Optimal Risk Sharing with Different Reference Probabilities

Beatrice Acciaio[*]   Gregor Svindland[†]

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Abstract

We investigate the problem of optimal risk sharing between agents endowed with cash-invariant choice functions which are law-invariant with respect to different reference probability measures. We motivate a discrete setting both from an operational and a theoretical point of view, and give sufficient conditions for the existence of Pareto optimal allocations in this framework. Our results are illustrated by several examples.

1 Introduction

The optimal exchange of risk between two parties is one of the major issues in mathematical economics and finance, and many authors have studied this problem, since the early works of Arrow [2], Borch [5], Du Mouchel [11], where the risk sharing is analyzed in the insurance and reinsurance context. The introduction of concepts like coherent and convex risk measures, by Artzner et al. [3] and Föllmer and Schied [17], recently paved the way to a further analysis of the problem of optimal sharing and allocation of risk. Several authors have considered the exchange of risk between agents endowed with these kinds of choice functions (see, e.g., Deprez and Gerber [9], Chateauneuf et al. [8], Barrieu and El Karoui [4], Jouini et al. [24],

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[*]Financial and Actuarial Mathematics, Vienna University of Technology, Wiedner Hauptstrasse 8-10/105-1, A-1040 Vienna, Austria, *acciaio@fam.tuwien.ac.at*

[†]Mathematics Institute, University of Munich, Theresienstrasse 39, D-80333 München, Germany, *Gregor.Svindland@mathematik.uni-muenchen.de*

Filipović and Kupper [13, 14], Burgert and Rüschendorf [6, 7]) or in a slightly more general setting (see, e.g., Acciaio [1], Filipović and Svindland [16]).

A natural property to require on those choice functions is indifference with respect to financial positions having the same distribution under some reference probability measure. This is the so-called law-invariance property, studied, e.g., by Kusuoka [25], Frittelli and Rosazza Gianin [19], Jouini et al. [23]. When all choice functions are assumed to be cash-invariant (or translation invariant) and law-invariant with respect to the same reference probability measure, the existence of optimal allocations has already been proved, see Jouini et al. [24], Filipović and Svindland [16] and Acciaio [1]. In this paper we study the risk sharing problem in the situation of two economic agents with different views of the world, that is, with different (subjective) reference probability measures. We consider them equipped with cash-invariant choice functions which are law-invariant in their respective worlds, i.e., with respect to their different reference probabilities. Manifold causes may motivate such a framework. In case of financial corporations, for instance, these different world views might stem from different internal models, from having access to different information, or from being subject to guidelines of different regulating agencies. We can consider, for example, the case of financial firms subject to stress tests, possibly managed by different supervising agencies. These tests are made to gauge the potential vulnerability of the firms to a given set of particular market events or stress scenarios, like a stock market crash or other market shocks. In this situation firms are interested in maintaining a 'good' position in case one of those shock scenarios were to occur, in order to 'pass' the test. This might amongst other things be achieved by exchanging risk in this set of events. We like to point out that our setting is in particular valid in a situation of both competition and cooperation. It is often observed that competing agents in a market do cooperate on the level of exchanging certain risks if they all believe to benefit from it. However, since in the end the agents are competitors, it is very unlikely that they will share full information with each other, that is agree on one reference probability, and thereby abandon advantages over their competitors which might result from access to some particular information. E.g. in insurance companies or banks the internal model is a crucial competitive advantage which is kept secret from other agents in the market. Within one firm, i.e. within one internal model, law-invariance of the choice function used is very likely. Hence, we end up in a situation as described above where the incomplete and different information is represented by the different reference probabilities of the agents which are a priori unrelated. Nevertheless, these competing agents may all benefit from trading/exchanging some of their risks on some prominent events which are common knowledge, like increase or decrease of interests, natural hazards, etc., but which are weighted differently according to different information.

In any case, any exchange of risk requires some kind of cooperation between the involved agents, who will have to agree on some set of (prominent) scenarios on which they want some

level of protection. As described above it is very likely that the agents weight these scenarios in a different way depending on their respective information. Moreover, it turns out that, in practice, such a set of scenarios is usually finite, thus reducing the optimal risk allocation problem to a finite dimensional risk exchange on these base scenarios. In this framework, and under some mild additional conditions, we show that there always exist Pareto optimal allocations for any aggregate risk. Our results are illustrated by several examples. In particular, we also give examples which show that Pareto optimal allocations do not exist in general if we drop some of our assumptions (Examples 4.3 and 4.4).

The remainder of the paper is organized as follows. In Section 2 we formalize the optimal risk sharing problem, and we state our main result on the existence of optimal allocations (Theorem 2.3). This result is then proved in Section 3. Our examples are collected in Section 4. We assume the reader to be familiar with basic convex duality as outlined in [26] or [12]. However, in the appendix we give a short review on some basic concepts and notation from convex analysis which are frequently used throughout the paper. Some known results are also postponed to the appendix.

2 Optimal Risk Sharing Problem

2.1 Framework

We consider a measurable space $(\Omega, \mathcal{F})$ and two probability measures $\mathbb{P}_1, \mathbb{P}_2$ on $(\Omega, \mathcal{F})$ such that $(\Omega, \mathcal{F}, \mathbb{P}_i), i = 1, 2$, are non-atomic standard probability spaces. The measures $\mathbb{P}_1, \mathbb{P}_2$ describe the view of two agents, say 1 and 2, on the world $(\Omega, \mathcal{F})$. The preferences of the i-th agent on $L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ are represented by a choice function $U_i : L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i) \to \mathbb{R}$, that throughout the paper is assumed to satisfy the following conditions:

(C1) concavity: $U_i(\alpha X + (1 - \alpha)Y) \geq \alpha U_i(X) + (1 - \alpha)U_i(Y)$ for all $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ and $\alpha \in (0, 1)$;

(C2) cash-invariance: $U_i(X + c) = U_i(X) + c$ for all $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ and $c \in \mathbb{R}$;

(C3) normalization: $U_i(0) = 0$;

(C4) $\mathbb{P}_i$-law-invariance: $U_i(X) = U_i(Y)$ whenever $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ are identically distributed under $\mathbb{P}_i$;

(C5) upper semi-continuity (u.s.c.): for any sequence $(X_n)_{n \in \mathbb{N}} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ converging to some $X \in L^\infty$, we have $U_i(X) \geq \limsup_n U_i(X_n)$.

If U_i in addition is monotone, i.e. $U_i(X) \geq U_i(Y)$ whenever $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ satisfy $X \geq Y$ $\mathbb{P}_i$-a.s., then U_i is a $\mathbb{P}_i$-law-invariant monetary utility function, i.e. $-U_i$ is a $\mathbb{P}_i$-law-invariant convex risk measure in the sense of Föllmer and Schied [18]. Note that (C5) is equivalent to the continuity of U_i because U_i is finitely-valued (see e.g. [12] Corollary 2.5), and that

any proper function on $L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ which is monotone and satisfies (C2) is automatically finitely-valued and 1-Lipschitz-continuous (see [18]). It is proved in [23] that the $\mathbb{P}_i$-law-invariance ensures the following dual representation for U_i (the so-called Fatou property):

$$U_i(X) = \inf_{Z \in L^1(\Omega, \mathcal{F}, \mathbb{P}_i)} \{V_i(Z) + E[ZX]\}, \quad X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i), \tag{2.1}$$

where V_i is the dual U_i^* of U_i (see (B.1)). Note how this representation of U_i can be seen as a worst-case evaluation, based on the set of test measures $\mathcal{Q}_i = \{\mathbb{Q} \, \sigma\text{-additive measure} : \mathbb{Q} \ll \mathbb{P}_i, V_i(\frac{d\mathbb{Q}}{d\mathbb{P}_i}) < \infty\}$, where to each test measure $\mathbb{Q} \in \mathcal{Q}_i$ is associated a penalty $V_i(\frac{d\mathbb{Q}}{d\mathbb{P}_i})$ which expresses the confidence of agent i on $\mathbb{Q}$ (the higher the penalty, the lower the confidence on that measure).

Here we fix the space $L^\infty(\Omega, \mathcal{F}, \mathbb{P}_i)$ of $\mathbb{P}_i$-essentially-bounded random variables as the set of possible financial positions considered by agent i. However, we can also think of the choice functions as defined on $L^{p_i}(\Omega, \mathcal{F}, \mathbb{P}_i)$, for any p_i in $[1, \infty]$ and possibly $p_1 \neq p_2$. Note that we do not require the $\mathbb{P}_i$'s to fulfill any absolute continuity- or even equivalence-relation. A priori the world views $\mathbb{P}_1$ and $\mathbb{P}_2$ are unrelated.

2.2 Formulation of the Problem

The problem we address in this paper is the optimal sharing and allocation of risk between two agents who have different views of the world, in the sense described in Section 2.1. As motivated in the Introduction, the discrete setting turns out to be a proper framework to formulate this problem. Roughly speaking, no matter how different is the world view of the two agents, we assume they agree on a finite set of possible scenarios. Therefore, any information they have about the preferences of the other, and any risk they consider or exchange, is relative to this set. To put this into mathematical terms, we let $A = \{A_1, \ldots, A_n\} \subset \mathcal{F}$ be a finite partition of Ω and $\mathcal{F}_A := \sigma(\{A_1, \ldots, A_n\})$ the σ-algebra it generates. The A_j's are the base events on which agents agree to exchange risk, and we assume that

$$\mathbb{P}_i(A_j) > 0 \text{ for all } j = 1, \ldots, n \text{ and } i = 1, 2. \tag{2.2}$$

This latter condition does not only seem natural, but it is in fact necessary for the existence of optimal allocations. Indeed, assume $0 = \mathbb{P}_1(A_j) < \mathbb{P}_2(A_j)$ (or vice versa, mutatis mutandis) for some $j \in \{1, \ldots, n\}$, then agent 2 could increase her wealth on A_j as much as she likes, and agent 1 would take all the risk on A_j. Hence, in this situation there cannot be any optimum. Moreover, let $\mathbb{Q}^+$ be the set of positive rationals, we assume that

$$\mathbb{P}_1(A_j) \in \mathbb{Q}^+ \text{ for all } j = 1, \ldots, n, \tag{2.3}$$

which is no restriction in the interesting cases. A finite partition $A = \{A_j\}_{j=1}^n$ of Ω such that $\{A_j\}_{j=1}^n \subset \mathcal{F}$ and (2.2), (2.3) hold will be called admissible. Now let $A = \{A_j\}_{j=1}^n$ be an

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admissible partition of Ω. Then the space of admissible financial positions which the agents consider in the exchange of risk, is the collection of all $\mathcal{F}_A$−measurable random variables, that we denote by $\mathcal{S}_A$. The optimal risk allocation problem, for any aggregate risk $X \in \mathcal{S}_A$, is therefore formulated as follows:

$$U_A(X) := \sup_{\substack{X_1, X_2 \in \mathcal{S}_A, \\ X_1 + X_2 = X}} \{U_1(X_1) + U_2(X_2)\}. \tag{2.4}$$

The solutions (X_1, X_2) to (2.4), i.e. $X_1, X_2 \in \mathcal{S}_A$ such that $X_1 + X_2 = X$ and $U_A(X) = U_1(X_1) + U_2(X_2)$, if exist, are called optimal allocations of X. Note that, due to cash-invariance, an allocation $(X_1, X_2) \in \mathcal{S}_A \times \mathcal{S}_A$ of X solves problem (2.4) if and only if it is optimal in the sense of Pareto: for all allocations $(Y_1, Y_2) \in \mathcal{S}_A \times \mathcal{S}_A$ of X s.t. $U_i(Y_i) \geq U_i(X_i), i = 1, 2$, then $U_i(Y_i) = U_i(X_i), i = 1, 2$.

Clearly the space $\mathcal{S}_A$ is isomorphic to $\mathbb{R}^n$: any $X \in \mathcal{S}_A$ admits a representation of the form $X = \sum_{j=1}^n x_j 1_{A_j}$, with $\{x_j\}_{j=1}^n \subset \mathbb{R}$, and is identified with the vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. In this way, the restriction of the agents' choice functions U_i on $\mathcal{S}_A$ can be read as defined on $\mathbb{R}^n$. In order to avoid confusion, we will use lowercase letters to denote functions on $\mathbb{R}^n$, i.e. $u_i(x_1, \ldots, x_n) \equiv U_i(\sum_{j=1}^n x_j 1_{A_j})$. Note that the functions u_i are concave, cash-invariant, normalized, and continuous (since concave and finitely-valued on $\mathbb{R}^n$, see e.g. [12] Corollary 2.3). Moreover, u_i is monotone whenever U_i is monotone. We may now rewrite the optimization problem (2.4) as follows:

$$u_1 \square u_2(x) = \sup_{\substack{x_1, x_2 \in \mathbb{R}^n, \\ x_1 + x_2 = x}} \{u_1(x_1) + u_2(x_2)\}, \tag{2.5}$$

where $u_1 \square u_2$ is the so-called convolution of u_1 and u_2 (see (B.5) and e.g. [26] for details on the convolution operation). The function $u_1 \square u_2 : \mathbb{R}^n \to (-\infty, \infty]$ inherits from u_i the concavity and cash-invariance properties. Moreover, due to concavity and the fact that $dom(u_1 \square u_2) = dom(u_1) + dom(u_2)$, we either have $u_1 \square u_2 \equiv +\infty$ or $u_1 \square u_2$ is finitely-valued and continuous on $\mathbb{R}^n$. If there exist optimal allocations of $x \in \mathbb{R}^n$, i.e. problem (2.5) admits solutions, then the convolution $u_1 \square u_2$ is said to be exact at x. From now on, the penalty functions or dual conjugates (see (B.1)) of $u_1, u_2, u_1 \square u_2$, also defined in $\mathbb{R}^n$, are denoted by v_1, v_2, v respectively. By $\mathbb{P}_i$-law-invariance and Proposition A.1 we obtain the following relation:

$$v_i(z) = V_i \left(\sum_{j=1}^n \frac{z_j}{\mathbb{P}_i(A_j)} 1_{A_j} \right), \quad \forall z \in \mathbb{R}^n, \ i = 1, 2, \tag{2.6}$$

which will turn out to be useful in the proof of the main theorem.

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2.3 Main Result

We work under the following assumptions.

Assumption 2.1. *Agent* 2 *gives a finite penalty to the reference probability measure of agent* 1, *i.e.*

$$\mathbb{P}_1 \in dom(v_2), \tag{2.7}$$

where $\mathbb{P}_1$ *is identified with the vector* $(p_1, \ldots, p_n)$, *with* $p_j = \mathbb{P}_1(A_j)$ *for all* $j = 1, \ldots, n$.

If we consider the dual representation of u_i on $\mathbb{R}^n$ via its conjugate v_i, which is the analogue of representation (2.1) of U_i, Assumption 2.1 says that the reference probability $\mathbb{P}_1$ of agent 1 belongs to the test measures set considered by agent 2.

Note that, by Proposition A.1 and the normalization property, we always have $\mathbb{P}_1 \in dom(v_1)$, with $v_1(\mathbb{P}_1) = 0$. Therefore (2.7) implies $\mathbb{P}_1 \in dom(v_1) \cap dom(v_2)$, which ensures that the convolution function $u_1 \square u_2$ is finitely-valued and continuous.

Assumption 2.2. *Either of the following two conditions holds:*

 (i) No Risk-Arbitrage (NRA), i.e. $u_1 \square u_2(0) = u_1(0) + u_2(0) = 0$,

 (ii) $\partial v_2(\mathbb{P}_1) \neq \emptyset$.

The requirement of (NRA) can be seen as a kind of no-arbitrage condition concerning risk, and it is exactly the equilibrium condition given in Burgert and Rüschendorf [7] (see also [6] and [22]). It says that, in a condition of balance between demand and supply, it is not possible to increase the utility of one agent without decreasing that of the other. This is equivalent to the existence of a no trade equilibrium premium, the notion of which is introduced in Deprez and Gerber [9], where Pareto optimal allocations are characterized for convex premium principles.

The alternative condition in Assumption 2.2 is more of technical nature. Its meaning is understood when going through the proof of Theorem 3.6. Examples 4.3,4.4 show that we cannot expect optimal allocations in case Assumption 2.2 is not satisfied.

The main result of the paper is the following theorem, which states the existence of optimal allocations for any risk $x \in \mathbb{R}^n$. Its proof is prepared in Sections 3.1 and 3.2 and finally presented in Section 3.3.

Theorem 2.3. *Let* $A = \{A_j\}_{j=1}^n$ *be an admissible partition of* Ω. *Then, under Assumptions 2.1, 2.2, the convolution* $u_1 \square u_2$ *in (2.5) is exact at any* $x \in \mathbb{R}^n$, *i.e. problem (2.4) admits solutions for every* $X \in \mathcal{S}_A$.

In Section 4, we compute optimal risk allocations for prominent classes of choice functions, like the Entropic Utility (Example 4.1) and the Mean Variance Choice Function (Example 4.2). In Examples 4.1 and 4.2 optimal allocations exist for every admissible partition

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of Ω. However, Examples 4.3 and 4.4 show that this is not the case in general, when Assumptions 2.1, 2.2 do not hold. In particular, with Example 4.4 we illustrate that the less information is exchanged, i.e. the less base scenarios we fix, the more likely are we to find optimal allocations. Clearly this is what we would expect.

3 Existence of Optimal Allocations

3.1 Preliminary Results on Convolution

In this section we provide results which form the basis for the proof of Theorem 2.3. Note that Lemmas 3.2, 3.3, and 3.4 hold true in more general settings, i.e., for more general classes of concave functions and far larger model spaces than $\mathbb{R}^n$. Here, for uniformity of notation, we enounce them in the present context.

Definition 3.1. *For any non empty convex set $C \subseteq \mathbb{R}^n$, the recession cone 0^+C is given by*

$$0^+C = \{y \in \mathbb{R}^n : x + ty \in C, \, \forall x \in C, \, \forall t \geq 0\}.$$

From now on, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_{u_1 \square u_2}$ denote the acceptance sets of u_1, u_2, and $u_1 \square u_2$ in $\mathbb{R}^n$ respectively, i.e. $\mathcal{A}_1 = \{x \in \mathbb{R}^n \mid u_1(x) \geq 0\}$ and similarly for u_2 and $u_1 \square u_2$.

Lemma 3.2. *For any $x \in 0^+\mathcal{A}_1 \cap -0^+\mathcal{A}_2$ and $y \in dom(v)$, we have $\langle y, x \rangle = 0$.*

Proof. If $dom(v) = \emptyset$, then there is nothing to prove. Otherwise, let x be in $0^+\mathcal{A}_1 \cap -0^+\mathcal{A}_2$ and $t \geq 0$. Then $tx \in \mathcal{A}_1$ and $-tx \in \mathcal{A}_2$, so that $sx \in \mathcal{A}_1 + \mathcal{A}_2 \subseteq \mathcal{A}_{u_1 \square u_2}$ for any $s \in \mathbb{R}$. This implies that for all $s \in \mathbb{R}$, $0 \leq u_1 \square u_2(sx) = \inf_{y \in \mathbb{R}^n}\{v(y) + \langle y, sx \rangle\}$, by (B.2). Hence, for any $y \in dom(v)$, $s\langle y, x \rangle = \langle y, sx \rangle \geq -v(y) \in \mathbb{R}$, $\forall s \in \mathbb{R}$, which gives $\langle y, x \rangle = 0$, as claimed. $\square$

In the lemmas that follow, we give necessary and sufficient conditions for the exactness of the convolution $u_1 \square u_2$.

Lemma 3.3. *The convolution $u_1 \square u_2$ is exact at every $x \in \mathbb{R}^n$ if and only if $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_{u_1 \square u_2}$.*

Proof. By definition of convolution, we always have $\mathcal{A}_1 + \mathcal{A}_2 \subseteq \mathcal{A}_{u_1 \square u_2}$. Suppose that $u_1 \square u_2$ is exact and take $x \in \mathcal{A}_{u_1 \square u_2}$. Then $0 \leq u_1 \square u_2(x) = u_1(y) + u_2(x - y)$ for some $y \in \mathbb{R}^n$. By cash-invariance we may assume that $u_1(y), u_2(x - y) \geq 0$, thus $x = y + (x - y) \in \mathcal{A}_1 + \mathcal{A}_2$. Conversely, let $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_{u_1 \square u_2}$ and fix some $x \in \mathbb{R}^n$. We have $x - u_1 \square u_2(x) \in \mathcal{A}_{u_1 \square u_2}$ and, by hypothesis, there exists $y \in \mathbb{R}^n$ s.t. $y \in \mathcal{A}_1$ and $(x - u_1 \square u_2(x) - y) \in \mathcal{A}_2$. This gives $u_1(y) + u_2(x - y) \geq u_1 \square u_2(x)$, where the inequality is indeed an equality by definition of convolution, and therefore the allocation $(y, x - y)$ is optimal for x. $\square$

Lemma 3.4. *The convolution $u_1 \square u_2$ is exact at every $x \in \mathbb{R}^n$ if and only if $\mathcal{A}_1 + \mathcal{A}_2$ is closed.*

Proof. One implication is immediate from the previous theorem. Indeed, for $u_1 \square u_2$ exact, the set $(\mathcal{A}_1 + \mathcal{A}_2)$ coincides with $\mathcal{A}_{u_1 \square u_2}$, which is closed by continuity of $u_1 \square u_2$. For the reverse implication, we assume that $\mathcal{A}_1 + \mathcal{A}_2$ is closed and claim that $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_{u_1 \square u_2}$. Since the inclusion $\mathcal{A}_1 + \mathcal{A}_2 \subseteq \mathcal{A}_{u_1 \square u_2}$ is obvious, we only have to prove the reverse inclusion. To this end, fix $x \in \mathcal{A}_{u_1 \square u_2}$ and consider some maximizing sequence for the convolution in (2.5): $(y_n, x - y_n)_n \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $u_1 \square u_2(x) = \lim_{n \to \infty} \{u_1(y_n) + u_2(x - y_n)\}$. If $u_1 \square u_2(x) > 0$, we can find an element $(y_{\bar{n}}, x - y_{\bar{n}})$ of the sequence such that $u_1(y_{\bar{n}}) + u_2(x - y_{\bar{n}}) > 0$. This gives $x = (y_{\bar{n}} - u_1(y_{\bar{n}})) + (x - y_{\bar{n}} + u_1(y_{\bar{n}})) \in \mathcal{A}_1 + \mathcal{A}_2$. On the other hand, if $u_1 \square u_2(x) = 0$, consider the sequence $(\eta_n^1, \eta_n^2)_n \subset \mathcal{A}_1 \times \mathcal{A}_2$ with $\eta_n^1 := y_n - u_1(y_n)$ and $\eta_n^2 := x - y_n - u_2(x - y_n)$. We have $\eta_n^1 + \eta_n^2 = x - u_1(y_n) - u_2(x - y_n) \xrightarrow[n \to +\infty]{} x$, which implies $x \in \mathcal{A}_1 + \mathcal{A}_2$ because $\mathcal{A}_1 + \mathcal{A}_2$ is closed by hypothesis. This proves the equality $\mathcal{A}_{u_1 \square u_2} = \mathcal{A}_1 + \mathcal{A}_2$. The exactness then follows from Lemma 3.3. $\square$

We close this section by stating a condition that ensures the closedness of the sum of two convex sets. The usefulness of this result is obvious by Lemma 3.4.

Lemma 3.5 (Corollary 9.1.2 in [26]). *Let C_1, C_2 be non-empty closed convex sets in $\mathbb{R}^n$. If there is no $x \neq 0$ such that $x \in 0^+ C_1$ and $-x \in 0^+ C_2$, then $C_1 + C_2$ is closed.*

3.2 Balanced Case

As preparatory result for the proof of Theorem 2.3, in this section we prove the exactness of $u_1 \square u_2$ in (2.5) in case the partition $A = \{A_1, \ldots, A_n\}$ of Ω is balanced w.r.to $\mathbb{P}_1$, i.e. $\mathbb{P}_1(A_j) = \frac{1}{n}, \forall j = 1, \ldots, n$.

Theorem 3.6. *Let $A = \{A_j\}_{j=1}^n$ be a $\mathbb{P}_1$-balanced admissible partition of Ω. Then, under Assumptions 2.1, 2.2, the convolution $u_1 \square u_2$ in (2.5) is exact at any $x \in \mathbb{R}^n$, i.e. problem (2.4) admits solutions for every $X \in \mathcal{S}_A$.*

Before we prove Theorem 3.6, we collect some helpful results. Let us first of all translate the concept of law-invariance into this balanced discrete setting. Let S_n be the set of all permutations in $\{1, \ldots, n\}$. Since, by assumption, partition A is balanced w.r.to $\mathbb{P}_1$, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\pi \in S_n$, we have that $\sum_{j=1}^n x_{\pi(j)} 1_{A_j}$ and $\sum_{j=1}^n x_j 1_{A_j}$ have the same law under $\mathbb{P}_1$. Therefore, the $\mathbb{P}_1$-law-invariance of U_1 ensures that the induced function u_1 is permutation invariant on $\mathbb{R}^n$: $u_1(x_\pi) = u_1(x)$ for all $\pi \in S_n$, where x_π is shorthand for $(x_{\pi(1)}, \ldots, x_{\pi(n)})$.

The following lemma will be used in the proof of Theorem 3.6. The assertion is evident and may be proved by induction.

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Lemma 3.7. *If for $x \in \mathbb{R}^n$, $n \geq 2$, there are $i, j \in \{1, \ldots, n\}$ such that $x_i > 0 > x_j$, then there exist permutations $\pi_1, \ldots, \pi_{n-1} \in S_n$ such that $x_{\pi_1}, \ldots, x_{\pi_{n-1}}$ are linearly independent.*

In what follows we denote by $\pi_{i,j}$ the transposition interchanging i and j. Moreover, E is the permutation invariant function $E : \mathbb{R}^n \to \mathbb{R}$ that operates as $E[x] := 1/n \cdot \sum_{i=1}^{n} x_i$, and $\mathcal{E}$ is its null set $\mathcal{E} := \{x \in \mathbb{R}^n : E[x] = 0\}$. Note that $E[x] = \mathbb{E}_{\mathbb{P}_1}\left[\sum_{j=1}^{n} x_j 1_{A_j}\right]$.

Proof of Theorem 3.6. If $n = 1$, exactness of $u_1 \square u_2$ follows from cash-invariance. Henceforth, let $n \geq 2$. If there is no $x \in \mathbb{R}^n \backslash \{0\}$ such that $x \in 0^+\mathcal{A}_1 \cap -0^+\mathcal{A}_2$, then the exactness follows from Lemma 3.5 and Lemma 3.4. Now suppose there exists $x \neq 0$ in $0^+\mathcal{A}_1 \cap -0^+\mathcal{A}_2$. From Assumption 2.1 and Lemma 3.2, we have that $E[x] = 0$. Consequently, there are $i, j \in \{1, \ldots, n\}$ such that $x_i > 0 > x_j$. Moreover, $x \in \mathcal{E} \cap 0^+\mathcal{A}_1$. We claim that this implies the existence of a vector $x' \in \mathcal{E} \cap 0^+\mathcal{A}_1$, $x' \neq 0$, having the following property:

$$\text{for all } \pi \in S_n \text{ there exists } \mu \in S_n \text{ such that } -x'_\pi = x'_\mu. \tag{3.1}$$

Suppose that such an x' exits. Then, since $E[x'] = 0$, there are $i, j \in \{1, \ldots, n\}$ such that $x'_i > 0 > x'_j$. Hence, by Lemma 3.7, there exist $n - 1$ linearly independent permutations $x'_{\pi_1}, \ldots, x'_{\pi_{n-1}}$ of x'. Therefore, $x'_{\pi_1}, \ldots, x'_{\pi_{n-1}} \in \mathcal{E} \cap 0^+\mathcal{A}_1$ form a basis of the $(n-1)$-dimensional subspace $\mathcal{E}$. Now, choose any $y \in \mathcal{E}$. For appropriate $\{a_i\}_{i=1}^{n-1} \subset \mathbb{R}$ and $\{\mu_i\}_{i=1}^{n-1} \subset S_n$, we have

$$y = \sum_{i=1}^{n-1} a_i x'_{\pi_i} = \sum_{i=1}^{n-1} |a_i| x'_{\mu_i} \in \mathcal{E} \cap 0^+\mathcal{A}_1,$$

due to (3.1) and the fact that $\mathcal{E} \cap 0^+\mathcal{A}_1$ is a permutation invariant convex cone. Thus $\mathcal{E} = \mathcal{E} \cap 0^+\mathcal{A}_1$. From Proposition A.1, $E[x] \geq u_1(x) \ \forall x \in \mathbb{R}^n$ and then $\mathcal{A}_1 \subseteq \mathcal{A}_E$. On the other hand, we always have $0^+\mathcal{A}_1 \subseteq \mathcal{A}_1$, so that $0^+\mathcal{A}_1 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_E$. Now we are going to prove that these are all equalities, showing that $\mathcal{A}_E \subseteq 0^+\mathcal{A}_1$. Indeed, fix $x \in \mathcal{A}_E$ and consider any $y \in \mathcal{A}_1$ and $t \geq 0$. Then $u_1(y + tx) = u_1(y + tE[x] + t(x - E[x])) \geq 0$ follows from $E[x] \geq 0$ and $(x - E[x]) \in \mathcal{E} \subseteq 0^+\mathcal{A}_1$. Therefore $x \in 0^+\mathcal{A}_1$ and $\mathcal{A}_E \subseteq 0^+\mathcal{A}_1$, which implies $u_1 = E$. Now, by (B.7) we have $u_1 \square u_2 = E + v_2(\mathbb{P}_1)$. Thus, if condition (i) of Assumption 2.2 holds, then $v_2(\mathbb{P}_1) = 0$ and $u_1 \square u_2 = E = u_1$, which in particular ensures the exactness of the convolution. On the other hand, if condition (ii) of Assumption 2.2 is satisfied, then for any $x \in \mathbb{R}^n$ and $y \in -\partial v_2(\mathbb{P}_1)$ we obtain $u_1 \square u_2(x) = E[x - y] + E[y] + v_2(\mathbb{P}_1) = u_1(x - y) + u_2(y)$, by (B.4). Therefore, the convolution is exact in this case too.

Finally, in order to verify that there indeed exists an $0 \neq x' \in \mathcal{E} \cap 0^+\mathcal{A}_1$ satisfying (3.1), let $0 \neq x \in \mathcal{E} \cap 0^+\mathcal{A}_1$ and note that (3.1) is always true for $n \leq 2$. In case $n > 2$, on the other hand, we consider the following algorithm:

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input: $0 \neq x \in \mathcal{E} \cap 0^{+}\mathcal{A}_1$

for $i = 3$ to n:

if $\forall\, \pi \in S_n\, \exists\, \mu \in S_n\colon -x_\pi = x_\mu$ then return $x' := x$ end,

else sort x in such a way that the output $\bar{x}$ satisfies $\bar{x}_{i-2} > 0$ and $\bar{x}_{i-1} < 0$, and additionally $\bar{x}_j = 0$ for all $j < i - 2$ in case $i > 3$.

$$\hat{x} := \bar{x} + \frac{-\bar{x}_{i-2}}{\bar{x}_{i-1}}\bar{x}_{\pi_{i-2,i-1}}$$

$$x := \hat{x}\,.$$

end

Since $\mathcal{E} \cap 0^{+}\mathcal{A}_1$ is a permutation invariant convex cone, sorting $0 \neq x \in \mathcal{E} \cap 0^{+}\mathcal{A}_1$ as described in the algorithm, the output $\bar{x}$ of each cycle is still an element of $\mathcal{E} \cap 0^{+}\mathcal{A}_1$, as is $\bar{x}_{\pi_{i-2,i-1}}$. From $\frac{-x_{i-2}}{x_{i-1}} > 0$, then also $\hat{x} \in \mathcal{E} \cap 0^{+}\mathcal{A}_1$, thus the algorithm never leaves the set $\mathcal{E} \cap 0^{+}\mathcal{A}_1$. Furthermore, in each cycle the algorithm either terminates or eliminates the $i-2$-nd entry, that is, it builds a vector x satisfying $x_{i-2} = 0$. Since for $j = i, \ldots, n$:

$$\bar{x}_j \neq 0 \quad \Rightarrow \quad \bar{x}_j - \frac{\bar{x}_{i-2}}{\bar{x}_{i-1}}\bar{x}_j \neq 0$$

and

$$\bar{x}_{i-1} - \frac{\bar{x}_{i-2}^2}{\bar{x}_{i-1}} = 0 \quad \Leftrightarrow \quad \bar{x}_{i-2} = -\bar{x}_{i-1},$$

our algorithm does not return the zero vector at any cycle. Indeed, suppose for the moment it did return the zero vector. Then the preceding relations tell us that $\bar{x}_j = 0$ for all $j \in \{1, \ldots, n\} \setminus \{i - 2, i - 1\}$ and $\bar{x}_{i-2} = -\bar{x}_{i-1} = a$ for some $a \neq 0$. Thus $\bar{x}$ is of type $(\bar{x}_1, \ldots, \bar{x}_{i-3}, \bar{x}_{i-2}, \bar{x}_{i-1}, \bar{x}_i, \ldots, \bar{x}_n) = (0, \ldots, 0, a, -a, 0, \ldots, 0)$, which cannot happen because this implies that the outcome of the previous cycle did satisfy the breaking condition (3.1), and thus the algorithm should already have terminated. Moreover, in case the algorithm does not terminate before all possible $n - 2$ cycles are through, it returns a vector of type $(0, \ldots, 0, a, -a)$ for some $a \in \mathbb{R}\setminus\{0\}$. So finally we find an $0 \neq x' \in \mathcal{E} \cap 0^{+}\mathcal{A}_1$ which does satisfy (3.1). $\qquad\square$

3.3 Proof of Theorem 2.3

We reduce the general discrete case to a balanced setting in order to apply Theorem 3.6. Remember that, by admissibility, the probabilities $a_i := \mathbb{P}_1(A_i)$ are in $\mathbb{Q}^{+}$ for all $i = 1, \ldots, n$, and consider the greatest rational number a s.t. a_i/a are all integers for $i = 1, \ldots, n$. By the non-atomicity of $(\Omega, \mathcal{F}, \mathbb{P}_1)$ and $(\Omega, \mathcal{F}, \mathbb{P}_2)$, for each $i = 1, \ldots, n$ we can find a partition $\{B_{i1}, \ldots, B_{im_i}\} \subset \mathcal{F}$ of the event A_i such that

$$\mathbb{P}_1(B_{ij}) = \frac{\mathbb{P}_1(A_i)}{m_i} = a \quad \text{and} \quad \mathbb{P}_2(B_{ij}) = \frac{\mathbb{P}_2(A_i)}{m_i}, \tag{3.2}$$

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where $m_i := a_i/a$ (see e.g. [10] Corollary 1.1). Therefore, we end up with a $\mathbb{P}_1$-balanced admissible partition $B = \{B_{ij}, j = 1, \ldots, m_i, i = 1, \ldots, n\}$ of Ω, which is a refinement of partition A and is composed of $M := 1/a$ sets. Denoting by $\mathcal{F}_B$ the σ-algebra generated by partition B, and $\mathcal{S}_B$ the space of $\mathcal{F}_B$-measurable random variables, we clearly have the inclusions $\mathcal{F}_A \subseteq \mathcal{F}_B \subseteq \mathcal{F}$ and $\mathcal{S}_A \subseteq \mathcal{S}_B$. Note that $\mathbb{P}_1$ and $\mathbb{P}_2$ restricted to $\mathcal{F}_B$ are equivalent (the same as in $\mathcal{F}_A$). Moreover, we have that the densities on $\mathcal{F}_A$ and $\mathcal{F}_B$ are respectively

$$f_A := \frac{d\mathbb{P}_2}{d\mathbb{P}_1}\Big|_{\mathcal{F}_A} = \sum_{i=1}^{n} \frac{\mathbb{P}_2(A_i)}{\mathbb{P}_1(A_i)} 1_{A_i}$$

and

$$f_B := \frac{d\mathbb{P}_2}{d\mathbb{P}_1}\Big|_{\mathcal{F}_B} = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\mathbb{P}_2(B_{ij})}{\mathbb{P}_1(B_{ij})} 1_{B_{ij}} = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\mathbb{P}_2(A_i)}{\mathbb{P}_1(A_i)} 1_{B_{ij}} = f_A,$$

by (3.2). Therefore, for any $\mathcal{F}_B$-measurable r.v. ξ we have

$$\mathbb{E}_{\mathbb{P}_2}[\xi|\mathcal{F}_A] = \mathbb{E}_{\mathbb{P}_1}\left[\frac{f_B}{f_A}\xi\Big|\mathcal{F}_A\right] = \mathbb{E}_{\mathbb{P}_1}[\xi|\mathcal{F}_A]. \tag{3.3}$$

Now fix $X \in \mathcal{S}_A$ and consider the maximization problem restricted to the $\mathcal{F}_B$-measurable pairs:

$$U_B(X) := \sup_{\substack{X_1, X_2 \in \mathcal{S}_B, \\ X_1 + X_2 = X}} \{U_1(X_1) + U_2(X_2)\}. \tag{3.4}$$

Suppose Assumptions 2.1 and 2.2 are satisfied w.r.to partition B. Then, from Theorem 3.6 we know that problem (3.4) admits solutions for any $\mathcal{F}_B$-measurable total risk, thus in particular for the $\mathcal{F}_A$-measurable r.v. X we have fixed. Let $(X_1, X_2) \in \mathcal{S}_B \times \mathcal{S}_B$ be such a solution. On the one hand, from equality (3.3) we have

$$\mathbb{E}_{\mathbb{P}_1}[X_1|\mathcal{F}_A] + \mathbb{E}_{\mathbb{P}_2}[X_2|\mathcal{F}_A] = \mathbb{E}_{\mathbb{P}_1}[X_1 + X_2|\mathcal{F}_A] = \mathbb{E}_{\mathbb{P}_1}[X|\mathcal{F}_A] = X.$$

On the other hand, by Proposition A.1, the $\mathbb{P}_i$-law-invariance of U_i implies $U_i(\mathbb{E}_{\mathbb{P}_i}[X_i|\mathcal{F}_A]) \geq U_i(X_i)$, so that $(\mathbb{E}_{\mathbb{P}_1}[X_1|\mathcal{F}_A], \mathbb{E}_{\mathbb{P}_2}[X_2|\mathcal{F}_A])$ is an $\mathcal{F}_A \times \mathcal{F}_A$-measurable allocation of X which is at least as good as (X_1, X_2). Thus we may assume that $(X_1, X_2) \in \mathcal{S}_A \times \mathcal{S}_A$. Hence,

$$U_1(X_1) + U_2(X_2) = U_B(X) \geq U_A(X) \geq U_1(X_1) + U_2(X_2),$$

so we obtain $U_A(X) = U_1(X_1) + U_2(X_2)$, i.e. the exactness of $u_1 \square u_2$ in (2.5). It remains to prove that Assumptions 2.1 and 2.2 are indeed satisfied by $u_i^B(x) \equiv U_i(\sum_{k=1}^{n} \sum_{j=1}^{m_k} x_{kj} 1_{B_{kj}})$, $x \in \mathbb{R}^M$, and $v_i^B = (u_i^B)^*$, $i = 1, 2$. To this end, note that, according to (2.6), by (3.2) we have

$$v_2^B(\mathbb{P}_1) = v_2^B(a, \ldots, a) = V_2\left(\sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{a}{\mathbb{P}_2(B_{kj})} 1_{B_{kj}}\right) = V_2\left(\sum_{k=1}^{n} \frac{m_k a}{\mathbb{P}_2(A_k)} 1_{A_k}\right) = v_2(\mathbb{P}_1),$$

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so $\mathbb{P}_1 \in dom(v_2^B)$ whenever $\mathbb{P}_1 \in dom(v_2)$. Moreover, it is easily verified that $y \in \partial v_2(\mathbb{P}_1)$ implies that the vector corresponding to $\sum_{k=1}^{n} \sum_{j=1}^{m_k} y_k 1_{B_{kj}}$ in $\mathbb{R}^M$ is an element of $\partial v_2^B(\mathbb{P}_1)$. Finally, $u_1 \square u_2(0) = 0$ implies $u_1^B \square u_2^B(0) = 0$, because otherwise there would be $Y \in \mathcal{S}_B$ such that $U_1(Y) + U_2(-Y) > 0$. But then again by Proposition A.1 and (3.3) we may assume that $(Y, -Y) \in \mathcal{S}_A \times \mathcal{S}_A$ which implies $u_1 \square u_2(0) > 0$ and this is a contradiction. Therefore, Assumptions 2.1 and 2.2 hold for the admissible partition B of Ω as well, and this concludes the proof. $\qquad\square$

4   Examples

In what follows we provide some examples of optimal risk sharing problems, including prominent choice functions such as the Entropic Utility and the Mean Variance Choice Function. One of the main ingredients we will use to study the existence of optimal allocations and, in case, to compute them explicitly, is Proposition A.2, which plays an important role whenever we have some information about the gradient of the convolution function.

In Examples 4.1, 4.2 the hypothesis of Theorem 2.3 are satisfied for every admissible partition of Ω, so there always exist Pareto optimal allocations, and we are able to compute them. Actually we can say even more: in these examples, we can formulate and solve the optimal risk sharing problem in continuous setting too, provided some strong link between the different world views.

Example 4.1. (Convolution of Entropic Utilities) The Entropic Utility w.r.to $\mathbb{P}_i$, with parameter $\beta > 0$ (see e.g. [20] and [21]) is given by

$$Entr_\beta^i(X) = -\beta \log \mathbb{E}_{\mathbb{P}_i}\left[e^{-\frac{X}{\beta}}\right] = \inf\left\{\mathbb{E}_{\mathbb{Q}}[X] + \beta H(\mathbb{Q}|\mathbb{P}_i) : \mathbb{Q} \ll \mathbb{P}_i\right\}, \quad X \in L^1(\Omega, \mathcal{F}, \mathbb{P}_i),$$

where, for any $\mathbb{Q} \ll \mathbb{P}_i$, $H(\mathbb{Q}|\mathbb{P}_i) = \mathbb{E}_{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}_i}\right)\right]$ denotes the relative entropy of $\mathbb{Q}$ w.r.to $\mathbb{P}_i$. The dual conjugate of $Entr_\beta^i$ is given by $V_\beta^i = \beta H(.|\mathbb{P}_i)$ on the set of probability measures $\mathbb{Q}$ s.t. $\mathbb{Q} \ll \mathbb{P}_i$, and $V_\beta^i = +\infty$ otherwise (see e.g. [18]). Let $\beta_1, \beta_2 > 0$ and $U_i = Entr_{\beta_i}^i$, $i = 1, 2$. Note that, for any admissible partition A of Ω, Assumptions 2.1 and 2.2 (ii) are satisfied. Hence, for every $X \in \mathcal{S}_A$ there is an optimal allocation $(X_1, X_2) \in \mathcal{S}_A \times \mathcal{S}_A$ and it is given in (4.2) below with $f = \frac{d\mathbb{P}_1}{d\mathbb{P}_2}|_{\mathcal{F}_A}$. However, for this kind of choice functions we can show even more. Suppose for the moment that $\mathbb{P}_1 \approx \mathbb{P}_2$ with density $\frac{d\mathbb{P}_1}{d\mathbb{P}_2}$ bounded and bounded away from 0. Then, $\forall p \in [1, \infty]$, $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P}_1) = L^p(\Omega, \mathcal{F}, \mathbb{P}_2)$ and the risk sharing problem is well defined on L^p as well, and formulated as

$$U_1 \square U_2(X) = \sup_{\substack{X_1, X_2 \in L^p, \\ X_1 + X_2 = X}} U_1(X_1) + U_2(X_2), \quad X \in L^p. \tag{4.1}$$

In the following we will show that (4.1) too admits solutions for every $X \in L^p$. Note that here we require a strong relation between the world views $\mathbb{P}_1$ and $\mathbb{P}_2$ which we do not in the discrete setting. Denote by f the density $\frac{d\mathbb{P}_1}{d\mathbb{P}_2}$ and consider the bidual $(U_1 \square U_2)^{**}$ of the convolution in (4.1) (see (B.2) and (B.6)):

$$
\begin{aligned}
(U_1 \square U_2)^{**}(X) &= \inf_{\mathbb{Q} \ll \mathbb{P}_i} \left\{ \mathbb{E}_\mathbb{Q}[X] + \beta_1 H(\mathbb{Q}|\mathbb{P}_1) + \beta_2 H(\mathbb{Q}|\mathbb{P}_2) \right\} \\
&= \inf_{\mathbb{Q} \ll \mathbb{P}_i} \left\{ \mathbb{E}_\mathbb{Q}[X] + \mathbb{E}_\mathbb{Q}\left[\beta_1 \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}_1}\right) + \beta_2 \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}_2}\right)\right] \right\} \\
&= \frac{\beta_1 + \beta_2}{\beta_1} \inf_{\mathbb{Q} \ll \mathbb{P}_i} \left\{ \mathbb{E}_\mathbb{Q}\left[\frac{\beta_1 X + \beta_1 \beta_2 \log(d\mathbb{P}_1/d\mathbb{P}_2)}{\beta_1 + \beta_2}\right] + \beta_1 \mathbb{E}_\mathbb{Q}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}_1}\right)\right] \right\} \\
&= \frac{\beta_1 + \beta_2}{\beta_1} U_1\left(\frac{\beta_1 X + \beta_1 \beta_2 \log(f)}{\beta_1 + \beta_2}\right) \\
&= \frac{\beta_1 + \beta_2}{\beta_2} U_2\left(\frac{\beta_2 X - \beta_1 \beta_2 \log(f)}{\beta_1 + \beta_2}\right).
\end{aligned}
$$

Now, for any aggregate risk X, by choosing

$$
X_1 := \frac{\beta_1 X + \beta_1 \beta_2 \log(f)}{\beta_1 + \beta_2}, \quad X_2 := X - X_1 = \frac{\beta_2 X - \beta_1 \beta_2 \log(f)}{\beta_1 + \beta_2}, \tag{4.2}
$$

we obtain $(U_1 \square U_2)^{**}(X) = U_1(X_1) + U_2(X_2)$, which in view of (B.3) implies $U_1 \square U_2(X) = U_1(X_1) + U_2(X_2)$. Therefore, the risk sharing problem (4.1) admits solutions for every $X \in L^p$, for any $p \in [1, \infty]$.

Clearly the very same computation can be done for the bidual of the convolution in the discrete setting (2.5), for any admissible partition A of Ω. Therefore, for any $X \in \mathcal{S}_A$, we have that an optimal allocation is given by (4.2), keeping in mind that in this case f means $\frac{d\mathbb{P}_1}{d\mathbb{P}_2}|_{\mathcal{F}_A}$. $\diamond$

Example 4.2. (Convolution of Mean Variance Choice Functions) The Mean Variance Choice Function w.r.to $\mathbb{P}_i$, with parameter $\gamma > 0$, is given by

$$
MV_\gamma^i(X) = \mathbb{E}_{\mathbb{P}_i}[X] - \gamma \mathbb{E}_{\mathbb{P}_i}[(X - \mathbb{E}_{\mathbb{P}_i}[X])^2], \quad X \in L^1(\Omega, \mathcal{F}, \mathbb{P}_i),
$$

and its dual by $V_\gamma^i(Z) = \frac{\mathbb{E}_{\mathbb{P}_i}[(Z-1)^2]}{4\gamma}$, for all $Z \in L^2(\Omega, \mathcal{F}, \mathbb{P}_i)$ with $\mathbb{E}_{\mathbb{P}_i}[Z] = 1$, and $V_\gamma^i(Z) = +\infty$ otherwise (see e.g. [1]). Let $\gamma_1, \gamma_2 > 0$ and $U_i = MV_{\gamma_i}^i$, $i = 1, 2$. As in the previous example, Assumptions 2.1 and 2.2 (ii) are satisfied for every admissible partition A of Ω, so the existence of optimal allocations follows by Theorem 2.3. Again, we can show even more. Indeed, if we consider the continuous setting described in Example 4.1, we can prove the exactness of the risk sharing problem in L^p as well, for all $p \in [1, \infty]$. In particular, by means of Proposition A.2, we have that an optimal allocation of any given total risk $X \in \mathcal{S}_A$ or $X \in L^p$ (i.e. when the problem is considered in discrete or in continuous setting) is given by

$$
X_1 := \frac{\gamma_2}{\gamma_2 + \gamma_1 f} X, \quad X_2 := \frac{\gamma_1 f}{\gamma_2 + \gamma_1 f} X,
$$

recalling the different meaning of f in the two cases. $\diamond$

The following example shows that, in general, we cannot expect existence of optimal allocations if Assumption 2.2 is not satisfied. Moreover, we present a continuous setting which does not allow for optimal allocations, whereas there are admissible partitions such that problem (2.4) admits solutions.

Example 4.3. Let $\mathbb{P}_1 \approx \mathbb{P}_2$ be such that $\mathbb{P}_2 \left(\frac{d\mathbb{P}_1}{d\mathbb{P}_2} = \frac{1}{2} \right) = \mathbb{P}_2 \left(\frac{d\mathbb{P}_1}{d\mathbb{P}_2} = \frac{3}{2} \right) = \frac{1}{2}$. Then, obviously, $L^p(\Omega, \mathcal{F}, \mathbb{P}_1) = L^p(\Omega, \mathcal{F}, \mathbb{P}_2) =: L^p$ for all $p \in [1, \infty]$. Now fix $p \in [1, \infty]$ and let

$$U_1(X) = \mathbb{E}_{\mathbb{P}_1}(X), \ X \in L^p \quad \text{and} \quad U_2(X) = \frac{1}{2} \left(\mathbb{E}_{\mathbb{P}_2}[X] + Entr_1^2(X) \right), \ X \in L^p.$$

The dual conjugate of U_1 is $V_1 = \delta(\cdot | \{\mathbb{P}_1\})$, which equals zero on $\mathbb{P}_1$ and $+\infty$ elsewhere. The dual of U_2 is given by

$$V_2(\mathbb{Q}) = \sup_{X \in L^p} \left\{ \mathbb{E}_{\mathbb{P}_2} \left[\left(\frac{1}{2} - Z_{\mathbb{Q}} \right) X \right] - \frac{1}{2} \log \mathbb{E}_{\mathbb{P}_2} \left[e^{-X} \right] \right\} = \frac{1}{2} \mathbb{E}_{\mathbb{P}_2}[(2Z_{\mathbb{Q}} - 1) \log(2Z_{\mathbb{Q}} - 1)]$$

if $Z_{\mathbb{Q}} := d\mathbb{Q}/d\mathbb{P}_2 \geq 1/2$, and $V_2(\mathbb{Q}) = +\infty$ elsewhere. Moreover, we have that $\partial V_2(\mathbb{Q}) = \{\log(2Z_{\mathbb{Q}} - 1) + c : c \in \mathbb{R}\}$ when well defined, and $\emptyset$ elsewhere. Hence, $\mathbb{P}_1 \in dom(V_2)$, but $\partial V_2(\mathbb{P}_1) = \emptyset$. Here the convolution in L^p leads to $U_1 \square U_2(X) = \mathbb{E}_{\mathbb{P}_1}(X) + V_2(\mathbb{P}_1)$, with $\partial U_1 \square U_2(X) = \{\mathbb{P}_1\} \ \forall X \in L^p$. Therefore, by Proposition A.2 there are no solutions to the risk sharing problem in L^p.

Now fix any admissible partition $A = \{A_1, \ldots, A_n\}$ of Ω. By (2.6) we have that

$$dom(v_2) = \left\{ z \in \mathbb{R}^n : \sum_{i=1}^n z_i = 1, z_i \geq \frac{\mathbb{P}_2(A_i)}{2} \ \forall i = 1, \ldots, n \right\}$$

and, for $z \in dom(v_2)$,

$$v_2(z) = \frac{1}{2} \sum_{i=1}^n \left(2z_i - \mathbb{P}_2(A_i) \right) \log \left(\frac{2z_i}{\mathbb{P}_2(A_i)} - 1 \right).$$

Thus $\partial v_2(z) \neq \emptyset$ if and only if $2z_i > \mathbb{P}_2(A_i)$ for all $i = 1, \ldots, n$, and, provided this holds,

$$\partial v_2(z) = \left\{ \left(\log \left(\frac{2z_1}{\mathbb{P}_2(A_1)} - 1 \right) + c, \ldots, \log \left(\frac{2z_n}{\mathbb{P}_2(A_n)} - 1 \right) + c \right) : c \in \mathbb{R} \right\}. \tag{4.3}$$

Since $\frac{d\mathbb{P}_1}{d\mathbb{P}_2} |_{\mathcal{F}_A} \geq \frac{1}{2}$, we always have $\mathbb{P}_1 \in dom(v_2)$. Moreover, by Proposition A.2 in conjunction with the fact that $\partial u_1 \square u_2(x) = \{\mathbb{P}_1\}$ for all $x \in \mathbb{R}^n$, we obtain that $u_1 \square u_2$ is exact if and only if condition (ii) of Assumption 2.2 holds. Hence, if there is one $j \in \{1, \ldots, n\}$ such that $A_j \subseteq \left\{ \frac{d\mathbb{P}_1}{d\mathbb{P}_2} = \frac{1}{2} \right\}$ a.s., then $\partial v_2(\mathbb{P}_1) = \emptyset$, so there are no optimal allocations. Otherwise, if there is no $j \in \{1, \ldots, n\}$ such that $A_j \subseteq \left\{ \frac{d\mathbb{P}_1}{d\mathbb{P}_2} = \frac{1}{2} \right\}$ a.s., then Assumptions 2.1 and 2.2 are satisfied, and according to (4.3) the optimal risk allocations of any $X \in \mathcal{S}_A$ are given by

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$(X - Y, Y)$ with $Y = -\log(2 \cdot \frac{d\mathbb{P}_1}{d\mathbb{P}_2}|_{\mathcal{F}_A} - 1) + c$, $c \in \mathbb{R}$. Note how the share of agent 2 does not depend on the total risk X. Indeed, what she takes is some measure of the difference between the world views. Clearly, this is equal to zero when $\frac{d\mathbb{P}_1}{d\mathbb{P}_2}|_{\mathcal{F}_A} \equiv 1$. $\diamond$

In Example 4.4 we motivate that the smaller the set of base scenarios, the more likely are we to find optimal solutions to the risk sharing problem. Obviously, if there is only one base scenario, i.e. if only cash is exchanged, then, due to cash-invariance, any allocation $(x - y, y)$, $y \in \mathbb{R}$, is an optimal allocation of $x \in \mathbb{R}$.

Example 4.4. Let $A = \{A_1, A_2, A_3\}$ be a $\mathbb{P}_1$-balanced partition of Ω and

$$\mathcal{C} := \left\{ \mathbb{Q} \ll \mathbb{P}_1 : \frac{d\mathbb{Q}}{d\mathbb{P}_1} = \sum_{i=1}^{3} 3a_i 1_{A_i}, \; \sum_{i=1}^{3} (a_i - \frac{1}{3})^2 \leq \frac{1}{6} \right\}.$$

Let $\Pi : \mathcal{C} \to \mathbb{R}^3$ be given by $\Pi(\mathbb{Q}) = (\mathbb{Q}(A_1), \mathbb{Q}(A_2), \mathbb{Q}(A_3))$ and consider the following function

$$\alpha(y) = \frac{1}{\sqrt{6}} - \left(\frac{1}{6} - \sum_{i=1}^{3} (y_i - \frac{1}{3})^2 \right)^{\frac{1}{2}}, \quad y \in \mathcal{B} := \left\{ z \in \mathbb{R}^3 : \sum_{i=1}^{3} (z_i - \frac{1}{3})^2 \leq \frac{1}{6} \right\}.$$

Note that α is not (sub)differentiable on the boundary of $\mathcal{B}$. Let U_1 be given by

$$U_1(X) := \inf_{\mathbb{Q} \ll \mathbb{P}_1} E_{\mathbb{Q}}[X] + V_1(\mathbb{Q}), \quad X \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}_1),$$

where $V_1(\mathbb{Q}) = \alpha(\Pi(\mathbb{Q})) + \delta(\mathbb{Q}|\mathcal{C})$. Moreover, let $\mathbb{P}_2$ be the probability measure given by $\frac{d\mathbb{P}_2}{d\mathbb{P}_1} = 2 1_{A_1} + \frac{1}{2} 1_{A_2} + \frac{1}{2} 1_{A_3}$ and let $U_2(X) := \mathbb{E}_{\mathbb{P}_2}(X)$, $X \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}_2)$. Suppose that agents 1 and 2, endowed with choice functions U_1 and U_2, agree on the base scenarios $\{A_1, A_2, A_3\}$. Then we have

$$u_1^A(x) = \inf_{z \in \mathbb{R}^3} \langle z, x \rangle + v_1^A(z) \quad \text{and} \quad u_2^A(x) = \frac{2}{3} x_1 + \frac{1}{6} x_2 + \frac{1}{6} x_3, \quad x \in \mathbb{R}^3,$$

where $v_1^A = \alpha + \delta(\cdot | \Pi(\mathcal{C}))$. Note that $\mathbb{P}_2 = (2/3, 1/6, 1/6) \in dom(v_1^A)$. Thus

$$u_1^A \square u_2^A(x) = u_2^A(x) + v_1^A(\mathbb{P}_2) \quad \text{and} \quad \partial u_1^A \square u_2^A(x) = \{\mathbb{P}_2\}, \; \forall x \in \mathbb{R}^3.$$

Since $(2/3, 1/6, 1/6)$ is a boundary point of $\mathcal{B}$, we have that $\partial v_1^A(\mathbb{P}_2) = \emptyset$ and then, by Proposition A.2, there are no optimal allocations (note that (NRA) is not satisfied because $v_1^A(\mathbb{P}_2) = 1/\sqrt{6}$). However, if we suppose that the agents agree on the scenarios $\widetilde{A} = \{A_1 \cup A_2, A_3\}$, then we obtain that

$$u_1^{\widetilde{A}}(x) = \sup_{z \in \mathbb{R}^2} \langle z, x \rangle + v_1^{\widetilde{A}}(z), \quad x \in \mathbb{R}^2,$$

where $v_1^{\widetilde{A}}(z) = v_1^A(\frac{z_1}{2}, \frac{z_1}{2}, z_2)$, and $u_2^{\widetilde{A}}(x) = \frac{5}{6} x_1 + \frac{1}{6} x_2$. Again $\mathbb{P}_2 = (5/6, 1/6) \in dom(v_1^{\widetilde{A}})$, but this time $\partial v_2^{\widetilde{A}}(5/6, 1/6) \neq \emptyset$ because α is differentiable at $(\frac{5}{12}, \frac{5}{12}, \frac{1}{6})$. Hence there are optimal allocations for any total risk $X \in \mathcal{S}_{\widetilde{A}}$. $\diamond$

Remark 4.5. Note that in Examples 4.1,4.2 the optimal allocations are obtained as functions of the total risk and of the density function. Indeed, with the same arguments as in the proof of Theorem 2.3, for any total risk X and any allocation (X_1, X_2) of X, we have that the allocation

$$(\mathbb{E}_{\mathbb{P}_1}[X_1|\sigma(X, f)], \mathbb{E}_{\mathbb{P}_2}[X_2|\sigma(X, f)])$$

of X outperforms (X_1, X_2), thus the optimization problem can be formulated over the $\sigma(X, f)$-measurable allocations. $\diamond$

As we have seen, in some circumstances it is possible to treat some continuous cases together with the discrete ones and use Proposition A.2 to possibly characterize optimal allocations. Clearly this is not the general situation (see e.g. Examples 4.3,4.4). However, there are cases where, even if we cannot state the existence of solutions or we are not able to compute them, our results can be used for approximations. Indeed, assume the density $d\mathbb{P}_1/d\mathbb{P}_2$ to be simple, $p \in [1, \infty)$ and the choice functions $U_i : L^p \to [-\infty, \infty)$ to satisfy (C1)–(C5) on L^p. Assumptions 2.1 and 2.2 here become $\mathbb{P}_1 \in dom(V_2)$, and $U_1 \square U_2(0) = 0$ or $\partial V_2(\mathbb{P}_1) \neq \emptyset$ respectively, with V_2 conjugate of U_2. Let U_1 be monotone and continuous, which ensures $U_1 \square U_2$ to be continuous as well. Fix $X \in L^p$ and consider simple r.v.'s $\{X_n\}_{n \in \mathbb{N}}$ converging to X in the L^p-norm. According to Remark 4.5, we obtain that the optimization problem in L^p is exact at X_n, with some simple optimal allocation (Y_1^n, Y_2^n), for all $n \in \mathbb{N}$. Therefore, by continuity of U_1 and $U_1 \square U_2$, we have $U_1(X - Y_2^n) + U_2(Y_2^n) \xrightarrow[n \to +\infty]{} U_1 \square U_2(X)$, i.e., the optimal value can be approximated by means of solutions to discrete problems. What is worth noticing here is the fact that we approach the optimal value by allocations that in many cases can be explicitly computed by means of standard algorithms (for convex optimization in $\mathbb{R}^n$).

A Some Useful Results

In this section we collect some known results which are used throughout the paper.

Proposition A.1 ([18],[15])**.** *Let $(\widetilde{\Omega}, \mathcal{G}, \mathbb{P})$ be a non-atomic standard probability space, $p \in [1, \infty]$ and $U : L^p(\widetilde{\Omega}, \mathcal{G}, \mathbb{P}) \to [-\infty, +\infty)$ be a $\mathbb{P}-law\text{-}invariant$ proper concave u.s.c. function. Then, for any sub-σ-algebra $\mathcal{B} \subseteq \mathcal{G}$,*

$$U(\mathbb{E}_{\mathbb{P}}[X|\mathcal{B}]) \geq U(X), \quad \forall X \in L^p(\widetilde{\Omega}, \mathcal{G}, \mathbb{P}).$$

In particular, $U(\mathbb{E}_{\mathbb{P}}(X)) \geq U(X)$.

Proposition A.2 ([24])**.** *Let $(\widetilde{\Omega}, \mathcal{G}, \mathbb{P})$ be a probability space, $p \in [1, \infty]$ and $U_1, U_2 : L^p(\widetilde{\Omega}, \mathcal{G}, \mathbb{P}) \to [-\infty, \infty)$ be proper concave u.s.c. functions. Then, for $X \in L^p(\widetilde{\Omega}, \mathcal{G}, \mathbb{P})$ s.t.*

$\partial U_1 \square U_2(X) \neq \emptyset$ and for any allocation $(X_1, X_2) \in L^p(\widetilde{\Omega}, \mathcal{G}, \mathbb{P}) \times L^p(\widetilde{\Omega}, \mathcal{G}, \mathbb{P})$ of X,

$$U_1 \square U_2(X) = U_1(X_1) + U_2(X_2) \iff \partial U_1 \square U_2(X) = \partial U_1(X_1) \cap \partial U_2(X_2).$$

B   Some Functional Analysis

In this section we recall some well-known concepts and results from convex analysis. Let $\mathcal{H}$ be some locally convex space. Given a concave function $\varphi : \mathcal{H} \to [-\infty, +\infty)$, its conjugate $\varphi^* : \mathcal{H}^* \to [\varphi(0), +\infty]$ is defined as

$$\varphi^*(\mu) := \sup_{X \in \mathcal{H}} \{\varphi(X) - \langle \mu, X \rangle\}, \tag{B.1}$$

and it is a convex and $\sigma(\mathcal{H}^*, \mathcal{H})$-lower semi-continuous function on $\mathcal{H}^*$ which is proper if and only if φ is proper. The conjugate $\varphi^{**} : \mathcal{H} \to [-\infty, +\infty)$ of φ^* is

$$\varphi^{**}(X) := \inf_{\mu \in \mathcal{H}^*} \{\varphi^*(\mu) + \langle \mu, X \rangle\}, \tag{B.2}$$

and it is a concave, $\sigma(\mathcal{H}, (\mathcal{H}^*)$-u.s.c. function on $\mathcal{H}$, with

$$\varphi^{**} \geq \varphi, \tag{B.3}$$

and $\varphi^{**} = \varphi$ if and only if φ is proper and $\sigma(\mathcal{H}, \mathcal{H}^*)$-u.s.c. or $\varphi = \pm\infty$.

The gradient of φ at $X \in \mathcal{H}$ is given by

$$\partial\varphi(X) = \{\mu \in \mathcal{H}^* : \varphi(Y) \leq \varphi(X) + \langle \mu, Y - X \rangle, \forall Y \in \mathcal{H}\}$$

and the gradient of φ^* at $\mu \in \mathcal{H}^*$ by

$$\partial\varphi^*(\mu) = \{X \in \mathcal{H} : \varphi^*(\nu) \geq \varphi^*(\mu) + \langle \nu - \mu, X \rangle, \forall \nu \in \mathcal{H}^*\}.$$

For φ proper, concave and u.s.c., the following chain of equivalences holds for any pair $(X, \mu) \in \mathcal{H} \times \mathcal{H}^*$:

$$\mu \in \partial\varphi(X) \iff X \in -\partial\varphi^*(\mu) \iff \varphi(X) = \varphi^*(\mu) + \langle \mu, X \rangle. \tag{B.4}$$

Let $\phi : \mathcal{H} \to [-\infty, +\infty)$ be another concave function. The convolution $\varphi \square \phi$ of φ and ϕ is given by

$$\varphi \square \phi(X) := \sup_{Y \in \mathcal{H}} \varphi(X - Y) + \phi(Y). \tag{B.5}$$

It inherits concavity from φ and ϕ. Moreover, its conjugate satisfies

$$(\varphi \square \phi)^* = \varphi^* + \phi^*. \tag{B.6}$$

In particular, if the convolution function is proper and u.s.c., then by (B.2) it admits the following dual representation

$$\varphi \square \phi(X) = \inf_{\mu \in \mathcal{H}^*} \{\varphi^*(\mu) + \phi^*(\mu) + \langle \mu, X \rangle\}. \tag{B.7}$$

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