FROM OPEN QUANTUM SYSTEMS TO OPEN QUANTUM MAPS

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1. Introduction and statement of the results

In this paper we show that for a class of open quantum systems satisfying a natural dynamical assumption (see §1.2) the study of the resolvent, and hence of scattering, and of resonances, can be reduced to the study of open quantum maps, that is of finite dimensional quantizations of canonical relations obtained by truncation of symplectomorphisms.

We first state the main result in a simplified setting. For that let

$$P = -h^2\Delta + V(x) - 1\,, \quad V \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)\,,$$

and let Φ^t be the corresponding classical flow on $T^*\mathbb{R}^n \ni (x,\xi)$:

$$\Phi^t(x,\xi) \stackrel{\rm def}{=} (x(t),\xi(t))\,,$$
$$x'(t) = 2\xi(t)\,, \quad \xi'(t) = -dV(x(t))\,, \quad x(0)=x\,, \quad \xi(0)=\xi\,.$$

This flow is generated by the classical Hamiltonian $p(x,\xi) = |\xi|^2 + V(x) - 1$. We asssume that $dp\restriction_{p^{-1}(0)} \neq 0$ and define the trapped set at energy 0 as

$$K_0 \stackrel{\rm def}{=} \{(x,\xi) \;:\; p(x,\xi)=0\,,\ \Phi^t(x,\xi) \text{ remains bounded for all } t\in\mathbb{R}\}\,. \tag{1.1}$$

The resolvent of P, $R(z) = (P-z)^{-1}$, continues meromorphically from $\operatorname{Im} z > 0$ to the disk $D(0,1)$, in the sense that $\chi R(z)\chi$, $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$, is a meromorphic family of operators, with poles independent of the choice of $\chi \not\equiv 0$.

Theorem 1. *Suppose that Φ^t is hyperbolic on K_0 and that K_0 is topologically one dimensional. Then for $\delta > 0$ small enough, there exists a family of matrices, $M(z,h)$, holomorphic in $z \in \mathcal{R}(\delta, Ch) \stackrel{\rm def}{=} [-\delta,\delta] + i[-Ch,Ch]$, and of rank comparable to h^{-n+1} such that the zeros of*

$$\zeta(z,h) \stackrel{\rm def}{=} \det(I - M(z,h))\,,$$

give the resonances of P in $\mathcal{R}(\delta, Ch)$ (with correct multiplicities). Moreover, $M(z,h)$ are open quantum maps in the sense that there exist projections Π_h of rank comparable to h^{-n+1}, and an h-Fourier integral operator, $\mathcal{M}(z,h)$, quantizing a certain Poincaré map (see §1.3.2), such that

$$M(z,h) = \Pi_h \mathcal{M}(z,h) \Pi_h + \mathcal{O}(h^L)\,, \tag{1.2}$$

where $M(z,h)$ can be constructed so that L is arbitrarily large.

The multiplicity of resonances away from 0 can be defined using the cutoff resolvent: for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\chi = 1$ in $B(0,R)$.

$$m_R(z) = \operatorname{rank} \oint_z \chi R(w) \chi \, dw \,, \quad z \in \mathcal{R}(\delta, Ch) \,,$$

where the integral is over a small positively oriented circle around z. The theorem then says that

$$\begin{aligned} m_R(z) &= \frac{1}{2\pi i} \oint_z \frac{\zeta'(w)}{\zeta(w)} dw \\ &= \frac{1}{2\pi i} \operatorname{tr} \oint_z (I - M(w)) M'(w) dw \,. \end{aligned} \tag{1.3}$$

A yet more precise global version, involving complex scaling and microlocally deformed spaces (see §2.4 and §2.5 respectively), will be given in Theorem 2 in §4.4. In particular Theorem 2 gives us a full control over both the cutoff resolvent of P, $\chi R(z) \chi$, and the full resolvent $(P_\theta - z)^{-1}$ of the complex scaled operator P_θ.

The mathematical applications of Theorem 1 and its refined version below include simpler proofs of fractal Weyl laws [38] and of the existence of resonance free strips [28]. The advantage lies in eliminating flows and reducing the dynamical analysis to that of maps. That provides an implicit second microlocalization without any technical complication (see [38, §5]). The key is a detailed understanding of $\mathcal{M}(z,h)$ in the statement of the theorem.

Relation to semiclassical trace formulæ. The notation $\zeta(z,h)$ in the above theorem hints at the resemblance between this determinant and a *semiclassical zeta function.* Various such functions have been introduced in the physics literature, to provide approximate ways of computing eigenvalues and resonances of quantum chaotic systems — see [42, 19, 10].

These semiclassical zeta functions are defined through formal manipulations starting from the Gutzwiller trace formula — see [37] for a mathematical treatment of the trace formula, and references therein. Zeta functions are given by sums, or Euler products, over periodic orbits, where each term, or factor is an asymptotic series in powers of h. Most studies have concentrated on the zeta function defined by the principal term, without h-corrections, which strongly resembles the Selberg zeta function defined for surfaces of constant negative curvature. However, unlike the case of the Selberg zeta function, there is no known rigorous connection between the zeroes of the semiclassical zeta function and the exact eigenvalues or resonances of the quantum system, even in the semiclassical limit. Nevertheless, numerical studies have indicated that the semiclassical zeta function admits a strip of holomorphy beyond the axis of absolute convergence, and that its zeroes there are close to actual resonances [10, 43].

The traces of $M(z,h)^k$, $k \in \mathbb{N}$ admit semiclassical expressions as sums over periodic points, which leads to a *formal* representation

$$\zeta(z,h) = \exp\Big\{ -\sum_{k=1}^{\infty} \frac{\operatorname{tr} M(z,h)^k}{k} \Big\}$$

as a product over periodic points. That gives it the same form as the semiclassical zeta functions in the physics literature. In this sense, the function $\zeta(z,h)$ is a resummation of these formal expressions. As will become clear from its construction below, the operator $M(z,h)$ is not unique: it depends on many arbitrary choices. However, the zeroes of $\zeta(z,h)$ in $\mathcal{R}(\delta, Ch)$ are the resonances of the quantum problem.

Comments on quantum maps in the physics literature. Similar methods of analysis have been introduced in the theoretical physics literature devoted to quantum chaos. The classical case involves a reduction to the boundary for obstacle problems: when the obstacle consists of several strictly convex bodies, none of which intersects a convex hull of any other two bodies, the flow is hyperbolic. The reduction can then be made to boundaries of the convex bodies, resulting with operators quantization Poincaré maps – see Gaspard and Rice [16], and for a mathematical treatment Gérard [17], in the case of two convex bodies, and [29, §5.1], for the general case. Fig.1 illustrates the trapped set in the case of three discs. The semiclassical analogue of the two convex obstacle, a system with one closed hyperbolic orbit, was treated by Gérard and the second author in [18]. The approach of that paper was also based on the quantization of the Poincaré map.

A reduction of a more complicated quantum system to a quantized Poincaré map was proposed in the physics literature by Bogomolny [4]. He studied a Schrödinger operator $P(h)$ with discrete spectrum, and constructed a family of energy dependent quantum transfer operators $T(E,h)$, which are integral operators acting on a hypersurface in the configuration space. These transfer operators are asymptotically unitary as $h \to 0$. The eigenvalues of $P(h)$ are then obtained, in the semiclassical limit, as the roots of the equation $\det(1 - T(E)) = 0$. Smilansky and co-workers derived a similar equation in the case of closed Euclidean 2-dimensional billiards [13], replacing $T(E)$ by a (unitary) scattering matrix $S(E)$ associated with the dual scattering problem. Bogomolny's method was also extended to study quantum scattering situations [15, 30].

Quantum open maps have first been defined in the quantum chaos literature as toy models for open quantized chaotic systems, independently of any Hamiltonian flow (see [26, §2.2], [27, §4.3] and references given there). They generalized the unitary quantum maps used to mimic bound chaotic systems [11]. Some examples of open quantum maps on the 2-dimensional torus or the cylinder, have been used as models in various physical setings: Chirikov's quantum standard map (or quantum kicked rotator) was first defined in the context of plasma physics, but then used as well to study ionization of atoms or molecules [9], as well as transport properties in mesoscopic quantum dots [41]. Other maps, like the open baker's map, were introduced as clean model systems, for which the classical

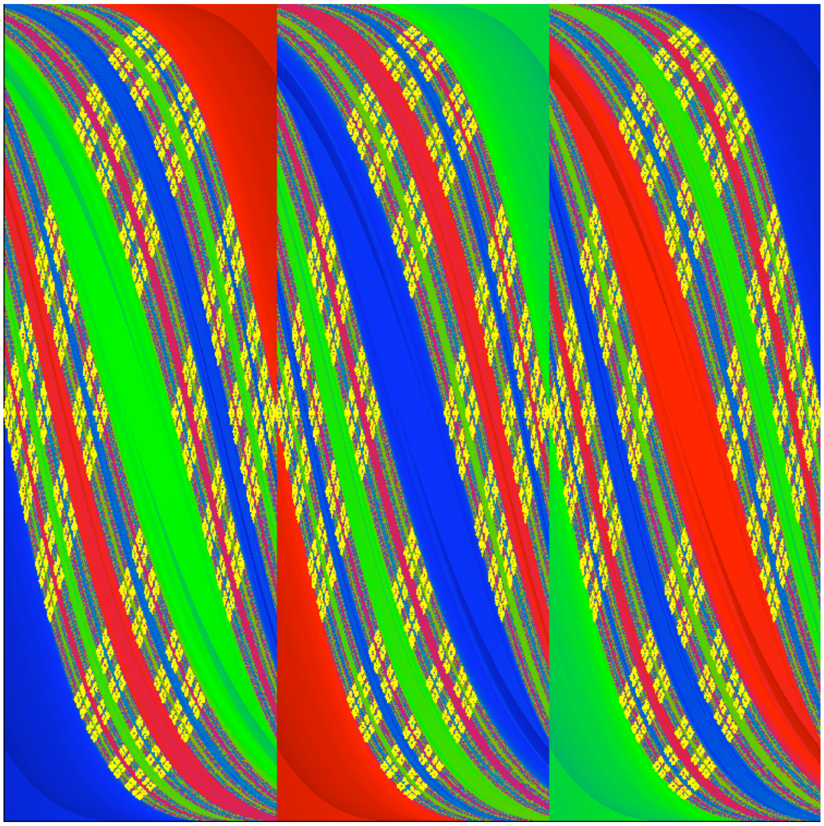

FIGURE 1. This figure, taken from [31], shows the Poincaré section for the symmetric three disc scattering problem. The section is the union of the three coball bundles of the circles parametrized by s (the length parameter on the circle, horizontal axis), and $\cos\varphi$, where φ is the angle between the velocity after impact and the tangent to the circle. Green, blue,red strips correspond to different regions of forward escape; they are bounded by components of the stable manifold. The trapped set, $\mathcal{T}$, shown in yellow, is the intersection of the latter with the unstable manifold.

dynamics is well understood [32, 27]. The popularity of quantum maps mostly stems from the much simplified numerical study they offer, both at the quantum and classical levels, compared with the case of Hamiltonian flows or the corresponding Schrödinger operators. For instance, the distribution of resonances and resonant modes has proven to be much easier to study numerically for open quantum maps, than for realistic flows [7, 33, 26, 23, 25]. Precise mathematical definitions of quantum maps are given in [26, §4.3-4.5].

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1.1. **Assumptions on the operator.** The results apply to operator satisfying general assumptions given in [28, §3] and [38, (1.5),(1.6)]. To simplify the presentation we will consider differential operators on $X = \mathbb{R}^n$ only, stressing that the results apply to operators on manifolds X, of the form,

$$X = K_R \sqcup \bigsqcup_{j=1}^{J} \left(\mathbb{R}^n \setminus \overline{B_{\mathbb{R}^n}(0, R)} \right) ,$$

where $R > 0$ is large and K_R is a compact subset of X. In the following we will use both notations $\mathbb{R}^n \equiv X$.

We assume that

$$P(h) = \sum_{|\alpha| \leq 2} a_\alpha(x, h)(hD_x)^\alpha , \tag{1.4}$$

where $a_\alpha(x, h)$ are bounded in $\mathcal{C}^\infty(\mathbb{R}^n)$, $a_\alpha(x, h) = a^0_\alpha(x) + \mathcal{O}(h)$ in $\mathcal{C}^\infty$, and $a_\alpha(x, h) = a_\alpha(x)$ is independent of h for $|\alpha| = 2$. Furthermore, for some $C_0 > 0$ the functions $a_\alpha(x, h)$ have holomorphic extensions to

$$\{x \in \mathbb{C}^n \ : \ |\operatorname{Re} z| > C_0 \, , \quad |\operatorname{Im} z| < |\operatorname{Re} z|/C_0\} \, , \tag{1.5}$$

they are bounded uniformly with respect to h, and $a_\alpha(x, h) = a^0_\alpha(x) + \mathcal{O}(h)$ on that set.

Let $P(x, \xi)$ denote the (full) Weyl symbol of the operator P, so that $P = P^w(x; hD; h)$ and assume

$$P(x, \xi; h) \to \xi^2 - 1 \tag{1.6}$$

when $x \to \infty$ in the set (1.5), uniformly with respect to $(\xi, h) \in K \times]0, 1]$ for any compact set $K \Subset \mathbb{R}^n$. We also assume that P is classically elliptic:

$$p_2(x, \xi) \stackrel{\text{def}}{=} \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \neq 0 \text{ on } T^*\mathbb{R}^n \setminus \{0\}, \tag{1.7}$$

and that P is self-adjoint on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$.

1.2. **Dynamical Assumptions.** The dynamical assumptions we need roughly mean that the flow Φ^t on the energy shell $p^{-1}(0) \subset T^*X$ can be encoded by a Poincaré section, the boundary of which does not intersect the trapped set K_0.

More precisely, we notice that

$$p(x, \xi) = \sum_{|\alpha| \leq 2} a^0_\alpha(x)\xi^\alpha \tag{1.8}$$

is the semi-classical principal symbol of the operator $P(x, hD; h)$, and let

$$H_p \stackrel{\text{def}}{=} \sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial}{\partial \xi_j} \,,$$

be the Hamilton vector field of p. Assume that the characteristic set of p is a simple hypersurface,

$$dp \neq 0 \text{ on } p^{-1}(0). \tag{1.9}$$

We will denote by

$$\Phi^t \stackrel{\text{def}}{=} \exp(tH_p) : T^*X \to T^*X$$

the flow generated by the Hamilton vector field H_p.

For E close to 0 (so that $dp\restriction_{p^{-1}(E)} \neq 0$) we define the trapped set K_E by

$$K_E \stackrel{\text{def}}{=} \{\rho \in p^{-1}(E);\ \Phi^{\mathbb{R}}(\rho) \text{ is bounded } \}. \tag{1.10}$$

We now assume that there exist a "nice" Poincaré section, namely finitely many compact contractible smooth hypersurfaces $\Sigma_k \subset p^{-1}(0)$, $k = 1, 2, \ldots, N$ with smooth boundaries, such that

$$\partial\Sigma_k \cap K_0 = \emptyset\,, \quad \Sigma_k \cap \Sigma_{k'} = \emptyset, \quad k \neq k', \tag{1.11}$$

$$H_p \text{ is transversal to } \Sigma_k \text{ uniformly up to the boundary,} \tag{1.12}$$

$$\begin{gathered} \text{For every } \rho \in K_0, \text{ there exist } \rho_- \in \Sigma_{j_-(\rho)}\,, \quad \rho_+ \in \Sigma_{j_+(\rho)} \\ \text{of the form } \rho_\pm = \Phi^{t_\pm(\rho)}(\rho), \text{ with } 0 < t_\pm(\rho) \leq t_{\max} < \infty, \text{ such that} \\ \{\Phi^t(\rho);\ -t_-(\rho) < t < t_+(\rho),\ t \neq 0\} \cap \Sigma_k = \emptyset\,, \quad \forall\, k\,. \end{gathered} \tag{1.13}$$

The functions $\rho \mapsto \rho_\pm(\rho)$, $\rho \mapsto t_\pm(\rho)$ are uniquely defined ($\rho_\pm(\rho)$ will be called respectively the successor and predecessor of ρ). They remain well-defined for ρ in some neighbourhood of K_0 in $p^{-1}(0)$) and, in such a neighbourhood, depend smoothly on ρ away from $\mathbf{\Sigma} \stackrel{\text{def}}{=} \sqcup_{k=1}^{N} \Sigma_k$. In order to simplify the presentation we also assume

$$\text{If } \rho \in \Sigma_k \cap K_0 \text{ for some } k, \text{ then } \rho_+(\rho) \in \Sigma_\ell \cap K_0 \text{ for some } \ell \neq k. \tag{1.14}$$

The section can always be enlarged to guarantee that this condition is satisfied. For instance, for K_0 consisting of one closed orbit we only need one transversal section to have (1.11)-(1.12). To fulfill (1.14) an additional transversal section has to be added.

We recall that that hypersurfaces in $p^{-1}(0)$ that are transversal to H_p are symplectic. In fact, a local application of Darboux's theorem (see for instance [22, §21.1]) shows that we can make a symplectic change of variables in which $p = \xi_n$ and $H_p = \partial_{x_n}$. If $\Sigma \subset \{\xi_n = 0\}$ is transversal to ∂_{x_n}, then $(x_1, \cdots x_{n-1}; \xi_1, \cdots, \xi_{n-1})$ can be chosen as coordinates on Σ. Since $\omega\restriction_{p^{-1}(0)} = \sum_{j=1}^{n-1} d\xi_j \wedge dx_j$, that means that $\omega\restriction_\Sigma$ is nondegenerate. The local normal form $p = \xi_n$ will be used further in the paper.

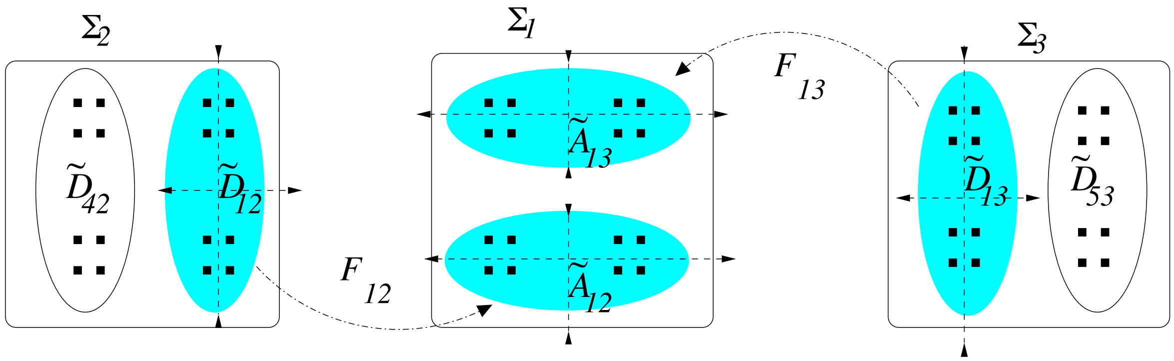

FIGURE 2. Schematic representation of the Poincaré return maps F_{ik} between the sets D_{ik} and A_{ik} (horizontal/vertical ellipses). The reduced trapped set $\mathcal{T}_i$ is represented by the black squares. The unstable/stable directions of the map are represented by the horizontal/vertical dashed lines.

The final assumption guarantees no topological or symplectic peculiarities:

$$\begin{gathered}\text{There exists a set } \widetilde{\Sigma}_k \Subset T^*\mathbb{R}^{n-1} \text{ with smooth boundary, and a symplectic} \\ \text{diffeomorphism } \kappa_k : \widetilde{\Sigma}_k \to \Sigma_k \text{ which is smooth up the boundary together} \\ \text{with its inverse. We assume that } \kappa_k \text{ extends to a neighbourhood of } \widetilde{\Sigma}_k \text{ in } T^*X.\end{gathered} \tag{1.15}$$

In other words, there exist symplectic coordinate charts on Σ_k, taking values in $\widetilde{\Sigma}_k$.

We recall the following result due to Bowen and Walters [8]:

Proposition 1.1. *Suppose that the assumptions of §1.1 hold, and that the flow* $\Phi^t\restriction_{K_0}$ *is uniformly hyperbolic in the standard sense of* [28, (3.11)]*. Then the existence of* $\mathbf{\Sigma} = \sqcup_{k=1}^N \Sigma_k$ *satisfying* (1.11)-(1.15) *is equivalent with* K_0 *being topologically one dimensional.*

In particular this shows that the assumptions of Theorem 1 imply the dynamical assumptions made in this section.

Remark. Bowen shows more, namely the fact that the sets $\{\Sigma_k \cap K_0\}$ can be chosen of small diameter, and constructed such as to form a *Markov partition*. We only need to ensure the properties (1.14) and (1.15) hold.

1.3. **The Poincaré map.** Here we will discuss the Poincaré map for the partition discussed in §1.2, and its semiclassical quantization.

1.3.1. *Classical analysis.* The assumptions in §1.2 imply the existence of an *open relation*, whose quantization is given by the operator $\mathcal{M}(z, h)$ introduced in the statement of Theorem 1.

More precisely, let us identify Σ_k's with $\widetilde{\Sigma}_k$ using κ_k given in (1.15). Then call

$$\mathbf{\Sigma} = \bigsqcup_{k=1}^{N} \Sigma_k \simeq \bigsqcup_{k=1}^{N} \widetilde{\Sigma}_k \subset \bigsqcup_{k=1}^{N} T^*\mathbb{R}^{n-1}$$

the full Poincaré section, and

$$\mathcal{T} \stackrel{\text{def}}{=} K_0 \cap \mathbf{\Sigma} = \bigsqcup_{k} \mathcal{T}_k \quad \text{the reduced trapped set.}$$

The map

$$f \ : \ \mathcal{T} \longrightarrow \mathcal{T} \,, \quad \rho \longmapsto f(\rho) \stackrel{\text{def}}{=} \rho_+(\rho)$$

(see the notation of (1.13)) is a Lipschitz bijection. The decomposition $\mathcal{T} = \bigsqcup_k \mathcal{T}_k$ allows us to define the *arrival* and *departure* subsets of $\mathcal{T}$:

$$\mathcal{D}_{ik} \stackrel{\text{def}}{=} \{\rho \in \mathcal{T}_k \subset \Sigma_k \ : \ \rho_+(\rho) \in \mathcal{T}_i\} = \mathcal{T}_k \cap f^{-1}(\mathcal{T}_i) \,,$$
$$\mathcal{A}_{ik} \stackrel{\text{def}}{=} \{\rho \in \mathcal{T}_i \subset \Sigma_i \ : \ \rho_-(\rho) \in \mathcal{T}_k\} = \mathcal{T}_i \cap f(\mathcal{T}_k) = f(\mathcal{D}_{ik}) \,,$$

For each k we call $J_+(k) \subset \{1, \ldots, N\}$ the set of indices i such that $\mathcal{D}_{ik}$ is not empty (that is, for which $\mathcal{T}_i$ is a successor of $\mathcal{T}_k$). Conversely, the set $J_-(i)$ refers to the predecessors of $\mathcal{T}_i$.

Using this notation, the map f obviously decomposes into a family of Lipschitz bijections $f_{ik} : \mathcal{D}_{ik} \to \mathcal{A}_{ik}$. Similarly to the maps $\rho_\pm$, each f_{ik} can be extended to a neighbourhood of $\mathcal{D}_{ik}$, to form a family of local smooth symplectomorphisms

$$F_{ik} \ : \ D_{ik} \longrightarrow \ F_{ik}(D_{ik}) \stackrel{\text{def}}{=} A_{ik} \,,$$

where D_{ik} (resp. A_{ik}) is a neighbourhood of $\mathcal{D}_{ik}$ in Σ_k (resp. a neighbourhood of $\mathcal{A}_{ik}$ in Σ_i). Since our assumption on K_0 is equivalent with the fact that the reduced trapped set $\mathcal{T}$ is *totally disconnected*, we may assume that the sets $\{D_{ik}\}_{i\in J_+(k)}$ (resp. the sets $\{A_{ik}\}_{k\in J_-(i)}$) are mutually disjoint. We will call

$$D_k \stackrel{\text{def}}{=} \sqcup_{i\in J_+(k)} D_{ik}, \qquad A_i \stackrel{\text{def}}{=} \sqcup_{k\in J_-(k)} A_{ik} \,.$$

Notice, however, that for any index i, the sets D_i, A_i both contain the reduced trapped set $\mathcal{T}_i$.

We will also define the *tubes* $T_{ik} \subset T^*X$ containing the trajectories between D_{ik} and A_{ik}:

$$T_{ik} \stackrel{\text{def}}{=} \{\Phi^t(\rho), : \ \rho \in D_{ik}, \ 0 \leq t \leq t_+(\rho)\} \,. \tag{1.16}$$

See Fig. 2 for a sketch of these definitions, and Fig. 3 for an artistic view of T_{ik} The maps F_{ik} will be grouped into the symplectic bijection F between $\bigsqcup_k D_k$ and $\bigsqcup_k A_k$. We will also call F the Poincaré map. We will sometimes identify the map F_{ik} with its action on subsets of $T^*\mathbb{R}^{n-1}$.

$$\tilde{F}_{ik} = \kappa_i^{-1} \circ F_{ik} \circ \kappa_k : \widetilde{D}_{ik} \longrightarrow \widetilde{A}_{ik} \,, \quad \widetilde{D}_{ik} \stackrel{\text{def}}{=} \kappa_k^{-1}(D_{ik}) \,, \quad \widetilde{A}_{ik} \stackrel{\text{def}}{=} \kappa_i^{-1}(A_{ik}) \,.$$

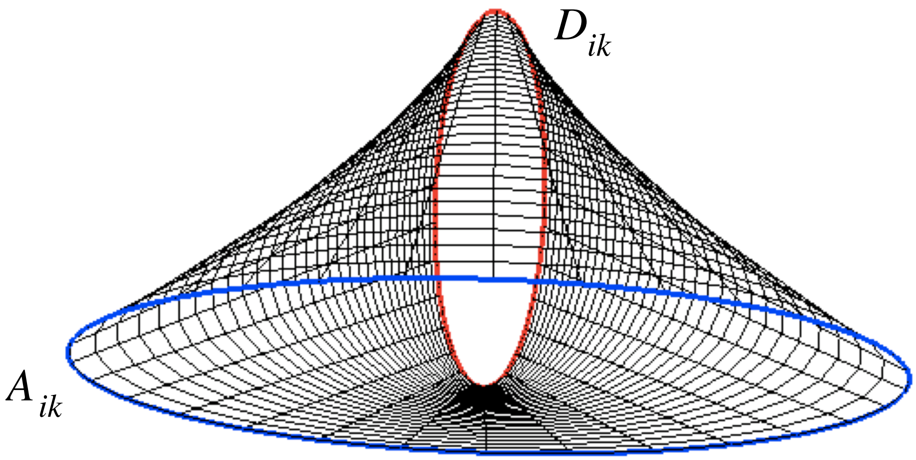

FIGURE 3. Trajectories linking the boundaries of the departure set $D_{ik} \subset \Sigma_k$ and the arrival set $A_{ik} \subset \Sigma_i$. Note the stretching and contraction implied by hyperbolicity. These trajectories and $D_{ik} \cup A_{ik}$ form the boundary of the tube T_{ik} defined by (1.16).

The above structure can be extended to a (small) energy interval $z \in [-\delta, \delta]$; the Poincaré maps for the flow in $p^{-1}(z)$ will be denoted by $F_z = (F_{ik,z})_{1 \le i,k \le N}$ (see §3.1.2 for details).

1.3.2. *Quantization of the Poincaré map.* Let us first focus on a single component $F_{ik} : \Sigma_k \supset D_{ik} \to A_{ik} \subset \Sigma_i$ of the Poincaré map. A quantization of F_{ik} (more precisely, of its pullback $\tilde{F}_{ik}$) is a semiclassical Fourier integral operator $\mathcal{M}_{ik} : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$, whose semiclassical wavefront set satisfies

$$\mathrm{WF}'_h(\mathcal{M}_{ik}) \Subset A_{ik} \times D_{ik} \,, \tag{1.17}$$

(WF'_h is defined in (2.8) below, and A_{ik}, D_{ik} are identified with their pullbacks on $T^*\mathbb{R}^{n-1}$ through (1.15)), and which is associated with the symplectic map F_{ik}, in the following sense: for any $a \in \mathcal{C}^\infty_{\mathrm{c}}(A_{ik})$ we have

$$\mathcal{M}_{ik}^* \mathrm{Op}_h^w(a) \mathcal{M}_{ik} = \mathrm{Op}_h^w(\alpha_{ik} F_{ik}^* a) + h\, \mathrm{Op}_h^w(b) \,, \tag{1.18}$$

where $\alpha_{ik} \in S_\delta$ is independent of a, $\alpha_{ik} = 1$ on some neighbourhood of $\mathcal{T}_k$ in Σ_k, and $b \in S_\delta$, for every $\delta > 0$. Here Op_h^w denotes the semiclassical Weyl quantization on $\mathbb{R}^{2(n-1)}$, and S_δ the symbol class defined in §2.1. The presence of δ in (1.18) comes from the slightly exotic nature of our Fourier integral operator, due to the presence of some mild exponential weights – see §2.5 below.

The property (1.18), which is a form of *Egorov's theorem*, characterizes $\mathcal{M}_{ik}$ as a semiclassical Fourier integral operator associated with F_{ik} (see [37, Lemma 2] and [14, §10.2] for that characterization):

We can then group together the $\mathcal{M}_{ik}$ into a single operator-valued matrix (setting $\mathcal{M}_{ik} = 0$ when $i \notin J_+(k)$):

$$\mathcal{M} \;:\; L^2(\mathbb{R}^{n-1})^N \longrightarrow L^2(\mathbb{R}^{n-1})^N\,, \quad \mathcal{M} = \big(\mathcal{M}_{ik}\big)_{1\leq i,k\leq N}\,.$$

We call this $\mathcal{M}$ a quantization of the Poincaré map F.

Comment on notation. Most of the estimates in this paper include error terms of the type $\mathcal{O}(h^\infty)$, which is natural in all microlocal statements. To simplify the notation we adopt the following convention (except in places where it could lead to confusion):

$$\begin{gathered} u \equiv v \iff \|u - v\| = \mathcal{O}(h^\infty)\|u\|\,, \\ \|Su\| \lesssim \|Tu\| + \|v\| \iff \|Su\| \leq \mathcal{O}(1)(\|Tu\| + \|v\|) + \mathcal{O}(h^\infty)\|u\|\,, \end{gathered} \tag{1.19}$$

with norms appropriate to context. Since most estimates involve functions u microlocalized to compact sets, in the sense that, $u - \chi(x, hD)u \in h^\infty\mathcal{S}(\mathbb{R}^n)$, for some $\chi \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n)$, the norms are almost exclusively L^2 norms, possibly with microlocal weights described in §2.5.

The notation $u = \mathcal{O}_V(f)$ means that $\|u\|_V = \mathcal{O}(f)$, and the notation $T = \mathcal{O}_{V\to W}(f)$ means that $\|Tu\|_W = \mathcal{O}(f)\|u\|_V$. Also, the notation

$$\mathrm{neigh}(A, B) \quad \text{for } A \subset B,$$

means an open neighbourhood of the set A inside the set B.

Starting with §3, we denote the Weyl quantization of a symbol a by the same letter $a = a^w(x, hD)$. This makes the notation less cumbersome and should be clear from the context.

Finally, we warn the reader that from §3 onwards the original operator P is replaced by the complex scaled operator $P_{\theta,R}$, whose construction is recalled in §2.4. Because of the formula (2.12), that does not affect the results formulated in this section.

2. Preliminaries

In this section we present background material and references needed for the proof of the theorems.

2.1. Semiclassical pseudodifferential calculus.

The class of symbols associated to order m is defined as

$$S^{m,k}_\delta(T^*\mathbb{R}^d) = \Big\{ a \in \mathcal{C}^\infty(T^*\mathbb{R}^d \times (0, 1]) : |\partial^\alpha_x\partial^\beta_\xi a(x, \xi; h)| \leq C_\alpha h^{-k-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{m-|\beta|} \Big\}\,.$$

Most of the time we will use the class with $\delta = 0$ in which case we drop the subscript. When $m = k = 0$, we simply write $S(T^*\mathbb{R}^d)$ or S for the class of symbols. In the paper $d = n$ (the dimension of the physical space) or $d = n-1$ (half the dimension of the Poincaré section), and occasionally (as in (1.17)) $d = 2n - 2$, depending on the context.

The quantization map, in its different notational guises, is defined as follows

$$
\begin{aligned}
a^w u &= \mathrm{Op}_h^w(a)u(x) = a^w(x, hD)u(x) \\
&\stackrel{\text{def}}{=} \frac{1}{(2\pi h)^d} \int\int a\big(\frac{x+y}{2}, \xi\big) e^{i\langle x-y,\xi\rangle/h} u(y) dy d\xi \,,
\end{aligned} \tag{2.1}
$$

and we refer to [12, Chapter 7] for a detailed discussion of semiclassical quantization (see also [35, Appendix]), and to [14, Appendix D.2] for the semiclassical calculus for the symbol classes given above).

We denote by $\Psi_\delta^{m,k}(\mathbb{R}^d)$ or $\Psi^{m,k}(\mathbb{R}^d)$ the corresponding classes of pseudodifferential operators, with surjective symbol maps:

$$
\sigma_h \;:\; \Psi_\delta^{m,k}(\mathbb{R}^d) \;\longrightarrow S_\delta^{m,k}(T^*\mathbb{R}^d)/S_\delta^{m-1,k-1+2\delta}(T^*\mathbb{R}^d)\,, \quad \sigma_h(A\circ B) = \sigma_h(A)\sigma_h(B)\,,
$$
$$
\sigma_h \circ \mathrm{Op}_h^w : S_\delta^{m,k}(T^*\mathbb{R}^d) \;\longrightarrow S_\delta^{m,k}(T^*\mathbb{R}^d)/S_\delta^{m-1,k-1+2\delta}(T^*\mathbb{R}^d)\,,
$$

where the last map is the natural projection.

The semiclassical Sobolev spaces, $H_h^s(\mathbb{R}^d)$ are defined using the semiclassical Fourier transform, $\mathcal{F}_h$:

$$
\|u\|_{H_h^s}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |\mathcal{F}_h u(\xi)|^2 d\xi\,, \quad \mathcal{F}_h u(\xi) \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^d} u(x) e^{-i\langle x,\xi\rangle/h} dx\,. \tag{2.2}
$$

Unless otherwise stated all norms in this paper, $\| \bullet \|$, are L^2 norms.

We recall that the operators in $\Psi(\mathbb{R}^d)$ are bounded on L^2 uniformly in h, and that they can be characterized using commutators by Beals's Lemma (see [12, Chapter 8] and [38, Lemma 3.5] for the S_δ case):

$$
A \in \Psi_\delta(X) \iff \left\{ \begin{array}{l} \| \mathrm{ad}_{\ell_N} \cdots \mathrm{ad}_{\ell_1} A\|_{L^2\to L^2} = \mathcal{O}(h^{(1-\delta)N}) \\ \text{for linear functions } \ell_j(x,\xi) \text{ on } \mathbb{R}^d\times\mathbb{R}^d, \end{array}\right. \tag{2.3}
$$

where $\mathrm{ad}_B A = [B, A]$.

For a given symbol $a \in S(T^*\mathbb{R}^d)$ we follow [37] and say that the *essential support* is equal to a given compact set $K \Subset T^*\mathbb{R}^d$,

$$
\text{ess-supp}_h\, a = K \Subset T^*\mathbb{R}^d\,,
$$

if and only if

$$
\forall\, \chi \in S(T^*\mathbb{R}^d)\,,\ \mathrm{supp}\,\chi \subset \complement K \implies \chi\, a \in h^\infty \mathcal{S}(T^*\mathbb{R}^d)\,.
$$

Here $\mathcal{S}$ denotes the Schwartz space. For $A \in \Psi(\mathbb{R}^d)$, $A = \mathrm{Op}_h^w(a)$, we call

$$
\mathrm{WF}_h(A) = \text{ess-supp}_h\, a\,. \tag{2.4}
$$

the semiclassical wavefront set of A. (In this paper we are concerned with a purely semiclassical theory and will only need to deal with *compact* subsets of $T^*\mathbb{R}^d$.)

2.2. **Microlocalization.** We will also consider spaces of L^2 functions (strictly speaking, of h-dependent families of functions) which are *microlocally concentrated* in an open set $V \Subset T^*\mathbb{R}^d$:

$$
\begin{aligned}
H(V) \stackrel{\text{def}}{=} \{u = (u(h) \in L^2(\mathbb{R}^d))_{h\in(0,1]}, \quad \text{such that} \\
\exists C_u > 0\,, \quad \|u(h)\|_{L^2(\mathbb{R}^d)} \leq C_u\,, \quad 0 < h < 1\,, \\
\exists\, \chi \in \mathcal{C}^\infty_{\rm c}(V)\,, \quad \chi^w(x, hD_x)\, u = u + \mathcal{O}_{\mathcal{S}}(h^\infty)\}\,.
\end{aligned} \tag{2.5}
$$

The semiclassical wave front set of $u \in H(V)$ is defined as:

$$
\mathrm{WF}_h(u) = \complement\big\{(x,\xi) \in T^*\mathbb{R}^d \;:\; \exists\, a \in S(T^*\mathbb{R}^d)\,, \quad a(x,\xi) = 1\,,\; \|a^w\, u\|_{L^2} = \mathcal{O}(h^\infty)\big\}\,. \tag{2.6}
$$

The condition (2.6) can be equivalently replaced with $a^w\, u = \mathcal{O}_{\mathcal{S}}(h^\infty)$, since we may always take $a \in \mathcal{S}(T^*\mathbb{R}^d)$. This set obviously satisfies $\mathrm{WF}_h(u) \Subset V$. Notice that the condition does not characterize the individual functions $u(h)$, but the full sequence as $h \to 0$.

We will say that an h-dependent family of operators $T = (T(h))_{h\in(0,1]} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^k)$ is *semiclassically tempered* if there exists $L \geq 0$ such that

$$
\|\langle x\rangle^{-L} T(h) u\|_{H_h^{-L}} \leq C\, h^{-L} \|\langle x\rangle^{L} u\|_{H_h^{L}}\,, \quad h \in (0,1)\,.
$$

Such a family of operators is *microlocally defined* on V if one only specifies (or considers) its action on states $u \in H(V)$, modulo $\mathcal{O}_{\mathcal{S}'\to\mathcal{S}}(h^\infty)$. For instance, T is said to be asymptotically uniformly bounded on $H(V)$ if

$$
\exists\, C_T > 0,\; \forall\, u \in H(V), \quad \exists\, h_{T,u}, \quad \|T(h)u(h)\|_{L^2(\mathbb{R}^k)} \leq C_T\, \|u\|_{L^2(\mathbb{R}^k)}\,, \quad h < h_{T,u}\,. \tag{2.7}
$$

Since we are dealing with families depending on h, without any uniformity assumptions, the statement that the bound holds for sufficiently small h depends on the family u.

If there exists an open subset $W \Subset T^*\mathbb{R}^k$ and $L \in \mathbb{R}$ such that T maps any $u \in H(V)$ into a state $Tu \in h^{-L}\, H(W)$, then we will write

$$
T = T(h) \;:\; H(V) \longrightarrow H(W)\,,
$$

and we say that T is defined microlocally in $W \times V$.

For such operators, we may define only the part of the (twisted) wavefront set which is inside $W \times V$:

$$
\begin{aligned}
\mathrm{WF}'_h(T) \cap (W \times V) \stackrel{\text{def}}{=} (W \times V) \setminus \{(\rho', \rho) \in W \times V \;:\; \exists\, a \in S(T^*\mathbb{R}^d),\; b \in S(T^*\mathbb{R}^k)\,, \\
a(\rho) = 1\,, b(\rho') = 1\,, \quad b^w\, T\, a^w = \mathcal{O}_{L^2\to L^2}(h^\infty)\}\,.
\end{aligned} \tag{2.8}
$$

If $\mathrm{WF}'_h(T) \cap (W \times V) \Subset W \times V$, then the family of operators $T(h)$ can be modified into a family of operators $\widetilde{T}(h) : L^2 \to L^2$, such that $\widetilde{T}$ is $\mathcal{O}_{\mathcal{S}'\to\mathcal{S}}(h^\infty)$ outside V, that is

$$
\widetilde{T} \circ a^w = \mathcal{O}(h^\infty) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^k)\,,
$$

for all $a \in S(T^*\mathbb{R}^d)$ and $\operatorname{supp} a \cap \overline{V} = \emptyset$, while the action of T and $\widetilde{T}$ on $H(V)$ are equal modulo $\mathcal{O}_{L^2\to L^2}(h^\infty)$. This extension is unique modulo $\mathcal{O}_{\mathcal{S}'\to\mathcal{S}}(h^\infty)$.

2.3. **Local h-Fourier integral operators.** We first present a global definition for a class of h-Fourier integral operators following [37] and [14, Chapter 10]. This global definition will then be used to define Fourier integral operators microlocally.

Thus let $A(t)$ be a smooth family of pseudodifferential operators,

$$A(t) = \mathrm{Op}_h^w(a(t))\,, \quad a(t) \in \mathcal{C}^\infty([-1,1]_t; S(T^*\mathbb{R}^d;\mathbb{R}))\,,$$

such that for all t, $\mathrm{WF}_h(A(t)) \Subset T^*\mathbb{R}^d$, in the sense of (2.4). We then define a family of operators

$$\begin{aligned} &U(t) \;:\; L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\,, \\ &hD_t U(t) + U(t)A(t) = 0\,, \quad U(0) = U_0 \in \Psi_h^{0,0}(\mathbb{R}^d)\,. \end{aligned} \tag{2.9}$$

An example is given by $A(t) = A$, independent of t, in which case $U(t) = \exp(-itA/h)$.

The family $(U(t))_{t\in[-1,1]}$ is an example of a family of unitary h-Fourier integral operators, associated to the family of canonical transformations $\kappa(t)$ generated by the (time-dependent) Hamilton vector fields $H_{a_0(t)}$. Here the real valued function $a_0(t)$ is the principal symbol of $A(t)$, and the canonical transformations $\kappa(t)$ are defined through

$$\frac{d}{dt}\kappa(t)(\rho) = (\kappa(t))_*(H_{a_0(t)}(\rho))\,, \quad \kappa(0)(\rho) = (\rho)\,, \quad \rho \in T^*\mathbb{R}^d\,.$$

If $U = U(1)$, say, and the graph of $\kappa(1)$ is denoted by C, we conform to the usual notation and write

$$U \in I_h^0(\mathbb{R}^d \times \mathbb{R}^d; C')\,, \quad \text{where} \quad C' = \{(x,\xi; y, -\eta) \;:\; (x,\xi) = \kappa(y,\eta)\}\,.$$

Here the twisted graph C' is a Lagrangian submanifold of $T^*(\mathbb{R}^d \times \mathbb{R}^d)$.

In words, U is a unitary h-Fourier integral operator associated to the canonical graph C (or the symplectomorphism $\kappa(1)$ defined by this graph). Locally all unitary h-Fourier Integral Operators associated to canonical graphs are of the form $U(1)$, since each local canonical transformation with a fixed point can be deformed to the identity, see [37, Lemma 3.2]. An operator of the form $U(1)\,\chi^w$, with $\chi \in S(T^*\mathbb{R}^d)$, is a (nonunitary) h-Fourier integral operator associated with C. This definition of the operator U, as an operator quantizing $\kappa = \kappa(1)$, depends only on $\kappa = \kappa(1)$, and not on the deformation from the identity to κ. This can be seen from the Egorov characterization of Fourier integral operators – see [37, Lemma 2] or [14, §10.2].

This definition can be generalized to graphs C associated with certain *relations* between phase spaces of possibly different dimensions. Namely, if a relation $C \subset T^*\mathbb{R}^d \times T^*\mathbb{R}^k$ is such that its twist

$$C' = \{(x,\xi; y, -\eta)\,; \;\; (x,\xi; y, -\eta') \in C\}$$

is a Lagrangian submanifold of $T^*(\mathbb{R}^d \times \mathbb{R}^k)$, then one can associate with this relation (microlocally in some neighbourhood) a family of Fourier Integral Operators $T : L^2(\mathbb{R}^k) \mapsto L^2(\mathbb{R}^d)$ [2, Definition 4.2]. This class of operators is denoted by $I_h^r(\mathbb{R}^d \times \mathbb{R}^k; C')$, with $r \in \mathbb{R}$.

The important property of these operators is that their composition is still a Fourier integral operator associated with the composed relations.

For an open set $V \Subset \mathbb{R}^d$ and κ a symplectomorphism defined in a neighbourhood $\widetilde{V}$ of V, we say that a tempered operator T satisfying

$$T \ : \ H(\widetilde{V}) \longrightarrow H(\kappa(\widetilde{V}))\,,$$

is a micrololocally defined *unitary* h-Fourier integral operator in V, if any point $\rho \in V$ has a neighbourhood $V_\rho \subset V$ such that

$$T \ : \ H(V_\rho) \longrightarrow H(\kappa(V_\rho))$$

is a unitary h-Fourier integral operator associated with $\kappa\restriction_{V_\rho}$, as defined by the above procedure.The microlocally defined operators can also be obtained by oscillatory integral constructions — see for instance [28, §4.1] for a brief self-contained presentation.

An example which will be used in §3.1 is given by the standard conjugation result, see [37, Proposition 3.5] or [14, Chapter 10] for self-contained proofs. Suppose that $P \in \Psi_h^{m,0}(\mathbb{R}^d)$ is a semi-classical *real principal type operator*: $p = \sigma(P)$ is real, independent of h, and $p = 0 \Longrightarrow dp \neq 0$. Then for any $\rho_0 \in p^{-1}(0)$, there exists a canonical transformation, κ, mapping $V = \mathrm{neigh}((0,0), T^*\mathbb{R}^d)$ to $\kappa(V) = \mathrm{neigh}(\rho_0, T^*\mathbb{R}^d)$, with $\kappa(0,0) = \rho_0$ and

$$p \circ \kappa(\rho) = \xi_n(\rho) \quad \rho \in V\,,$$

and a unitary microlocal h-Fourier integral operator $U : H(V) \to H(\kappa(V))$ associated to κ, such that

$$U^* P U \equiv h D_{x_n} : H(V) \to H(V)\,.$$

2.4. **Complex scaling.** We briefly recall the complex scaling method of Aguilar-Combes [1] – see [36, 34], and references given there. In this section we consider h as a fixed parameter which plays no rôle in the definition of resonances and let P be an operator satisfying the assumptions above.

For any $0 \leq \theta \leq \theta_0$ and $R > 0$, we define $\Gamma_{\theta,R} \subset \mathbb{C}^n$ to be a totally real deformation of $\mathbb{R}^n$, with the following properties:

$$\begin{gathered} \Gamma_\theta \cap B_{\mathbb{C}^n}(0,R) = B_{\mathbb{R}^n}(0,R)\,, \\ \Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,2R) = e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,2R)\,, \\ \Gamma_\theta = \{x + i f_{\theta,R}(x) \ : \ x \in \mathbb{R}^n\}\,, \quad \partial_x^\alpha f_{\theta,R}(x) = \mathcal{O}_\alpha(\theta)\,. \end{gathered} \tag{2.10}$$

If R is large enough, the coefficients of P continue analytically outside of $B(0,R)$, and we can define a dilated operator:

$$P_{\theta,R} \stackrel{\text{def}}{=} \widetilde{P}\restriction_{\Gamma_{\theta,R}}\,, \quad P_{\theta,R} u = \widetilde{P}(\tilde{u})\restriction_{\Gamma_{\theta,R}}\,,$$

where $\widetilde{P}$ is the holomorphic continuation of the operator P, and $\tilde{u}$ is an almost analytic extension of $u \in \mathcal{C}_c^\infty(\Gamma_{\theta,R})$ from the totally real submanifold $\Gamma_{\theta,R}$ to $\mathrm{neigh}(\Gamma_{\theta,R}, \mathbb{C}^n)$.

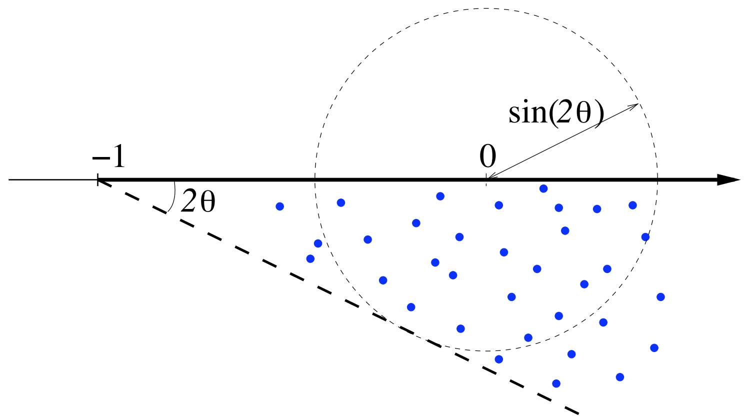

FIGURE 4. The complex scaling in the z-plane used in this paper.

The operator $P_{\theta,R} - z$ is a Fredholm operator for $2\theta > \arg(z+1) > -2\theta$. That means that the resolvent, $(P_{\theta,R}-z)^{-1}$, is meromorphic in that region, the spectrum of $P_{\theta,R}$ in that region is independent of θ and R, and consists of the *quantum resonances* of P.

To simplify notations we identify $\Gamma_{\theta,R}$ with $\mathbb{R}^n$ using the map, $S_{\theta,R} : \Gamma_{\theta,R} \to \mathbb{R}^n$,

$$\Gamma_{\theta,R} \ni x \longmapsto \operatorname{Re} x \in \mathbb{R}^n \,, \tag{2.11}$$

and using this identification, consider $P_{\theta,R}$ as an operator on $\mathbb{R}^n$, defined by $(S_{\theta,R}^{-1})^* P_{\theta,R} S_{\theta,R}^*$ (here S^* means the pullback through S) We note that this identificaton satisfies

$$C^{-1} \|u(h)\|_{L^2(\mathbb{R}^n)} \leq \|S_{\theta,R}^* u(h)\|_{L^2(\Gamma_{\theta,R})} \leq C \|u(h)\|_{L^2(\mathbb{R}^n)} \,,$$

with C independent of θ if $0 \leq \theta \leq \theta_0$.

The identification of the eigenvalues of $P_{\theta,R}$ with the poles of the meromorphic continuation of

$$(P-z)^{-1} \ : \ \mathcal{C}_{\rm c}^\infty(\mathbb{R}^n) \ \longrightarrow \ \mathcal{C}^\infty(\mathbb{R}^n)$$

from $\{\operatorname{Im} z > 0\}$ to $D(0,\sin(2\theta))$, and in fact, the existence of such a continuation, follows from the following formula (implicit in [34], and discussed in [40]): if $\chi \in \mathcal{C}_{\rm c}^\infty(\mathbb{R}^n)$, $\operatorname{supp}\chi \Subset B(0,R)$, then

$$\chi(P_{\theta,R}-z)^{-1}\chi = \chi(P-z)^{-1}\chi \,. \tag{2.12}$$

This is initially valid for $\operatorname{Im} z > 0$ so that the right hand side is well defined, and then by analytic continuation in the region where the left hand side is meromorphic. The reason for the Fredholm property of $(P_{\theta,R}-z)$ in $D(0,\sin(2\theta))$ comes from the properties of the principal symbol of $P_{\theta,R}$ – see Fig. 4. Here for convenience, and for applications to our setting, we consider $P_{\theta,R}$ as a semiclassical operator on $L^2(\mathbb{R}^n)$ using the identification above. The principal symbol is given by

$$p_{\theta,R}(x,\xi) = p(x+if_{\theta,R}(x), [(1+idf_{\theta,R}(x))^t]^{-1}\xi) \,, \tag{2.13}$$

where the complex arguments are allowed due to the analyticity of $p(x,\xi)$ outside of a compact set — see §1.1. We have the following properties

$$\begin{aligned} &\operatorname{Re} p_{\theta,R}(x,\xi) = p(x,\xi) + \mathcal{O}(\theta^2)\langle\xi\rangle^2 \,, \\ &\operatorname{Im} p_{\theta,R}(x,\xi) = -d_\xi p(x,\xi)[df_{\theta,R}(x)^t\xi] + d_x p(x,\xi)[f_{\theta,R}(x)] + \mathcal{O}(\theta^2)\langle\xi\rangle^2 \,. \end{aligned} \tag{2.14}$$

This implies, for R large enough,

$$|p(x,\xi)| \leq \delta \,, \quad |x| \geq 2\,R \implies \operatorname{Im} p_{\theta,R}(x,\xi) \leq -C\theta \,. \tag{2.15}$$

We can take θ to be h dependent: as long as $\theta > ch\log(1/h)$ the estimates above guarantee the Fredholm property of $(P_{\theta,R} - z)$ for $z \in D(0,\theta/C)$, by providing approximate inverses near infinity.

2.5. **Microlocally deformed spaces.** Microlocal deformations using exponential weights have played an important role in the theory of resonances since [20]. Here we take an intermediate point of view [24, 38] by combining compactly supported weights with complex scaling. We should stress however that the full power of [20] would allow more general behaviours at infinity, for instance potentials growing in some directions at infinity.

Let us consider an h-independent function $G_0 \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^d)$, and take

$$G(x,\xi) = Mh\log(1/h)G_0(x,\xi) \,. \tag{2.16}$$

For $A \in \Psi^{m,0}(\mathbb{R}^d)$, we consider the conjugated operator

$$\begin{aligned} e^{-G^w(x,hD)/h}Ae^{G^w(x,hD)/h} &= e^{-\operatorname{ad}_{G^w(x,hD)}/h}A \\ &= \sum_{\ell=0}^{L-1} \frac{(-1)^\ell}{\ell!}\left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^\ell A + R_L \,, \end{aligned} \tag{2.17}$$

where

$$R_L = \frac{(-1)^L}{L!}\int_0^1 e^{-tG^w(x,hD)}\left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^L Ae^{tG^w(x,hD)}dt \,.$$

The semiclassical calculus and (2.16) show that

$$\left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^\ell A \in (Mh\log(1/h))^\ell\, \Psi^{-\infty,0}_{h,0}(\mathbb{R}^d) \,, \quad \ell > 0 \,.$$

Since $\|G_0^w\|_{L^2\to L^2} \leq C_0$, functional calculus shows that

$$\exp(\pm tG^w(x,hD)) = \mathcal{O}_{L^2\to L^2}(h^{-tC_0M}) \,,$$

so we obtain the bound

$$R_L = \mathcal{O}_{L^2\to L^2}(h^{L-2tC_0M-L\delta}) \,.$$

Applying this bound and (2.3), we obtain (2.19). It is then justified to write (2.17) as

$$e^{-G^w(x,hD)/h}Ae^{G^w(x,hD)/h} \sim \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!}\left(\frac{1}{h}\operatorname{ad}_{G^w(x,hD)}\right)^\ell A \in \Psi^{m,0}(\mathbb{R}^d) \,. \tag{2.18}$$

This expansion, combined with Beals's characterization of pseudodifferential operators (2.3), shows that

$$\exp(G^w(x,hD)/h) \in \Psi_\delta^{0,C_0 M}(\mathbb{R}^d)\,, \quad \forall \delta > 0\,. \tag{2.19}$$

Using the weight function G, we can now define weighted spaces which, as we will explain, can be interpreted as to geometric deformation. Let $H_h^k(\mathbb{R}^d)$ be the semiclassical Sobolev spaces defined in (2.2). We put

$$H_G^k(\mathbb{R}^d) = e^{G^w(x,hD)/h} H_h^k(\mathbb{R}^d)\,, \quad \|u\|_{H_G^k} \stackrel{\text{def}}{=} \|e^{-G^w(x,hD)/h} u\|_{H_h^k}\,. \tag{2.20}$$

As a vector space, $H_G^k(\mathbb{R}^d)$ is identical with $H_h^k(\mathbb{R}^d)$, but the Hilbert norms are different. In the case of L^2, that is of $k=0$, we simply put $H_G^0 = H_G$.

The mapping properties of $P = p^w(x,hD)$ on $H_G(\mathbb{R}^d)$ are governed by the properties of the symbol p_G of $P_G \stackrel{\text{def}}{=} e^{-G^w/h} P\, e^{G^w/h}$: formula (2.18) shows that

$$p_G = p - iH_pG + \mathcal{O}(h^2 \log^2(1/h))\,. \tag{2.21}$$

At this moment it is convenient to introduce a notion of *leading symbol*, which is adapted to the stuy of conjugated operators such as p_G^w. For a given $Q \in S(T^*\mathbb{R}^d)$, we say that $q \in S(T^*\mathbb{R}^d)$ is a leading symbol of $Q^w(x,hD)$, if

$$\forall \alpha \in \mathbb{N}^{2d}, \quad h^{-\gamma}\partial^\alpha(Q-q) = \mathcal{O}_\alpha(1)\,, \quad \forall\, 0 < \delta < 1\,, \tag{2.22}$$

that is, $(Q-q) \in S^{0,-\gamma}(T^*\mathbb{R}^d)$ for any $\gamma \in (0,1)$. This property is obviously an equivalence relation inside $S(T^*\mathbb{R}^d)$, which is weaker than the equivalence relation defining the symbol map on $\Psi^{m,k}$ (see §2.1). In particular, this allows us to look at symbols modulo terms of the size $h\log(1/h)$. For example, the leading symbols of p_G and p are the same. If we can find q independent of h, then it is unique.

For future use we record the following:

Lemma 2.1. *Suppose*

$$Q^w(x,hD)\ :\ H_G(\mathbb{R}^d) \longrightarrow H_G(\mathbb{R}^d)\,, \quad Q \in S(T^*\mathbb{R}^d)\,,$$

is self-adjoint. Then this operator admits a real leading symbol (in the sense of (2.22)*). Conversely, if $q \in S(T^*\mathbb{R})$ is real then there exists $Q \in S(T^*\mathbb{R}^d)$ with leading symbol q, such that $Q^w(x,hD)$ is self-adjoint on $H_G(\mathbb{R}^d)$.*

Proof. This follows from noting that

$$Q_G^w \stackrel{\text{def}}{=} e^{-G^w/h} Q^w(x,hD) e^{G^w/h}\,,$$

has the same leading symbol as $Q^w(x,hD)$, and that self-adjointness of Q^w on H_G is equivalent to self-adjointness of Q_G^w on L^2. □

The weighted spaces can also be microlocalized in the sense of §2.2: for $V \Subset T^*\mathbb{R}^d$, we define the space

$$H_G(V) \stackrel{\text{def}}{=} \{u = u(h) \ : \exists\, \chi \in \mathcal{C}^\infty_{\rm c}(V)\,, \quad \exists C_u > 0\,, \\ \chi^w u = u + \mathcal{O}_{\mathcal{S}}(h^\infty)\,,\ \|u(h)\|_{H_G(\mathbb{R}^d)} \leq C_u\}\,. \tag{2.23}$$

In other words, $H_G(V) = e^{G^w(x,hD)/h} H(V)$. This definition depends only on the values of G in V.

For future reference we state the following

Lemma 2.2. *Suppose $T : H(V) \to H(\kappa(V))$ is an h-Fourier integral operator associated to a symplectomorphism κ (in the sense of §2.3), and is asymptotically uniformly bounded (in the sense of (2.7)). Take $G_0 \in \mathcal{C}^\infty(\mathrm{neigh}(\kappa(V)))$, $G = Mh\log(1/h)G_0$.*
Then the operator

$$T \ : \ H_{\kappa^*G}(V) \to H_G(\kappa(V))$$

is also asymptotically uniformly bounded with respect to the deformed norms.

Proof. Since the statement is microlocal we can assume that V is small enough so that $T = T_0 A$ where T_0 is unitary on $L^2(\mathbb{R}^d)$ and $A \in \Psi_h$. We then have to check that

$$\begin{aligned} T_0^{-1} e^{-G^w(x,hD)/h} T_0 e^{(\kappa^*G)^w(x,hD)/h} &= e^{-M\log(1/h)(T_0^{-1}G_0^w(x,hD)T_0} e^{M\log(1/h)(\kappa^*G_0)^w(x,hD)} \\ &= e^{-M\log(1/h)((\kappa^*G_0)^w + \mathcal{O}(h))}\, e^{M\log(1/h)(\kappa^*G_0)^w(x,hD)}\,, \end{aligned}$$

and this is asymptotically uniformly bounded in the sense of (2.7). Here, the second identity resulted from Egorov's theorem, and the last one from semiclassical calculus. □

2.6. **Escape function away from the trapped set.** In this section we recall the construction of the specific weight function G which, up to some further small modifications, will be used to prove Theorems 1 and 2.

Let $K_\delta \subset p^{-1}(\delta)$ be the trapped set in the δ-energy surface (defined by (1.1) with 0 replaced by δ), and define

$$\widehat{K} = \widehat{K}_\delta \stackrel{\text{def}}{=} \bigcup_{|\lambda|\leq\delta} K_\lambda\,. \tag{2.24}$$

The construction of the weight function is based on the following result of [18, Appendix]: for any open neighbourhoods U, V of $\widehat{K}$, $\overline{U} \subset V$, there exists $G_1 \in \mathcal{C}^\infty(T^*X)$, such that

$$G_1 \restriction_U \equiv 0\,, \quad H_p G_1 \geq 0\,, \quad H_p G_1 \restriction_{p^{-1}([-2\delta,2\delta])} \leq C\,, \quad H_p G_1 \restriction_{p^{-1}([-\delta,\delta])\setminus V} \geq 1\,. \tag{2.25}$$

These properties mean that G_1 is an *escape function*: it increases along the flow, and *strictly increases* along the flow on $p^{-1}([-\delta,\delta])$ away from $\widehat{K}$ (as specified by the neighbourhood V). Furthermore, $H_p G$ is bounded in a neighbourhood of $p^{-1}(0)$.

Since such a function G_1 is necessarily of unbounded support, we need to modify it to be able to use H_G-norms defined in §2.5 (otherwise methods of [20] could be used and that alternative would allow more general behaviours at infinity, for instance a wide class of polynomial potentials). For that we follow [38, §§4.1,4.2,7.3] and [28, §6.1]: G_1 is modified to a compactly supported G_2 in a way allowing complex scaling estimates (2.15) to compensate for the wrong sign of H_pG_2. Specifically, [28, Lemma 6.1] states that for any large $R > 0$ and $\delta_0 \in (0, 1/2)$ we can construct G_2 with the following properties: $G_2 \in \mathcal{C}^\infty_c(T^*X)$ and

$$
\begin{aligned}
&H_pG_2 \geq 0 && \text{on } T^*_{B(0,3R)}X, \\
&H_pG_2 \geq 1 && \text{on } T^*_{B(0,3R)}X \cap (p^{-1}([-\delta, \delta]) \setminus V), \\
&H_pG_2 \geq -\delta_0 && \text{on } T^*X.
\end{aligned}
\tag{2.26}
$$

Let

$$
G \stackrel{\text{def}}{=} Mh\log(1/h)G_2\,, \quad \text{with } M > 0 \text{ a fixed constant.}
$$

Then, in the notations of §2.5, we will be interested in the operator

$$
P_{\theta,R} \,:\, H^2_G(\mathbb{R}^n) \longrightarrow H_G(\mathbb{R}^n)\,.
$$

Inserting the above estimates in (2.21), we get

$$
|\operatorname{Re} p_{\theta,R}\restriction_{\Lambda_G}(\rho)| < \delta/2\,, \quad \operatorname{Re}\rho \notin V\,, \implies \operatorname{Im} p_{\theta,R}\restriction_{\Lambda_G}(\rho) \leq -\theta/C_1\,,
\tag{2.27}
$$

provided that we choose [28, §6.1]

$$
\frac{M}{C} \geq \frac{\theta}{h\log(1/h)} \geq \frac{\delta_0 M}{C}, \qquad \text{for some } C > 0\,,
\tag{2.28}
$$

2.7. **Grushin problems.** In this section we recall some linear algebra facts related to the Schur complement formula. Any invertible square matrix decomposed into 4 blocks, we have

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \implies a^{-1} = \alpha - \beta\delta^{-1}\gamma\,,
$$

provided that δ^{-1} exists. As reviewed in [39] this formula, applied to *Grushin problems*

$$
\begin{bmatrix} P & R_- \\ R_+ & 0 \end{bmatrix} \,:\, \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+\,,
$$

is able to reduce the spectral problem for P to a nonlinear spectral problem of lower dimension. If $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, we write

$$
\begin{bmatrix} P - z & R_- \\ R_+ & 0 \end{bmatrix}^{-1} = \begin{bmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{bmatrix},
$$

and the invertibility of $(P - z) : \mathcal{H}_1 \to \mathcal{H}_2$ is equivalent to the invertibility of the finite dimensional matrix $E_{-+}(z)$, more precisely $\dim\ker(P - z) = \dim\ker E_{-+}(z)$ (for this reason, the latter is often called an *effective Hamiltonian*).

We illustrate this scheme with a simple lemma which will be useful later in §4.3.

Lemma 2.3. *Suppose that*

$$\mathcal{P} \stackrel{\text{def}}{=} \begin{bmatrix} P & R_- \\ R_+ & 0 \end{bmatrix} \; : \; \mathcal{H}_1 \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+ \,,$$

where $\mathcal{H}_j$ *and* $\mathcal{H}_\pm$ *are Banach spaces. If* $P^{-1} : \mathcal{H}_2 \to \mathcal{H}_1$ *exists then*

$$\mathcal{P} \text{ is a Fredholm operator } \iff R_+ P^{-1} R_- : \mathcal{H}_- \to \mathcal{H}_+ \text{ is a Fredholm operator} ,$$

and

$$\operatorname{ind} \mathcal{P} = \operatorname{ind} R_+ P^{-1} R_- \,.$$

Proof. We proceed by constructing a Grushin problem for $\mathcal{P}$:

$$\widetilde{\mathcal{P}} \stackrel{\text{def}}{=} \begin{bmatrix} P & R_- & 0 \\ R_+ & 0 & \mathrm{Id}_{\mathcal{H}_+} \\ 0 & \mathrm{Id}_{\mathcal{H}_-} & 0 \end{bmatrix} \; : \; \mathcal{H}_1 \oplus \mathcal{H}_- \oplus \mathcal{H}_+ \longrightarrow \mathcal{H}_2 \oplus \mathcal{H}_+ \oplus \mathcal{H}_- \,.$$

This Grushin problem is well posed, with the inverse given by

$$\widetilde{\mathcal{P}}^{-1} = \begin{bmatrix} P^{-1} & 0 & -P^{-1} R_- \\ 0 & 0 & \mathrm{Id}_{\mathcal{H}_-} \\ -R_+ P^{-1} & \mathrm{Id}_{\mathcal{H}_+} & R_+ P^{-1} R_- \end{bmatrix} .$$

Hence the effective Hamiltonian for the Grushin problem for $\mathcal{P}$ is given by $R_+ P^{-1} R_-$ and we can apply [39, Proposition 2.2]. □

3. A microlocal Grushin problem

In this section we recall and extend the analysis of [37] to treat an ensemble of Poincaré sections associated to a trapped set K_E satisfying the assumptions in §1.2. In [37] a Poincaré section associated to a single closed orbit was considered. The results presented here are purely microlocal in the sense of §2.2, first near a given component Σ_k of the section, and then near the trapped set K_0. In this section P is the original operator, but it could be replaced by $P_{\theta,R}$ since the complex deformation described in §2.4 takes place away from K_0. Also, when no confusion is likely to occur, we will often denote the Weyl quantization χ^w of a symbol $\chi \in S(T^*\mathbb{R}^d)$ by the same letter: $\chi = \chi^w$.

3.1. **Microlocal study near Σ_k.**

First we focus on a single component Σ_k of the Poincaré section, for some arbitrary $k \in \{1, ..., N\}$. Most of the time we will then drop the subscript k. Our aim is to construct a microlocal Grushin problem for the operator

$$\frac{i}{h}(P - z) \,,$$

near $\Sigma = \Sigma_k$, where $|\operatorname{Re} z| \leq \delta$, $\operatorname{Im} z = \mathcal{O}(h)$, and δ will be chosen small enough so that the flow on $\Phi^t \restriction_{K_{\operatorname{Re} z}}$ is a small perturbation of $\Phi^t \restriction_{K_0}$.

3.1.1. *A normal form near* Σ_k. Using the assumption (1.15) and a version of Darboux's theorem (see for instance [22, Theorem 21.2.3]), we may extend the map $\kappa_k = \kappa : \widetilde{\Sigma}_k \to \Sigma_k$ to a canonical transformation $\widetilde{\kappa}_k$ defined in a neighbourhood of $\widetilde{\Sigma}_k$ in $T^*\mathbb{R}^n$,

$$\widetilde{\Omega}_k \stackrel{\text{def}}{=} \{(x,\xi) \in T^*\mathbb{R}^n; (x',\xi') \in \widetilde{\Sigma}_k,\ \ |x_n| \leq \epsilon, \ , |\xi_n| \leq \delta\} ,$$

such that

$$\widetilde{\kappa}_k(x',0,\xi',0) = \kappa_k(x',\xi') \in \Sigma_k , \qquad p \circ \widetilde{\kappa}_k = \xi_n . \tag{3.1}$$

We call $\Omega_k = \widetilde{\kappa}_k(\widetilde{\Omega}_k)$ the neighbourhood of Σ_k in T^*X in the range of $\widetilde{\kappa}_k$. The "width along the flow" $\epsilon > 0$ is taken small enough, so that the sets $\{\Omega_k,\ k = 1,\ldots,N\}$ are mutually disjoint, and it takes at least a time 20ϵ for a point to travel between any Ω_k and its successors.

The symplectic maps $\widetilde{\kappa}_k$ allow us to construct a Poincaré section in the neighbouring energy layers $p^{-1}(z)$, $z \in [-\delta,\delta]$. Let us denote

$$\kappa_{k,z} \stackrel{\text{def}}{=} \widetilde{\kappa}_k\restriction (\widetilde{\Omega}_k \cap \{\xi_n = z\}) .$$

Then, if $\delta > 0$ is taken small enough, then for $|\operatorname{Re} z| \leq \delta$ the sets

$$\Sigma_k(z) = \kappa_{k,z}(\widetilde{\Sigma}_k) = \{\widetilde{\kappa}(x',0;\xi',z),\ (x',\xi') \in \widetilde{\Sigma}_k\}$$

are still transversal to the flow, and $\partial\Sigma_k(z) \cap K_z = \emptyset$. Besides, decreasing δ if necessary, we can assume that the sets $D_k(z) \stackrel{\text{def}}{=} \widetilde{\kappa}_{k,z}(\widetilde{D}_k) \subset \Sigma_k(z)$ are still open neighbourhoods of the reduced trapped set $\mathcal{T}_j(z) = \Sigma(z) \cap K_z$, and that the Poincaré maps $F_{jk,z}$ still map bijectively their connected components $D_{jk}(z) = \widetilde{\kappa}_{k,z}(\widetilde{D}_{jk})$ to arrival sets $A_{jk}(z) \subset \Sigma_j(z)$. Notice that for $z \neq 0$ these arrival sets $A_{jk}(z)$ are in general different from $\kappa_{j,z}(\widetilde{A}_{jk})$, or equivalently $\widetilde{A}_{jk}(z) = \kappa_{j,z}^{-1}(A_{jk}(z))$ is generally different from $\widetilde{A}_{jk}(0)$. The tube connecting $D_{jk}(z)$ with $A_{jk}(z)$ is denoted by $T_{jk}(z)$ – see Fig.3. For any set $S(z) = S(\operatorname{Re} z)$ depending on the energy in the interval $\operatorname{Re} z \in [-\delta,\delta]$, we use the notation

$$\widehat{S} \stackrel{\text{def}}{=} \bigcup_{|z|\leq\delta} S(z) . \tag{3.2}$$

3.1.2. *Microlocal solutions near* Σ. Let us now restrict ourselves to the neighbourhood of Σ_k, and drop the index k. The canonical transformation $\widetilde{\kappa}$ can be locally quantized using the procedure reviewed in §2.3, resulting in a microlocally defined unitary Fourier integral operator

$$U \ :\ H(\widetilde{\Omega}) \longrightarrow H(\Omega) , \quad U^* \, P \, U \equiv hD_{x_n} , \text{ microlocally in } \widetilde{\Omega}. \tag{3.3}$$

For $z \in \mathcal{R}(\delta, Ch)$, we consider the microlocal Poisson operator

$$\mathrm{K}(z) : L^2(\mathbb{R}^{n-1}) \to L^2_{\mathrm{loc}}(\mathbb{R}^n) , \quad [\mathrm{K}(z)\, v_+](x',x_n) = e^{ix_nz/h}\, v_+(x') , \tag{3.4}$$

which obviously solves the equation $(hD_{x_n} - z)\,\mathrm{K}(z)\, v_+ = 0$.

For v_+ which is microlocally concentrated in a compact set, the wavefront set of $\mathrm{K}(z)\,v_+$ is not localized in the flow direction. On the other hand, the Fourier integral operator U is well-defined and unitary only from $\widetilde{\Omega}$ to Ω. Therefore, we use a smooth cutoff function χ_Ω, $\chi_\Omega = 1$ in Ω, $\chi_\Omega = 0$ outside Ω' a small open neighbourhood of Ω, and define the Poisson operator

$$K(z) \stackrel{\text{def}}{=} \chi_\Omega^w\, U\, \mathrm{K}(z) : H(\widetilde{\Sigma}) \to H(\Omega')\,.$$

This operator maps any state $v_+ \in H(\widetilde{\Sigma}) \subset L^2(\mathbb{R}^{n-1})$, to a microlocal solution of the equation $(P-z)u = 0$ in Ω, with $u \in H(\Omega')$. As we will see below, the converse holds: each microlocal solution in Ω is parametrized by a function $v_+ \in H(\widetilde{\Sigma})$.

In a sense, the solution $u = K(z)v_+$ is an extension along the flow of the *transverse data* v_+. More precisely, $K(z)$ is a microlocally defined Fourier integral operator associated with the graph

$$C_- = \{(\widetilde{\kappa}(x', x_n, \xi', \operatorname{Re} z); x', \xi'),\ \ (x', \xi') \in \widetilde{\Sigma},\ |x_n| \leq \epsilon\} \subset T^*(X \times \mathbb{R}^{n-1})\,. \tag{3.5}$$

Equivalently, this relation associates to each point $(x', \xi') \in \widetilde{\Sigma}$ a short trajectory segment through the point $\widetilde{\kappa}(x', 0; \xi', \operatorname{Re} z) \in \Sigma(\operatorname{Re} z)$. We use the notation C_- since this relation is associated with the operator R_- defined in (3.13) below.

Back to the normal form hD_{x_n}, let us consider a smoothed out step function,

$$\chi_0 \in \mathcal{C}^\infty(\mathbb{R}_{x_n}),\ \ \chi(x_n) = 0 \text{ for } x_n \leq -\epsilon/2,\ \ \chi_0(x_n) = 1 \text{ for } x_n \geq \epsilon/2\,.$$

We notice that the commutator $(i/h)[hD_{x_n}, \chi_0] = \chi_0'(x_n)$ is localized in the region of the step and integrates to 1 in x_n: this implies the normalization property

$$\langle (i/h)[hD_{x_n}, \chi_0]\mathrm{K}(z)v_+, \mathrm{K}(\bar{z})v_+\rangle = \|v_+\|^2_{L^2(\mathbb{R}^{n-1})}\,, \tag{3.6}$$

where $\langle \bullet, \bullet \rangle$ is the usual Hermitian inner product on $L^2(\mathbb{R}^n)$. Notice that the right hand side is independent of the precise choice of χ_0.

We now bring this expression to the neighbourhood of Σ through the Fourier integral operator $\chi_\Omega^w U$. This implies that the Poisson operator $K(z)$ satisfies:

$$\langle (i/h)[P, \chi^w]K(z)v_+, K(\bar{z})v_+\rangle \equiv \|v_+\|^2 \qquad \text{for any } v_+ \in H(\widetilde{\Sigma})\,. \tag{3.7}$$

Here the symbol χ is such that $\chi^w \equiv U\,\chi_0^w\,U^*$ inside Ω, so χ is equal to 0 before $\Phi^{-\epsilon}(\Sigma)$ and equal to 1 after $\Phi^{\epsilon}(\Sigma)$ (in the following we will often use this time-like terminology referring to the flow Φ^t). In (3.7), we are only concerned with $[P, \chi^w]$ microlocally near Ω, since the operator $\chi_\Omega^w U$ is microlocalized in $\Omega' \times \widetilde{\Omega}'$. Hence, at this stage we can ignore the properties of the symbol χ outside Ω'.

The expression (3.7) can be written

$$K(\bar{z})^*\,[(i/h)P, \chi^w]K(z) = Id\ :\ H(\widetilde{\Sigma}) \to H(\widetilde{\Sigma})\,. \tag{3.8}$$

Fixing such a cutoff function $\chi = \chi_f$ (where f is for *forward*), we define the operator

$$R_+(z) \stackrel{\text{def}}{=} K(\bar{z})^*\,[(i/h)P, \chi_f] = \mathrm{K}(\bar{z})^*\,U^*\chi_\Omega^w\,[(i/h)P, \chi_f] \tag{3.9}$$

(from here on we denote $\chi = \chi^w$ in similar expressions). This operator "projects" any $u \in H(\Omega)$ to a certain transversal function $v_+ \in H(\widetilde{\Sigma})$. But it is important to notice that $R_+(z)$ is also well-defined on states u microlocalized in a small neighbourhood of $\widehat{K}$: the operator $\chi_\Omega^w\,[(i/h)P, \chi_f]$ cuts off the components of u outside Ω. Hence, we may write

$$R_+(z) : H(\mathrm{neigh}(\widehat{K})) \to H(\widetilde{\Sigma})\,.$$

The equation (3.8) shows that this projection is compatible with the above extension of the transversal function:

$$R_+(z)\,K(z) = Id \;:\; H(\widetilde{\Sigma}) \to H(\widetilde{\Sigma})\,. \tag{3.10}$$

This shows that transversal functions $v_+ \in H(\widetilde{\Sigma})$ and microlocal solutions to $(P-z)u = 0$ are bijectively related. Since $\mathrm{Im}\, z = \mathcal{O}(h)$, it is clear that $K(z)$ is uniformly bounded in the L^2 sense, and from the definition above we get the same property for $R_+(z)$. Just as $K(\bar{z})^*$, $R_+(z)$ is a microlocally defined Fourier integral operator associated with the relation

$$C_+ = \{x', \xi'; (\widetilde{\kappa}(x', x_n, \xi', \mathrm{Re}\, z)),\quad (x', x_n, \xi', \mathrm{Re}\, z) \in \widetilde{\Omega}\} \subset T^*(\mathbb{R}^{n-1} \times X)\,, \tag{3.11}$$

namely the inverse of C_- given in (3.5). In words, this relation consists of taking any $\rho \in \Omega \cap p^{-1}(\mathrm{Re}\, z)$ and projecting it along the flow on the section $\Sigma(z)$.

We now select a second cutoff function χ_b with properties similar with χ_f, and satisfying also the nesting

$$\chi_b = 1 \;\text{ in a neighbourhood of }\; \mathrm{supp}\, \chi_f\,. \tag{3.12}$$

With this new cutoff, we define the operator

$$R_-(z)u_- = [(i/h)P, \chi_b]\,K(z) \;:\; H(\widetilde{\Sigma}) \to H(\Omega)\,. \tag{3.13}$$

Starting from a transversal data $u_- \in H(\widetilde{\Sigma})$, this operator creates a microlocal solution in Ω and truncates by applying a pseudodifferential operator with symbol $H_p\chi_b$. Same as $K(z)$, it is a microlocally defined Fourier integral operator associated with the graph C_-.

3.1.3. *Solving a Grushin problem.* We are now equipped to define our microlocal Grushin problem in Ω. Given $v \in H(\Omega)$, $v_+ \in H(\widetilde{\Sigma})$, we want to solve the system

$$\begin{cases} (i/h)(P-z)u + R_-(z)u_- & = v, \\ R_+(z)u & = v_+\,, \end{cases} \tag{3.14}$$

with $u \in L^2(X)$ a forward solution, and $u_- \in H(\widetilde{\Sigma})$.

Let us show how to solve this problem. First let $\widetilde{u}$ be the forward solution of $(i/h)(P-z)\widetilde{u} = v$, microlocally in Ω. That solution can be obtained using the Fourier integral operator U in (3.3) and the easy solution for hD_{x_n}. We can also proceed using the propagator to define a forward parametrix:

$$\widetilde{u} \stackrel{\text{def}}{=} E(z)\, v, \qquad E(z) \stackrel{\text{def}}{=} \int_0^T e^{-it(P-z)/h}\, dt. \tag{3.15}$$

The time T is such that $\Phi^T(\Omega) \cap \Omega = \emptyset$ (from the above assumption on the separation between the Ω_k we may take $T = 5\epsilon$). By using the model operator hD_{x_n}, one checks that the parametrix $E(z)$ transports the wavefront set of v as follows:

$$\mathrm{WF}_h(E(z)v) \subset \mathrm{WF}_h(v) \cup \Phi^T(\mathrm{WF}_h(v)) \cup \bigcup_{0 \le t \le T} \Phi^t(\mathrm{WF}_h(v) \cap p^{-1}(\mathrm{Re}\, z))\,. \tag{3.16}$$

In general, $\widetilde{u}$ does not satisfy $R_+(z)\widetilde{u} = v_+$, so we need to correct it. For this aim, we solve the system

$$\begin{cases} (i/h)(P-z)\widehat{u} + R_-(z)u_- & \equiv 0\,, \\ R_+(z)\widehat{u} & \equiv v_+ - R_+(z)\widetilde{u} \end{cases} \tag{3.17}$$

through the Ansatz

$$\begin{cases} u_- & = -v_+ + R_+(z)\widetilde{u}\,, \\ \widehat{u} & = -\chi_b\, K(z)\, u_-\,. \end{cases} \tag{3.18}$$

Indeed, the property $(P-z)\, K(z) \equiv 0$ ensures that $(i/h)(P-z)\widehat{u} = -R_-(z)u_-$. We then obtain the identities

$$\begin{aligned} R_+(z)\widehat{u} &= -K(\bar{z})^*\, [(i/h)P, \chi_f]\, \chi_b\, K(z)\, u_- \\ &\equiv -K(\bar{z})^*\, [(i/h)P, \chi_f]\, K(z)\, u_- \\ &\equiv -u_-\,. \end{aligned}$$

The second identity uses the nesting assumption $(H_p\chi_f)\chi_b = H_p\chi_f$, and the last one results from (3.8). This shows that the Ansatz (3.18) solves the system (3.17). Finally, $(u = \widetilde{u} + \widehat{u}, u_-)$ solves (3.14) microlocally in $\Omega \times \widetilde{\Sigma}$, for $v \in H(\Omega)$ and $v_+ \in H(\widetilde{\Sigma})$ respectively. Furthermore, these solutions satisfy the norm estimate

$$\|u\| + \|u_-\| \lesssim \|v\| + \|v_+\|\,. \tag{3.19}$$

The form of the microlocal construction in this section is an important preparation for the construction of our Grushin problem in the next section. In itself, it only states that $(i/h)(P-z)u = v$ can be solved microlocally near Σ in the forward direction, for v microlocalized near Σ.

3.2. **Microlocal solution near $\widehat{K}$.** We will now extend the construction of the Grushin problem near each Σ_k, described in §3.1, to obtain a microlocal Grushin problem near the full trapped set $\widehat{K}$. This will be achieved by relating the construction near Σ_k to the one near the successor sections Σ_j. We now need to restore all indices $k \in \{1, \dots, N\}$ in our notations.

3.2.1. *Setting up the Grushin problem.* We recall that $H(\widetilde{\Sigma}_k) \subset L^2(\mathbb{R}^{n-1})$ is the space of functions microlocally concentrated in $\widetilde{\Sigma}_k$ (see (2.5)). For $u \in L^2(X)$ microlocally concentrated in $\operatorname{neigh}(\widehat{K}, T^*X)$, we define

$$R_+(z)u = (R_+^1(z)u, ..., R_+^N(z)u) \in H(\widetilde{\Sigma}_1) \times ... \times H(\widetilde{\Sigma}_N)\,, \tag{3.20}$$

where each $R_+^k(z) : H(\operatorname{neigh}(\widehat{K})) \to H(\widetilde{\Sigma}_k)$ was defined in §3.1 using a cutoff $\chi_f^k \in \mathcal{C}_c^\infty(T^*X)$ realizing a smoothed-out step from 0 to 1 along the flow near Σ_k.

Similarly, we define

$$\begin{aligned} R_-(z) \,&: H(\widetilde{\Sigma}_1) \times \ldots \times H(\widetilde{\Sigma}_N) \to H(\cup_{k=1}^N \Omega_k), \\ R_-(z)u_- &= \sum_1^N R_-^j(z)u_-^j, \qquad u_- = (u_-^1, ..., u_-^N). \end{aligned} \tag{3.21}$$

Each $R_-^k(z)$ was defined in (3.13) in terms of a cutoff function $\chi_b^k \in \mathcal{C}_c^\infty(T^*X)$ which also changes from 0 to 1 along the flow near Σ_k, and does so before χ_f^k. Below we will impose more restrictions on the cutoffs χ_b^k.

With these choices, we now consider the microlocal Grushin problem

$$\begin{cases} (i/h)(P-z)u + R_-(z)u_- & \equiv v\,, \\ R_+(z)u & \equiv v_+\,. \end{cases} \tag{3.22}$$

The aim of this section is to construct a solution (u, u_-) microlocally concentrated in a small neighbourhood of

$$K_0 \times \kappa_1^{-1}(\mathcal{T}_1) \times ... \times \kappa_N^{-1}(\mathcal{T}_N)\,,$$

provided (v, v_+) is concentrated in a sufficiently small neighbourhood of the same set.

To this aim we need to put more constraints on the cutoffs χ_b^k. We assume that each $\chi_b^k \in \mathcal{C}_c^\infty(T^*X)$ is supported near the direct outflow of $\mathcal{T}_k$. To give a precise condition, let us slightly modify the energy-thick tubes $\widehat{T}_{jk}$ (see (1.16), (3.2)) by removing or adding some parts near their ends:

$$\widehat{T}_{jk}^{s_1 s_2} \stackrel{\text{def}}{=} \{\Phi^t(\rho) \,:\, \rho \in \widehat{D}_{jk},\ -s_2\, 2\epsilon < t < t_+(\rho) + s_1\, 2\epsilon\}\,, \qquad s_i = \pm\,.$$

With this definition, the short tubes $\widehat{T}_{jk}^{--}$ do not intersect the neighbourhoods Ω_k, Ω_j, while the long tubes $\widehat{T}_{jk}^{++}$ intersect both.

We then assume that

$$\chi_b^k(\rho) = 1 \quad \text{for} \quad \rho \in \bigcup_{j \in J_+(k)} \widehat{T}_{jk}^{--}\,, \tag{3.23}$$

and $\operatorname{supp} \chi_b^k$ is contained in a small neighbourhood of that set. Furthermore, we want the cutoffs $\{\chi_b^k\}_{k=1,...,N}$ to form a *microlocal partition of unity* near K_0: there exists a

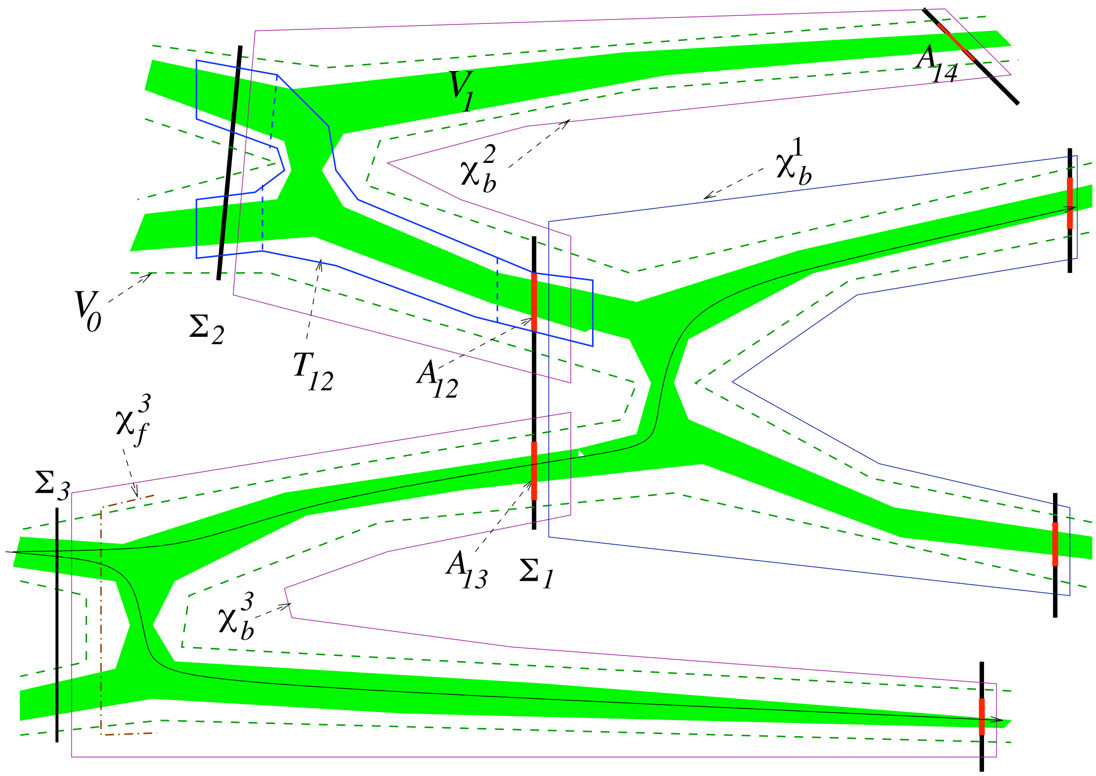

FIGURE 5. Schematic representation of (part of) the neighbourhoods $V_1 \subset V_0$ of K_0 (resp. green shade and green dashed contour), some sections Σ_k (thick black) and arrival sets $A_{kj} \subset \Sigma_k$ (red). We also show the tubes $T_{12}^{\pm\pm}$ connecting Σ_2 with A_{12} (the dashed lines indicate the boundaries of T_{12}^{--}), the supports of the cutoffs χ_b^k and χ_f^3 (dot-dashed line), and two trajectories in K_0 (full lines inside V_1).

neighbourhood V_0 of $\widehat{K}$ containing all long tubes:

$$V_0 \supset \bigcup_{k,j} \widehat{T}_{jk}^{++} \,, \tag{3.24}$$

and such that

$$\sum_{k=1}^{N} \chi_b^k(\rho) \equiv 1 \quad \text{for} \quad \rho \in V_0 \,. \tag{3.25}$$

These conditions on χ_b^k can be fulfilled thanks to the assumption (1.14) on the section $\mathbf{\Sigma}$. A schematic representation of these sets and cutoffs is shown in Fig. 5.

3.2.2. *Solving the homogeneous Grushin problem.* Let us first solve (3.22) when $v \equiv 0$. The wavefront set $\mathrm{WF}_h(v_+^k) \subset \widetilde{\Sigma}_k$ is mapped through $\kappa_{k,z}$ to a subset of $\Sigma(z)$. The microlocal solution $K_k(z)v_+^k$, initially concentrated inside the neighbourhood Ω_k', can be extended along the flow to a larger set Ω_k^+, which intersects the successors $\Sigma_j(z)$ of $\Sigma_k(z)$ and contains the union of tubes $\bigcup_{j\in J_+(k)} \widehat{T}_{jk}^{++}$ (we remind that $j \neq k$ according to assumption (1.14)). This can be done by extending the symplectomorphism $\widetilde{\kappa}_k$, the associated unitary Fourier integral operator U_k, and replace the cutoff function χ_{Ω_k} by a function $\chi_{\Omega_k^+}$ supported in the set Ω_k^+; we can then define the extended Poisson operator as:

$$K_k^+(z) = \chi_{\Omega_k^+}^w \, U_k \, \mathrm{K}(z) : H(\widetilde{\Sigma}) \to H(\Omega_k^+) \,.$$

Assuming $\kappa_{k,z}(\mathrm{WF}_h(v_+^k))$ is contained in the departure set $D_k(z) \subset \Sigma_k(z)$, the extended microlocal solution $K_k^+(z)v_+^k$ is concentrated in the union of tubes $\cup_{j\in J_+(k)} T_{jk}^{++}(z)$. In that case, we take as our Ansatz

$$u_k \stackrel{\mathrm{def}}{=} \chi_b^k \, K_k^+(z) \, v_+^k \,. \tag{3.26}$$

Due to the assumption (3.23), the cutoff χ_b^k effectively truncates the solution only near the sections $\Sigma_k(z)$ and $\Sigma_j(z)$, $j \in J_+(k)$, but not on the "sides" of $\operatorname{supp} \chi_b^k$. Hence, the expression

$$(i/h)(P - z)u_k \equiv [(i/h)P, \chi_b^k] \, K_k^+(z) \, v_+^k \tag{3.27}$$

can be decomposed into one component $R_-^k(z)v_+^k$ supported near $D_k(z)$, and other components supported near the arrival sets $A_{jk}(z) \subset \Omega_j$, due to the "step down" of χ_b^k near $A_{jk}(z)$. The assumption (3.25) ensures that

$$[(i/h)P, \chi_b^k] \equiv -[(i/h)P, \chi_b^j] \quad \text{microlocally near } A_{jk}(z) \,, \tag{3.28}$$

so the expression in (3.27) reads

$$(i/h)(P - z)u_k \equiv R_-^k(z)v_+^k - \sum_{j\in J_+(k)} -[(i/h)P, \chi_b^j] \, K_k^+(z) \, v_+^k \,. \tag{3.29}$$

Now, for each $j \in J_+(k)$ we notice that $K_k^+(z) \, v_+^k$ is a solution of $(P - z)u = 0$ near $A_{jk}(z)$, so this solution can also be parametrized by some transversal data "living" on the section $\Sigma_j(z)$ (see the discussion before (3.5)). This data obviously depends linearly on v_+^k, which defines the *monodromy operator* $\mathcal{M}_{jk}(z)$:

$$K_k^+(z)v_+^k \equiv K_j(z) \, \mathcal{M}_{jk}(z)v_+^k \,, \quad \text{microlocally near } A_{jk}(z) \,. \tag{3.30}$$

The operators $\mathcal{M}_{jk}(z)$ are microlocally defined from $\widetilde{D}_k \subset \widetilde{\Sigma}_k$ to $\widetilde{A}_{jk}(z) \subset \widetilde{\Sigma}_j$, they are zero on $H(\widetilde{D}_{\ell k})$ for $\ell \neq j$. The identity (3.8) provides an explicit formula:

$$\mathcal{M}_{jk}(z) = K_j(\bar{z})^* \, [(i/h)P, \chi_f^j] K_k^+(z) = R_+^j(z) K_k^+(z) \,. \tag{3.31}$$

Before further describing these operators, let us complete the solution of our Grushin problem. Combining (3.29) with (3.30), we obtain

$$(i/h)(P-z)u_k \equiv R_-^k(z)v_+^k - \sum_{j\in J_+(k)} R_-^j(z)\mathcal{M}_{jk}(z)v_+^k\,. \tag{3.32}$$

This shows that the problem (3.22) in the case $v=0$ and a single v_k^+, $\mathrm{WF}_h(v_+^k)\subset \widetilde{D}_k$ is solved by

$$u \equiv \chi_b^k\, K_k^+(z)\, v_+^k, \quad u_-^k = -v_+^k, \quad u_-^j = \mathcal{M}_{jk}(z)v_+^k, \;\; j\in J_+(k)\,.$$

We now consider the Grushin problem with $v=0$, $v_+ = (v_+^1,\ldots,v_+^N)$ with each v_+^k microlocalized in $\widetilde{D}_k$. By linearity, this problem is solved by

$$\begin{aligned} u &\equiv \sum_k \chi_b^k\, K_k^+(z)\, v_+^k,\\ u_-^j &\equiv -v_+^j + \sum_{k\in J_-(j)} \mathcal{M}_{jk}(z)v_+^k\,. \end{aligned} \tag{3.33}$$

From the above discussion, u is microlocalized in the neighbourhood V_0 of $\widehat{K}$, while u_-^j is microlocalized in $\widetilde{D}_j\cup\widetilde{A}_j(z)$.

Let us now come back to the monodromy operators. The expression (3.31) shows that $\mathcal{M}_{jk}(z)$ is a microlocal Fourier integral operator. Since we have extended the solution $K_k(z)\,v_+^k$ beyond Ω_k, the relation associated with the restriction of $K_k^+(z)$ on $H(\widetilde{D}_{jk})$ is a modification of (3.5), of the form

$$C_-^{jk} = \{(\Phi^t(\widetilde{\kappa}_{k,z}(\rho));\rho),\;\rho\in\widetilde{D}_{jk},\quad -\epsilon\le t\le t_{\max}+\epsilon\}\,,$$

such that the trajectories cross Σ_j. On the other hand, the relation C_+ associated with $R_-^j(z)$ is identical with (3.11). By the composition rules, the relation associated with $\mathcal{M}_{jk}(z)$ is

$$C^{jk} = \{(\rho',\rho),\quad \rho\in\widetilde{D}_{jk},\; \rho' = \kappa_{j,z}^{-1}\circ F_{jk,z}\circ\kappa_{k,z}(\rho) = \tilde{F}_{jk,z}(\rho)\}\,.$$

This is exactly the graph of the Poincaré map $F_{jk,z}: D_{jk}(z)\to A_{jk}(z)$, seen through the coordinates charts $\kappa_{k,z}$, $\kappa_{j,z}$.

When z is real, the identity (3.8) implies that $\mathcal{M}_{jk}(z) : H(\widetilde{D}_{jk})\to H(\widetilde{A}_{jk}(z))$ is microlocally unitary. Also, the definition (3.31) shows that this operator depends holomorphically of z in the rectangle $\mathcal{R}(\delta, Ch)$. To lowest order, the z-dependence takes the form

$$\mathcal{M}_{jk}(z) = \mathcal{M}_{jk}(0)\,\mathrm{Op}_h^w(\exp(iz\tilde{t}_+/h)) + \mathcal{O}(h)\,,$$

where $\tilde{t}_+ : \widetilde{D}_k\to\mathbb{R}_+$ is the return time expressed in the coordinate chart $\widetilde{\kappa}_k$.

3.2.3. *Solving the inhomogeneous Grushin problem.* It remains to discuss the inhomogeneous problem

$$(i/h)(P-z)u + R_- u_- \equiv v\,, \tag{3.34}$$

for v microlocalized in a neighbourhood V_1 of $\widehat{K}$, which satistfies

$$V_1 \subset \bigcup_{j,k} \widehat{T}_{jk}^{-+}\,. \tag{3.35}$$

(each tube $\widehat{T}_{jk}^{-+}$ intersects Ω_k only near $\widehat{D}_k$, see figure 5).

Let us first assume that v is microlocally concentrated inside a short tube $\widehat{T}_{jk}^{--}$. Using the forward parametrix $E(z)$ of $(i/h)(P-z)$ given in (3.15), we propose the Ansatz

$$u \stackrel{\text{def}}{=} \chi_b^k\, E(z)\, v\,. \tag{3.36}$$

According to the transport property (3.16), $E(z)v$ is microlocalized in the outflow of $\widehat{T}_{jk}^{--}$, so the cutoff χ_b^k effectively truncates $E(z)v$ only near $A_{jk}(z) \subset \Omega_j$. The partition of unity (3.25) then implies that

$$(i/h)(P-z)u \equiv v + [(i/h)P, \chi_b^k]\, E(z)\, v \equiv v - [(i/h)P, \chi_b^j]\, E(z)\, v\,.$$

Also, $E(z)v$ is a microlocal solution of $(P-z)u = 0$ near $A_{jk}(z)$, so

$$E(z)v \equiv K_j(z) R_+^j(z) E(z) v \quad \text{microlocally near } A_{jk}(z)\,.$$

Thus, we can solve (3.34) by taking

$$u_-^j \equiv R_+^j(z) E(z) v\,, \quad u_-^\ell = 0, \quad \ell \neq j\,.$$

The propagation of wavefront sets given in (3.16) shows that $u_-^j \in H(\widetilde{A}_{jk}(z))$, and that $\mathrm{WF}_h(u) \subset \widehat{T}_{jk}^{+-}$ does not intersect the "step up" region of the forward cutoffs χ_f^ℓ, so that $R_+^\ell(z) u \equiv 0$ for all $\ell = 1, \ldots, N$.

If v is microlocally concentrated in $V_1 \cap \cup_{|t|\leq\epsilon} \Phi^t(\widehat{D}_k)$, we can replace the cutoff χ_b^k in (3.36) by

$$\chi_b^k + \sum_{\ell \in J_-(k)} \chi_b^\ell\,,$$

and apply the same construction. The only notable difference is the fact that $R_+^k(z)u$ may be a nontrivial state concentrated in $\cup_{|t|\leq\epsilon} \widehat{D}_k$.

In both cases, we see that $\|u\| + \|u_-\| \lesssim \|v\|$, so $\|R_+ u\| \lesssim \|v\|$. By linearity, the above procedure allows to solve (3.34) for any v microlocalized inside the neighbourhood V_0.

We summarize the construction of our microlocal Grushin problem in the following

Proposition 3.1. *For $\delta > 0$ small enough, there exist neighbourhoods of $\widehat{K} = \widehat{K}_\delta$ in T^*X, V_+ and V_-, and neighbourhoods of $\widetilde{\kappa}_j^{-1}(\widehat{T}_j)$ in $\widetilde{\Sigma}_j$, V_+^j, and V_-^j, $j = 1, \cdots N$, such that for any*

$$(v, v_+) \in H(V_+) \times H(V_+^1) \times \cdots H(V_+^N)\,,$$

we can find

$$(u, u_-) \in H(V_-) \times H(V_-^1) \times \cdots H(V_-^N)\,,$$

satisfying

$$\frac{i}{h}(P - z)u + R_-(z)u_- \equiv v\,, \qquad R_+(z)u \equiv v_+ \quad \text{microlocally in } V_+ \times V_+^1 \times \cdots V_N^+\,.$$

Here $R_\pm(z)$ are given by (3.20) and (3.21). Furthermore, the solutions satisfy the norm estimates

$$\|u\| + \|u_-\| \lesssim \|v\| + \|v_+\|\,.$$

One possible choice for the above sets is

$$V_+ = V_1, \quad V_- \stackrel{\text{def}}{=} V_0, \quad V_+^k = \widetilde{D}_k, \quad V_-^k = \widetilde{D}_k \cup \bigcup_{\operatorname{Re} z \leq \delta} \widetilde{A}_k(z)\,.$$

Proof. Take $v \in H(V_1)$, and call $(\widetilde{u}, \widetilde{u}_-)$ the solution for the inhomogeneous problem (3.34). Then the propagation estimate (3.16) implies that $\widetilde{u}$ is concentrated inside the larger neighbourhood $V_0 \subset \cup_{j,k} \widehat{T}_{jk}^{++}$ (see (3.24)), while $\widetilde{u}_-^j \in H(\widetilde{A}_j(z))$.

We have $R_+^k(z)\widetilde{u} \in H(\widetilde{D}_k)$ so, provided the data satisfies $v_+^k \in \widetilde{D}_k$, the computations of §3.2.2 show how to solve the homogeneous problem with data $(v_+ - R_+(z)\widetilde{u})$. That solves the full problem. The expressions (3.33) show that the solutions to the homogeneous problem $(\widehat{u}, \widehat{u}_-^k)$ are microlocalized, respectively, in V_0 and in $\widetilde{D}_k \cup \widetilde{A}_k(z)$. □

Remark. The proof of the proposition shows that the neighbourhoods V_+^k and V_-^k are different. For given data (v, v^+), the solutions (u, u_-) will not in general be concentrated in the same small set as the initial data. This, of course, reflects the fact that a neighbourhood V of K_0 is not invariant under the forward flow, but escapes along the unstable direction. In order to transform the microlocal Grushin problem described in this proposition into a well-posed problem, we need to take care of this escape phenomenon. This will be done using escape functions in order to deform the norms on the spaces $L^2(X)$ (as described in §2.5), but also on the auxiliary spaces $L^2(\mathbb{R}^{n-1})$.

4. A well posed Grushin problem

The difficulty described in the remark at the end of §3 will be resolved by modifying the norms on the space $L^2(X) \times L^2(\mathbb{R}^{n-1})^N$, through the use of exponential weight functions as described in §2.5. These weight functions will be based on the construction described in §2.6.

In most of this section we will consider the scaled operator $P_{\theta,R}$ globally, so we cannot replace it by P any longer. To alleviate notation, we will write this operator

$$P = P_{\theta,R}\,, \quad \theta = M_0 h \log(1/h)\,, \quad R \gg C_0\,, \tag{4.1}$$

where C_0 is the constant appearing in (1.5), and $M_0 > 0$ is some constant (which will be required to satisfy (2.28) once we fix the weight G).

We will first discuss the local behaviour construction each Σ_k and then, as in the previous section, adapt it to construct a global Grushin problem.

Our first task is still microlocal: we explain how a deformation of the norm on $L^2(X)$ by a suitable weight function G can be used to deform the norms on the N auxiliary spaces $L^2(\mathbb{R}^{n-1})$, microlocally near $\widetilde{\Sigma}_k$.

4.1. **Exponential weights near Σ_k.** As in §3.1, in this subsection we work microlocally in the neighbourhood Ω_k of one component Σ_k (Ω_k is the neighbourhood described in §3.1); we drop the index k in our notations. Notice that the complex scaling has no effect in this region, so $P \equiv P_{\theta,R}$. We will impose a constraint on the weight function G near Σ, and construct a weight functions g on $\widetilde{\Sigma}$. The construction of the local solution performed in §3.1 will then be studied in these deformed spaces.

Take a function $g^0 \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^{n-1})$, and use it to define $\widetilde{G}_0 \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$, so that

$$\widetilde{G}_0(x', x_n, \xi', \xi_n) = g^0(x', \xi') \quad \text{in } \widetilde{\Omega}'.$$

Then, using the Fourier Integral Operator U given in (3.3), one can construct a weight function $G_0 \in S(T^*X)$ such that

$$G_0^w \equiv U\,(\widetilde{G}_0)^w\,U^* \quad \text{microlocally near } \Omega.$$

Notice that G_0 now depends on h through an asymptotic expansion

$$G_0(h) \sim \sum_{j\geq 0} h^j\, G_{0,j}\,, \quad G_{0,j} \in \mathcal{C}^\infty_{\rm c}(T^*X) \text{ independent of } h\,. \tag{4.2}$$

This weight satisfies $G_{0,0} = \widetilde{G}_0 \circ \widetilde{\kappa}^{-1}$ in Ω, and the invariance property

$$[P(h), G_0^w(x, hD)] \equiv 0 \quad \text{microlocally in } \Omega\,. \tag{4.3}$$

As in §2.5, we rescale these weight functions by

$$G \stackrel{\rm def}{=} M h \log(1/h)\, G_0, \qquad g \stackrel{\rm def}{=} M h \log(1/h)\, g^0\,. \tag{4.4}$$

Still using the model hD_{x_n}, one can easily check the intertwining property

$$\begin{aligned} &G^w(x, hD_x; h)\, K(z) \equiv K(z)\, g^w(x', hD_{x'}; h) : H(\widetilde{\Sigma}) \to H(\Omega')\,, \\ &e^{-G^w(x,hD_x;h)/h}\, K(z) \equiv K(z)\, e^{-g^w(x',hD_{x'};h)/h} : H(\widetilde{\Sigma}) \to H(\Omega')\,. \end{aligned} \tag{4.5}$$

Using the weights G and g we define the microlocal Hilbert spaces $H_G(\Omega')$ and $H_g(\widetilde{\Sigma})$ by the method of §2.5. We need to check that the construction of a microlocal solution performed in §3.1.2 remains under control with respect to these new norms.

Lemma 4.1. *The operators*

$$K(z) \; : \; H_g(\widetilde{\Sigma}) \to H_G(\Omega') \,, \quad z \in \mathcal{R}(\delta, Ch)$$

satisfy the analogue of (3.7). *Namely, taking a cutoff* χ *jumping from* 0 *to* 1 *near* Σ *as in* §*3.1.2, then any* $v_+ \in H_g(\widetilde{\Sigma})$ *will satisfy*

$$\langle [(i/h)P, \chi^w] \, K(z) \, v_+, K(\bar{z}) \, v_+ \rangle_{H_G} \equiv \|v\|^2_{H_g} \,. \tag{4.6}$$

Proof. From the cutoff χ we define the deformed symbol χ_G through

$$\chi^w_G(x, hD) \stackrel{\text{def}}{=} e^{-G^w(x,hD)/h} \, \chi^w(x, hD) \, e^{G^w(x,hD)/h} \,.$$

The symbol calculus of §2.5 shows that χ_G also jumps from 0 to 1 near Σ, so that (returning to the convention of using χ for χ^w)

$$\begin{aligned} \langle [(i/h)P, \chi] K(z) v_+, K(\bar{z}) v_+ \rangle_{H_G} &\equiv \langle e^{-G/h} [(i/h)P, \chi] K(z) v_+, e^{-G/h} K(\bar{z}) v_+ \rangle_{L^2} \\ &\equiv \langle K(\bar{z})^* [(i/h)P_G, \chi_G] \, K(z) \, e^{-g/h} \, v_+, e^{-g/h} \, v_+ \rangle_{L^2} \\ &\equiv \langle K(\bar{z})^* [(i/h)P, \chi_G] \, K(z) \, e^{-g/h} \, v_+, e^{-g/h} \, v_+ \rangle_{L^2} \\ &\equiv \|e^{-g/h} \, v_+\|^2 \equiv \|v_+\|^2_{H_g} \,. \end{aligned}$$

In the second line we used (4.5), the third line results from $P \equiv P_G$, due to (4.3), and the last one from (3.7) applied to χ_G. ☐

Equation (4.5) shows that, for $z \in \mathcal{R}(\delta, Ch)$, the operator

$$K(z) : H_g(\widetilde{\Sigma}) \to H_G(\Omega) \quad \text{is asymptotically uniformly bounded} \,. \tag{4.7}$$

The above Lemma implies that the operators $R_+(z)$, $R_-(z)$ defined respectively in (3.9) and (3.13), are also asymptotically uniformly bounded with respect to the new norms:

$$\|R_+(z)\|_{H_G(\Omega) \to H_g(\widetilde{\Sigma})} = \mathcal{O}(1) \,, \qquad \|R_-(z)\|_{H_g(\widetilde{\Sigma}) \to H_G(\Omega)} = \mathcal{O}(1) \,. \tag{4.8}$$

The arguments presented in §3.1 carry over to the weighted spaces, and the microlocal solution to the problem (3.14) constructed in §3.1.3 satisfies the norm estimates

$$\|u\|_{H_G} + \|u_-\|_{H_g} \lesssim \|v\|_{H_G} + \|v_+\|_{H_g} \,. \tag{4.9}$$

Given a function $G_{0,0}(x, \xi)$ satisfying $H_p G_{0,0} = 0$ in Ω, one can iteratively construct a full symbol G_0 of the form (4.2), such that (4.3) holds. Now, the lower order terms in G_0 may change the norms only by factors $\big(1 + \mathcal{O}(hM \log(1/h))\big)$, so the same norm estimates hold if we replace G_0 by its principal symbol $G_{0,0}$ in the definition of the new norms. As a result, we get the following

Proposition 4.2. *Take* $\widetilde{G}_0 \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^{n-1})$, $G_0 \in \mathcal{C}^\infty_{\rm c}(X)$ *satisfying* $G_0 = \widetilde{G}_0 \circ \widetilde{\kappa}$ *in* Ω, *and*

$$G = Mh\log(1/h)\, G_0, \qquad g = Mh\log(1/h)\, \widetilde{G}_0 \,.$$

Then, the estimates (4.7–4.9) hold in the spaces $H_G(\Omega)$, $H_g(\widetilde{\Sigma})$.

4.2. **Globally defined operators and finite rank weighted spaces.** In this section we transform our microlocal Grushin problem into a globally defined one. This will require transforming all the microlocally defined operators ($R_\pm(z)$, $\mathcal{M}_{jk}(z)$) into globally defined operators acting on $L^2(X)$ or $L^2(\mathbb{R}^{n-1})$. Because our analysis took place near the trapped set K_0, we will need to restrict our auxiliary operators to some subspaces of $L^2(\mathbb{R}^{n-1})$ obtained as images of some finite rank projectors. These subspaces are composed of functions microlocalized near K_0. To show that the resulting Grushin problem is well-posed (invertible), the above construction must be performed using appropriately deformed norms on the spaces $L^2(X)$, $L^2(\mathbb{R}^{n-1})$, obtained by using globally defined weight functions G, g_j. Our first task is thus to complete the constructions of these global weights, building on §2.6 and §4.1.

4.2.1. *Global weight functions.* We will now construct global weight functions $G \in \mathcal{C}^\infty_{\rm c}(X)$, $g_j \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^{n-1})$ (one for each section Σ_j). For this, we will use the construction of an escape function away from K_0 presented in §2.6, and modify it near the Poincaré section so that it takes the form required in Proposition 4.2, and allows us to define auxiliary escape functions g_j. These weight functions will allow us to to define *finite rank* realizations of the microlocally defined operators $R_\pm(z)$ and $\mathcal{M}(z)$.

Our escape function $G_0 \in S(T^*X)$ is obtained through a slight modification of the weight $G_2(x,\xi)$ described in (2.26). The modification only takes place near the trapped set $\widehat{K}$, and in particular near the sections Σ_j. The following lemma is easy to verify.

Lemma 4.3. *Let* $\{\Omega_j,\}_{j=1,\dots,K}$ *be the neighbourhoods of* Σ_j *described in §3.1.1,* Ω'_j *and* Ω''_j *be small neighbourhoods of* Ω_j, $\Omega_j \Subset \Omega'_j \Subset \Omega''_j$, *and let* V *be a small neighbourhood of* $\widehat{K}_\delta$ *(see (2.24)). Then there exists* $G_0 \in \mathcal{C}^\infty_{\rm c}(T^*X)$ *such that*

$$\begin{aligned} & H_pG_0 \geq 1 \quad \text{on} \quad T^*_{B(0,3R)}X \cap p^{-1}([-\delta,\delta]) \setminus W, \quad W \stackrel{\rm def}{=} V \cup \bigcup_{j=1}^N \Omega''_j\,, \\ & H_pG_0 = 0 \quad \text{on} \quad \Omega'_j, \\ & H_pG_0 \geq 0 \quad \text{on} \quad T^*_{B(0,3R)}X, \\ & H_pG_0 \geq -\delta_0 \quad \text{on}\ T^*X. \end{aligned} \tag{4.10}$$

Besides, using the coordinate charts $\widetilde{\kappa}_j : \widetilde{\Omega}'_j \to \Omega'_j$ *(see §3.1.1), we can construct* G_0 *such that* $G_0 \circ \widetilde{\kappa}_j \restriction \widetilde{\Omega}'_j$ *is independent of the energy variable* $\xi_n \in [-\delta,\delta]$.

The last assumption (local independence on ξ_n) is not strictly necessary, but it simplifies our construction below, making the auxiliary functions g_j independent of z — see Proposition 4.2.

For the set V we assume that $V \Subset V_1' \Subset V_1$ (here V_1 is the set defined in (3.35)), and satisfies the following property. Consider the time $T > 0$ used to define the parametrix $E(z)$ in 3.15. Then there exists $t_1 > 0$ such that, for any $\rho \in p^{-1}([-\delta, \delta]) \setminus V_1'$, the trajectory segment $\{\Phi^t(\rho),\ 0 \leq t \leq T\}$ spends a time $t \geq t_1$ *outside of* W. The main consequence of this property is that

$$\forall \rho \in T^*_{B(0,2R)} \cap p^{-1}([-\delta, \delta]) \setminus V_1', \quad G_0(\Phi^T(\rho)) - G_0(\rho) \geq t_1 \,. \tag{4.11}$$

(Here we use the fact that T is small enough, so that a particle of energy $z \approx 0$ starting inside $T^*_{B(0,2R)}$ at $t = 0$ will remain inside $T^*_{B(0,3R)}$ up to $t = T$.) The set V will be further characterized in the next subsection.

From now on, we will take for weight function $G = Mh \log h\, G_0$ with such a function G_0, and use it to define a global Hilbert norm $\| \bullet \|_{H^k_G(X)}$ as in (2.20). As in Proposition 4.2, we define, for each $j = 1, \ldots, N$, the auxiliary weight

$$g_j(x', \xi') \stackrel{\text{def}}{=} Mh \log(1/h)\, G_0 \circ \widetilde{\kappa}_j(x', 0, \xi', 0)\,, \quad (x', \xi') \in \widetilde{\Sigma}_j\,, \tag{4.12}$$

and extend it to an element of $\mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^{(n-1)})$, so that the deformed Hilbert norm

$$\|v\|_{H_{g_j}} = \|e^{-g_j^w(x', hD'_x)/h}\, v\|_{L^2(\mathbb{R}^{n-1})}$$

is globally well-defined. Proposition 4.2 shows that our microlocal construction near Σ_j satisfies nice norm estimates with respect to the spaces $H_G(X)$, H_{g_j}.

To see the advantages of having weights which are escape function we state the following lemma which results from applying Lemma 2.2 to the case $T = \exp(-itP/h)$:

Lemma 4.4. *Suppose that* $\rho_1 = \Phi^t(\rho_0)$ *for some* $t > 0$, *and that*

$$G_0(\rho_1) > G_0(\rho_0)\,.$$

Suppose also that $\chi_j \in \mathcal{C}^\infty_{\rm c}(T^*X)$, $j = 0, 1$, *have their supports in small neighbourhoods of* ρ_j*'s. Then*

$$\|e^{-itP}\, \chi_0^w\|_{H_G \to H_G} \leq h^{M/C}\,, \qquad \|\chi_1^w\, e^{-itP}\|_{H_G \to H_G} \leq h^{M/C}\,, \tag{4.13}$$

for some $C > 0$ *independent of* M.

4.2.2. *Finite dimensional projections.* We want to construct a *finite dimensional* subspace of the Hilbert space $H_{g_j}(\mathbb{R}^{n-1})$, such that the microlocal spaces $H_{g_j}(V^j_\pm)$ are both approximated by it modulo $\mathcal{O}(h^\infty)$.

For each $j = 1, \ldots, N$, let S_j', S_j be two families of open sets with smooth boundaries in $T^*\mathbb{R}^{n-1}$, satisfying

$$\widetilde{\kappa}_j^{-1}(\widehat{\mathcal{T}}_j) \Subset S_j' \Subset S_j \subset \widetilde{D}_j\,, \quad j = 1, \ldots, N\,. \tag{4.14}$$

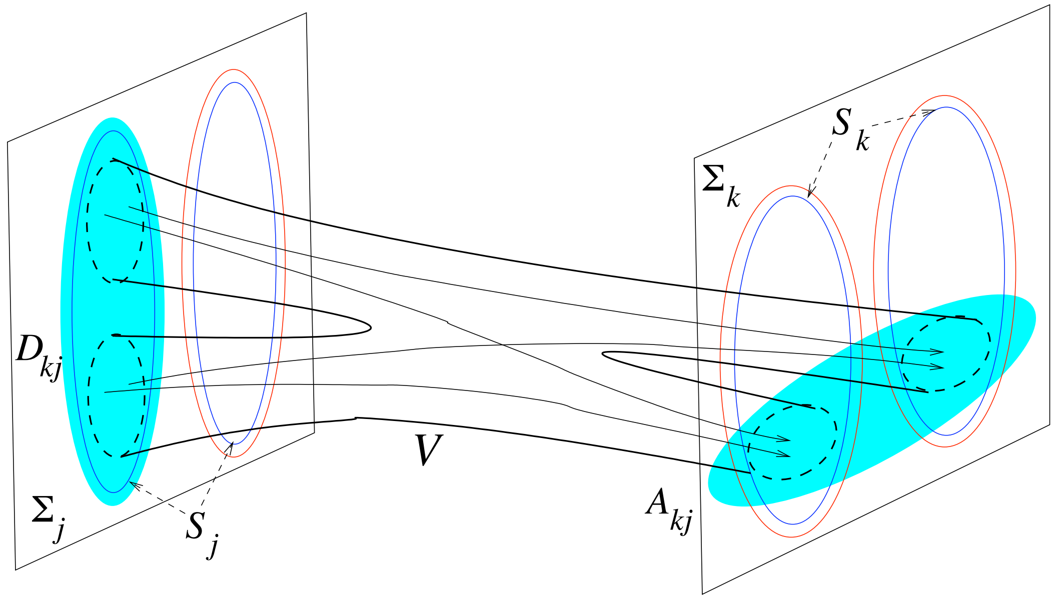

FIGURE 6. Schematic representation (inside some energy layer $p^{-1}(z)$) of the neighbourhood V. The departure/arrival sets D_{kj}, A_{kj} are similar to the ones appearing in figure 2. The sets S_k, S_j are represented through their images in Σ_k, Σ_j through $\kappa_{k,z}$, $\kappa_{j,z}$

In particular, each S_j, S'_j splits into disjoint components $S'_{kj} \Subset S_{kj} \subset \widetilde{D}_{kj}$.

Once these sets are chosen, we need to choose the set V in Lemma 4.3 to be thin enough:

Lemma 4.5. *For $\delta > 0$ small enough, there exists $V = \mathrm{neigh}(\widehat{K}_\delta, T^*X)$ and $t_0 > 0$ such that the following property holds.*

For any indices $j = 1, \ldots, N$, $k \in J_+(j)$, any $z \in [-\delta, \delta]$ and any point $\rho \in \widetilde{D}_{kj} \cap S_j$ such that its successor $\tilde{F}_{kj,z}(\rho)$ does not belong to S'_k, then the trajectory between $\kappa_{j,z}(\rho)$ and $F_{kj,z}(\kappa_{j,z}(\rho))$ spends a time $t \geq t_0$ outside of $W = V \cup \bigcup_{j=1}^N \Omega''_j$.

Notice that such a set V is necessarily contained in the union of the tubes $\widehat{T}_{kj}$ connecting the $D_{kj}(z)$ with the $A_{kj}(z)$ (see figure 6) for a sketch). Now, let

$$Q_j = Q_j(x', \xi'; h) \in S(T^*\mathbb{R}^{n-1}),$$

with leading symbol q_j independent of h (the leading symbol is meant in the sense of (2.22)). We choose that leading symbol to be real and have the following properties:

$$\begin{aligned} &q_j(\rho) < 0, \quad \rho \in S_j, \\ &q_j(\rho) > 0, \quad \rho \in T^*\mathbb{R}^{n-1} \setminus \overline{S}_j, \quad \liminf_{\rho \to \infty} q_j(\rho) > 0. \end{aligned} \tag{4.15}$$

Lemma 2.1 shows that one can choose Q_j so that

$$Q_j^w(x', hD_{x'}) \ : \ H_{g_j}(\mathbb{R}^{n-1}) \longrightarrow H_{g_j}(\mathbb{R}^{n-1}) \quad \text{is self-adjoint.}$$

Under the assumptions (4.15), we know that Q_j has discrete spectrum in a fixed neighbourhood of $\mathbb{R}_-$ when $h > 0$ is small enough. Let

$$\mathcal{H}_j \stackrel{\text{def}}{=} \Pi_j\big(H_{g_j}(\mathbb{R}^{n-1})\big) \,, \quad \text{where } \Pi_j \stackrel{\text{def}}{=} 1\!\!1_{\mathbb{R}_-}\big(Q_j^w(x', hD_{x'})\big) \,, \tag{4.16}$$

that is, Π_j is the spectral projection corresponding to the negative spectrum of Q_j^w. In particular,

$$\|\Pi_j\|_{H_{g_j}\to H_{g_j}} = 1, \qquad \dim(\mathcal{H}_j) \sim c_j\, h^{1-n} \,, \quad c_j > 0 \,. \tag{4.17}$$

The space $\mathcal{H}_j$ will be equipped with the norm $\| \bullet \|_{H_{g_j}}$. For future reference we record the following lemma based on functional calculus of pseudodifferential operators (see for instance [12, Chapter 7]):

Lemma 4.6. *For any uniformly bounded family of states* $u = (u(h) \in L^2(\mathbb{R}^{n-1}))_{h\to 0}$,

$$\mathrm{WF}_h(u) \Subset S_j \implies \|u - \Pi_j u\|_{H_{g_j}} = \mathcal{O}(h^\infty)\|u\|_{H_{g_j}} \,.$$

In §4.1 we used the microlocally defined operators

$$R_+^j(z) : H_G(\Omega_j) \to H_{g_j}(\widetilde{\Sigma}_j) \,.$$

Renaming them $R_{+,m}^j(z)$ (where m stands for *microlocal*) we now define

$$R_+^j(z) \stackrel{\text{def}}{=} \Pi_j\, R_{+,m}^j : H_G(X) \to \mathcal{H}_j \,. \tag{4.18}$$

The estimate (4.8) together with the above Lemma shows that

$$\|R_+^j(z)\|_{H_G(X)\to\mathcal{H}_j} = \mathcal{O}(1) \,, \quad z \in \mathcal{R}(\delta, Ch) \,. \tag{4.19}$$

The operators $R_+^j(z)$ are globally well-defined once we choose a specific realization of $R_{+,m}^j(z)$, which gives a unique definition mod $\mathcal{O}(h^\infty)$. We have thus obtained a family of uniformly bounded operators

$$R_+(z) \stackrel{\text{def}}{=} (R_+^1, \ldots, R_+^N) \ : \ H_G(X) \longrightarrow \mathcal{H}_1 \times \cdots \mathcal{H}_N \,.$$

In turn, the operators

$$R_-^j(z) \tag{4.20}$$

are obtained by selecting a realization of the microlocally defined operator $R_{-,m}^j(z)$ on $H_{g_j}(\widetilde{\Sigma}_j)$, and restricting that realization to $\mathcal{H}_j$:

$$R_-^j(z) = R_{-,m}^j(z)\,\Pi_j \ : \ \mathcal{H}_j \longrightarrow H_G(X) \,.$$

Again, these operators are well defined mod $\mathcal{O}(h^\infty)$. Putting together (4.8) with (4.17) ensures that

$$\|R_-^j(z)\|_{\mathcal{H}_j\to H_G} = \mathcal{O}(1) \,.$$

We group these operators into

$$R_-(z)\ :\ \mathcal{H}_1\times\cdots\mathcal{H}_N\longrightarrow H_G(X) \tag{4.21}$$
$$R_-(z)u_-=\sum_{j=1}^N R_-^j(z)\,u_-^j\,,\qquad u_-=(u_-^1,\dots,u_-^N)\,.$$

4.3. **A well posed Grushin problem.** With these definitions we consider the following Grushin problem:

$$\mathcal{P}(z)\ :\ H_G^2\times\mathcal{H}\to H_G\times\mathcal{H},\qquad \mathcal{H}\stackrel{\text{def}}{=}\mathcal{H}_1\times\cdots\mathcal{H}_N\,, \tag{4.22}$$
$$\mathcal{P}(z)\stackrel{\text{def}}{=}\begin{pmatrix}(i/h)(P_{\theta,R}(h)-z) & R_-(z)\\ R_+(z) & 0\end{pmatrix},\qquad z\in\mathcal{R}(\delta,Ch)\,.$$

Since $P_{\theta,R}(h)-z$ (which we will denote by $P-z$ for short) is a Fredholm operator, so is $\mathcal{P}(z)$, as we have only added finite dimensional spaces. For $\operatorname{Im} z>0$ the operator $(P-z)$ is invertible, so Lemma 2.3 shows that the index of $\mathcal{P}(z)$ is 0. Hence, in order to prove that $\mathcal{P}(z)$ is bijective it suffices to to construct an approximate right inverse and then use a Neumann series. The rest of this section will be devoted to the proof of this (approximate) right invertibility of $\mathcal{P}(z)$.

4.3.1. *A well-posed homogeneous problem.* As before we first consider the homogeneous problem

$$(i/h)(P-z)u+R_-(z)u_-=0\,,\qquad R_+(z)u=v_+\,, \tag{4.23}$$

where only one component v_+^k is nonzero (we may assume that $\|v_+^k\|_{\mathcal{H}_1}=1$). For that we adapt the methods of §3.2.2. We construct an approximate solution using the extended Poisson operator $K_k^+(z)$ (that operator acts on the microlocal space $H_{g_k}(\widetilde{\Sigma}_k)$, so its action on $\mathcal{H}_k$ is well-defined modulo $\mathcal{O}(h^\infty)$), and take

$$u=\chi_b^k\,K_k^+(z)\,v_+^k\,,$$

where χ_b^k is the backwards cutoff function with properties given in (3.12),(3.23) and (3.25).

The microlocally defined operator satisfies

$$R_{+,m}^k(z)\,u\equiv v_+^k+\mathcal{O}_{H_{g_k}}(h^\infty)\,,\qquad R_{+,m}^j(z)\,u=\mathcal{O}_{H_{g_j}}(h^\infty),\quad j\neq k\,.$$

As a result, projecting the left hand side onto $\mathcal{H}^k$ has a negligible effect:

$$R_+^k(z)\,u=\Pi_k(v_+^k+\mathcal{O}(h^\infty))=v_+^k+\mathcal{O}_{\mathcal{H}^k}(h^\infty)\,.$$

Following (3.27) we write

$$(i/h)(P-z)u\equiv[(i/h)P,\chi_b^k]\,K_k^+(z)v_+\in H_G(X)\,. \tag{4.24}$$

As noticed in §3.2.2, the transport properties of $K_k^+(z)$ show that u is microlocalized inside the union of tubes $\cup_{j\in J_+(k)}T_{jk}^{++}(z)$, so the right hand side in (4.24) splits into a component concentrated near $\widetilde{D}_k$, and other components concentrated near the arrival sets $A_{jk}(z)$, $j \in J_+(k)$. We rewrite (3.32) for the present data:

$$(i/h)(P-z)u \equiv R^k_{-,m}(z)v^k_+ - \sum_{j\in J_+(k)} R^j_{-,m}(z)\mathcal{M}_{jk}(z)v^k_+ \,. \tag{4.25}$$

Each state $\mathcal{M}_{jk}(z)v^k_+$ is microlocalized inside the arrival set $\widetilde{A}_{jk}(z) \subset \widetilde{\Sigma}_j$, which is not contained in S_j in general – see the remark at the end of §3.

Consequently one could fear that replacing the operators $R^j_{-,m}(z)$ by the truncated operators $R^j_-(z)$ would drastically modify the above right hand side. The microlocally weighted spaces H_G, H_{g_j} have been constructed precisely to avoid this problem. The mechanism is a direct consequence of the relative properties of the sets S_j and V explained in Lemma 4.5. Namely, a point $\rho_k \in S_{jk}$ is either "good", if its image $\rho_j = F_{jk,z}(\rho_k) \in S'_j$, or "bad", in which case

$$G_0(\rho_j) - G_0(\rho_k) \geq t_0 \,, \tag{4.26}$$

Let us choose a cutoff

$$\chi_j \in \mathcal{C}^\infty_{\rm c}(S_j), \quad \chi_j = 1 \ \text{ on } S'_j \,, \quad \chi_j = 0 \ \text{ outside } \operatorname{neigh}(S'_j, S_j) \,. \tag{4.27}$$

Since the Fourier integral operator $\mathcal{M}_{jk}(z) : H(\widetilde{D}_k) \to H(\widetilde{A}_{jk}(z))$ is uniformly bounded, (4.26) implies the norm estimate (see Lemma 4.4)

$$\forall v^k_+ \in \mathcal{H}_k \,, \qquad \|(1-\chi^w_j)\,\mathcal{M}_{jk}(z)\,v^k_+\|_{H_{g_j}} \lesssim h^{Mt_0}\,\|v^k_+\|_{\mathcal{H}_k}.$$

On the other hand, $\chi^w_j\,\mathcal{M}_{jk}(z)\,v^k_+$ is microlocalized inside $\operatorname{neigh}(S'_j, S_j)$, so Lemma 4.6 implies that $(\Pi_j - 1)\chi^w_j\,\mathcal{M}_{jk}(z)\,v^k_+ = \mathcal{O}_{H_{g_j}}(h^\infty)$. Putting these estimates altogether, we find that

$$\forall v^k_+ \in \mathcal{H}_k, \qquad \mathcal{M}_{jk}(z)\,v^k_+ = \Pi_j\,\mathcal{M}_{jk}(z)\,v^k_+ + \mathcal{O}(h^{Mt_0})\,\|v^k_+\| \,. \tag{4.28}$$

This crucial estimate shows that the projection of $\mathcal{M}_{jk}(z)\,v^k_+$ on $\mathcal{H}_j$ has a negligible effect. We now define the finite rank operators

$$\widetilde{M}_{jk}(z) \stackrel{\rm def}{=} \begin{cases} \Pi_j\,\mathcal{M}_{jk}(z)\,\Pi_k : \mathcal{H}_k \to \mathcal{H}_j \,, & j \in J_+(k) \,, \\ 0 \quad \text{otherwise} \,. \end{cases} \tag{4.29}$$

Using these operators, and remembering that the operators $R^j_- : \mathcal{H}_j \to H_G(X)$ are uniformly bounded, we rewrite (4.25) as

$$(i/h)(P-z)u \equiv R^k_-(z)v^k_+ - \sum_{j\in J_+(k)} R^j_-(z)\widetilde{M}_{jk}(z)\,v^k_+ + \mathcal{O}(h^{Mt_0})\,\|v^k_+\| \,.$$

Generalizing the initial data to arbitrary $v_+ \in \mathcal{H}_1 \times \cdots \times \mathcal{H}_N$, we obtain the

Proposition 4.7. *Assume $z \in \mathcal{R}(\delta, Ch)$. Let $v_+ \in \mathcal{H}$. Then there exists $(u, u_-) \in H^2_G(X) \times \mathcal{H}$ such that*

$$(i/h)(P - z)u + R_-(z)\, u_- = \mathcal{O}(h^{cM})\|v_+\|_{\mathcal{H}} \qquad in\ H_G(X), \tag{4.30}$$

$$R_+(z)u = v_+ + \mathcal{O}(h^\infty)\, \|v_+\|_{\mathcal{H}} \qquad in\ \mathcal{H}, \tag{4.31}$$

$$\|u\|_{H_G(X)} = \mathcal{O}(1)\|v_+\|_{\mathcal{H}}, \qquad \|u_-\|_{\mathcal{H}} = \mathcal{O}(1)\|v_+\|_{\mathcal{H}}. \tag{4.32}$$

The second part of the solution, u_-, is of the form

$$u_- = (\widetilde{M}(z) - Id)v_+\,, \quad \|\widetilde{M}(z)\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(1)\,,$$

where $\widetilde{M}(z) = (\widetilde{M}_{jk}(z))_{j,k=1,..,N}$ is the matrix of operators defined in (4.29).

We collect some properties of the operators $\widetilde{M}_{jk}(z)$, $j \in J_+(k)$:

- $\widetilde{M}_{jk}(z) = O(1) : \mathcal{H}_k \to \mathcal{H}_j$ and $\mathrm{WF}'_h(\widetilde{M}_{jk}(z)) \subset \overline{S}_j \times \overline{S}_k$.
- take $\rho_k \in \overline{S}_k$, $\rho_j = \tilde{F}_{jk,z}(\rho_k) \in \overline{S}_j$:
  (1) if the trajectory segment connecting the points $\kappa_{k,z}(\rho_k)$, $\kappa_{j,z}(\rho_j)$ is contained in W, then microlocally near (ρ_j, ρ_k), $\widetilde{M}_{jk}(z)$ is an h-Fourier integral operator of order zero with associated canonical transformation $\tilde{F}_{jk,z} = \kappa_{j,z}^{-1} \circ F_{jk,z} \circ \kappa_{k,z}$
  (2) if furthermore the above segment is disjoint from the support of G, then $\widetilde{M}_{jk}(z)$ is microlocally unitary near (ρ_j, ρ_k).
  (3) if, on the opposite, this segment contains a part outside W, then there exist $\chi_j \in \mathcal{C}^\infty_c(\mathrm{neigh}(\rho_j))$, $\chi_k \in \mathcal{C}^\infty_c(\mathrm{neigh}(\rho_k))$, equal to 1 near ρ_j and ρ_k respectively, such that
  $$\chi_j^w\, \widetilde{M}_{jk}(z)\chi_k^w = \mathcal{O}(h^{Mt_0}) : H_{g_k} \to H_{g_j}\,,$$
  with $t_0 > 0$ independent of M.

4.3.2. *A well-posed inhomogeneous problem.* Let us now consider the inhomogeneous problem

$$(i/h)(P_{\theta,R} - z)u + R_-(z)u_- = v \qquad v \in H_G(X)\,. \tag{4.33}$$

We will use a partition of unity to decompose v to several component.

Take $\psi_\delta \in S(T^*X)$, $\psi_\delta = 1$ near $p^{-1}([-\delta/2, \delta/2])$, and $\psi_\delta = 0$ outside $p^{-1}([-\delta, \delta])$. The operator $(P_{\theta,R} - z)$ is elliptic outside $p^{-1}[-\delta/2, \delta/2]$. Taking $\widetilde{\psi}_\delta$ similar with ψ_δ but with $\operatorname{supp} \widetilde{\psi}_\delta \subset p^{-1}([-\delta/2, \delta/2])$, the operator

$$L \stackrel{\text{def}}{=} (P_{\theta,R} - z - i\widetilde{\psi}_\delta^w) : H^2_G \to H_G$$

is invertible, with uniformly bounded inverse $L^{-1} \in \Psi^0_h$. Hence, by taking

$$u = (h/i)L^{-1}\,(1 - \psi_\delta^w)\, v\,,$$

we find

$$(i/h)(P_{\theta,R} - z)u = (i/h)(P_{\theta,R} - z - i\widetilde{\psi}_\delta^w)u + \mathcal{O}(h^\infty)\, \|u\| = (1 - \psi_\delta^w)\, v + \mathcal{O}(h^\infty)\|v\|\,,$$

which solves our problem for the data $(1-\psi_\delta^w)\,v$. The first equality uses pseudodifferential calculus and the fact that $\psi_\delta \equiv 1$ on the support of $\widetilde{\psi}_\delta$:

$$\widetilde{\psi}_\delta^w L^{-1}(1-\psi_\delta^w) = \mathcal{O}_{\mathcal{S}'\to\mathcal{S}}(h^\infty)\,.$$

Let us now consider the data $\psi_\delta^w v$ microlocalized in $p^{-1}([-\delta,\delta])$. We split this state using a cutoff $\psi_R \in \mathcal{C}_{\rm c}^\infty(X)$, such that $\psi_R = 1$ in $B(0,R)$, $\psi_R = 0$ outside $B(0,2R)$. To solve the equation

$$(i/h)(P_{\theta,R}-z)u = \widetilde{v}, \qquad \widetilde{v} = (1-\psi_R)\,\psi_\delta^w\,v\,, \tag{4.34}$$

we take the Ansatz

$$u = E(z)\,\widetilde{v}\,, \tag{4.35}$$

with $E(z)$ the parametrix of (3.15) (with P replaced by $P_{\theta,R}$). It satisfies

$$(i/h)(P_{\theta,R}-z)u = \widetilde{v} - e^{-iT(P_{\theta,R}-z)/h}\,\widetilde{v}\,. \tag{4.36}$$

The time T is chosen small enough, so that

$$\Phi^t\left(p^{-1}([-\delta,\delta])\setminus T^*B(0,R)\right)\cap T^*B(0,R/2) = \emptyset\,, \quad 0\le t\le T\,.$$

Hence, the states

$$\widetilde{v}(t) \stackrel{\rm def}{=} e^{-it(P_{\theta,R}-z)/h}\widetilde{v}$$

are all microlocalized outside $T^*B(0,R/2)$ for $t\in[0,T]$. The estimate (2.27) (adapted to the weight G_0) then implies that [28, Lemma 6.4]

$$\partial_t\|\widetilde{v}(t)\|^2_{H^G} = \frac{2}{h}\,\mathrm{Im}\langle P_{\theta,R}\widetilde{v}(t),\widetilde{v}(t)\rangle_{H_G} \le -c\,M\,\log(1/h)\,, \quad \forall t\in[0,T]\,,$$

where $c>0$ is independent of the choice of M. This shows that

$$\|e^{-iT(P_{\theta,R}-z)/h}\,\widetilde{v}\|_{H_G} \le C\,h^{cMt_2}\|\widetilde{v}\|_{H_G}\,,$$

so the problem (4.34) is solved modulo a remainder $\mathcal{O}(h^{cMt_2})$.

We now consider the component $\psi_R\psi_\delta^w v$ microlocalized in $T^*_{B(0,2R)}\cap p^{-1}([-\delta,\delta])$. We split it again using a cutoff $\psi_{V_1}\in\mathcal{C}_{\rm c}^\infty(V_1)$, $\psi_{V_1}=1$ in the set $V_1'\Subset V_1$ (see the discussion after Lemma 4.3). To solve the problem for the inhomogeneous data

$$\widetilde{v} = (1-\psi_{V_1}^w)\psi_R\psi_\delta^w v\,,$$

we use the Ansatz (4.35), resulting in (4.36). The microlocalization of $\widetilde{v}$ outside of V_1', together with the assumption (4.11), implies the norm estimate (see Lemma 4.4)

$$\|e^{-iT(P_{\theta,R}-z)/h}\,\widetilde{v}\|_{H_G} \le C\,h^{cMt_1}\|\widetilde{v}\|_{H_G}.$$

We finally consider the data $\widetilde{v} = \psi_{V_1}^w\psi_R\psi_\delta^w v$ microlocalized inside V_1. For this data, we can use the microlocal analysis of §3.2.3. If $\mathrm{WF}_h(\widetilde{v})$ is contained inside $V_1\cap\widehat{T}_{jk}^{--}$, then $\mathrm{WF}_h(\chi_b^k\,E(z)\,v)$ (see Ansatz (3.36)) will intersect Σ_j inside the arrival set $\widetilde{A}_{jk}(z)$, but not necessarily inside S_j. However, the same phenomenon as in Lemma 4.5 occurs: there exists a time $t_3 > 0$ such that, for any $z \in [-\delta, \delta]$ and any $\rho \in V_1 \cap \widehat{T}_{jk}^{--}$,

$$\rho_+(\rho) \in \widehat{\Sigma}_j \setminus \kappa_{j,z}(S'_j) \Longrightarrow G_0(\rho_+(\rho)) - G_0(\rho) \geq t_3 \,. \tag{4.37}$$

If we decompose $R^j_{+,m}(z)E(z)\widetilde{v}$ using the cutoff χ_j of (4.27), the property (4.37) implies that

$$\|(1-\chi_j^w)\, R^j_{+,m}(z)E(z)\widetilde{v}\|_{H_{g_j}} = \mathcal{O}(h^{Mt_3})\|\widetilde{v}\|_{H_G} \,.$$

Hence, if we set

$$u_-^j = R^j_+(z)\chi_j^w\, E(z)\widetilde{v} = R^j_{+,m}(z)\chi_j^w E(z)\widetilde{v} + \mathcal{O}(h^\infty) = R^j_{+,m}(z)E(z)\widetilde{v} + \mathcal{O}(h^{Mt_3})\,,$$

we end up with a solution of (4.33) modulo a remainder $\mathcal{O}(h^{Mt_3})\|\widetilde{v}\|_{H_G}$.

We have thus shown that the problem (4.33) admits a solution for any $v \in H_G$, up to some remainder $\mathcal{O}(h^{cM})$. We may then apply Proposition 4.7 to solve the resulting homogeneous problem, and get an approximate solution for the full problem (4.22). We summarize this solution in the following

Proposition 4.8. *Assume $z \in \mathcal{R}(\delta, Ch)$. Let $(v, v_+) \in H_G \times \mathcal{H}$. Then there exists $(u, u_-) \in H^2_G \times \mathcal{H}$ such that*

$$\begin{cases} (i/h)(P-z)u + R_- u_- &= v + \mathcal{O}(h^{cM})(\|v\|_{H_G} + \|v_+\|_{\mathcal{H}}) \qquad \text{in } H_G(X)\,, \\ R_+(z)u &= v_+ + \mathcal{O}(h^\infty)\ (\|v\|_{H_G} + \|v_+\|_{\mathcal{H}}) \qquad \text{in } \mathcal{H}\,, \end{cases} \tag{4.38}$$

$$\|u\|_{H^2_G} + \|u_-\|_{\mathcal{H}} = \mathcal{O}(1)\,(\|v\|_{H_G} + \|v_+\|_{\mathcal{H}})\,. \tag{4.39}$$

4.4. **Invertibility of the Grushin problem.** We can transform this approximate solution into an exact one. The system (4.38) can be expressed as an approximate inverse of $\mathcal{P}(z)$:

$$\begin{gathered} \begin{pmatrix} u \\ u_- \end{pmatrix} = \widetilde{\mathcal{E}}(z) \begin{pmatrix} v \\ v_+ \end{pmatrix}, \\ \mathcal{P}(z)\, \widetilde{\mathcal{E}}(z) = I + \mathcal{R}(h) : H_G \times \mathcal{H} \longrightarrow H_G \times \mathcal{H}\,, \quad \|\mathcal{R}(h)\| = \mathcal{O}(h^{cM})\,. \end{gathered} \tag{4.40}$$

For h small enough the operator $I + \mathcal{R}(h)$ can be inverted by a Neumann series, so we obtain an exact right inverse of $\mathcal{P}(z)$,

$$\mathcal{E}(z) = \widetilde{\mathcal{E}}(z)\,(I + \mathcal{R}(z))^{-1}\,.$$

Since $\mathcal{P}(z)$ is of index zero, $\mathcal{E}(z)$ is also a left inverse, which proves the well-posedness of our Grushin problem (4.22).

Theorem 2. *We consider $h > 0$ small enough, and $z \in \mathcal{R}(\delta, Ch)$. For every $(v, v_+) \in H_G \times \mathcal{H}$, there exists a unique $(u, u_-) \in H^2_G \times \mathcal{H}$ such that*

$$\begin{cases} (i/h)(P_{\theta,R} - z)u + R_-(z)u_- &= v \\ R_+(z)u &= v_+\,, \end{cases} \tag{4.41}$$

where $R_\pm(z)$ are defined by (4.18) *and* (4.20)*. The estimates (4.39) hold, so if we write*

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \mathcal{E}(z) \begin{pmatrix} v \\ v_- \end{pmatrix}, \qquad \mathcal{E}(z) = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix},$$

then the following operator norms (between the appropriate Hilbert spaces) are uniformly bounded:

$$\|E\|, \quad \|E_+\|, \quad \|E_-\|, \quad \|E_{-+}\| = \mathcal{O}(1). \tag{4.42}$$

Moreover, we have a precise expression for the effective Hamiltonian:

$$E_{-+}(z) = -I + \widetilde{M}(z) + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^{cM}) \stackrel{\text{def}}{=} -I + M(z,h), \tag{4.43}$$

where $\widetilde{M}(z)$ is the matrix of "open quantum maps" defined in (4.29) *and described after Proposition 4.7.*

Theorem 1 and the formula (1.3) follow from this more precise result. In fact, the equality (2.12) shows that

$$\operatorname{rank} \oint_z \chi R(w) \chi \, dw = \operatorname{rank} \oint_z \chi R_{\theta,R}(w) \chi \, dw = \frac{1}{2\pi i} \operatorname{tr} \oint_z R_{\theta,R}(w) \, dw, \tag{4.44}$$

and the well-posedness of our Grushin problem implies [39, Proposition 4.1] that the above right hand side is equal to

$$-\frac{1}{2\pi i} \operatorname{tr} \oint_z E_{-+}(w)^{-1} E'_{-+}(w) \, dw,$$

which in view of (4.43) gives (1.3).

References

[1] J. Aguilar, J.M. Combes, *A class of analytic perturbations for one-body Schrdinger Hamiltonians,* Comm. Math. Phys. 22(1971), 269–279.

[2] I. Alexandrova, *Semi-Classical Wavefront Set and Fourier Integral Operators,* Can. J. Math **60**(2008), 241–263.

[3] N.L. Balazs and A. Voros, *The quantized baker's transformation* , Ann. Phys. (NY) **190** (1989) 1–31

[4] E. B. Bogomolny, *Semiclassical quantization of multidimensional systems*, Nonlinearity **5** (1992) 805–866

[5] F. Borgonovi, I. Guarneri and D.L. Shepelyansky, *Statistics of quantum lifetimes in a classically chaotic system*, Phys. Rev. **A 43** (1991) 4517–4520

[6] J.-M. Bony and J.-Y. Chemin, *Espaces fonctionnels associés au calcul de Weyl-Hörmander,* Bull. Soc. math. France, **122**(1994), 77-118.

[7] F. Borgonovi, I. Guarneri and D. L. Shepelyansky, *Statistics of quantum lifetimes in a classically chaotic system*, Phys. Rev. **A 43** (1991) 4517–4520

[8] R. Bowen and P. Walters, *Expansive One-parameter Flows*, J. Diff. Equ. **12** (1972), 180–193

[9] B.V. Chirikov, F.M. Izrailev, D.L. Shepelyansky, *Dynamical stochasticity in classical and quantum mechanics.* Mathematical physics reviews, **2** (1981) 209–267, Soviet Sci. Rev. Sect. **2 C**: Math. Phys. Rev., 2, Harwood Academic, Chur.

[10] P. Cvitanović, P. Rosenquist, G. Vattay and H.H. Rugh, *A Fredholm determinant for semiclassical quantization*, CHAOS **3** (1993) 619–636

[11] M. Degli Esposti and S. Graffi (eds), *The mathematical aspects of quantum maps*, Springer, Heidelgerg, 2003.

[12] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the semi-classical limit,* Cambridge University Press, 1999.

[13] E. Doron and U. Smilansky, *Semiclassical quantization of chaotic billiards: a scattering theory approach*, Nonlinearity **5** (1992) 1055–1084; C. Rouvinez and U. Smilansky, *A scattering approach to the quantization of Hamiltonians in two dimensions – application to the wedge billiard*, J. Phys. **A 28** (1995) 77–104

[14] L.C. Evans and M. Zworski, *Lectures on Semiclassical Analysis,* `http://math.berkeley.edu/∼zworski/semiclassical.pdf`

[15] B. Georgeot and R. E. Prange, *Fredholm theory for quasiclassical scattering*, Phys. Rev. Lett. **74** (1995) 4110-4113

[16] P. Gaspard and S.A. Rice, *Semiclassical quantization of the scattering from a classically chaotic repellor,* J. Chem. Phys. **90**(1989), 2242-2254.

[17] C. Gérard, *Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes.* Mémoires de la Société Mathématique de France Sér. 2, **31**(1988), 1–146.

[18] C. Gérard and J. Sjöstrand, *Semiclassical resonances generated by a closed trajectory of hyperbolic type,* Comm. Math. Phys. **108**(1987), 391-421.

[19] M. Gutzwiller, *Chaos in classical and quantum mechanics*, Springer, New York, 1990

[20] B. Helffer, J. Sjöstrand, *Résonances en limite semi-classique,* Mém. Soc. Math. France (N.S.) 24–25(1986),

[21] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. I, II, Springer-Verlag, Berlin, 1983.

[22] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. III, IV, Springer-Verlag, Berlin, 1985.

[23] J.P. Keating, M. Novaes, S.D. Prado and M. Sieber, *Semiclassical structure of quantum fractal eigenstates*, Phys. Rev. Lett. **97** (2006) 150406

[24] A. Martinez, *Resonance free domains for non globally analytic potentials,* Ann. Henri Poincaré 3(4)(2002), 739–756. Erratum: Ann. Henri Poincar 8(7)(2007), 1425–1431.

[25] S. Nonnenmacher and M. Rubin, *Resonant eigenstates for a quantized chaotic system*, Nonlinearity **20** (2007) 1387–1420.

[26] S. Nonnenmacher and M. Zworski, *Distribution of resonances for open quantum maps,* Comm. Math. Phys. **269**(2007), 311–365.

[27] S. Nonnenmacher and M. Zworski, *Fractal Weyl laws in discrete models of chaotic scattering,* Journal of Physics A, **38** (2005), 10683-10702.

[28] S. Nonnenmacher and M. Zworski, *Quantum decay rates in chaotic scattering,* Acta Math. **203** (2009) 149–233.

[29] S. Nonnenmacher, J. Sjöstrand and M. Zworski, *Hyperbolic quantum monodromy operators,* in preparation.

[30] A. M. Ozorio de Almeida and R. O. Vallejos, *Decomposition of Resonant Scatterers by Surfaces of Section*, Ann. Phys. (NY) **278** (1999) 86–108

[31] L. Poon, J. Campos, E. Ott, and C. Grebogi, *Wada basin boundaries in chaotic scattering,* Int. J. Bifurcation and Chaos **6**(1996), 251–266.

[32] M. Saraceno and R.O. Vallejos, *The quantized D-transformation*, Chaos **6** (1996) 193–199

[33] H. Schomerus and J. Tworzydlo and, *Quantum-to-classical crossover of quasi-bound states in open quantum systems*, Phys. Rev. Lett. Phys. Rev. Lett. **93** (2004) 154102

[34] J. Sjöstrand, *A trace formula and review of some estimates for resonances*, in *Microlocal analysis and spectral theory* (Lucca, 1996), 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Kluwer Acad. Publ., Dordrecht, 1997.
[35] J. Sjöstrand, *Eigenvalue distribution for non-self-adjoint operators with small mult iplicative random perturbations,* arXiv:0802.3584
[36] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles*, Journal of AMS **4**(1991), 729–769
[37] J. Sjöstrand and M. Zworski, *Quantum monodromy and semiclassical trace formulae,* J. Math. Pure Appl. **81**(2002), 1–33.
[38] J. Sjöstrand and M. Zworski, *Fractal upper bounds on the density of semiclassical resonances,* Duke Math. J. **137**(2007), 381–459.
[39] J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems,* Annales de l'Institut Fourier, **57**(2007), 2095–2141.
[40] S.H. Tang and M. Zworski, *From quasimodes to resonances,* Math. Res. Lett. **5**(1998), 261-272.
[41] J. Tworzydlo, A. Tajic, H. Schomerus and C.W. Beenakker, *Dynamical model for the quantum-to-classical crossover of shot noise*, Phys. Rev. B **68** (2003) 115313
[42] A. Voros, *Unstable periodic orbits and semiclassical quantisation*, J. Phys. **A 21** (1988) 685–692
[43] A. Wirzba, *Quantum Mechanics and Semiclassics of Hyperbolic n-Disk Scattering Systems*, Phys. Rep. **309** (1999), 1-116

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