ON THE HODGE DECOMPOSITION IN $\mathbb{R}^n$

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ABSTRACT. We prove a version of L^p-Hodge decomposition for differential forms in Euclidean space and a generalization to the class of Lizorkin currents. Using these tools, we also compute $L_{q,p}$-cohomology of $\mathbb{R}^n$.

1. INTRODUCTION

The classical *Hodge decomposition Theorem* is a fundamental result in differential geometry. It says that on a compact Riemannian manifold without boundary, any L^2-differential form can be uniquely decomposed as sum of an exact form plus a coexact and a harmonic form. The standard proof is obtained by constructing an inverse G to the Laplacian Δ on the L^2-orthogonal complement to the space of harmonic forms (see, e.g., [14, Chap. 5.8]). The map G is called the *Green operator*. In 1995, Chad Scott proved that this decomposition also holds for L^p-differential forms on a closed manifold [11]. For non compact, complete Riemannian manifolds, the L^2-Hodge decomposition holds under some technical hypothesis (which can be written in the language of L^2-cohomology, see [5, Theorem 14.3]), but the situation for L^p-forms is still an open problem.

The main goal of the present paper is to provide a rigorous statement and proof of the Hodge decomposition theorem for L^p-differential forms in Euclidean space. Our proof uses standard techniques from Fourier analysis, but also the more specialized notion of *Lizorkin currents* (see Sections 4 and 8.2 below and the books [7], [9]). In order to keep the paper readable for non specialists we give all the necessary background starting from the classic notion of tempered distribution, which are dual objects to rapidly decreasing smooth functions.

Our techniques also provide a Hodge decomposition for *temperate currents*, which are differential forms on $\mathbb{R}^n$ with coefficients in the space of temperate distributions (besides a short mention in the last section of the book [10], temperate currents seem to have been left aside in the literature). Let us denote by $\mathcal{S}'(\mathbb{R}^n, \Lambda^k)$ the topological vector space of temperate currents of degree k, we then have the following theorem:

Theorem 1.1. *There is an exact sequence*

$$0 \to \mathcal{H}(\mathbb{R}^n, \Lambda^k) \to \mathcal{S}'(\mathbb{R}^n, \Lambda^k) \xrightarrow{\Delta} \mathcal{S}'(\mathbb{R}^n, \Lambda^k) \to 0, \tag{1.1}$$

where $\mathcal{H}(\mathbb{R}^n, \Lambda^k)$ is the space of differential forms on $\mathbb{R}^n$ whose coefficients are harmonic polynomials.

This theorem is contained in Corollary 7.3 below. We also have an exact sequence

$$0 \to \mathcal{H}(\mathbb{R}^n, \Lambda^k) \to \mathcal{P}(\mathbb{R}^n, \Lambda^k) \xrightarrow{\Delta} \mathcal{P}(\mathbb{R}^n, \Lambda^k) \to 0, \tag{1.2}$$

where $\mathcal{P}(\mathbb{R}^n, \Lambda^k)$ is the space of differential forms of degree k on $\mathbb{R}^n$ with polynomial coefficients.

It follows from the sequence (1.1) and the identity $\Delta = (d\delta + \delta d)$ that any $\theta \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$ can be written as

$$\theta = \Delta\omega = d(\delta\omega) + \delta(d\omega) \tag{1.3}$$

for some $\omega \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$. In particular, any temperate current $\theta \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$ can be decomposed as sum of an exact current plus a coexact current; this is the *Hodge–Kodaira decomposition* for such currents. A similar statement holds for polynomial differential forms.

If one formally introduces the operators

$$U = \delta \circ \Delta^{-1}, \quad U^* = d \circ \Delta^{-1},$$

then the Hodge decomposition (1.3) writes

$$\theta = d(U\theta) + \delta(U^*\theta). \tag{1.4}$$

Of course, due to the kernel $\mathcal{H}(\mathbb{R}^n, \Lambda^k)$ in the exact sequence (1.1), the operator Δ^{-1}, as well as U and U^*, is not really well defined. However if one restricts our attention to the space $L^p(\mathbb{R}^n, \Lambda^k) \subset \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$ of differential forms with coefficients in $L^p(\mathbb{R}^n)$, then

$$L^p(\mathbb{R}^n, \Lambda^k) \cap \mathcal{H}(\mathbb{R}^n, \Lambda^k) = \{0\}$$

for any $1 \leqslant p < \infty$ because $L^p(\mathbb{R}^n)$ does not contains any non zero polynomials. The Laplacian $\Delta \colon L^p(\mathbb{R}^n, \Lambda^k) \to \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$ is thus injective and the operators U and U^* can be properly defined on appropriate subspaces of $\mathcal{S}'(\mathbb{R}^n, \Lambda^k)$.

We can now state the Hodge–Kodaira decomposition for the space $L^p(\mathbb{R}^n, \Lambda^k)$:

Theorem 1.2. *Let $1 < p < \infty$. The space $L^p(\mathbb{R}^n, \Lambda^k)$ admits the following direct sum decomposition*

$$L^p(\mathbb{R}^n, \Lambda^k) = EL^p(\mathbb{R}^n, \Lambda^k) \oplus E^*L^p(\mathbb{R}^n, \Lambda^k), \tag{1.5}$$

where $EL^p(\mathbb{R}^n, \Lambda^k) = L^p(\mathbb{R}^n, \Lambda^k) \cap d\mathcal{S}'(\mathbb{R}^n, \Lambda^{k-1}T^*\mathbb{R}^n)$ is the space of exact currents belonging to L^p and $E^*L^p(\mathbb{R}^n, \Lambda^k) = L^p(\mathbb{R}^n, \Lambda^k) \cap \delta\mathcal{S}'(\mathbb{R}^n, \Lambda^{k+1}T^*\mathbb{R}^n)$ is the space of coexact currents belonging to L^p. Furthermore:

  (i) *$EL^p(\mathbb{R}^n, \Lambda^k)$ and $E^*L^p(\mathbb{R}^n, \Lambda^k)$ are closed subspaces;*
  (ii) *the projections $E \colon L^p(\mathbb{R}^n, \Lambda^k) \to EL^p(\mathbb{R}^n, \Lambda^k)$ and $E^* \colon L^p(\mathbb{R}^n, \Lambda^k) \to E^*L^p(\mathbb{R}^n, \Lambda^k)$ are bounded operators;*
  (iii) *these operators satisfy*

$$E^2 = E, \quad E^{*2} = E^*, \quad E + E^* = \mathrm{Id};$$

(iv) *the projection E is self-adjoint, meaning that if $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\langle E\theta, \phi \rangle = \langle \theta, E\phi \rangle,$$

for any $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$ and $\phi \in L^q(\mathbb{R}^n, \Lambda^k)$ (where $\langle \theta, \phi \rangle = \int_{\mathbb{R}^n} \theta \wedge \star\phi$);
(v) *the same property holds for E^*.*

We proove this result in Section 7.4.

Corollary 1.3. *If $\frac{1}{p} + \frac{1}{q} = 1$, then $E^* L^q(\mathbb{R}^n, \Lambda^k)$ and $EL^p(\mathbb{R}^n, \Lambda^k)$ are orthogonal, meaning that*

$$\langle \theta, \phi \rangle = \int_{\mathbb{R}^n} \theta \wedge \star\phi = 0,$$

for any $\theta \in EL^p(\mathbb{R}^n, \Lambda^k)$ and $\phi \in E^* L^q(\mathbb{R}^n, \Lambda^k)$.

Proof. Since E and E^* are projectors on complementary subspaces, we have $E\theta = \theta$ for any $\theta \in EL^p(\mathbb{R}^n, \Lambda^k)$ and $E\phi = 0$ for any $\phi \in E^* L^q(\mathbb{R}^n, \Lambda^k)$, hence $\langle \theta | \phi \rangle = \langle E\theta | \phi \rangle = \langle \theta | E\phi \rangle = \langle \theta | 0 \rangle = 0$. $\square$

The previous theorem implies that any differential form $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$ admits a Hodge–Kodaira decomposition $\theta = d\alpha + \delta\beta$, where $d\alpha = E\theta$ and $\delta\beta = E^*\theta$ belong to $L^p(\mathbb{R}^n, \Lambda^k)$. The forms α and β are in general just temperate distributions, but more can be said if $1 < p < n$:

Theorem 1.4. *Let $1 < p < n$ and $q = \frac{np}{n-p}$. There are bounded linear operators*

$$U^*: L^p(\mathbb{R}^n, \Lambda^k) \to L^q(\mathbb{R}^n, \Lambda^{k-1}) \quad and \quad U: L^p(\mathbb{R}^n, \Lambda^k) \to L^q(\mathbb{R}^n, \Lambda^{k+1}),$$

such that $E = d \circ U: L^p(\mathbb{R}^n, \Lambda^k) \to EL^p(\mathbb{R}^n, \Lambda^k)$ and $E^* = \delta \circ U^*: L^p(\mathbb{R}^n, \Lambda^k) \to E^* L^p(\mathbb{R}^n, \Lambda^k)$. In particular, any differential form $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$ can be uniquely decomposed as a sum of an exact form $d\alpha$ plus a co-exact form $d\beta$ with $\alpha = U\theta \in L^q(\mathbb{R}^n, \Lambda^{k-1})$ and $\beta = U^*\theta \in L^q(\mathbb{R}^n, \Lambda^{k+1})$:

$$\theta = E\,\theta + E^*\,\theta = d(U\theta) + \delta(U^*\theta) = d\alpha + \delta\beta. \tag{1.6}$$

This theorem is also proved in Section 7.4. We will also see that such a decomposition exists only if $1 < p < n$ and $q = \frac{np}{n-p}$.

As said before, our results are proved using various known facts from harmonic analysis and symbolic calculus in the context of differential forms and currents. We present all necessary notions in a self contained way, and the paper is organized as follows: In Section 2, we recall some basic facts about the space $\mathcal{S}'$ of tempered disitribution, the Fourier transform and the notion of convolution. In Section 3 we recall the characterization of polynomials as elements in $\mathcal{S}'$ annihilating some power of the Laplacian. In Section 4 we introduce the Lizorkin distributions, which are tempered distributions modulo the space of polynomials. In Section 5, we develop some symbolic calculus for Lizorkin distributions and apply it to the Riesz potential and Riesz transform, and in Section 6 we recall some basic facts from L^p-harmonic analysis.

In Sections 2–6, only functions are investigated. In Section 7, we recall some basic facts about differential forms in $\mathbb{R}^n$ and we prove all the results stated in the present introduction. Section 8 is devoted to some applications of the previous results; in

particular, we prove three fundamental inequalities concerning differential forms in $\mathbb{R}^n$ and we give a necessary and sufficient condition for the vanishing of the L_{qp}-cohomology of $\mathbb{R}^n$. The paper ends with a technical appendix devoted to a calculation of the Fourier transform of the Riesz kernel.

2. The Space of Tempered Distributions

We will work with the Fourier transform of tempered distributions as they are developed, e.g., in [10], [13], [14]. Recall that the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions is the space of smooth functions $f \colon \mathbb{R}^n \to \mathbb{C}$ such that

$$[f]_{m,\alpha} = \|(1+|x|)^m \partial^\alpha f\|_{L^\infty(\mathbb{R}^n)} < \infty$$

for all $m \in \mathbb{N}$ and all multi-indices $\alpha \in \mathbb{N}^n$. This is a Frechet space for the topology induced by the collection of all semi-norms $[\cdot]_{m,\alpha}$, it is dense in $L^p(\mathbb{R}^n)$ for any $1 \leqslant p < \infty$ and it is also a pre-Hilbert space for the inner product $(f|g) = \langle f, \bar{g} \rangle$, where

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) g(x) \, dx. \tag{2.1}$$

Recall also that $\mathcal{S}$ is an algebra for the multiplication and for the convolution product, it is closed under translation, differentiation and multiplication by polynomials.

Of basic importance is the fact that the Fourier transform[1]

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix\cdot\xi} \, dx$$

is an isomorphism $\mathcal{F} \colon \mathcal{S} \to \mathcal{S}$, with inverse

$$\mathcal{F}^{-1}(g)(x) = \check{g}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, dx.$$

Some of the basic properties of the Fourier transform are

(i) $\mathcal{F}(f*g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$ (here and below $*$ is the convolution product);
(ii) $\mathcal{F}(f \cdot g) = \frac{1}{(2\pi)^n} \mathcal{F}(f) * \mathcal{F}(g)$;
(iii) $\mathcal{F}(\partial_j f)(\xi) = -i\xi_j \cdot \mathcal{F}(f)(\xi)$;
(iv) $\mathcal{F}(\bar{g}) = (2\pi)^n \overline{\mathcal{F}^{-1}(g)}$;
(v) $\mathcal{F}(f \circ A) = \frac{1}{|\det A|} \mathcal{F}(f) \circ (A^{-1})^t$ for any $A \in \mathrm{GL}_n(\mathbb{R})$.

From Fubini's Theorem, we have

$$\langle \mathcal{F}f, g \rangle = \langle f, \mathcal{F}g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(y) e^{ixy} \, dx \, dy. \tag{2.2}$$

This identity can also be written as

$$(\mathcal{F}f|g) = (2\pi)^n \left(f|\mathcal{F}^{-1}g \right) \quad \text{or} \quad (\mathcal{F}f|\mathcal{F}h) = (2\pi)^n (f|h) \tag{2.3}$$

(just set $h = \mathcal{F}^{-1}g$ in the previous identity). The latter formula is the *Parseval–Plancherel identity*.

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[1]There are different conventions for this definition, this affects some constants in the following formulas. Here, we follow [13].

The topological dual of $\mathcal{S}$ is called the space of *tempered distributions* and is denoted by $\mathcal{S}'$, and if $w \in \mathcal{S}'$ and $f \in \mathcal{S}$, the evaluation of w on f will be denoted by

$$\langle w, f \rangle \in \mathbb{C}.$$

Any measurable function f such that $|f(x)| \leqslant C(1 + |x|^m)$ for some $m > 0$ and any function in $f \in L^p(\mathbb{R}^n)$ defines a tempered distributions[2] by the formula (2.1). Distributions with compact support also belong to $\mathcal{S}'$.

The space $\mathcal{S}'$ is a complete locally convex topological vector space when equipped with the weak* topology, i.e. the smallest topology for which the linear form

$$w \mapsto \langle w, \varphi \rangle$$

is continuous for any $\varphi \in \mathcal{S}$ (note that $\mathcal{S}'$ is not a Frechet space).

Lemma 2.1. *If $A \subset \mathbb{R}^n$ is a non empty closed subset, then*

$$\mathcal{S}'_A = \{w \in \mathcal{S}' \colon \operatorname{supp}(w) \subset A\}$$

is a closed subset in $\mathcal{S}'$.

Proof. Suppose that $w_0 \notin \mathcal{S}'_A$, then, by definition, there exists a function $\varphi \in \mathcal{S}$ such that $\operatorname{supp}(\varphi) \cap A = \varnothing$ and $s = \langle w_0, \varphi \rangle > 0$. Consider now the set $\mathcal{W} \subset \mathcal{S}'$ defined by

$$\mathcal{W} = \left\{ w \in \mathcal{S}' \colon \langle w, \varphi \rangle > \frac{s}{2} \right\}.$$

By definition of the weak* topology, $\mathcal{W}$ is open in $\mathcal{S}'$. It is clear that $\mathcal{W} \cap \mathcal{S}'_A = \varnothing$. We have thus found, for any $w_0 \notin \mathcal{S}'_A$, an open set such that

$$w_0 \in \mathcal{W} \subset \mathcal{S}' \setminus \mathcal{S}'_A.$$

This means that the complement of $\mathcal{S}'_A$ is an open subset in $\mathcal{S}'$. $\qquad\square$

The differential operator ∂_i acts continuously on $\mathcal{S}'$ by duality:

$$\langle \partial_i w, f \rangle = -\langle w, \partial_i f \rangle.$$

We can also define the Fourier transform by

$$\langle \mathcal{F}w, f \rangle = \langle w, \mathcal{F}f \rangle$$

and its inverse by

$$\langle \mathcal{F}^{-1}w, f \rangle = \langle w, \mathcal{F}^{-1}f \rangle.$$

These are continuous isomorhisms $\mathcal{F}, \mathcal{F}^{-1} \colon \mathcal{S}' \to \mathcal{S}'$ which are inverse to each other. Some important examples of Fourier transforms are

$$\mathcal{F}(e^{-\frac{|x|^2}{2}})(\xi) = (2\pi)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{2}}, \quad \mathcal{F}(1) = (2\pi)^n \delta_0, \quad \mathcal{F}(\delta_0) = 1,$$

where $\delta_0 \in \mathcal{S}'$ is the Dirac measure.

The convolution of two tempered distributions is in general not defined, but we can define a convolution product

$$* \colon \mathcal{S} \times \mathcal{S}' \to \mathcal{S}'$$

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[2]More generally, a complex Borel measure μ on $\mathbb{R}^n$ belongs to $\mathcal{S}'$ if and only if $|\mu(B(0, R))| \leqslant C \cdot (1 + R)^N$ for some $N \in \mathbb{Z}$ and all $R > 0$.

by the formula

$$\langle f * w, \, g \rangle = \langle w, \, \tilde{g} * f \rangle,$$

where $w \in \mathcal{S}'$ and $f, g \in \mathcal{S}$. Here $\tilde{f}(x) = f(-x)$. Observe that this formula is consistent with Fubini theorem in the case $w \in \mathcal{S}$. The Dirac measure $\delta_0 \in \mathcal{S}'$ is the convolution identity in the sense that

$$f * \delta_0 = f.$$

for all $f \in \mathcal{S}$.

3. The Laplacian and Polynomials

Let uss denote by $\mathcal{P}$ the space of all polynomials $P \colon \mathbb{R}^n \to \mathbb{C}$. It is a subspace of $\mathcal{S}$ and it has the following important characterization (see [14, Proposition 4.5]):

Proposition 3.1. *A tempered distribution $f \in \mathcal{S}'$ is a polynomial if and only if the support of its Fourier transform is contained in $\{0\}$:*

$$\mathcal{P} = \{ f \in \mathcal{S}' \colon \operatorname{supp} \hat{f} \subset \{0\} \}.$$

Corollary 3.2. $\mathcal{P}$ *is a closed subspace of $\mathcal{S}'$.*

Proof. This follows directly from the previous proposition and Lemma 2.1. $\qquad\square$

The Laplacian on $\mathbb{R}^n$ is the partial differential operator $\Delta = -\sum_{j=1}^{n} \partial_j^2$. For a distribution $w \in \mathcal{S}'$, we define $\Delta w \in \mathcal{S}'$ by

$$\langle \Delta w, \, \varphi \rangle = \langle w, \, \Delta \varphi \rangle$$

for any $\varphi \in \mathcal{S}$. The relation with the Fourier transform is given by

$$\mathcal{F}(\Delta w)(\xi) = |\xi|^2 \mathcal{F}(w).$$

A distribution $w \in \mathcal{S}'$, is called *harmonic* if $\Delta w = 0$ and we denote by $\mathcal{H}$ the space of harmonic tempered distributions, i.e., the kernel of Δ:

$$\mathcal{H} = \{ w \in \mathcal{S}' \colon \Delta w = 0 \}.$$

Proposition 3.3. *A tempered distribution $f \in \mathcal{S}'$ is a polynomial if and only if $\Delta^m f = 0$ for some $m \in \mathbb{N}$.*

Proof. It is obvious that if $f \in \mathcal{P}$ is a polynomial of degree m, then $\Delta^m f = 0$. Conversely, if $\Delta^m f = 0$, then

$$0 = \mathcal{F}(\Delta^m f)(\xi) = (-1)^m |\xi|^{2m} \hat{f}(\xi),$$

hence $\operatorname{supp} \hat{f} \subset \{0\}$. $\qquad\square$

We just proved that $\mathcal{H} = \ker \Delta \subset \mathcal{P}$. A consequence of this result is the following generalization of Liouville's theorem:

Corollary 3.4.

$$L^\infty(\mathbb{R}^n) \cap \ker \Delta = \mathbb{R} \quad and \quad L^p(\mathbb{R}^n) \cap \ker \Delta = \{0\} \quad for\ 1 \leqslant p < \infty. \qquad\square$$

Observe however that not every globally defined harmonic function in $\mathbb{R}^n$ is a polynomial, for instance the function $h(x) = \sin(x_1)\sinh(x_2)$ is harmonic. Of course $h \notin \mathcal{S}'$.

Theorem 3.5. *The Laplacian $\Delta\colon \mathcal{P} \to \mathcal{P}$ is surjective; we thus have an exact sequence*

$$0 \to \mathcal{H} \to \mathcal{P} \xrightarrow{\Delta} \mathcal{P} \to 0.$$

Proof. A computation shows that if $m \in \mathbb{N}$ and $h \in \mathcal{P}$ is a homogenous function of degree ν (i.e., $h(tx) = t^\nu h(x)$ for $t > 0$), then

$$\Delta(|x|^{2m+2}h(x)) = |x|^{2m+2}\Delta h(x) + c_{n,m,\nu}|x|^{2m}h(x), \tag{3.1}$$

where $c_{n,m,\nu} = 2(m+1)(2m+2\nu+n)$ (use Euler's formula for homogenous functions: $\sum x_i\partial_i h = \nu\cdot h$). On the other hand, a basic result about polynomials (see [2]) says that any $f \in \mathcal{P}$ can be written as a finite sum

$$f(x) = \sum_{m,\nu} a_{m,\nu}|x|^{2m}h_{m,\nu}(x), \tag{3.2}$$

where $h_{m,\nu} \in \mathcal{H}$ is a homogenous polynomial of degree ν Now it is clear from (3.1) that $f = \Delta g$ with

$$g(x) = \sum_{m,\nu} \frac{a_{m,\nu}}{c_{n,m,\nu}}|x|^{2m+2}h_{m,\nu}(x). \tag{3.3}$$

The surjectivity of $\Delta\colon \mathcal{P} \to \mathcal{P}$ follows. $\qquad\square$

Remark. The paper [2] gives an explicit procedure to compute the decomposition (3.2), the proof thus shows that the inverse Laplacian $\Delta^{-1}\colon \mathcal{P} \to \mathcal{P}$ given by (3.3) is algorithmically computable.

4. The Lizorkin Space and its Fourier Image

Definition. We introduce two subspaces Φ and Ψ of $\mathcal{S}$ defined as follow:

$$\Phi = \bigcap_{m=0}^{\infty} \Delta^m(\mathcal{S}) \quad \text{and} \quad \Psi = \{\psi \in \mathcal{S}\colon \partial^\mu\psi(0) = 0, \text{ for any } \mu \in \mathbb{N}^n\}.$$

The space Φ is called the *Lizorkin space*, basic references on this space are [7], [9]. We shall see below that Ψ is the Fourier dual of Φ, that is the image of Φ under the Fourier transform.

Theorem 4.1. *The restriction of the Laplacian to the Lizorkin space is a bijection $\Delta\colon \Phi \to \Phi$.*

Proof. The Laplacian is injective on $\mathcal{S}$ because $\ker\Delta \cap \mathcal{S} \subset \mathcal{P} \cap \mathcal{S} = \{0\}$ by proposition 3.3. To prove the surjectivity, consider an arbitrary element $\varphi \in \Phi$. By definition, for any $m \in \mathbb{N}$ there exists $g_m \in \mathcal{S}$ such that $\Delta^m g_m = \varphi$. Observe that $\Delta(\Delta^m g_{m+1} - g_1) = \varphi - \varphi = 0$. Since Δ is injective on $\mathcal{S}$, we have $g_1 = \Delta^m g_{m+1} \in \Delta^m(\mathcal{S})$. It follows that $g_1 \in \Phi$ and therefore $\varphi = \Delta g_1 \in \Delta\Phi$. $\qquad\square$

Proposition 4.2. *For any rapidly decreasing function $\psi \in \mathcal{S}$, the following conditions are equivalent*:

(a) $\psi \in \Psi$;

(b) $\partial^\mu \psi(\xi) = o(|\xi|^t)$ *as* $|\xi| \to 0$ *for any multi-indice* $\mu \in \mathbb{N}^n$ *and any* $t > 0$;

(c) $|\xi|^{-2m}\psi \in \mathcal{S}$ *for any* $m \in \mathbb{N}$.

Proof. The implication (b) $\Rightarrow$ (a) is obvious and (a) $\Rightarrow$ (b) is clear by Taylor expansion.

To prove that (b) $\Rightarrow$ (c), observe that condition (b), together with the Leibniz rule, implies that the function $|\xi|^{-2m}\psi$ vanishes at the origin and is continuous as well as all its derivatives. It is then clear that $|\xi|^{-2m}\psi \in \mathcal{S}$.

To prove (c) $\Rightarrow$ (a), observe that condition (c) says that $\psi = |\xi|^{2m}\rho$ for some function $\rho \in \mathcal{S}$. By the Leibniz rule, we then have $\partial^m \psi(0) = \partial^m(|\xi|^{2m}\rho)(0) = 0$. $\square$

Proposition 4.3. *We have* $\mathcal{F}(\Phi) = \Psi$.

Proof. For any $\varphi \in \Phi$ and $m \in \mathbb{N}$ there exists $\varphi_m \in \mathcal{S}$ such that $\Delta^m \varphi_m = \varphi$. The Fourier transform of this relation writes $\widehat{\varphi} = (-1)^m |\xi|^{2m}\widehat{\varphi}_m$, thus $|\xi|^{-2m}\widehat{\varphi} = (-1)^m \widehat{\varphi}_m \in \mathcal{S}$ for any integer m and it follows from condition (c) in the previous proposition that $\widehat{\varphi} \in \Psi$, hence $\mathcal{F}(\Phi) \subset \Psi$.

To prove the opposite inclusion, we consider a function $\psi \in \Psi$. Using again condition (c) in the previous proposition, we know that for any $m \in \mathbb{N}$, we can write $\psi = |\xi|^{2m}\psi_m$ for some function $\psi_m \in \mathcal{S}$. We then have

$$\mathcal{F}^{-1}(\psi) = \mathcal{F}^{-1}(|\xi|^{2m}\psi_m) = (-1)^m \Delta^m(\mathcal{F}^{-1}(\psi_m)),$$

hence $\mathcal{F}^{-1}(\psi) \in \bigcap_{m=0}^\infty \Delta^m(\mathcal{S}) = \Phi$. $\square$

Corollary 4.4. *For any* $\varphi \in \mathcal{S}$, *we have*

$$\varphi \in \Phi \iff \langle P, \varphi \rangle = 0 \text{ for any polynomial } P \in \mathcal{P}.$$

Proof. For any $\varphi \in \mathcal{S}$ and $\mu \in \mathbb{N}^n$, we have

$$\langle x^\mu, \varphi \rangle = \int_{\mathbb{R}^n} x^\mu \varphi(x)\, dx = i^{-|\mu|} \int_{\mathbb{R}^n} (ix)^\mu \varphi(x) e^{-ix \cdot 0}\, dx = i^{-|\mu|} \partial^\mu \hat{\varphi}(0).$$

Thus $\langle P, \varphi \rangle = 0$ for any polynomial if and only if $\partial^\mu \widehat{\varphi}(0) = 0$ for any $\mu \in \mathbb{N}^n$, i.e., if $\widehat{\varphi} \in \Psi$ and we conclude by the previous proposition. $\square$

Proposition 4.5. Ψ *is a closed ideal of* $\mathcal{S}$.

Proof. It is clear from the Leibniz rule that if $\psi \in \Psi$ and $f \in \mathcal{S}$, then $\partial^\mu(f\psi)(0) = 0$ for any $\mu \in \mathbb{N}^n$, hence $\Psi \subset \mathcal{S}$ is an ideal. To show that $\Psi \subset \mathcal{S}$ is closed, let us consider a sequence $\{\psi_j\} \subset \Psi$ converging to $\psi \in \mathcal{S}$. This means that for any $\mu \in \mathbb{N}^n$, $m \in \mathbb{N}$, we have $\sup_{x \in \mathbb{R}^n}(1 + |x|)^m |\partial^\mu \psi - \partial^\mu \psi_j| \to 0$ as $j \to \infty$. But then $\partial^\mu \psi(0) = \lim_{j \to \infty} \partial^\mu \psi_j(0) = 0$. $\square$

Since $\mathcal{F} \colon \mathcal{S} \to \mathcal{S}$ is a linear homeomorphism sending Φ to Ψ, we immediately conclude that

Corollary 4.6. Φ *is a closed subspace of* $\mathcal{S}$ *and it is an ideal for the convolution.*

The statement that $\Phi \subset \mathcal{S}$ is a convolution ideal means that if $\varphi \in \Phi$ and $f \in \mathcal{S}$, then $\varphi * f = f * \varphi \in \Phi$. This can also be seen directly from the definition of Φ, indeed, if $\varphi \in \Phi$, then for any $m \in \mathbb{N}$ there exists $g_m \in \mathcal{S}$ such that $\Delta^m g_m = \varphi$ and we have

$$\Delta^m(f * g_m) = f * (\Delta^m g_m) = f * \varphi.$$

Thus $f * \varphi \in \Delta^m(\mathcal{S})$, for any m, i.e., $f * \varphi \in \Phi$.

Corollary 4.7. $\Delta \colon \Phi \to \Phi$ *is a homeomorphism.*

Proof. We already know that $\Delta \colon \Phi \to \Phi$ is bijective. The inverse $\Delta^{-1} \colon \Phi \to \Phi$ is given by the formula

$$\Delta^{-1}(\varphi) = \mathcal{F}^{-1}(|\xi|^{-2} \mathcal{F}\varphi).$$

Since the map $\psi \to |\xi|^2 \psi$ is clearly a self-homeomorphism of Ψ, we obtain the continuity of $\Delta^{-1} \colon \Phi \to \Phi$. $\qquad\square$

Proposition 4.8. (A) *The topological dual Φ' of Φ is the quotient of the space of tempered distribution modulo the polynomials*

$$\Phi' = \mathcal{S}'/\mathcal{P}.$$

(B) *The topological dual Ψ' of Ψ is the quotient of the space of tempered distribution modulo the Fourier transforms of polynomials*

$$\Psi' = \mathcal{S}'/\mathcal{F}(\mathcal{P}).$$

Proof. The closed subspace $\mathcal{P} \subset \mathcal{S}'$ coincides with $\Phi^\perp = \{w \in \mathcal{S}' \colon w(\Phi) = 0\}$ and $\Psi^\perp = \mathcal{F}(\mathcal{P})$. The Proposition follows now from standard results from functional analysis (see, e.g., [3, Chap. V, Th. 2.3]). $\qquad\square$

An element $w \in \Phi'$ is thus represented by a tempered distribution which is only well defined up to a polynomial. The Fourier transform $\mathcal{F} \colon \mathcal{S}' \to \mathcal{S}'$ gives an isomorphism between these quotients which we continue to denote by $\mathcal{F} \colon \Phi' \to \Psi'$. We have

$$\langle \mathcal{F}w, \, \varphi \rangle = \langle w, \, \mathcal{F}\varphi \rangle$$

for any $w \in \Phi'$ and $\varphi \in \Phi$.

5. Some Symbolic Calculus

5.1. Operators on Ψ' and multipliers. In this section, we study the operators $M \colon \Psi' \to \Psi'$ which can be represented by a multiplication.

Definitions. 1. By an *operator* $M \colon \Psi' \to \Psi'$ we mean a continuous linear map. Concretely, an operator associates to an element $w \in S'$ another tempered distribution $Mw \in S'$ which is well defined modulo $\mathcal{F}(\mathcal{P})$. The linearity means that $M(a_1 w_1 + a_2 w_2) = a_1 M(w_1) + a_2 M(w_2)$ modulo $\mathcal{F}(\mathcal{P})$ for any $a_1, a_2 \in \mathbb{C}$, $w_1, w_2 \in \mathcal{S}'$ and the continuity means that $\langle w_i, \psi \rangle \to \langle w, \psi \rangle$ for any $\psi \in \Psi$ implies $\langle Mw_i, \varphi \rangle \to \langle Mw, \varphi \rangle$. If M has a continuous inverse, then we say that it is an *isomorphism.*

2. We denote by $\mathrm{Op}(\Psi')$ the algebra of all operators $\Psi' \to \Psi'$.

We will discuss a special class of operators on Ψ, obtained by multiplication with a suitable function which we now introduce:

Definition. Let $\mathcal{M}_{\Psi'}$ be the space of all functions $\sigma \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{C})$ such that for any multi-index $\mu \in \mathbb{N}^n$, there exists constants $m \in \mathbb{N}$ and $C > 0$ with

$$|\partial^\mu \sigma(\xi)| \leqslant C(|\xi|^m + |\xi|^{-m})$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$. An element of $\mathcal{M}_{\Psi'}$ is called a Ψ'-*multiplier.*

It is that clear that $\mathcal{S} \subset \mathcal{M}_{\Psi'}$ and $\mathcal{P} \subset \mathcal{M}_{\Psi'}$, other typical elements of $\mathcal{M}_{\Psi'}$ are the functions $\log |\xi|$ and $|\xi|^\alpha$ (for any $\alpha \in \mathbb{C}$). Observe also that $\mathcal{M}_{\Psi'}$ is a commutative algebra.

The units in $\mathcal{M}_{\Psi'}$, i.e., the group of invertible elements, will be denoted by $\mathcal{U}\mathcal{M}_{\Psi'}$, hence

$$\mathcal{U}\mathcal{M}_{\Psi'} = \left\{ \sigma \in \mathcal{M}_{\Psi'} : \frac{1}{\sigma} \in \mathcal{M}_{\Psi'} \right\}.$$

Elements in $\mathcal{M}_{\Psi'}$ are not tempered distributions, however, we have the following important lemma:

Lemma 5.1. Ψ *is a module over the algebra* $\mathcal{M}_{\Psi'}$, *that is for any* $\sigma \in \mathcal{M}_{\Psi'}$ *and any* $\psi \in \Psi$, *we have* $\sigma \cdot \psi \in \Psi$.

Proof. By Proposition 4.2, we know that an element $\psi \in \mathcal{S}$ belongs to Ψ if and only if $\partial^\mu \psi(\xi) = o(|\xi|^t)$ as $|\xi| \to 0$ for any multi-indice $\mu \in \mathbb{N}^n$ and any $t > 0$. The proof of the lemma follows now easily from the Leibniz rule. $\qquad\square$

By duality, we can now associate to any $\sigma \in \mathcal{M}_{\Psi'}$ an operator $M_\sigma \in \mathrm{Op}(\Psi')$ defined by

$$\langle M_\sigma g, \psi \rangle = \langle g, \sigma\psi \rangle \tag{5.1}$$

for any $g \in \Psi'$ and $\psi \in \Psi$.

Lemma 5.2. *This correspondence defines a map*

$$M \colon \mathcal{M}_{\Psi'} \to \mathrm{Op}(\Psi')$$
$$\sigma \mapsto M_\sigma$$

which is a continuous homomorphism of algebras. In particular $M_{\sigma_1 \sigma_2} = M_{\sigma_1} \circ M_{\sigma_2}$ *and* M_σ *is invertible if and only if* $\sigma \in \mathcal{U}\mathcal{M}_{\Psi'}$

The proof is elementary. $\qquad\square$

Definition. An operator $M_\sigma \in \mathrm{Op}(\Psi')$ of this type is called a *multiplier in* Ψ'; the set of those multipliers is denoted by $\mathcal{M}\,\mathrm{Op}(\Psi')$, it is a commutative subalgebra of $\mathrm{Op}(\Psi')$.

Observe that, by Lemma 5.1, $\Psi \subset \Psi'$ is invariant under any multiplier in Ψ' (i.e., $M_\sigma(\Psi) \subset \Psi$ for any $M_\sigma \in \mathcal{M}\,\mathrm{Op}(\Psi')$). The converse is in fact also true. The multiplication $M_\sigma(\psi) = \sigma \cdot \psi$ is a continuous operator on Ψ' if and only if $\sigma \in \mathcal{M}_{\Psi'}$ (see [8]).

5.2. Operators on Φ' and their symbols. In this section, we study the operators $T \colon \Phi' \to \Phi'$ which can be represented on the Fourier side by a multiplication. We already know the Laplacian:

$$\mathcal{F}(\Delta w) = |\xi|^2 \mathcal{F}(w).$$

Definitions. 1. An *operator* $T \colon \Phi' \to \Phi'$, is a continuous linear map, it associates to an element $w \in S'$ another tempered distribution $Tw \in S'$ which is well defined up to a polynomial.

2. We denote by $\mathrm{Op}(\Phi')$ the algebra of all operators $\Phi' \to \Phi'$.

An obvious consequence of the previous section is the following

Proposition 5.3. *For any $\sigma \in \mathcal{M}_{\Psi'}$, the map $T_\sigma \colon \Phi' \to \Phi'$ defined by*

$$T_\sigma = \mathcal{F}^{-1} \circ M_\sigma \circ \mathcal{F}$$

belongs to $\mathrm{Op}(\Phi')$. If $\sigma \in \mathcal{UM}_{\Psi'}$, then T_σ is an isomorphism of Φ'. $\qquad\square$

Definition. Operators of this type are called *Fourier multipliers in Φ'*. We denote the set of those operators by $\mathcal{FM}\,\mathrm{Op}(\Phi')$.

If $T = T_\sigma = \mathcal{F}^{-1} \circ M_\sigma \circ \mathcal{F} \in \mathcal{FM}\,\mathrm{Op}(\Phi')$, then the function $\sigma \in \mathcal{M}_{\Psi'}$ is the *symbol* of T_σ and we write

$$\sigma = \mathrm{Smb}(T).$$

Thus, to say that $T \in \mathcal{FM}\,\mathrm{Op}(\Phi')$ means that for any Lizorkin distribution $f \in \Phi'$ and any $\varphi \in \Phi$, we have

$$\langle Tf,\, \varphi \rangle = \langle f,\, \mathcal{F}\left(\sigma \cdot \mathcal{F}^{-1}(\varphi)\right)\rangle,$$

where $\sigma = \mathrm{Smb}(T)$. Observe that $\mathcal{FM}Op(\Phi')$ is a commutative algebra and the map $\mathcal{M}_{\Psi'} \to \mathcal{FM}\,\mathrm{Op}(\Phi')$ given by $\sigma \mapsto T_\sigma$ is an isomorphism whose inverse is give by the symbol map:

$$\mathrm{Smb} \colon \mathcal{FM}\,\mathrm{Op}(\Phi') \to \mathcal{FM}_{\Psi'}$$
$$\sigma \mapsto T_\sigma.$$

Examples. (i) The symbol of the identity is 1;
  (ii) $\mathrm{Smb}(T \circ U) = \mathrm{Smb}(T) \cdot \mathrm{Smb}(U)$;
  (iii) The derivative $\partial_j \in \mathcal{FM}Op(\Phi')$ and $\mathrm{Smb}(\partial_j) = -i\xi_j$;
  (iv) The symbol of the Laplacian is $\mathrm{Smb}(\Delta) = |\xi|^2$;
  (v) More generally, T is a partial differential operator with constant coefficients if and only if $P = \mathrm{Smb}(T) \in \mathcal{P}$;
  (vi) If $T(w) = \varphi * w$ for some $\varphi \in \mathcal{S}$, then $\mathrm{Smb}(T) = \widehat{\varphi}$.

Any operator $T_\sigma \in \mathcal{FM}\,\mathrm{Op}(\Phi')$ is self-adjoint in the following sense:

Proposition 5.4. *For any $T_\sigma \in \mathcal{FM}\,\mathrm{Op}(\Phi')$, we have $T_\sigma(\Phi) \subset \Phi$ and*

$$\langle T_\sigma w,\, \varphi \rangle = \langle w,\, T_\sigma \varphi \rangle$$

for all $w \in \Phi'$ and $\varphi \in \Phi$.

Proof. The fact that $T_\sigma(\Phi) \subset \Phi$ follows from Lemma 5.1 and we have

$$
\begin{aligned}
\langle T_\sigma w, \, \varphi \rangle &= \langle \mathcal{F}^{-1} \circ M_\sigma \circ \mathcal{F}(w), \, \varphi \rangle \\
&= (2\pi)^{-n} \langle M_\sigma \circ \mathcal{F}(w), \, \mathcal{F}(\varphi) \rangle \\
&= (2\pi)^{-n} \langle \mathcal{F}(w), \, M_\sigma \circ \mathcal{F}(\varphi) \rangle \\
&= \langle w, \, \mathcal{F}^{-1} \circ M_\sigma \circ \mathcal{F}(\varphi) \rangle \\
&= \langle w, \, T_\sigma \varphi \rangle \, . \qquad\qquad \square
\end{aligned}
$$

5.3. The Riesz potential and the Riesz operator

Definitions. The *Riesz potential* on Φ' of order $\alpha \in \mathbb{R}$ is the operator $I^\alpha \in \mathcal{F}\mathcal{M}\,\mathrm{Op}(\Phi')$ whose symbol is

$$
\mathrm{Smb}(I^\alpha) = |\xi|^{-\alpha}.
$$

Theorem 5.5. $\Delta \colon \Phi' \to \Phi'$ *is an isomorphism with inverse* $I^2 \colon \Phi' \to \Phi'$.

Proof. We have $\mathrm{Smb}(\Delta) = |\xi|^2$, hence $\mathrm{Smb}(I^2 \circ \Delta) = |\xi|^{-2} \cdot |\xi|^2 = 1$. $\qquad \square$

Corollary 5.6. $\Delta \colon \mathcal{S}' \to \mathcal{S}'$ *is surjective and we we thus have an exact sequence*

$$
0 \to \mathcal{H} \to \mathcal{S}' \xrightarrow{\Delta} \mathcal{S}' \to 0.
$$

Proof. The previous theorem says that for any $f \in \mathcal{S}'$, we can find a distribution $g \in \mathcal{S}'$ such that

$$
\Delta g = f \quad \text{in} \quad \Phi' = \mathcal{S}'/\mathcal{P}.
$$

This means that there exists a polynomial $P \in \mathcal{P}$ such that $\Delta g = f + P$ in $\mathcal{S}'$. By Theorem 3.5, we can find a polynomial $Q \in \mathcal{P}$ such that $\Delta Q = P$ and it is now clear that

$$
\Delta(g - Q) = f \quad \text{in} \quad \mathcal{S}'.
$$

This proves that $\Delta(\mathcal{S}') = \mathcal{S}'$. $\qquad\qquad \square$

Remark. The distribution g in the above reasoning is only well defined in Φ' (by the formula $g = I^2 f$). In the space $\mathcal{S}'$ it is only well defined up to a polynomial and we have no constructive inverse map $\Delta^{-1} \colon \mathcal{S}' \to \mathcal{S}'$.

Definition. The *Riesz operator* in direction j is the operator $\mathcal{R}_j \in \mathcal{F}\mathcal{M}\,\mathrm{Op}(\Phi')$ defined by

$$
\mathcal{R}_j := -I^1 \circ \partial_j = -\partial_j \circ I^1.
$$

Its symbol is

$$
\mathrm{Smb}(\mathcal{R}_j) = -\,\mathrm{Smb}(I^1)\,\mathrm{Smb}(\partial_j) = i\frac{\xi_j}{|\xi|}.
$$

Proposition 5.7. *The Riesz potential and the Riesz operator enjoy the following properties*:

    (i) $I^0 = \mathrm{Id}$;
    (ii) $I^\alpha \circ I^\beta = I^\beta \circ I^\alpha = I^{\alpha+\beta}$;
    (iii) $I^{-2} = \Delta = \sum_j \partial_j^2$;
    (iv) $\Delta \circ I^\alpha = I^\alpha \circ \Delta = I^{\alpha-2}$;

(v) $\mathcal{R}_i \circ \mathcal{R}_j = \mathcal{R}_j \circ \mathcal{R}_i = I^2 \partial_i \partial_j$;

(vi) $\sum_{j=1}^{n} \mathcal{R}_j^2 = -\mathrm{Id}$;

(vii) $\langle I^\alpha \varphi, \eta \rangle = \langle \varphi, I^\alpha \eta \rangle$.

The proof is straightforward. $\qquad\square$

The Riesz potential $I^\alpha \in \mathcal{FM}\,\mathrm{Op}(\Phi')$ is sometimes denoted by $I^\alpha = \Delta^{-\alpha/2}$, the previous lemma justifies this notation.

5.4. Convolution operators in $\mathcal{S}$. Let $T = T_\sigma \in \mathcal{FM}\,\mathrm{Op}(\Phi')$ be an operator such that $\sigma = \mathrm{Smb}(T) \in \mathcal{S}' \cap \mathcal{M}$, then we can define another operator $\widetilde{T} \colon \mathcal{S} \to \mathcal{S}'$ by the convolution

$$\widetilde{T}\varphi = (\mathcal{F}^{-1}\sigma) * \varphi.$$

The next lemma is easy to check.

Lemma 5.8. *The relation between T and $\widetilde{T}$ is given by*

$$T(\varphi) = \widetilde{T}(\varphi) \qquad (\mathrm{mod}\ \mathcal{P})$$

for any $\varphi \in \Phi$. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\ \widetilde{T}\ } & \mathcal{S}' \\[2pt]
\cup & & \downarrow \\[2pt]
\Phi & \xrightarrow{\ T\ } & \Phi'.
\end{array}
$$
$\qquad\square$

Theorem 5.9. *The symbol of the Riesz potential I^α of order α belongs to $\mathcal{S}'$ if $\alpha < n$.*

If $0 < \alpha < n$, then I^α defines a convolution operator $\mathcal{S} \to \mathcal{S}'$ by

$$I^\alpha \varphi = k_\alpha * \varphi,$$

where k_α is the Riesz kernel

$$k_\alpha(x) = \frac{1}{\gamma(n,\,\alpha)} |x|^{\alpha - n}.$$

The proof is given in the appendix. $\qquad\square$

The Riesz potential $I^\alpha \colon \mathcal{S} \to \mathcal{S}'$ is thus given by the explicit formula

$$I^\alpha \varphi(x) = \frac{1}{\gamma(n,\,\alpha)} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x - y|^{n - \alpha}}\, dy \qquad\qquad (5.2)$$

if $0 < \alpha < n$, and

$$I^\alpha \varphi(x) = \frac{1}{\gamma(n,\,\alpha)} \int_{\mathbb{R}^n} \varphi(y) \log \frac{1}{|x|}\, dy \qquad\qquad (5.3)$$

if $\alpha = n$.

6. The Lizorkin Space and Harmonic Analysis in $L^p(\mathbb{R}^n)$

Proposition 6.1. *The subspace $\Phi \subset L^p(\mathbb{R}^n)$ is dense $1 < p < \infty$.*

The proof can be found in [9, Theorem 2.7].

Proposition 6.2. *The space $L^p(\mathbb{R}^n)$ injects in Φ' for any $1 \leqslant p < \infty$. More generally, $L^p(\mathbb{R}^n) + L^q(\mathbb{R}^n)$ injects in Φ' if $p + q < \infty$.*

This result is a direct consequence of the following lemma:

Lemma 6.3. *Let f, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'$ be two locally integrable functions such that*
$$\lambda^n\{x \in \mathbb{R}^n \colon |f(x) - g(x)| \geqslant a\} < \infty,$$
for some $a \geqslant 0$ (here λ^n is the Lebesgue measure). Assume that f and g coincide in Φ', i.e.,
$$\int_{\mathbb{R}^n} f\varphi \, dx = \int_{\mathbb{R}^n} g\varphi \, dx$$
for any $\varphi \in \Phi$. Then $(f - g)$ is almost everywhere constant in $\mathbb{R}^n$.

Proof. This is Lemma 3.8 in [7]. We repeat the proof, which is very short. Since f and g coincide in Φ', we have $P = (f - g) \in \mathcal{P}$; by hypothesis, we have $\lambda^n\{x \in \mathbb{R}^n \colon |P(x)| \geqslant a\} < \infty$, this is only possible if $P = c$ is a constant such that $|c| < a$. $\qquad\square$

Remark. The argument also shows that $L^\infty(\mathbb{R}^n)/\mathbb{R}$ injects in Φ'.

We summarize the known inclusions in the following lemma:

Lemma 6.4. *We have the following inclusions $(1 < p < \infty)$*
$$\Phi \subset \mathcal{S} \subset L^p(\mathbb{R}^n) \subset \Phi' = \mathcal{S}'/\mathcal{P}.$$
Furthermore Φ is dense in L^p (for the L^p norm) and in Φ' (for the weak topology). $\qquad\square$

Lemma 6.5. *If $0 < \alpha < n$, then the Riesz kernel k_α is a tempered distribution. In fact*
$$k_\alpha \in (L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)) \subset \mathcal{S}'$$
for any r, $s \geqslant 1$ such that
$$0 < \frac{1}{s} < 1 - \frac{\alpha}{n} < \frac{1}{r} \leqslant 1.$$

Proof. Let χ_B be the characteristic function of the unit ball and set $K_1 = \chi_B k_\alpha$ and $K_2 = (k_\alpha - K_1) = (1 - \chi_B) k_\alpha(x)$. It is easy to check that $K_1 \in L^r(\mathbb{R}^n)$ for any $1 \leqslant r < \frac{n}{n-\alpha}$ and $K_2 \in L^s(\mathbb{R}^n)$ for any $\frac{n}{n-\alpha} < s < \infty$, thus
$$k_\alpha = K_1 + K_2 \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n). \qquad\square$$

Corollary 6.6. *If $0 < \alpha < n$, then the Riesz potential*
$$I^\alpha \varphi(x) = k_\alpha * \varphi(x) = \frac{1}{\gamma(n, \alpha)} \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x - y|^{n-\alpha}} \, dy$$

defines a bounded operator

$$I^\alpha \colon L^p(\mathbb{R}^n) \to (L^{q_1}(\mathbb{R}^n) + L^{q_2}(\mathbb{R}^n))$$

for any $1 \leqslant p \leqslant \infty$ and $1 \leqslant q_1 < q_2 < \infty$ such that $\frac{1}{q_2} < \frac{1}{p} - \frac{\alpha}{n} < \frac{1}{q_1}$.

In particular, I^α is a continuous operator $I^\alpha \colon L^p(\mathbb{R}^n) \to \Phi'$ for any $0 < \alpha < n$.

Proof. Let us set $r = (pq_1 - p - q)/(pq_1)$ and $s = (pq_s - p - q)/(pq_2)$, we then have

$$\frac{1}{r} = 1 + \frac{1}{q_1} - \frac{1}{p} > 1 - \frac{\alpha}{n} \quad \text{and} \quad \frac{1}{s} = 1 + \frac{1}{q_2} - \frac{1}{p} < 1 - \frac{\alpha}{n}. \qquad (6.1)$$

By the previous lemma, we may write $k_\alpha = K_1 + K_2$ with $K_1 \in L^r(\mathbb{R}^n)$ and $K_2 \in L^s(\mathbb{R}^n)$.

The Young inequality for convolutions says that under the condition (6.1), we have

$$\|K_1 * f\|_{L^{q_1}} \leqslant \|K_1\|_{L^r} \|f\|_{L^p} \quad \text{and} \quad \|K_2 * f\|_{L^{q_2}} \leqslant \|K_2\|_{L^s} \|f\|_{L^p}$$

Since $k_\alpha = K_1 + K_2$, we conclude that

$$\|k_\alpha * f\|_{L^{q_1}} \leqslant (\|K_1\|_{L^r} + \|K_2\|_{L^s}) \|f\|_{L^p}. \qquad \square$$

If $p \in (1, n/\alpha)$, then we have the following much deeper result:

Theorem 6.7 (Hardy–Littlewood–Sobolev). *The Riesz potential defines a bounded operator*

$$I^\alpha \colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

if and only if $\alpha \in (0, n)$, $1 < p < n/\alpha$ and $q = \frac{np}{n - p\alpha}$. The formulas (5.2) and (5.3) still hold in this case.

References for this important result are [12, page 119], [9, Th. 2.2] or [1, Th. 3.14].

Recall that we defined the Riesz transform in direction j to be the operator $\mathcal{R}_j = -I^1 \circ \partial_j$. Its symbol is $\rho_j = i|\xi|^{-1}\xi_j$, and for any $\varphi \in \mathcal{S}$, we thus have

$$\mathcal{F}(\mathcal{R}_j(\varphi)) = \rho_j \mathcal{F}(\varphi).$$

The Riesz transform of a Lizorkin distribution $f \in \Phi$ is characterized by

$$\langle \mathcal{R}_j(\varphi)(f), \varphi \rangle = \langle f, \mathcal{F}^{-1}(\rho_j \mathcal{F}(\varphi)) \rangle$$

for any $\varphi \in \mathcal{S}$.

The function ρ_j does not belong to the Schwartz space $\mathcal{S}$ and thus its (inverse) Fourier transform $\check{\rho}_j = \mathcal{F}^{-1}(\rho_j)$ is not a priori well defined. We can therefore not write the Riesz transform as a convolution. However, $\mathcal{R}_j$ can be represented as a singular integral:

Theorem 6.8 (Calderon–Zygmund–Cotlar). *The Riesz transform $\mathcal{R}_j \colon \mathcal{S} \to \mathcal{S}'$ is given by the formula*

$$\mathcal{R}_j(\varphi(x)) = \lim_{\delta \to 0} \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \int_{|y| > \delta} \frac{(x_j - y_j)}{|x - y|} \varphi(y) \, dy. \qquad (6.2)$$

Furthermore, $\mathcal{R}_j$ extends as a bounded operator

$$\mathcal{R}_j \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

for all $1 < p < \infty$ and the formula (6.2) still holds in this case.

This deep result is a consequence of Sections II.4.2, III.1.2 and III.3.3 in the book of Stein [12], see also [1]. $\square$

Let us denote by $c_p = \|\mathcal{R}_j\|_{L^p \to L^p}$ the norm of the operator $\mathcal{R}_j \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, it is clearly independant of j. The exact value of c_p is known, see [6, page 304]; let us only stress that

$$\lim_{p \to 1} c_p = \lim_{p \to \infty} c_p = \infty.$$

Remark. For $p = 1$ and $p = \infty$, the Riesz transform is still a bounded operator in appropriate function spaces, namely

$$\mathcal{R}_j \colon L^1(\mathbb{R}^n) \to \text{weak } L^1(\mathbb{R}^n)$$
$$\mathcal{R}_j \colon L^\infty(\mathbb{R}^n) \to BMO(\mathbb{R}^n)$$

are bounded operators. There are also results on weighted L^p spaces satisfying a Muckenhoupt condition.

6.1. Applications of these L^p bounds. To illustrate the power of the two previous theorems, we give below very short proofs of two important results for functions in $\mathbb{R}^n$ (cf. [12, pages 59 and 126]).

Theorem 6.9 (Sobolev–Gagliardo–Nirenberg)**.** *Let $1 < p, q < \infty$ be such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. There exists a finite constant $C < \infty$ such that for any $f \in L^p(\mathbb{R}^n)$, we have*

$$\|f\|_{L^q(\mathbb{R}^n)} \leqslant C \sum_{j=1}^n \|\partial_j f\|_{L^p(\mathbb{R}^n)}. \tag{6.3}$$

Remarks. 1. This inequality also holds for $p = 1$, see [12, Chap. V §2.5 pp. 128–130], but not for $p = \infty$.

2. A homogeneity argument shows that the inequality (6.3) cannot hold with a finite constant if $\frac{1}{p} - \frac{1}{q} \neq \frac{1}{n}$ (see the argument in the proof of Theorem 8.2 below).

Proof. Combining the identity $-\text{Id} = \sum_j \mathcal{R}_j^2 = \sum_j I^1 \circ \mathcal{R}_j \circ \partial_j$ with Theorems 6.7 and 6.8, we obtain

$$\|f\|_{L^q(\mathbb{R}^n)} \leqslant a_p \sum_{j=1}^n \|\mathcal{R}_j \partial_j f\|_{L^p(\mathbb{R}^n)} \leqslant a_p c_p \sum_{j=1}^n \|\partial_j f\|_{L^p(\mathbb{R}^n)}. \qquad \square$$

Theorem 6.10 (A priori estimates for the Laplacian)**.** *The following inequality holds for any $f \in \Phi'$ and any $\mu, \nu = 1, \ldots, n$:*

$$\|\partial_\mu \circ \partial_\nu f\|_{L^p(\mathbb{R}^n)} \leqslant c_p^2 \|\Delta f\|_{L^p(\mathbb{R}^n)}. \tag{6.4}$$

Recall that $c_p < 0$ if and only if $1 < p < \infty$.

Proof. This result is an obvious consequence of the definition of c_p and the identity

$$\partial_\mu \circ \partial_\nu = -\mathcal{R}_\mu \circ \mathcal{R}_\nu \circ \Delta. \qquad \square$$

Remark. This result holds for any $f \in L^s(\mathbb{R}^n)$, $1 \leqslant s \leqslant \infty$, since $L^s(\mathbb{R}^n) \subset \Phi'$. But it does not hold for arbitrary functions $f \in \mathcal{S}'(\mathbb{R}^n)$, for instance the harmonic polynomial $f(x, y) = xy \in \mathcal{H}$ satisfies $\Delta f = 0$, but $\partial_x \partial_y f = 1 \neq 0$.

7. APPLICATIONS TO DIFFERENTIAL FORMS

7.1. Differential forms in $\mathbb{R}^n$. We denote by $\Lambda^k = \Lambda^k(\mathbb{R}^{n*})$ the vector space of antisymmetric multinear k-forms on $\mathbb{R}^n$. Recall that $\dim(\Lambda^k) = \binom{n}{k}$ and a basis of this space is given by

$$\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} : i_1 < i_2 < \cdots < i_k\}.$$

A *smooth differential form θ of degree k* on $\mathbb{R}^n$ is simply a smooth function on $\mathbb{R}^n$ with values in Λ^k. It is thus uniquely represented as

$$\theta = \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1 \ldots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \tag{7.1}$$

where the coefficients $a_{i_1 \ldots i_k}$ are smooth functions. We denote by $C^\infty(\mathbb{R}^n, \Lambda^k)$ the space of smooth differential forms of degree k on $\mathbb{R}^n$.

We will also consider later other spaces of differential forms on $\mathbb{R}^n$ such as $L^p(\mathbb{R}^n, \Lambda^k)$ or $\mathcal{S}(\mathbb{R}^n, \Lambda^k)$. The form (7.1) belongs to $\mathcal{S}(\mathbb{R}^n, \Lambda^k)$ if its coefficients $a_{i_1 \ldots i_k}$ are rapidly decreasing functions and $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$ if all $a_{i_1 \ldots i_k} \in L^p(\mathbb{R}^n)$.

We shall study a number of operators on differential forms. Observe first that the operators $\partial_\mu = \frac{\partial}{\partial x_\mu}$, I^α and $\mathcal{R}_\mu$ are well defined on appropriate classes of differential forms by acting on the coefficients $a_{i_1 \ldots i_k}$ of the form (7.1).

The *Hodge star operator* is the linear map $\star \colon \Lambda^k \to \Lambda^{n-k}$ defined by the condition

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge \star (dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

for any $i_1 < i_2 < \ldots i_k$; observe that

$$\star \star = (-1)^{k(n-k)} \mathrm{Id} \quad \text{on } \Lambda^k. \tag{7.2}$$

The $\star$ operator naturally extends to the space of differential forms with any kind of coefficients.

The *interior product* of the k-form θ with the vector X is the $(k-1)$-form defined by

$$\iota_X \theta(v_1, \ldots, v_{k-1}) = \theta(X, v_1, \ldots, v_{k-1}).$$

We denote by $\iota_\mu = \iota_{\frac{\partial}{\partial x_\mu}}$ the interior product with $\frac{\partial}{\partial x_\mu}$ and by ε_μ the exterior product with dx_μ:

$$\varepsilon_\mu \theta = dx_\mu \wedge \theta.$$

Lemma 7.1. *The following holds on k-forms*:

$$\iota_\mu = (-1)^{nk+n} \star \varepsilon_\mu \star. \tag{7.3}$$

Proof. We first show that for an arbitrary differential form α, we have

$$\iota_\mu(\star \alpha) = \star(\alpha \wedge dx_\mu). \tag{7.4}$$

It is enough to prove this identity for $\alpha = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$. Observe that if $\mu = j_r$ for some $r \in \{1, 2, \ldots, k\}$, then both sides of the equation (7.4) trivially vanish: we thus assume $\mu \neq j_r$ for all r and set $\beta = \star(\alpha \wedge dx_\mu)$. Then, by definition

$$\alpha \wedge dx_\mu \wedge \beta = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n,$$

this relation clearly implies $\star \alpha = dx_\mu \wedge \beta$, and the equation (7.4) is now easy to check:

$$\boldsymbol{\iota}_\mu(\star \alpha) = \boldsymbol{\iota}_\mu(dx_\mu \wedge \beta) = \beta = \star(\alpha \wedge dx_\mu).$$

Let us now consider an arbitrary $k-$form θ, and let $\alpha = (-1)^{k(n-k)} \star \theta$, i.e., $\star \alpha = \theta$. Using (7.4), we have

$$\begin{aligned}
\boldsymbol{\iota}_\mu(\theta) = \boldsymbol{\iota}_\mu(\star \alpha) &= \star(\alpha \wedge dx_\mu) = (-1)^{n-k} \star (dx_\mu \wedge \alpha) \\
&= (-1)^{n-k} \star (\varepsilon_\mu \, \alpha) = (-1)^{n-k}(-1)^{k(n-k)} \star (\varepsilon_\mu \star \theta) \\
&= (-1)^{kn+n} \star (\varepsilon_\mu \star \theta). \qquad\qquad\qquad\qquad\qquad \square
\end{aligned}$$

We now define the *exterior differential operator* by

$$d = \sum_{\mu=1}^{n} \varepsilon_\mu \circ \partial_\mu = \sum_{\mu=1}^{n} \partial_\mu \circ \varepsilon_\mu, \qquad (7.5)$$

and the *codifferential operator* by

$$\delta = -\sum_{\mu=1}^{n} \boldsymbol{\iota}_\mu \circ \partial_\mu = -\sum_{\mu=1}^{n} \partial_\mu \circ \boldsymbol{\iota}_\mu. \qquad (7.6)$$

It follows from Lemma 7.1, that for any k-form θ, we have

$$\delta\theta = (-1)^{nk+n+1} \star d \star \theta. \qquad (7.7)$$

If θ has the representation (7.1) then $d\theta$ is given by

$$d\theta = \sum_{i_1 < \cdots < i_k} da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_k},$$

and $\delta\theta$ is given by

$$\delta\theta = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^{k} (-1)^j \frac{\partial a_{i_1 \ldots i_k}}{\partial x_{i_j}} \, dx_{i_1} \wedge \cdots \wedge \widehat{dx}_{i_j} \wedge \cdots \wedge dx_{i_k};$$

A direct computation show that these operators enjoy the following properties:

$$d \circ d = \delta \circ \delta = 0$$

and

$$\Delta = (d+\delta)^2 = d \circ \delta + \delta \circ d = -\sum_{\mu=1}^{n} \partial_\mu^2.$$

7.2. Temperate and Lizorkin currents

Definitions. A *rapidly decreasing* differential form of degree k is an element $\theta \in \mathcal{S}(\mathbb{R}^n, \Lambda^k)$, i.e., a differential form with coefficients in the Schwartz space $\mathcal{S}$.

A *temperate current* f of degree k is a continuous linear form on $\mathcal{S}(\mathbb{R}^n, \Lambda^k)$, see [10]. The evaluation of the temperate current f on the differential form $\theta \in \mathcal{S}(\mathbb{R}^n, \Lambda^k)$ is denoted by

$$\langle f, \theta \rangle \in \mathbb{C}.$$

the continuity of f means that if $\{\theta_i\} \subset \mathcal{S}(\mathbb{R}^n, \Lambda^k)$ is a sequence of rapidly decreasing differential forms such that θ_i converges to $\theta \in \mathcal{S}(\mathbb{R}^n, \Lambda^k)$ (i.e., all coefficients converge in the Schwartz space $\mathcal{S}$), then

$$\langle f, \theta \rangle = \lim_{i \to \infty} \langle f, \theta_i \rangle.$$

The space of temperate currents is denoted by $\mathcal{S}'(\mathbb{R}^n, \Lambda^k)$. Any differential form $f \in L^p(\mathbb{R}^n, \Lambda^k)$ determines a temperate current by the formula

$$\langle f, \theta \rangle = \int_{\mathbb{R}^n} f \wedge *\theta.$$

This formula defines an embedding $L^p(\mathbb{R}^n, \Lambda^k) \subset \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$.

Another important class of temperate currents is given by the space $\mathcal{P}(\mathbb{R}^n, \Lambda^k)$ of differential forms with polynomial coefficients. We may thus define the space of *Lizorkin forms* as

$$\Phi(\mathbb{R}^n, \Lambda^k) = \{\phi \in \mathcal{S}(\mathbb{R}^n, \Lambda^k) \colon \langle P, \phi \rangle = 0 \text{ for all } P \in \mathcal{P}(\mathbb{R}^n, \Lambda^k)\}.$$

The dual space is called the space of *Lizorkin currents*, it coincides with the quotient

$$\Phi'(\mathbb{R}^n, \Lambda^k) = \mathcal{S}'(\mathbb{R}^n, \Lambda^k)/\mathcal{P}(\mathbb{R}^n, \Lambda^k),$$

we can think of a Lizorkin current as a differential forms with coefficients in Φ'.

Given a Lizorkin current $f \in \Phi'(\mathbb{R}^n, \Lambda^k)$, we define its differential df, its codifferential δf, its Laplacian and its Riesz potential of order α by the following formulas

$$\langle df, \varphi \rangle = \langle f, \delta \varphi \rangle, \qquad \langle \delta f, \varphi \rangle = \langle f, d\varphi \rangle,$$
$$\langle \Delta f, \varphi \rangle = \langle f, \Delta \varphi \rangle, \qquad \langle I^\alpha f, \varphi \rangle = \langle f, I^\alpha \varphi \rangle$$

for all $\varphi \in \Phi(\mathbb{R}^n, \Lambda^k)$. These are continuous operators $\Phi'(\mathbb{R}^n, \Lambda^k) \to \Phi'(\mathbb{R}^n, \Lambda^k)$.

Observe that the Riesz potential commutes with δ and d:

$$\delta I^2 = I^2 \delta \qquad \text{and} \qquad dI^2 = I^2 d,$$

and that we have $\Delta I^2 = I^2 \Delta = \text{Id}$. In particular, we have the

Theorem 7.2. *The Laplacian* $\Delta \colon \Phi'(\mathbb{R}^n, \Lambda^k) \to \Phi'(\mathbb{R}^n, \Lambda^k)$ *is an isomorphism with inverse* I^2.

Proof. This follows immediately from Theorem 5.5. $\square$

Corollary 7.3. *We have the exact sequences*

$$0 \to \mathcal{H}(\mathbb{R}^n, \Lambda^k) \to \mathcal{P}(\mathbb{R}^n, \Lambda^k) \xrightarrow{\Delta} \mathcal{P}(\mathbb{R}^n, \Lambda^k) \to 0,$$

and

$$0 \to \mathcal{H}(\mathbb{R}^n, \Lambda^k) \to \mathcal{S}'(\mathbb{R}^n, \Lambda^k) \xrightarrow{\Delta} \mathcal{S}'(\mathbb{R}^n, \Lambda^k) \to 0.$$

In particular, any temperate current $\theta \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$ *is the sum of an exact plus a coexact current, more precisely we have*

$$\theta = \Delta \omega = \delta(d\omega) + d(\delta \omega)$$

for some $\omega \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$, *well defined up to a harmonic current.*

Proof. The first exact sequence follows from Theorem 3.5 and the second one is proved as in Corollary 5.6, using the previous theorem. $\square$

Remark. Observe that the situation is very different from the L^2-theory (or any other Hilbert space model); in L^2, we have $\overline{\mathrm{Im}\,\Delta} = (\ker\Delta)^\perp$; in particular $\ker\Delta = 0$ if Δ is onto, but in $\mathcal{S}'(\mathbb{R}^n, \Lambda^k)$, the Laplacian is onto while $\ker\Delta = \mathcal{H}(\mathbb{R}^n, \Lambda^k) \neq 0$.

7.3. The Riesz transform on currents

Definitions. We define the *Riesz transform* on $\Phi'(\mathbb{R}^n, \Lambda^k)$ by

$$\mathcal{R} = d \circ I^1 = I^1 \circ d = \sum_{\mu=1}^{n} \boldsymbol{\varepsilon}_\mu \circ \mathcal{R}_\mu \tag{7.8}$$

and its adjoint

$$\mathcal{R}^* = \delta \circ I^1 = I^1 \circ \delta = -\sum_{\mu=1}^{n} \boldsymbol{\iota}_\mu \circ \mathcal{R}_\mu. \tag{7.9}$$

We also define four additional operators $E, E^*, U, U^* \colon \Phi'(\mathbb{R}^n, \Lambda^k) \to \Phi'(\mathbb{R}^n, \Lambda^k)$ by

$$E = d \circ \delta \circ I^2 = \mathcal{R} \circ \mathcal{R}^*, \quad E^* = \delta \circ d \circ I^2 = \mathcal{R}^* \circ \mathcal{R}, \tag{7.10}$$

and

$$U = I^1 \circ \mathcal{R}^* = I^2 \circ \delta, \quad U^* = I^1 \circ \mathcal{R} = I^2 \circ d. \tag{7.11}$$

Proposition 7.4. *These operators are continuous on* $\Phi'(\mathbb{R}^n, \Lambda^k)$. *They enjoy the following properties*:

(a) $E + E^* = \mathcal{R} \circ \mathcal{R}^* + \mathcal{R}^* \circ \mathcal{R} = \Delta \circ I^2 = \mathrm{Id}$;
(b) $E = 0$ *on* $\ker\delta$ *and* $E^* = 0$ *on* $\ker d$;
(c) $E \circ E^* = E^* \circ E = 0$;
(d) $E \circ E = E$ *and* $E^* \circ E^* = E^*$;
(e) $E = \mathrm{Id}$ *on* $\ker d$ *and* $E^* = \mathrm{Id}$ *on* $\ker\delta$;
(f) $\mathrm{Im}\,E = \ker(E^*) = \mathrm{Im}\,d = \ker d$ *and* $\mathrm{Im}\,E^* = \ker E = \mathrm{Im}\,\delta = \ker\delta$;
(g) $(\mathcal{R}, \mathcal{R}^*)$ *and* (U, U^*) *are adjoint pairs, i.e.*,

$$\langle \mathcal{R}\theta, \varphi \rangle = \langle \theta, \mathcal{R}^*\varphi \rangle \quad \textit{and} \quad \langle \mathcal{R}^*\theta, \varphi \rangle = \langle f\theta, \mathcal{R}\varphi \rangle,$$

for any $\theta \in \Phi'(\mathbb{R}^n, \Lambda^k)$ *and* $\varphi \in \Phi(\mathbb{R}^n, \Lambda^k)$, *and likewise for* U;
(h) E *and* E^* *are self-adjoint*;
(i) $E = d \circ U$ *and* $E^* = \delta \circ U^*$.

Proof. (a) Follows from the definitions and $\mathrm{Id} = \Delta \circ I^2 = (d\delta + \delta d) \circ I^2 = E + E^*$.

(b) If $\delta\theta = 0$, then $E\theta = d\,\delta\,I^2\,\theta = I^2\,d\,\delta\,\theta = 0$, hence $E^* = 0$ on $\ker d$. A similar argument shows that $E^* = 0$ on $\ker d$.

(c) By definition $E \circ E^* = d\,\delta\,I^2\,\delta\,d\,I^2 = I^4\,d\,\delta^2 d = 0$. The proof that $E^* \circ E = 0$ is the same.

(d) This follows from (a) and (c), since

$$E = E \circ \mathrm{Id} = E \circ E + E \circ E^* = E \circ E.$$

(e) This follows immediately from (a) and (b).

(f) From $E \circ E = E$, we have $\theta \in \operatorname{Im} E$ if and only if $\theta = E\theta$. Since $E + E^* = \operatorname{Id}$, we have $E^*\theta = (\operatorname{Id} - E)\theta = 0$, thus $\operatorname{Im} E = \ker E^*$. Furthermore, using $E = d\delta I^2$ and Property (e), we see that

$$\operatorname{Im} E \subset \operatorname{Im} d \subset \ker d = E(\ker d) \subset \operatorname{Im} E.$$

This shows that $\operatorname{Im} E = \operatorname{Im} d = \ker d$. The proof that $\operatorname{Im} E^* = \ker E = \operatorname{Im} \delta = \ker \delta$ is similar.

(g) We have

$$\langle \mathcal{R}\theta,\, \varphi \rangle = \langle dI^1\theta,\, \varphi \rangle = \langle I^1\theta,\, \delta\varphi \rangle = \langle \theta,\, I^1\delta\varphi \rangle = \langle \theta,\, \mathcal{R}^*\varphi \rangle.$$

The proof that $(U,\, U^*)$ is an adjoints pair is similar, using $U = I^2\delta$.

(h) It follows that E is selfadjoint, for

$$\langle E\theta,\, \varphi \rangle = \langle \mathcal{R}\mathcal{R}^*\theta,\, \varphi \rangle = \langle \mathcal{R}^*\theta,\, \mathcal{R}^*\delta\varphi \rangle = \langle \theta,\, \mathcal{R}\mathcal{R}^*\varphi \rangle = \langle \theta,\, E\varphi \rangle,$$

and likewise for E^*.

(i) We have $dU = dI^2\delta = d\delta I^2 = E$ and $\delta U^* = \delta I^2 d = \delta dI^2 = E^*$. $\qquad\square$

Let us denote by

$$E\Phi'(\mathbb{R}^n,\, \Lambda^k) = d\Phi'(\mathbb{R}^n,\, \Lambda^{k-1})$$

the space of *exact Lizorkin currents* of degree k and by

$$E^*\Phi'(\mathbb{R}^n,\, \Lambda^k) = \delta\Phi'(\mathbb{R}^n,\, \Lambda^{k+1})$$

the space of *coexact Lizorkin currents* of degree k.

Corollary 7.5. *These subspaces can be expressed as*

$$E\Phi'(\mathbb{R}^n,\, \Lambda^k) = \operatorname{Im}(E) = \ker E^* = \ker[d \colon \Phi'(\mathbb{R}^n,\, \Lambda^k) \to \Phi'(\mathbb{R}^n,\, \Lambda^{k+1})]$$

and

$$E^*\Phi'(\mathbb{R}^n,\, \Lambda^k) = \operatorname{Im}(E^*) = \ker E = \ker[\delta \colon \Phi'(\mathbb{R}^n,\, \Lambda^k) \to \Phi'(\mathbb{R}^n,\, \Lambda^{k-1})].$$

In particular $E\Phi'(\mathbb{R}^n,\, \Lambda^k)$ and $E^*\Phi'(\mathbb{R}^n,\, \Lambda^k)$ are closed subspaces in $\Phi'(\mathbb{R}^n,\, \Lambda^k)$ and we have a direct sum decomposition

$$\Phi'(\mathbb{R}^n,\, \Lambda^k) = E\Phi'(\mathbb{R}^n,\, \Lambda^k) \oplus E^*\Phi'(\mathbb{R}^n,\, \Lambda^k).$$

Proof. This is obvious from the previous proposition. $\qquad\square$

Remarks. 1. Thus E and E^* are the projections of $\Phi'(\mathbb{R}^n,\, \Lambda^k)$ onto $E\Phi'(\mathbb{R}^n,\, \Lambda^k)$ and $E^*\Phi'(\mathbb{R}^n,\, \Lambda^k)$ respectively. One says that $E\theta$ is the *exact part* of $\theta \in \Phi'(\mathbb{R}^n,\, \Lambda^k)$ and $E^*\theta$ is its *coexact part*. The formula

$$\theta = E\theta + E^*\theta$$

is the *Hodge–Kodaira decomposition* of the Lizorkin distribution θ.

2. The last part says in particular that there is no cohomology in $\Phi'(\mathbb{R}^n, \Lambda^k)$, i.e.,

$$\cdots \to \Phi'(\mathbb{R}^n, \Lambda^{k-1}) \xrightarrow{d} \Phi'(\mathbb{R}^n, \Lambda^k) \xrightarrow{d} \Phi'(\mathbb{R}^n, \Lambda^{k+1}) \to \cdots$$

is an exact sequence.

3. Using the equalities $U = I^2 \circ \delta$ and $U^* = I^2 \circ d$, and observing that $E = d \circ U$ and $E^* = \delta \circ U^*$, one can write the Hodge–Kodaira decomposition of $\theta \in \Phi'(\mathbb{R}^n, \Lambda^k)$ as

$$\theta = d(U\theta) + \delta(U^*\theta).$$

7.4. Proof of Theorems 1.2 and 1.4. The operators defined in the previous section are well behaved on L^p:

Theorem 7.6. *The Riesz transform and its dual*

$$\mathcal{R}, \mathcal{R}^* \colon L^p(\mathbb{R}^n, \Lambda^k) \to L^p(\mathbb{R}^n, \Lambda^{k-1})$$

are bounded operators on L^p for any $1 < p < \infty$.

Proof. The boundedness of these operators on L^p follows from Theorem 6.8 and the expansions (7.8) and (7.9). $\square$

Theorem 7.7. *The operators U^* and U restrict as bounded operators*

$$U^*, U \colon L^p(\mathbb{R}^n, \Lambda^k) \to L^q(\mathbb{R}^n, \Lambda^{k-1}),$$

if and only if $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$.

Proof. This follows from Theorems 6.7. $\square$

We can now prove the theorems stated in the introduction.

Proof of Theorem 1.2. The equation (1.5) is a trivial consequence of the equality $E + E^* = \mathrm{Id}$ and $\mathrm{Im}(E) \cap \mathrm{Im}(E^*) = \{0\}$. We know from Theorem 7.6, that the Riesz transform and its dual are well defined bounded operators $\mathcal{R}, \mathcal{R}^* \colon L^p(\mathbb{R}^n, \Lambda^k) \to L^p(\mathbb{R}^n, \Lambda^{k-1})$. The operators $E = \mathcal{R} \circ \mathcal{R}^*$ and $E^* = \mathcal{R}^* \circ \mathcal{R}$ are then also clearly bounded on $L^p(\mathbb{R}^n, \Lambda^{k-1})$. The algebraic properties (iii), (iv) and (v) in Theorem 1.2 are proved in Proposition 7.4, and we know that $EL^p(\mathbb{R}^n, \Lambda^k) \subset L^p(\mathbb{R}^n, \Lambda^k)$ is a closed subspace, since it coincides with the kernel of the bounded operator $E^* \colon L^p(\mathbb{R}^n, \Lambda^k) \to L^p(\mathbb{R}^n, \Lambda^k)$. Likewise $E^* L^p(\mathbb{R}^n, \Lambda^k) = \ker E$ is also bounded in $L^p(\mathbb{R}^n, \Lambda^k)$. $\square$

Proof of Theorem 1.4. By Theorem 7.7, the operators U^*, U restrict as bounded operators on $L^q(\mathbb{R}^n, \Lambda^{k-1})$. The relations $dU = E$ and $\delta U^* = E^*$ are given in Proposition 7.4. $\square$

8. Some Additional Applications

8.1. The Gaffney inequality

Theorem 8.1. *Assume* $1 < p < \infty$. *There exists a constant* C_p *such for any* $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$ *and any* $\mu = 1, 2, \ldots n$, *we have*

$$\|\partial_\mu \theta\|_{L^p(\mathbb{R}^n)} \leqslant C_p \left(\|d\theta\|_{L^p(\mathbb{R}^n)} + \|\delta\theta\|_{L^p(\mathbb{R}^n)} \right). \tag{8.1}$$

Proof. By Theorem 6.8 , we know that $\mathcal{R}_\mu$ is a bounded operator on L^p and by Theorem 7.6, it is also the case of $\mathcal{R}$ and $\mathcal{R}^*$. The Theorem follows now immediately from the following identity:

$$\partial_\mu = \mathcal{R}_\mu \circ \mathcal{R} \circ \delta + \mathcal{R}_\mu \circ \mathcal{R}^* \circ d. \tag{8.2}$$

The latter formula is a consequence of the relations $\Delta \circ I^2 = \mathrm{Id}$ and $\mathcal{R}_\mu = I^1 \circ \partial_\mu$, indeed:

$$\begin{aligned}
\partial_\mu &= \Delta \circ I^2 \circ \partial_\mu \\
&= (I^1 \circ \partial_\mu) \circ (\Delta \circ I^1) \\
&= \mathcal{R}_\mu \circ (d\delta + \delta d) \circ I^1 \\
&= \mathcal{R}_\mu \circ (d \circ I^1) \circ \delta + \mathcal{R}_\mu \circ (\delta \circ I^1) \circ d \\
&= \mathcal{R}_\mu \circ \mathcal{R} \circ \delta + \mathcal{R}_\mu \circ \mathcal{R}^* \circ d. \qquad\qquad \square
\end{aligned}$$

8.2. A Sobolev inequality for differential forms.
We have the following Sobolev–Gagliardo–Nirenberg inequality for differential forms on $\mathbb{R}^n$:

Theorem 8.2. *Let* $1 < p, q < \infty$. *There exists a constant* $C < \infty$ *such that for any* $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$, *we have*

$$\|\theta\|_{L^q(\mathbb{R}^n)} \leqslant C \left(\|d\theta\|_{L^p(\mathbb{R}^n)} + \|\delta\theta\|_{L^p(\mathbb{R}^n)} \right). \tag{8.3}$$

if and only if $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$.

Proof. We have $\theta = E\theta + E^*\theta = U(d\theta) + U^*(\delta\theta)$. By Theorem 1.2, we know that $U, U^* \colon L^p(\mathbb{R}^n, \Lambda^{k-1}) \to L^q(\mathbb{R}^n, \Lambda^{k-1})$ are bounded operators if and only if $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, hence the inequality (8.3) holds with

$$C = \max\{\|\mathcal{U}\|_{L^p \to L^q}, \|\mathcal{U}^*\|_{L^p \to L^q}\}.$$

We need to show in the converse direction, that the inequality (8.3) cannot hold with a finite constant if $\frac{1}{p} - \frac{1}{q} \neq \frac{1}{n}$. To do that, we consider a non zero k-form $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$, observe that either $d\theta \neq 0$ or $\delta\theta \neq 0$, for otherwise the form θ would have constant coefficients, which is impossible for a non zero form in $L^p(\mathbb{R}^n)$. The following quantity is therefore well defined:

$$Q(t) = \frac{\|h_t^* \theta\|_{L^q(\mathbb{R}^n)}}{\|h_t^* d\theta\|_{L^p(\mathbb{R}^n)} + \|h_t^* \delta\theta\|_{L^p(\mathbb{R}^n)}},$$

where h_t is the 1-parameter group of linear dilations in $\mathbb{R}^n$ given by $h_t(x) = t \cdot x$.

A calculation shows that for any $\omega \in L^s(\mathbb{R}^n, \Lambda^m)$, we have

$$\|h_t^* \omega\|_{L^s(\mathbb{R}^n)} = t^{m - \frac{n}{s}} \|\theta\|_{L^s(\mathbb{R}^n)}, \tag{8.4}$$

since $dh_t^* \theta = h_t^* d\theta$ is a $(k+1)$-form, we obtain

$$\|h_t^* d\theta\|_{L^p(\mathbb{R}^n)} = t^{1+k-\frac{n}{p}}\|d\theta\|_{L^p(\mathbb{R}^n)}. \tag{8.5}$$

Now be careful, because $\delta h_t^* \neq h_t^* \delta$. In fact $\delta h_t^* \theta = t^2 h_t^* \delta\theta$; this is a $(k-1)$-form and thus (8.4) implies that

$$\|h_t^* \delta\theta\|_{L^p(\mathbb{R}^n)} = t^2 t^{k-1-\frac{n}{p}}\|\delta\theta\|_{L^p(\mathbb{R}^n)} = t^{1+k-\frac{n}{p}}\|\delta\theta\|_{L^p(\mathbb{R}^n)}. \tag{8.6}$$

The last three identities give us

$$Q(t) = \frac{t^{k-\frac{n}{q}}}{t^{1+k-\frac{n}{p}}}\, Q(1) = t^{\frac{n}{p}-\frac{n}{q}-1}\, Q(1).$$

If $\frac{1}{p} - \frac{1}{q} - \frac{1}{n} < 0$, then $\lim_{t\to 0} Q(t) = \infty$ and if $\frac{1}{p} - \frac{1}{q} - \frac{1}{n} > 0$, then $\lim_{t\to\infty} Q(t) = \infty$. We conclude that the Sobolev inequality (8.3) cannot hold if $\frac{1}{p} - \frac{1}{q} - \frac{1}{n} \neq 0$. $\quad\square$

8.3. The L^p *a priory* estimates for the Laplacian on forms

Theorem 8.3. *The following inequality holds for any* $\theta \in \Phi'(\mathbb{R}^n, \Lambda^k)$, $1 < p < \infty$ *and* $\mu, \nu = 1, 2, \ldots, n$:

$$\|\partial_\mu \partial_\nu \theta\|_{L^p(\mathbb{R}^n)} \leqslant c_p^2\, \|\Delta\theta\|_{L^p(\mathbb{R}^n)} \tag{8.7}$$

where c_p *is the norm of the operator* $\mathcal{R}_j : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

Observe that this estimate is actually a scalar estimate.

Proof. By Theorem 6.8, we know that $\mathcal{R}_\mu$ is a bounded operator on L^p. It is also clearly the case of $\mathcal{R}$ and $\mathcal{R}^*$. The corollary now follows from the following lemma and the definition of c_p. $\quad\square$

Lemma 8.4. *The following identity holds in* $\operatorname{Op}\Phi'(\mathbb{R}^n, \Lambda^k)$:

$$\partial_\mu \partial_\nu = \mathcal{R}_\mu \circ \mathcal{R}_\nu \circ \Delta. \tag{8.8}$$

Proof. We have

$$\begin{aligned}
\partial_\mu \partial_\nu &= \Delta \circ I^2 \circ \partial_\mu \circ \partial_\nu \\
&= \Delta \circ (I^1 \circ \partial_\mu) \circ (I^1 \circ \partial_\nu) \\
&= \Delta \circ \mathcal{R}_\mu \circ \mathcal{R}_\nu \\
&= \mathcal{R}_\mu \circ \mathcal{R}_\nu \circ \Delta. \qquad\qquad\square
\end{aligned}$$

8.4. The $L_{q,p}$-cohomology of $\mathbb{R}^n$. The set of closed forms in $L^p(\mathbb{R}^n, \Lambda^k)$ is denoted by

$$Z_p^k(\mathbb{R}^n) = L^p(\mathbb{R}^n, \Lambda^k) \cap \ker d,$$

and the set of exact forms in $L^p(\mathbb{R}^n, \Lambda^k)$ which are differentials of forms in L^q is denoted by

$$B_{q,p}^k(\mathbb{R}^n) = d(L^q(\mathbb{R}^n, \Lambda^{k-1})) \cap L^p(\mathbb{R}^n, \Lambda^k).$$

Lemma 8.5. $Z_p^k(\mathbb{R}^n) \subset L^p(\mathbb{R}^n, \Lambda^k)$ *is a closed linear subspace.*

Proof. By definition, a form $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$ belongs to $Z_p^k(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} \theta \wedge \varphi = 0$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n, \Lambda^{n-k})$. Suppose now that $\theta \in L^p(\mathbb{R}^n, \Lambda^k)$ is in the closure of $Z_p^k(\mathbb{R}^n)$. This means that there exists a sequence $\theta_i \in Z_p^k(\mathbb{R}^n)$ converging to θ for the L^p norm. Using the Hölder inequality with $q = p/(p-1)$, we have

$$\left| \int_{\mathbb{R}^n} \theta \wedge \varphi \right| = \lim_{i \to \infty} \left| \int_{\mathbb{R}^n} (\theta - \theta_i) \wedge \varphi \right| \leqslant \lim_{i \to \infty} \|\theta - \theta_i\|_{L^p} \|\varphi\|_{L^q} = 0,$$

and therefore $\theta \in Z_p^k(\mathbb{R}^n)$. $\qquad\square$

Remark. It is clear that $B_{q,p}^k(\mathbb{R}^n) \subset EL^p(\mathbb{R}^n, \Lambda^k) \subset Z_p^k(\mathbb{R}^n)$. By Proposition 7.4 (a), 7.4 (b) and Theorem 1.2, we have in fact $Z_p^k(\mathbb{R}^n) = EL^p(\mathbb{R}^n, \Lambda^k)$.

Definition. The $L_{q,p}$-*cohomology* of $\mathbb{R}^n$ is the quotient

$$H_{q,p}^k(\mathbb{R}^n) = Z_p^k(\mathbb{R}^n)/B_{q,p}^k(\mathbb{R}^n).$$

The next result computes this cohomology:

Theorem 8.6. *For any* $p, q \in (1, \infty)$ *and* $1 \leqslant k \leqslant n$, *we have*

$$H_{q,p}^k(\mathbb{R}^n) = 0 \iff \frac{1}{p} - \frac{1}{q} = \frac{1}{n}.$$

Proof. Assume first that $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. By Proposition 7.4, we have for any $\theta \in Z_p^k(\mathbb{R}^n)$

$$\theta = E\theta + E^*\theta = E\theta = d(U\theta),$$

because $E^* = 0$ on $\ker d$. By Theorem 1.2, we know that $U \colon L^p(\mathbb{R}^n, \Lambda^{k-1}) \to L^q(\mathbb{R}^n, \Lambda^{k-1})$ is a bounded operator. Hence $\theta = d(U\theta) \in B_{q,p}^k(\mathbb{R}^n)$; but since $\theta \in Z_p^k(\mathbb{R}^n)$ is arbitrary, we have

$$H_{q,p}^k(\mathbb{R}^n) = Z_p^k(\mathbb{R}^n)/B_{q,p}^k(\mathbb{R}^n) = 0.$$

To prove the converse direction, we use the interpretation of the $L_{q,p}$-cohomology in terms of Sobolev inequalities. In particular, it is proven in [5, Theorem 6.2] that if $H_{q,p}^k(\mathbb{R}^n) = 0$, then there exists a constant C such that for any $\phi \in L^q(\mathbb{R}^n, \Lambda^{k-1})$, there exists a closed form $\zeta = \zeta(\phi) \in Z_q^{k-1}(\mathbb{R}^n)$ such that

$$\|\phi - \zeta\|_{L^q} \leqslant C \|d\phi\|_{L^p}. \tag{8.9}$$

Let us fix a form $\phi \in L^q(\mathbb{R}^n, \Lambda^{k-1})$ which is *not* closed and apply the above inequality to $h_t^* \phi$, where $h_t(x) = t \cdot x$. It says in this case that for any $t \in \mathbb{R}$, there exists $\zeta_t \in Z_q^{k-1}(\mathbb{R}^n)$ such that

$$\|h_t^* \phi - \zeta_t\|_{L^q} \leqslant C \|h_t^* d\phi\|_{L^p}. \tag{8.10}$$

Using the identity (8.4) with $s = q$, $m = (k-1)$ and $s = p$, $m = k$, we obtain the inequality

$$\|\phi - h_{-t}^* \zeta_t\|_{L^q} \leqslant Ct^\gamma \|d\phi\|_{L^p} \tag{8.11}$$

with $\gamma = 1 + \frac{n}{q} - \frac{n}{p}$. The right hand side of this inequality converges to zero as $t \to 0$ if $\gamma < 0$ or as $t \to \infty$ if $\gamma > 0$. Since $h_{-t}^* \zeta_t \in Z_q^{k-1}(\mathbb{R}^n)$ for any t and $Z_q^{k-1}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n, \Lambda^{k-1})$ is closed, it follows that $\phi \in Z_q^{k-1}(\mathbb{R}^n)$. But ϕ is not closed by hypothesis, we thus conclude that $\gamma = 0$. To sum up, the argument shows that if $H_{q,p}^k(\mathbb{R}^n) = 0$, then $\gamma = 1 + \frac{n}{q} - \frac{n}{p} = 0$. $\qquad\square$

Remark. Theorem 6.2 in [5] says in fact that there exists a constant C such that (8.9) holds if and only if $T_{q,p}^k(\mathbb{R}^n) = 0$ (provided $1 < q,\ p < \infty$). Here $T_{q,p}^k$ is the *torsion*, which is defined to be the quotient $\overline{B_{q,p}^k}/B_{q,p}^k$. We have thus also proved that $T_{q,p}^k(\mathbb{R}^n) = 0 \Leftrightarrow \frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. In particular $H_{q,p}^k(\mathbb{R}^n)$ is infinite dimensional if $\frac{1}{p} - \frac{1}{q} \neq \frac{1}{n}$.

Appendix: Computation of the Fourier Transform of the Riesz Kernel

Definition. The *Riesz kernel* of order $\alpha \in (0,\ n)$ is the function k_α defined on $\mathbb{R}^n$ by

$$k_\alpha(x) = \frac{1}{\gamma(n,\ \alpha)}|x|^{\alpha-n},$$

where the normalizing constant is given by

$$\gamma(n,\ \alpha) = 2^\alpha \pi^{n/2} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

Theorem 8.7. *The Fourier transform of the Riesz kernel of order $\alpha \in (0,\ n)$ is given by*

$$\mathcal{F}(k_\alpha) = |\xi|^{-\alpha}.$$

Proof. We will use the fact that the Gaussian function $g(x) = e^{-s|x|^2}$ belongs to $\mathcal{S}$ for any $s > 0$ and that its Fourier transform is given by

$$\mathcal{F}(e^{-s|x|^2})(\xi) = \left(\frac{\pi}{s}\right)^{n/2} e^{-|\xi|^2/4s}, \tag{8.12}$$

(this is a well known fact; see, e.g., [4, Proposition 8.24] or [13, page 38]).

To compute the Fourier transform of k_α, we start from the formulas

$$\Gamma(z)\,a^{-z} = \int_0^\infty s^{z-1}e^{-as}\,ds \quad \text{and} \quad \Gamma(w)\,b^{-w} = \int_0^\infty s^{-w-1}e^{-b/s}\,ds, \tag{8.13}$$

which hold for any $a,\ b \in (0,\ \infty)$ and any $z,\ w \in \mathbb{C}$ such that $\mathrm{Re}(z),\ \mathrm{Re}(w) > 0$.

To check these formulas, use the substitution $t = as$ (for the first identity) and $t = b/s$ (for the second identity) in the definition $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ of the Γ-function.

We will use the first formula with $a = |x|^2$ and apply the Fourier transform; keeping in mind the identity (8.12), we have

$$\begin{aligned}
\mathcal{F}(\Gamma(z)|x|^{-2z}) &= \mathcal{F}\left(\int_0^\infty s^{z-1}e^{-|x|^2 s}\,ds\right) \\
&= \int_0^\infty s^{z-1}\mathcal{F}(e^{-|x|^2 s})\,ds \qquad \text{(Fubini)} \\
&= \int_0^\infty s^{z-1}\left(\frac{\pi}{s}\right)^{n/2} e^{-|\xi|^2/4s}\,ds \\
&= \pi^{n/2}\int_0^\infty s^{z-\frac{n}{2}-1}e^{-|\xi|^2/4s}\,ds.
\end{aligned}$$

Setting $b = |\xi|^2/4$ and $w = \frac{n}{2} - z$, we obtain from the second identity in (8.13)

$$\Gamma(z)\mathcal{F}(|x|^{-2z}) = \pi^{n/2} \int_0^\infty s^{-w-1}e^{-b/s}\,ds = \pi^{n/2}\Gamma(w)4^w\,|\xi|^{-2w}.$$

Let us set $\alpha = n - 2z$, thus $z = \frac{n-\alpha}{2}$ and $w = \frac{n}{2} - z = \frac{\alpha}{2}$; we write this formula as

$$\mathcal{F}(|x|^{\alpha-n}) = \gamma(n,\,\alpha)|\xi|^{-\alpha},$$

where

$$\gamma(n,\,\alpha) = \pi^{n/2}2^\alpha \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

The above calculation assumes $\mathrm{Re}(z),\ \mathrm{Re}(w) > 0$, which is equivalent to $0 < \alpha < n$. $\square$

Remark. Using the Fourier transform in the Lizorkin sense, it is possible to extend the Riesz kernel k_α of order α for any real number $\alpha > 0$ (and in fact any complex number with $\mathrm{Re}\,\alpha > 0$). We define it as follow

$$k_\alpha = \frac{1}{\gamma(n,\,\alpha)}\,|x|^{\alpha-n}$$

if $\alpha \neq n + 2m$ for any $m \in \mathbb{N}$, and by

$$k_\alpha = \frac{1}{\gamma(n,\,\alpha)}\,|x|^{\alpha-n}\,\log\frac{1}{|x|}$$

if $\alpha = n + 2m$ for some $m \in \mathbb{N}$.

With this definition, the previous result is still valid

Proposition 8.8. *The Fourier transform of k_α, $\alpha \in \mathbb{C}$ is given by*

$$\mathcal{F}(k_a) = |\xi|^{-\alpha}. \hspace{3cm} \square$$

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