AVI as a mechanical tool for studying dynamic and static Euler-Bernoulli beam structures

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Abstract. The equilibrium position of an Euler-Bernoulli beam is investigated. Using the discrete Euler-Lagrange and Lagrange-d'Alembert principles, the behavior of the beam is simulated using variational integrators and, in particular, AVIs (Asynchronous Variational Integrators). Special emphasis is placed on the discrete mechanics and geometric structure underlying stress resultant beam models.

Keywords: Variational integrators, AVI, discrete mechanics, Euler-Bernoulli beam, B-spline.
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INTRODUCTION

Discrete variational mechanics and associated numerical integrators have seen a major development in recent years. There has been a growing realization that stability of numerical methods can be improved for certain systems by algorithms that are compatible with variational and geometric structures, such as preservation of the symplectic form on phase space and of the momentum maps arising from the symmetries of the system. Discrete mechanics has been developed as a result of the interplay of classical theoretical mechanics, numerical analysis, and computer science, which has become increasingly important in concrete applications. Remarkably, to our knowledge, there is no major application of these discrete mechanics techniques to civil engineering.

The longer term aim of this work is to apply structure preserving algorithms to concrete problems in construction, like thin-shells with irregular surfaces. Our goal in this paper is to provide variational integrators associated with mathematical models of beams and to carry out dynamic and static two-dimensional simulations.

DEFORMATIONS AND MECHANICS

Notations and definitions

In this paper we shall regard a body $\mathscr{B} \subseteq \mathbf{R}^3$ as a smooth orientable Riemannian manifold endowed with a metric $\mathbf{G}$. The space $\mathscr{S} \subseteq \mathbf{R}^3$ in which the body moves is also taken to be a smooth orientable Riemannian manifold with a metric $\mathbf{g}$. The deformed body $\varphi(\mathscr{B})$ inherits the Riemannian structure of $\mathscr{S}$. We shall call $\mathscr{B}$ the *reference configuration* and $\mathscr{S}$ the *ambient space*.

Let $T\mathscr{B}$, $T\mathscr{S}$ be the tangent bundles of $\mathscr{B}$ and $\mathscr{S}$, respectively, and let $T^*\mathscr{B}$, $T^*\mathscr{S}$ be their cotangent bundles. So we think of $\mathscr{B}$ and $\mathscr{S}$ as manifolds with a vector space attached at each point.

Let $\{Z^I\}$ denote the Euclidean coordinates of a point $X \in \mathscr{B}$ relative to the standard basis $\{\widehat{\mathbf{I}}_I\}$ of $\mathbf{R}^3$. Similarly, $\{z^i\}$ are the Euclidean coordinates of a point $x \in \mathscr{S}$ relative to the standard basis $\{\widehat{\mathbf{i}}_i\}$ of $\mathbf{R}^3$. Thus, the coordinate system $\{x^i\}$ on $\mathscr{S}$ is a C^∞ map

$$\left(z^1, z^2, z^3\right) \longmapsto \left(x^1\left(z^1, z^2, z^3\right), x^2\left(z^1, z^2, z^3\right), x^3\left(z^1, z^2, z^3\right)\right)$$

with C^∞ inverse. The coordinate bases $\{\mathbf{E}_I(X)\}$, $\{\mathbf{e}_i(x)\}$ associated to the coordinates $\{X^I\}$, $\{x^i\}$ are defined by

$$\mathbf{E}_I = \frac{\partial Z^J}{\partial X^I}\widehat{\mathbf{I}}_J, \qquad \mathbf{e}_i = \frac{\partial z^j}{\partial x^i}\widehat{\mathbf{i}}_j, \qquad I, J, i, j = 1, 2, 3.$$

A *configuration* $\phi : \mathscr{B} \to \mathscr{S}$ is, by definition, an orientation preserving diffeomorphism between $\mathscr{B}$ and its embedded image $\phi(\mathscr{B}) \subseteq \mathscr{S}$. The *configuration space* is defined to be $\mathscr{C} := \{\phi : B \to S \mid \phi \text{ a } C^\infty \text{ embedding }\}$. A *motion* of the body is a curve $t \in \mathbf{R} \mapsto \phi_t \in \mathscr{C}$, where $\phi_t(X) = \phi(X,t)$ for $t \in \mathbf{R}$ fixed, and $\phi_0 = \text{Identity}$. This motion is assumed to occur due to the action of body forces $\mathbf{b}$ per unit mass and surface traction forces $\mathbf{t}$ per unit area of the boundary $\partial\mathscr{B}$.

In order to characterize the deformation, we consider $\mathbf{F} = T\phi_t : T\mathscr{B} \to T\mathscr{S}$, the *deformation gradient*, which is a vector bundle map, that is, its restriction to each tangent space $T_X\mathscr{B}$ is a linear map $T_X\phi_t : T_X\mathscr{B} \to T_{\phi_t(X)}\mathscr{S}$. The *Cauchy-Green deformation tensor* is defined by $\mathbf{C}(X) := \mathbf{F}(X)^T\mathbf{F}(X)$ and the *Lagrangian strain* by $\mathbf{E} := (\mathbf{C} - \mathbf{G})/2$.

We postulate the existence of a vector field $\mathbf{t}(x,t,\mathbf{n})$, the *Cauchy stress vector*, depending on time t, the spatial point x, and a unit vector $\mathbf{n}$ normal to a surface element. Due to the Cauchy stress theorem and *balance of momentum*, $\mathbf{t}(x,t,\mathbf{n}) = \langle \sigma(x,t), \mathbf{n} \rangle$, where σ is *Cauchy stress tensor*, and the symbol $\langle\, , \rangle$ is the natural duality pairing.

The *Lagrangian* $L : T\mathscr{S} \to \mathbf{R}$ is defined on tangent bundle $T\mathscr{S}$, and *Hamiltonian* $H : T^*\mathscr{S} \to \mathbf{R}$ on cotangent bundle $T^*\mathscr{S}$. To pass from the Lagrangian description to the Hamiltonian formulation of the problem, we use the *Legendre transformation*

$$\mathbf{F}L : T\mathscr{S} \to T^*\mathscr{S}, \quad (x,\dot{x}) \mapsto (x,p) = \left(x, \frac{\partial L}{\partial \dot{x}}\right).$$

The Lagrangian L is said to be *regular* if $\mathbf{F}L$ is locally an isomorphism of vector bundles. If L is the difference between a kinetic and a potential energy, $\mathbf{F}L$ is globally invertible.

The *Euler-Lagrange equations*

$$\frac{\partial L}{\partial x^i}(x,\dot{x}) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}(x,\dot{x})\right) = 0.$$

are a consequence of *Hamilton's variational principle*

$$\delta \int_0^T L(x,\dot{x},t)dt = 0,$$

over all trajectories $x(t)$ is joining two given points in $\mathscr{S}$ over a fixed time interval $[0,T]$. Under suitable smoothness assumptions, the converse is also true.

The discrete point of view [1], [2]

Given a smooth regular Lagrangian L, two points $x_0, x_1 \in \mathscr{S}$, and a time h, if x_1 and x_0 are close and $|h|$ is sufficiently small then there exists a unique trajectory $x : \mathbf{R} \to \mathscr{S}$ which is a solution of the Euler-Lagrange equations for L satisfying $x(0) = x_0$ and $x(h) = x_1$ (see e.g. [3]). Given this trajectory x, define an *exact discrete Lagrangian* by

$$L_d^E(x_0,x_1,h) = \int_0^h L(x,\dot{x},t)dt,$$

for x_0 and x_1 close and h sufficiently small. However, the exact discrete Lagrangian is generally hard to compute. In order to avoid this problem, we approximate the exact discrete Lagrangian L_d^E by a discrete Lagrangian L_d satisfying

$$L_d\,(x_0,x_h,h) = L_d^E\,(x_0,x_h,h) + O(h^r);$$

L_d is said to approximate L_d^E to order r.

Let $\phi(\mathscr{B}) \times \phi(\mathscr{B})$ be the discrete configuration space associated to the deformed body $\phi(\mathscr{B})$ and define the discrete path space by $\mathscr{C}_d(\phi(\mathscr{B})) := \{x_d = \{x_k\}_{k=0}^N \mid x_k \in \phi(\mathscr{B}),\ x_k = x_d(t_k),\ t_k = kh,\ t_k \in [0,T]\}$ with time step h. A discrete path $x_d \in \mathscr{C}_d$ is said to be a solution of the discrete Euler-Lagrange equations if

$$D_2L_d(x_{k-1},x_k,h) - D_1L_d(x_k,x_{k+1},h) = 0, \tag{1}$$

where D_1 and D_2 denote the first and second partial derivatives of L_d.

The points $\{x_k\}$ are iteratively defined by the *one-step integrator* $F_{L_d} : (x_{k-1},x_k) \mapsto (x_k,x_{k+1})$ which has two important structure preserving properties. First, F_{L_d} is *symplectic* which implies area preservation in phase-space or, more precisely, $\left(F_{L_d}\right)^* \Omega_{L_d} = \Omega_{L_d}$, where Ω_{L_d} is a discrete Lagrangian 2-form. Second, if L_d is invariant under

Lie algebra action, the discrete Lagrangian momentum map J_{L_d} is a conserved quantity: $J_{L_d} \circ F_{L_d} = J_{L_d}$. Examples of momentum maps are the classical linear and angular momentum.

In order to achieve conservation of energy we also consider the time interval $[0,T]$ and define the *extended configurations* by $\widetilde{\varphi} : \widetilde{\mathscr{B}} \to \widetilde{\mathscr{S}}$, where $\widetilde{\mathscr{B}} := \mathbf{R} \times \mathscr{B}$, $\widetilde{\mathscr{S}} := \mathbf{R} \times \mathscr{S}$, and $\mathbf{R}$ is time axis. Thus we get a new condition that ensures conservation of discrete energy:

$$D_3 L_d(x_{k-1}, x_k, h_{k-1}) - D_3 L_d(x_k, x_{k+1}, h_k) = 0, \quad \text{where} \;\; h_k = t_{k+1} - t_k \tag{2}$$

Consequently, we get an implicit algorithm giving the value of the time step h_k for each k; the integrator is said to be an *Asynchronous Variational Integrator* (AVI). However, the implicit algorithm (2) is not always very easy to calculate. Fortunately, the total energy oscillates around a constant value for very long time without overall growth or decay, if we choose a particular time step for each cell, in relation both with the geometry of the mesh, and with material properties. In addition, convergence of the AVI for time steps vanishing toward zero is known for fixed spatial discretization (see [1]).

Application of AVI to triangulation of the reference configuration $\mathscr{B}$

Given a potential energy $V(x)$, we choose the discrete Lagrangian

$$L_d(x_0, x_1, h) = h \left(\frac{1}{2} \left(\frac{x_1 - x_0}{h} \right)^T M \left(\frac{x_1 - x_0}{h} \right) - V(x_0) \right),$$

which gives an explicit integrator, where M is a symmetric, positive definite matrix. Let the triangulation $\mathscr{T}$ of $\mathscr{B}$ be composed of cells K with nodes a of mass $m_{a,K}$ such that $\sum_{a \in K} m_{a,K} = M_K$, where M_K is the mass of K. The vector positions of K and a at times $t_{K,j}$ and $t_{a,i}$ are denoted by $x_{K,j}$ and $x_{a,i}$. In this way, we get one-step integrator (1) giving the speed $v_{a,i}$ of node a at time $t_{a,i} = t_{K,j}$

$$v_{a,i} = \left(\frac{x_{a,i+1} - x_{a,i}}{t_{a,i+1} - t_{a,i}} \right) = \left(\frac{x_{a,i} - x_{a,i-1}}{t_{a,i} - t_{a,i-1}} \right) - \frac{h_K}{m_a} \frac{\partial V_K}{\partial x_{K,j}} (x_{K,j}), \tag{3}$$

with time step h_K and the potential energy V_K of the cell K.

EULER-BERNOULLI BEAM

For the Euler-Bernoulli beam the assumptions of infinitesimal deformations are applied and the material is assumed to be hyperelastic and isotropic. Assume that the configuration ϕ is of the form

$$\phi : \mathscr{B} \to \phi(\mathscr{B}) \subset \mathscr{S}, \quad \mathbf{X} \mapsto \mathbf{x} = \phi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}),$$

where $\mathbf{u}(\mathbf{X})$ is the displacement. The cross-section $\mathscr{A} \subset \mathbf{R}^2$ of the beam is an open set with compact closure. Let $\{\xi^i\}$ be the coordinate system associated to the standard basis $\{\widehat{\mathbf{I}}_i\}$ in $\mathscr{B}$. Points in $\mathscr{A}$ are written as $\mathbf{s} = \xi^1 \widehat{\mathbf{I}}_1 + \xi^2 \widehat{\mathbf{I}}_2$ in the reference configuration. Let $\{\mathbf{d}_I\}$ be the standard orthonormal basis of $\mathscr{S} = \mathbf{R}^3$ after deformation. The cross-section is now determined by $\mathbf{d}_1(\xi^3)$ and $\mathbf{d}_2(\xi^3)$. The spatial representation of the beam is $\mathbf{x} = \phi(\mathbf{X}) = \varphi_0(\xi^3) + \xi^1 \mathbf{d}_1(\xi^3) + \xi^2 \mathbf{d}_2(\xi^3)$, where $\xi^3 \in I \subset \mathbf{R} \mapsto \varphi_0(\xi^3) \in \mathbf{R}^3$ is the line of centroids. By assumption, the cross-section remains planar and normal to the line of centroids upon bending the Bernoulli beam. For small deformations, with slope θ of the cross-section, we take the displacement $\mathbf{u}(\mathbf{X})$ to have the form

$$u^1\left(\xi^1, \xi^2, \xi^3\right) = \text{const}, \qquad u^2\left(\xi^1, \xi^2, \xi^3\right) = u^2\left(\xi^3\right), \qquad u^3\left(\xi^1, \xi^2, \xi^3\right) \approx -\xi^2 \theta = -\xi^2 \frac{\partial u^2\left(\xi^3\right)}{\partial \xi^3}. \tag{4}$$

The local internal energy $W(\mathbf{e})$ depends only on the strain tensor $\mathbf{e} = \frac{1}{2}\left(\nabla \mathbf{u}(\mathbf{X}) + (\nabla \mathbf{u}(\mathbf{X}))^{\mathsf{T}}\right)$, the linearization of the Lagrangian strain tensor $\mathbf{E}$. To get the Cauchy stress tensor σ we use Young's modulus E which provides a relationship between the applied axial stress and the measured strain as $\sigma = E\mathbf{e}$. The bending energy function of the displacement field $\mathbf{u}_K$ for a beam element K of volume V and length l is given by

$$U(\mathbf{u}_K) = \frac{1}{2}\int_V \sigma_{33} e_{33} dv = \frac{1}{2}\int_0^l EI \left(\frac{\partial^2 u^2}{\partial \xi^3 \partial \xi^3}\right)^2 d\xi^3,$$

where E is Young's modulus and I is the second moment of the cross section.

The AVI method is used to carry out dynamic and static two-dimensional simulations. For static simulations we consider a non-conservative system with dissipation to find the equilibrium position and use the discrete version of the Lagrange-d'Alembert principle. The associated forced discrete Euler-Lagrange equations are then

$$D_2 L_d(q_k, q_{k+1}) + D_1 L_d(q_{k+1}, q_{k+2}) + F_d^+(q_k, q_{k+1}) + F_d^-(q_{k+1}, q_{k+2}) = 0, \quad (5)$$

where F_d^- and F_d^+ are left and right discrete forces.

In numerical experiments we used two kinds of shape functions: Hermitian cubic shape functions and cubic B-splines. This second choice prepares for future work on thin-shells [4]. We compare our experimental results to analytical results for several boundary conditions (simply supported, fixed-ends, cantilever), different lengths of the beam (0.1 to 3 meters), different thickness and materials, all in the framework of Euler-Bernoulli beam hypotheses. As displayed in the limited set of results presented in Figure 1, simulated beams at equilibrium are very close to analytical results. The amplitude behavior shown in Figure 2 hints at good conservation of energy by the AVI integrator. Several improvements, such as the understanding of the influence of several possible simplex mass matrices or the precise calculation of total energy of the systems in the context of asynchronous time stepping, are currently under investigation.

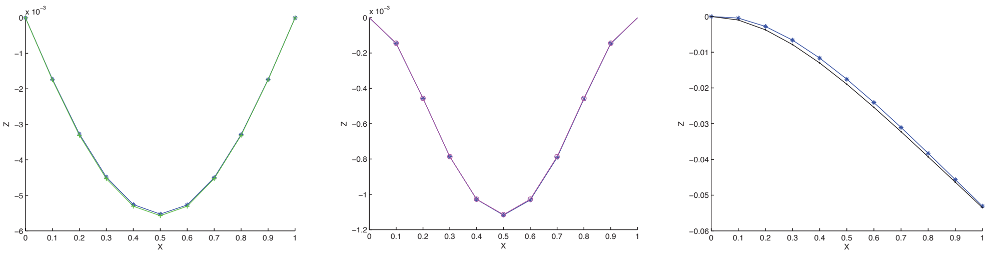

FIGURE 1. Position at equilibrium of a simulated Bernoulli beam using the AVI framework with dissipative Lagrangian and quadratic B-spline shape functions. From left to right: simply-supported beam, fixed-ends beam, cantilevered beam; $L = 1m$, $h = 0.01m$, width$= 0.02m$, $E = 1.1E09$, 10 elements.

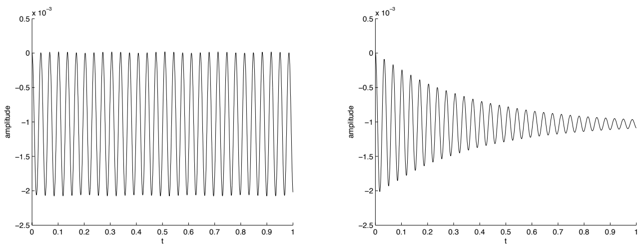

FIGURE 2. Amplitude of a B-spline control point for (left) non-dissipative Lagrangian, and (right) dissipative Lagrangian

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