

# EXPONENTIAL-POLYNOMIAL FAMILIES AND THE TERM STRUCTURE OF INTEREST RATES

DAMIR FILIPOVIĆ

DEPARTMENT OF MATHEMATICS, ETH, RÄMISTRASSE 101, CH-8092 ZÜRICH,  
SWITZERLAND. E-MAIL: FILIPO@MATH.ETHZ.CH

**ABSTRACT.** Exponential-polynomial families like the Nelson–Siegel or Svensson family are widely used to estimate the current forward rate curve. We investigate whether these methods go well with inter-temporal modelling. We characterize the consistent Itô processes which have the property to provide an arbitrage free interest rate model when representing the parameters of some bounded exponential-polynomial type function. This includes in particular diffusion processes. We show that there is a strong limitation on their choice. Bounded exponential-polynomial families should rather not be used for modelling the term structure of interest rates.

*Keywords:* consistent Itô process, diffusion process, exponential-polynomial family, forward rate curve, interest rate model, inverse problem

## 1. INTRODUCTION

The current term structure of interest rates contains all the necessary information for pricing bonds, swaps and forward rate agreements of all maturities. It is used furthermore by the central banks as indicator for their monetary policy.

There are several algorithms for constructing the current forward rate curve from the (finitely many) prices of bonds and swaps observed in the market. Widely used are splines and parameterized families of smooth curves  $\{F(\cdot, z)\}_{z \in \mathcal{Z}}$ , where  $\mathcal{Z} \subset \mathbb{R}^N$ ,  $N \geq 1$ , denotes some finite dimensional parameter set. By an optimal choice of the parameter  $z$  in  $\mathcal{Z}$  an optimal fit of the forward curve  $x \mapsto F(x, z)$  to the observed data is attained. Here  $x \geq 0$  denotes *time to maturity*. In that sense  $z$  represents the current state of the economy taking values in the state space  $\mathcal{Z}$ .

Examples are the Nelson–Siegel [8] family with curve shape

$$F_{NS}(x, z) = z_1 + (z_2 + z_3x)e^{-z_4x}$$

and the Svensson [11] family, an extension of Nelson–Siegel,

$$F_S(x, z) = z_1 + (z_2 + z_3x)e^{-z_5x} + z_4xe^{-z_6x}.$$

Table 1 gives an overview of the fitting procedures used by some selected central banks. It is taken from the documentation of the Bank for International Settlements [1].

Despite the flexibility and low number of parameters of  $F_{NS}$  and  $F_S$ , their choice is somewhat arbitrary. We shall discuss them from an inter-temporal point of view: A lot of cross-sectional data, i.e. daily estimations of  $z$ , is available. Therefore it

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TABLE 1. Forward rate curve fitting procedures

central bank	curve fitting procedure
Belgium	Nelson–Siegel, Svensson
Canada	Svensson
Finland	Nelson–Siegel
France	Nelson–Siegel, Svensson
Germany	Svensson
Italy	Nelson–Siegel
Japan	smoothing splines
Norway	Svensson
Spain	Nelson–Siegel (before 1995), Svensson
Sweden	Svensson
UK	Svensson
USA	smoothing splines

would be natural to ask for the stochastic evolution of the parameter  $z$  over time. But then there exist economic constraints based on no arbitrage considerations.

Following [2], instead of  $F_{NS}$  and  $F_S$  we consider general exponential-polynomial families containing curves of the form

$$F(x, z) = \sum_{i=1}^K \left( \sum_{\mu=0}^{n_i} z_{i,\mu} x^\mu \right) e^{-z_{i,n_i+1} x}.$$

Hence linear combinations of exponential functions  $\exp(-z_{i,n_i+1}x)$  over some polynomials of degree  $n_i \in \mathbb{N}_0$ . Obviously  $F_{NS}$  and  $F_S$  are of this type. We replace then  $z$  by an Itô process  $Z = (Z_t)_{t \geq 0}$  taking values in  $\mathcal{Z}$ . The following questions arise:

- Does  $F(\cdot, Z)$  provide an arbitrage free interest rate model?
- And what are the conditions on  $Z$  for it?

Working in the Heath–Jarrow–Morton [5] – henceforth HJM – framework with deterministic volatility structure, Björk and Christensen [2] showed that the exponential-polynomial families are in a certain sense too large to carry an interest rate model. This result has been generalized for the Nelson–Siegel family in [4], including stochastic volatility structure. Expanding the methods used in there, we give in this paper the general result for bounded exponential-polynomial families.

The paper is organized as follows. In Section 2 we introduce the class of Itô processes consistent with a given parameterized family of forward rate curves. Consistent Itô processes provide an arbitrage free interest rate model when driving the parameterized family. They are characterized in terms of their drift and diffusion coefficients by the HJM drift condition.

By solving an inverse problem we get the main result for consistent Itô processes, stated in Section 3. It is shown that they are remarkably limited. The proof is divided into several steps, given in Sections 4, 5 and 6.

In Section 7 we extend the notion of consistency to e-consistency when  $\mathbb{P}$  is not a martingale measure.

The main result reads much clearer when restricted to diffusion processes, as shown in Section 8. It turns out that e-consistent diffusion processes driving

bounded exponential-polynomial families like Nelson–Siegel or Svensson are very limited: most of the factors are either constant or deterministic. It is shown in Section 9, that there is no non-trivial diffusion process which is e-consistent with the Nelson–Siegel family. Furthermore we identify the diffusion process which is e-consistent with the Svensson family. It contains just one non deterministic component. The corresponding short rate model is shown to be the generalized Vasicek model.

We conclude that bounded exponential-polynomial families, in particular  $F_{NS}$  and  $F_S$ , should rather not be used for modelling the term structure of interest rates.

## 2. CONSISTENT ITÔ PROCESSES

For the stochastic background and notations we refer the reader to [9] and [6]. Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, \mathbb{P})$  be a filtered complete probability space, satisfying the usual conditions, and let  $W = (W_t^1, \dots, W_t^d)_{0 \leq t < \infty}$  denote a standard  $d$ -dimensional  $(\mathcal{F}_t)$ -Brownian motion,  $d \geq 1$ .

Let  $Z = (Z^1, \dots, Z^N)$  denote an  $\mathbb{R}^N$ -valued Itô process,  $N \geq 1$ , of the form

$$Z_t^i = Z_0^i + \int_0^t b_s^i ds + \sum_{j=1}^d \int_0^t \sigma_s^{i,j} dW_s^j, \quad i = 1, \dots, N, \quad 0 \leq t < \infty,$$

where  $Z_0$  is  $\mathcal{F}_0$ -measurable, and  $b, \sigma$  are progressively measurable processes with values in  $\mathbb{R}^N$ , resp.  $\mathbb{R}^{N \times d}$ , such that

$$\int_0^t (|b_s| + |\sigma_s|^2) ds < \infty, \quad \mathbb{P}\text{-a.s.}, \quad \text{for all finite } t.$$

Let  $F(x, z)$  be a function in  $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$ , i.e.  $F$  and the partial derivatives  $(\partial F / \partial x)$ ,  $(\partial F / \partial z_i)$ ,  $(\partial^2 F / \partial z_i \partial z_j)$ , which exist for  $1 \leq i, j \leq N$ , are continuous functions on  $\mathbb{R}_+ \times \mathbb{R}^N$ . Interpreting  $Z_t$  as the state of the economy at time  $t$ , we let  $x \mapsto F(x, Z_t)$  stand for the corresponding term structure of interest rates. Meaning that  $F(x, Z_t)$  denotes the *instantaneous forward rate* at time  $t$  for date  $t + x$ .

Notice that

$$G(x, z) := \exp \left( - \int_0^x F(\eta, z) d\eta \right)$$

is in  $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$  too. Therefore the price processes for *zero coupon T-bonds*

$$P(t, T) := G(T - t, Z_t), \quad 0 \leq t \leq T < \infty, \quad (1)$$

and the process of the *savings account*

$$B(t) := \exp \left( - \int_0^t \frac{\partial}{\partial x} G(0, Z_s) ds \right), \quad 0 \leq t < \infty,$$

form continuous semimartingales.

Let  $\mathcal{Z}$  denote an arbitrary subset of  $\mathbb{R}^N$ . The function  $F$  generates in a canonical way a parameterized set of forward curves  $\{F(\cdot, z)\}_{z \in \mathcal{Z}}$ . We shall refer to  $\mathcal{Z}$  as the state space of the economy.

**Definition 2.1.**  $Z$  is called consistent with  $\{F(\cdot, z)\}_{z \in \mathcal{Z}}$ , if the support of  $Z$  is contained in  $\mathcal{Z}$  and

$$\left( \frac{P(t, T)}{B(t)} \right)_{0 \leq t \leq T} \quad (2)$$

is a  $\mathbb{P}$ -martingale, for all  $T < \infty$ .

Set  $a := \sigma\sigma^*$ , where  $\sigma^*$  denotes the transpose of  $\sigma$ , i.e.  $a_t^{i,j} = \sum_{k=1}^d \sigma_t^{i,k} \sigma_t^{j,k}$ , for  $1 \leq i, j \leq N$  and  $0 \leq t < \infty$ . Then  $a$  is a progressively measurable process with values in the symmetric nonnegative definite  $N \times N$ -matrices.

Using Itô's formula, the dynamics of (2) can be decomposed into finite variation and local martingale part. Requiring consistency the former has to vanish. This is the well known HJM drift condition and is stated explicitly in the following proposition.

**Proposition 2.2.** *If  $Z$  is consistent with  $\{F(\cdot, z)\}_{z \in \mathcal{Z}}$  then*

$$\begin{aligned} \frac{\partial}{\partial x} F(x, Z) &= \sum_{i=1}^N b^i \frac{\partial}{\partial z_i} F(x, Z) \\ &+ \frac{1}{2} \sum_{i,j=1}^N a^{i,j} \left( \frac{\partial^2}{\partial z_i \partial z_j} F(x, Z) - \frac{\partial}{\partial z_i} F(x, Z) \int_0^x \frac{\partial}{\partial z_j} F(\eta, Z) d\eta \right. \\ &\quad \left. - \frac{\partial}{\partial z_j} F(x, Z) \int_0^x \frac{\partial}{\partial z_i} F(\eta, Z) d\eta \right), \end{aligned} \quad (3)$$

for all  $x \geq 0$ ,  $dt \otimes d\mathbb{P}$ -a.s.

*Proof.* Analogous to the proof of [4, Proposition 3.2].  $\square$

### 3. EXPONENTIAL-POLYNOMIAL FAMILIES

In this section we introduce a particular class of functions  $F$ . As the main result we characterize the corresponding consistent Itô processes.

Let  $K$  denote a positive integer and let  $n = (n_1, \dots, n_K)$  be a vector with components  $n_i \in \mathbb{N}_0$ , for  $1 \leq i \leq K$ . Write  $|n| := n_1 + \dots + n_K$ . For a point

$$z = (z_{1,0}, \dots, z_{1,n_1+1}, z_{2,0}, \dots, z_{2,n_2+1}, \dots, z_{K,0}, \dots, z_{K,n_K+1}) \in \mathbb{R}^{|n|+2K} \quad (4)$$

define the polynomials  $p_i(z)$  as

$$p_i(z) = p_i(x, z) := \sum_{\mu=0}^{n_i} z_{i,\mu} x^\mu, \quad 1 \leq i \leq K.$$

The function  $F$  is now defined as

$$F(x, z) := \sum_{i=1}^K p_i(x, z) e^{-z_i n_i + 1x}. \quad (5)$$

Obviously  $F \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^{|n|+2K})$ . Hence the preceding section applies with  $N = |n| + 2K$ .

From an economic point of view it seems reasonable to restrict to bounded forward rate curves. Let therefore  $\mathcal{Z}$  denote the set of all  $z \in \mathbb{R}^N$  such that  $\sup_{x \in \mathbb{R}_+} |F(x, z)| < \infty$ .

**Definition 3.1.** *The exponential-polynomial family  $EP(K, n)$  is defined as the set of forward curves  $\{F(\cdot, z)\}_{z \in \mathbb{R}^N}$ .*

*The bounded exponential-polynomial family  $BEP(K, n) \subset EP(K, n)$  is defined as the set of forward curves  $\{F(\cdot, z)\}_{z \in \mathcal{Z}}$ .*

Clearly  $F_{NS}(x, z) \in BEP(2, (0, 1))$  and  $F_S(x, z) \in BEP(3, (0, 1, 1))$ , if at each case the parameter  $z$  is chosen such that the curve is bounded. From now on, the Nelson–Siegel and Svensson families are considered as subsets of  $BEP(2, (0, 1))$  and  $BEP(3, (0, 1, 1))$  respectively.

If two exponents  $z_{i, n_i+1}$  and  $z_{j, n_j+1}$  coincide, the sum (5) defining  $F$  reduces to a linear combination of  $K - 1$  exponential functions. Thus for  $z \in \mathbb{R}^N$  we introduce the equivalence relation

$$i \sim_z j : \iff z_{i, n_i+1} = z_{j, n_j+1} \quad (6)$$

on the set  $\{1, \dots, K\}$  and denote by  $[i] = [i]_z$  the equivalence class of  $i$ . We will use the notation

$$\begin{aligned} n_{[i]} &= n_{[i]}(z) := \max\{n_j \mid j \in [i]_z\} \\ \mathcal{I}_{[i], \mu} &= \mathcal{I}_{[i], \mu}(z) := \{j \in [i]_z \mid n_j \geq \mu\}, \quad 0 \leq \mu \leq n_{[i]}(z) \\ z_{[i], \mu} &= z_{[i], \mu}(z) := \sum_{j \in \mathcal{I}_{[i], \mu}(z)} z_{j, \mu}, \quad 0 \leq \mu \leq n_{[i]}(z) \\ p_{[i]}(z) &:= \sum_{j \in [i]_z} p_j(z). \end{aligned} \quad (7)$$

In particular  $p_{[i]}(z) = \sum_{\mu=0}^{n_{[i]}} z_{[i], \mu} x^\mu$  and (5) reads now

$$F(x, z) = \sum_{[i] \in \{1, \dots, K\} / \sim} p_{[i]}(z) e^{-z_{i, n_i+1} x}.$$

Observe that for  $z \in \mathcal{Z}$  we have

$$\begin{aligned} z_{i, n_i+1} &= 0, \quad \text{only if } p_{[i]}(z) = z_{[i], 0} \\ z_{i, n_i+1} &< 0, \quad \text{only if } p_{[i]}(z) = 0. \end{aligned} \quad (8)$$

We shall write the  $\mathbb{R}^N$ -valued Itô process  $Z$  with the same indices which we use for a point  $z \in \mathbb{R}^N$ , see (4),

$$Z_t^{i, \mu} = Z_0^{i, \mu} + \int_0^t b_s^{i, \mu} ds + \sum_{\lambda=1}^d \int_0^t \sigma_s^{i, \mu; \lambda} dW_s^\lambda, \quad 0 \leq \mu \leq n_i + 1, \quad 1 \leq i \leq K. \quad (9)$$

It's diffusion matrix  $a$  consists of the components

$$a^{i, \mu; j, \nu} = \sum_{\lambda=1}^d \sigma^{i, \mu; \lambda} \sigma^{j, \nu; \lambda}, \quad 0 \leq \mu \leq n_i + 1, \quad 0 \leq \nu \leq n_j + 1, \quad 1 \leq i, j \leq K.$$

Notice that for  $1 \leq i \leq K$

$$\begin{aligned} \{z \mid p_{[i]}(z) = 0\} &= \bigcup_{\substack{J \subset \{1, \dots, K\} \\ J \ni i}} \left( \{z \mid z_{j, n_j+1} = z_{i, n_i+1} \text{ for all } j \in J\} \right. \\ &\quad \cap \bigcap_{\mu=0}^{\max\{n_j \mid j \in J\}} \left. \{z \mid \sum_{\substack{j \in J \\ n_j \geq \mu}} z_{j, \mu} = 0\} \right. \\ &\quad \left. \setminus \bigcup_{l \in J^c} \{z \mid z_{l, n_l+1} = z_{i, n_i+1}\} \right) \end{aligned} \quad (10)$$

is not closed in general but nevertheless a Borel set in  $\mathbb{R}^N$ . We introduce the following, thus optional, random sets of singular points  $(t, \omega)$

$$\begin{aligned}\mathcal{A}_i &:= \{p_i(Z) = 0 \text{ or } p_{[i]}(Z) = 0\}, \quad 1 \leq i \leq K \\ \mathcal{B} &:= \bigcup_{\substack{i,j=1 \\ i \neq j}}^K \{Z^{i,n_i+1} = Z^{j,n_j+1}\} \\ \mathcal{C} &:= \bigcup_{\substack{i,j=1 \\ i \neq j}}^K \{2Z^{i,n_i+1} = Z^{j,n_j+1}\}\end{aligned}$$

and the optional random sets of regular points  $(t, \omega)$

$$\begin{aligned}\mathcal{D} &:= (\mathbb{R}_+ \times \Omega) \setminus \left( \bigcup_{i=1}^K \mathcal{A}_i \cup \mathcal{B} \cup \mathcal{C} \right) \\ \mathcal{D}' &:= (\mathbb{R}_+ \times \Omega) \setminus (\mathcal{B} \cup \mathcal{C}).\end{aligned}$$

Let us recall that for  $S$  and  $T$  stopping times, a stochastic interval like  $[S, T]$  is a subset of  $\mathbb{R}_+ \times \Omega$ . Hence  $[S] = [S, S]$  is the restriction of the graph of the mapping  $S : \Omega \rightarrow [0, \infty]$  to the set  $\mathbb{R}_+ \times \Omega$ .

For any stopping time  $\tau$  with  $[\tau] \in (\mathbb{R}_+ \times \Omega) \setminus \mathcal{A}_i$  we define

$$\tau'(\omega) := \inf\{t > \tau(\omega) \mid (t, \omega) \in \mathcal{A}_i\},$$

the debut of the optional set  $[\tau, \infty[ \cap \mathcal{A}_i$ . Observe that in general it is not true that  $\tau' > \tau$  on  $\{\tau < \infty\}$ . This can be seen from the following example: For

$$F(x, z) = z_{1,0}e^{-z_{1,1}x} + z_{2,0}e^{-z_{2,1}x} + z_{3,0}e^{-z_{3,1}x} \in BEP(3, (0, 0, 0))$$

let  $Z_t^{1,0} = Z_t^{3,0} = 1$ ,  $Z_t^{2,0} = -1$ ,  $Z_t^{3,1} = 1+t$  and  $Z_t^{1,1} = Z_t^{2,1} = 1$  for  $t \in [0, 1]$ . Then  $p_1(Z_0) = p_{[1]}(Z_0) = 1$  and  $p_{[1]}(Z_t) = 0$  for all  $t \in (0, 1]$ . Hence  $[0] \in (\mathbb{R}_+ \times \Omega) \setminus \mathcal{A}_1$ , but  $\tau' = 0$ . However, by continuity of  $Z$  we always have

$$\tau < \tau' \quad \mathbb{P}\text{-a.s. on } \{\omega \mid (\tau(\omega), \omega) \in \mathcal{D}'\}. \quad (11)$$

Recall the fact that there is a one to one correspondence between the Itô processes  $Z$  starting in  $Z_0$  (up to indistinguishability) and the equivalence classes of  $b$  and  $\sigma$  with respect to the  $dt \otimes d\mathbb{P}$ -nullsets in  $\mathcal{R}_+ \otimes \mathcal{F}$ . Hence we may state the following *inverse problem* to equation (3): Given a family of forward curves. For which choices of coefficients  $b$  and  $\sigma$  do we get a consistent Itô process  $Z$  starting in  $Z_0$ ?

The main result is the following characterization of all consistent Itô processes, which is remarkably restrictive. The proof of the theorem will be given in Sections 5 and 6.

**Theorem 3.2.** *Let  $K \in \mathbb{N}$ ,  $n = (n_1, \dots, n_K) \in \mathbb{N}_0^K$  and  $Z$  as above. If  $Z$  is consistent with  $BEP(K, n)$ , then necessarily for  $1 \leq i \leq K$*

$$a^{i,n_i+1;i,n_i+1} = 0, \quad \text{on } \{p_i(Z) \neq 0\}, \quad dt \otimes d\mathbb{P}\text{-a.s.} \quad (12)$$

$$b^{i,n_i+1} = 0, \quad \text{on } \{p_i(Z) \neq 0\} \cap \{p_{[i]}(Z) \neq 0\}, \quad dt \otimes d\mathbb{P}\text{-a.s.} \quad (13)$$

Consequently,  $Z^{i,n_i+1}$  is constant on intervals where  $p_i(Z) \neq 0$  and  $p_{[i]}(Z) \neq 0$ . That is, for  $\mathbb{P}$ -a.e.  $\omega$

$$Z_t^{i,n_i+1}(\omega) = Z_u^{i,n_i+1}(\omega) \quad \text{for } t \in [u, v],$$

if  $p_i(Z_t(\omega)) \neq 0$  and  $p_{[i]}(Z_t(\omega)) \neq 0$  for  $t \in (u, v)$ .

For a stopping time  $\tau$  with  $[\tau] \subset \mathcal{D}'$  let  $\tau'(\omega) := \inf\{t \geq \tau(\omega) \mid (t, \omega) \notin \mathcal{D}'\}$  denote the debut of the optional random set  $(\mathcal{B} \cup \mathcal{C}) \cap [\tau, \infty[$ . Then it holds furthermore that  $\tau < \tau'$  on  $\{\tau < \infty\}$  and

$$\begin{aligned} Z_{\tau+t}^{i,\mu} &= Z_{\tau}^{i,\mu} e^{-Z_{\tau}^{i,n_i+1} t} + Z_{\tau}^{i,\mu+1} t e^{-Z_{\tau}^{i,n_i+1} t} \\ Z_{\tau+t}^{i,n_i} &= Z_{\tau}^{i,n_i} e^{-Z_{\tau}^{i,n_i+1} t} \end{aligned}$$

on  $[0, \tau' - \tau[$ , for  $0 \leq \mu \leq n_i - 1$  and  $1 \leq i \leq K$ , up to evanescence.

If  $\mathcal{D}'$  above is replaced by  $\mathcal{D}$  and  $\tau'$  is the debut of  $(\cup_{i=1}^K \mathcal{A}_i \cup \mathcal{B} \cup \mathcal{C}) \cap [\tau, \infty[$ , then  $\tau' = \infty$  and in addition

$$Z_{\tau+t}^{i,n_i+1} = Z_{\tau}^{i,n_i+1}$$

for  $1 \leq i \leq K$ ,  $\mathbb{P}$ -a.s. on  $\{\tau < \infty\}$ .

**Remark 3.3.** It will be made clear in the proof of the theorem that it is actually sufficient to assume  $Z$  to be consistent with  $EP(K, n)$  for (12) to hold.

As an immediate consequence we may state the following corollaries. The notation is the same as in the theorem.

**Corollary 3.4.** If  $Z$  is consistent with  $BEP(K, n)$  and if the optional random sets  $\{p_i(Z) = 0\}$  and  $\{p_{[i]}(Z) = 0\}$  have  $dt \otimes d\mathbb{P}$ -measure zero, then the exponent  $Z^{i,n_i+1}$  is indistinguishable from  $Z_0^{i,n_i+1}$ ,  $1 \leq i \leq K$ .

*Proof.* If  $\{p_i(Z) = 0\}$  and  $\{p_{[i]}(Z) = 0\}$  have  $dt \otimes d\mathbb{P}$ -measure zero, then  $\{p_i(Z) \neq 0\} \cap \{p_{[i]}(Z) \neq 0\} = \mathbb{R}_+ \times \Omega$  up to a  $dt \otimes d\mathbb{P}$ -nullset. The claim follows using (12) and (13).  $\square$

**Corollary 3.5.** If  $Z$  is consistent with  $BEP(K, n)$  and if the following three points are  $\mathbb{P}$ -a.s. satisfied

- i)  $p_i(Z_0) \neq 0$ , for all  $1 \leq i \leq K$ ,
- ii) there exists no pair of indices  $i \neq j$  with  $Z_0^{i,n_i+1} = Z_0^{j,n_j+1}$ ,
- iii) there exists no pair of indices  $i \neq j$  with  $2Z_0^{i,n_i+1} = Z_0^{j,n_j+1}$ ,

then  $Z$  and hence the interest rate model  $F(x, Z)$  is quasi deterministic, i.e. all randomness remains  $\mathcal{F}_0$ -measurable. In particular the exponents  $Z^{i,n_i+1}$  are indistinguishable from  $Z_0^{i,n_i+1}$ , for  $1 \leq i \leq K$ .

*Proof.* If i), ii) and iii) hold  $\mathbb{P}$ -a.s. then  $[0] \subset \mathcal{D}$ . The claim follows from the second part of the theorem setting  $\tau = 0$ .  $\square$

#### 4. AUXILIARY RESULTS

For the proof of the main result we need three auxiliary lemmas, presented in this section. First there is a result on the identification of the coefficients of Itô processes.

**Lemma 4.1.** Let

$$\begin{aligned} X_t &= X_0 + \int_0^t \beta_s^X ds + \sum_{j=1}^d \int_0^t \gamma_s^{X,j} dW_s^j \\ Y_t &= Y_0 + \int_0^t \beta_s^Y ds + \sum_{j=1}^d \int_0^t \gamma_s^{Y,j} dW_s^j \end{aligned}$$

be two Itô processes. Then  $dt \otimes d\mathbb{P}$ -a.s.

$$\begin{aligned} 1_{\{X=Y\}} \sum_{j=1}^d (\gamma^{X,j})^2 &= 1_{\{X=Y\}} \sum_{j=1}^d \gamma^{X,j} \gamma^{Y,j} = 1_{\{X=Y\}} \sum_{j=1}^d (\gamma^{Y,j})^2 \\ 1_{\{X=Y\}} \beta^X &= 1_{\{X=Y\}} \beta^Y. \end{aligned}$$

*Proof.* We write  $\langle \cdot, \cdot \rangle$  for the scalar product in  $\mathbb{R}^d$ . Then

$$|\langle \gamma^X, \gamma^X \rangle - \langle \gamma^X, \gamma^Y \rangle| = |\langle \gamma^X, \gamma^X - \gamma^Y \rangle| \leq \sqrt{\langle \gamma^X, \gamma^X \rangle} \sqrt{\langle \gamma^X - \gamma^Y, \gamma^X - \gamma^Y \rangle}$$

By the occupation times formula, see [9, Corollary (1.6), Chapter VI],

$$\int_0^t 1_{\{X_s=Y_s\}} \langle \gamma_s^X - \gamma_s^Y, \gamma_s^X - \gamma_s^Y \rangle ds = 0, \quad \text{for all } t < \infty, \mathbb{P}\text{-a.s.}$$

Hence by Hölder inequality

$$\begin{aligned} &\int_0^t 1_{\{X_s=Y_s\}} |\langle \gamma_s^X, \gamma_s^X \rangle - \langle \gamma_s^X, \gamma_s^Y \rangle| ds \\ &\leq \int_0^t 1_{\{X_s=Y_s\}} \sqrt{\langle \gamma_s^X, \gamma_s^X \rangle} \sqrt{\langle \gamma_s^X - \gamma_s^Y, \gamma_s^X - \gamma_s^Y \rangle} ds \\ &\leq \left( \int_0^t 1_{\{X_s=Y_s\}} \langle \gamma_s^X, \gamma_s^X \rangle ds \right)^{\frac{1}{2}} \left( \int_0^t 1_{\{X_s=Y_s\}} \langle \gamma_s^X - \gamma_s^Y, \gamma_s^X - \gamma_s^Y \rangle ds \right)^{\frac{1}{2}} \\ &= 0, \quad \text{for all } t < \infty, \mathbb{P}\text{-a.s.} \end{aligned}$$

Thus by symmetry

$$1_{\{X=Y\}} \langle \gamma^X, \gamma^X \rangle = 1_{\{X=Y\}} \langle \gamma^X, \gamma^Y \rangle = 1_{\{X=Y\}} \langle \gamma^Y, \gamma^Y \rangle, \quad dt \otimes d\mathbb{P}\text{-a.s.}$$

By continuity of the processes  $X$  and  $Y$  there are sequences of stopping times  $(S_n)$  and  $(T_n)$ ,  $S_n \leq T_n$ , with  $[S_m, T_m] \cap [S_n, T_n] = \emptyset$  for all  $m \neq n$  and

$$\{X = Y\} = \bigcup_{n \in \mathbb{N}} [S_n, T_n], \quad \text{up to evanescence.}$$

To see this, let  $n \in \mathbb{N}$  and let  $S(n, 1) := \inf\{t > 0 \mid |X_t - Y_t| = 0\}$ . Define  $T(n, p) := \inf\{t > S(n, p) \mid |X_t - Y_t| > 0\}$  and inductively

$$S(n, p+1) := \inf\{t > S(n, p) \mid |X_t - Y_t| = 0 \text{ and } \sup_{S(n, p) \leq s \leq t} |X_s - Y_s| > 2^{-n}\}.$$

Then by continuity we have  $\lim_{p \rightarrow \infty} S(n, p) = \infty$  for all  $n \in \mathbb{N}$  and it follows that  $\{X = Y\} = \bigcup_{n, p \in \mathbb{N}} [S(n, p), T(n, p)]$ . Now proceed as in [6, Lemma I.1.31] to find the sequences  $(S_n)$  and  $(T_n)$  with the desired properties.

From above we have  $1_{\{X=Y\}} (\gamma^X - \gamma^Y)^2 = 0$ ,  $dt \otimes d\mathbb{P}$ -a.s. For any  $0 \leq t < \infty$  therefore  $\int_{S_n \wedge t}^{T_n \wedge t} (\gamma_s^X - \gamma_s^Y) dW_s = 0$ ,  $\mathbb{P}$ -a.s. Hence

$$0 = (X - Y)_{T_n \wedge t} - (X - Y)_{S_n \wedge t} = \int_{S_n \wedge t}^{T_n \wedge t} (\beta_s^X - \beta_s^Y) ds, \quad \mathbb{P}\text{-a.s.}$$

We conclude

$$\int_0^t 1_{\{X_s=Y_s\}} (\beta_s^X - \beta_s^Y) ds = \sum_{n \in \mathbb{N}} \int_{S_n \wedge t}^{T_n \wedge t} (\beta_s^X - \beta_s^Y) ds = 0, \quad \text{for } 0 \leq t < \infty, \mathbb{P}\text{-a.s.}$$

Using the same arguments as in the proof of [4, Proposition 3.2], we derive the desired result.  $\square$



Secondly there are listed two results in matrix algebra.

**Lemma 4.2.** *Let  $\gamma = (\gamma_{i,j})$  be a  $N \times d$ -matrix and define the symmetric nonnegative definite  $N \times N$ -matrix  $\alpha := \gamma\gamma^*$ , i.e.  $\alpha_{i,j} = \alpha_{j,i} = \sum_{\lambda=1}^d \gamma_{i,\lambda}\gamma_{j,\lambda}$ . Let  $I$  and  $J$  denote two arbitrary subsets of  $\{1, \dots, N\}$ . Define*

$$\alpha_{I,J} = \alpha_{J,I} := \sum_{j \in J} \sum_{i \in I} \alpha_{i,j}.$$

*Then it holds that  $\alpha_{I,I} \geq 0$  and  $|\alpha_{I,J}| \leq \sqrt{\alpha_{I,I}}\sqrt{\alpha_{J,J}}$ .*

*Proof.* For  $1 \leq \lambda \leq d$  define  $\gamma_{I,\lambda} := \sum_{i \in I} \gamma_{i,\lambda}$ . Then by definition

$$\alpha_{I,J} = \sum_{j \in J} \sum_{i \in I} \sum_{\lambda=1}^d \gamma_{i,\lambda}\gamma_{j,\lambda} = \sum_{\lambda=1}^d \left( \sum_{i \in I} \gamma_{i,\lambda} \right) \left( \sum_{j \in J} \gamma_{j,\lambda} \right) = \sum_{\lambda=1}^d \gamma_{I,\lambda}\gamma_{J,\lambda}.$$

Hence

$$\alpha_{I,I} = \sum_{\lambda=1}^d (\gamma_{I,\lambda})^2 \geq 0$$

and by Schwarz inequality

$$|\alpha_{I,J}| \leq \sqrt{\sum_{\lambda=1}^d (\gamma_{I,\lambda})^2} \sqrt{\sum_{\lambda=1}^d (\gamma_{J,\lambda})^2} = \sqrt{\alpha_{I,I}}\sqrt{\alpha_{J,J}}.$$

□

**Lemma 4.3.** *Let  $\alpha = (\alpha_{i,j})$  be a  $n \times n$ -matrix,  $n \in \mathbb{N}$ , which is diagonally dominant from the right, i.e.*

$$\begin{aligned} |\alpha_{i,i}| &\geq \sum_{\substack{j=1 \\ j \neq i}}^n |\alpha_{i,j}| \\ |\alpha_{i,i}| &> \sum_{j=i+1}^n |\alpha_{i,j}|, \quad \left( \text{set } \sum_{j=n+1}^n \dots := 0 \right), \end{aligned}$$

*for all  $1 \leq i \leq n$ . Then  $\alpha$  is regular.*

*Proof.* The proof is a slight modification of an argument given in [10, Theorem 1.5].

Gaussian elimination: by assumption  $|\alpha_{1,1}| > \sum_{j=2}^n |\alpha_{1,j}| \geq 0$ , in particular  $\alpha_{1,1} \neq 0$ . If  $n = 1$  we are done. If  $n > 1$ , the elimination step

$$\alpha_{i,j}^{(1)} := \alpha_{i,j} - \frac{\alpha_{i,1}}{\alpha_{1,1}}\alpha_{1,j}, \quad 2 \leq i, j \leq n,$$

leads to the  $(n-1) \times (n-1)$ -matrix  $\alpha^{(1)} = (\alpha_{i,j}^{(1)})_{2 \leq i, j \leq n}$ . We show that  $\alpha^{(1)}$  is diagonal dominant from the right. If  $\alpha_{i,1} = 0$ , there is nothing to prove for the  $i$ -th row. Let  $\alpha_{i,1} \neq 0$ , for some  $2 \leq i \leq n$ . We have

$$|\alpha_{i,j}^{(1)}| \geq |\alpha_{i,j}| - \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| |\alpha_{1,j}|, \quad 2 \leq j \leq n.$$

Therefore

$$\begin{aligned}
\sum_{j=i+1}^n |\alpha_{i,j}^{(1)}| &\leq \sum_{\substack{j=2 \\ j \neq i}}^n |\alpha_{i,j}^{(1)}| = \sum_{\substack{j=2 \\ j \neq i}}^n \left| \alpha_{i,j} - \frac{\alpha_{i,1}}{\alpha_{1,1}} \alpha_{1,j} \right| \leq \sum_{\substack{j=2 \\ j \neq i}}^n |\alpha_{i,j}| + \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| \sum_{\substack{j=2 \\ j \neq i}}^n |\alpha_{1,j}| \\
&= \sum_{\substack{j=1 \\ j \neq i}}^n |\alpha_{i,j}| - |\alpha_{i,1}| + \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| \left( \sum_{j=2}^n |\alpha_{1,j}| - |\alpha_{1,i}| \right) \\
&< |\alpha_{i,i}| - |\alpha_{i,1}| + \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| \left( |\alpha_{1,1}| - |\alpha_{1,i}| \right) \\
&= |\alpha_{i,i}| - \left| \frac{\alpha_{i,1}}{\alpha_{1,1}} \right| |\alpha_{1,i}| \leq |\alpha_{i,i}^{(1)}|.
\end{aligned}$$

Proceed inductively to  $\alpha^{(2)}, \dots, \alpha^{(n-1)}$ .  $\square$

## 5. THE CASE $BEP(1, n)$

We will treat the case  $K = 1$  separately, since it represents a key step in the proof of the general  $BEP(K, n)$  case. For simplicity we shall skip the index  $i = 1$  and write  $n = n_1 \in \mathbb{N}_0$ ,  $p = p_1$ ,  $b^j = b^{1,j}$ ,  $a^{i,j} = a^{1,i;1,j}$ , etc. In particular we use the notation of Section 2 with  $N = n + 2$ .

**Lemma 5.1.** *Let  $n \in \mathbb{N}_0$  and  $Z$  be as above. If  $Z$  is consistent with  $BEP(1, n)$ , then necessarily*

$$\begin{aligned}
Z_t^i &= Z_0^i e^{-Z_0^{n+1}t} + Z_0^{i+1} t e^{-Z_0^{n+1}t} \\
Z_t^n &= Z_0^n e^{-Z_0^{n+1}t} \\
Z_t^{n+1} &= Z_0^{n+1} + \left( \int_0^t b_s^{n+1} ds + \sum_{j=1}^d \int_0^t \sigma_s^{n+1,j} dW_s^j \right) 1_{\Omega_0},
\end{aligned}$$

for  $0 \leq i \leq n-1$  and  $0 \leq t < \infty$ ,  $\mathbb{P}$ -a.s., where  $\Omega_0 := \{p(Z_0) = 0\}$ .

Consequently, if  $Z$  is consistent with  $BEP(1, n)$ , then  $\{p(Z) = 0\} = \mathbb{R}_+ \times \Omega_0$ . Hence  $\{Z^{n+1} \neq Z_0^{n+1}\} \subset \{p(Z) = 0\}$ . Therefore we may state

**Corollary 5.2.** *If  $Z$  is consistent with  $BEP(1, n)$ , then  $Z$  is as in the lemma and*

$$F(x, Z) = p(x, Z) e^{-Z_0^{n+1}x}.$$

Hence the corresponding interest rate model is quasi deterministic, i.e. all randomness remains  $\mathcal{F}_0$ -measurable.

*Proof of Lemma 5.1.* Let  $n \in \mathbb{N}_0$  and let  $Z$  be an Itô process, consistent with  $BEP(1, n)$ . Fix a point  $(t, \omega)$  in  $\mathbb{R}_+ \times \Omega$ . For simplicity we write  $z_i$  for  $Z_t^i(\omega)$ ,  $a_{i,j}$  for  $a_t^{i,j}(\omega)$  and  $b_i$  for  $b_t^i(\omega)$ . The proof relies on expanding equation (3) in the point

$z = (z_0, \dots, z_{n+1})$ . The involved terms are

$$\frac{\partial}{\partial x} F(x, z) = \left( \frac{\partial}{\partial x} p(x, z) - z_{n+1} p(x, z) \right) e^{-z_{n+1} x} \quad (14)$$

$$\frac{\partial}{\partial z_i} F(x, z) = \begin{cases} x^i e^{-z_{n+1} x}, & 0 \leq i \leq n \\ -x p(x, z) e^{-z_{n+1} x}, & i = n + 1 \end{cases} \quad (15)$$

$$\frac{\partial^2 F(x, z)}{\partial z_i \partial z_j} = \frac{\partial^2 F(x, z)}{\partial z_j \partial z_i} = \begin{cases} 0, & 0 \leq i, j \leq n \\ -x^{i+1} e^{-z_{n+1} x}, & 0 \leq i \leq n, j = n + 1 \\ x^2 p(x, z) e^{-z_{n+1} x}, & i = j = n + 1. \end{cases} \quad (16)$$

Finally it's useful to know the following relation for  $m \in \mathbb{N}_0$

$$\int_0^x \eta^m e^{-z_{n+1} \eta} d\eta = \begin{cases} -q_m(x) e^{-z_{n+1} x} + \frac{m!}{z_{n+1}^{m+1}}, & z_{n+1} \neq 0 \\ \frac{x^{m+1}}{m+1}, & z_{n+1} = 0, \end{cases} \quad (17)$$

where  $q_m(x) = \sum_{k=0}^m \frac{m!}{(m-k)!} \frac{x^{m-k}}{z_{n+1}^{k+1}}$  is a polynomial in  $x$  of order  $m$ .

Let's suppose first that  $z_{n+1} \neq 0$ . Thus, subtracting  $\frac{\partial}{\partial x} F(x, Z)$  from both sides of (3) we get a null equation of the form

$$q_1(x) e^{-z_{n+1} x} + q_2(x) e^{-2z_{n+1} x} = 0, \quad (18)$$

which has to hold simultaneously for all  $x \geq 0$ . The polynomials  $q_1$  and  $q_2$  depend on the  $z_i$ 's,  $b_i$ 's and  $a_{i,j}$ 's. Equality (18) implies  $q_1 = q_2 = 0$ . This again yields that all coefficients of the  $q_i$ 's have to be zero.

To proceed we have to distinguish the two cases  $p(z) \neq 0$  and  $p(z) = 0$ . Let's suppose first the former is true. Then there exists an index  $i \in \{0, \dots, n\}$  such that  $z_i \neq 0$ . Set  $m := \max\{i \leq n \mid z_i \neq 0\}$ . With regard to (15)–(17) it follows that  $\deg q_2 = 2m + 2$ . In particular

$$q_2(x) = a_{n+1, n+1} \frac{z_m^2}{z_{n+1}} x^{2m+2} + \dots,$$

where  $\dots$  denotes terms of lower order in  $x$ . Hence  $a_{n+1, n+1} = 0$ . But the matrix  $a$  has to be nonnegative definite, so necessarily

$$a_{n+1, j} = a_{j, n+1} = 0, \quad \text{for all } 1 \leq j \leq n + 1.$$

In view of Lemma 4.1 (setting  $Y = 0$ ), since we are characterizing  $a$  and  $b$  up to  $dt \otimes d\mathbb{P}$ -nullsets, we may assume  $a_{i,j} = a_{j,i} = 0$ , for  $0 \leq j \leq n + 1$ , for all  $i \geq m + 1$ . Thus the degree of  $q_2$  reduces to  $2m$ . Explicitly

$$q_2(x) = \frac{a_{m,m}}{z_{n+1}} x^{2m} + \dots$$

Hence  $a_{m,m} = 0$  and so  $a_{m,j} = a_{j,m} = 0$ , for  $0 \leq j \leq n + 1$ . Proceeding inductively for  $i = m - 1, m - 2, \dots, 0$  we finally get that the diffusion matrix  $a$  is equal to zero and hence  $q_2 = 0$  is fulfilled.

Now we determine the drift  $b$ . By Lemma 4.1, we may assume  $b_i = 0$  for  $m + 1 \leq i \leq n$ . With regard to (14) and (15),  $q_1$  reduces therefore to

$$q_1(x) = -b_{n+1} z_m x^{m+1} + \dots$$

It follows  $b_{n+1} = 0$  and it remains

$$\begin{aligned} q_1(x) &= (b_m + z_{n+1}z_m)x^m + \sum_{i=0}^{m-1} (b_i - z_{i+1} + z_{n+1}z_i)x^i \\ &= (b_n + z_{n+1}z_n)x^n + \sum_{i=0}^{n-1} (b_i - z_{i+1} + z_{n+1}z_i)x^i. \end{aligned}$$

We now turn to the singular cases. If  $p(z) = 0$ , that is  $z_0 = \dots = z_n = 0$ , we may assume  $a_{i,j} = a_{j,i} = b_i = 0$ ,  $0 \leq j \leq n+1$ , for all  $i \leq n$ . But this means that  $q_1 = q_2 = 0$ , independently of the choice of  $b_{n+1}$  and  $a_{n+1,n+1}$ .

For the case where  $z_{n+1} = 0$  we need the boundedness assumption  $z \in \mathcal{Z}$ . By (8) it follows that  $z_1 = \dots = z_n = 0$ . So by Lemma 4.1 again  $a_{i,j} = a_{j,i} = b_i = 0$ ,  $0 \leq j \leq n+1$ , for all  $i \geq 1$ . Thus in this case equation (3) reduces to

$$0 = b_0 - a_{0,0}x$$

and therefore  $b_0 = a_{0,0} = 0$ .

Summarizing all cases we conclude that necessarily

$$\begin{aligned} b_i &= -z_{n+1}z_i + z_{i+1}, \quad 0 \leq i \leq n-1 \\ b_n &= -z_{n+1}z_n \\ a_{i,j} &= 0, \quad \text{for } (i,j) \neq (n+1, n+1). \end{aligned}$$

Whereas  $b_{n+1}$  and  $a_{n+1,n+1}$  are arbitrary real, resp. nonnegative real, numbers whenever  $p(z) = 0$ . Otherwise  $b_{n+1} = a_{n+1,n+1} = 0$ .

The rest of the proof is analogous to the proof of [4, Proposition 4.1].  $\square$

## 6. THE GENERAL CASE $BEP(K, n)$

Using again the notation of Section 3 we give the proof of the main result for the case  $K \geq 2$ . The exposure is somewhat messy, which is due to the multi-dimensionality of the problem. The idea however is simple: For a fixed point  $(t, \omega) \in \mathbb{R}_+ \times \Omega$  we expand equation (3), which turns out to be a linear combination of linearly independent exponential functions over the ring of polynomials, equaling zero. Consequently many of the coefficients have to vanish, which leads to our assertion.

The difficulty is that some exponents may coincide. This causes a considerable number of singular cases which require a separate discussion.

*Proof of Theorem 3.2.* Let  $K \geq 2$ ,  $n = (n_1, \dots, n_K) \in \mathbb{N}_0^K$ , and let  $Z$  be consistent with  $BEP(K, n)$ . As in the proof of Lemma 5.1 we fix a point  $(t, \omega)$  in  $\mathbb{R}_+ \times \Omega$  and use the shorthand notation  $z_{i,\mu}$  for  $Z_t^{i,\mu}(\omega)$ ,  $a_{i,\mu;j,\nu}$  for  $a_t^{i,\mu;j,\nu}(\omega)$  and  $b_{i,\mu}$  for  $b_t^{i,\mu}(\omega)$ , etc. Since we are characterizing  $a$  and  $b$  up to a  $dt \otimes d\mathbb{P}$ -nullset, we assume that  $(t, \omega)$  is chosen outside of an exceptional  $dt \otimes d\mathbb{P}$ -nullset. In particular the lemmas from Section 4 shall apply each time we use them.

The strategy is the same as for the case  $K = 1$ . Thus we expand equation (3) in the point  $z = (z_{1,0}, \dots, z_{K,n_K+1})$  to get a linear combination of (ideally) linearly

independent exponential functions over the ring of polynomials

$$\sum_{i=1}^K q_i(x) e^{-z_{i,n_i+1}x} + \sum_{1 \leq i \leq j \leq K} q_{i,j}(x) e^{-(z_{i,n_i+1} + z_{j,n_j+1})x} = 0. \quad (19)$$

Consequently, all polynomials  $q_i$  and  $q_{i,j}$  have to be zero. The main difference to the case  $K = 1$  is that representation (19) may not be unique due to the possibly multiple occurrence of the following singular cases

- i)  $z_{i,n_i+1} = z_{j,n_j+1}$ , for  $i \neq j$ ,
- ii)  $2z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$ ,
- iii)  $2z_{i,n_i+1} = z_{j,n_j+1}$ ,
- iv)  $z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$ ,

for some indices  $1 \leq i, j, k \leq K$ . However, the lemmas in Section 4 and the boundedness assumption  $z \in \mathcal{Z}$  are good enough to settle these four cases.

Let's suppose first that  $p_i(z) \neq 0$ , for all  $i \in \{1, \dots, K\}$ . To settle Case i), let  $\sim$  denote the equivalence relation defined in (6). After re-parametrization if necessary we may assume that

$$\{1, \dots, K\} / \sim = \{[1], \dots, [\tilde{K}]\}$$

and  $z_{1,n_1+1} < \dots < z_{\tilde{K},n_{\tilde{K}+1}}$  for some integer  $\tilde{K} \leq K$ . Write  $I := \{1, \dots, \tilde{K}\}$ . In view of Lemma 4.1 we may assume

$$a_{j,n_j+1; j, n_j+1} = a_{i,n_i+1; i, n_i+1} \text{ and } b_{j,n_j+1} = b_{i,n_i+1} \text{ for all } j \in [i], i \in I. \quad (20)$$

The proof of (12) and (13) is divided into four claims.

**Claim 1.**  $a_{i,n_i+1; i, n_i+1} = 0$ , for all  $i \in I$ .

Expression (19) takes the form

$$\sum_{i \in I} \tilde{q}_i(x) e^{-z_{i,n_i+1}x} + \sum_{\substack{i, j \in I \\ i \leq j}} \tilde{q}_{i,j}(x) e^{-(z_{i,n_i+1} + z_{j,n_j+1})x} = 0, \quad (21)$$

for some polynomials  $\tilde{q}_i$  and  $\tilde{q}_{i,j}$ . Taking into account Cases ii)–iv), this representation may still not be unique. However if for an index  $i \in I$  there exist no  $j, k \in I$  such that  $2z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$  or  $2z_{i,n_i+1} = z_{j,n_j+1}$  (in particular  $z_{i,n_i+1} \neq 0$ ) then we have

$$\tilde{q}_{i,i}(x) = a_{i,n_i+1; i, n_i+1} \frac{\sum_{j \in \mathcal{I}_{[i], \mu_m}} z_{j, \mu_m}^2 x^{2\mu_m+2} + \dots,}{z_{i,n_i+1}}$$

where  $\mu_m := \max\{\nu \mid \nu \leq n_j \text{ and } z_{j,\nu} \neq 0 \text{ for some } j \in [i]\} \in \mathbb{N}_0$ . Hence  $a_{i,n_i+1; i, n_i+1} = 0$  and Claim 1 is proved for the regular case.

For the singular cases observe first that  $z_{i,n_i+1} = 0$  implies  $a_{i,n_i+1; i, n_i+1} = 0$ , which follows from Lemma 4.1. Now we split  $I$  into two disjoint subsets  $I_1$  and  $I_2$ , where

$$\begin{aligned} I_1 &:= \{i \in I \mid z_{i,n_i+1} \neq 0 \text{ and there exist } j, k \in I, \text{ such that} \\ &\quad 2z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1} \text{ or } 2z_{i,n_i+1} = z_{j,n_j+1}\} \\ I_2 &:= I \setminus I_1. \end{aligned}$$

Observe that  $z_{\tilde{K},n_{\tilde{K}+1}} > 0$  implies  $\tilde{K} \in I_2$  and  $z_{1,n_1+1} < 0$  implies  $1 \in I_2$ . Since at least one of these events has to happen, the set  $I_2$  is not empty. We have shown

above that  $a_{i,n_i+1;i,n_i+1} = 0$ , for  $i \in I_2$ . If  $I_1$  is not empty, we will show that for each  $i \in I_1$ , the parameter  $z_{i,n_i+1}$  can be written as a linear combination of  $z_{j,n_j+1}$ 's with  $j \in I_2$ . From this it follows by Lemmas 4.1 and 4.2 that  $a_{i,n_i+1;i,n_i+1} = 0$  for all  $i \in I_1$  and Claim 1 is completely proved. We proceed as follows. Write  $I_1 = \{i_1, \dots, i_r\}$  with  $z_{i_1,n_{i_1}+1} < \dots < z_{i_r,n_{i_r}+1}$ . For each  $i_k \in I_1$  there exists one linear equation of the form

$$(*, \dots, *, 2, *, \dots, *) \begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ z_{i_k,n_{i_k}+1} \\ \vdots \\ z_{i_r,n_{i_r}+1} \end{pmatrix} = \alpha_k,$$

where  $*$  stands for 0 or  $-1$ , but at most one  $-1$  on each side of 2. The  $\alpha_k$  on the right hand side is either 0 or  $z_{i,n_i+1}$  or  $z_{i,n_i+1} + z_{j,n_j+1}$  for some indices  $i, j \in I_2$ . Hence we get the system of linear equations

$$\begin{pmatrix} 2 & * & \dots & * \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ * & \dots & * & 2 \end{pmatrix} \begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ \vdots \\ z_{i_r,n_{i_r}+1} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix}.$$

By Lemma 4.3, the matrix on the left hand side is invertible, from which follows our assertion.

**Claim 2.**  $a_{j,n_j+1;k,\nu} = a_{k,\nu;j,n_j+1} = 0$ , for  $0 \leq \nu \leq n_k$ , for all  $1 \leq j, k \leq K$ .

In view of (20), Claim 2 follows immediately from Claim 1 and Lemma 4.2.

Analogous to the notation introduced in (7) we set

$$\begin{aligned} b_{[i],\mu} &:= \sum_{j \in \mathcal{I}_{[i],\mu}} b_{j,\mu} \\ \sigma_{[i],\mu;\lambda} &:= \sum_{j \in \mathcal{I}_{[i],\mu}} \sigma_{j,\mu;\lambda} \\ a_{[i],\mu;k,\nu} &:= \sum_{j \in \mathcal{I}_{[i],\mu}} a_{j,\mu;k,\nu}, \end{aligned}$$

for  $0 \leq \mu \leq n_{[i]}$ ,  $0 \leq \nu \leq n_k$ ,  $1 \leq k \leq K$ ,  $1 \leq \lambda \leq d$ ,  $i \in I$ , and

$$a_{[i],\mu;[k],\nu} := \sum_{l \in \mathcal{I}_{[k],\nu}} \sum_{j \in \mathcal{I}_{[i],\mu}} a_{j,\mu;l,\nu},$$

for  $0 \leq \mu \leq n_{[i]}$ ,  $0 \leq \nu \leq n_{[k]}$ ,  $i, k \in I$ .

**Claim 3.** If  $z_{[i],\mu} = 0$ , for  $i \in I$  and  $\mu \in \{0, \dots, n_{[i]}\}$ , then

$$b_{[i],\mu} = a_{[i],\mu;[i],\mu} = a_{[i],\mu;k,\nu} = a_{k,\nu;[i],\mu} = 0,$$

for all  $0 \leq \nu \leq n_k$ ,  $1 \leq k \leq K$ .

Notice that  $a_{[i],\mu;[i],\mu} = \sum_{\lambda=1}^d \sigma_{[i],\mu;\lambda}^2$ . Hence Claim 3 follows by Lemma 4.1 and Lemma 4.2.

**Claim 4.**  $b_{i,n_i+1} = 0$ , for all  $i \in I$  such that  $p_{[i]}(z) \neq 0$ .

Suppose first that  $z_{i,n_i+1} \neq 0$ , for all  $i \in I$ . Let  $i \in I$  such that  $p_{[i]}(z) \neq 0$ , and let's assume there exist no  $j, k \in I$  with  $z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$ . How does the polynomial  $\tilde{q}_i$  in (21) look like? With regard to (20), Claim 2, Lemma 4.1 and equalities (14)–(17) the contributing terms are

$$\begin{aligned} \frac{\partial}{\partial x} p_j(x, z) e^{-z_{j,n_j+1}x} &= \left( \left( \sum_{\mu=1}^{\mu_m \wedge n_j} z_{j,\mu} x^{\mu-1} \right) - z_{i,n_i+1} \left( \sum_{\mu=0}^{\mu_m \wedge n_j} z_{j,\mu} x^{\mu} \right) \right) e^{-z_{i,n_i+1}x}, \\ \sum_{\mu=0}^{n_j+1} b_{j,\mu} \frac{\partial}{\partial z_{j,\mu}} F(x, z) &= \left( \left( \sum_{\mu=0}^{\mu_m \wedge n_j} b_{j,\mu} x^{\mu} \right) - b_{i,n_i+1} \left( \sum_{\mu=0}^{\mu_m \wedge n_j} z_{j,\mu} x^{\mu+1} \right) \right) e^{-z_{i,n_i+1}x} \end{aligned} \quad (22)$$

and

$$\begin{aligned} &- 2 \frac{1}{2} \left( \sum_{\mu=0}^{n_j} a_{j,\mu;k,\nu} \frac{\partial}{\partial z_{j,\mu}} F(x, z) \int_0^x \frac{\partial}{\partial z_{k,\nu}} F(\eta, z) d\eta \right) \\ &= - \left( \sum_{\mu=0}^{\mu_m \wedge n_j} a_{j,\mu;k,\nu} \frac{n_k!}{z_{k,n_k+1}^{\mu}} x^{\mu} \right) e^{-z_{i,n_i+1}x} - \left( \text{polynomial in } x \right) e^{-(z_{i,n_i+1} + z_{k,n_k+1})x}, \end{aligned} \quad (23)$$

for  $0 \leq \nu \leq n_k$ , for all  $1 \leq k \leq K$  and  $j \in [i]$ . We have used the integer

$$\mu_m := \max\{\lambda \mid \lambda \leq n_l \text{ and } z_{l,\lambda} \neq 0 \text{ for some } l \in [i]\}.$$

Define  $\tilde{\mu}_m := \max\{\lambda \mid \lambda \leq n_{[i]} \text{ and } z_{[i],\lambda} \neq 0\} \in \mathbb{N}_0$ . Obviously  $\tilde{\mu}_m \leq \mu_m$ . By Claim 3 we have  $a_{[i],\mu;k,\nu} = 0$ , for all  $\tilde{\mu}_m < \mu \leq n_{[i]}$ . Thus summing up the above expressions over  $j \in [i]$  we get

$$\tilde{q}_i(x) = -b_{i,n_i+1} z_{[i],\tilde{\mu}_m} x^{\tilde{\mu}_m+1} + \dots \quad (24)$$

Consequently  $b_{i,n_i+1} = 0$  in the regular case.

For the singular cases the boundedness assumption  $z \in \mathcal{Z}$  is essential. We split  $I$  into two disjoint subsets  $J_1$  and  $J_2$ , where

$$\begin{aligned} J_1 := \{i \in I \mid \text{there exist } j, k \in I, \text{ such that } z_{i,n_i+1} &= z_{j,n_j+1} + z_{k,n_k+1} \\ &\text{and } z_{j,n_j+1} > 0 \text{ and } z_{k,n_k+1} > 0\} \end{aligned}$$

$$J_2 := I \setminus J_1.$$

Notice that in any case  $1 \in J_2$ . We have shown above that for each  $i \in J_2$  such that  $z_{i,n_i+1}$  is not the sum of two other  $z_{j,n_j+1}$ 's it follows that  $b_{i,n_i+1} = 0$ . We will now show that  $b_{i,n_i+1} = 0$  for all  $i \in J_2$ . Let  $i \in J_2$  and assume there exist  $j, k \in I$  with  $z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$ . Then necessarily one of the summands is strictly less than zero. Without loss of generality  $z_{j,n_j+1} < 0$ . Since  $z \in \mathcal{Z}$ , we have  $p_{[j]}(z) = 0$ , see (8). Thus  $a_{[j],\mu;[j],\mu} = 0$  by Claim 3 and therefore  $a_{[j],\mu;k,\nu} = 0$ , for all  $0 \leq \mu \leq n_{[j]}$ ,  $0 \leq \nu \leq n_k$ ,  $1 \leq k \leq K$ . The contributing terms to the polynomial in front of  $e^{-z_{i,n_i+1}x}$ , i.e.  $\tilde{q}_i + \tilde{q}_{j,k} + \dots$ , are those in (22) and (23) and also

$$\begin{aligned} &- 2 \frac{1}{2} a_{l,\mu;m,\nu} \left( \frac{\partial F(x, z)}{\partial z_{l,\mu}} \int_0^x \frac{\partial F(\eta, z)}{\partial z_{m,\nu}} d\eta + \frac{\partial F(x, z)}{\partial z_{m,\nu}} \int_0^x \frac{\partial F(\eta, z)}{\partial z_{l,\mu}} d\eta \right) \\ &= -a_{l,\mu;m,\nu} \left( x^{\mu} e^{-z_{j,n_j+1}x} \int_0^x \eta^{\nu} e^{-z_{k,n_k+1}\eta} d\eta + x^{\nu} e^{-z_{k,n_k+1}x} \int_0^x \eta^{\mu} e^{-z_{j,n_j+1}\eta} d\eta \right), \end{aligned} \quad (25)$$

for  $0 \leq \mu \leq n_l$ ,  $0 \leq \nu \leq n_m$ ,  $l \in [j]$ ,  $m \in [k]$ . However, summing up – for fixed  $\mu$ ,  $m$  and  $\nu$  – the right hand side of (25) over  $l \in \mathcal{I}_{[j],\mu}$  gives zero. Hence the terms in (25) actually don't contribute to the meant polynomial. The same conclusion can be drawn for all  $j, k \in I$  with the property that  $z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}$ . It finally follows as in the regular case that  $b_{i,n_i+1} = 0$  for all  $i \in J_2$ .

If  $J_1$  is not empty, we show that for each  $i \in J_1$ , the parameter  $z_{i,n_i+1}$  can be written as a linear combination of  $z_{j,n_j+1}$ 's with  $j \in J_2$ . From this it follows by Lemma 4.1 that  $b_{i,n_i+1} = 0$  for all  $i \in J_1$ . We proceed as follows. Write  $J_1 = \{i_1, \dots, i_{r'}\}$  with  $z_{i_1,n_{i_1}+1} < \dots < z_{i_{r'},n_{i_{r'}}+1}$ . For each  $i_k \in J_1$  there exists one linear equation of the form

$$(*, \dots, *, 1, 0, \dots, 0) \begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ z_{i_k,n_{i_k}+1} \\ \vdots \\ z_{i_{r'},n_{i_{r'}}+1} \end{pmatrix} = \alpha'_k,$$

where  $*$  stands for 0 or  $-1$ , but at most two of them are  $-1$ . The  $\alpha'_k$  on the right hand side is either 0 or  $z_{i,n_i+1}$  or  $z_{i,n_i+1} + z_{j,n_j+1}$  for some indices  $i, j \in J_2$  with  $z_{i,n_i+1} > 0$  and  $z_{j,n_j+1} > 0$ . Obviously  $\alpha'_1$  is of the latter form. Hence we get the system of linear equations

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ * & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & 1 \end{pmatrix} \begin{pmatrix} z_{i_1,n_{i_1}+1} \\ \vdots \\ \vdots \\ z_{i_{r'},n_{i_{r'}}+1} \end{pmatrix} = \begin{pmatrix} z_{i,n_i+1} + z_{j,n_j+1} \\ \alpha'_2 \\ \vdots \\ \alpha'_{r'} \end{pmatrix},$$

for some  $i, j \in J_2$ . On the left hand side stands a lower-triangular matrix, which is therefore invertible. Hence Claim 4 is proved in the case where  $z_{i,n_i+1} \neq 0$  for all  $i \in I$ .

Assume now that there exists  $i \in I$  with  $z_{i,n_i+1} = 0$ . Then  $i \in J_2$ . We have to make sure that also in this case  $b_{j,n_j+1} = 0$ , for all  $j \in J_2$ . Clearly  $b_{i,n_i+1}$  is zero by Lemma 4.1. The problem is that  $z_{j,n_j+1} = z_{i,n_i+1} + z_{j,n_j+1}$  for all  $j \in J_2$ . But following the lines above it is enough to show  $a_{[i],\mu;[i],\mu} = 0$ , for all  $0 \leq \mu \leq n_{[i]}$ . From the boundedness assumption  $z \in \mathcal{Z}$  we know that  $p_{[i]}(z) = z_{[i],0}$ , see (8). Hence  $a_{[i],\mu;[i],\mu} = 0$ , for  $1 \leq \mu \leq n_{[i]}$ . Suppose there is no pair of indices  $j, k \in I \setminus \{i\}$  with  $z_{j,n_j+1} + z_{k,n_k+1} = 0$ . Summing up the contributing terms in (22) and (23) over  $j \in [i]$  we get the polynomial in front of  $e^0$ , i.e.

$$\tilde{q}_i(x) + \tilde{q}_{i,i}(x) = -a_{[i],0;[i],0}x + \dots, \quad (26)$$

hence  $a_{[i],0;[i],0} = 0$ . If there exist a pair of indices  $j, k \in I \setminus \{i\}$  with  $z_{j,n_j+1} + z_{k,n_k+1} = 0$ , then one of these summands is strictly less than zero. Arguing as before, the polynomial in front of  $e^0$  remains of the form (26) and again  $a_{[i],0;[i],0} = 0$ . Thus Claim 4 is completely proved.

Up to now we have established (12) and (13) under the hypothesis that  $p_i(z) \neq 0$ , for all  $i \in \{1, \dots, K\}$ . Suppose now, there is an index  $i \in \{1, \dots, K\}$  with  $p_i(z) = 0$ . By Lemma 4.1, we may assume  $a_{i,\mu;i,\mu} = b_{i,\mu} = 0$ , for all  $0 \leq \mu \leq n_i$ . But then Lemma 4.2 tells us that none of the terms including the index  $i$  appears



in (19). In particular  $a_{i,n_i+1;i,n_i+1}$  and  $b_{i,n_i+1}$  can be chosen arbitrarily without affecting equation (19). This means that we may skip  $i$  and proceed, after a reparametrization if necessary, with the remaining index set  $\{1, \dots, K-1\}$  to establish Claims 1–4 as above.

This all has to hold for  $dt \otimes d\mathbb{P}$ -a.e.  $(t, \omega)$ . Hence (12) and (13) are fully proved. A closer look to the proof of (12), i.e. Claim 1, shows that the boundedness assumption  $z \in \mathcal{Z}$  was not explicitly used there. Whence Remark 3.3.

Next we prove that the exponents  $Z^{i,n_i+1}$  are locally constant on intervals where  $p_i(Z)$  and  $p_{[i]}(Z)$  do not vanish. Let  $v \geq 0$  be a rational number and let  $T_v := \inf\{t > v \mid p_i(Z_t) = 0 \text{ or } p_{[i]}(Z_t) = 0\}$  denote the debut of the optional set  $[v, \infty[\cap \mathcal{A}_i$ . By (12) and (13) and the continuity of  $Z$  we have that  $Z^{i,n_i+1}$  is  $\mathbb{P}$ -a.s. constant on  $[v, T_v]$ , hence  $\mathbb{P}$ -a.s. constant on every such interval  $[v, T_v]$ . Since every open interval where  $p_i(Z_t) \neq 0$  or  $p_{[i]}(Z_t) \neq 0$  is covered by a countable union of intervals  $[v, T_v]$  and by continuity of  $Z$  the assertion follows and the first part of the theorem is proved.

For establishing the second part of the theorem let  $\tau$  be a stopping time with  $[\tau] \in \mathcal{D}'$  and  $\mathbb{P}(\tau < \infty) > 0$ . Define the stopping time  $\tau'(\omega) := \inf\{t \geq \tau(\omega) \mid (t, \omega) \notin \mathcal{D}'\}$ . By continuity of  $Z$ , we conclude that  $\tau < \tau'$  on  $\{\tau < \infty\}$ . Choose a point  $(t, \omega)$  in  $[\tau, \tau'[\mathcal{D}'$ . We use shorthand notation as above.

By definition of  $\mathcal{D}'$  we can exclude the singular cases  $z_{i,n_i+1} = z_{j,n_j+1}$  or  $2z_{i,n_i+1} = z_{j,n_j+1}$ , for  $i \neq j$ . In particular  $\tilde{K} = K$ , hence  $I = \{1, \dots, K\}$ . First we show that the diffusion matrix for the coefficients of the polynomials  $p_i(z)$  vanishes.

**Claim 5.**  $a_{i,\mu;j,\nu} = a_{j,\nu;i,\mu} = 0$ , for  $0 \leq \mu \leq n_i$ , for  $0 \leq \nu \leq n_j$ , for all  $i, j \in I$ .

By Lemma 4.1 it's enough to prove that the diagonal  $a_{i,\mu;i,\mu}$  vanishes for  $0 \leq \mu \leq n_i$  and  $i \in I$ . If there is an index  $i \in I$  with  $p_i(z) = 0$  then argued as above  $a_{i,\mu;i,\mu} = b_{i,\mu} = 0$ , for all  $0 \leq \mu \leq n_i$ , and we may skip the index  $i$ . Hence we assume now that there is a  $K' \leq K$  such that  $p_i(z) \neq 0$  (and thus  $z_{i,n_i+1} \geq 0$ , since  $z \in \mathcal{Z}$ ) for all  $1 \leq i \leq K'$ . Let  $I' := \{1, \dots, K'\}$ . For handling the singular cases, we split  $I'$  into two disjoint subsets  $I'_1$  and  $I'_2$ , where

$$\begin{aligned} I'_1 &:= \{i \in I' \mid z_{i,n_i+1} > 0 \text{ and there exist } j, k \in I' \\ &\quad \text{such that } 2z_{i,n_i+1} = z_{j,n_j+1} + z_{k,n_k+1}\} \\ I'_2 &:= I' \setminus I'_1. \end{aligned}$$

Hence  $z_{i,n_i+1} = 0$  for  $i \in I'$  implies  $i \in I'_2$ . We have already shown in the proof of Claim 4 that in this case  $a_{i,\mu;i,\mu} = 0$ , for all  $0 \leq \mu \leq n_i$ . The same follows for  $i \in I'_2$  with  $z_{i,n_i+1} > 0$ , as it was demonstrated for the case  $K = 1$ .

Now let  $i \in I'_1$  and let  $l, m \in I'$ , such that  $l \leq m$  and  $2z_{i,n_i+1} = z_{l,n_l+1} + z_{m,n_m+1}$ . Thus the polynomial in front of  $e^{-2z_{i,n_i+1}x}$  is  $q_{i,i} + q_{l,m} + \dots$ , and among the contributing terms are also those in (25). If  $l$  or  $m$  is in  $I'_2$ , those are all zero. Write  $I'_1 = \{i_1, \dots, i_{r''}\}$  with  $z_{i_1,n_{i_1}+1} < \dots < z_{i_{r''},n_{i_{r''}}+1}$ . Then necessarily  $l \in I'_2$  in the above representation for  $z_{i_1,n_{i_1}+1}$ . Thus the polynomial in front of  $e^{-2z_{i_1,n_{i_1}+1}x}$  is  $q_{i_1,i_1}$ . It follows  $a_{i_1,\mu;i_1,\mu} = 0$ , for all  $0 \leq \mu \leq n_{i_1}$ , as it was demonstrated for the case  $K = 1$ . Proceeding inductively for  $i_2, \dots, i_{r''}$ , we derive eventually that  $a_{i,\mu;i,\mu} = 0$ , for all  $0 \leq \mu \leq n_i$  and  $i \in I'$ . This establishes Claim 5.

We are left with the task of determining the drift of the coefficients in  $p_i(z)$ . By (13), we have  $b_{i,n_i+1} = 0$  for all  $i \in I'$ . Straightforward calculations show that (19) reduces to

$$\sum_{i=1}^{K'} q_i(x) e^{-z_{i,n_i+1}x} = 0,$$

with

$$q_i(x) = (b_{i,n_i} + z_{i,n_i+1}z_{i,n_i})x^{n_i} + \sum_{\mu=0}^{n_i-1} (b_{i,\mu} - z_{i,\mu+1} + z_{i,n_i+1}z_{i,\mu})x^\mu.$$

We conclude that for all  $1 \leq i \leq K$  (in particular if  $p_i(z) = 0$ )

$$\begin{aligned} b_{i,\mu} &= z_{i,\mu+1} - z_{i,n_i+1}z_{i,\mu}, & 0 \leq \mu \leq n_i - 1 \\ b_{i,n_i} &= -z_{i,n_i+1}z_{i,n_i}. \end{aligned} \tag{27}$$

By continuity of  $Z$ , Claim 5 and (27) hold pathwise on the semi open interval  $[\tau(\omega), \tau'(\omega)[$  for almost every  $\omega$ . Therefore  $Z_{\tau+}$  is of the claimed form on  $[0, \tau' - \tau[$ .

Now replace  $\mathcal{D}'$  by  $\mathcal{D}$  and proceed as above. By (11) we have  $\tau < \tau'$  on  $\{\tau < \infty\}$ , and since  $\mathcal{D} \subset \mathcal{D}'$ , all the above results remain valid. In addition  $p_i(z) = p_{[i]}(z) \neq 0$  and thus  $a_{i,n_i+1;i,n_i+1} = b_{i,n_i+1} = 0$ , for all  $1 \leq i \leq K$ , by (12) and (13). Hence  $Z_{\tau+}^{i,n_i+1} = Z_{\tau}^{i,n_i+1}$  on  $[0, \tau' - \tau[$ , for all  $1 \leq i \leq K$ , up to evanescence. But this again implies  $\tau' = \infty$  by the continuity of  $Z$ .  $\square$

## 7. E-CONSISTENT ITÔ PROCESSES

An Itô process  $Z$  is by definition consistent with a family  $\{F(\cdot, z)\}_{z \in \mathcal{Z}}$  if and only if  $\mathbb{P}$  is a martingale measure for the discounted bond price processes. We could generalize this definition and call a process  $Z$  *e-consistent* with  $\{F(\cdot, z)\}_{z \in \mathcal{Z}}$  if there exists an equivalent martingale measure  $\mathbb{Q}$ . Then obviously consistency implies e-consistency, and e-consistency implies the absence of arbitrage opportunities, as it is well known.

In case where the filtration is generated by the Brownian motion  $W$ , i.e.  $(\mathcal{F}_t) = (\mathcal{F}_t^W)$ , we can give the following stronger result:

**Proposition 7.1.** *Let  $K \in \mathbb{N}$  and  $n = (n_1, \dots, n_K) \in \mathbb{N}_0^K$ . If  $(\mathcal{F}_t) = (\mathcal{F}_t^W)$ , then any Itô process  $Z$  which is e-consistent with  $BEP(K, n)$ , is of the form as stated in Theorem 3.2.*

*Proof.* Let  $Z$  be an e-consistent Itô process under  $\mathbb{P}$ , and let  $\mathbb{Q}$  be an equivalent martingale measure. Since  $(\mathcal{F}_t) = (\mathcal{F}_t^W)$ , we know that all  $\mathbb{P}$ -martingales have the representation property relative to  $W$ . By Girsanov's theorem it follows therefore that  $Z$  remains an Itô process under  $\mathbb{Q}$ , which is consistent with  $BEP(K, n)$ . The drift coefficients  $b^{i,\mu}$  change under  $\mathbb{Q}$  into  $\tilde{b}^{i,\mu}$ . Whereas  $b^{i,\mu} = \tilde{b}^{i,\mu}$  on  $\{a^{i,\mu;i,\mu} = 0\}$ ,  $dt \otimes d\mathbb{P}$ -a.s. The diffusion matrix  $a$  remains the same. Therefore and since the measures  $dt \otimes d\mathbb{Q}$  and  $dt \otimes d\mathbb{P}$  are equivalent on  $\mathbb{R}_+ \times \Omega$ , the Itô process  $Z$  is of the form as stated in Theorem 3.2.  $\square$

Notice that in this case the expression quasi deterministic, i.e.  $\mathcal{F}_0$ -measurable, in Corollaries 3.5 and 5.2 can be replaced by purely deterministic.

## 8. THE DIFFUSION CASE

The main result from Section 3 reads much clearer for diffusion processes. In all applications the generic Itô process  $Z$  on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t < \infty}, \mathbb{P})$  given by (9) is rather the solution of a stochastic differential equation

$$Z_t^{i,\mu} = Z_0^{i,\mu} + \int_0^t b^{i,\mu}(s, Z_s) ds + \sum_{\lambda=1}^d \int_0^t \sigma^{i,\mu;\lambda}(s, Z_s) dW_s^\lambda, \quad (28)$$

for  $0 \leq \mu \leq n_i + 1$  and  $1 \leq i \leq K$ , where  $b$  and  $\sigma$  are some Borel measurable mappings from  $\mathbb{R}_+ \times \mathbb{R}^N$  to  $\mathbb{R}^N$ , resp.  $\mathbb{R}^{N \times d}$ .

The coefficients  $b$  and  $\sigma$  could be derived by statistical inference methods from the daily observations of the diffusion  $Z$  made by some central bank. These observations are of course made under the objective probability measure. Hence  $\mathbb{P}$  is not a martingale measure in applications of this kind.

On the other hand we want a model for pricing interest rate sensitive securities. Thus the diffusion has to be e-consistent. If we assume that  $(\mathcal{F}_t) = (\mathcal{F}_t^W)$ , the last section applies. To stress the fact that  $\mathcal{F}_0^W$ -measurable functions are deterministic, we denote the initial values of the diffusion in (28) with small letters  $z_0^{i,\mu}$ .

Since all reasonable theory for stochastic differential equations requires continuity properties of the coefficients, we shall assume in the sequel that  $b(t, z)$  and  $\sigma(t, z)$  are continuous in  $z$ . The main result for e-consistent diffusion processes is divided into the two following theorems. The first one only requires consistency with  $EP(K, n)$ .

**Theorem 8.1.** *Let  $K \in \mathbb{N}$ ,  $n = (n_1, \dots, n_K) \in \mathbb{N}_0^K$ ,  $(\mathcal{F}_t) = (\mathcal{F}_t^W)$ , the diffusion  $Z$ ,  $b$  and  $\sigma$  as above. If  $Z$  is e-consistent with  $BEP(K, n)$  or with  $EP(K, n)$ , then necessarily the exponents are constant*

$$Z^{i, n_i+1} \equiv z_0^{i, n_i+1},$$

for all  $1 \leq i \leq K$ .

*Proof.* The significant difference to the proof of Theorem 3.2 is that now the diffusion matrix  $a$  and the drift  $b$  depend continuously on  $z$ .

First observe that the following sets of singular values

$$\mathcal{M} := \bigcup_{i=1}^K \{z \in \mathbb{R}^N \mid p_i(z) = 0 \text{ or } p_{[i]}(z) = 0\}$$

$$\mathcal{N} := \{z \in \mathbb{R}^N \mid z_{i, n_i+1} = z_{j, n_j+1} + z_{k, n_k+1} \text{ for some } 1 \leq i, j, k \leq K\}$$

are contained in a finite union of hyperplanes of  $\mathbb{R}^N$ , see (10). Hence  $(\mathcal{M} \cup \mathcal{N}) \subset \mathbb{R}^N$  has Lebesgue measure zero. Thus the topological closure of  $\mathcal{G} := \mathbb{R}^N \setminus (\mathcal{M} \cup \mathcal{N})$  is  $\mathbb{R}^N$ .

Now let  $Z$  be the diffusion in (28), which is e-consistent either with  $BEP(K, n)$  or  $EP(K, n)$ . A closer look to the proof of Claim 4 shows that the boundedness assumption  $z \in \mathcal{Z}$  was not used for the regular case  $z \in \mathcal{G}$ , see (24). Combining this with (12), (13) and Remark 3.3 we conclude that for any  $1 \leq i \leq K$  and  $1 \leq \lambda \leq d$

$$b^{i, n_i+1}(t, Z_t(\omega)) = \sigma^{i, n_i+1; \lambda}(t, Z_t(\omega)) = 0, \quad \text{for } (t, \omega) \in \{Z \in \mathcal{G}\} \setminus N,$$

where  $N$  is a  $\mathcal{R}_+ \otimes \mathcal{F}$ -measurable  $dt \otimes d\mathbb{P}$ -nullset. By the very definition of the product measure

$$0 = \int_N 1 dt \otimes d\mathbb{P} = \int_{\mathbb{R}_+} \mathbb{P}[N_t] dt,$$

where  $N_t := \{\omega \mid (t, \omega) \in N\} \in \mathcal{F}$ . Consequently  $\mathbb{P}[N_t] = 0$  for a.e.  $t \in \mathbb{R}_+$ . Hence by continuity of  $b(t, \cdot)$  and  $\sigma(t, \cdot)$

$$b^{i, n_i+1}(t, \cdot) = \sigma^{i, n_i+1; \lambda}(t, \cdot) = 0 \quad (29)$$

on  $\text{supp}(Z_t) \cap \mathcal{G}$ , for a.e.  $t \in \mathbb{R}_+$ . Here  $\text{supp}(Z_t)$  denotes the support of the (n.b. regular) distribution of  $Z_t$ , which is by definition the smallest closed set  $A \subset \mathbb{R}^N$  with  $\mathbb{P}[Z_t \in A] = 1$ . Thus again by continuity of  $b(t, \cdot)$  and  $\sigma(t, \cdot)$  equality (29) holds for a.e.  $t \in \mathbb{R}_+$  on the closure of  $\text{supp}(Z_t) \cap \mathcal{G}$ , which is  $\text{supp}(Z_t)$ . Hence we may replace the functions  $b^{i, n_i+1}(t, \cdot)$  and  $\sigma^{i, n_i+1; \lambda}(t, \cdot)$  by zero for almost every  $t$  without changing the diffusion  $Z$ , whence the assertion follows.  $\square$

The sum of two real valued diffusion processes with coefficients continuous in some argument is again a real valued diffusion with coefficients continuous in that argument. Consequently we may assume that the exponents  $z_0^{i, n_i+1}$  of the above e-consistent diffusion are mutually distinct. Since otherwise we add the corresponding polynomials to get in a canonical way an  $\mathbb{R}^{\tilde{N}}$ -valued diffusion  $\tilde{Z}$  which is e-consistent with  $BEP(\tilde{K}, \tilde{n})$ , resp.  $EP(\tilde{K}, \tilde{n})$ , for some  $\tilde{K} < K$ ,  $\tilde{N} < N$  and some  $\tilde{n} \in \mathbb{N}_0^{\tilde{K}}$ . Clearly  $\tilde{Z}$  provides the same interest rate model as  $Z$  and its coefficients are continuous in  $z$ .

For the second theorem we have to require e-consistency with  $BEP(K, n)$ . After a re-parametrization if necessary we may thus assume that

$$0 \leq z_0^{1, n_1+1} < \dots < z_0^{K, n_K+1},$$

see (8). The sequel of Theorem 8.1 reads now

**Theorem 8.2.** *If  $Z$  is e-consistent with  $BEP(K, n)$ , then it is non-trivial only if there exists a pair of indices  $1 \leq i < j \leq K$ , such that*

$$2z_0^{i, n_i+1} = z_0^{j, n_j+1}.$$

*Proof.* If there is no pair of indices  $1 \leq i < j \leq K$  such that  $2z_0^{i, n_i+1} = z_0^{j, n_j+1}$ , then  $\mathcal{D}' = \mathbb{R}_+ \times \Omega$ . But then  $Z$  is deterministic by the second part of Theorem 3.2.  $\square$

The message of Theorem 8.1 is the following: There is no possibility for modelling the term structure of interest rates by exponential-polynomial families with varying exponents driven by diffusion processes. From this point of view there is no use for daily estimations of the exponents of some exponential-polynomial type functions like  $F_{NS}$  or  $F_S$ . Once the exponents are chosen, they have to be kept constant. Furthermore there is a strong restriction on this choice by Theorem 8.2. It will be shown in the next section what this means for  $F_{NS}$  and  $F_S$  in particular.

**Remark 8.3.** *The boundedness assumption in Theorem 8.2 – that is e-consistency with  $BEP(K, n)$  – is essential for the strong (negative) result to be valid. It can be easily checked, that  $F(x, z) = z_0 + z_1 x \in EP(1, 1)$  allows for a non-trivial consistent diffusion process, see [3, Section 7].*

**Remark 8.4.** *The choice of an infinite time horizon for traded bonds is not a restriction, see (1). Indeed, we can limit our considerations on bonds  $P(t, T)$  which mature within a given finite time interval  $[0, T^*]$ . Consequently, the HJM drift condition (3) can only be deduced for  $x \in [0, T^* - t]$ , for  $dt \otimes d\mathbb{P}$ -a.e.  $(t, \omega) \in [0, T^*] \times \Omega$ . But the functions appearing in (3) are analytic in  $x$ . Hence whenever  $t < T^*$ , relation (3) extends to all  $x \geq 0$ . All conclusions on e-consistent Itô processes  $(Z_t)_{0 \leq t \leq T^*}$  can now be drawn as before.*

## 9. APPLICATIONS

In this section we apply the results on e-consistent diffusion processes to the Nelson–Siegel and Svensson families.

### 9.1. The Nelson–Siegel family.

$$F_{NS}(x, z) = z_1 + (z_2 + z_3x)e^{-z_4x}$$

In view of Theorem 8.1 we have  $z_4 > 0$ . Hence it's immediate from Theorem 8.2 that there is no non-trivial e-consistent diffusion. This result has already been obtained in [4] for e-consistent Itô processes.

### 9.2. The Svensson family.

$$F_S(x, z) = z_1 + (z_2 + z_3x)e^{-z_5x} + z_4xe^{-z_6x}$$

By Theorems 8.1 and 8.2 there remain the two choices

- i)  $2z_6 = z_5 > 0$
- ii)  $2z_5 = z_6 > 0$

We shall identify the e-consistent diffusion process  $Z = (Z^1, \dots, Z^6)$  in both cases. Let  $\mathbb{Q}$  be an equivalent martingale measure. Under  $\mathbb{Q}$  the diffusion  $Z$  transforms into a consistent one. Now we proceed as in the proof of Theorem 3.2. The expansion (19) reads as follows

$$\begin{aligned} Q_1(x) + Q_2(x)e^{-z_5x} + Q_3(x)e^{-z_6x} \\ + Q_4(x)e^{-2z_5x} + Q_5(x)e^{-(z_5+z_6)x} + Q_6(x)e^{-2z_6x} = 0, \end{aligned}$$

for some polynomials  $Q_1, \dots, Q_6$ . Explicitly

$$\begin{aligned} Q_1(x) &= -a_{1,1}x + \dots \\ Q_2(x) &= -a_{1,3}x^2 + \dots \\ Q_3(x) &= -a_{1,4}x^2 + \dots \\ Q_4(x) &= \frac{a_{3,3}}{z_5}x^2 + \dots \\ Q_6(x) &= \frac{a_{4,4}}{z_6}x^2 + \dots \end{aligned}$$

where  $\dots$  denotes terms of lower order in  $x$ . Hence  $a_{1,1} = 0$  in any case. By the usual arguments (the matrix  $a$  is nonnegative definite) the degree of  $Q_2$  and  $Q_3$  reduces to at most 1. Thus in both Cases i) and ii) it follows  $a_{3,3} = a_{4,4} = 0$ . It

remains

$$\begin{aligned}
Q_1(x) &= b_1 \\
Q_2(x) &= (b_3 + z_3 z_5)x + b_2 - z_3 - \frac{a_{2,2}}{z_5} + z_2 z_5 \\
Q_3(x) &= (b_4 + z_4 z_6)x - z_4 \\
Q_4(x) &= \frac{a_{2,2}}{z_5}
\end{aligned} \tag{30}$$

while  $Q_5 = Q_6 = 0$ . Since in Case i) it must hold that  $Q_4 = 0$ , we have  $a_{2,2} = 0$  and  $Z$  is deterministic. We conclude that there is no non-trivial e-consistent diffusion in Case i).

In Case ii) the condition  $Q_3 + Q_4 = 0$  leads to

$$a_{2,2} = z_4 z_5. \tag{31}$$

Hence a possibility for a non deterministic consistent diffusion  $Z$ . We derive from (30) and (31)

$$\begin{aligned}
b_1 &= 0 \\
b_2 &= z_3 + z_4 - z_5 z_2 \\
b_3 &= -z_5 z_3 \\
b_4 &= -2z_5 z_4.
\end{aligned}$$

Therefore the dynamics of  $Z^1, Z^3, \dots, Z^6$  are deterministic. In particular

$$\begin{aligned}
Z_t^1 &\equiv z_0^1 \\
Z_t^3 &= z_0^3 e^{-z_0^5 t} \\
Z_t^4 &= z_0^4 e^{-2z_0^5 t},
\end{aligned} \tag{32}$$

while  $Z_t^5 \equiv z_0^5$  and  $Z_t^6 \equiv 2z_0^5$ . Denoting by  $\tilde{W}$  the Girsanov transform of  $W$ , we have under the equivalent martingale measure  $\mathbb{Q}$

$$Z_t^2 = z_0^2 + \int_0^t \left( \Phi(s) - z_0^5 Z_s^2 \right) ds + \sum_{\lambda=1}^d \int_0^t \sigma^{2,\lambda}(s) d\tilde{W}_s^\lambda, \tag{33}$$

where  $\Phi(t)$  and  $\sigma^{2,\lambda}(t)$  are deterministic functions in  $t$ , namely

$$\Phi(t) := z_0^3 e^{-z_0^5 t} + z_0^4 e^{-2z_0^5 t}$$

and

$$\sum_{\lambda=1}^d \left( \sigma^{2,\lambda}(t) \right)^2 = z_0^4 z_0^5 e^{-2z_0^5 t}.$$

By Lévy's characterization theorem, see [9, Theorem (3.6), Chapter IV], the real valued process

$$W_t^* := \sum_{\lambda=1}^d \int_0^t \frac{\sigma^{2,\lambda}(s)}{\sqrt{z_0^4 z_0^5 e^{-2z_0^5 s}}} d\tilde{W}_s^\lambda, \quad 0 \leq t < \infty,$$

is an  $(\mathcal{F}_t)$ -Brownian motion under  $\mathbb{Q}$ . Hence the corresponding *short rates*  $r_t = F_S(0, Z_t) = z_0^1 + Z_t^2$  satisfy

$$dr_t = \left( \phi(t) - z_0^5 r_t \right) dt + \tilde{\sigma}(t) dW_t^*,$$

where  $\phi(t) := \Phi(t) + z_0^1 z_0^5$  and  $\tilde{\sigma}(t) := \sqrt{z_0^4 z_0^5} e^{-z_0^5 t}$ . This is just the generalized Vasicek model. It can be easily given a closed form solution for  $r_t$ , see [7, p. 293].

Summarizing Case ii) we have found a non-trivial e-consistent diffusion process, which is identified by (32) and (33). Actually  $\Phi$  has to be replaced by a predictable process  $\tilde{\Phi}$  due to the change of measure. Nevertheless this is just a one factor model. The corresponding interest rate model is the generalized Vasicek short rate model. This is very unsatisfactory since Svensson type functions  $F_S(x, z)$  have six factors  $z_1, \dots, z_6$  which are observed. And it is seen that after all just one of them – that is  $z_2$  – can be chosen to be non deterministic.

## 10. CONCLUSIONS

Bounded exponential-polynomial families like the Nelson–Siegel or Svensson family may be well suited for daily estimations of the forward rate curve. They are rather not to be used for inter-temporal interest rate modelling by diffusion processes. This is due to the facts that

- the exponents have to be kept constant
- and moreover this choice is very restricted

whenever you want to exclude arbitrage possibilities. It is shown for the Nelson–Siegel family in particular that there exists no non-trivial diffusion process providing an arbitrage free model. However there is a choice for the Svensson family, but still a very limited one, since all parameters but one have to be kept either constant or deterministic.

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