On the capacity of erasure relay channel: Multi-relay case

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***Abstract*— We consider here a single sender-destination multi-relay channel. The links connecting the nodes are supposed to be erasure where symbols are received correctly without any error, or lost. We consider that the nodes are not able to use any interference cancellation mechanism. The interference might be suppressed through using separated physical channel or thought a time-sharing mechanism. This model is realistic for many practical scenarios in the context of wireless networks.**

In previous works, the capacity region of broadcast erasure channels as well as the capacity of the single-sender relay channel (under degraded and non-degraded hypothesis) has been derived. This paper extends the previous results to the more general case of multi-relay channels. We derive the cut-set bound for a general (stationary ergodic) multi-relay erasure channel, and we show that it can be reached through a practical linear coding scheme based on MDS codes.

I. Introduction

Formally, a relay channel is a network consist of senders, receivers and a number of intermediate nodes which participate in the communication by relaying the packets from the sender to the receiver. The capacity region of the relay channels remains unknown in the general case. Until now, most of the researches have focused on the case of error channel with interferences between sender and relay transmission. The capacity of the simple case of relay channel, composed of a unique intermediate node, is presented in [2], [14] under the physically degraded hypothesis. The extension of the result to the multi-relay channel, under the degraded hypothesis, is presented in [6]. In these works interference cancellation mechanisms is used by nodes to attain the capacity region.

However; in many wireless channel architectures interference cancellation mechanism can not be used and the classical approaches as proposed in [14], [7] are not applicable. Moreover, from the viewpoint of the packet layer, where applications stand, channel appear as an erasure channel. Sender sends packets that might be received by destination nodes or be erased because of transmission errors, collisions, or buffer overflows. This important fact is sometime overlooked in the information theoretical literature and it results in a lot of simplification in analysis. In [5] a simple form capacity bound is derived under perfect side information hypothesis at the decoder. The side information is provided in the form of the exact erasure pattern over **every link** in the network. The capacity bound is achieved through a random coding scheme, but it seems that the achievability is only valid under degraded hypothesis. In [13] the capacity of a general stationary and ergodic broadcast erasure channel is derived which leads to a simple linear capacity bound. This capacity bound can be achieved optimally through a simple time sharing mechanism called Priority Encoding Technique. In [10] the capacity of the single relay erasure channel under degraded hypothesis is derived and a coding scheme based on a practical MDS code is proposed to achieve this capacity without need to any side information. In [9], the capacity of the single relay erasure channel without the restriction of degraded hypothesis is derived using image size theorem [3] and it is shown that a simple variation of the same coding scheme achieve the capacity. This means that the capacity of the single relay erasure channel is known to be achievable under general hypothesis of stationarity and ergodicity, and without any degraded hypothesis or side information. Finally in [11], capacity region for the single relay case is derived under cheap relay hypothesis [12], where the nodes cannot receive simultaneously from more than one source and use a temporal scheduling to suppress interference.

In this paper, we extend the previous works to the situation of single sender-receiver multi-relay channels for two different scenario : frequency assignment scenario which channel separation is obtained through using different physical channels and time devision scenario which a temporal scheduling is used to suppress interference. A cut-set bound is derived for these different scenarios and it is shown that almost all point of this bound can be reached by a simple and practical coding scheme. Cut-set bound can not be achieved in the frequency assignment scenario and where the transfer rate is limited by the receiver side bound. We believe that in this specific case, the cut-set bound is a looser bound and a tighter bound have to be driven. However, it is out of objective of this paper and it might be presented in a longer paper. We first present the theoretical bound in section II and then a coding scheme achieving the theoretical bounds is presented in section III.

II. Information Theoretical bounds

In this section we show the cut-set bound of a single sender-destination multi-relay channel. We use the subscripts to denote node and superscripts to denote time. Let's consider a set of N nodes, $\mathcal{N} = \{1, .., N\}$, communicating over a general

Vector Erasure Channel. For the special case of non-correlated erasure channel where erasure are not spatially correlated, and under the hypothesis that erasure probability do not depend on the sent symbol, the channel could be characterized by an erasure probability matrix $\mathbb{P}^u = (p_{ij}^u)_{N\times N}$, where p_{ij}^u is the erasure probability of a transmission between nodes i and j at the u-th transmission, $p_{ij}^u = \text{Prob}\{Y_{ji}^u = \mathtt{e}\}$. This comes from the erasure nature of the channel and the fact that symbols sent over the channel are not interfering. The received symbols separation is obtained though using different physical channel (interface card) or through a time-sharing mechanism (that could be centralized or distributed as CSMA/CA access mechanisms). As explained in [1] we might drop the superscript u for the stationary and ergodic erasure channels without loss of generality.

Let's suppose that there is a single communication taking place between a single sender-destination pair (s, d). Corresponding to this sender-destination pair is a message W. All the nodes in $\mathcal{N} - \{s, d\}$ are the relay nodes and are to be used to aid the communication. Now let's assume that nodes in $\mathcal{N}$ are divided into two sets, $\mathcal{S}$ and the complement $\mathcal{S}^c$. n_s (resp. n_{s^c}) is the number of nodes in $\mathcal{S}$ (resp. $\mathcal{S}^c$) and, l_i^s ,$i = 1,..,n_s$, (resp. $l_j^{s^c}$,$j = 1,..,n_{s^c}$) is node i in $\mathcal{S}$ (resp. j in $\mathcal{S}^c$) where, $l_1^s = s$ and $l_{n^{s^c}}^{s^c} = d$. We define X_i^{us} (resp. $X_j^{us^c}$) as the message sent by node i in $\mathcal{S}$ (resp. j in $\mathcal{S}^c$) at time u. Therefore, $\mathbf{X}^{us} = \{X_i^{us}, i \in [1, n^s]\}$ (resp. $\mathbf{X}^{us^c} = \{X_j^{us^c}, j \in [1, n^{s^c}]\}$) is the message vector sent by nodes in $\mathcal{S}$ (resp. $\mathcal{S}^c$) at time u. Let $Y_{i,l_j^s}^{u\ s^c}$,$i = 1,..,n_{s^c}$ and $j = 1,..,n_s$, the message received by node i in $\mathcal{S}^c$ if X_j^{us} is sent over the channel. $\mathbf{Y}_{l_j^s}^{u\ s^c} = \{Y_{i,l_j^s}^{u\ s^c}, i \in [1, n_{s^c}]\}$ is therefore the vector message received by nodes in $\mathcal{S}^c$ if X_j^{us} is sent and, $\mathbf{Y}^{us^c}$ is the vector message received by nodes in $\mathcal{S}^c$ if $\mathbf{X}^u = \{\mathbf{X}^{us}, \mathbf{X}^{us^c}\}$ are sent over the channel. The rate of flow of information from sender to the destination is bounded by the following theorem.

Theorem 1 (Cut Set bound of multi-relay channel) *The cut-set bound of a relay channel with a single source-destination pair (s, d) is bounded by :*

$$R \leq \min_{\mathcal{C}}\{R^c\}$$

for the frequency assignment scenario; such that R^c is defined as :

$$R^c \leq \sum_{i=0}^{n_s} I(X_i^s; \mathbf{Y}_{l_i^s}^{s^c})$$

and

$$R \leq \max_t \left[\min_{\mathcal{C}}\{R^c\}\right]$$

for the time division scenario where, the maximization occurs over all possible mode proportion $\{t_i\}_{i=1}^{n_c}$. t_i is the asymptotic proportional of time that the node l_i^s acts as a sender. Moreover, R^c is defined as :

$$R^c \leq \sum_{i=0}^{n_s} t_i \cdot I(X_i^s; \mathbf{Y}_{l_i^s}^{s^c})$$

$\mathcal{C}$ determines an $s - d$ cut.

See the appendices for the proof □.

The cut-set bound might be simplified thanks to the erasure nature of channels. The Shearer theorem presented thereafter is very helpful for the analysis of erasure channels.

Theorem 2 (Shearer Theorem [1]) *Let X^n be a collection of n random variables and Z^n be a collection of n boolean random variable, such that for each i, $1 \leq i \leq n$, $E\{Z_i\} = 1 - \tilde{C}$. If $X^n(Z^n)$ is a sub-collection containing the i^th random variable X_i if $Z_i = 1$. Then $E\{H(X^n(Z^n))\} \geq \tilde{C}H(X^n)$ □.*

The theorem can be extended to conditional entropy as well. It can be shown thanks to this theorem that the mutual information over a stationary and ergodic point to point erasure channel with an erasure process Z have a very simple form given by [13]:

$$I(X^n; Y^n) = n\tilde{C}H(X) \tag{1}$$

where, $E\{Z_i\} = 1 - \tilde{C}$ is the average erasure probability on the channel. In other word the capacity of a stationary and ergodic channel is simply $\tilde{C}$ symbol. Thanks to this result we have :

Theorem 3 (Cut-set bound of erasure multi-relay channel) *The capacity region of a single sender-destination erasure channel is bounded by :*

$$R \leq \min_{\mathcal{C}}\{R^c\}$$

for the frequency assignment scenario such that R^c is defined as :

$$R^c \leq \sum_{i=0}^{n_s}(1 - \prod_{j=1}^{n_{s^c}} p_{ij})$$

and

$$R \leq \max_t \left[\min_{\mathcal{C}}\{R^c\}\right]$$

for the time devision scenario where, R^c is defined as :

$$R^c \leq \sum_{i=0}^{n_s} t_i \cdot (1 - \prod_{j=1}^{n_{s^c}} p_{ij})$$

$p_{i,j}$ is the loss probability from node i in the set $\mathcal{S}$ to the node j in the set $\mathcal{S}^c$ and $\mathcal{C}$ determines an $s - d$ cut. □

III. ACHIEVABILITY AND CODING METHOD

In this section a coding scheme which attains the cut-set bound of theorem3 is presented. We first describe the simplest case of one intermediate node, as shown in figure1. we will then show that how this coding scheme can be generalized to the more general case of multi-relay. For the sake of simplification let's assume p_1, p_2, and p as the packet loss rate between sender-relay, relay-receiver, and sender-receiver. The cut-set bound can be expressed as : $R = \min\{(1-p_1p), (1-p) + (1-p_2)\}$ for the frequency assignment scenario and $R = \sup_{t_1} \min\{t_1 \cdot (1-p_1p), t_1 \cdot (1-p) + t_2 \cdot (1-p_2)\}$ for the time division scenario. We first describe the coding

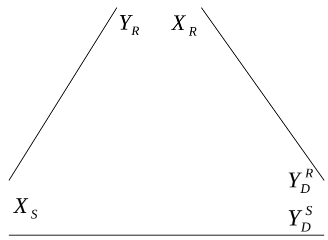

Fig. 1. Relay Channel

scheme for the degraded hypothesis when the relay node $\mathcal{R}$ has to receive all information send by the sender. We will then extend it to the non-degraded hypothesis when the relay node receives only a part of information and it is not able to decode initial symbol sent by the sender.

Before going to the details let's describe the characteristics of the class of Maximal Distance Separable (MDS) codes [15] that will be used thereafter. Let's suppose a systematic (n, k) MDS code taking k information symbols and generating n encoded symbols. Now this code has the property that the initial k information symbols can be retrieved from any combination of k encoded symbols out of the n symbols constituting the block, *i.e* the MDS code can retrieve up to $n - k$ erasure in a block of n packets. A MDS code can achieve the capacity of a stationary and ergodic point to point erasure channel asymptotically with a block size $n \to \infty$ if its rate R is less than the capacity of the channel. Maximal Distance Separable (MDS) code leads to sphere packing codes for erasure channels.

A. Coding scheme description: Degraded case

Lets suppose that we have designed an $(k + m + l, k)$ MDS code with an encoding matrix $[I_{k\times k}|A_{k\times m}|B_{k\times l}]$. At the sender we encode these packet with the $(n = k + m, k)$ MDS code with encoding matrix $[I_{k\times k}|A_{k\times m}]$ leading to m redundant packet and a rate $R^s = \frac{k}{n}$. These redundant packets will help the receiver and the relay to retrieve the erased packets over the channel. Under the condition that $R^s < (1 - p_1)$ a MDS code will asymptotically with large k and n, ensure perfect communication between the sender and relay, as the MDS codes achieve the capacity of the erasure channel. Such a code will therefore validate the conditions of the degraded situation.

Following the coding structure proposed in [14] the relay should transfer only side information's (indices) reducing the ambiguity at the receiver. Every packet of an MDS code can be used as an index that will reduce the ambiguity about the initial message. We therefore generate at the relay some redundant packets coming from multiplying the message by the encoding matrix $B_{k\times l}$. At the receiver side we will receive asymptotically with large n, around $n(1 - p)$ packets from the sender and $l(1 - p_2)$ packets from relay. The receiver have to decode the MDS code with generator matrix $[I_{k\times k}|A_{k\times m}|B_{k\times l}]$. The k message packets can be recovered if we receive from sender and relay at least k packets. Using this coding scheme relay only send useful index information to the receiver. The fact that $[I_{k\times k}|A_{k\times m}|B_{k\times l}]$ is the generator of an MDS code guarantee that every packet received from the relay will reduce the ambiguity about the initial message. The decoding process can be hold on the receiver if $n(1-p)+l(1-p_2) > k$. The rate of information can be transfered by the channel is therefore bounded by : $R = R^s = \frac{k}{n} \le \min\{(1-p_1), (1-p_2)+(1-p)\}$ for the frequency assignment scenario (maximum can be reached by choosing $l = n$) and by : $R = t_1 \cdot R^s = \sup_t\{\min\{t_1(1 - p_1), t_1(1 - p_2) + t_1(1 - p)\}\}$ for the time devision scenario. t_1 is the proportion of time that $\mathcal{S}$ acts as a sender and t_2 is the proportion of time that $\mathcal{R}$ transmit over channel.

B. Coding scheme description: Non Degraded case

In this section we will present coding schemes applicable without the degraded assumption. The main difference between this situation with the degraded case is that in the non-degraded situation the receiver might have some information that have not been received by the relay, where under the degraded assumption all information available at receiver are also available at the relay. We will show that in the time devision scenario the cut-set bound is achievable, however; for the frequency assignment scenario the bound can be attained if $(1 - p_1p) \le \{(1 - p) + p(1 - p_2)\}$, *i.e.*, $p_2 \le p_1$.

Let's suppose that the sender use a (n, k) MDS code $\left[I_{k\times k}|A_{k\times(n-k)}\right]$. Under the general erasure relay channel scenario defined previously if an asymptotically large number n of packet is sent by the sender, a number $n(1 - p)$ (resp. $n(1 - p_1)$) packets are received at receiver (resp. relay). Out of these packets $n(1 - p)(1 - p_1)$ are received at both the relay and sender and $np_1(1 - p)$ (resp. $np(1 - p_1)$) only at the receiver (resp. the relay). Relay has to forward a sufficient number of packets only received by it to eliminate ambiguity at receiver. However, the relay and receiver are not aware of which packets they respectively received. The solution consists of forwarding all received packets at the relay to the receiver, *i.e.*, the relay assumes that the $n(1 - p_1)$ packets received at the relay are information packets.

However, the relay is not able to send all of these packets to the receiver as it can not send with a rate more than $(1 - p_2)$ over the relay to destination channel. Moreover, it only received $n(1-p_1)$ packets from the sender. The solution is that the relay chooses randomly $k^* = \min\{n(1 - p_1), n(1 - p_2)\}$ packets from its received packets and sends them over the channel using a (n, k^*) MDS code $\left[B_{k^*\times(n-k^*)}\right]$. The receiver could retrieve packets sent by relay if ,asymptotically with large k^* and n, $k^* < n(1-p_2)$. As said before, only p percent of these packets has not been yet received by the receiver. Asymptotically for large n and k, *i.e.* k^*, if $n(1-p)+k^*p > k$ the initial packets sent by sender can be recovered successfully at the destination.

$R = \frac{k}{n} = \min\{(1 - p_1p), (1 - p) + p(1 - p_2)\}$ for the frequency assignment scenario and $R = \min\{t_1(1 - p_1p), t_1(1 - p) + t_2(1 - p_2)\}$ for the time devision scenario. Therefore the cut-set bound can be attained for the time devision scenario while, for the frequency assignment scenario the cut-set bound is achievable if $p_2 \leq p_1$.

C. Multi-relay

The presented scheme for the single relay hypothesis can be extended in a straightforward way to the multi-relay hypothesis. We just do the same things for all nodes in the channel. Let's consider that the sender has k information symbol to send over the channel. It uses the encoding matrix $B^0_{k\times n}$ (known by all the nodes) and send X^n over the channel.

If a relay node receives sufficient numbers (more exactly k) of independent codeword packets from the environment it is able to decode the information symbols sent by the sender. This node is therefore in the degraded situation. The decoded block is re-encoded by using a random linear encoding function where, the coefficients of the encoding function are drawn uniformly from $\mathbb{F}_q$. The encoding vector (the vector of coefficients of the encoding function) of codeword packets are sent along with them, in their headers. This information would help the decoding and encoding process in the other nodes of the channel. In order to reduce the possibility of sending useless codeword packets over the channel, the encoding vector of the generated codeword packets have to be chosen independent from those received by the node.

If a relay node has not receives sufficiently number of independent codewords from the environment the decoding process can not be hold. This node is therefore in the non-degraded situation. In this case the relay node assumes that the received packets are information packets. It chooses randomly k_i^* packets of the received packets and thereafter re-encodes them by using a random linear encoding function and sends the results over the channel. k_i^* depends on the amount of the independent codewords received at the node.

This process is continued until the final destination that will be able to decode information symbols sent by the sender if it receives at least k packets from the channel.

It is worth notable that in the single relay hypothesis the codeword packets sent by the sender are independent. Therefore; each codeword received at the relay is an independent linear equation of the information symbols. However; For multi-relay channel the probability that all the transmitted codewords by nodes in channel are independent is not actually one. To handle this problem relay nodes constraints a reception matrix where, rows of this matrix are encoding vectors of each received packet. If the reception of a packet increases the rank of this matrix it is an independent codeword packet and therefore is useful. A packet which does not increase the rank of this matrix contains information previously received by the relay and it has to be dropped. By this scheme the probability that two nodes have the same set of blocks and pick the same set of coefficients and construct the same block goes to zero if the size of field $\mathbb{F}_q$ is sufficiently large (for the proof see [8]).

IV. CONCLUSION

We presented the capacity region of the erasure multi-relay channel and show that it is achievable by a practical coding scheme based on MDS codes. Reed Solomon codes are MDS codes and can be used for applying the proposed coding scheme. This scheme can be easily implemented in actual WIFI based wireless networks and it does not need any new physical layer architecture as needed for interference cancellation method.

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APPENDIX A

We first proof the first part of theorem for frequency assignment scenario. Let's consider that W is uniformly distributed over his respective range $\{1, 2, .., 2^{nR^c}\}$.

$$
\begin{aligned}
nR^c &= H(W) \\
&= I(W;\mathbf{Y}^{1^{s^c}},..,\mathbf{Y}^{n^{s^c}}) + H(W|\mathbf{Y}^{1^{s^c}},..,\mathbf{Y}^{n^{s^c}}) \\
&\overset{(a)}{\leq} I(W;\mathbf{Y}^{1^{s^c}},..,\mathbf{Y}^{n^{s^c}}) + n\epsilon_n \\
&\overset{(b)}{=} \textstyle\sum_{u=1}^n I(W;\mathbf{Y}^{u^{s^c}}|\mathbf{Y}^{1^{s^c}},..,\mathbf{Y}^{u-1^{s^c}}) + n\epsilon_n \\
&\overset{(c)}{=} \textstyle\sum_{u=1}^n H(\mathbf{Y}^{u^{s^c}}|\mathbf{Y}^{1^{s^c}},..,\mathbf{Y}^{u-1^{s^c}}) \\
&\quad -H(\mathbf{Y}^{u^{s^c}}|\mathbf{Y}^{1^{s^c}},..,\mathbf{Y}^{u-1^{s^c}},W) + n\epsilon_n \\
&\overset{(d)}{\leq} \textstyle\sum_{u=1}^n H(\mathbf{Y}^{u^{s^c}}) - H(\mathbf{Y}^{u^{s^c}}| \\
&\quad \mathbf{Y}^{1^{s^c}},..,\mathbf{Y}^{u-1^{s^c}},W,\mathbf{X}^{u^s},\mathbf{X}^{u^{s^c}}) + n\epsilon_n \\
&\overset{(e)}{=} \textstyle\sum_{u=1}^n H(\mathbf{Y}^{u^{s^c}}) - H(\mathbf{Y}^{u^{s^c}}|\mathbf{X}^{u^{s^c}},\mathbf{X}^{u^s}) + n\epsilon_n \\
&= \textstyle\sum_{u=1}^n I(\mathbf{X}^{u^s};\mathbf{Y}^{u^{s^c}}|\mathbf{X}^{u^{s^c}}) + n\epsilon_n
\end{aligned}
$$

where (a) follows from Fano's inequality, (b) from the chain rule (c) from the definition of mutual information, (d) from the fact that removing conditioning increase the first term and conditioning reduces the second term, and (e) from the fact that $\mathbf{Y}^{us^c}$ depends only on $\mathbf{X}^{us}$ and $\mathbf{X}^{us^c}$ property of the channel [2]. Moreover; $I(\mathbf{X}^{us};\mathbf{Y}^{us^c}|\mathbf{X}^{us^c})$ can be simplified as following :

$$\begin{aligned} I(\mathbf{X}^{us};\mathbf{Y}^{us^c}|\mathbf{X}^{us^c}) &= I(X_1^{us},..,X_{n_s}^{u\ s};\mathbf{Y}^{us^c}|\mathbf{X}^{us^c}) \\ &\overset{(f)}{=} I(X_1^{us},..,X_{n_s}^{u\ s};\mathbf{Y}_{l_1^s}^{u\ s^c},..,\mathbf{Y}_{l_{n_s}^s}^{u\ \ s^c} \\ &\quad |\mathbf{X}^{us^c}) \\ &\overset{(g)}{=} I(X_1^{us},..,X_{n_s}^{u\ s};\mathbf{Y}_{l_1^s}^{u\ s^c},..,\mathbf{Y}_{l_{n_s}^s}^{u\ \ s^c}) \\ &= I(\mathbf{X}^{us};\mathbf{Y}^{us^c}) \\ &\overset{(h)}{\leq} \textstyle\sum_{i=0}^{n_s} I(X_i^{us};\mathbf{Y}_{l_i^s}^{u\ s^c}) \end{aligned}$$

(f) follows from the definition of $\mathbf{Y}_{l_i^s}^{u\ s^c}$, (g) from the fact that $\mathbf{Y}_{l_i^s}^{u\ s^c}$ is the received vector message if X_i^{us} is sent over the channel. Moreover; based on the definition of relay channel [4] $\mathbf{X}^{us^c}$ only depends on the past received symbols of $\mathbf{Y}^{s^c}$. Therefore at the transmission time u, the received vector $\mathbf{Y}_{l_i^s}^{u\ s^c}$ only depends on X_i^{us} and is conditionally independent from $\mathbf{X}^{us^c}$ (also note that $\mathbf{X}^{us^c}$ is sent over a channel different from X_i^{us} and so they have not any interfere), and (h) as said before, the nodes send out over the physically separated channels. From the point of view of the nodes therefore, the relay channel seems as a set of point to point channels. In such a scenario, it is not difficult to show that the maximum of $I(\mathbf{X}^{us};\mathbf{Y}^{us^c})$ achieves if X_j^s, sent by l_j^s, is independent from X_i^s, sent by l_i^s, for $i \neq j \in [0, n_c - 1]$. Thus the mutual information can be then written by the summation.

By replacing $I(\mathbf{X}^{us};\mathbf{Y}^{us^c}|\mathbf{X}^{us^c})$ in the previous inequality we have :

$$\begin{aligned} nR^c &\leq \textstyle\sum_{i=0}^{n_s}\sum_{u=1}^{n} I(X_i^{us};\mathbf{Y}_{l_i^s}^{u\ s^c}) + n\epsilon_n \\ &= \textstyle\sum_{i=0}^{n_s}\sum_{u=1}^{n} I(X_{i\,q}^{\ s};\mathbf{Y}_{l_i^s\,q}^{\ \ s^c}|Q=u) + n\epsilon_n \\ &\overset{(i)}{=} \textstyle\sum_{i=0}^{n_s} nI(X_{i\,Q}^{\ s};\mathbf{Y}_{l_i^s\,Q}^{\ \ s^c}|Q) + n\epsilon_n \\ &\overset{(j)}{=} \textstyle\sum_{i=0}^{n_s} nI(X_i^{\ s};\mathbf{Y}_{l_i^s}^{\ \ s^c}|Q) + n\epsilon_n \\ &\overset{(k)}{\leq} \textstyle\sum_{i=0}^{n_s} nI(X_i^{\ s};\mathbf{Y}_{l_i^s}^{\ \ s^c}) \end{aligned}$$

(i) follows from the introduction of a new time sharing random variable Q uniformly distributed on $\{1, 2, ..., n\}$, (j) follows by defining $X_i^{\ s} \triangleq X_{i\,Q}^{\ s}$, $\mathbf{Y}_{l_i^s}^{\ \ s^c} \triangleq \mathbf{Y}_{l_i^s\,Q}^{\ \ s^c}$ as the new random variable where $Q \to X_i^{\ s} \to \mathbf{Y}_{l_i^s}$, and (k) from the Markov chain properties. The proof is therefore completed for the frequency assignment scenario. It is notable that this bound can be also derived by using the image size theorem [3].

APPENDIX B

Now let's assume the time division scenario. In this case each node in the channel might be in one of these two states: sending (S) and receiving (R). The scheduling algorithm defines different transmission mode for the relay network by assigning to each node a state (R or S). Collisions occur in a mode if two or more senders are assigned to the S state by the scheduler. For the sake of simplification let's consider the simplest case of single-relay channel where, $\mathcal{S}$ is the sender, $\mathcal{R}$ is the relay, and $\mathcal{D}$ is the destination (see figure 1). In this case, the scheduler defines four different modes. In mode m_1 node $\mathcal{S}$ send messages and nodes $\mathcal{D}$ and $\mathcal{R}$ receive the message with a specified probability that depends on the distance and transmission power. In this mode the relay channel appears as an erasure broadcast channel $\left(\mathbf{X}_{\mathcal{S}(m_1)};\mathbf{Y}_{\mathcal{R}(m_1)},\mathbf{Y}_{\mathcal{D}(m_1)}^{\mathcal{S}}\right)$ as defined in [13]. In the second mode m_2 $\mathcal{R}$ acts as the sender and $\mathcal{S}$ and $\mathcal{D}$ are in the receiving mode. In this case the channel appears as a point to point erasure channel $(\mathbf{X}_{\mathcal{R}(m_2)};\mathbf{Y}_{\mathcal{D}(m_2)}^{\mathcal{R}})$. In mode m_3, $\mathcal{S}$ and $\mathcal{R}$ are in sending state and collision occurs. In this mode the nodes do not receive any information. In the last mode m_4, all the nodes are in receiving state and no information is transferred through the packets. Let suppose that the t_i is the proportion of time that the wireless network is in state m_i. Clearly $\sum_{m=1}^{M} t_i = 1$, meaning that the scheduling mechanism do a time-sharing between the different modes.

The previous description might be generalized in a straightforward way to a network with more than one intermediate nodes. In this setting any possible assignment of state value to each node define a transmission mode m_i that is active a proportion t_i of time. We will assume for the sake of tractable theoretical analysis that the scheduling is deterministic, meaning that the scheduling is defined in advance independently of transmission results. In [12] a cut-set bound is derived for the achievable rate over a general multi-mode relay channels :

Theorem 4 (Cheap relay channel Cut-set Bound [12]) *Consider a general network with N nodes and a finite number of states, M. Now suppose that state of network is a deterministic function for every network use v as m_v and is fixed and is known to all nodes then*

$$\sum_{i\in S, j\in S^c} R^{ij} \leq \sum_{m=1}^{M} t_m I\left(X_{(m)}^{S}, Y_{(m)}^{S^c}|X_{(m)}^{S^c}\right)$$

for all $S \subset \{1, 2, .., N\}$, where the set of all nodes are partitioned into two disjoint set S and S^c by cut-set. The portion of time that network operate in mode m is defined as $t_m = \lim_{k\to\infty} n_m(v)/v$. For any state m, $n_m(v)$ is defined as the number of state which is equal to m in the first v network uses$\square$.

Lets consider t_1 as the proportion of time that $\mathcal{S}$ is int he sending mode and t_2 is the proportion of time that $\mathcal{R}$ is the transmitter. For the cuts C_1 and C_2 shown in figure1 we have :

$$\begin{aligned} R^{c_1} &\leq \{\ t_1 I(\mathbf{X}_{(m_1)}^{s};\mathbf{Y}_{(m_1)}^{s^{c_1}}|\mathbf{X}_{(m_1)}^{s^{c_1}}) \\ &\qquad + t_2 I(\mathbf{X}_{(m_2)}^{s},\mathbf{Y}_{(m_2)}^{s^{c_1}}|\mathbf{X}_{(m_2)}^{s^{c_1}}\ \} \\ R^{c_2} &\leq \{\ t_1 I(\mathbf{X}_{(m_1)}^{s};\mathbf{Y}_{(m_1)}^{s^{c_2}}) + t_2 I(\mathbf{X}_{(m_2)}^{s};\mathbf{Y}_{(m_2)}^{s^{c_2}})\ \} \end{aligned}$$

For C_1 and in the mode m_1 (resp. m_2), $\{l_1^s = \mathcal{S}\}$ (resp. $\{l_1^{s^{c_1}} = \mathcal{R}\}$) acts as sender and $\{\mathcal{R}, l_2^{s^{c_1}} = \mathcal{D}\}$ (resp. $\{\mathcal{S},\mathcal{D}\}$) are the receivers. For C_2 and in the mode m_1 (resp. m_2), $\{l_1^s = \mathcal{S}\}$ (resp. $\{l_2^s = \mathcal{R}\}$) acts as the sender and $\{\mathcal{R}, l_1^{s^{c_2}} = \mathcal{D}\}$ (resp. $\{\mathcal{S},\mathcal{D}\}$) are the receivers. Therefore we have :

$$\begin{aligned} R^{c_1} &\leq t_1 \cdot I(X_1^s;\mathbf{Y}_{l_1^s}^{s^{c_1}}) \\ R^{c_2} &\leq t_1 \cdot I(X_1^s;\mathbf{Y}_{l_1^s}^{s^{c_2}}) + t_2 \cdot I(X_2^s;\mathbf{Y}_{l_2^s}^{s^{c_2}}) \end{aligned}$$

The same arguments can be used for the multi-relay case which leads us to the cut-set bound.