPM}: \max_{\boldsymbol{q}_i \geq \boldsymbol{0}} v_i = \max_{\boldsymbol{q}_i \geq \boldsymbol{0}} u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right) - \sum_{j=1}^{J} p_j q_{ij} \tag{4.6}$$

Lemma 4.4. *For every maximizer $\boldsymbol{q}_i$ of the UPM problem, $\sum_{j=1} c_{ij} q_{ij} = x^*_i$, for a unique nonnegative value of x^*_i. Furthermore, for any j such that $q_{ij} > 0$, $\frac{p_j}{c_{ij}} = \min_{k \in \mathcal{J}} \frac{p_k}{c_{ik}}$.*

Proof is given in Appendix 4.B. We remark that, together, the unique x^*_i's from Lemma 4.4 form $\boldsymbol{x}^*$, which is equal to $\boldsymbol{x}^*$, the unique maximizer of SWO from Lemma 4.1. We omit the proof for brevity.

Definition 4.5 (Preference set). *For any price vector $\boldsymbol{p}$, user i's preference set $\mathcal{J}_i(\boldsymbol{p})$ includes each provider $j \in \mathcal{J}$ with $\frac{p_j}{c_{ij}} = \min_{k \in \mathcal{J}} \frac{p_k}{c_{ik}}$.*

In light of Lemma 4.4, $\mathcal{J}_i$ is the set of providers from which user i might request a strictly positive amount of resource. Users can again be partitioned to decided and undecided based on the cardinality of their preference sets, analogous to the distinction made in Section 4.2. The preference set of a decided user i contains a singleton, and there is a unique vector $\boldsymbol{q}_i$ that maximizes his payoff. By contrast, the preference set of an undecided user i contains more than one provider, and any choice of $\boldsymbol{q}_i \geq \boldsymbol{0}$ such that $x^*_i = \sum_{j \in \mathcal{J}_i} q_{ij} c_{ij}$ maximizes his payoff.

There is a close relationship between the support sets from Section 4.2 and preference sets defined here. Facing prices $\boldsymbol{p}$, a user i *may* request positive resource only from providers who are in his preference set $\mathcal{J}_i$. By definition, he *actually* requests positive resource from providers who are in his support set $\hat{\mathcal{J}}_i$. So the support set of a user is a subset of his preference set: $\hat{\mathcal{J}}_i(\boldsymbol{q}(\boldsymbol{p})) \subset \mathcal{J}_i(\boldsymbol{p})$. We can construct a BGR based on the preference sets and show that this BGR also has no loops with probability 1. The proof is nearly identical to proving that the BGR based on support sets has no loops (see Appendix 4.A).

Suppose that Lagrange multipliers $\boldsymbol{p}^*$ were announced as prices. Since all users have access to complete information, each of them can calculate all users' preference sets, and can construct the corresponding BGR. Undecided users can now uniquely determine their demand vector by independently running the same BGR decoding algorithm. The demand found through BGR decoding is unique as all demand vectors are considered at one time and the equality of supply and demand is taken into account. The demand found in this way is only one of an undecided user's infinitely many best responses under prices $\boldsymbol{p}^*$. However, only the demands given by the BGR decoding algorithm will balance the supply and demand for each provider at the optimal price $\boldsymbol{p}^*$, and we will later show that this is the only subgame perfect equilibrium of the provider competition game.

Equilibrium strategy of the providers in Stage I

The optimal choice of prices for the providers depends on users' utility functions. A utility function u_i can be characterized by its *coefficient of relative risk aversion* [1], i.e. $k^i_{RRA} = -\frac{x u_i''(x)}{u_i'(x)}$. This quantity characterizes how a user's demand changes with respect to the price. Here we focus on a class of utility functions characterized in Assumption 4.4.

Assumption 4.4 (**Coefficient of relative risk aversion**). *The coefficient of relative risk aversion of the utility function of user i is less than 1, for all $i \in \mathcal{I}$.*

Assumption 4.4 is satisfied by some commonly used utility functions, such as $\log(1+x)$ and the $\alpha-$fair utility functions $\frac{x^{1-\alpha}}{1-\alpha}$, for $\alpha \in (0,1)$ [37]. Under Assumption 4.4, a monopolistic provider will sell all of its resource Q_j to maximize its revenue. Intuitively, when a provider lowers the price, the demand of the users increases significantly enough that the change in revenue is positive. This encourages the provider to lower the price further such that eventually total demand equals total supply. In the case of multiple providers, Assumption 4.4 also ensures that all providers are willing to sell all their resources to maximize their revenues.

We call the prices that achieve equality of demand and supply *market clearing prices.*

Theorem 4.5 (**Unique equilibrium of the provider competition game**). *Under Assumption 4.4, the unique socially optimal demand vector $\boldsymbol{q}^*$ and the associated Lagrangian multiplier vector $\boldsymbol{p}^*$ of the SWO problem constitute the unique sub-game perfect equilibrium of the provider competition game.*

The proof is given in Appendix 4.B.

It is interesting to see that the competition of providers does not reduce social efficiency. This is not a simple consequence of the strict concavity of the users' utility functions; it is also related to the elasticity of users' demands. Assumption 4.4 ensures that the demands are elastic enough such that a small decrease in price leads to significant increase in demand and thus a net increase in revenue.

Under the optimal prices $\boldsymbol{p}^*$ announced by the providers in the first stage, the users in the second stage will determine the unique demand vector $\boldsymbol{q}^*$ using BGR decoding. On the other hand, if the providers charge prices other than $\boldsymbol{p}^*$, no best-response from the users will make the demand equals to the supply, which is a necessary condition for an equilibrium.

In light of Theorem 4.5, we will refer to the unique subgame perfect equilibrium $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ of the provider competition game as **the equlibrium**.

The number of undecided users

Since the presence of undecided users makes the analysis challenging, it is interesting to understand how many undecided users there can be in a given game. It turns out that such number is upperbounded by the number of providers J in the network.

Lemma 4.6. *Under any given price vector $\boldsymbol{p}$ in the first stage, the number of undecided users in the second stage is strictly less than J.*

The proof is given in Appendix 4.B. The main idea is that if the number of undecided user nodes in a BGR is not smaller than the number of provider nodes, then there exists a loop in the BGR. This, however, occurs with zero probability, as shown in Section 4.2.

Given a set of prices, the decided users can calculate the unique demand vector that maximizes their payoffs, while the undecided users have an infinite number of such vectors. In particular, calculating the equilibrium maximizing demand vectors for undecided users may require cooperation between different providers, which may pose a challenge.

On the other hand, the number of undecided users is small, i.e., not larger than J, and it does not grow with the number of users. Future systems may have user action replaced by the actions of software agents in charge of connection and handover between different providers. In this case, splitting over different providers may become feasible. This is not very much unlike soft handoff (soft handover), a feature used by CDMA and WCDMA standards [38]. In addition, when the number of users is large, the impact of a single user on the price may be small. Hence, operating at a non-equilibrium price as the result of the decisions of a few undecided users may not have a great impact on the experienced quality of service, although the exact loss remains to be quantified.

In the model we consider, we observe the "locally monopolistic" nature of wireless commerce, which does not exist for most other traditional goods. Namely, a user that has a strong channel to some provider, but a weak one to others, is willing to pay a higher price to the provider with the strong channel, and is thus not influenced by moderate price changes during the price competition. On the other hand, users with similar channel gains to all providers will be more sensitive to price competition. This local monopoly is in contrast to some other wireless resource allocation models where users' association is based solely on the price, so given the choice between wireless providers, *all* users go to the cheaper one.

4.4 Primal-dual algorithm

The previous analysis of the subgame perfect equilibrium has assumed that every player (provider or user) knows the complete information of the system. This may not be true in practice. In this section we present a distributed primal-dual algorithm where providers and users only know local information and make local decisions in an iterative fashion. We show that such algorithm globally converges to the unique equilibrium discussed in Theorem 4.5 under mild conditions on the updating rates.

The key proof idea is to show that the primal-dual algorithm converges to a set containing the optimal solution of SWO. We can further show that this set contains only the unique optimal solution in most cases, regardless of the values of the updating rates.

We first present the algorithm, and then the proof of its convergence.

Algorithm definition

In this section we will consider a continuous-time algorithm, where all the variables are functions of time. For compactness of exposition, we will sometimes write q_{ij} and p_j when we mean $q_{ij}(t)$ and $p_j(t)$, respectively. Their time derivatives $\frac{\partial q_{ij}}{\partial t}$ and $\frac{\partial p_j}{\partial t}$ will often be denoted by $\dot{q}_{ij}$ and $\dot{p}_j$. We denote by $\boldsymbol{q}^*$ and $\boldsymbol{p}^*$ the unique maximizer of SWO and the corresponding Lagrange multiplier vector, respectively. As shown in Theorem 4.5, $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ is also the unique subgame perfect equilibrium of the provider competition game. These values are constant.

To simplify the notation, we denote by $f_{ij}(t)$ or simply f_{ij} the marginal utility of user i with respect to q_{ij} when his demand vector is $\boldsymbol{q_i}(t)$:

$$f_{ij} = \frac{\partial u_i(\boldsymbol{q_i})}{\partial q_{ij}} = c_{ij} \frac{\partial u_i(x)}{\partial x}\Big|_{x=x_i=\sum_{j=1}^{J} q_{ij}c_{ij}}. \tag{4.7}$$

We will use f^*_{ij} to denote the value of $f_{ij}(t)$ evaluated at $\boldsymbol{q}^*_i$, the maximizing demand vector of user i. So, f^*_{ij} is a constant that is equal to a user's equilibrium marginal utility as opposed to $f_{ij}(t)$ which indicates marginal utility at any given time. We also define $\nabla u_i(\boldsymbol{q_i}) = [f_{i1} \cdots f_{iJ}]^T$ and $\nabla u_i(\boldsymbol{q^*_i}) = [f^*_{i1} \cdots f^*_{iJ}]^T$, where all the vectors are column vectors.

We define $(x)^+ = \max(0, x)$ and

$$(x)^+_y = \begin{cases} x & y > 0 \\ (x)^+ & y \leq 0. \end{cases}$$

Another way to think of this notation is $(x)^+_y = x(1 - \mathbb{1}_{(-\infty,0]}(x)\mathbb{1}_{(-\infty,0]}(y))$, where $\mathbb{1}$ is the indicator function, i.e., $\mathbb{1}_A(x) = 1$ if $x \in A$, and 0 otherwise.

Motivated by the work in [39], we consider the following standard **primal-dual variable update algorithm**:

$$\dot{q}_{ij} = k^q_{ij}\left(f_{ij} - p_j\right)^+_{q_{ij}}, \; i \in \mathcal{I}, j \in \mathcal{J} \tag{4.8}$$

$$\dot{p}_j = k^p_j \left(\sum_{i=1}^{I} q_{ij} - Q_j \right)^+_{p_j}, \; j \in \mathcal{J}. \tag{4.9}$$

Here k^p_{ij}, k^p_j are the constants representing update rates. The update rule ensures that, when a variable of interest (q_{ij} or p_j) is already zero, it will not become negative even when the direction of the update (i.e. quantity in the parenthesis) is negative. The tuple $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ controlled by equations (4.8) and (4.9) will be referred to as the *solution trajectory* of the differential equations system defined by (4.8) and (4.9).

The motivation for the proposed algorithm is quite natural. A provider increases its price when the demand is higher than its supply and decreases its price when the demand is lower. A user decreases his demand when a price is higher than his marginal utility and increases it when a price is lower. In essence, the algorithm is following the natural direction of the market forces.

One key observation is that these updates can be implemented in a distributed fashion. The users only need to know the prices proposed by the providers. The providers only need to know the demand of the users for their own resource, and not for the resource of other providers (as was the case in the analysis of Section 4.3). In particular, only user i needs to know the channel offset parameters c_{ij}, $j \in \mathcal{J}$.

The first step to prove the algorithm's convergence is to construct a lower-bounded La Salle function $V(\boldsymbol{q}(t), \boldsymbol{p}(t))$ and show that $V(t)$ is non-increasing for any solution trajectory $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ that satisfies (4.8) and (4.9). This will ensure that $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ converge to a set of values that keeps $V(\boldsymbol{q}(t), \boldsymbol{p}(t))$ constant.

Convergence of the primal-dual to an invariant set

We consider the following La Salle function:

$$V(\boldsymbol{q}(t), \boldsymbol{p}(t)) = V(t) = \sum_{i,j} \frac{1}{k_{ij}^q} \int_0^{q_{ij}(t)} (\beta - q_{ij}^*) d\beta + \sum_j \frac{1}{k_j^p} \int_0^{p_j(t)} (\beta - p_j^*) d\beta,$$

It can be shown that $V(\boldsymbol{q}(t), \boldsymbol{p}(t)) \geq V(\boldsymbol{q}^*, \boldsymbol{p}^*)$, i.e., V is bounded from below. This ensures that if the function V is non-increasing, it will eventually reach a constant value (which may or may not be the global minimum $V(\boldsymbol{q}^*, \boldsymbol{p}^*)$).

The derivative of V *along the solution trajectories of the system*, $\frac{\partial V}{\partial t}$, denoted by $\dot{V}$, is given by:

$$\dot{V}(t) = \sum_{i,j} \frac{\partial V}{\partial q_{ij}} \dot{q}_{ij} + \sum_j \frac{\partial V}{\partial p_j} \dot{p}_j.$$

Lemma 4.7. *The value of the La Salle function V is non-increasing along the solution trajectory, defined by (4.8) and (4.9), i.e. $\dot{V}(t) \leq 0$.*

Proof. Proof is given in Appendix 4.C. The proof manipulates the expression for $\dot{V}$ and shows that it can be reduced to the following form:

$$\dot{V} \leq \sum_i \left(\sum_j (q_{ij}(t) - q_{ij}^*)(f_{ij}(t) - f_{ij}^*) \right) + \sum_{i,j} (q_{ij}(t) - q_{ij}^*)(f_{ij}^* - p_j^*). \tag{4.10}$$

Then, using concavity of $u_i's$ and properties of the equilibrium point $\boldsymbol{q}^*, \boldsymbol{p}^*$, we show that individual elements of the summations in (4.10) are non-positive. □

Combining Lemma 4.7 and the La Salle's *invariance principle* (Theorem 4.4 of [40], p. 128) we can prove the following:

Proposition 4.8. *The pair $\boldsymbol{q}(t), \boldsymbol{p}(t)$ converges to the invariant set $V_L = \{\boldsymbol{q}(t), \boldsymbol{p}(t) : \dot{V}(\boldsymbol{q}(t), \boldsymbol{p}(t)) = 0\}$ as $t \to \infty$.*

It is clear that the invariant set V_L contains the solution trajectory that has the value of the unique maximizer of SWO $(\boldsymbol{q}^*(t), \boldsymbol{p}^*(t)) = (\boldsymbol{q}^*, \boldsymbol{p}^*)$ for all t, since $\dot{V}(\boldsymbol{q}^*, \boldsymbol{p}^*) = 0$. However it may contain other points as well. When the trajectory $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ enters the invariant set, it either reaches its minimum (i.e., by converging to the unique equilibrium point $(\boldsymbol{q}^*, \boldsymbol{p}^*)$), or it gets stuck permanently in some limit cycle. In either case, the trajectory will be confined to a subset of $V_L = \{(\boldsymbol{q}(t), \boldsymbol{p}(t)) : \dot{V}(\boldsymbol{q}(t), \boldsymbol{p}(t)) = 0\}$.

The remainder of this section is to show that the invariant set V_L contains only the equilibrium point $\boldsymbol{q}^*, \boldsymbol{p}^*$. This will be done in two steps. First, we show that the set V_L has only one element for the majority of provider competition instances, without any restrictions on the variable update rates. Second, we provide a sufficient condition on the update rates so that the global convergences to the unique equilibrium point is also guaranteed in the remaining instances.

Convergence when providers have decided customers

In the following two sections we consider the properties of the solution trajectory **on the invariant set** V_L.

The proof of Lemma 4.7 shows that individual terms on the right-hand side of (4.10) are non-positive. Combined with Proposition 4.8, we get the following result:

Corollary 4.9. *On the invariant set V_L, $\boldsymbol{q}(t), \boldsymbol{p}(t)$ are such that:*

$$\sum_j (q_{ij}(t) - q_{ij}^*)(f_{ij}(t) - f_{ij}^*) = (\nabla u_i(\boldsymbol{q}_i)(t) - \nabla u_i(\boldsymbol{q}_i^*))^T(\boldsymbol{q}_i(t) - \boldsymbol{q}_i^*) = 0, \forall i \in \mathcal{I} \quad (4.11)$$

$$(q_{ij}(t) - q_{ij}^*)(f_{ij}^* - p_j^*) = 0, \text{ for all } i \in \mathcal{I},\ j \in \mathcal{J}, \quad (4.12)$$

where we recall that $\nabla u_i(\boldsymbol{q_i})(t) = [f_{i1} \cdots f_{iJ}]^T$ and $\nabla u_i(\boldsymbol{q_i^*}) = [f_{i1}^* \cdots f_{iJ}^*]^T$.

Expressions (4.11) and (4.12) give basic properties of the solution trajectories $\boldsymbol{q}(t), \boldsymbol{p}(t)$ on the invariant set. We next prove two intermediate results that give further characterization of $\boldsymbol{q}(t), \boldsymbol{p}(t)$ on V_L.

Lemma 4.10. *For any point in the invariant set V_L, we have $f_{ij}(t) = f_{ij}^*$ for all $i \in \mathcal{I}, j \in \mathcal{J}$. In other words, any user i's marginal utility with respect to its demand of any provider j equals to the corresponding value at the unique equilibrium. In addition, $q_{ij}(t)(f_{ij}^* - p_j^*) = 0$ on the invariant set V_L.*

The proof is given in Appendix 4.C.

An equivalent of saying that all users attain the equilibrium marginal utility is that $x_i(t) = x_i^*$ on V_L:

$$\sum_j q_{ij}(t)c_{ij} = \sum_j q_{ij}^* c_{ij}, \text{ for all } i \in \mathcal{I}. \quad (4.13)$$

The second part of Lemma 4.10, claims that $q_{ij}(t)(f_{ij}^* - p_j^*) = 0$. From the proof of Lemma 4.4 we know $q_{ij}^*(f_{ij}^* - p_j^*) = 0$. So for i, j such that $f_{ij}^* < p_j^*$, we see that $q_{ij}^* = 0$ implies that $q_{ij}(t) = 0$. This is good news, since from Lemma 4.6 most users have zero demand to all but one provider at the unique equilibrium. Now we know that the same holds on the invariant set V_L. Similarly, $q_{ij}(t) > 0$ only if $f_{ij}^* = p_j^*$.

Lemma 4.10 does not preclude the possibility that a demand for a user with $f_{ij}^* = p_j^*$ may oscillate between being zero and being strictly positive. The following Lemma shows that this is not possible:

Lemma 4.11. *The set $\hat{\mathcal{J}}_i(t) = \{j \in \mathcal{J} : q_{ij}(t) > 0\}$ does not change over time on the invariant set. In addition, $p_j(t) > 0$ on the invariant set for all $j \in \mathcal{J}$.*

The proof is given in Appendix 4.C. Lemma 4.11 implies that if $q_{ij}(t) = 0$ on the invariant set V_L, then $\dot{q}_{ij}(t) = 0$ on V_L. Also, if $q_{ij}(t) > 0$ then $\dot{q}_{ij}(t) = k^q_{ij}\left(f^*_{ij} - p_j\right)^+_{q_{ij}} = k^q_{ij}\left(f^*_{ij} - p_j\right)$, as strictly positive demand stays strictly positive on the invariant set.

We are now ready to claim the main result of this subsection:

Theorem 4.12. *A demand vector of a decided user $i \in \mathcal{I}$ converges to the equilibrium demand vector, i.e. $\lim_{t\to\infty} \boldsymbol{q}_i(t) = \boldsymbol{q}^*_i$. The price p_j of any provider j who has at least one decided user at the equilibrium of the provider competition game, converges to the equilibrium price, i.e. $\lim_{t\to\infty} p_j(t) = p^*_j$.*

Proof. Consider an arbitrary user i who at the equilibrium has only one preferred provider $\bar{\jmath}$, i.e., $f^*_{i\bar{\jmath}} = p^*_{\bar{\jmath}}$ for $q^*_{i\bar{\jmath}} > 0$, and $f^*_{ij} < p^*_j$ for $j \neq \bar{\jmath}$. By Lemma 4.10, this implies $q_{i\bar{\jmath}}(t) > 0$, $q_{ij}(t) = 0$ for all $j \neq \bar{\jmath}$ and $\dot{q}_{ij}(t) = 0$, for all $j \in \mathcal{J}$. Combined with (4.13), this means that user i's demand vector converges to its equilibrium value. For provider $\bar{\jmath}$, $q_{i\bar{\jmath}} > 0$ so by Lemma 4.11:

$$0 = \dot{q}_{i\bar{\jmath}}(t) = k^q_{i\bar{\jmath}}\left(f^*_{i\bar{\jmath}} - p_{\bar{\jmath}}(t)\right)^+_{q_{i\bar{\jmath}}} = k^q_{i\bar{\jmath}}\left(f^*_{i\bar{\jmath}} - p_{\bar{\jmath}}(t)\right).$$

From this it follows that $p_{\bar{\jmath}}(t) = f^*_{i\bar{\jmath}} = p^*_{\bar{\jmath}}$. Further differentiation yields $\dot{p}_{\bar{\jmath}}(t) = 0$, meaning that the prices have also converged, which completes the proof. □

Theorem 4.13. *If every provider has at least one decided customer in the unique equilibrium of the provider competition game, the primal dual algorithm converges to this equilibrium.*

Proof. By Theorem 4.12, the prices of all providers and the demand vectors of all decided users converge on the invariant set. It remains to be shown that the demand vectors of undecided users also converge. By an argument similar to the proof of Lemma 4.2, we can draw a BGR for the undecided users. The demand vectors of undecided users that satisfy the constraints on the BGR are unique, and thus also converge. □

In most practical cases, where the number of users is much larger than the number of providers, all providers will have at least one decided user, and hence the convergence of the primal-dual algorithm is virtually guaranteed[3]. Next we study the more complicated case where some providers do not have any associated decided users. In that case, we can still prove the convergence of the primal-dual algorithm under mild conditions of the variable update rates.

Convergence when providers have no decided customers

Without loss of generality, we now focus on the problem where all the users are undecided at the equilibrium. If we can prove that the algorithm converges in this case, then we can also prove convergence in the more general case where some providers have decided

[3] A priori, there is a small, but finite probability that some providers have only undecided users, regardless of the number of users.

users. Let I be the number of undecided users, and J the number of providers. From Lemma 4.6 we know that $I < J$.

Theorem 4.14. *Let $I < J$, and suppose that at the equilibrium $|\{j \in \mathcal{J} : q^*_{ij} > 0\}| > 1$ for all $i \in \mathcal{I}$. The primal-dual algorithm converges to the unique equilibrium if the price update rates k^p_j are not integer multiples of each other and the demand update rates are equal, i.e. $k^q_{ij} = k\ \forall i \in \mathcal{I}, \forall j \in \mathcal{J}$.*

Proof. We first define some matrices to facilitate the proof. Let C be the $I \times IJ$ matrix of channel offsets c_{ij}. Let $\mathbb{I}_J$ be the identity matrix of size $J \times J$. Define matrix A to be the $IJ \times J$ matrix of I vertically stacked identity matrices. Let $K^p = diag(k^p_j, j \in \mathcal{J})$ be the $J \times J$ diagonal matrix containing price update rates. Let the $IJ \times IJ$ diagonal matrix K^q be the matrix of demand update rates whose y^{th} entry is $K^q_{y,y} = k^q_{i,j}$, where $i = \lfloor (y-1)/J \rfloor + 1$ and $j = (y-1) \mod J + 1$, and zero otherwise, where $\lfloor x \rfloor$ is the largest integer not larger than x.

For example, with $I = 2$, $J = 3$:

$$C = \begin{bmatrix} \boldsymbol{c}_1^T & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{c}_2^T & \dots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{c}_I^T \end{bmatrix}, \ K^q = \begin{bmatrix} k^q_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & k^q_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & k^q_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & k^q_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & k^q_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & k^q_{23} \end{bmatrix}, \ K^p = \begin{bmatrix} k^p_1 & 0 \\ 0 & k^p_2 \end{bmatrix}.$$

By (4.13), we know that the service acquired of all users has converged on the invariant set V_L. We rewrite this as $\boldsymbol{c}_i^T(t)\boldsymbol{q}_i = \boldsymbol{c}_i^T \boldsymbol{q}_i^*$ for all $i \in \mathcal{I}$, or $C\boldsymbol{q}(t) = C\boldsymbol{q}^*$.

We now want to express the primal-dual update algorithm (4.8) and (4.9) in matrix form. The final hurdle is getting rid of the $(x)^+_y$ operation. From Lemma 4.11 we know that $\boldsymbol{p}(t) > 0$ and that the support sets of vectors $\boldsymbol{q}(t)$ do not change on the invariant set. Hence, we can write $\dot{q}_{ij} = k^q_{ij}\left(f_{ij} - p_j\right)^+_{q_{ij}} = k^q_{ij}\left(f^*_{ij} - p_j\right)\mathbb{1}_{(0,\infty)}(q_{ij})$ (recall that $f_{ij} = f^*_{ij}$ on the invariant set). This enables us to revise the definition of the update rates matrix to be $\hat{K}^q = diag(k^q_{ij}\mathbb{1}_{(0,\infty)}(q_{ij}), i \in \mathcal{I}, j \in \mathcal{J})$.

For example, if $q_{11}, q_{12}, q_{22}, q_{23} > 0$ and $q_{13}, q_{21} = 0$ then

$$\hat{K}^q = \begin{bmatrix} k^q_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & k^q_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k^q_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & k^q_{23} \end{bmatrix}.$$

Then, (4.8) and (4.9) can be written as:

$$\dot{\boldsymbol{q}} = \hat{K}^q\left(\boldsymbol{f}^* - A\boldsymbol{p}(t)\right) \tag{4.14}$$

$$\dot{\boldsymbol{p}} = K^p\left(A^T\boldsymbol{q}(t) - Q\right). \tag{4.15}$$

Notice that (4.14) and (4.15) form a system of linear equations, so the non-linear primal dual defined (4.8) and (4.9) becomes linear on the invariant set. The following result paves the way to showing that $\boldsymbol{p}(t)$ is constant on V_L.

Lemma 4.15. *Let E and $\mathcal{B}$ be constant matrices, where $\mathcal{B} = \left[B; BD; \cdots; BD^{J-1}\right]$ ($\mathcal{B}$ is a tall matrix), $B = C\hat{K}^q A$ and $D = K^p A^T \hat{K}^q A$. The dimensions of $\mathcal{B}$, B and D are $IJ \times J$, $I \times J$ and $J \times J$, respectively. On the invariant set, $\mathcal{B}\boldsymbol{p}(t) = E$.*

Proof. The proof is obtained by repeatedly differentiating Equations (4.14) and (4.15) with respect to time. The more detailed calculation is given in Appendix 4.C. □

If we can prove that the rank of $\mathcal{B}$ is J, then we could write $\mathcal{B}_J\boldsymbol{p}(t) = E$, where $\mathcal{B}_J$ is a $J \times J$ matrix constructed by taking J linearly independent rows of $\mathcal{B}$. Then $\boldsymbol{p}(t) = E\mathcal{B}_J^{-1}$, which implies that $\boldsymbol{p}(t)$ converges on the invariant set. To show that the rank of matrix $\mathcal{B}$ is J, we use Theorem 6.01 from [41] (its proof is similar to that of Theorem 6.1 in [41]):

Theorem 4.16. *A tall (IJ by J) matrix $\mathcal{B} = \left[B; BD; \cdots; BD^{J-1}\right]$ has full column rank if and only if matrix $G = \begin{bmatrix} B \\ D - \lambda_j \mathbb{I} \end{bmatrix}$ has full column rank for all eigenvalues λ_j, $j \in \mathcal{J}$ of D.*

We now provide a sufficient condition to ensure convergence of the primal-dual algorithm.

Lemma 4.17. *Let matrices B and D be defined as in Lemma 4.15. Matrix $G = \begin{bmatrix} B \\ D - \lambda \mathbb{I} \end{bmatrix}$ has full column rank for all eigenvalues λ of D if $k_j^p \neq a k_{j'}^p$ $\forall j, j' \in \mathcal{J}$ and all $a \in \mathbb{N}^+$ (i.e., as long as the price update rates are not integer multiples of each other) and $k_{ij}^q = k$, for all $i \in \mathcal{I}, j \in \mathcal{J}$.*

Proof. The proof is given in Appendix 4.C. □

Combining Lemma 4.16 and Lemma 4.17, we see that a unique vector $\boldsymbol{p}(t)$ can be computed from equation $\mathcal{B}\boldsymbol{p}(t) = E$, meaning that $\boldsymbol{p}(t)$ takes a single value on the invariant set and does not change with time. This also means that the demand vector $\boldsymbol{q}(t)$ does not change on the invariant set. So since $\dot{\boldsymbol{p}} = 0$ and $\dot{\boldsymbol{q}} = 0$, the primal dual algorithm converges to an equilibrium point, which we call a dynamic equilibrium point. It can be shown that the dynamic equilibrium point is constrained by the same set of equations as the unique equilibrium of the provider competition game (refer to the proof of Theorem 4.5). Hence, there is only one element in the invariant set V_L, and it corresponds to the equilibrium of the provider competition game, $(\boldsymbol{q}^*, \boldsymbol{p}^*)$. This concludes the proof of Theorem 4.14. □

Note that that the condition on the update rates in Theorem 4.14 is sufficient but not necessary. In fact, by looking at the form of the D matrix from the proof of Lemma 4.17, we can see that a sufficient condition on the price update rates is that $k_j^p(\sum_{i \in \mathcal{I}_j} k_{ij})$, where $\mathcal{I}_j = \{j \in \mathcal{J} : q_{ij} > 0\}$, has a different value for each $j \in \mathcal{J}$. This condition can be satisfied with probability 1, e.g. by drawing k_j^p's and k_{ij}^q's independently from some continuous distribution.

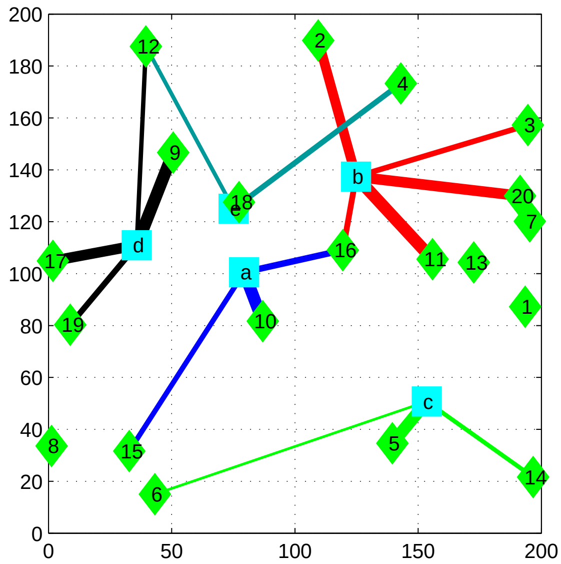

Figure 4.2: Example of equilibrium user association

4.5 Numerical results

For numerical results, we extend the setup from Example 4.1, where the resource being sold is the fraction of time allocated to exclusive use of the providers' frequency band, i.e., $Q_j = 1$ for $j \in \mathcal{J}$. We take the bandwidth of the providers to be $W_j = 20MHz$, $j \in \mathcal{J}$. User i's utility function is $a_i \log(1 + \sum_{j=1}^{J} q_{ij}c_{ij})$, where we compute the spectral efficiency c_{ij} from the Shannon formula $\frac{1}{2}W \log(1 + \frac{E_b/N_0}{W}|h_{ij}|^2)$, q_{ij} is the allocated time fraction, E_b/N_0 is the ratio of transmit power to thermal noise, and a_i is the individual willingness to pay factor taken to be the same across users. The channel gain amplitudes $|h_{ij}| = \frac{\xi_{ij}}{d_{ij}^{\alpha/2}}$ follow Rayleigh fading, where ξ_{ij} is a Rayleigh distributed random variable with parameter 1, and $\alpha = 3$ is the outdoor power distance loss. We choose the parameters so that the c_{ij} of a user is on average around $3.5Mbps$ when the distance is $50m$, and around $60Mbps$ when the distance is $5m$. The average signal-to-noise ratio $E_b/(N_0d^{\alpha})$ at $5m$ is around $25dB$. We assume perfect modulation and coding choices such that the communication rates come from a continuum of values. The users are uniformly placed in a $200m$ by $200m$ area. We want to emphasize that the above parameters are chosen for illustrative purposes only. Our theory applies to any number of providers, any number of users, any type of channel attenuation models, and arbitrary network topologies.

We first consider a single instantiation with 20 users and 5 providers. In Figure 4.2, we show the user-provider association at the equilibrium for a particular realization of channel gains, where the thickness of the link indicates the amount of resource purchased. The users are labeled by numbers (1-20), and the providers are labeled by letters (a-e). This figure shows two undecided users (12 and 16), and that certain users (1,7,13, and 8) do not purchase any resource at equilibrium. For the same realization of channel parameters, Figure 4.3 shows the channel quality, user demand, and rate at the equilibrium, for users 5, 10, 12, 15, and 18. We see that user 15 has better channel with provider d than with provider a, but in the equilibrium all of his demand is towards provider a. This can

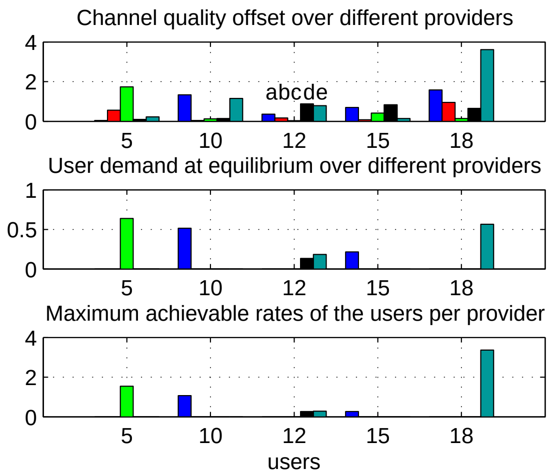

Figure 4.3: Channel quality offset, demand and service acquired (rate) at equilibrium

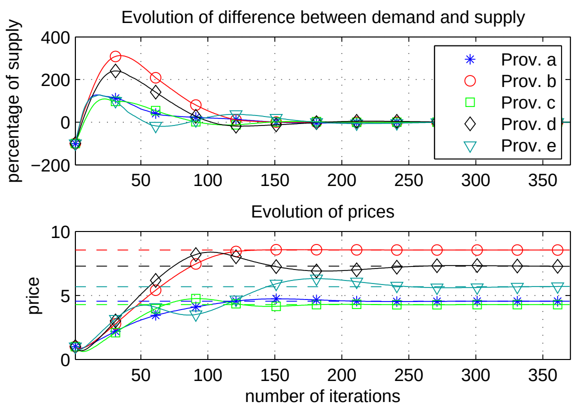

Figure 4.4: Evolution of the primal-dual algorithm

be explained by looking at the bottom part of Figure 4.4, where the dashed lines indicate equilibrium prices. We see that provider a announces a smaller price than provider d. The equilibrium prices reflect the competition among users: in Figure 4.2 we see that provider b has the most customers, so it is not surprising that its price is the highest, as seen in Figure 4.4.

We next consider the convergence time of the discrete time version of the primal-dual algorithm. We fix the number of providers to be 5, and change the number of users from 20 to 100. For each parameter, we run 200 experiments with randomly generated user and provider locations and plot the average speed of convergence. The convergence is defined as the number of iterations after which the difference between supply and demand is no larger than ϵQ_j. Figure 4.5 shows the average convergence time for different values of ϵ. In general, 200 to 400 iterations are needed for convergence with $\epsilon = 10^{-2}$, and 100-200 more iterations to get to 10^{-3}. In Figure 4.6, we compare the average convergence time for different number of providers. Here we take the stopping criterion to be $\epsilon = 10^{-3}$. The convergence time depends on the update rates used for the primal-dual: if the update rates are too high, then the variables will tend to oscillate, so the algorithm will take

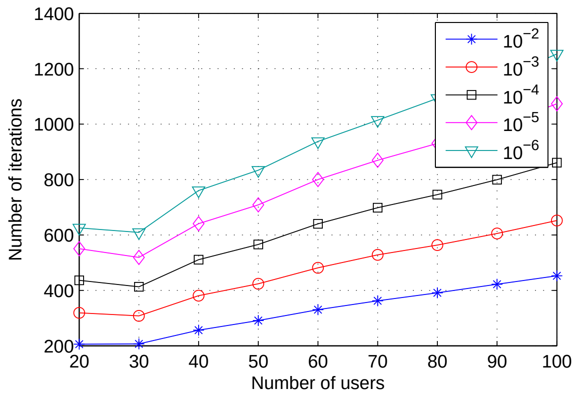

Figure 4.5: Average time to convergence, varying ϵ

too long to converge. On the other hand, if the rates are too small, the variables may converge too slowly. According to our theoretical analysis from Section 4.4, we randomly assigned update variables to ensure global convergence of the algorithm. In general a very small or very large number of users per provider means that the algorithm will take longer to converge. Finally, Figure 4.7 shows the average convergence time for 5 providers with the standard deviation. The convergence time variance does not change with the number of users, except for the case of 20 users. The algorithm is sensitive to the choice of update rates when the ratio of users per provider is smaller than 4. In such cases the prices and demands may oscillate and take a long time to converge, which can be seen in Figure 4.6.

4.6 Related work

In this chapter we have considered a linear-usage pricing scheme, which has been widely adopted in the literature (see e.g. [33, 42]). Analyzing such pricing yields various insights: for example the existing TCP protocol can be interpreted as a usage-based pricing scheme that solves a network utility maximization problem [33]. In practice, however, providers charge monthly subscription fees. For both voice and data plans, these subscriptions are sometimes combined with linear pricing beyond a predefined usage threshold. Pure linear pricing based on instantaneous channel conditions is generally not used, although it has recently received renewed attention due to near-saturation of some mobile networks (see, e.g. [43]).

There exists a rich body of related literature on using pricing and game theory to study provider resource allocation and interactions of service providers. The related research in the wireless setting can be divided into several categories: optimization-based resource allocation of one provider (e.g., [14, 44, 45, 46, 47, 48]), game theoretic study of interactions between the users of one provider (e.g., [49, 50, 51, 52]), competition of different service providers on behalf of the users (e.g., [53, 54]), and providers' price competition to attract users (e.g., [55, 56, 57, 58, 59, 60, 20, 61, 62]). Our work falls into the last category.

In our work, we have simultaneously considered several factors that reflect diverse

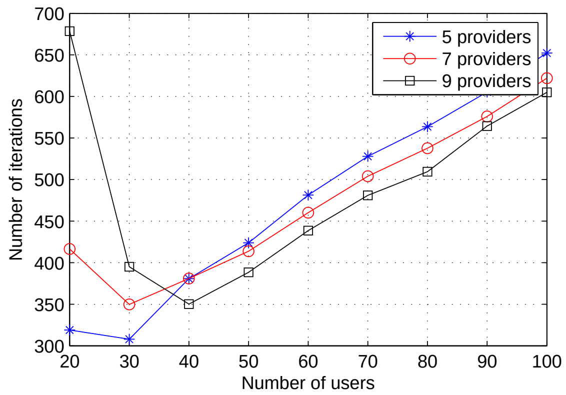

Figure 4.6: Average time to convergence for different numbers of providers

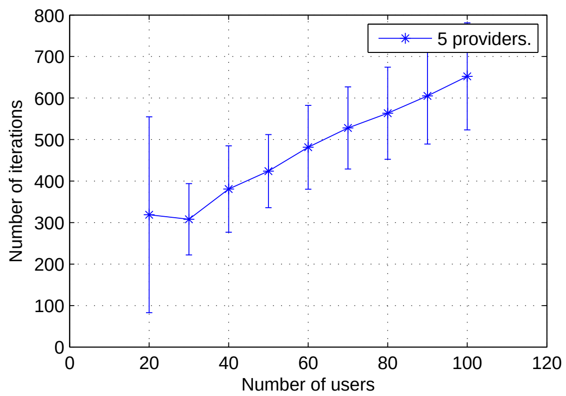

Figure 4.7: Average time to convergence with standard deviation

wireless network scenarios: an arbitrary number of wireless providers compete for an arbitrary number of atomic users, where the users are heterogenous both in channels gains and in willingness to pay. In related work where providers price-compete to attract users [55, 57, 58], purchasing a unit of resource from different providers brings the same amount of utility to a user; in our work a user's utility still depends on the channel gain to the provider. In other work where the Wardrop equilibrium concept is used (e.g., [63]), users are infinitesimal and non-heterogenous; in our work users are atomic and have different willingness to pay. One of the early work that explicitly takes into account the channel differences for different users on a line is [60], for infinitesimal users and distance-based channel gains. Recently, a model similar to ours was used to treat a three-tier system [59], although for specific utility functions. The multiple-seller multiple-buyer dynamics in a cognitive radio setting was studied in [20] using evolutionary game theory. Finally, [61] and [62] consider price competition in a multi-hop wireless network scenario.

The design and proof of the decentralized algorithm were inspired by Chen et al. [39], with several key differences. First, their work considers the optimal resource allocation of a single OFDM cell. Second, it studies a system where each user has a total power constraint. Third, there are often infinitely many global optimal solutions in [39] and finally, our convergence results are proved with a set of conditions that are less stringent than those of [39].

4.7 Conclusions

We provide an overview of the relationship between different concepts used throughout this chapter in Figure 4.8.

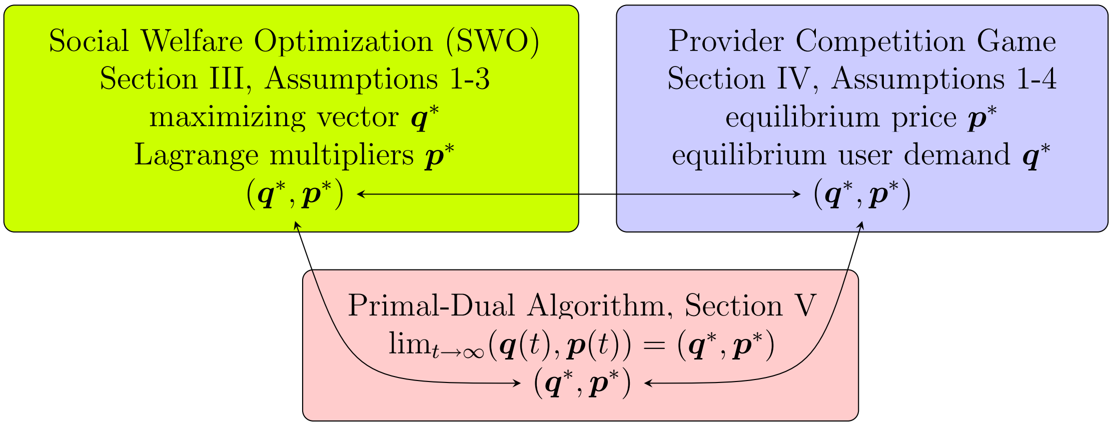

Figure 4.8: Relationship between different concepts

In this chapter we have studied the competition of an arbitrary number of wireless service providers, who want to serve a group of atomic users who are heterogenous in both willingness to pay and channel quality. We have modeled this interaction as a two-stage wireless provider game, and have characterized its unique equilibrium. We have shown that the provider competition leads to a unique socially optimal resource allocation for a broad class of utility functions and a generic channel model. Our results show that some users need to purchase their resource from several providers at the equilibrium, although the number of such users is upper bounded by the number of providers. We have also developed a decentralized algorithm which converges to the equilibrium prices as well as the equilibrium demand vectors using only local knowledge.

Appendices

4.A Social optimality proofs

Proof of BGR Property 3

We first examine the properties of the optimal demand vector of the SWO problem. We will express the SWO in terms of the demand vector $\boldsymbol{q}$ only, by substituting directly equation (4.3) into equation (4.2). Let $\boldsymbol{p} = [p_1 \cdots p_J]$ be the vector of Lagrangian multipliers.

The Lagrangian for SWO is then

$$L(\boldsymbol{q},\boldsymbol{p}) = \sum_{i=1}^{I} u_i \left(\sum_{j=1}^{J} q_{ij} c_{ij} \right) + \sum_{j=1}^{J} p_j \left(Q_j - \sum_{i=1}^{I} q_{ij} \right). \tag{4.16}$$

It is easy to check that the SWO problem satisfies the Slater's condition [64], and thus the sufficient and necessary KKT conditions for an optimal solution $(\boldsymbol{q},\boldsymbol{p})$ are as follows:

$$\begin{aligned}
\frac{\partial u_i(x_i)}{\partial x_i} c_{ij} - p_j \leq& 0,\ j \in \mathcal{J};\ i \in \mathcal{I} && (4.17)\\
q_{ij}\left(\frac{\partial u_i(x_i)}{\partial x_i} c_{ij} - p_j\right) =& 0,\ j \in \mathcal{J};\ i \in \mathcal{I} && (4.18)\\
\sum_{j=1}^{J} q_{ij} c_{ij} =& x_i,\ i \in \mathcal{I} && (4.19)\\
\sum_{i=1}^{I} q_{ij} =& Q_j,\ j \in \mathcal{J} && (4.20)\\
p_j > 0,\ q_{ij} \geq& 0\ j \in \mathcal{J};\ i \in \mathcal{I} && (4.21)
\end{aligned}$$

where with some abuse of notation we use $\frac{\partial u_i(x_i)}{\partial x_i}$ to denote $\left.\frac{\partial u_i(x)}{\partial x}\right|_{x=x_i=\sum_{j=1}^{J} q_{ij}c_{ij}}$.

The following characterizes the relationship between the prices of any two networks from which a user has strictly positive demand.

Recall the support set definition $\hat{\mathcal{J}}_i(\boldsymbol{q}_i) = \{j \in \mathcal{J} : q_{ij} > 0\}$. From (4.17) we see that $\frac{\partial u_i(x_i)}{\partial x_i} \leq \min_{k\in\mathcal{J}} \frac{p_k}{c_{ik}}$. Then, from (4.18) we can see that $q_{ij} > 0$ only when $\frac{\partial u_i(x_i)}{\partial x_i} = \frac{p_j}{c_{ij}}$. Hence $\frac{p_j}{c_{ij}} = \min_{k\in\mathcal{J}} \frac{p_k}{c_{ik}}$ is a necessary condition for $q_{ij} > 0$ for all $i \in \mathcal{I}, j \in \mathcal{J}$. Then, $q_{ij} > 0$ and $q_{ij'} > 0$ implies $\frac{p_j}{c_{ij}} = \frac{p_{j'}}{c_{ij'}} = \min_{k\in\mathcal{J}} \frac{p_k}{c_{ik}}$. In particular, $q_{ij} > 0$ and $q_{ij'} > 0$ implies

$$\frac{c_{ij}}{c_{ij'}} = \frac{p_j}{p_{j'}}. \tag{4.22}$$

We now consider the BGR defined by the support sets $\{\hat{\mathcal{J}}_i\}_{i=1}^{I}$ of undecided users. For any two edges (i,j) and (i,k) of BGR, where i is the user index and j,k are the provider indices, $q_{ij} > 0$ and $q_{ik} > 0$ so by (4.22) we have $\frac{c_{ij}}{c_{ik}} = \frac{p_j}{p_k}$.

Suppose that a loop exists in BGR (refer to Figure 4.9 for this part of the proof). Then, a sequence of nodes $i_1, j_1, i_2, j_2, \ldots,\ i_n, j_n, i_1$ exists, where $i_1, \ldots, i_n$ are the user nodes and $j_1, \ldots, j_n$ are the provider nodes, such that (i_k, j_k) and (j_{k-1}, i_k) are edges in BGR for $k = 1, \ldots n$ (with i_0 defined as i_n). We assume that the members of the sequence are distinct otherwise there is already a smaller loop inside. Since both (i_k, j_k) and (j_{k-1}, i_k) are edges, then $\frac{c_{ik-1}}{c_{ik}} = \frac{p_{k-1}}{p_k}$, based on (4.22). A loop in the BGR implies:

$$\frac{c_{i_1 n}}{c_{i_1 1}} \frac{c_{i_2 1}}{c_{i_2 2}} \cdots \frac{c_{i_{n-1} n-2}}{c_{i_{n-1} n-1}} \frac{c_{i_n n-1}}{c_{i_n n}} = \frac{p_n}{p_1} \frac{p_1}{p_2} \cdots \frac{p_{n-2}}{p_{n-1}} \frac{p_{n-1}}{p_n} = 1.$$

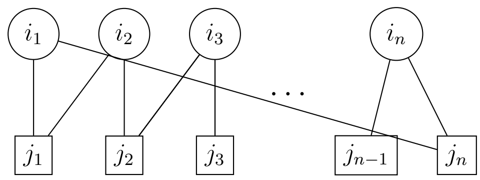

Figure 4.9: A bipartite graph representation loop

Since $\frac{c_{i_1 n}}{c_{i_1 1}} \frac{c_{i_2 1}}{c_{i_2 2}} \cdots \frac{c_{i_{n-1} n-2}}{c_{i_{n-1} n-1}} \frac{c_{i_n n-1}}{c_{i_n n}}$ is a function of independent continuous random variables, it is also a continuous random variable itself. The probability that the product of independent continuous random variables equals a constant is zero, so we can conclude that a BGR has loops with probability zero. In other words, a BGR has no loop with probability one.

BGR algorithm

Let E be the set of edges, and $\hat{\mathcal{I}}$ and $\hat{\mathcal{J}}$ be the set of all user and provider nodes, respectively, present in the BGR. The demand of undecided users can be found using Algorithm 1.

Table 4.1: BGR decoding

1: For each undecided node $i \in \hat{\mathcal{I}}$, calculate the checksum $P_i \leftarrow x_i^*$
2: For each provider $j \in \hat{\mathcal{J}}$ calculate the checksum $S_j \leftarrow Q_j - \sum_{i:(i,j)\notin\mathcal{G}} q_{ij}^*$, $\forall j \in \hat{\mathcal{J}}$
3: For each $q_{ij}^* > 0$, add edge (i,j) to the edge set E
4: **while** E $\neq \emptyset$ **do**
5:     **find** a leaf node l and associated edge (i, j)
6:     **if** the leaf node is a user node **then**
7:         $q_{ij}^* \leftarrow \frac{P_i}{c_{ij}}$
8:     **else**
9:         $q_{ij}^* \leftarrow S_i$
10:     **end if**
11:     $P_i \leftarrow \left(P_i - q_{ij}^* c_{ij}\right)$ and $S_j \leftarrow \left(S_j - q_{ij}^*\right)$
12:     remove edge (i, j)
13: **end while**

We now give an informal description of an algorithm that finds the optimal and unique values of $\boldsymbol{q}_i^*$ for undecided users. Since BGR has no loops, it is a (unrooted) tree. Hence, we can run a simple iterative algorithm which removes a leaf node (node with a single incoming edge) and its associated edge at each iteration. We begin by finding a leaf node in the BGR. We then determine the demand of the edge associated to the leaf node either from BGR Property 1) or 2). Using this value we update the check-sum value of its parent node. Then we remove the leaf node and the associated edge. This completes one iteration. We repeat the process until there are no more edges in the graph.

The key for Algorithm 4.1 to work is that the BGR has no loops, so a leaf node can always be found in line 5. Notice that in the last iteration, there will be only one user node i and one provider node j left connected by an edge with value q^*_{ij}. The checksums for these two nodes are P_i and S_j, which satisfy $P_i = S_j c_{ij}$ since $P_i = q^*_{ij} c_{ij}$ and $S_j = q^*_{ij}$. Upon completion of the algorithm, the demand of undecided users is uniquely defined.

Proof of Theorem 4.3

Assume there exist two optimal demand vectors of SWO $\boldsymbol{q}^*$ and $\boldsymbol{q}'$. By Lemma 4.2, $\boldsymbol{q}^*$ and $\boldsymbol{q}'$ have different support sets $\{\hat{\mathcal{J}}^*_i\}_{i=1}^I$ and $\{\hat{\mathcal{J}}'_i\}_{i=1}^I$ almost surely. Next, consider a convex combination demand vector $\boldsymbol{q}^\lambda = \lambda \boldsymbol{q}^* + \bar{\lambda} \boldsymbol{q}'$ where $\lambda \in (0,1)$, $\bar{\lambda} = 1-\lambda$. Since $x^*_i = \sum_{j=1}^J q^*_{ij} c_{ij} = \sum_{j=1}^J q'_{ij} c_{ij}$, then $\sum_{j=1}^J q^\lambda_{ij} c_{ij} = \lambda \sum_{j=1}^J q^*_{ij} c_{ij} + \bar{\lambda} \sum_{j=1}^J q'_{ij} c_{ij} = x^*_i$, so it follows that $\boldsymbol{q}^\lambda$ is also a maximizing solution of SWO for any $\lambda \in (0,1)$. Then, the support set $\hat{\mathcal{J}}^\lambda_i(\boldsymbol{q}^\lambda) = \{j \in \mathcal{J} : q^\lambda_{ij} = \lambda q^*_{ij} + \bar{\lambda} q'_{ij} > 0\}$ for user i is $\hat{\mathcal{J}}^\lambda_i = \hat{\mathcal{J}}^*_i \cup \hat{\mathcal{J}}'_i$, for all $\lambda \in (0,1)$. In particular, the support sets $\{\hat{\mathcal{J}}^\lambda_i\}_{i=1}^I$ are the same for all $\lambda \in (0,1)$, which is a contradiction to Lemma 4.2.

4.B Provider competition game proofs

Proof of Lemma 4.4

It can be verified that UPM satisfies Slater's conditions [64]. The necessary and sufficient KKT conditions for an optimal solution $\boldsymbol{q}_i \geq \boldsymbol{0}$ of UPM of user i are as follows:

$$u'_i(x_i) c_{ij} \leq p_j, \ j \in \mathcal{J} \tag{4.23}$$

$$q_{ij} \left(u'_i(x_i) c_{ij} - p_j \right) = 0, \ j \in \mathcal{J} \tag{4.24}$$

$$\text{where } x_i = \sum_{j=1}^J q_{ij} c_{ij}, \ \boldsymbol{q}_i \geq \boldsymbol{0} \tag{4.25}$$

The expression (4.23) implies $u'_i(x_i) \leq \alpha$, where $\alpha = \min_{k \in \mathcal{J}} \frac{p_k}{c_{ik}}$. Based on the utility function of user i, there are two cases: $u'_i(0) < \alpha$ and $u'_i(0) \geq \alpha$.

In the first case $u'_i(0) c_{ij} - p_j < 0$, so $u'_i(x_i) c_{ij} - p_j < 0$ for all $x_i \geq 0$, for all $j \in \mathcal{J}$ since, by Assumption 4.1, the marginal utility $u'_i(\cdot)$ is a strictly decreasing function. Thus, by (4.24), $q_{ij} = 0$ for all $j \in \mathcal{J}$. Therefore, $\boldsymbol{q_i} = \boldsymbol{0}$ and by (4.25) $x^*_i = 0$. So, (4.23)-(4.25) hold for a unique value $x^*_i = 0$.

In the second case, $u'_i(0) \geq \alpha$. Then, because $u'_i(\cdot)$ decreases to zero (Assumption 4.1), there is a unique $\hat{x}_i \geq 0$ such that $u'_i(\hat{x}_i) = \alpha$. We first check that there is a $\boldsymbol{q}_i$ such that equations (4.23)-(4.25) hold with $x_i = \hat{x}_i$. Equation (4.23) holds because $u'_i(\hat{x}_i) = \alpha \leq p_j / c_{ij}$ for all $j \in \mathcal{J}$. Next, by (4.24), for any j such that $p_j / c_{ij} > \alpha = u'_i(\hat{x}_i)$ we have $q_{ij} = 0$. For any other j, $p_j / c_{ij} = \alpha = u'_i(\hat{x}_i)$, thus, with respect to (4.24), q_{ij} can take any non-negative value. In particular, for the set $\{j \in \mathcal{J} : p_j / c_{ij} = \alpha\}$ we can choose q_{ij}'s so that (4.25) holds.

Note that q_{ij} is positive only when $p_j / c_{ij} = \alpha$, which proves the last part of the lemma. It remains to show that $\hat{x}_i$ is the only value of x_i for which a $\boldsymbol{q}_i$ satisfying (4.23)-(4.25) exists. For any $x_i < \hat{x}_i$, $u'_i(x_i) > \alpha$ so (4.23) is violated for $j \in \operatorname{argmin} p_k / c_{ik}$. For any $x_i > \hat{x}_i$, $u_i(x_i) < \alpha$ so $u_i(x_i) c_{ij} - p_j < 0$ for all $j \in \mathcal{J}$. Then, (4.24) implies that $q_{ij} = 0$ for all $j \in \mathcal{J}$, meaning that $x_i = 0$, which contradicts $x_i > \hat{x}_i > 0$. Therefore $\hat{x}_i$ is the unique searched value x^*_i.

Proof of Theorem 4.5

Assume that the providers charge prices $\boldsymbol{p} = [p_1 \ldots p_J]$ to the users. Then, each user faces a local maximization problem $\mathrm{UPM}_i(\boldsymbol{p})$, as defined in (4.6).

Equations (4.23) - (4.25), together with $\boldsymbol{q}_i \geq \mathbf{0}$ for all $i \in \mathcal{I}$, are equivalent to equations (4.17)-(4.19) and (4.21). Furthermore, under assumption 4.4, for $\boldsymbol{q}$ to be an SPE of the provider competition game, the demand to each provider must equal its supply, i.e., $\sum_{i=1}^{I} q_{ij} = Q_j$ for all $j \in \mathcal{J}$ (i.e., equations (4.20)). Hence, the SPE is a price vector tuple $\boldsymbol{p}, \boldsymbol{q}$ that satisfies KKT conditions (4.17)-(4.21). But, the KKT conditions (4.17)-(4.21) are necessary and sufficient for any vector tuple $\boldsymbol{p}, \boldsymbol{q}$ to be the maximizing solution of SWO. Hence, we have established formal equivalence between the SPE of the provider competition game and the maximizing demand vector and Lagrangian multipliers of the SWO problem $(\boldsymbol{q}^*, \boldsymbol{p}^*)$. Hence, $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ form the unique SPE of the provider competition game.

Proof of Lemma 4.6

Given an arbitrary price vector $\boldsymbol{p}$, we can construct the corresponding $\mathrm{BGR}(\boldsymbol{p})$ from the users' preference sets. First consider a BGR that is connected (single component). We start drawing the graph with a single undecided user node, and then add the provider nodes that are connected to this user node. There should be at least two such provider nodes. Then we add another undecided user node which shares one common provider node with the existing undecided user node. This new undecided user node will bring at least one new provider node into the graph, otherwise it leads to a loop in the graph. We repeat this process iteratively. Since the number of provider nodes is upper-bounded by J, the total number of undecided user nodes is upper-bounded by $J-1$. The "-1" is due to the fact that the first undecided user node is connected to (at least) two new provider nodes in the graph. If we consider a BGR with b disconnected subgraphs, it can be shown that the total number of undecided users is bounded by $J-b$.

4.C Primal-dual algorithm supporting proofs

Proof of Lemma 4.7

For the optimal demand vector $\boldsymbol{q}^*$ of the SWO problem and the associated Lagrange multipliers $\boldsymbol{p}^*$, we see that $f_{ij}^* = p_j^*$ whenever $q_{ij}^* > 0$ from equation (4.18), and $f_{ij}^* \leq p_j^*$ when $q_{ij}^* = 0$ from equations (4.17) and (4.18). Similarly, equation (4.18) ensures that

$f_{ij}^* < p_j^*$ implies $q_{ij}^* = 0$. This fact will be used shortly. For our La Salle function:

$$
\begin{aligned}
\dot{V} &\overset{(a)}{=} \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij} - p_j)^+_{q_{ij}} + \sum_j (p_j - p_j^*)(\sum_i q_{ij} - Q_j)^+_{p_j} \\
&\overset{(b)}{\leq} \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij} - p_j) + \sum_j (p_j - p_j^*)(\sum_i q_{ij} - Q_j) \\
&\overset{(c)}{=} \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij} - p_j) + \sum_j (p_j - p_j^*)(\sum_i q_{ij} - \sum_i q_{ij}^*) + \sum_j (p_j - p_j^*)(\sum_i q_{ij}^* - Q_j) \\
&\overset{(c)}{=} \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij} - p_j) + \sum_j (p_j - p_j^*)(\sum_i q_{ij} - \sum_i q_{ij}^*) \\
&\overset{(d)}{=} \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij} - p_j) + \sum_{i,j} (p_j - p_j^*)(q_{ij} - q_{ij}^*) \\
&\overset{(e)}{=} \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij} - p_j^*) \\
&\overset{(f)}{=} \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij} - f_{ij}^*) + \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij}^* - p_j^*) \\
&\overset{(c)}{=} \sum_i (\sum_j (q_{ij} - q_{ij}^*)(f_{ij} - f_{ij}^*)) + \sum_{i,j}(q_{ij} - q_{ij}^*)(f_{ij}^* - p_j^*),
\end{aligned}
$$

where (a) follows from the definition of V and $\dot{V}$, (b) can be readily verified by examining all the cases, (c) follows since $Q_j = \sum_i q_{ij}^*$ for the maximizer of SWO, (d), (e), (f) and (g) are obtained by rearranging the terms. The expression for $\dot{V}$ is now in such a form that we can prove $\dot{V} \leq 0$.

First, the following is true for any two vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ due to concavity (see, e.g. Section 3.1.3 of [64]):

$$
\nabla u_i^T(\boldsymbol{q}_1)(\boldsymbol{q}_2 - \boldsymbol{q}_1) \geq u_i(\boldsymbol{q}_2) - u_i(\boldsymbol{q}_1), \tag{4.26}
$$

$$
\nabla u_i^T(\boldsymbol{q}_2)(\boldsymbol{q}_1 - \boldsymbol{q}_2) \geq u_i(\boldsymbol{q}_1) - u_i(\boldsymbol{q}_2). \tag{4.27}
$$

Substituting $\boldsymbol{q}_1 = \boldsymbol{q}_i(t)$ and $\boldsymbol{q}_2 = \boldsymbol{q}_i^*$ in (4.26) and (4.27) gives:

$$
\nabla u_i^T(\boldsymbol{q}_i)(\boldsymbol{q}_i^* - \boldsymbol{q}_i) \geq \nabla u_i^*(\boldsymbol{q}_i)(\boldsymbol{q}_i^* - \boldsymbol{q}_i),
$$

which can be rewritten as

$$
\sum_j (q_{ij}(t) - q_{ij}^*)(f_{ij}(t) - f_{ij}^*) \leq 0, \text{ for all } i \in \mathcal{I}, \tag{4.28}
$$

which we recognize as one of the components of the first term in (4.10). Considering a component of the second term in (4.10), $(q_{ij}(t) - q_{ij}^*)(f_{ij}^* - p_j^*)$, we first recall from the KKT conditions (4.18) that either $f_{ij}^* = p_j^*$, in which case $(q_{ij} - q_{ij}^*)(f_{ij}^* - p_j^*) = 0$, or $f_{ij}^* < p_j^*$, in which case $q_{ij}^* = 0$ so $(q_{ij} - q_{ij}^*)(f_{ij}^* - p_j^*) \leq 0$. Hence,

$$
(q_{ij}(t) - q_{ij}^*)(f_{ij}^* - p_j^*) \leq 0, \text{ for all } i \in \mathcal{I},\ j \in \mathcal{J}. \tag{4.29}
$$

which completes the proof that $\dot{V} \leq 0$.

Proof of Lemma 4.10

Recall that for user i, $f_{ij}(\boldsymbol{q}_i) = \frac{\partial u_i(x_i)}{\partial x_i} c_{ij} = u_i'(x_i) c_{ij}$ for all $j \in \mathcal{J}$, where $x_i(t) = \sum_j q_{ij}(t) c_{ij}$ and $u_i'(x_i(t))$ is a **scalar** function of time. Hence,

$$\nabla u_i^T(\boldsymbol{q}_i) = u_i'(x_i)[c_{i1} \cdots c_{iJ}] = u_i'(x_i)\boldsymbol{c}_i^T.$$

We now consider equations (4.11), which can be rewritten as:

$$\begin{aligned} \nabla u_i^T(\boldsymbol{q}_i)(\boldsymbol{q}_i - \boldsymbol{q}_i^*) &= \nabla u_i^T(\boldsymbol{q}_i^*)(\boldsymbol{q}_i - \boldsymbol{q}_i^*), \text{ or,} \\ u_i'(x_i)\boldsymbol{c}_i^T(\boldsymbol{q}_i - \boldsymbol{q}_i^*) &= u_i'(x_i^*)\boldsymbol{c}_i^T(\boldsymbol{q}_i - \boldsymbol{q}_i^*), \end{aligned}$$

which leads to $u_i'(x_i)\Delta x_i(t) = u_i'(x_i^*)\Delta x_i(t)$, where $\Delta x_i(t) = \boldsymbol{c}_i^T(\boldsymbol{q}_i(t) - \boldsymbol{q}_i^*)$. Then, either $\Delta x_i(t) = 0$, or $u_i'(x_i(t)) = u_i'(x_i^*)$. The necessary and sufficient condition for both is that that $x_i(t) = x_i^*$, for $\boldsymbol{q}_i(t)$, $i \in \mathcal{I}$ on the invariant set V_L. An immediate corollary is that $f_{ij}(t) = f_{ij}^*$ on V_L for all $i \in \mathcal{I}, j \in \mathcal{J}$.

The second part of the Lemma is simpler to prove. Equation (4.12) gives $q_{ij}(t)(f_{ij}^* - p_j^*) = q_{ij}^*(f_{ij}^* - p_j^*)$, which is equal to zero by equation (4.18). This concludes the proof.

Proof of Lemma 4.11

We prove that the set of positive demands does not change over time on the invariant set by contradiction. Suppose that $q_{ij}(t) > 0$ but $q_{ij}(t+\tau) = 0$ for all τ such that $0 < \tau < \epsilon$, where ϵ is a small number. Then $\dot{q}_{ij}(t) < 0$ but $\dot{q}_{ij}(t+\tau) = 0$, so we have

$$\lim_{\tau \to 0} \sum_{j \in \mathcal{J}_i} q_{ij}(t)c_{ij} - q_{ij}(t+\tau)c_{ij} < 0$$

On the other hand differentiating equation (4.13) with respect to time yields $\sum_{j \in \mathcal{J}_i} q_{ij}(t)c_{ij} = 0$ and $\sum_{j \in \mathcal{J}_i} q_{ij}(t+\tau)c_{ij} = 0$, which is a contradiction. So a non-zero q_{ij} stays non-zero. Suppose now that $q_{ij}(t) = 0$ but $q_{ij}(t+\tau_{ij}) > 0$ for some $\tau_{ij} > 0$. After time τ_{ij} variable q_{ij} becomes non-zero and stays that way forever (according to the argument we made earlier in the proof). Then no q_{ij} escapes from the boundary after time $t + \tau^*$, where $\tau^* = \max_{i,j} \tau_{ij}$.

Now we prove the second part of the Lemma. Similar to the previous argument, once the $p_j(t) > 0$ on the invariant set, $p_j(t+\tau) > 0$ for all $\tau > 0$. It remains to show that $p_j(t) > 0$ on the invariant set. But, if $p_j(t) = 0$ were true, then $\dot{q}_{ij} > 0$ would imply $\lim_{t\to\infty} q_{ij}(t) = \infty$, which would violate (4.13).

Proof of Lemma 4.15

Since $C\boldsymbol{q}(t) = C\boldsymbol{q}^*$ is a constant on V_L, then taking the time derivative of both sides gives $C\dot{\boldsymbol{q}}(t) = 0$. Then:

$$\begin{aligned} &0 = C\dot{\boldsymbol{q}}(t) = C\hat{K}^q\left(\boldsymbol{f}^* - A\boldsymbol{p}(t)\right) \\ &\Leftrightarrow C\hat{K}^q A\boldsymbol{p}(t) = C\hat{K}^q\boldsymbol{f}^*, \end{aligned}$$

which means that $C\hat{K}^q A\boldsymbol{p}(t)$ equals a constant on the invariant set. Taking the time derivative one more time yields

$$\begin{aligned} 0 = C\hat{K}^q A\dot{\boldsymbol{p}(t)} =& C\hat{K}^q AK^p\left(A^T\boldsymbol{q}(t) - Q\right) \\ \Leftrightarrow\ C\hat{K}^q AK^p A^T\boldsymbol{q}(t) =& C\hat{K}^q AK^p Q = \text{constant} \end{aligned}$$

Repeating the derivative operation $2n+1$ times will yield

$$\begin{aligned} C\hat{K}^q A(K^pA^T\hat{K}^qA)^n\boldsymbol{p} =& C\hat{K}^q A(K^pA^T\hat{K}^qA)^{n-1}K^pA^T\hat{K}^q\boldsymbol{f}^*, \\ = C\hat{K}^q AD^n\boldsymbol{p} =& C\hat{K}^q AD^{n-1}K^pA^T\hat{K}^q\boldsymbol{f}^* = \text{const} \end{aligned}$$

where we defined $D = K^pA^T\hat{K}^qA$. Note that D^n here stands for "D to the n^{th} power". Let $B = C\hat{K}^qA$, then we can write:

$$\begin{bmatrix} B & BD & \cdots & BD^{J-1} \end{bmatrix}^T \boldsymbol{p}(t) = \mathcal{B}\boldsymbol{p}(t) = \text{constant},$$

which completes the proof.

Proof of Lemma 4.17

Consider matrix $D = K^pA^T\hat{K}^qA$. It can be verified that

$$\begin{aligned} D =& \begin{bmatrix} k_1^p(\sum_{i\in\mathcal{I}_1} k_{i1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_J^p(\sum_{i\in\mathcal{I}_J} k_{iJ}) \end{bmatrix} \\ \overset{(a)}{=}& k\begin{bmatrix} k_1^p|\mathcal{I}_1| & 0 & \cdots & 0 \\ 0 & k_2^p|\mathcal{I}_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_J^p|\mathcal{I}_J| \end{bmatrix}, \end{aligned}$$

where $\mathcal{I}_j = \{i \in \mathcal{I} : q_{ij}(t) > 0\}$ is the set of users of provider j, where (a) follows from $k_{ij}^q = k$. Without loss of generality, take $k = 1$. Now, assuming that k_j^p's are not integer multiples of each other, we can see that at most one row of $D - \lambda\mathbb{I}$ can be all-zero vector for any eigenvalue λ of D (indeed, exactly one since D is a diagonal matrix). So, we can always choose $J-1$ rows of $D - \lambda\mathbb{I}$ that are non-zero and also linearly independent.

$$G' = \begin{bmatrix} k_1^p|\mathcal{I}_1| - \lambda_J & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & k_{J-1}^p|\mathcal{I}_{J-1}| - \lambda_J & 0 \\ k_{i1}^q c_{i1}\mathbb{1}_{\mathbb{R}^+}(q_{i1}) & \cdots & k_{iJ-1}^q c_{iJ-1}\mathbb{1}_{\mathbb{R}^+}(q_{iJ-1}) & k_{iJ}^q c_{iJ} \end{bmatrix}$$

Without loss of generality, assume that we choose rows 1 through $J-1$ (i.e., we consider the J^{th} eigenvalue $\lambda_J = k_J^p|\mathcal{I}_J|$). Now, consider matrix $B' = CK^qA$. It can be verified that $B'_{i,j} = k_{ij}^q c_{i,j}$. The entries of matrix $B = C\hat{K}^qA$ are then $k_{ij}^q c_{ij}\mathbb{1}_{(0,\infty)}(q_{ij}(t))$. Since $q_{iJ}(t) > 0$ for at least one $i = \hat{\imath}$, $k_{\hat{\imath}J}^q c_{\hat{\imath}J}\mathbb{1}_{(0,\infty)}(q_{\hat{\imath}J}(t)) = k_{\hat{\imath}J}^q c_{\hat{\imath}J}$, so the $\hat{\imath}$ row has an entry at the J^{th} column. We then append this row to the $J-1$ rows we took from

$D - \lambda_J \mathbb{I}$, forming a submatrix of G, denoted G'. The submatrix G' is then a lower triangular matrix. Lower triangular matrices have full rank, in this case J. Since the submatrix of G has the full column rank, then G also has the full column rank J. The procedure works in the same way for any eigenvalue λ_j of D we may choose, since we can always relabel the providers. This completes the proof.

CHAPTER SUMMARY

- **In this chapter we study the price competition of service providers for wireless users.**
- ***General Heterogeneous Wireless Network Model*: We consider a general model that captures the heterogeneity of wireless users and service providers. The users have different utility functions, the providers have different resource constraints, the channel gains between users and providers are independent and arbitrarily distributed, and the numbers of users and providers can be arbitrary.**
- ***Unique Socially Optimal Allocation*: We show that when channel parameters are randomly drawn from continuous distributions, there exists a *unique* optimal solution to the problem with probability 1, despite the non-strict convexity of the optimization problem.**
- ***Existence, Uniqueness, and Zero Efficiency Loss of Equilibrium*: We prove existence and uniqueness of the subgame perfect Nash equilibrium in the two-stage game, under an easily verifiable sufficient condition on the users' utility functions. Moreover, we show that the unique equilibrium maximizes the social welfare, despite the selfish nature of the providers and users.**
- ***Primal-Dual Algorithm converging to Equilibrium*: We provide a decentralized algorithm that results in an equilibrium of the provider competition game.**

5 Pricing for Wireless Resources: Results and Open Questions

In this chapter we discuss some results related to the subject of wireless service providers. We consider profitability of two different pricing mechanisms, the subject of admission control, and we revisit provider competition.

In the previous chapter we considered price competition of wireless service providers for users who are free to choose their provider. One of the novelties of that work is that providers are charging for the resource used, while the users are ultimately interested in the service acquired, which is resource multiplied by a user-provider specific constant. The advantage of such pricing turns out to be the social optimality of the market equilibrium allocation. In this chapter we revisit the pricing-for-resource assumption by considering pricing for provided service[1]. We are motivated by two main reasons.

The first one is that pricing for resources used is not a common pricing scheme in the wireless communications business, where flat monthly fees are preferred for their simplicity. Lately, the proliferation of smartphones started to strain the wireless networks and forced companies to revisit the idea of usage-based pricing [43], but for services provided. For example, recently AT&T introduced hybrid price plans, consisting of a flat rate fee for a certain amount of data, and linear pricing beyond that limit [65]. Usage-based pricing is likely to become commonplace, where the providers will be charging per unit of data provided (i.e. service acquired by the users), for example a dollar per Gb, but not per unit of resource that the company had to use to provide this data to the user. Hence, even though charging for resource is socially optimal, it may not be technologically and socially feasible to implement, at least not for the time being. For now, providers are leaning towards pricing of services provided.

The second reason to question the pricing-for-resource assumption is that it is not

[1] We will use the term "service acquired" when taking a user's point of view, and the term "provided service" when taking the point of view of a provider.

clear that it is in the best interest of wireless providers to charge for resource directly. In particular, a provider may be making more profit by charging for the service acquired, than by charging for the resource. If this is the case, the additional profit would be at the detriment of the user experience, since directly charging for resources is socially optimal. However, it is the providers who decide on the pricing method, so it is an interesting question to know which pricing scheme they have an incentive to use.

In Section 5.1, we will compare two different pricing methods for a monopolistic wireless service provider: charging for resources used and charging for provided service. We assume wireless users who have logarithmic utility functions. It turns out that in some special cases it is better to charge for used resources, and in others for provided service. For most other cases, however, such a comparison can only be performed by explicitly calculating the profits for the two different pricings.

In Section 5.2, we remark that charging for provided service has an additional degree of complexity for a monopolistic provider, in the form of admission control. Namely, a provider may strictly improve his profit by refusing to serve users with particularly bad channels. We demonstrate this explicitly, and provide an optimal admission control algorithm for a special case of the users' willingness to pay. In Section 5.3 we briefly discuss some of the issues for the competition of wireless providers charging for provided service, and point to other promising research directions in this field.

5.1 Is it better to charge for resources used or for provided service?

In this section we consider the profit of a monopolistic provider evaluating two different pricing methods. The quantities related to charging for the provided service will be indicated by a hat ($\hat{}$) operator.

Under pricing for resource, the utility of the user is a function of the service that it acquires, and the payment is a function of the resource that it uses:

$$v_i = u_i(q_i c_i) - p q_i.$$

When the provider is charging for provided service, the payoff function of a user is:

$$\begin{aligned}\hat{v}_i =& u_i(\hat{q}_i c_i) - \hat{p} c_i \hat{q}_i \\ =& u_i(\hat{x}_i) - p\hat{x}_i,\end{aligned}$$

where we have defined the provided service as $\hat{x}_i = c_i \hat{q}_i$. Both cases are subject to the same constraint: $\sum_i q_i \leq Q$ and $\sum_i \hat{q}_i \leq Q$. If we think of c_i as rate per unit of resource, then the service acquired $\hat{x}_i = \hat{q}_i c_i$ can be thought of as achieved rate.

For the remainder of this section, we will assume that $u_i(x) = a_i \log(1 + x)$, where a_i is the subjective willingness to pay parameter of a user. This utility function leads to closed-form expression for user demand, and tractable expressions for market-clearing price and profit.

The users have the following payoff functions for the two different pricing models:

$$\begin{aligned}v_i =& a_i \log(1 + x_i) - p q_i \\ \hat{v}_i =& a_i \log(1 + \hat{x}_i) - \hat{p}\hat{x}_i,\end{aligned}$$

where $x_i = q_i c_i$.

The payoff maximizing quantities, which we also call demands, for a user in the two pricing schemes are:

$$q_i^* = \left(\frac{a_i}{p} - \frac{1}{c_i}\right)^+ \tag{5.1}$$

$$\hat{x}_i^* = \left(\frac{a_i}{\hat{p}} - 1\right)^+. \tag{5.2}$$

Note that the demand $\hat{x}_i^*$ does not depend on the channel quality of a user, but simply on his willingness to pay factor and price. This is sensible: for example a user who connects to a Wi-Fi provider does not consider his channel quality to that provider.

The conditions for the demand to equal the supply are:

$$\sum_{i=1}^{I} q_i^* = Q \tag{5.3}$$

$$\sum_{i=1}^{I} \hat{q}_i^* = \frac{\hat{x}_i^*}{c_i} = Q. \tag{5.4}$$

We focus on market clearing prices, i.e., prices that result in equality of demand and supply. Let $\mathcal{I} = \{1, \ldots, I\}$ be the set of users who are interested in purchasing wireless resource or service, depending on the pricing scheme employed. Appendix 5.A.1 provides an algorithm to calculate market clearing prices for a given set of users, here we only provide the final expression:

$$p^* = \frac{\sum_{i \in \mathcal{I}^+(p^*)} a_i}{Q + \sum_{i \in \mathcal{I}^+(p^*)} \frac{1}{c_i}} \tag{5.5}$$

$$\hat{p}^* = \frac{\sum_{i \in \hat{\mathcal{I}}^+(\hat{p}^*)} \frac{a_i}{c_i}}{Q + \sum_{i \in \hat{\mathcal{I}}^+(\hat{p}^*)} \frac{1}{c_i}}, \tag{5.6}$$

where the sets $\mathcal{I}^+(p^*)$ and $\hat{\mathcal{I}}^+(\hat{p}^*)$ contain the users with strictly positive demands at prices p^* and $\hat{p}^*$, respectively, under resource pricing and provided service pricing, respectively. For any given set of parameters $(\{a_i\}_{i=1}^I, \{c_i\}_{i=1}^I)$, these sets may be different. We restrict ourselves to the case where $\mathcal{I}^+(p^*) = \hat{\mathcal{I}}^+(\hat{p}^*) = \mathcal{I}$. This can be ensured by assuming that Q is sufficiently high.

Assumption 5.1. *We assume that*

$$Q > \max\left(\frac{\sum_{i \in \mathcal{I}} a_i}{\bar{ac}} - \sum_{i \in \mathcal{I}} \frac{1}{c_i}, \frac{\sum_{i \in \mathcal{I}} \frac{a_i}{c_i}}{\bar{a}} - \sum_{i \in \mathcal{I}} \frac{1}{c_i}\right),$$

where $\bar{ac} = \min_{i \in \mathcal{I}} a_i c_i$ *and* $\bar{a} = \min_{i \in \mathcal{I}} a_i$.

Under Assumption 5.1, $p^* \leq \bar{ac}$ and $\hat{p}^* \leq \bar{a}$, meaning that $q_i^* > 0$ and $\hat{x}_i^* > 0$, for all $i \in \mathcal{I}$, i.e. all users have positive demand in both pricing schemes under market-clearing prices.

In general the amount of resource of a provider will be fixed and not something that can be easily changed, so the condition above may not always be fulfilled. However, it is needed for the following discussion to apply.

The total profit under market clearing prices and payoff maximizing demands is:

$$\Pi = p^* \sum_k q_k^* = \sum a_k - \frac{\sum_k \frac{1}{c_k} \sum_i a_i}{Q + \sum_i \frac{1}{c_i}}$$
$$\hat{\Pi} = p^* \sum_k \hat{x}_k^* = \sum a_k - \frac{\sum_k \sum_i \frac{a_i}{c_i}}{Q + \sum_i \frac{1}{c_i}}.$$

The provider will maximize the profit by choosing the option that minimizes the numerator of the second term. Unfortunately, there is no simple inequality relation between the two numerators. The following lemma captures the special conditions under which the provider knows which pricing scheme is more profitable.

Proposition 5.1. *Without loss of generality, order users so that $c_i > c_{i+1}$, for all $i \in \mathcal{I}$. Then, when $a_i > a_{i+1}$ for all $i \in \mathcal{I}$, it is more profitable for the provider to charge for the provided service then for used resource, i.e. $\hat{\Pi} > \Pi$. On the other hand, when $a_i < a_{i+1}$ for all $i \in \mathcal{I}$, $\Pi > \hat{\Pi}$.*

The proof is a direct consequence of the following lemma.

Lemma 5.2. *Let I be a positive integer, and let $\boldsymbol{x} = [x_1, \ldots, x_I]$ and $\boldsymbol{y} = [y_1, \ldots, y_I]$ be two real vectors with strictly positive components. Without loss of generality, assume that $x_1 > \ldots > x_I$. Then*

$$\sum_i x_i \sum_k y_k - I \sum_i x_i y_i \begin{cases} < 0 & \textit{when } y_1 > \ldots > y_I \\ > 0 & \textit{when } y_1 < \ldots < y_I. \end{cases}$$

Proof. The proof is by induction. For $I = 2$:

$$\begin{aligned}(x_1 + x_2)(y_1 + y_2) - 2(x_1 y_1 + x_2 y_2) =& x_1 y_2 + x_2 y_1 - x_1 y_1 - x_2 y_2 \\ =& (y_2 - y_1)(x_1 - x_2),\end{aligned}$$

which satisfies the claim. Now assume that the claim is true for I when $y_1 > \ldots > y_I > y_{I+1}$:

$$\sum_{i=1}^{I} x_i \sum_{k=1}^{I} y_k - I \sum_{i=1}^{I} x_i y_i < 0.$$

We need to show that the claim is true for $I+1$. Consider

$$\begin{aligned}
&\sum_{i=1}^{I+1} x_i \sum_{k=1}^{I+1} y_k - (I+1) \sum_{i=1}^{I+1} x_i y_i \\
=&\left(\sum_{i=1}^{I} x_i + x_{I+1}\right)\left(\sum_{i=1}^{I} y_i + y_{I+1}\right) - I\sum_{i=1}^{I} x_i y_i - \sum_{i=1}^{I} x_i y_i - (I+1) x_{I+1} y_{I+1} \\
\leq&\sum_{i=1}^{I} x_{I+1} y_i + \sum_{i=1}^{I} x_i y_{I+1} - \sum_{i=1}^{I} x_i y_i - I x_{I+1} y_{I+1} \\
=&\sum_{i=1}^{I} x_{I+1}(y_i - y_{I+1}) + \sum_{i=1}^{I} x_i (y_{I+1} - y_i) \\
=&\sum_{i=1}^{I} (x_{I+1} - x_i)(y_i - y_{I+1}) = \sum_{i=1}^{I} (x_i - x_{I+1})(y_{I+1} - y_i) \\
<&0.
\end{aligned}$$

which completes the proof (all the steps are identical for the case $y_1 < \ldots < y_I < y_{I+1}$ with the inequality being in the other direction). □

The final step of the proof also shows that there is little else we can say about the merits of one pricing versus the other, except in the case of monotonous ordering of parameters.

We interpret the Proposition 5.1 by analyzing the following situation. Consider a group of customers with high values of a_i who reside close to a provider's antenna, and a group of customers with low values of a_i live further away from the antenna, with the proximity of a residence indicating the quality of the channel parameter c_i. High value of a_i implies high demand. For such a situation, Proposition 5.1 tells us that the profit-maximazing strategy is to charge for the provided service. Since here users with high demand also have good channel quality, and users with low demand have bad channel quality, charging for provided service means that all users take up approximately the same amount of resource. The demand-hungry consumers actually pay more money for the resource they are using, so they are in a sense subsidizing the users with bad channel quality.

On the other hand, consider the reverse situation where the demand-hungry consumers have bad channel gains, and the users with low demand have good channels. Then it pays out more to charge for the resources used. This is because charging for provided service would result in demand-hungry consumers with poor channels taking a lot of the resource. This would imply little service acquired for everyone, leading to poor revenue. Users with poor channel qualities are the inefficient consumers, and the profit can be maximized only if they have low demand. This is obtained by charging for the used resource, which has the added benefit of increasing the demand of users with good channel gains.

Charging for resource directly maximizes social welfare as well. The message is that when the affluent consumers are further away, the social goal of sharing is aligned with the providers goal of profit maximization. However, when the affluent consumers are nearby, it is more profitable to focus on giving resources to those who not only want to spend more, but also have more efficient channels.

Finally, we remark that both pricing methods have hidden benefits that are difficult to measure. Charging for provided service has the benefit of being simple, as it is easier for users to purchase what they care about, without thinking about how getting what they want impacts the others, or the providers. On the other hand, charging for resources used has the benefit of being the socially optimal pricing. Hence, a provider using this pricing could advertise himself as being the most user-friendly provider. In the long run, users would notice the difference and choose the provider who gives them a better wireless experience.

5.2 Admission control when pricing provided service

In the section above we compared the pricing of provided service to the sale of resources used under the assumption that all users will have positive demand, both for the service and for the resource. However, we did not consider whether the provider is interested in selling the service to all users. For example, consider two users who have identical willingness to pay, demand the same amount of service, according to equation (5.2), and bring the same profit to the provider. However, one of them has a much worse channel parameter, meaning that a lot of the resource is needed to give that user what he wants, which further means that the provider may not be able to provide a lot of service acquired to both users. Then, it may be better to ignore the user with the bad channel, and sell only to the user with the good channel.

We will show that users whose channel is significantly worse than those of other participants are lowering the profit of a provider, if admitted. Hence, for a provider to maximize the obtained profit, some form of admission control is needed. This is in contrast to the pricing of the resource where no admission control is needed. There, all users pay the same price per unit resource, and hence internalize the quality of their channel into their purchasing decision.

Lemma 5.3. *Let $\mathcal{I}_{n+1} = \{1, \ldots, n+1\}$, and let $\hat{p}^*_{n+1} = \hat{p}^*(\mathcal{I}_{n+1})$ be the market clearing price for user set $\mathcal{I}_{n+1}$ given by Equation (5.6). Without loss of generality, assume that $c_i \geq c_{i+1}$ for all $i \in \mathcal{I}_{n+1}$. In addition, assume that $a_i > \hat{p}^*_{n+1}$ for all $i \in \mathcal{I}_{n+1}$. Then, the profit of a provider is increased by removing user $n+1$ and charging market clearing price $\hat{p}^*_n$ if and only if*

$$\sum_{i=1}^{n} \left(\frac{1}{c_{n+1}} - \frac{1}{c_i} \right) > Q. \tag{5.7}$$

Proof. By rejecting user $n+1$ the market clearing price falls to $\hat{p}_n$, since the demand of the remaining users should increase to compensate for the demand lost by the departure of user $n+1$. The profit obtained by a provider from user i at price $\hat{p}^*_n$ is $\Pi_i = \hat{x}^*_i \hat{p}^*_n = (a_i - \hat{p}^*_n)^+$. The total profit with user $n+1$ is

$$\Pi_{n+1} = \sum_{i=1}^{n+1} (a_i - \hat{p}^*_{n+1})^+ = \sum_{i=1}^{n+1} a_i - (n+1)\hat{p}^*_{n+1},$$

If the user $n+1$ is rejected, the profit becomes

$$\Pi_n = \sum_{i=1}^{n}(a_i - \hat{p}_n^*) = \sum_{i=1}^{n} a_i - n\hat{p}_n^*.$$

The difference in profit is then

$$\begin{aligned}\Delta\Pi_{n+1} \leq \Pi_{n+1} - \Pi_n =& a_{n+1} - \hat{p}_{n+1}^* - n(\hat{p}_{n+1}^* - \hat{p}_n^*) \\ =& a_{n+1} - \hat{p}_n^* - (n+1)(\hat{p}_{n+1}^* - \hat{p}_n^*).\end{aligned}$$

Using equation (5.6) we find an exact expression for market clearing prices $\hat{p}_n^*$ and $\hat{p}_{n+1}^*$.

$$\begin{aligned}\hat{p}_n^* =& \frac{\sum_{i=1}^{n} \frac{a_i}{c_i}}{Q + \sum_{i=1}^{n} \frac{1}{c_i}} \\ \hat{p}_{n+1}^* =& \frac{\sum_{i=1}^{n+1} \frac{a_i}{c_i}}{Q + \sum_{i=1}^{n+1} \frac{1}{c_i}}\end{aligned} \tag{5.8}$$

The difference between the market clearing prices can now be found as:

$$\begin{aligned}\hat{p}_{n+1}^* - \hat{p}_n^* =& \frac{\sum_{i=1}^{n+1} \frac{a_i}{c_i}}{Q + \sum_{i=1}^{n+1} \frac{1}{c_i}} - \frac{\sum_{i=1}^{n} \frac{a_i}{c_i}}{Q + \sum_{i=1}^{n} \frac{1}{c_i}} \\ =& \frac{a_{n+1} - \hat{p}_n^*}{c_{n+1}(Q + \sum_{i=1}^{n} \frac{1}{c_i} + \frac{1}{c_{n+1}})},\end{aligned}$$

after minor algebraic manipulations.

Combining with the expression for the difference of the profits (5.8) yields:

$$\begin{aligned}\Delta\Pi_{n+1} \leq& (a_{n+1} - \hat{p}_n^*)\left(1 - \frac{n+1}{c_{n+1}(Q + \sum_{i=1}^{n} \frac{1}{c_i}) + 1}\right) \\ & (a_{n+1} - \hat{p}_n^*)\frac{c_{n+1}(Q + \sum_{i=1}^{n} \frac{1}{c_i}) - n}{c_{n+1}(Q + \sum_{i=1}^{n} \frac{1}{c_i}) + 1}.\end{aligned} \tag{5.9}$$

The provider should reject user $n+1$ as a customer if the expression (5.9) above is positive. The term $(a_{n+1} - \hat{p}_n^*)$ is positive by assumption since user $n+1$ has positive demand under price $\hat{p}_{n+1}^*$, and hence under price $\hat{p}_n^* < \hat{p}_{n+1}^*$ as well. So, the profitability depends on the sign of the term $c_{n+1}(Q + \sum_{i=1}^{n} \frac{1}{c_i}) - n$, which is indeed Equation (5.7). □

Corollary 5.4. *If it is profitable for a provider to reject user k, it is profitable to reject users $k+1, k+2, \ldots, n+1$, where $c_i > c_{i+1}$, for all $i \in \mathcal{I}_{n+1}$.*

The proof is by successive application of Lemma 5.3.

Lemma 5.3 is a relatively simple criterion to verify whether a provider can improve profit by not serving a user. The profitability depends on the channel of other users as well as their number. Surprisingly, profitability of accepting or rejecting a single user does not depend on his willingness to pay, although the actual profit does. Indeed, it may seem that the provider could calculate the market clearing price, reject users who fulfill Equation (5.7), and the profit will be maximized. Unfortunately, the result of Lemma 5.3 holds only when $a_i > \hat{p}^*_{n+1}$ for all $i \in \mathcal{I}_{n+1}$.

We will demonstrate that Lemma 5.3 does not hold in general with the following example, in which we dispose of the assumption of Lemma 5.3 that $a_i > \hat{p}^*_{n+1}$ for all users.

Example 5.1. *Consider a situation where a provider is charging for provided service to* 9 *customers who are ordered in decreasing order of channel quality* c_i*. Refer to Figures 5.1-5.4 throughout the exposition. When the provider is charging price* p*, the profit brought to him by user* i *is equal to* $(a_i - p)^+$*. For example, Figure 5.1 shows the profit as the sum of the areas above the market clearing price* $p_1(\{1,2,3,9\}) = \hat{p}^*(\mathcal{I}_9)$*. The top bar in Figure 5.2 illustrates how the resource is distributed in order to provide users with the service they required. User* 9*, due to his poor channel, takes up almost half of the resource under price* p_1*.*

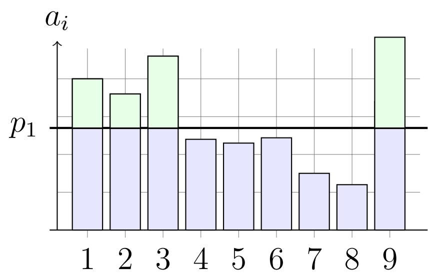

Figure 5.1: $p_1 = \hat{p}^*(\{1,2,3,9\})$

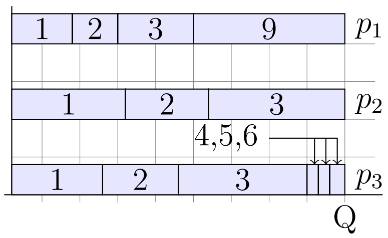

Figure 5.2: Distribution of resource across users

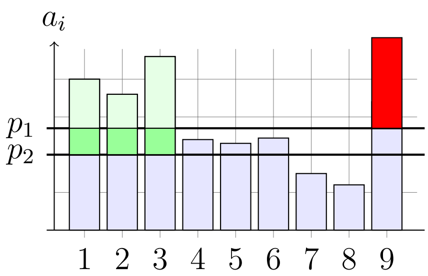

Figure 5.3: $p_2 = \hat{p}^*(\{1,2,3\})$

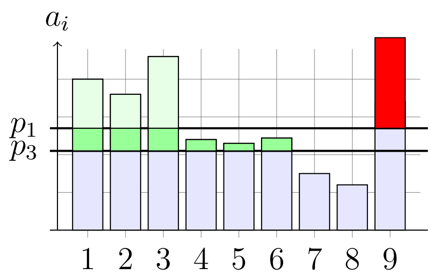

Figure 5.4: $p_3 = \hat{p}^*(\{1,2,3,4,5,6\})$

Then, the provider may check if $\sum_{i=1}^{3}(\frac{1}{c_9} - \frac{1}{c_i}) > Q$ *and see whether it is feasible to reject user* 9*. This is equivalent to calculating market clearing price* $p_2 = \hat{p}^*(\{1,2,3\})$*, and comparing whether the losses incurred by losing user* 9 *are outweighed by the gains brought by the increase in demand of users* 1, 2 *and* 3*. This is illustrated by darker areas in Figure 5.3. The change in the resource required to support the extra demand is shown by the middle bar in Figure 5.2. In this example, it is* ***not*** *profitable to reject user* 9*.*

However, Equation (5.7) does not consider users 4, 5 *and* 6 *for which* $p_2 < a_i < p_1$*. These are users for which participation is too expensive when user* 9 *is present, but becomes affordable if user* 9 *is rejected. Furthermore, these users have a better channel than user* 9*, so providing them with service does not cost the provider too much in terms* *of resource. Then, it is profitable to include those users, as illustrated by the dark-shaded profit-gain areas in Figure 5.4. The market clearing price for users* 1 *through* 6*, which we label* $p_3 = \hat{p}^*(\{1, \dots, 6\})$*, is necessarily greater than* p_2*, although the exact value depends on the values of* a_i *and* c_i*.*

Several comments on the above example are in order.

- There is no need to check the impact of users 7 or 8 as $a_8 < a_7 < p_2$.
- Including users $4, 5, 6$ is not necessarily profit-optimal. User 6 may need to be rejected, and perhaps even user 5. On the other hand, users $4, 5$ and 6 cannot be all rejected as we already know that adding them to users $1, 2$ and 3 is better than only keeping users $1, 2, 3$.
- One can construct an example where adding users $4, 5$ and 6 leads to no profit gain, but adding only users 4 and 5 leads to profit gain. Hence, just checking for adding those users who have positive demand under a new price is not enough. In fact, one would need to check for adding any subset of the users who have positive demand before declaring if an addition is profitable or not.

The above example illustrates that in general it is not possible to use Lemma 5.3 to find the profit maximizing set of users. When $a_i > \hat{p}^*(\mathcal{I})$ for all $i \in \mathcal{I}$, the Lemma 5.3 is a sufficient and necessary condition for finding the maximum profit. However, in general, Lemma 5.3 is only a sufficient condition showing the unprofitability of a single user. In fact, in general it need not be profit-optimal to remove users one by one. An example can be constructed where removing one user is not profitable, but removing two or more of them is. Hence, barring extensive search, there is currently no simple algorithm to find the set of users who guarantee profit maximizing for a provider charging for provided service. This is an interesting area for future work.

5.3 Some results on competition of providers charging for provided service

In this section we consider the competition of wireless providers selling service to free-roaming users. The setting is identical to the previous chapter: there are J providers, I users, each provider has Q_j of resource, and users are rational entities who want to maximize their payoff functions. The payoff of a user is the difference between the utility of having service acquired $x_i = \sum_j x_{ij}$, where $x_{ij} = c_{ij} q_{ij}$, and the payment $\sum_j p_j x_{ij}$ for that service acquired. The users may purchase the service from several providers at once. The quality of the channel between user i and the antenna of provider j is given by c_{ij}, which can be thought of as the rate per unit of resource, so that resource x_i can be thought of as obtained communication rate.

In the previous chapter we saw that when providers are charging for resource directly, users decide which provider to join based on both price and channel quality. When providers are charging for provided service, the users no longer need to take channel quality into account, all that matters is willingness to pay and price. This can easily be seen from the expression for the payoff $\hat{v}$ of a user, given in (5.2). Then if one provider charges a smaller price than the others, he would have all the customers. So the other providers would have an incentive to match this low price and obtain at least some customers. Hence, we expect the prices of all providers to be the same at equilibrium.

At the same time, a provider cannot reduce his price all the way down to zero, as a low price would imply high demand from the users. A provider may lower the price so that the demand is at most as high as his supply. So, at equilibrium, the unique price charged by the providers should be the market-clearing price for the users who are associated with each customer. This is where the analysis gets less clear: how do users decide which provider to associate with, if all the providers are offering the same price? One idea is that each user goes to the provider to whom he has the best channel, but a quick inspection shows that this cannot be a solution in general: simply consider users who all have the best channel to the same provider.

The following example gives a situation where the outcome of the provider competition can be found.

Example 5.2. *Consider users $\mathcal{I} = \{1, \ldots, I\}$ who are interested in buying service from providers 1 and 2, who have Q_1 and Q_2 amount of resource at their disposal. Assume $c_{i1} > c_{(i+1)1}$ and $c_{i2} < c_{(i+1)2}$ for all $i \in \mathcal{I}$.*

Then, there exists a price $\hat{p}^*$ such that $\hat{p}^* = \hat{p}_1^*(\{1, \ldots, k-1, \epsilon k\})$ and $\hat{p}^* = \hat{p}_2^*(\{(1-\epsilon)k, k+1, \ldots, I\})$, where ϵk and $(1-\epsilon)k$ indicate a splitting of demand of user k between the two providers.

We claim that in the example above, charging $\hat{p}^*$ by both providers is the equilibrium outcome of the competition, and call such a price doubly market-clearing. Both providers are charging market-clearing prices, and it can be shown that all users are getting payoff maximizing amount of service acquired. It can also be shown that the complexity of finding a unique ϵ for which $\hat{p}^*$ is a doubly market clearing price is not too high, i.e. it is polynomial in the number of users. We omit the proofs as they are tedious and bring little insight. But the interesting thing is that each provider gets the users with the best channel quality towards him that he can possibly get. As a consequence, the resource that each provider has at his disposal is used efficiently.

On the other hand, it can be shown that the partitioning of users in $\mathcal{I}_1$ and $\mathcal{I}_2$, as done in Example 5.2, is not the only only way to split the user set $\mathcal{I}$ that leads to a doubly market clearing price. For example, by assigning to providers those users with the worst channel quality, a similar partitioning could be found, although most likely with a different "border user" k' and with a different value of ϵ' to split the demand. However, this arrangement would be highly inefficient, as the users would now need more resource to operate, and as a result all users would have acquired less service at equilibrium. Ironically, the new price $\hat{p}^*$ would be higher, indicating more competition, but it would bring less profit to the providers[2]. Indeed, any other association of users would lead to the outcome where all participants, i.e. both users and providers, would be worse off than with the association given in Example 5.2.

It is the assumption on the ordering of channel qualities that permits us to find an association that is best for everyone in Example 5.2. In general, users cannot be ordered in such a way and it is far from clear how the equilibrium association should arise. One

[2]Here the price can be considered a measure of scarcity. If users have bad channels, they need a lot of resource to communicate, and hence the resource becomes scarce.

possible solution would be to find the user association that leads to the lowest multiple market clearing price. It can be argued that such a price ensures that users have the highest possible service acquired demand at equilibrium. Even though it is not clear that this is in the best interest of providers, such a price is in the best interest of users, so it makes sense to assume that users would associate themselves with providers in such a way that the equilibrium price is the lowest.

Finding the lowest market-clearing price in general remains an open problem. Under the complete information assumption, each user can solve a problem separately, find the optimal solution (if such a solution is unique), and associate with an appropriate provider. Alternatively, there could be a central authority that finds the optimal price, and the user-provider association leading to that price. Needless to say, both of these solutions are unsatisfactory. Furthermore, this argument does not take into account any potential admission control. This remains an interesting area for future research.

Appendices

5.A Market clearing price

5.A.1 Market clearing price computation

In this section we show how the market clearing prices $p^*(\mathcal{I})$ and $\hat{p}^*(\mathcal{I})$ can be computed for any given set of users $\mathcal{I}$. We focus on the calculation of $p^*(\mathcal{I})$, but the procedure is the same for $\hat{p}^*(\mathcal{I})$. The demand of user i under price p is $q_i = \left(\frac{a_i}{p} - \frac{1}{c_i}\right)^+$. For users who have $\frac{a_i}{p} \geq \frac{1}{c_i}$, $q_i = \frac{a_i}{p} - \frac{1}{c_i}$. The set of such users is $\mathcal{I}^+(p)$. The market clearing condition is that demand equals to the supply:

$$\sum_{i\in\mathcal{I}} q_i = \sum_{i\in\mathcal{I}^+(p)} \left(\frac{a_i}{p} - \frac{1}{c_i}\right) = Q,$$

from which we get the expression for the market clearing price:

$$p^*(\mathcal{I}) = \frac{\sum_{\mathcal{I}^+(p^*)} a_i}{\sum_{\mathcal{I}^+(p^*)} \frac{1}{c_i} + Q}. \tag{5.10}$$

In order to calculate the market-clearing price, we need to know the set of users who get positive demand from it. But to know this set, we need to know the market-clearing price. We seem to be facing a "chicken and egg" problem. To overcome this issue, consider the following price function

$$p^f(\mathcal{I}) = \frac{\sum_{\mathcal{I}} a_i}{\sum_{\mathcal{I}} \frac{1}{c_i} + Q}, \tag{5.11}$$

which is the solution to the equation $pQ = \sum_{i\in\mathcal{I}} \left(a_i - p\frac{1}{c_i}\right)$, i.e. it would be the market-clearing price if users were allowed to purchase negative amount of resource. This solution has no physical meaning. We will call $p^f(\mathcal{I})$ the *fictitious* price function. Unlike the market-clearing price function, the fictitious price function is easily calculated from (5.11). Furthermore, we can use the fictitious price function to calculate the market-clearing price. Before defining the algorithm rigorously, we give a high level explanation. The idea is to find a subset of users for which the fictitious price is equal to the market-clearing price.

The fictitious price equals the market-clearing price if and only if $\frac{a_i}{p^f(\mathcal{I})} - \frac{1}{c_i} \geq 0$ for all $i \in \mathcal{I}$. In general, this is not the case since some users will have negative demand. These users are not interested in purchasing the resource. Then, we proceed by removing those "uninterested" users from the set $\mathcal{I}$ and by recalculating the fictitious price for the users with positive valuation only (the users "interested" in purchasing the resource). Removing uninterested users increases the fictitious price so the set of potentially interested users can only get smaller. We terminate the procedure when we find the fictitious price for which there are no more uninterested users. In each step we remove at least one user, and there has to be at least one interested user remaining. Hence this algorithm terminates in at most $I-1$ steps with the set of only interested users who have positive demand at the market-clearing price. The formal proof follows.

We begin by calculating $p^f(\mathcal{I})$. Then, we check whether all of the users in $\mathcal{I}$ have positive demand (i.e. if $\frac{a_i}{p^f(\mathcal{I})} - \frac{1}{c_i} \geq 0$ for all $i \in \mathcal{I}$). If yes, then $\frac{a_i}{p^f(\mathcal{I})} - \frac{1}{c_i} = (\frac{a_i}{p^f(\mathcal{I})} - \frac{1}{c_i})^+$ for all $i \in \mathcal{I}$ and the fictitious price is equal to the market-clearing price. If not, there exists at least one user k who has negative demand. We compare $p^f(\mathcal{I})$ and $p^f(\mathcal{I} \setminus \{k\})$. To simplify notation, we denote $\mathcal{I}' = \mathcal{I} \setminus \{k\}$ to be the set of users without user k:

$$\begin{aligned} p^f(\mathcal{I}') - p^f(\mathcal{I}) =& \frac{\sum_{i\in\mathcal{I}'} a_i}{\sum_{i\in\mathcal{I}'} \frac{1}{c_i} + Q} - \frac{\sum_{i\in\mathcal{I}} a_i}{\sum_{i\in\mathcal{I}} \frac{1}{c_i} + Q} \\ =& \frac{p^f(\mathcal{I}')\left(\frac{1}{c_k} - \frac{a_k}{p^f(\mathcal{I}')}\right)}{\sum_{i\in I} \frac{1}{c_i} + Q} > 0, \end{aligned} \tag{5.12}$$

where the last inequality follows since user k is an uninterested user, i.e. $\frac{a_k}{p^f(\mathcal{I})} - \frac{1}{c_k} < 0$. Hence, removing a user with negative demand increases the fictitious price. Notice that, since $p^f(\mathcal{I}') > p^f(\mathcal{I})$, then $\frac{a_k}{p^f(\mathcal{I}')} - \frac{1}{c_k} < \frac{a_k}{p^f(\mathcal{I})} - \frac{1}{c_k} < 0$, so a user with negative demand at price $p^f(\mathcal{I})$ will still have negative demand at price $p^f(\mathcal{I}')$. The same statement holds if we remove several uninterested users at the same time, since we can always think of this as removing uninterested users one by one. At the same time, if the price is changed to $p^f(\mathcal{I}')$, all interested users will have their demand decreased, and some may see their demand go negative, effectively becoming uninterested users. Hence, additional users may need to be removed and price recaculated. These steps are repeated until the price is such that there are no users with negative demand.

This procedure is guaranteed to terminate with at least one user. To see this, note that a single user always has positive demand when he is the network's only customer.

Now we can define the market-clearing price computation algorithm. We initiate $\mathcal{I}^* = \mathcal{I}$ at the beginning of the algorithm.

Algorithm 5.1. *Price computation algorithm:*
Begin

1. *Calculate* $p^f(\mathcal{I}^*)$.

2. Find $\hat{\mathcal{I}} = \{i \in \mathcal{I}^* : \frac{a_i}{p^f(I)} - \frac{1}{c_i} < 0\}$

3. If $|\hat{\mathcal{I}}| > 0$ *set* $\mathcal{I}^* := \mathcal{I}^* \setminus \hat{\mathcal{I}}$ *and go to step 1.*

4. $p^*(\mathcal{I}) = p^f(\mathcal{I}^*)$.

End

Notice that, at the end of the algorithm,

$$\begin{aligned}\sum_{i\in\mathcal{I}} \left(\frac{a_i}{p^*(\mathcal{I})} - \frac{1}{c_i}\right)^+ =&\sum_{i\in\mathcal{I}^*} \left(\frac{a_i}{p^*(\mathcal{I})} - \frac{1}{c_i}\right)^+ + \sum_{i\in\mathcal{I}\setminus\mathcal{I}^*} \left(\frac{a_i}{p^*(\mathcal{I})} - \frac{1}{c_i}\right)^+ \\ =&\sum_{i\in\mathcal{I}^*} \left(\frac{a_i}{p^f(\mathcal{I}^*)} - \frac{1}{c_i}\right),\end{aligned}$$

since $\left(\frac{a_i}{p^f(\mathcal{I}^*)} - \frac{1}{c_i}\right)^+ = 0$ for all $i \in \mathcal{I} \setminus \mathcal{I}^*$. Hence, the fictitious price for $\mathcal{I}^*$ is the market-clearing price for $\mathcal{I}$.

5.A.2 Monotonicity of the market clearing price

Lemma 5.5. *Let* $\bar{\mathcal{I}}$ *and* $\mathcal{I}$ *be two user sets, where* $\mathcal{I} \subset \bar{\mathcal{I}}$. *Then* $p^*(\mathcal{I}) \leq p^*(\bar{\mathcal{I}})$.

Proof. We first prove that adding a new user to the user set can only increase the market-clearing price. Suppose that we already computed $p^*(\mathcal{I})$ and we wish to compute $p^*(\mathcal{I}')$, where $\mathcal{I}' = \mathcal{I} \cup \{k\}$. Without loss of generality, assume that $\frac{a_i}{p^*(\mathcal{I})} - \frac{1}{c_i} \geq 0$ for all $i \in \mathcal{I}$, otherwise, we can always restrict ourselves to such a set of users. Notice that adding user $\{k\}$ does not change anything if $\frac{a_k}{p^*(\mathcal{I})} - \frac{1}{c_k} \leq 0$, as then user k has zero demand, and the market-clearing price is unchanged. This is different than the fictitious price from Appendix 5.A.1 which decreases if user with negative demand is added. So, we assume $\frac{a_k}{p^*(\mathcal{I})} - \frac{1}{c_k} > 0$. Then, following (5.12)

$$\begin{aligned}p^f(\mathcal{I}') - p^*(\mathcal{I}) =&\frac{\sum_{i\in\mathcal{I}'} a_i}{\sum_{i\in\mathcal{I}'} \frac{1}{c_i} + Q} - \frac{\sum_{i\in\mathcal{I}} a_i}{\sum_{i\in\mathcal{I}} \frac{1}{c_i} + Q} \\ =&\frac{p^*(\mathcal{I})\left(\frac{a_k}{p^*(\mathcal{I})} - \frac{1}{c_k}\right)}{\sum_{i\in\mathcal{I}'} \frac{1}{c_i} + Q} > 0.\end{aligned} \tag{5.13}$$

In principle, in order to find $p^*(\mathcal{I}')$ we need to run Algorithm 5.1. Here, it suffices to notice that in each step of Algorithm 5.1 we remove only uninterested users, which can only increase the fictitious price. Hence, $p^*(\mathcal{I}) \geq p^f(\mathcal{I}')$ and therefore $p^*(\mathcal{I}') > p^*(\mathcal{I})$ if $\frac{a_k}{p^*(\mathcal{I})} - \frac{1}{c_k} > 0$. In general, for $\frac{a_k}{p^*(\mathcal{I})} - \frac{1}{c_k} \in \mathbb{R}$ we have $p^*(\mathcal{I}') \geq p^*(\mathcal{I})$.

Hence, the market-clearing price function is non-decreasing if we add a single user. Since any group of users can be added by adding users one by one, the market-clearing price is monotone in the set of users. □

CHAPTER SUMMARY

- **In the previous section we considered wireless providers who are charging for resource used by the users. In this chapter we study providers who are charging for the provided service to the wireless users.**
- **We compare the two kinds of pricing for a special case of logarithmic utility functions. We describe the special cases when one pricing is preferable to the other. However, in general the profit from the two pricing schemes needs to be explicitly calculated before concluding that one of them is more profitable.**
- **We consider optimal pricing of provided service by introducing admission control for users with poor channel gains. We provide a sufficient condition for excluding certain users from the pricing scheme, and describe when this condition is necessary as well.**
- **We consider the competition of providers charging for services provided, and describe associated challenges.**

6 General Related Work

The past decade saw an impressive number of publications focusing on the interaction of users in communication settings, both wireless and wireline. The goal of this chapter is to provide an overview of the work treating the wireless setting[1]. The main theme in the work mentioned in this chapter is decentralized resource allocation, in some form. The wireless participants are in a situation where they want to use limited wireless resources, and the main objective is to understand how these resources are distributed, depending on the specific situation. How do users make decisions in this setting, and what are the properties of the outcomes? If there exists a desirable outcome, can it be reached by selfish users who make independent decisions? If the outcome is undesirable, is there any way to improve it? How do these questions depend on the underlying assumptions?

Naturally, game theory and microeconomic theory are the tools employed for the analysis, as is the case in this thesis. Strictly speaking, microeconomic theory, as used in communication literature, is an application of game theory. However, it is often helpful to make the distinction between the two, especially in the games that involve pricing. In particular, there exist models that treat interactions between two or three fundamentally different types of participants in a wireless setting. For example, in the interaction between service providers and users, the former sell their service to the latter and the interaction is via pricing. We contrast these situations to those where the interaction takes place between the participants of one type, such as end-users who interfere with each other. These situation are treated using either cooperative or non-cooperative game theory without pricing. Of course, the borders between the two cases may be blurred, for example when pricing is introduced to improve the efficiency of equilibria that arise when selfish end-users interact.

This chapter is divided into five sections with the intent that they be as disjoint as

[1]For an overview of work in wireline communication, see [66].

possible, although some overlaps still remain.

Section 6.1 considers work on the interaction of users whose set of available options is given by information theoretic bounds. The majority of references consider the interference channel, although considerable work exists on the multiple access channel as well (such as our work in Chapters 2 and 3). One of the major issues in this section is that the actions of users (e.g. choosing transmission power) may not translate to a unique outcome in terms of rate, which is what the users are typically interested in. In such cases, a simplifying assumption may be made, such as assuming a specific decoding order at the receiver in the multiple access channel case. Similarly, the interference channel settings often do not even consider the end rate, and the main focus is on choosing the correct transmit power levels.

Section 6.2 considers user interaction in the presence of infrastructure, such as base-stations and cell towers. The most common example is power control in multi-user CDMA. The relationship between the user decisions, in terms of power, and the outcomes, in terms of rate, is clear as users either treat each other as noise (uplink CDMA) or are orthogonal (downlink CDMA). Most work is differentiated based on the kind of utility function that is chosen (e.g., voice based, data based, sigmoid, quadratic, probability of error, efficiency etc).

The case of user-behavior in ad-hoc networks is treated in Section 6.3. Most work considers the single-hop case where several transmitter-receiver pairs interact, although we mention some multi-hop references as well. Depending on the communication layer under consideration, different tools are used and different types of conclusions made. We briefly mention the physical layer, medium access, and the network layer (there is a plethora of work on the transport layer in the wireline context, which we do not consider here).

The work on cognitive radio and in general spectrum sharing is considered in Section 6.4 in the context of mechanism design, and more specifically auctions. Since the spectrum can be thought of as a divisible resource, interested users submit bids to the authority selling it, under some pre-defined mechanism. Different auction types are presented, depending on the context, goal of the seller, available information, etc.

The work on competition of wireless service providers is treated in Section 6.5. The distinguishing feature of this work is that end-users typically do not belong to a specific provider, but are free to join the provider that currently offers the best trade-off of parameters. This is also the subject of our work in Chapters 4 and 5. This user-provider dynamics is often coupled with the third-layer interaction, where service providers dynamically obtain the wireless spectrum from a spectrum-selling authority (typically a government agency), before selling it to end-users.

To recap, Section 6.1 considers the interaction of users where the set of available choices is given by information theoretic bounds. Section 6.2 treats the interaction of users in the presence of infrastructure. Section 6.3 considers users without the support of infrastructure, and is divided in subsections based on the communication layer being considered. The use of auctions to obtain wireless spectrum is treated in Section 6.4. The competition of wireless service providers is the subject of Section 6.5. Finally, some non-standard references that do not fit in the five main sections are given in Section 6.6.

6.1 User interaction in an information theoretic setting

In the following, we give a description of some of the main work concerning the interaction of users on the physical layer. In this section, the choice of the users is typically the information-theoretic capacity or best known achievable rate region. By contrast, the choice of users in situations treated in Sections 6.2 and 6.3 typically depend on a pre-specified communication scheme (where the resulting operating point lies in a subset of the information theoretic capacity region). The situations in this section may or may not be infrastructure based but the analysis is different than that presented in Sections 6.2 and 6.3, so we treat this work separately.

6.1.1 Zero-sum games in compound channels

The early work treated nature and a single user as opponents in a zero-sum game, where the gain of a user is the loss of the nature and vice versa.

The first paper to introduce game theoretic considerations in a communication setting was [67], for a compound channel setting. A user is free to choose a communication rate, while the "malicious" nature is free to choose a channel. The capacity of the possible channels is already known for different values of nature's choice. Depending on who chooses first, the game yields different outcomes. The cases treated are when the user chooses rate first, when the nature chooses the channel first, and when the two decisions are made at the same time.

The work in [68] is treating a multiple-input-multiple-output (MIMO) fading compound channel, using a similar approach as [67]. It is assumed that there is no channel-state-information (CSI), and even the channel distribution is unknown to the user. Here, the user chooses a power distribution, which is followed by nature's choice of the fading channel from a discrete set known to the user. It is shown that the best strategy of a user is to allocate power uniformly across the transmit antennas. The situation is viewed as a maxmin problem, where the user and nature are players in a zero-sum game and mutual information is considered as payoff. A similar approach is taken for the multiple user setting, where the users are still playing against nature, not against each other.

6.1.2 Game theory for the multiple access channel

An N-user AWGN multiple access channel is treated in [21]. Rational users are interested in maximizing their communication rates, where each user has a maximum power constraint. We consider the same communication setting in Chapter 2 of this thesis using non-cooperative game theory. The game-theoretic concept used is coalitional game theory, which is a sub-field of cooperative game theory. The users are free to form coalitions, and threaten users outside of the coalition with jamming, in order to obtain a better operating point for the coalition. A guaranteed rate is found for a user being affronted with a jamming threat.

The behavior of users in a multiple access channel using cooperative game theory is also the subject of [69]. The authors consider linear multiuser detection, in contrast to the optimal receiver used in [21]. The grand coalition of MMSE receivers is stable and sum-rate maximizing, while for the decorrelating multiuser receiver this is only true in the high SNR regime.

The setting in [70] is a vector (MIMO) multiple access Gaussian channel. The main result of the work is game-theoretic in nature, although there is no explicit game-theoretic treatment. If the users are trying to maximize the sum of rates, then the optimal input power covariance matrix for each user is the same as in a single-user water-filling solution to a channel where the power of other users is treated as noise. An algorithm is proposed where users adapt their power according to the best-response to the power of other users. The algorithm converges to the power profile which is a fixed point of the best responses of the users, which is indeed a Nash equilibrium. Here the strategic decisions of the users only determine the capacity region, but not the rate vector of the users. The rate vector still needs to be chosen, perhaps in a centralized way.

The work in [11] considers the interaction of users in a fading multiple access channel, where rational users are assumed to be limited by average power constraints. A water-filling game is proposed, and it is shown that the sum-rate optimal point on the boundary of the multiple-access channel capacity region is the unique Nash Equilibrium of the corresponding water-filling game. The base-station is then introduced as a player interested in maximizing a weighted sum of the individual rates. A Stackelberg formulation is proposed, in which the base-station is the designated game leader, and it decides which user is decoded first. It turns out that this formulation allows for achieving all the corner points of the capacity region, in addition to the sum-rate optimal point. The optimal sum-rate point is achieved in a time-sharing manner, where only one user transmits for a given realization of fading coefficients. On the negative side, there is no strategy that achieves other points on the dominant face of the capacity region. To overcome this limitation, a repeated game approach is used which achieves the capacity region of the fading multiple access channel.

While [11] considers a single-input single-output (SISO) fast fading MAC with channel state informations known at all receivers and transmitters, a few other cases were considered in literature. The work in [71] treats a MIMO case with channel state information known at all receivers, but only channel state distribution known at the transmitters. User interaction in a fast fading MIMO MAC is also considered in [72], where the transmitters only have distribution information for the channel gains. However, they have access to a simple coordination mechanism which tells them their decoding order at the receiver. The players are free to choose their power allocation in order to maximize their individual transmission rates.

The case of limited information in a fading multiple access channel was treated in [73]. Each user knows his own channel state, but only knows the probability distribution from which the channel gains of other users are drawn. The interaction can be modeled as a Bayesian game, where a unique Bayesian equilibrium of the game is found.

6.1.3 Game theory for the (frequency selective) interference channel

Interaction of two users in a Gaussian Interference channel is considered in [50]. A Nash Equilibrium is found for the one-shot game in which the users are only interested in maximizing their own rate. It is shown that this equilibrium is Pareto inefficient in the set of all possible rates. Then, a repeated game is treated, in which it is assumed that the same users play the game infinitely many times. Applying the Folk theorem leads to a Pareto optimal Nash Equilibrium. Essentially, the users can pre-agree to any outcome strictly better than the single-shot game outcome. Then, no user has an incentive to deviate since the other user could punish such a deviation by reverting to the inefficient strategy from the single-shot game. Depending on the interference strength and channel gain coefficients the repeated-game equilibrium may be socially optimal, or not too far off from the optimal.

In [74], Bargaining Theory (a subfield of cooperative game theory [75]) is applied to an interference channel. Bargaining theory treats the situation where users are supposed to come to an agreement on the outcome of resource/good sharing, while both users are selfish, and have some negotiating power. For example, as shown in [50], users can revert to an inefficient strategy if they are not satisfied by the outcome of the negotiation. Here, Nash bargaining solution is used as a solution concept for the 2x2 interference channel. The frequency-division-multiplexing achievable region is given as the set of achievable operating points.

In [76], the two-by-two interference channel is considered for senders who have multiple antennas. It is shown that if the two senders are competing, and not cooperating, the resulting Nash equilibrium is inefficient. Cooperative game theory is then employed in order to find the Nash bargaining solution, which is close to being sum-rate optimal.

A frequency-selective interference channel model is proposed for the digital subscriber lines in [77]. The problem of power allocation is treated as a non-cooperative game, and a Nash Equilibrium was found to be the outcome of iterative waterfilling. This equilibrium is not efficient, but it has a benefit of being distributed, which is important in a setting where users are thought not to be in a position to communicate.

The work in [78] looks into a 2-user interference channel. As already mentioned, iterative water-filling is optimal for the Gaussian multiple access vector channel [70], but it is suboptimal for the Gaussian vector interference channel [77]. In [78], the authors are explaining the inefficiency of the NE using the Prisoner's dilemma. The users are facing a dilemma of whether to employ frequency division multiplexing (cooperate), or waterfilling (defect). The resulting rate payoffs suggest that the user will behave like in the Prisoner's dilemma, i.e they will defect, which results in inefficiencies.

A generalization of iterative waterfilling results for a MIMO interference channel is given in [79], where it is assumed that the number of transmit and receive antennas are not equal. The authors characterize the channel matrix that guarantees uniqueness of Nash equilibrium, and prove the convergence of iterative waterfilling algorithm to this equilibrium.

The work in [49] studies the interaction of N transmitter-receiver pairs that share the same spectrum using several non-cooperative games of incomplete information. Such scenarios are captured using both static and dynamic Bayesian games. A similar setting is considered in [80], where coalitional game theory is used to model a variety of spectrum sharing situations. Another related work is [81], which specifically considers the core of cooperative games for several information-theoretic settings, not only for interference channels.

The tutorial [82] provides an overview of game-teoretic work on the interference channel and the frequency selective interference channel. Below is the breakdown of various work on the subject.

Work on the interference channel:

- The early work focused on the non-cooperative games (e.g. [77, 50]), followed by the work on the weak interference channel [83], an algorithm involving pricing in [84], and generalizations for the MIMO case [85, 79]. The interference channel was likened to the prisoner's dilemma in [78], showing that the rationality of users leads to inefficiencies.

- To combat the inefficiencies, cooperative game theory was applied using the concept of Nash bargaining solution in [74, 76, 86] for the SISO case, in [87, 88, 89] for the MISO case, and in [90] for the MIMO case. Some of the results were extended to the log-convex utility functions in [91].

- The case of multimedia distribution networks was treated in [92]. Finally, coalition game theory was applied to the interference channel in [93, 80], and more generally in various information-theoretic settings in [81].

Work on the frequency selective interference channel:

- Initially, [94] studied the Nash Bargaining in the frequency selective channel under FDM/TDM strategies and total power constraint. The proposed algorithm was inefficient, as well as in the case of the average power constraint, which was treated in [95].

- Introducing a power spectral density constraint eliminated inefficiencies in [96]. In [97], an algorithm was introduced that converts the PSD limited case to the total power constraint case.

- The problem of convergence of iterative waterfilling algorithms in both interference channels and frequency selective channels was treated in [15, 98, 99, 100, 101], and finally generalized in [102].

- Other work involving game theoretic approach: [103], [104], [105], [106].

It is worth noting that most of the work on the interaction in the interference channel requires the presence of some way to share information. For example the users need to exchange information about the waterfilling levels, channel gains, etc. In a wireless setting, this may be a somewhat strong assumption. Accordingly, a lot of the work in this setting focuses on cooperative game theory where prior agreement is needed, which supposes already some way for users to exchange information. This is in contrast to non-cooperative game theory, where decisions are made independently by the users without the need to exchange information during the decision-making process.

6.1.4 Various

Mutual information games in multi-user channels with correlated jamming was considered in [107].

6.2 Strategic interaction of users in the presence of infrastructure

Some of the first work applying game theory in wireless communications considers a CDMA setting where N power-limited users are trying to communicate to their respective destinations. A common sub-case is **uplink**, when all users communicate to the same base-station. The utility function of each user is an increasing function of their transmit power, either through bit probability of error or Signal to Interference Ratio (SIR). In such a situation, each user has an incentive to increase their transmit power to the maximum, even though this generates interference to other users. The resulting Nash equilibrium is typically inefficient, since users do not take into account the interference they are causing to other users.

To combat the inefficiencies, the infrastructure (base-station, network, wireless service provider) introduces pricing for the power used. Since the users need to pay for the power, they have an incentive to use less of it. This reduces the interference experienced by others, and results in an efficient allocation. The prices may be chosen so that the sum of user utilities is maximized, or to maximize the provider's profit, or in some other way. As we mentioned in the introduction to this chapter, the pricing is typically introduced *to the users by the infrastructure*, in order to obtain a certain goal. On the other hand, the analysis that does not involve pricing is typically concerned with the interaction *between the users themselves.*

In the **downlink** the base-station communicates to the users so there is a total power constraint (e.g. [108]). This is in contrast to the individual user power constraints in the uplink case. In the downlink, typically the resources are attributed so that the users are orthogonal within a single cell, although they may interfere with neighboring cells. In the downlink case the Nash equilibrium is typically efficient, but difficult to obtain without the explicit knowledge of the utility functions, which are usually a private knowledge to the users. To combat this issue, many references develop algorithms which use local information and typically converge either to an optimal operating point, or close to one.

The work in this section typically has a single entity (cell manager, provider, base-station) that is in charge of pricing once it is determined that users end up in an inefficient state on their own. The pricing itself can be linear in the transmit power, linear in rate, non-linear (convex or concave function of power of rate), or even fixed, where users pay a fee to access the spectrum. Some of the work considers several wireless providers in the system, where some user-provider association method is assumed in order to determine which user is connected to which provider. The user-provider association can be based on the SIR, channel gain, or something else. The case where different providers compete for wireless users through pricing is treated separately in Section 6.5.

6.2.1 Power control games

We now consider in more details the work on power control games in wireless communications. Due to the extensive number of publications, we explain in more detail only some of them, while others are only briefly mentioned.

Utility as a function of bit probability of error

In one of the earliest work applying game-theoretic analysis and pricing in a wireless setting, Shah et al. [109] develop a distributed power control framework for wireless data services. The interaction of users is modeled as a non-cooperative game in which users are rational entities who maximize the utility gained from the allocation of transmission power. Each user is causing interference to others. The utility in this work is a concave function of bit error rate, which, in addition to the SIR, also depends on the coding and modulation of the users. It is shown that there is a unique, Pareto inefficient Nash equilibrium in which the received powers at the base station are equal for all users. Next, a linear pricing function is introduced, where the sum paid by each user is proportional to their transmitted power, in order to give users an incentive to reduce their power. This results in an improvement in the overall efficiency. The model is further developed in subsequent work [110, 111, 112, 113, 114, 14, 115] which considers various model extensions and includea additional factors in the analysis.

For example, Goodman and Mandayam [111] consider a pricing power-control scheme for data services in CDMA networks, where users are N transmitter-receiver pairs using the same spectrum, (originally treated in [12] with linear utility functions). This work considers the power allocation objectives in voice and data services. For voice services, the utility function can be modeled as a step function; users need to attain a minimum SIR value, beyond which their valuation remains constant. Therefore, the transmit power levels are set so that all users have the same target SIR measured at the base station. In contrast, the utility function for data services is an increasing concave function of the SIR level, so a different power allocation method is needed to efficiently allocate power between data users. A linear pricing function is then introduced, so users maximize their net payoff, i.e., the difference between their utility and the cost to be paid for the transmit power. For any fixed price, an equilibrium can be found, where all terminals operate at lower power, but obtain higher utility than in the absence of pricing. The total utility in the system is maximized for a unique price.

Utility as a function of achievable rate, uplink case

In [13], Alpcan et al. consider a price-based power control scheme for data users in an uplink CDMA network. M users request network resources for the transmission between their mobile device and a single base station. Their interaction is modeled as a non-cooperative game where the goal of each user is to maximize the difference between a logarithmic function of the SIR and a linear function of the transmit power. This type of utility function models users' satisfaction as a function of rate. Since rate is arguably what wireless users care about the most, this type of utility function became standard for power control in wireless data traffic. Based on the individual maximizations, the optimal power allocation for all users can be identified. Some users may choose zero power at equilibrium if transmission power is too expensive. In addition to the uniform pricing for all users, the authors introduce a differentiated pricing scheme where users are charged a price proportional to their channel gain. Two different update algorithms are introduced that make the system converge to a stable Nash equilibrium. In [116], the single cell power control scheme of [13] is extended to multiple cells and to a broader class of cost functions. In [117], this Nash equilibrium result is generalized for a strictly convex pricing function with respect to transmit power, using potential games [118].

In [119] the uplink power control setting is considered with sigmoidal utility functions and linear costs on the interference received. In contrast, [120] considers the linear cost that depends on the interference caused. Similar to the uplink case [111], the interaction of users in a single-hop ad-hoc network (i.e., N transmitter-receiver pairs) was treated in [121], with users having multi-step utility functions (i.e., piece-wise constant, non-decreasing). Joint rate and power control in a game-theoretic setting for N transmitter-receiver pairs was considered in [122].

In [123] the impact of different receivers in an uplink CDMA is investigated in a flat fading scenario. Users are assumed to have a utility function that measures the energy efficiency of the communication. The Nash Equilibrium is described as a point where all users are achieving the SIR ratio that maximizes their personal utility function. It is shown that users can myopically adjust their power to improve their utility until they reach the desired SIR. The users themselves do not need to have any strategic thinking in order to arrive to this NE, they simply play a best response strategy.

A similar power allocation problem in CDMA is treated in [16] for a frequency selective channel using game theoretic arguments. The authors investigate the asymptotic case where the number of users is going to infinity, and find simple expressions for the optimal power allocations. One of the key ingredients is that, since the number of users is large, the impact of one user on others is negligible. The power allocation found this way (Wardrop Equilibrium) is proven not to be too far off from the optimum, found through the Nash Equilibrium. The results obtained for an infinite number of users are close to those obtained for a moderately high finite number of users.

Utility as a function of achievable rate, downlink case

A downlink case for a single wireless cell is considered in [44]. Users have varying channel quality and utility functions, and are interested in obtaining a part of a perfectly divisible resource. For a time-slotted system the resource is a time fraction, and for a CDMA system the resource is the fraction of the base-station's transmit power. First, revenue optimization is compared to social welfare optimization, and the former is concluded to be more realistic since it is in the interest of the network and does not require the knowledge of the users utility functions, which is usually private information to the users. Then, a price-driven auction mechanism for the allocation of transmission times is proposed. Users bid by submitting a price based on which the base station allocates resources to maximize its revenue. If the network has perfect knowledge about the users' demand functions, the optimal solution would allow it to obtain profit equal to the entire user surplus. In the imperfect information case, which is more realistic, the obtained revenue is lower, although it converges to the optimum in the limit of infinitely many users.

The Nash equilibrium found for the downlink case in [108] is socially optimal, and differs from Pareto inefficient outcomes for the uplink case (e.g. [12, 110]), although even here the (centralized) algorithm can only converge to a point near the Nash equilibrium. The downlink voice traffic is treated in [124] where one network is maximizing either profit or social welfare over multiple-cells through pricing. A multi-class system for the CDMA downlink is considered in [108].

Extending [125], which only considered voice users, a combination of users interested in voice service and in data service is considered in [126] for downlink traffic of two CDMA cells on a line. The optimal solution, which maximizes the sum of utilities over both cells, is parametrized by the prices for both power and rate. In several subsequent papers, the basic model is extended to other settings [127, 126, 128, 129, 53]. A single CDMA cell in which uplink and downlink resources are allocated separately based on the user's utility function is considered in [130]. Two different user utility functions are considered depending on the service type; a concave and a sigmoid utility function, which allows for the modeling of a minimum bandwidth requirement.

In [131], a single sender is receiving auction bids from multiple receivers, whose utility is linear in their channel gains. For certain channel gain distributions, such a system results in a Pareto efficient Nash equilibrium which is within 25% of the social optimum. In [132], auctions are used to share spectrum between a group of users subject to an interference constraint at a measurement point (see Section 6.4 for more details).

The multi-cell case where N randomly placed users can be served by one of the K base-stations is considered in [113]. Users choose their transmit power, and their utility is an increasing function of the obtained SIR. The channel strengths are a function of the distance between the users and the base-station. Users are assigned to base-stations either based on maximum SIR, or maximum received signal strength. The Nash equilibrium of the user game without pricing is compared to that of two different pricing methods: uniform across base-stations and one price for each base-station. Here, uniform pricing can model the case where all base-stations belong to the same provider, as opposed to the competition of many providers, where it is reasonable to expect different prices. In both cases, pricing helps improve the sum of utilities.

A channel-based pricing mechanism for CDMA-based wireless networks using an auction is presented in [133]. Each user chooses a utility function according to the service he is running. For example, a user may have a preference for continuous transmission of smaller data units while another user prefers larger data bursts. The predefined utility functions are matched with the existing service classes, and the allocation performed at the base station automatically on behalf of the users by a Generalized Vickrey Auction. The experimental results indicate that users are either served with few violations of the requested scheduling scheme or are not served at all. In subsequent work [134], the authors adapt the model to the High-Speed Downlink Packet Access (HSDPA) extension of the WCDMA standard. Again, different predefined utility functions allow users to set different priorities for the scheduling of the packets.

A game theoretic approach to spectrum sharing is considered in [135]. There is one spectrum provider (primary user) and N spectrum users (secondary users), whose interaction is modeled as an oligopoly (i.e. Cournot game [1]). The primary user assigns a single price, and the strategies of the secondary users is a bid for the piece of the spectrum.

Recently, power control for M transmitter-receiver pairs was treated in [136] using measurement centers to control interference.

6.2.2 Wi-Fi

The pricing of users in 802.11 networks was treated in [137, 138, 47].

The allocation of channels in competing WiFi base-stations is analyzed as a game in

[139]. The authors propose a graph-theoretic model and discuss the price of anarchy under various topology conditions such as different channel numbers and bargaining strategies.

6.3 User interaction in ad-hoc networks

The content of this section is mainly based on the survey work in [140].

6.3.1 Physical layer

The **power control games** in Section 6.2.1 consider infrastructure-based wireless networks, involving communication of many users with a single base-station, although often their interaction is considered without taking into accout the influence of the central authority. The exceptions are [111, 121] and [122], where N transmitter-receiver pairs interact. Similar setting is treated in [141] using potential games. In this section we consider additional work on interaction of users without a central authority.

The interaction of users in a MIMO ad-hoc network is considered in [142]. The authors design a utility function that takes into account power management, and propose a mechanism for turning off links that have low throughput but generate interference to other nodes. The proposed mechanism improves data rate and energy efficiency for the network.

Authors of [51] consider a distributed power control scheme for wireless ad hoc networks, where each user announces a price that reflects compensation paid to it by other users for their interference. A distributed algorithm for updating power levels and prices is interpreted as a best response updates in a fictitious game. The convergence of the algorithm is then characterized using using supermodular game theory. This work is one of the rare ones that introduces game theory to prove a convergence result. Typically it is the other way around: algorithms are introduced to prove convergence to an equilibrium of the game. The users are still considered cooperative, as they report the true interference prices. To overcome this assumption, [143] proposes a reinforcement learning algorithm, where the pricing is not required.

In [61], the interaction of self-interested nodes is considered in a multi-hop relay network. The relaying nodes charge the senders in order to relay data, but they themselves need to pay to the nodes further down the route. It is shown that the minimum network cost routing can always be induced as a Nash equilibrium of the pricing game. On the negative side, the selfish nature of the system leads to unbounded price of anarchy for the case of convex marginal cost functions, although the price of anarchy is limited for concave cost functions. Interestingly, it is the very multi-hop structure of the network which is to blame for the inefficiencies, as the nodes cannot control the upstream traffic.

Another decentralized network context that received attention is **interference avoidance**, i.e. the selection of a waveform by a node such that the interference at its receiver is reduced. Game theory was used to show that for a single receiver system with two players, any combination of the metric (such as Mean Square Error or SINR) and receiver types (such as a correlator or MSINR receiver) results in a game with convergent Nash equilibrium solutions [144]. A game-theoretic framework to analyze power control and signature sequence adaptation in a single-cell synchronous CDMA systems is also presented in [145] (this issue was first considered in [146]). For the power and waveform adaptation game, it is shown that multiple Nash equilibria exist. In addition, properties of each user's utility function that ensure the existence of a Nash equilibrium are characterized.

In [147], the problem is formulated in a multicell scenario. It is shown that greedy sequence adaptation methods may lead to instability if applied directly to cellular systems. To combat this, a new interference measure for multicell systems is proposed. Although the Nash equilibria cannot be constructed explicitly, it is proven that at least one of them exists. A framework based on potential game theory for multicell systems that gives rise to convergent waveform adaptation games in a multicell scenario was considered in [148]. Recently, interference avoidance was considered in a cellular networks context [149] and in multi-cell OFDMA systems [150].

6.3.2 Medium access layer

In the medium access control, many users contend for the same medium. Users have an incentive to make choices that increase their utility and obtain an unfair share of access to the channel. This is to the detriment of other users.

The earliest applications of game theory to a medium access control problem is the work of Zander in [151, 152], where nodes in a network running Aloha protocol cooperate to forward a packet that a jammer is trying to block. On the other hand, in [153, 154, 155], MacKenzie and Wicker consider the slotted Aloha medium access control protocol to be a game between users contending for the channel. Users attempt to maximize the discounted sum of their payoffs over time. Each user receives one unit of payoff when they transmit successfully. In the absence of a cost, users always transmit, resulting in frequent collisions and extremely low throughput. However, if a transmission cost is introduced, the users start being more selective with their transmission attempts. This is similar to the effect of pricing in power control games from Section 6.2.1. The cost may be introduced by the network, or it may be internal, modeling, for example, users who try to conserve energy inherent in every transmission attempt. For optimal values of the cost parameter, the throughput of non-cooperative slotted-Aloha system can be as high as the cooperative one. Authors of [156] consider a two-by-two random access interference channel where transmitters independently and simultaneously choose a transmission probability. The cases of complete and partial information about the transmitters' backlogs are considered.

Contrary to [155], the work in [157] considers a model in which the number of backlogged users is unknown. However, here the total number of users in the system is known and the users retransmit probabilities cannot change over time. Similar to [155], the achievable throughput in the presence of cost in the non-cooperative case may coincide with the throughput obtained by cooperative users.

In [158], users attempt to obtain a target throughput by updating their transmit probabilities in best-response to observed activity of other users. Using a potential function, the set of throughput points that are achievable through this update scheme is characterized. The work in [159] considers the situations where non-cooperative nodes are introduced into a network of mostly cooperative users. In [160], nodes selfishly adjust their random back-off timers in order to increase throughput in a CSMA/CA system. A Pareto optimal point of operation for such a situation is found, and then shown that it is a Nash equilibrium of a repeated game. A game theoretic approach to exploiting diversity in slotted Aloha was considered in [161].

6.3.3 Network layer

The network layer is responsible for establishing and updating routes, and for forwarding of packets along those routes. Game theory is used to investigate the effect of node behavior on issues in this context, e.g. the influence of selfish nodes that prefer to drop packets rater than route them. This is the goal of work in [162, 163, 164, 165, 166, 167]. They all have in common that the equilibrium for the one-shot game is that none of the nodes cooperate in forwarding packets.

This is not unexpected: single-shot games, starting with the Prisoner's dilemma, are notorious for their lack of cooperation. Indeed, cooperation starts showing up in repeated interactions, which are appropriate for an ad-hoc network where users need to forward packets for each other over a longer time-period. In a repeated game, strategies such as tit-for-tat and its variants [163, 165, 166] result in desirable Nash equilibria. In tit-for-tat, if a node is forwarding packets for other nodes, the other nodes will do the same and if the node drops packets, so will the other nodes. Hence, a node has an incentive to behave in a socially desirable manner out of pure selfishness.

Some other approaches include the grim-trigger strategy investigated in [168], where users cooperate in file sharing as long as everyone else does, and stop cooperating forever as soon as someone deviates. The equilibrium in a game with the grim-trigger strategy may depend on the number of nodes and their knowledge about the duration of the game. In [164], the interaction of nodes in an ad-hoc network is first analyzed using cooperative game theory, and then a decision-history based mechanism is introduced that gives nodes the incentive to cooperate. In [167], nodes are incentivized to cooperate using a punishment mechanism.

A single relay channel was considered in [169] using stochastic games. The transmitter and relay are competing over a collision channel to deliver packets to the common receiver. Cooperation is induced using a reward mechanism.

The representative of limited work on game theory in sensor networks is [170]. This work considers a network of mobile sensors who track multiple targets. A game theory based algorithm is used to model negotiations of sensors as to which targets to track.

6.3.4 Cooperation enforcement mechanisms in ad-hoc networks

In this section we mention work whose emphasis is on providing mechanisms for enforcing cooperation of users in an ad-hoc network, as opposed to analyzing their free interaction.

The most common mechanism for ensuring cooperation is to adopt some kind of **pricing and reward system** [171, 172, 173, 174, 175]. The "account" of a node improves when it acts in the interest of the network, while it gets reduced when the node is using the services of others. Such a scheme is implemented using virtual currency, or "tokens" in [171], where nodes earn tokens by forwarding packets and spend them when their packets are forwarded by others. For this scheme to function, tamper-proof hardware is required so that nodes cannot falsely report their number of tokens.

The work in [172, 173, 174, 175] considers classical mechanism design to develop incentive-compatible, cheat-proof mechanisms to enforce node collaboration for routing in ad hoc networks. The work in [176] focuses on multicast routing. The works in [177, 178, 179] form a small sample of references where **pricing** is used to enforce desirable behavior, although there are many more we do not cite.

Section 6.3.3 already mentions some of the work that uses past behavior of users to ensure cooperation. Some other work employing **reputation mechanisms** includes [164, 180, 181, 182].

6.4 Auction-based spectrum sharing

In this section we consider work implementing mechanism design, in particular various types of auctions, in a wireless communication setting. We focus in particular on cognitive radio and bandwidth/spectrum allocation. These are similar problems in the sense that most of the work in cognitive radio resource allocation is concerned with spectrum allocation. However, some cognitive radio situations contain specific constraints, such as a limited interference at a measurement point, so we treat the two separately.

6.4.1 Cognitive radio

The setting where N spread spectrum cognitive-radio transmit-receiver pairs (users) interfering with each other is considered in [132]. Users are constrained by the interference temperature *at a single measurement point*, but not by their maximum power constraint, which is considered to be big enough. The utility of users is a function of the received SINR. The resource being sold is a portion of the interference temperature that a user can get at the measurement point. Users submit one-dimensional bids instead of the entire utility. Weighted proportional allocation rule is used, where a user's power is proportional to his bid. The auctioneer submits a reserve bid in order to guarantee a unique outcome. Two auction mechanisms are considered.

- Users are charged for received SINR. When utility functions are logarithmic, this implies weighted max-min fairness.

- Users are charged for power, which maximizes total utility *when the bandwidth is large enough* and the receivers are collocated (effectively turning the setting from the interference channel to a multiple access channel).

Both auctions are socially optimal in the limit as $N \to \infty$. In addition, an iterative, distributed algorithm that leads to the Nash equilibrium of the auction is provided.

An integrated pricing, allocation, and billing system for cognitive radio was proposed in [183]. In this system, the problem of pricing negotiation between the operator and the service users was formulated as a multi-unit sealed-bid auction. A real-time spectrum auction framework with interference constraints was proposed in [184] to get a conflict-free allocation. This work considers revenue maximization under interference constraints. A belief-assisted distributive pricing algorithm based on double auction mechanisms was proposed in [185] to achieve efficient dynamic spectrum allocation. In a similar setting,

[186] shows that the pricing game is resistant to users forming coalitions (i.e. it is collusion resistant).

Collusion resistant auctions in a cognitive framework for multiple winners were considered in [187, 188]. Here, the set of winners is determined by a binary linear programming problem which guarantees spectrum efficiency, and the pricing strategy is modeled as a convex optimization problem with constraints precluding user collusion. In addition to preventing user collusion, the proposed strategies improve the primary users revenue.

In [189], the authors investigate two game-theoretical mechanism design methods to suppress cheating and collusion behavior of selfish users: a self-enforcing truth-telling mechanism for unlicensed spectrum sharing, and a collusion-resistant multi-stage dynamic spectrum pricing game for licensed spectrum sharing. For the unlicensed spectrum case (spectrum sharing), three different games are treated: single shot game, repeated game (where Folk theorem leads to an efficient solution), and a mechanism that achieves the Bayesian Nash equilibrium. The utility of the users is a logarithmic function of the SINR, and users interfere with each other. On the other hand, for the licensed spectrum case (cognitive radio), primary users pay some fee to get access to the spectrum, and then sell the unused part of the spectrum to the secondary users. The selling is performed through a collusion resistant double auction mechanism.

The work in [190] considers users who bid in sequential time-slots, and at the end of the session the bids are accumulated and the user who values the resources the most gets them. The efficiency is similar to that obtained using first-price or second-price auctions.

6.4.2 Traditional spectrum allocation

The work in [58] considers a three-tier system where a spectrum server is auctioning off spectrum to the spectrum providers, who are then charging it to the end-users. The competition between the providers is modeled as a game, and an iterative bidding scheme is first proposed for a single customer, that results in a Nash equilibrium of the game. The work is then extended to the case where the providers are competing for several customers, although they still compete for one customer at a time. It is shown that a spectrum policy server scheme improves the bandwidth utilization compared to the scheme where bandwidth is equally shared between the users.

In [131], multiple receivers are bidding at each time slot for the sole use of a channel to the single transmitter. Then, the highest bidder wins the auction, and pays the second highest bid. Throughput is a linear function of channel state, and the channel state comes from some known distribution. A user gets throughput equal to its channel gain if he wins the auction, otherwise he gets zero and spends zero. A receiver knows his own realization of the random variable, but not those of other receivers. The problem gets abstracted to one where a player is maximizing the expected average of a random variable over an average budget constraint. The strategy of a user is a function that maps a realization of the channel gain r.v. to the value of the bid.

The service providers compete for spectrum using a winner determining sealed bid knapsack auction in [56, 191, 192]. The users then choose the provider that offers the best payoff, where the payoff of a user is the familiar logarithmic function of their rate reduced by the payment, minus a congestion-based term (a function of the rate of all users of a single provider) that accounts for the queueing delay: $a_{ij} \log(1+q_{ij}) - p_j q_{ij} - \xi(\boldsymbol{q}_j)$. Users have diverse demand, and are heterogenous towards various providers. Providers have orthogonal resources, although they compete to obtain them initially from the spectrum dealer. This work is treating a partial information case, since the providers need to know the users utility functions, as well as some of the parameters.

Selling spectrum (bandwidth or power) to competing transmitters via a sequential auction is considered in [193]. The second-price auction is no longer optimal when the resource is sold in repeater iterations. For two users, the worst-case efficiency losses of the unique auction outcome are specified, although simulations indicate that the outcome is usually efficient. For more than two transmitters there are typically multiple equilibria, and the efficiency decreases with the number of users.

Revenue generation for spectrum auction implementable in truthful strategies is studied for a Bayesian setting in [194]. The optimal incentive compatible auction uses a VCG mechanism to maximize expected revenue, assuming that the distribution of user valuation is known. A truthful suboptimal auction is then developed to reduce complexity.

A market-based framework for decentralized radio resource management in environments populated by multiple, possibly heterogeneous, access points (APs) is treated in [195]. The provided service for the users consists of file transfers. Resources (transmission time) are partitioned between users through a proportionally fair divisible auction. The user (trade-agent), then needs to determine how much resources he should purchase from the different APs in order to maximize his utility.

Bandwidth allocation under budget constraints is studied in [196]. Wireless users demand bandwidth from the base stations by submitting bids. Users are price anticipating, and split their wealth across different base-stations to maximize their payoff. Once the bidding process is complete, a base station distributes its bandwidth to the users in proportion to their bids. The properties of Nash Equilibria for this bandwidth allocation game are studied. For the special case where each user can access all base stations, there exists a unique NE at which the bandwidth obtained by any user is proportional to its wealth.

The case where multiple spectrum sellers are designing an auction in the presence of multiple buyers is treated in [197]. The existence of a symmetric mixed strategy Nash equilibrium is shown. It is proven that the sellers have an incentive to form coalitions in order to maximize profits. The interaction between sellers is modeled using cooperative game theory. Here, the core of the game is non-empty, meaning that there is a way for the sellers to share profits in a coalition such that no seller has an incentive to deviate.

6.4.3 Multimedia

In [133], mini auctions in 2.5/3G networks are performed in each time-slot, while users' utility is a function of utility obtained over longer time periods, modeling multimedia service. The utility is a linear function of the utilities obtained in individual slots. The mini-auctions are all VCG auctions so bids are considered to be truthful.

Dynamic bidding in cognitive radio scenario for delay sensitive applications, such as multimedia, is considered in [198]. There is one auctioneer implementing a utility maximizing rule, for users who bid for resources iteratively. The modeling approach considers two different types of learning: reinforcement learning and best response learning.

Auction-based resource allocation for video transmission is treated in [199], while multimedia streaming over cognitive radio networks is considered in [200].

6.4.4 Ad-hoc and relay networks

In [201], the authors adopt the pay for service model of co-operation, and propose an auction-based incentive scheme to enable cooperative packet forwarding behavior in mobile ad-hoc networks. Each flow pays the market price of packet forwarding service to the intermediate routers. The resource allocation mechanism in the scheme is based on the generalized Vickrey auction with reserve pricing.

A more recent example of auctions in ad-hoc networks is [202], which focuses on a situation where a relay is available to a number of transmitter-receiver pairs. The work investigates when relaying is beneficial, how to choose a relay, and how much of the traffic the relay should carry. A single relay case is considered first. Similar to [132], there are two auctions (again SINR and power). Whether relaying is beneficial is decided by a threshold policy, while which relay to choose and how much to get depends on a weighted allocation policy. The multiple relay case is decoupled into multiple single relay cases that are treated separately. Finally, an iterative algorithm that leads to the Nash equilibrium is described.

6.4.5 Various

The authors of [203] apply distributed auction theory in order to improve the achievable secrecy rate in a wireless setting.

6.5 Competition of service providers for wireless users

6.5.1 Price competition

One of the main subjects treated in this thesis is the competition of service providers for wireless users (Chapters 4 and 5). Surprisingly, there is relatively little work done so far in this field. We summarize it in this section.

One of the first work to treat the competition on service providers for wireless users is [58], which we already talked about in Section 6.4. There, two providers who purchase bandwidth from a spectrum server, offer it to a customer at a certain cost. The customer then may accept the offer with certain probability, where the probability is higher if the conditions are more favorable. The goal of each provider is to maximize his profit, while the user has a sigmoidal utility function of the rate. The Nash equilibrium is found, and then the analysis is extended to multiple customers, where the customers are treated one by one. There is no competition of users for the spectrum.

One of the first works to treat the competition of providers for heterogeneous users is [60]. There, two providers are in a price war for the users on a line between them. Users choose a provider if some function of price and distance is larger for that provider than for the other. Users are homogenous in their willingness to pay, but heterogenous in their channels (which are distance based).

Competition of two providers for heterogeneous users is considered in [57]. Similar to [58], the service providers obtain their bandwidth from the spectrum holder, where the price of the bandwidth is given by a convex function, but this time the competition of users is considered as well. The end-users are modeled with quadratic utility functions. It is proved that a unique Nash equilibrium exists for a two-stage game of complete information. For the case with incomplete information, price adjustment algorithms are proposed that converge close to the NE from the static case.

Multiple self-interested providers compete for users in [204] as well. The providers have different technologies and costs, and users are heterogenous, in the sense that they can be price sensitive, quality sensitive, or anywhere in between. It turns out that providers enter a price war when they are catering to price-sensitive buyers. The effect of the spectrum cost on the profit is characterized. Under limited information, a stochastic learning algorithm is proposed that allows providers to converge to the optimal price.

Another work that considers three-layered architecture is [56], which considers separately the interaction between the spectrum providers and the spectrum broker, and the spectrum providers and the end-users.

Recently, the competition of providers for heterogenous users in a three-layer architecture was treated in [59], quite similar to our model in in Chapter 4. Users decide to purchase from a network based on the ratio of price and spectrum efficiency (which is called channel offset parameter in Chapter 4). The main difference between this work and our work is that the amount of resource in [59] is flexible, where the service provider obtains the spectrum dynamically from the spectrum dealer. On the other hand, the provider competition is analyzed only for two providers, and for users with a specific utility function.

Competition of heterogenous providers for heterogenous users with quadratic utility functions is considered in [205]. The providers are assumed to support two types of connections: premium and best effort. The premium connection has a standard price while the best-effort is dynamic and depends on the behavior of other providers. The users are considered to have aggregate demand for each provider. A simultaneous move game where all providers propose prices at the same time is considered first and the Nash equilibrium is found for this case. The situation where one of the providers is a Stackelberg leader and the others are followers is investigated next, and a Stackelberg equilibrium is found. Finally, coalition forming of providers for maximized profit is treated using cooperative game theory.

The following work is somewhat orthogonal to the work considering three-layer architecture. The competition is taking place in a shared medium, and there is no spectrum to be sold.

In [206], providers enter price competition for wireless users who are operating in a congestion limited scenario (e.g., 802.11) where adding more and more users leads to a graceful performance degradation. Users are heterogenous in willingness to pay since utility function is $a_i \log(1 + r_i)$, where r_i is the throughput. The interaction is modeled by a stochastic evolutionary game.

The work in [63] also considers a duopoly price competition for users. It is assumed that the number of users is large, so they are represented by an aggregate demand function. As such, all users go for the cheaper provider, and the only way to have non-zero demand for both providers is to have equal prices. The presented model can be applied in wireline as well as wireless situations. Users are using a common spectrum for their packets, so if the link capacity is exceeded, some packets are lost. In this case, the users pay a price per sent packet, while they are interested in received packets, leading to the concept of perceived price, which is also present in [59].

A similar duopoly price competition is treated in [207]. Each provider has its own (fixed) number of degrees of freedom to sell, but there is also a fixed amount of shared spectrum that they compete for. There is no cost to using the shared spectrum, the only difference is that a provider cannot be certain about how much of it it can use since the other provider is also there. The users are infinitesimal, and their aggregate demand function is considered. Users are not interfering, but when the demand is larger than the supply, it is considered that only some packets (proportional to the ration of supply and demand) go through. Hence, even though there is no interference, users are also not completely orthogonal. Users are charged for sent packets, so the perceived price, i.e., the price per successfully received packet, increases during congestion period, which gives an incentive to send packets less frequently.

Finally, we mention that the thesis work of [208], [209] and [210] contains a lot of insight on the issue of revenue maximization of competing wireless providers, each from a different point of view.

6.5.2 Competition of service providers on behalf of users

One of the rare examples of work where service providers compete on behalf of users who belong to their network is [54], where base-stations adjust access probabilities so that their users get better throughput.

6.5.3 Diverse service provider competition references

An example of work on competition of providers that does not include any profit maximization is given by [211, 212], where cellular providers compete by adjusting the power of the pilot signal in order to attract more customers.

The work of [213] investigates admission schemes in a CDMA setting using game theory. The providers are not competing against each other, but against the users who may or may not join them (or decide to quit the current provider). The providers want to stop users from leaving, but they also need to protect the quality of service for the existing users. The users are heterogenous in their channel gain, leading to different probabilities for packets to be dropped. Users are also heterogenous in their willingness to pay, where they are assumed to be in K service classes. The provider's strategy is not the price, which is fixed, but the choice of an admission policy.

6.6 Various

Recently, a tutorial paper on the use of coalitional game theory, a sub-category of cooperative game theory, applied to wireless networks was written by Saad et al [214]. In addition to the coalitional game theory already mentioned in this chapter, the authors cite the multitude of their recent work in the context of:

- unmanned aerial vehicles [215]
- distributed algorithm for fair cooperation in wireless networks [216]

- distributed coalition formation in wireless networks [217]
- collaborative spectrum sensing in cognitive radio networks [218]
- physical layer security [219]
- voice over IP services in 802.16j networks [220, 221].

Wireless social community networks were considered as a replacement to the traditional content delivery networks in [222].

The work in [223] explains the problems in applying game theory to a practical communication system. The analysis can be useful to anyone who considers such an application of game theory.

Conclusions

The focus of this thesis is the decision-making of individuals in wireless communication settings, in situations where the decisions of one individual influence the choices available to others. We focused on two scenarios: the independent interaction of users in a Gaussian multiple access channel, and the price competition of service providers for wireless users. From our study we were able to draw mixed conclusions. On one hand, the independent interaction of users leads to inefficient outcomes, as users do not coordinate their actions. On the other, the independent behavior of competing providers and users leads to a socially optimal equilibrium outcome. Here we briefly discuss these conclusions and point to some open questions.

User interaction in Gaussian MAC

As we saw in Chapter 2, independent decision-making by users in the Gaussian MAC results in inefficiencies due to the multiplicity of desirable outcomes. This is in contrast to the inefficiencies of the type present in the prisoner's dilemma, which arise due to selfish behavior of individuals. Lack of coordination is easier to overcome, for example the use of a correlating device results in an efficient equilibrium for the Gaussian MAC game. From this we understand that the use of communication resources cannot always be left solely up to the users, and some central intervention is required. In this case, the central intervention is quite mild. It is provided in the form of a coordinating schedule that the users are invited (but not obliged) to follow. Such a solution can be considered easy to implement, as it is not imposed to the users. Even though the users are not forced to follow the schedule, it is certainly in their best interest to do so.

In Chapter 3, we consider again users in the Gaussian MAC, this time in the presence of a benevolent central authority. We assume that the users can transmit their preferences to this central authority, who then makes a decision that maximizes the overall satisfaction in the system. The solution is obtained through the famous VCG mechanism, which requires that users pay a certain amount of money to ensure the truthful revealing of preferences. Such an approach has numerous caveats. A VCG mechanism requires that each user reveals his entire utility function to the central planner, which is not feasible in practice (even if we assume that users know their own utility functions). We manage to circumvent this problem by assuming that users' utility functions need to conform to a parametrized logarithmic function. Hence, users can simply transmit a vector of parameters which uniquely defines their preferences. Even then, the central authority needs to solve $I + 1$ maximization problems[2], which need not be trivial. The issue of finding a benevolent central planner also remains. Finally, it is not clear that wireless users would like to participate in any scheme where the cost is unknown to them beforehand. This problem is also present in the analysis of the provider competition.

Competition of wireless service providers

One of the major positive results of our analysis is that the competitive user-provider interaction can lead to socially optimal resource allocation. The equilibrium of this interaction presumes complete knowledge on behalf of all participants, which is typically a problem as such knowledge is rarely available. For example, we cannot assume that each user knows the utility functions of all other users. For this reason, an equilibrium in such games is considered stable if it can be achieved, but the question of how to reach that equilibrium remains open. To this end, we provide a decentralized algorithm that converges to the equilibrium of the user-provider interaction, using only local and private knowledge.

One of the novelties of our work is that the resource providers are pricing the commodity that limited their operation, and not the commodity that the users are interested in. In wireless communications, service providers are typically limited by their transmit power, bandwidth, number of degrees of freedom in the system, etc., while the users are usually interested in their communication rate. Users typically pay for access to the wireless service with certain rate guarantees, where the payment does not depend on the amount of network resources required to support their connection. Hence, one barrier to introducing socially optimal pricing is that users are not used to it. Any such pricing would be likely seen as too complex for most users, who instead prefer simple flat-rate fees, or per minute/per gigabyte charges.

Another issue is that the socially optimal pricing is highly variable. Even in a slowly varying environment, over time some users would disconnect, while new users would connect. We can imagine that a new set of prices would have to be computed every time there is a change in the number of users or in the preferences of some users. The issue is then how users would react to changing prices. Similar to the problem of linear pricing, users tend to prefer pricing schemes where one does not have to worry about consumption all the time.

Open questions and future work

The applications of game theory and microeconomics in communications are numerous, as witnessed by the extraordinary number of publications on this subject in the past two decades. One of the challenges in applying game theory is, perhaps contradictorily, its great variety. The “standard” procedure is the following: a communication scenario is considered, together with some assumptions on the behavior and information available to the participants. Then an appropriate game-theoretic tool is invoked to handle the underlying assumption. A change in one assumption often means that a different game-theoretic tool needs to be employed, which often yields significantly different results. For example, analyzing the behavior of users in a Gaussian MAC yields different results based

[2]One for each user’s VCG tax, and one for the network utility maximization.

on whether we assume that users make independent decisions with complete information (non-sooperative game theory), whether they can make off-line agreements (coalitional game theory), or they interact through a central planner (mechanism design). Hence, it is unlikely that a single wireless scenario yields a unique outcome in terms of user behavior, as the behavior of users is ultimately defined by the underlying game-theoretic tools and assumptions.

On the other hand, sometimes seemingly different communication scenarios are analyzed using similar game-theoretic tools, and similar results arise. It would be interesting to see if further work in this field can lead to unification of such results. One such example is [81], which considers coalitional game theory for several different information-theoretic situations. In this thesis, we consider the competition of service providers and users which we modeled as a two-stage multi-leader-follower game. Can other wireless settings be modeled using the same game, reaching the same conclusions? Our impression is that the answer to this question should be yes. For example, the model and most of the reasoning from Chapter 4 can be extended to some multipath routing and MIMO power control scenarios. This is one of the potential directions for future research.

In addition, we feel that there is much more work to be done to model the interaction of service-providers, end-users, and possibly spectrum brokers. Even after taking into account continuing deregulation trend, the future wireless users will likely have access to the wireline backbone through some entity (such as a base-station or a service provider), whose service is going to be priced in some way. This may be true even for public networks, as completely unregulated use often results in the tragedy of the commons (as witnessed by most references treated in Section 6.2.1). Pricing schemes that are used in practice are often very different from those treated in theory. One direction of future research is to form an understanding of how the two compare.

Ultimately, the user decision-making models that are used in theory should be tested-out in practice. Of course, this may not be feasible for many situations, but any such experiment would be greatly beneficial for the understanding of the behavior of users. People do not always make rational decisions, and do not always have access to full information. How do people react to novel pricing schemes? What kind of decisions do they make? What is the optimal mechanism for allocating resources in this case? Hopefully some of these questions will be answered in the future.

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